

**STUDIES ON SOME GRAPH
OPERATORS AND RELATED TOPICS**

Thesis submitted to the
Cochin University of Science and Technology
for the award of the degree of
DOCTOR OF PHILOSOPHY
under the Faculty of Science

By

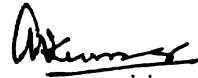
MANJU K. MENON

Department of Mathematics
Cochin University of Science and Technology
Cochin - 682 022

March 2009

Certificate

This is to certify that the thesis entitled '**Studies on some graph operators and related topics**' submitted to the Cochin University of Science and Technology by Ms. Manju K. Menon for the award of the degree of Doctor of Philosophy under the Faculty of Science is a bonafide record of studies carried out by her under my supervision in the Department of Mathematics, Cochin University of Science and Technology. This report has not been submitted previously for considering the award of any degree, fellowship or similar titles elsewhere.



Dr. A. Vijayakumar

(Supervisor)

Reader

Department of Mathematics

Cochin University of Science and Technology

Cochin- 682022, Kerala.

Cochin-22

12/03/09.

Declaration

I, Manju K. Menon hereby declare that this thesis entitled '**Studies on some graph operators and related topics**' contains no material which had been accepted for any other Degree, Diploma or similar titles in any University or institution and that to the best of my knowledge and belief, it contains no material previously published by any person except where due references are made in the text of the thesis.



Manju K. Menon

Research Scholar

Registration No.2488

Department of Mathematics

Cochin University of Science and Technology

Cochin-682022, Kerala.

Cochin-22

12/03/09.

Acknowledgement

'Gurur Brahma Gurur Vishnu Gurur Devo Maheswara
Gurur saakshat parahbrahma Thasmai sree gurave Namah' with
this prayer in my mind, let me express my gratitude to the God
Almighty for the completion of my thesis.

I would like to thank very sincerely all those who have helped
me in many different ways to complete my research work. It
gives me immense pleasure to thank all my teachers, relatives
and friends who directly or indirectly helped me in my research
work.

The greatest thanks I owe is to my supervisor and guide Dr.
A. Vijayakumar, Reader, Department of Mathematics, Cochin
University of Science and Technology for keeping my spirits up
and my hamstrings in working order. He is such a guide one
could hope for. I am indebted to the bright ideas and sharp
eyes of him. His timely help, keen guidance and criticism with
love and affection helped me a lot to complete my research work.

I am also grateful to Dr. R. S. Chakravarti, Head, Department of Mathematics, Cochin University of Science and Technology for his helps during my research work. I also thank teachers, office staff, librarian and former librarian of the Department of Mathematics for their helps of various kinds. My gratitude also goes to the authorities of Cochin University of Science and Technology for the facilities they provided and also to the National Board for Higher Mathematics for providing the research fellowship during the initial years of my research work.

I wish to express my sincere thanks to Prof. S. B. Rao, Director, C. R. Rao Advanced Institute of Mathematics, Hyderabad, India for spending his valuable time for the discussions with us. Also I express my feeling of gratitude to Dr. Daniela Ferrero, Department of Mathematics, Texas University, USA for introducing us the notion of the wide diameter and related topics and also for permitting to include our joint work in this thesis.

I acknowledge with gratitude Rev. Dr. Douglas Pinheiro,

Manager, Prof. P. J. Roy, Principal and Prof. K. V. N. Sarma, Head, Department of Mathematics of St. Paul's College, Kalamassery for their encouragement and support. I also remember with thanks our former manager Rev. Fr. Jose Chelangara, former principal Prof. V. X. Sebastian and former Head of the Department Prof. Titus Paul for their good wishes. I am also thankful to my colleagues in the Department of Mathematics of St. Paul's College, Kalamassery for their whole hearted support.

I express my feeling of gratitude to my Graph Theory friends Dr. G. Indulal, Dr. K. S. Parvathy, Dr. B. Radhakrishnan Nair, Ms. Aparna Lakshmanan, Ms. Nisha Mary Thomas, Ms. Seema Varghese, Mr. Pravas K. for their interest in my research work and they were always ready to share their bright ideas with me. I want to express my warmest thanks to my fellow research scholars Ms. Anu V., Ms. Chithra M. R., Dr. Babu S., Ms. Lalitha Haridas, Ms. Lexy Alexander, Mr. Manikandan R., Mr. Pramod P. K., Dr. Rajeswaridevi M. B., Mr. Ratheesh R., Mr. Santhosh Pandey, Mr. Sathian M. K., Mr. Shinoj K. M., Ms. Shiny Philip, Mr. Tonny K. B., Ms. Tresa Chacko and Ms. Viji M. for their inspiration on my work.

Some friends deserve special mention for their long term help in matters professional and personal. These good souls are Ms. Pramada Ramachandran and Ms. Savitha K. S.

My special thanks to Mrs. Prema Vijayakumar for her good loving words and encouragement.

I gratefully acknowledge my beloved father, Parents in-law, and my grand uncle Mr. C. N. U. Nair for their love, expectation, prayers and encouraging words. Also I would like to thank my brother, brother in law and sisters in law for giving me strength and encouragement through out my research. Finally but immensely I express my thanks to my husband Mr. Benoy P. and my son Master Mohith B. Nair for their love and support. But words are not enough to express my feelings towards them.

Manju K. Menon

**STUDIES ON SOME
GRAPH OPERATORS
AND RELATED TOPICS**

Contents

1	Introduction	1
1.1	Basic definitions	7
1.2	Basic lemmas and theorems	24
1.3	New definitions	27
1.4	A survey of results	29
1.5	Summary of the thesis	34
1.6	List of publications	38
2	The P_3 intersection graph of a graph	41

2.1	The P_3 intersection graph of a graph	42
2.2	Forbidden subgraph characterizations	50
2.3	The chromatic number of $P_3(G)$	53
2.4	Some other graph parameters of $P_3(G)$	58
3	The edge C_4 graph of a graph	69
3.1	The edge C_4 graph of a graph	70
3.2	Diameter, Radius and Center	72
3.3	Some domination parameters in $E_4(G)$	76
3.4	$E_4(G)$ and some graph classes	86
4	Dynamics of $P_3(G)$ and $E_4(G)$	97
4.1	Dynamics of $P_3(G)$	98
4.2	Dynamics of $E_4(G)$	103
4.3	The touching number	105

