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Bifurcation structure and Lyapunov exponents of a modulated logistic map

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Abstract. We have studied the bifurcation structure of the logistic map with a time dependant control parameter. By introducing a specific nonlinear variation for the parameter, we show that the bifurcation structure is modified qualitatively as well as quantitatively from the first bifurcation onwards. We have also computed the two Lyapunov exponents of the system and find that the modulated logistic map is less chaotic compared to the logistic map.

Keywords. Modulated logistic map; period-doubling; bifurcation structure; Lyapunov exponents.

PACS Nos 05·40; 02·50

1. Introduction

Nonlinear difference equations have become effective mathematical models in our understanding of the transition from regular to chaotic behaviour in physical systems (Feigenbaum 1978, 1979; Bai-Lin 1984). Several examples are known in which a system shows the period doubling route to chaos (May 1976), where the details of the transition from periodic to chaotic behaviour is represented by a bifurcation structure. Kapral and Mandel (1985) studied the bifurcation structure of a nonautonomous quadratic map in which the control parameter is assumed to vary linearly in time, and showed that the time dependence delayed the onset of bifurcations in the system.

In this paper, we study the bifurcation structure of the logistic map with the control parameter λ made a function of time. In particular, we have considered a situation where the value of λ at any instant depends on its value at the previous instant in a nonlinear way. This map shows some very interesting features which are absent in the case of the logistic map. Now, there are several experimental situations where the parameter is made time-dependent and this time dependence has been shown to induce dramatic changes in the bifurcation diagram (Mandel and Erneux 1984). Moreover, during the last few years, there has been an increased interest in the study of different kinds of laser systems by modulating one of its physical parameters (Tredicce *et al* 1985). Several authors have analysed the bifurcation structure and have shown that the output can be periodic as well as chaotic depending on the strength of the modulation (Brun *et al* 1984; Midavaine *et al* 1985).

In § 2 we introduce a modulated logistic map, compute its bifurcation structure and discuss the new features that appear in the diagram. In § 3 we obtain analytically the fixed points of the map up to the 2-cycle and perform a linear stability analysis to confirm some of our numerical results. In § 4, we analyse the chaotic region of the map by computing the two Lyapunov exponents of the system. Discussions are presented in § 5.

2. A modulated logistic map: bifurcation structure

We consider the following two-dimensional map:

$$X_{t+1} = 4\lambda_t X_t(1 - X_t),$$
$$\lambda_{t+1} = 4\mu\lambda_t(1 - \lambda_t). \tag{1}$$

Here, essentially we modulate the parameter of the logistic map. Now, the role of the control parameter is played by μ, which represents the strength of the modulation. We can consider (1) as a map in the space $\mathbf{R}^1 \otimes \mathbf{R}^1$ and we call it a modulated logistic map.

For $0 < \mu < 0.75$ we have a single attracting fixed point for λ_t and hence for X_t also. Therefore, in order to determine the bifurcation structure, we start from the parameter value $\mu = 0.7$ and increase it by steps of 0·01, always using an initial condition for (X_t, λ_t) in the interval [0, 1], say (0·3, 0·3). The important asymptotic values for λ_t, X_t and the corresponding parameter μ are collected in table 1.