5	The wide diameter and diameter variability	109
5.1	The w -wide diameter	110
5.2	The diameter variability of some graph operators	115
5.3	The diameter variability of some graph operations	119
	•	
	List of problems	125
	List of Symbols	127
	List of some special graphs	131
	Bibliography	133
	Index	143

Chapter 1

Introduction

Graph theory as a mathematical discipline was initiated by the renowned Swiss mathematician Leonhard Euler (1707 - 1783) in his famous discussion of the Königsberg Bridge problem entitled ‘The solution of a problem relating to the geometry of position’. It was presented at the St.Petersberg Academy on 26th August, 1735. Unfortunately, this article of Euler, published in 1736, remained an isolated contribution for nearly a hundred years. However, in the middle of the nineteenth century, there was a resurgence of interest in the area of graph theory. The natural sciences exercised their influence through investigations of

electrical networks and models for crystals and molecular structure and theoretically, the development of formal logic led to the study of binary relations in the form of graphs.

Today, graph theory is a branch of mathematics which finds applications in many areas - anthropology, architecture, biology, chemistry, computer science, economics, physics, psychology, sociology and telecommunications, to name a few. The applications of graph theory in operations research, social science, psychology and physics are detailed in C. W. Marshall [61]. J. L. Gross and J. Yellen [34] discuss a variety of graph classes with numerous illuminating examples which are of topological relevance. The development of graph theory with its applications to electrical networks, flows and connectivity are included in [11] and [22]. Ramsey theory is an interesting branch of graph theory which relates to the number theory. In [22], R. Diestel covers all major developments in the subject. More recently, the exciting notion of 'Web graphs' [6] has been finding applications in very many different areas. Such graphs are examples of large, dynamic, distributed graphs and share many properties with several other complex graphs [64] found in a variety of

systems ranging from social organizations to biological systems. The best barometer to indicate the growth of interest in graph theory is the explosion in the number of pages that Section 05: Combinatorics occupies in the Mathematical Reviews.

Volumes have been written on the rich theory and the very many applications of graphs. To name a few, [5], [9], [10], [32], [34], [35], [49], [71], [78]. This thesis entitled '*Studies on some graph operators and related topics*' is a humble attempt at making a small addition to the vast ocean of results in graph theory.

'Graph operator' is a mapping $T : \mathcal{G} \rightarrow \mathcal{G}'$ where \mathcal{G} and \mathcal{G}' are families of graphs. Krausz [52] introduced the concept of the line graph and also that of 'graph operators'. He also gave a characterization of line graphs. Whitney [79] showed that every finite connected graph except C_3 has at most one connected L-root.

The study of graph operators gained increasing importance

due to the study of its dynamics as detailed by E. Prisner [66]. The beginning of graph dynamics dates back to 1960s, with the publication [36] by Harary and Norman, and also the three problems of great influence posed by Ore in his monograph [65], namely

1. Determine all graphs isomorphic to their interchange graph (line graph).
2. When the interchange graph is given, is the original graph uniquely determined?
3. Investigate the repeated interchange graphs.

In the graph dynamics terminology, the first problem deals with the ‘fixedness’ and the second and third will lead to the ‘1-periodicity’ and ‘convergence’ or ‘divergence’.

The 1960s were mainly devoted to the investigation of the line graph and the line digraph operators. Several solutions to Ore’s problems for the line graph appeared in [8], [15], [72], [73]. The question of periodicity was considered only for the fixed graphs till 1970s. General periods were investigated by

Escalante [25] and it was studied for the line digraphs by Hemminger [42]. The transition number was first explicitly defined in [1].

While dealing with graph classes, a main source is the classical book by M. C. Golumbic [32]. Since then many interesting new graph classes have been studied as discussed in detail by A.Brandstädt, et.al. [13].

By applying graph operators also, we get some graph classes. The line graphs, Gallai and the anti-Gallai graphs, the cycle graphs and the edge graphs are some of the graph classes obtained by choosing appropriate graph operators. In fact, the intersection graphs form a sub collection of the graph classes obtained by using graph operators. The intersection graph is a very general notion in which objects are assigned to the vertices of a graph and two distinct vertices are adjacent if the corresponding objects have a non empty intersection. A variety of well studied graph classes such as the line graphs, the clique graphs and the block graphs are actually special types of inter-

section graphs. J. L. Szwarcfiter has made an excellent survey of the clique graphs [75]. The block graph [37], the square [38] and the complement [70] are some well studied graph operators.

Several graph operators and the dynamical behavior of these operators are extensively studied in [66].

It is interesting to study when the graph operators belong to some special graph classes. The inclusions between graph classes can be identified from their forbidden subgraph characterizations. The cographs, the split graphs, the threshold graphs and the line graphs are some of the interesting graph classes which admit finite forbidden subgraph characterizations and the perfect graphs, the distance hereditary graphs, the comparability graphs and the chordal graphs are some of the other interesting graph classes defined by forbidding an infinite collection of induced subgraphs.

While studying a graph operator, the study of its parameters such as clique number, independence number, chromatic num-

ber, domination number, diameter, radius, eccentricity, center etc are important. It is quite interesting to study the relationship between these parameters of G and those under graph operators.

This thesis is mainly concerned with the graph operators - the ' P_3 intersection graph' and the 'edge C_4 graph'.

1.1 Basic definitions

The basic notations, terminology and definitions are from ([5], [14], [32], [66] and [78]) and the basic results are from ([13], [39], [45] and [77]).

Definition 1.1.1. A graph $G = (V, E)$ consists of a collection of points, V called its vertices and a set of unordered pairs of distinct vertices, E called its edges. If $|V|$ is finite, then G is a **finite graph**. The unordered pair of vertices $\{u, v\} \in E$ are called the **end vertices** of the edge $e = \{u, v\}$. When u and v

are end vertices of an edge, then u and v are **adjacent**. If the vertex v is an end vertex of an edge e , then e is **incident** on v . Two edges which are incident with a common vertex are said to be adjacent edges. The cardinality of V is called the **order** of G and the cardinality of E is called the **size** of G . A graph G of order n and size m is also denoted by $G = (n, m)$. A graph is the **null graph**, denoted by ϕ if it has no vertices and **trivial** if it has no edges.