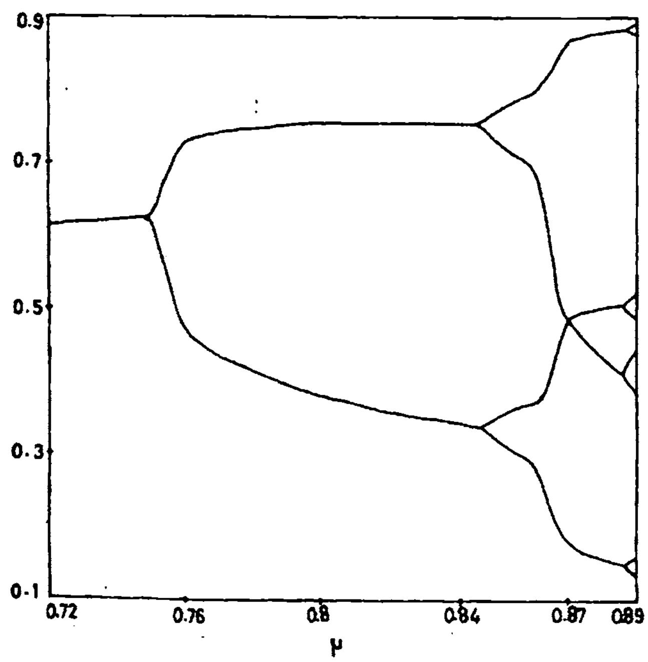

Figure 1. Bifurcation structure of the modulated logistic map projected in the (X, μ) plane. The branches cross over each other in the 4-cycle region resulting in a complete modification of the structure. Note that the second bifurcation is much earlier than in the case of the logistic map and also the asymmetry of the figure.

Table 1. Asymptotic values of λ_t and X_t.

μ	λ_t	X_t
0·7	0·6428571	0·6111111
0·74	0·6621622	0·6224488
0·75	0·6683250	0·6202936
	0·6649998	0·6296404
0·78	0·5475750	0·7489202
	0·7729381	0·4118614
0·8	0·5130444	0·7539450
	0·7994550	0·3807031
0·84	0·4623756	0·7534620
	0·8352432	0·3435580
		0·7534756
		0·3435452
0·85	0·4519632	0·7340323
	0·8421544	0·3529470
		0·7693097
		0·3208439
0·87	0·3950656	0·8688380
	0·8316804	0·1800840
	0·4871584	0·4912024
	0·8694256	0·4870072
0·89	0·3488245	0·8894232
	0·8086390	0·1372268
	0·5508806	0·3829568
	0·8807835	0·5206940
	0·3738138	0·8792732
	0·8333148	0·1587240
	0·4944891	0·4450920
	0·8898920	0·4885256

When we plot these values against μ, a three-dimensional bifurcation diagram results. But the essential modification in the bifurcation structure and the new features that appear due to the modulation of the parameter can be clearly shown by taking a two-dimensional projection of the diagram in the (X_t, μ) plane, which is shown in figure 1.

It is clear from the figure that even from the first bifurcation onwards the behaviour is quite different from that of the logistic map. One novel aspect of the diagram is that the inner bifurcation branches cross over each other in the 4-cycle region. Although the branches appear to cut each other, it is not so because we are only considering the projection of the 3-dimensional diagram in a 2-dimensional plane. The crossing over of the bifurcation branches is the result of a significant change in the asymptotic behaviour of the system from the normal one, in a small range of the parameter, say $\mu=0{\cdot}86$ to 0·87. In this small range, the branches appear to be very steep indicating that the asymptotic behaviour of the system in this region is very sensitive to small changes in the parameter values. To have a closer look at this region, we calculated the orbits separately in that range by increasing μ in steps of 0·002, and is shown in figure 2. From the figure we see that as μ is increased from 0·861 to 0·865, the lower

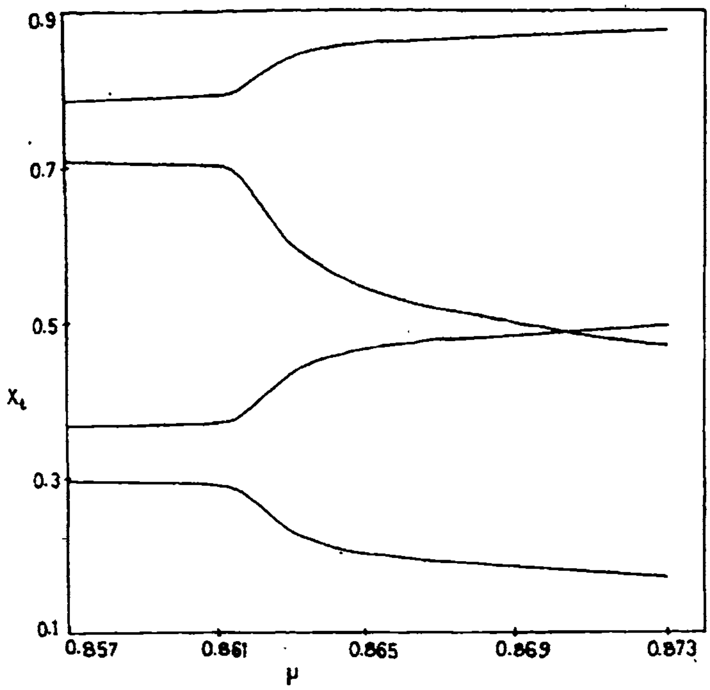

Figure 2. The 4-cycle region of the figure 1 is shown magnified. The branches are very steep in a small parameter range $\mu = 0{\cdot}861$ to $0{\cdot}865$ which results in the cross over of the bifurcation branches at a value of μ around 0·87.

branch shoots up and the upper branch sharply comes down to eventually cross each other at some value around $\mu = 0{\cdot}87$.

Another important observation is that the bifurcations occur earlier than in the case of the logistic map from the second bifurcation onwards. This is clearly evident in the case of the 2-cycle (see figure 1 or table 1) which becomes unstable at $\mu = 0{\cdot}86225$ for the logistic map, whereas, for our map it is somewhere around $\mu = 0{\cdot}845$ with the value of the parameter λ still lower. The difference becomes less pronounced as we go to the higher and higher bifurcations. This result is exactly opposite to the one obtained by Kapral and Mandel (1985) and implies that linear and nonlinear variations of the parameter can have entirely different effect on the asymptotic behaviour of the system.

Finally, we note a peculiar aspect of our bifurcation diagram. It is easily seen that figure 1 lacks the symmetry of the bifurcation structure of the logistic map. There is a marked difference in the bifurcations between the upper and lower arms of our bifurcation tree. All the bifurcations of the upper branch appear to be asymmetric whereas that of the lower branch are somewhat symmetric.

3. Linear stability analysis

In this section, we will try to confirm some of the results obtained above by determining the fixed points of the map analytically and then performing a linear stability analysis (Greene 1968; Bountis 1981). Taking (X^*, λ^*) as the fixed point of the map (1), we get:

$$X^* = [(3\mu - 1)/(4\mu - 1)], \tag{2}$$

and
$$\lambda^* = (4\mu - 1)/4\mu \tag{3}$$

apart from the trivial solution $(X^*, \lambda^*) = (0, 0)$. It is evident that, for $0 < \mu < 1/4$, (0, 0) is the only stable fixed point of the map (1). As μ increases beyond 1/4, (3) becomes stable whereas (2) becomes stable only at a higher value of μ, say μ'. The value of μ' is determined by the condition

$$(4\mu' - 1)/4\mu' = 1/4 \tag{4}$$

which gives $\mu' = 1/3$.

Therefore, for $1/4 < \mu < 1/3$, the stable 1-cycle is given by

$$(X^*, \lambda^*) = (0, (4\mu - 1)/4\mu) \tag{5}$$

whereas the nontrivial fixed point given by (2) and (3) becomes stable in the range $1/3 < \mu < 3/4$ and at $\mu = 3/4$, we have the first bifurcation. Now, what we are really interested is in the region $\mu > 3/4$. But it is impractical to obtain the fixed points analytically beyond the 2-cycle.