Definition 1.1.2. The **degree** of a vertex v , denoted by $d(v)$ is the number of edges incident on v . A graph G is **k -regular** if $d(v) = k$ for every vertex $v \in V$. A vertex of degree zero is an **isolated vertex** and of degree one is a **pendant vertex**. The edge incident on a pendant vertex is a **pendant edge**. A vertex of degree $n - 1$ is called a **universal vertex**. In a graph G , the maximum degree of vertices is denoted as $\Delta(G)$ and the minimum degree of vertices is denoted as $\delta(G)$.

Definition 1.1.3. A $v_0 - v_k$ **walk** in a graph G is a finite list $v_0, e_1, v_1, e_2, v_2, \dots, e_k, v_k$ of vertices and edges such that for $1 \leq i \leq k$, the edge e_i has end vertices v_{i-1} and v_i . In the $v_0 - v_k$ walk, v_0 is the **origin**, v_k is the **terminus** and v_1, v_2, \dots, v_{k-1} are its **internal vertices**. If the vertices v_0, v_1, \dots, v_k of the above

walk are distinct, then it is called a **path**. A path from a vertex u to a vertex v is called a **$u - v$ path**. A path on n vertices is denoted by P_n . If the edges e_1, e_2, \dots, e_k of the walk are distinct, it is called a **trail**. A graph G is **Eulerian** if it has a closed trail containing all the edges. A nontrivial closed trail is called a **cycle** if its origin and internal vertices are distinct. A cycle with n vertices is denoted by C_n . The **length** of a walk, a path or a cycle is its number of edges. A graph containing exactly one cycle is called a **unicyclic graph**. A graph is **acyclic** if it does not contain cycles. The **girth** of G , $g(G)$ is the length of a shortest cycle in G . An acyclic graph has infinite girth. The **circumference** of G , $c(G)$ is the length of any longest cycle in G .

Definition 1.1.4. A graph $H = (V', E')$ is called a **subgraph** of G if $V' \subseteq V$ and $E' \subseteq E$. A subgraph H is a **spanning subgraph** if $V' = V$. The graph H is called an **induced subgraph** of G if E' is the collection of all edges in G which has both its end vertices in V' . $\langle V' \rangle$ denotes the induced subgraph with vertex set V' . A spanning 1-regular graph is called a **1-factor** or **perfect matching**. A graph G is **H-free** if it does not contain H as an induced subgraph.

Definition 1.1.5. A graph G is **connected** if for every $u, v \in V$, there exists a $u - v$ path. If G is not connected then it is **disconnected**. The **components** of G are its maximal connected subgraphs. A connected acyclic graph is called a **tree**.

Definition 1.1.6. The **distance** between two vertices u and v of a connected graph G , denoted by $d(u, v)$ or $d_G(u, v)$ is the length of a shortest $u - v$ path in G . The **eccentricity** of a vertex u , $e(u) = \text{maximum } \{d(u, v) / v \in V(G)\}$. The **radius** $\text{rad}(G)$ and the **diameter** $\text{diam}(G)$ are respectively the minimum and the maximum of the vertex eccentricities. The **center** of a graph G , $C(G)$ is the subgraph induced by the vertices of minimum eccentricity.

Definition 1.1.7. A **chord** of a cycle C is an edge not in C whose end points lie in C . A graph G is **chordal** if every cycle of length at least four in G has a chord.

Definition 1.1.8. A **complete graph** is a graph in which each pair of distinct vertices is joined by an edge. A complete graph on n vertices is denoted by K_n . The graph obtained by deleting any edge of K_n is denoted by $K_n - \{e\}$. K_3 is called a **triangle** and a **paw** is a triangle with a pendant edge. A **clique** is a

maximal complete subgraph. The size of the largest clique in G is the **clique number** $\omega(G)$. A clique of size k is called a **k -clique**.

Definition 1.1.9. A cycle C of G is a **b -cycle** of G if C is not contained in a complete subgraph of G . The **bulge** of G , $b(G)$ is the minimum length of a b -cycle in G if G contains a b -cycle and is ∞ otherwise.

Definition 1.1.10. The set of all vertices adjacent to a vertex v is called **open neighborhood** of v , denoted by $N(v)$. The **closed neighborhood** of v , $N[v] = N(v) \cup \{v\}$.

Definition 1.1.11. Assigning colors to the vertices of a graph is called a **vertex coloring**. If no two adjacent vertices receive the same color, then such a coloring is called a **proper vertex coloring**. The minimum number of colors required for a proper vertex coloring of a graph G is called its **chromatic number**, denoted by $\chi(G)$.

Definition 1.1.12. A property P of a graph G is **vertex hereditary** if every induced subgraph of G has the property P . A graph H is a **forbidden subgraph** for a property P , if any graph G which satisfies the property P cannot have H as an

induced subgraph.

Definition 1.1.13. A graph $G = (V, E)$ is **isomorphic** to a graph $H = (V', E')$ if there exists a bijection from V to V' which preserves adjacency. If G is isomorphic to H , we write $G \cong H$.

Definition 1.1.14. Let G be a graph. The **complement** of G , denoted by G^c is the graph with vertex set same as that of V and any two vertices are adjacent in G^c if they are not adjacent in G . K_n^c is called **totally disconnected**. A graph G is **self complementary** if $G \cong G^c$.

Definition 1.1.15. A graph G is **bipartite** if the vertex set can be partitioned into two non-empty sets U and U' such that every edge of G has one end vertex in U and the other in U' . A bipartite graph in which each vertex of U is adjacent to every vertex of U' is called a **complete bipartite graph**. If $|U| = m$ and $|U'| = n$, then the complete bipartite graph is denoted by $K_{m,n}$. The complete bipartite graph $K_{1,n}$ is called a **star**. A graph G is **complete multipartite** if the vertices can be partitioned into sets so that $\{u, v\} \in E$ if and only if u and v belong to different sets of the partition. A complete k -partite

graph with partite sets of cardinalities n_1, n_2, \dots, n_k is denoted by K_{n_1, n_2, \dots, n_k} .

Definition 1.1.16. A subset $I \subseteq V$ of vertices is **independent** if no two vertices of I are adjacent. The maximum cardinality of an independent set is called the **independence number** and is denoted by $\alpha(G)$. A subset $F \subseteq E$ of edges is said to be an **independent set of edges** or a **matching** if no two edges in F have a vertex in common. The maximum cardinality of a matching set of edges is the **matching number** and is denoted by $\beta(G)$.