The 2-cycle, say (X_1^*, λ_1^*) and (X_2^*, λ_2^*), is defined by the following set of four equations:

$$X_2^* = 4\lambda_1^* X_1^*(1 - X_1^*), \tag{6.1}$$

$$\lambda_2^* = 4\mu\lambda_1^*(1 - \lambda_1^*), \tag{6.2}$$

$$X_1^* = 4\lambda_2^* X_2^*(1 - X_2^*), \tag{6.3}$$

$$\lambda_1^* = 4\mu\lambda_2^*(1 - \lambda_2^*). \tag{6.4}$$

Since (2) and (3) constitute a trivial solution of (6), we can solve them completely to get:

$$\lambda_1^* = (1/8\mu)(4\mu + 1) + (1/8\mu)[(4\mu + 1)(4\mu - 3)]^{1/2}, \tag{7}$$

$$X_1^* = \frac{(5\mu - 1)}{2(4\mu - 1)} - \frac{1}{2}\left[\frac{(5\mu - 1)^2}{(4\mu - 1)^2} + \frac{8\mu(4\mu - 1)(\mu^2 - 4\mu - 1)}{\beta(4\mu + 1)(3\mu - 1)}\right]^{1/2}, \tag{8}$$

$$\lambda_2^* = (1/8\mu)(4\mu + 1) - (1/8\mu)[(4\mu + 1)(4\mu - 3)]^{1/2}, \tag{9}$$

$$X_2^* = \frac{(4\mu - 1)(4\mu + 1 - \mu^2)}{(4\mu + 1)(3\mu - 1)} - \frac{\beta X_1^*}{2(4\mu - 1)}, \tag{10}$$

where
$$\beta = (4\mu + 1) + [(4\mu + 1)(4\mu - 3)]^{1/2}. \tag{11}$$

From (7) and (9), we see that the bifurcation branches of the logistic map are exactly symmetric whereas, (8) and (10) suggest that such a symmetry is lacking in our case. Here, the asymptotic values X_1^* and X_2^* depend on each other resulting in the observed asymmetry of the bifurcation branches.

Now, to study the stability of the 2-cycle we consider the Jacobian of map (1):

$$J=\begin{bmatrix} 4\lambda_t(1-2X_t) & 4X_t(1-X_t) \\ 0 & 4\mu(1-2\lambda_t) \end{bmatrix}$$

Taking $M=\prod_{i=1}^{2} J(X_i^*,\lambda_i^*)$, we have

$$\operatorname{Tr} M=16\lambda_1^*\lambda_2^*(1-2X_1^*)(1-2X_2^*)+16\mu^2(1-2\lambda_1^*)(1-2\lambda_2^*). \tag{12}$$

For the 2-cycle to be stable, we have the condition $|\operatorname{Tr} M|\leqslant 2$. Substituting (7)–(10) in (12), the required μ values can be obtained numerically. We see that the 2-cycle becomes stable at $\mu\sim 0{\cdot}754$ and becomes unstable at $\mu\sim 0{\cdot}858$, which is in agreement with our numerical result that the bifurcations are earlier from the second bifurcation onwards.

4. Lyapunov exponents

In this section, we attempt to study the chaotic region of the map by estimating the Lyapunov exponent (LE) which plays a crucial role in the theory of dynamical systems. For a chaotic system, it measures the rate of exponential divergence of nearby trajectories in phase space, whereas, for stable periodic orbits, it measures the rate of convergence towards the stable attractor. The dependence of LE of stable and unstable periodic orbits of the logistic map on the control parameter has been studied by Giesel *et al* (1981), using an approximate renormalization procedure. They take negative values for stable periodic orbits, whereas in the chaotic region, their values become positive. Huberman and Rudnick (1980) have studied the LE for chaotic bands and found a power law behaviour as a function of the control parameter.

Since map (1) is a two-dimensional map, we can define two LEs, σ_1 and σ_2, one for the X-degree of freedom and the other for the λ-degree of freedom (Lichtenberg and Lieberman 1983). Also, the LE for the λ-degree of freedom, say σ_2, will be same as that of the logistic map. We now compute σ_1 and σ_2 numerically for several values of μ in the range $\mu=0{\cdot}84$ to $\mu=1$. The method we use has been frequently employed in the literature (Benettin *et al* 1976; Lichtenberg and Lieberman 1983) for the numerical estimation of the LE.

Similar to the case of one-dimensional maps, we can define LEs of map (1) as follows: Let M be the matrix given by the product of the Jacobians $J(x,\lambda)$ of map (1):

$$M=\prod_{i=1}^{N} J(X_i,\lambda_i). \tag{13}$$

Then the LEs of the map are given by

$$\sigma_1=\lim_{N\to\infty}\frac{1}{N}\ln|\Lambda_1|, \tag{14.1}$$

$$\sigma_2=\lim_{N\to\infty}\frac{1}{N}\ln|\Lambda_2|, \tag{14.2}$$

where Λ_1 and Λ_2 are the eigenvalues of M given by

$$\Lambda_1 = \prod_{i=1}^{N} 4\lambda_i(1-2X_i), \tag{15.1}$$

$$\Lambda_2 = \prod_{i=1}^{N} 4\mu(1-2\lambda_i). \tag{15.2}$$

Note that σ_2 reduces to the LE of the logistic map.

Now, in order to evaluate these numerically, we take a particular μ value for the map. Starting from an initial value (X_1, λ_1), we iterate the map N times and calculate the Jacobian at all these N iterates. We then evaluate M and the corresponding eigenvalues Λ_1 and Λ_2 and calculate the quantities $(1/N)\ln|\Lambda|$. Repeating this for several N values, we plot these quantities against N separately. The same is repeated for other μ values and the results are shown in figures 3 and 4. From the behaviour of the graph, we see that $(1/N)\ln|\Lambda_1|$ and $(1/N)\ln|\Lambda_2|$ settle down to almost constant values as $N\to\infty$, which serves as an approximate estimate of the LE and can be directly read from the graphs.

A comparison of figures 3 and 4 reveals some interesting features. σ_2 is negative in the regular region but becomes positive as μ crosses over to the chaotic phase. At $\mu=0{\cdot}96$, σ_2 is negative indicating the existence of a periodic window, where the

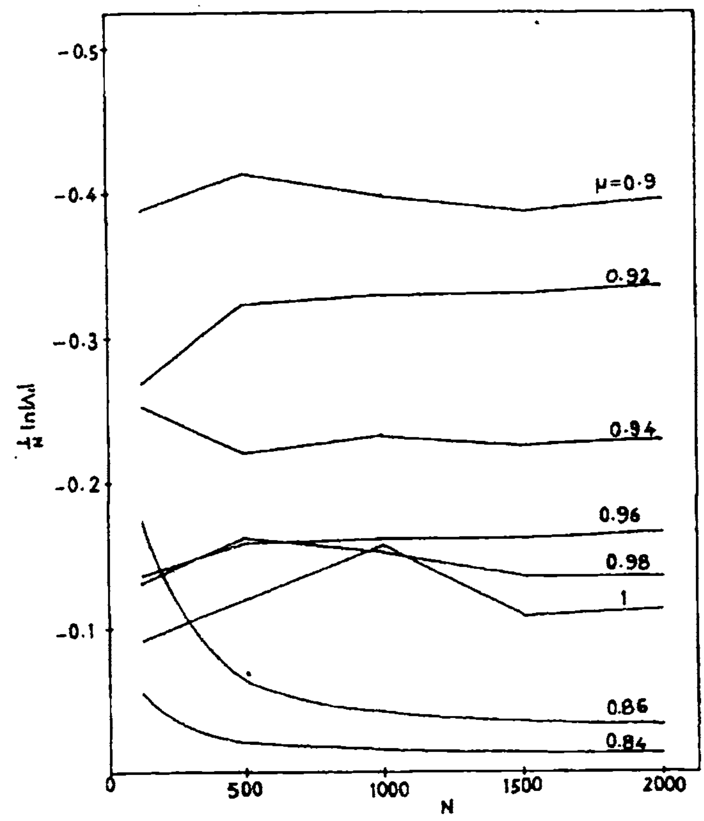

Figure 3. An approximate numerical estimation of the Lyapunov exponent of the modulated logistic map for the X-degree of freedom. Note that it is negative for all values of μ.