Definition 1.1.17. A subset $K \subseteq V$ is called a **vertex cover** of G if every edge of G is incident with at least one vertex of K . The minimum cardinality of a vertex cover is the **vertex covering number** $\alpha_0(G)$.

Definition 1.1.18. For a graph G , a subset V' of $V(G)$ is a **k -vertex cut** of G if the number of components in $G - V'$ is greater than that of G and $|V'| = k$. The **vertex connectivity** of G , $\kappa(G)$ is the smallest number of vertices in G whose deletion from G increases the number of components of G . A graph is **n -connected** if $\kappa(G) \geq n$. A vertex v of G is a **cut vertex**

of G if $\{v\}$ is a vertex cut of G . If G has no cut vertices, then G is a **block**. For $\{u, v\} \in V(G)$, the $u - v$ **cut** is a set $S \subseteq V(G) - \{u, v\}$ such that $G - S$ has no $u - v$ path. The **edge connectivity** of a graph G , $\kappa'(G)$ is the least number of edges whose deletion increases the number of components of G .

Definition 1.1.19. A vertex x dominates a vertex y if $N(y) \subseteq N[x]$. If x dominates y or y dominates x , then x and y are **comparable**. Otherwise, they are **incomparable**. The **Dilworth number** of a graph G , $dilw(G)$ is the largest number of pairwise incomparable vertices of G .

As an example, $dilw(C_4) = 2$.

Definition 1.1.20. A subset $S \subseteq V$ of vertices is a **dominating set** if each vertex of G that is not in S is adjacent to at least one vertex of S . If S is a dominating set then $N[S] = V$. A dominating set of minimum cardinality in G is called a **minimum dominating set**, its cardinality is called the **domination number** of G and it is denoted by $\gamma(G)$.

Definition 1.1.21. A dominating set S is an **independent dominating set** if S is an independent set. The **independent domination number** of a graph G , $\gamma_i(G)$ is the minimum cardinality

of an independent dominating set in G . The minimum cardinality of a maximal independent set of vertices in G is also the same as $\gamma_i(G)$. A subset $S \subseteq V$ is a total (open) dominating set if $N(S) = V$. The **total (open) domination number** of a graph G , $\gamma_t(G)$ is the minimum cardinality of a total dominating set in G . A dominating set S is a connected dominating set if $\langle S \rangle$ is a connected subgraph of G and the corresponding domination number is the **connected domination number** $\gamma_c(G)$. A dominating set S is a paired dominating set if $\langle S \rangle$ has a perfect matching and the corresponding domination number is the **paired domination number** $\gamma_{pr}(G)$. The paired domination number exists for all graphs with out isolated vertices. A dominating set S is a clique dominating set if $\langle S \rangle$ is a complete graph. The minimum cardinality of a clique dominating set, if it exists is the **clique domination number** $\gamma_{cl}(G)$. A **clique dominated graph** is a graph that contains a dominating clique.

Definition 1.1.22. The subgraph weakly induced by a set S of vertices is the graph $\langle S \rangle_w$ whose vertex set is $N[S]$ and whose edge set consists of those edges in $E(G)$ with at least one vertex, and possibly both, in S . A dominating set S is called a weakly connected dominating set if $\langle S \rangle_w$ is connected. The corre-

sponding domination number is **weakly connected domination number**, $\gamma_w(G)$. The cardinality of the weakly connected independent dominating set is the **weakly connected independent domination number**, denoted by $i_w(G)$.

For example:-

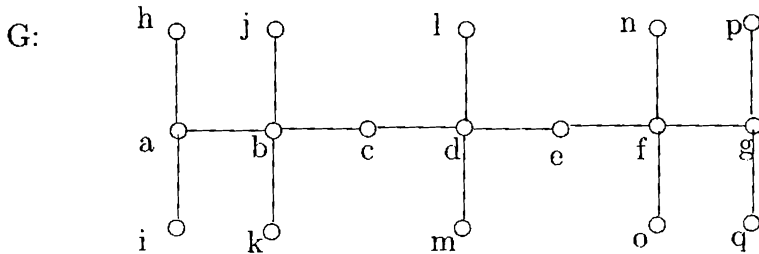


Figure 1.1.1

Then, $\gamma(G) = 5$ ($\{a, b, d, f, g\}$ is a dominating set of minimum cardinality),

$\gamma_i(G) = 7$ ($\{h, i, b, d, f, p, q\}$ is an independent dominating set of minimum cardinality),

$\gamma_t(G) = 6$ ($\{a, b, d, e, f, g\}$ is a total dominating set of minimum cardinality),

$\gamma_c(G) = 7$ ($\{a, b, c, d, e, f, g\}$ is a connected dominating set of

minimum cardinality),

$\gamma_{pr}(G) = 6$ ($\{a, b, c, d, f, g\}$ is a paired dominating set of minimum cardinality),

$\gamma_w(G) = 5$ ($\{a, b, d, f, g\}$ is a weakly connected dominating set of minimum cardinality),

$i_w(G) = 7$ ($\{h, i, b, d, f, p, q\}$ is a dominating set of minimum cardinality).

The graph G is not a clique dominated graph.

Definition 1.1.23. A subset $S' \subseteq E$ is an edge dominating set if every edge not in S' is adjacent to some edge in S' . The **edge domination number** $\gamma'(G)$ of G is the minimum cardinality of all edge dominating sets of G . A subset $S' \subset E$ is an efficient edge dominating set for G if each edge in E is dominated by exactly one edge in S' . The **efficient edge domination number** of G is denoted by $\gamma'_e(G)$.

Definition 1.1.24. The **intersection graph** is a graph whose vertex set is a collection of objects and any two vertices are adjacent if the corresponding objects intersect. The intersection graph of all the edges of G is the **line graph** of G denoted by $L(G)$. Thus, the line graph $L(G)$ of a graph G is a graph that

has a vertex for every edge of G , and two vertices of $L(G)$ are adjacent if and only if they correspond to two edges of G with a common end vertex.