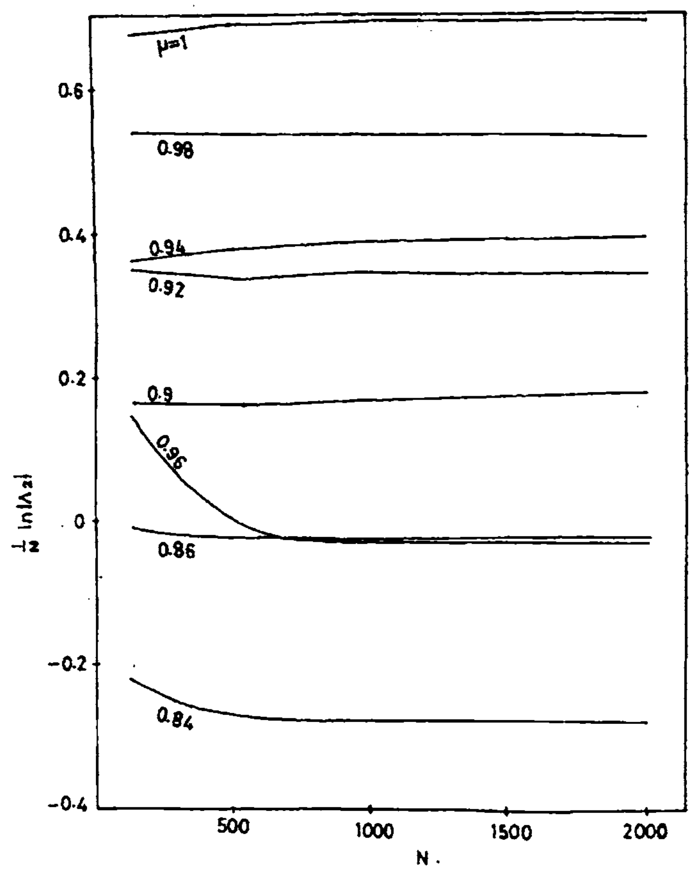

Figure 4. Lyapunov exponent for the λ-degree of freedom which is same as that of the logistic map. As μ increases to μ_∞, and above, the LE cross over from a negative to positive value reaching ln 2 for $\mu = 1$. At $\mu = 0{\cdot}96$ the value is negative indicating the presence of a periodic window.

behaviour is once again regular. As we increase μ, σ_2 increases steadily and reaches a value $\sim \ln 2$ corresponding to $\mu = 1$. (There are several periodic windows in between with negative σ_2 values which cannot be seen in our graph). The behaviour of σ_1 is quite different. It is always negative irrespective of whether we are in the regular region or chaotic region. It has a small negative value in the regular region and reaches its maximum when the system just crosses over to the chaotic state. With further increase in μ, its value is reduced.

The fact that the LE for the X-degree of freedom is always negative can be understood in the following way. The modulated logistic map is a two-dimensional noninvertible map of a square interval [0, 1] on to itself. A bounded chaotic motion, as shown by the system, is not possible if it has positive LEs for both the degrees of freedom because, in that case, an initial phase volume will expand for ever making the system unstable. This naturally restricts σ_1 to negative values.

Now, LE is a measure of the exponential separation of nearby points in phase space and hence is proportional to the rate of loss of information regarding the state of the system (Shuster 1984). Also, the degree of chaos in a system can be measured in terms of this rate of loss of information. The logistic map has a single positive LE in the chaotic phase whereas the modulated logistic map has by definition two LEs σ_1 and

σ_2. Such a system turns chaotic when at least one LE becomes positive. Since σ_1 is always negative, the rate of loss of information contained in a cell of phase space of the modulated logistic map is less compared to that of the logistic map. In other words, we can say that the system which results by modulating the parameter of the logistic map is less chaotic than the logistic map itself. This is in agreement with a recent observation by Tomita (1984).

In order to get a better comparison of the chaotic regions of the two systems, we have computed the power spectrum for the modulated and unmodulated cases, for $\mu=0{\cdot}98$, which are shown in figures 5 and 6 respectively. It is easily seen that figure 6 contains more frequency components than figure 5 which supports our conclusion.

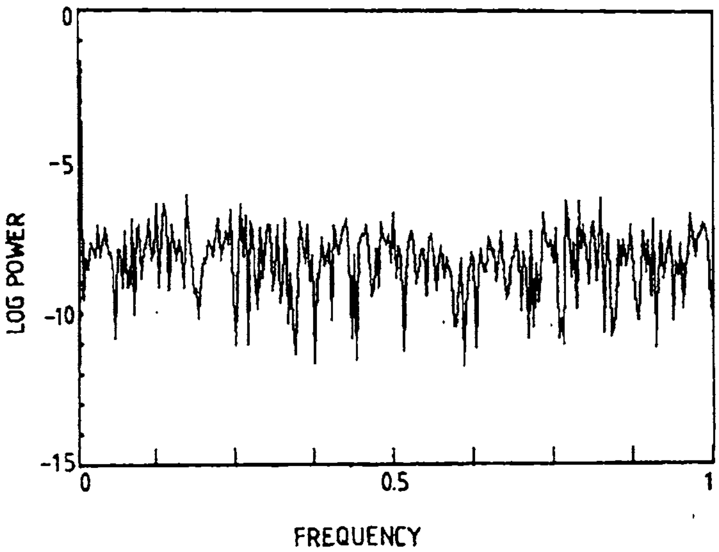

Figure 5. Power spectrum of the modulated logistic map for $\mu=0{\cdot}98$.

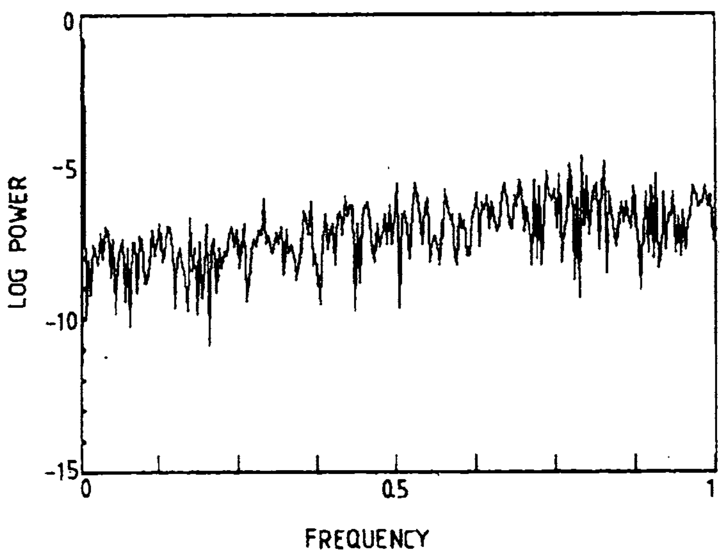

Figure 6. Power spectrum of the logistic map for $\mu=0{\cdot}98$.

5. Discussion and summary

We have considered the logistic map, which is a very well-studied model applicable in several physical situations, with a time-dependent control parameter. We observe that by introducing a specific nonlinear variation for the parameter, several new features appear in the bifurcation diagram and it is modified qualitatively as well as quantitatively. For example, though the 8-cycle region appear similar to that of the logistic map, it is not because the inner branches are interchanged as a result of the crossing over of the branches. There may be more such crossings as we approach the chaotic region and so the structure soon becomes more complicated than that of the logistic map.

Before concluding, let us note a final aspect of our bifurcation diagram which may be of mathematical interest. It is known that the ordering of the iterates for any stable period obey certain allowed sequences called the MSS sequences (Derrida *et al* 1979) and this ordering is unchanged throughout the entire stability zone. But in our diagram, it seems that this is no longer satisfied, as the ordering is necessarily changed when the bifurcation branches cross over.

All these novel aspects of the diagram must be considered as the effect of the nonlinear modulation of the parameter since they are absent when the parameter is varied linearly with time.

To summarize, we have shown that a simple type of nonlinear modulation of the control parameter can change the asymptotic behaviour of the logistic map significantly.

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