Illustration:

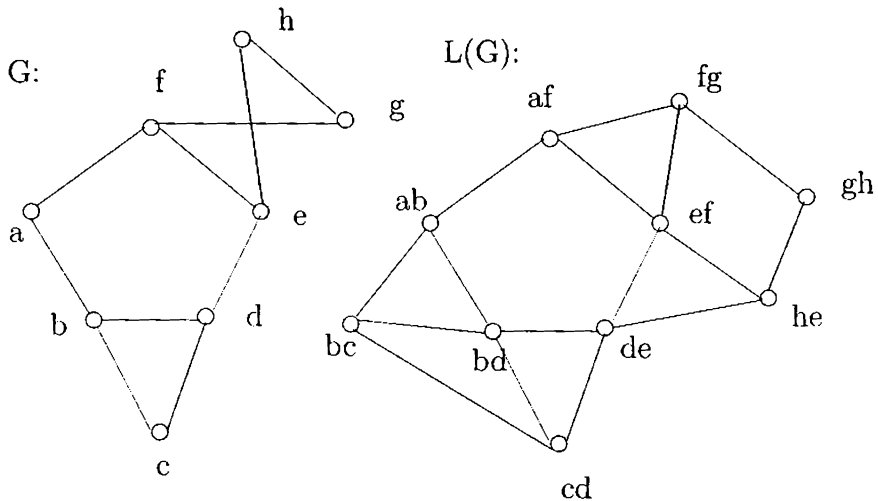


Figure 1.1.2

Definition 1.1.25. The k -**path graph** corresponding to a graph G has the set of all paths of length k as vertices and two vertices in the k -path graph are adjacent whenever the intersection of the corresponding paths form a path of length $k - 1$ in G and their union forms either a cycle or a path of length $k + 1$ in G .

Definition 1.1.26. For any graph G , the n^{th} iterated graph under the operator Φ is iteratively defined as $\Phi^1(G) = \Phi(G)$ and $\Phi^n(G) = \Phi(\Phi^{n-1}(G))$ for $n > 1$. A graph G is Φ^n -complete if $\Phi^n(G)$ is a complete graph. We say that G is **convergent** under Φ if $\{\Phi^n(G), n \in N\}$ is finite. If G is not convergent under Φ , then G is **divergent** under Φ . A graph G is **periodic** if there is some natural number n with $G = \Phi^n(G)$. The smallest such number n is called the **period** of G . A graph G is **Φ -fixed** if the period of G is one. The **transition number** $t(G)$ of a convergent graph G is zero if G is periodic and is the smallest number n such that $\Phi^n(G)$ is periodic otherwise. A graph G is **mortal** if for some $n \in N$, $\Phi^n(G) = \phi$, the null graph. A **semibasin** is any subset B of the class of graphs \mathcal{G} with $\Phi(B) \subseteq B$. A **basin** is a semibasin B if its complement is also a semibasin.

Definition 1.1.27. The **touching number** of a cycle is the cardinality of the set of all edges having exactly one of its end vertices on the cycle. For every integer $n \geq 3$, the n -touching number $t_n(G)$ of a graph G is the supremum of all touching numbers of C_n , provided G contains some C_n . If G contains no C_n then $t_n(G)$ is undefined. The **vertex touching number** of an induced C_k is the cardinality of the set of all vertices which

are adjacent to exactly one vertex of the C_k . The vertex touching number of a graph $vt_k(G)$ is the supremum of all vertex touching numbers of induced C_k , provided G contains some induced C_k .

For example, for the graph G in Figure 1.1.3, $t_5(G) = 7$, $t_3(G) = 5$, $vt_5(G) = 1$, $vt_3(G) = 5$.

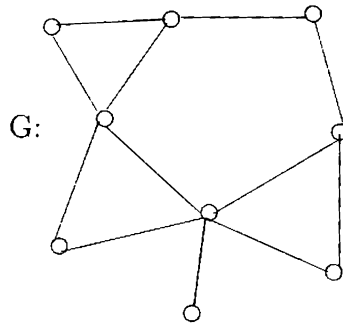


Figure 1.1.3

Definition 1.1.28. A graph G whose vertex set can be partitioned into an independent set and a clique is a **split graph**.

Definition 1.1.29. A graph G is **perfect** if $\chi(H) = \omega(H)$ for every induced subgraph H of G .

Definition 1.1.30. A graph G is a **threshold graph** if it can be obtained from K_1 by recursively adding isolated vertices and universal vertices.

Definition 1.1.31. A connected graph is a **block graph** if every maximal 2-connected subgraph (block) is complete. A graph is a **geodetic graph** if for every pair of vertices there is a unique path of minimum length between them and a graph is **weakly geodetic** if for every pair of vertices of distance two, there is a unique common neighbor.

Definition 1.1.32. A graph that can be reduced to the trivial graph by taking complements within components is called a **co-graph**.

Definition 1.1.33. For every integer w : $1 \leq w \leq \delta(G)$, a **w -container** between any two distinct vertices u and v of G is a set of ' w ' internally vertex disjoint paths between them. Let $C_w(u, v)$ denote a w -container between u and v . In $C_w(u, v)$, the parameter w is the **width** of the container. The **length** of the container is the longest path in $C_w(u, v)$. The **w -wide diameter** of G , $D_w(G)$ is the minimum number l such that there is a $C_w(u, v)$ of length l between any pair of distinct vertices u and v .

For example:-

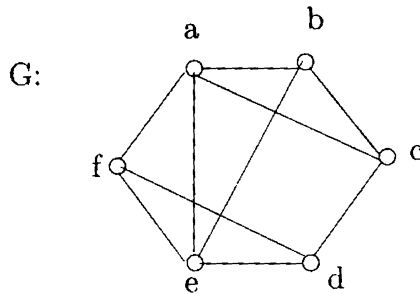


Figure 1.1.4

For this graph G , $C_3(a, b) = \{a - b, a - c - b, a - e - b\}$. Length of this $C_3(a, b) = 2$. $D_3(G) = 3$.

Definition 1.1.34. For any k , the **diameter variability** arising from the change of edges of a graph G are as follows.

$D^{-k}(G)$: The least number of edges whose addition to G decreases the diameter by (at least) k .

$D^{+0}(G)$: The maximum number of edges whose deletion from G does not change the diameter.

$D^{+k}(G)$: The least number of edges whose deletion from G increases the diameter by (at least) k .

Definition 1.1.35. The graph obtained from G by subdividing

each edge of G exactly once is called the **subdivision of G** and is denoted by $S(G)$.

Definition 1.1.36. The **union** of two vertex disjoint graphs G and H denoted by $G \cup H$ is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.

Definition 1.1.37. The **join** of two graphs G and H denoted by $G \vee H$ is the graph obtained from the union $G \cup H$ by adding the edges $\{u - v : u \in V(G) \text{ and } v \in V(H)\}$. The graph $K_1 \vee 2K_2$ is called a **bow**. The **moth** [58] graph is $K_1 \vee \{P_3 \cup 2K_1\}$.

Definition 1.1.38. The **corona** of two graphs $G_1 = (n_1, m_1)$ and $G_2 = (n_2, m_2)$, denoted by $G_1 \circ G_2$, is the graph obtained by taking one copy of G_1 and n_1 copies of G_2 , and then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2 .

Definition 1.1.39. The **cartesian product** of two graphs G and H denoted by $G \times H$ is the graph with $V(G \times H) = \{(u, v) : u \in V(G) \text{ and } v \in V(H)\}$ and any two vertices $(u_1, v_1), (u_2, v_2) \in G \times H$ are adjacent if one of the following holds.

- (i) $u_1 = u_2$ and $v_1 - v_2 \in E(H)$
- (ii) $u_1 - u_2 \in E(G)$ and $v_1 = v_2$.

1.2 Basic lemmas and theorems

Lemma 1.2.1. [8] *The line graph $L(G)$ has nine forbidden subgraphs.*

Figure 1.2.1 gives the nine forbidden subgraphs of $L(G)$.

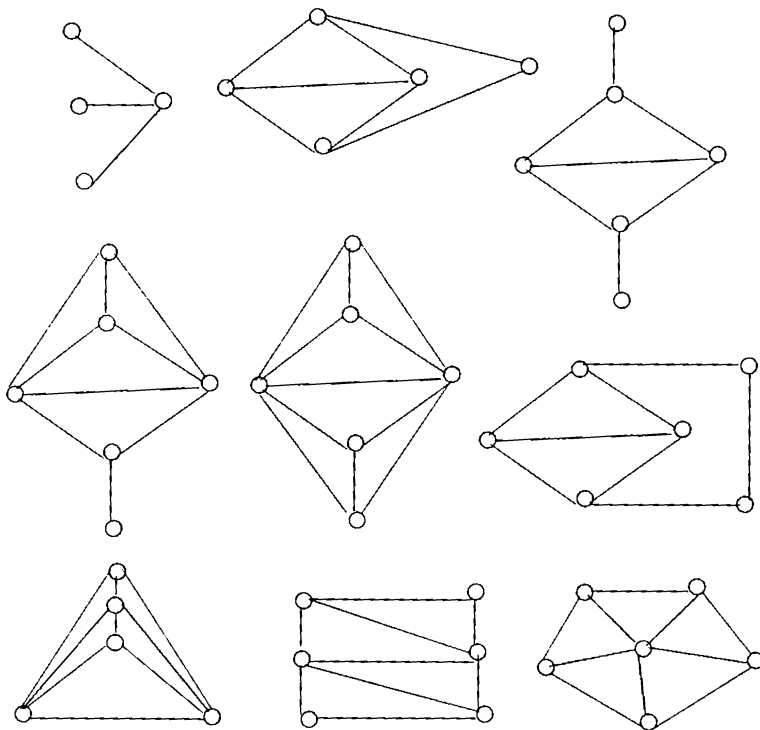


Figure 1.2.1

Lemma 1.2.2. [19] *G is a cograph if and only if G is P_4 -free.*

Lemma 1.2.3. [68] *If G is a cograph, then the domination number of G is at most two.*

Lemma 1.2.4. [29] *If G is a graph without isolated vertices, then $\gamma(G) \leq \text{dilw}(G)$.*

Lemma 1.2.5. [16] *A graph G is a threshold graph if and only if $\text{dilw}(G) = 1$.*

Lemma 1.2.6. [16] *A graph G is a threshold graph if and only if G contains no induced $\{2K_2, C_4\}$ and no P_4 .*

Lemma 1.2.7. [44], [48] *A graph G is a block graph if and only if $b(G) = \infty$.*

Lemma 1.2.8. [44], [48] *A graph G is weakly geodetic if and only if $b(G) \geq 5$.*

Lemma 1.2.9. [30] *A graph G is a split graph if and only if G contains no induced $2K_2, C_4$ and no C_5 .*

Lemma 1.2.10. [77] *For a connected graph G , $D^{+i}(G) \leq \kappa'(G)$.*

Lemma 1.2.11. [26], [31] *A connected graph G is Eulerian if and only if the degree of each vertex of G is even.*

Theorem 1.2.12. [15] *For any graph $G = (n, m)$, $\gamma'(G) \leq \lfloor n/2 \rfloor$.*

Theorem 1.2.13. [4] *For any connected graph G of even order n , $\gamma'(G) = n/2$ if and only if G is isomorphic to K_n or $K_{n/2, n/2}$.*

Theorem 1.2.14. [4] *For any tree T of order $n \neq 2$, $\gamma'(T) \leq (n - 1)/2$, equality holds if and only if T is isomorphic to the subdivision of a star.*

Theorem 1.2.15. [4] *Let $G = (n, m)$ be a connected unicyclic graph. Then $\gamma'(G) = \lfloor n/2 \rfloor$ if and only if G is isomorphic to either C_4 , C_5 , C_7 , $C_{3,k}$ or $C_{4,k}$ for some $k \geq 0$.*

Theorem 1.2.16. [41] *For a connected graph G , $\text{diam}(G) - 1 \leq \gamma_c(G) \leq 2\beta(G)$.*

Theorem 1.2.17. [23] *For a connected graph G , $\gamma_c(G) \leq 2\alpha(G) - 1$.*

Theorem 1.2.18. (Whitney's theorem) [79] *Let G be a simple graph with at least three vertices. Then G is 2-connected if and only if for each pair of distinct vertices u and v of G there are two internally disjoint $u - v$ paths in G .*

Theorem 1.2.19. (Menger's theorem) [62], [21] *Let u and v be two non adjacent vertices of a graph G . Then the maximum number of internally disjoint $u - v$ paths in G is the minimum number of vertices in a $u - v$ separating set.*

Theorem 1.2.20. (Generalized Whitney's theorem) [17]

A simple graph G is n -connected if and only if, given any pair of distinct vertices u and v of G , there are at least n internally disjoint $u - v$ paths in G .

1.3 New definitions

Definition 1.3.1. [59] The P_3 intersection graph of a graph G , $P_3(G)$ is the intersection graph of all induced 3-paths in G . That is, $P_3(G)$ has the induced paths on three vertices in G as its vertices and two distinct vertices in $P_3(G)$ are adjacent if the corresponding induced 3-paths in G intersect. If $a_1 - a_2 - a_3$ is an induced 3-path in G then the corresponding vertex in $P_3(G)$ is denoted by $a_1a_2a_3$.

Definition 1.3.2. A graph G is a P_3 intersection graph if there exists a graph H such that $G \cong P_3(H)$.

In Figure 1.3.1 a graph G and its $P_3(G)$ are shown.

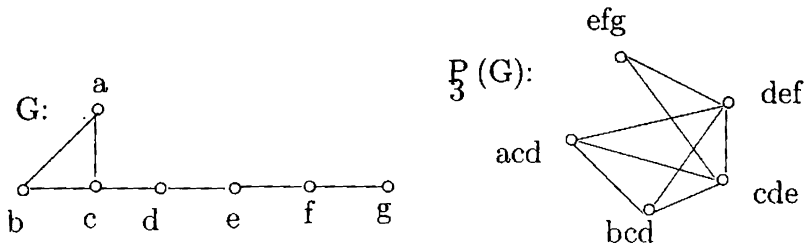


Figure 1.3.1

Definition 1.3.3. [57] The edge C_4 graph of a graph, $E_4(G)$ is a graph whose vertices are the edges of G and two vertices in $E_4(G)$ are adjacent if the corresponding edges in G are either incident or are opposite edges of some C_4 in G . This graph class is also known by the name edge graph in [66].

In $E_4(G)$ any two vertices are adjacent if the union of the corresponding edges in G induce any one of the graphs $P_3, C_3, C_4, K_4 - \{e\}, K_4$. If $a_1 - a_2$ is an edge in G , the corresponding vertex in $E_4(G)$ is denoted by a_1a_2 .

Definition 1.3.4. A graph G is an edge C_4 graph if there exists a graph H such that $G \cong E_4(H)$.

In Figure 1.3.2 a graph G and its $E_4(G)$ are shown.

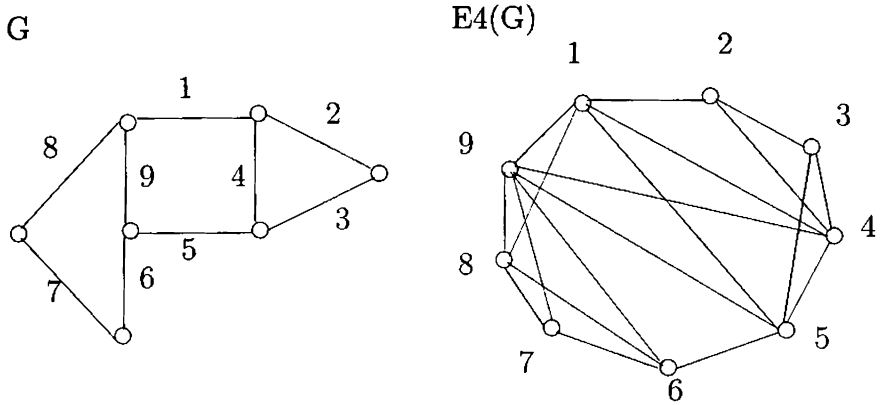


Figure 1.3.2

1.4 A survey of results

This section is a survey of results related to that of ours.

The H - **intersection graph** $Int_H(G)$ [66] is the intersection graph of all subgraphs of G that are isomorphic to H . If H is K_2 then $Int_H(G)$ is the line graph. Trotter [76] characterized the graphs for which $Int_{K_2}(H)$ is perfect. The 3-edge graph is the intersection graph of the set of all 3-edges of G [67]. The K_3 intersection graph is the 3-edge graph provided every edge

lies in some triangle [66]. In [2], Akiyama and Chvátal have characterized the graphs for which $IntP_3(G)$ is perfect.

In [27], Daniela derived some properties of the girth and connectivity of the path graphs. In [51], Knor and Niepel characterized the graphs isomorphic to their path graphs.

In [7], Bandelt and others proved that a bipartite graph is dismantlable if and only if its edge C_4 graph is dismantlable and a bipartite graph is neighborhood-Helly if and only if its edge C_4 graph is neighborhood-Helly. For any given graph G , the edge graph is a supergraph of $L(G)$. In [50] it has been shown that for any graph G without isolated vertices, there is a graph H such that $C(H) = G$ and $C(L(H)) = L(G)$.

Many types of dominations and their characteristics are discussed in [24], [39], [40]. In [18], efficient algorithms are developed for finding a minimum cardinality of connected dominating set and a minimum cardinality Steiner tree in permutation graphs. In [20], forbidden subgraph conditions sufficient to im-

ply the existence of a dominating clique are given.

The concept of edge domination was introduced by Mitchell and Hedetniemi in [63]. The edge dominating sets and the bounds for the edge domination number γ' are studied in [47]. In [15], an upper bound for $\gamma'(G)$ is obtained. Again, these bounds are modified in [4] for a connected graph G of even order, tree and a connected unicyclic graph. In [23], [41] a bound for the connected domination number of a graph G with regard to the diameter of the graph, the vertex independence number and the matching number of a graph are obtained.

In [29], it is observed that for graphs G without isolated vertices, $\gamma(G) \leq dilw(G)$. Threshold graphs were introduced by Chvátal.V and Hammer.P.L in [16], where different characterizations for such graphs are given. Block graphs, geodetic graphs and weakly geodetic graphs are studied in detail in [44], [48]. Stemple et.al [74] showed that a graph is geodetic if and only if each of its block is geodetic. It is known that block graphs \subseteq geodetic graphs \subseteq weakly geodetic graphs [13]. In

[19], eight characterizations of cographs which include the recursive characterization and the forbidden subgraph characterization are given. The median and the anti-median of cographs are discussed in [69]. The recent Ph.D thesis by Ms. Aparna Lakshmanan [3] contains results regarding cographs and other graph classes such as the Gallai and the anti- Gallai graphs, the clique irreducible graphs, the clique vertex irreducible graphs and the weakly clique irreducible graphs.

The concept of wide diameter has been discussed and used in distributed and parallel computer networks [45]. In [43], Hou and Wang defined generalized wide diameter and calculated it for any k - regular k -connected graph. A generalized p -cycle is a digraph whose set of vertices is partitioned into p parts that can be ordered in such a way that a vertex is adjacent only to the vertices in the next part. The bounds for the wide diameter of the generalized p -cycle is obtained in [28]. The wide diameter of butterfly networks is studied in [53]. Bolian Liu and Xiankun Zhang studied some problems on the relations between $D_w(G)$ and $diam(G)$ in [54]. In this paper they characterized the graphs G for which $D_w(G) = diam(G), w > 1$.

The diameter of a graph is an important factor for communication as it determines the maximum communication delay between any pair of processors in a network. The diameter of a graph may be affected by the addition or deletion of edges. In [33], Graham and Harary studied this aspect in hypercubes and proved that $D^{-1}(Q_n) = 2$, $D^{+1}(Q_n) = n - 1$ and $D^{+0}(Q_n) \geq (n-3)2^{n-1} + 2$. Bouabdallah et.al [12] improved the lower bound of $D^{+0}(Q_n)$ and furthermore gave an upper bound, $(n-2)2^{n-1} - {}^n C_{\lfloor n/2 \rfloor} + 2 \leq D^{+0}(Q_n) \leq (n-2)2^{n-1} - \lceil 2^{n-1}/2n - 1 \rceil + 1$.

The diameter variability arising from the addition or deletion of edges of a graph G is defined in [77] and in this paper, Wang et.al proved that $D^{-1}(C_m) \geq 2$, $D^{-1}(T_{m,n}) \geq 2$, $D^{-2}(T_{m,n}) = 2$ for $m \geq 14$ and $m \neq 15$. Also they obtained the exact value of $D^{+1}(T_{m,n})$.

1.5 Summary of the thesis

This thesis entitled '*Studies on some graph operators and related topics*' is divided into five chapters including an introductory chapter. We shall now give a summary of each chapter.

The first chapter is an introduction and contains literature on graph operators. It also includes some basic definitions and terminology used in this thesis.

In the second chapter, the P_3 intersection graph of a graph G which is the intersection graph of all induced 3-paths in G is studied in detail. The following are some of the results proved:

- For a connected graph G , $P_3(G)$ is bipartite if and only if G is P_3 , P_4 , $K_4 - \{e\}$ or a paw.
- $K_{1,4}$ is a forbidden subgraph for a graph to be the P_3 intersection graph.
- There exist only a finite family of forbidden subgraphs for the P_3 intersection graphs to be H-free for any finite graph

H.

- For a connected graph G , $\chi(P_3(G)) \geq \chi(G) - 1$. The equality holds if and only if G is either $K_n - \{e\}$ or a complete graph with a pendant vertex attached to it.
- The relationship between the chromatic number, clique number, connectivity, independence number, domination number, the radius and the diameter of a graph and its P_3 intersection graph.

The third chapter is the study of another graph operator - the edge C_4 graph of a graph. If G does not contain C_4 as a subgraph, then the edge C_4 graph of a graph coincides with its line graph. So if G is an Eulerian graph which does not contain C_4 as a subgraph, then $E_4(G)$ is Eulerian. Following are some of the results obtained:

- There exist infinitely many pairs of non isomorphic graphs whose edge C_4 graphs are isomorphic.
- Characterizations for $E_4(G)$ being connected, complete, bipartite etc.

- There is no forbidden subgraph characterization for $E_4(G)$.
- The relationships between the diameter, radius, center, domination number of G and those of $E_4(G)$.
- Relationships between different types of dominations of G and that of $E_4(G)$.
- For any connected graph G , $\text{diam}(G) - 2 \leq \gamma_c(E_4(G)) \leq 2\beta(G) - 1$.
- A bound for the domination number of $E_4(G)$ in terms of the order of G . Further for a graph G , which is a tree or a unicyclic graph, characterization is obtained for the strict bound of the domination number of $E_4(G)$.
- Conditions for the $E_4(G)$ being a clique dominated graph, threshold graph, cograph, geodetic graph, weakly geodetic graph and block graph.

The dynamics such as convergence, divergence, periodicity, fixedness etc of the P_3 intersection graph and the edge C_4 graph are included in chapter four. The following are some of the results proved:

- There are no P_3 -periodic graphs.
- If a graph G is P_3 -convergent, then it is $P_3^n(G)$ -complete for some $n \geq 1$ and hence all the P_3 -convergent graphs are P_3 -mortal graphs.
- The relationship between the touching number of $P_3(G)$ and the vertex touching number of G .
- Characterization of the E_4 -convergent graphs.
- The relationship between the touching number of G and that of $E_4(G)$.

In chapter five of this thesis, the diameter variability and the w -wide diameter of the three graph operators - the P_3 intersection graph, the edge C_4 graph and the line graph and some graph operations such as join and corona are studied. Some of the results are listed below:

- Corresponding to a w -container in G , there exists w -containers in $P_3(G)$ and $E_4(G)$.
- Strict bound for the w -wide diameter of $P_3(G)$, $L(G)$ and $E_4(G)$.

- Strict bounds for D^{+i} of $P_3(G)$, $E_4(G)$ and $L(G)$.
- The diameter variability of join and corona of two graphs.

All the graphs considered in this thesis are finite, undirected and simple. Some results of this thesis are included in [55] - [60]. We conclude the thesis with some suggestions for further study and a bibliography.

1.6 List of publications

Papers presented

- (1) The P_3 intersection graph, 20th Annual Conference of Ramanujan Mathematical Society, July 27 – 30, 2005, University of Calicut, Calicut, India.
- (2) The edge C_4 graph of a graph, International Conference on Discrete Mathematics, December 15 – 18, 2006, IISc, Bangalore, India.

- (3) The Dynamics of the P_3 intersection graph, National Seminar on Algebra and Discrete Mathematics, November 14 – 16, 2007, University of Kerala, Trivandrum, India.
- (4) The edge C_4 graph of some graph classes, International Conference on Discrete Mathematics, June 6 – 10, 2008, University of Mysore, Mysore, India.
- (5) Some Domination parameters in $E4(G)$, National Seminar on Discrete Mathematics and its Applications, August 7 – 9, 2008, St.Pauls College, Kalamassery, India.
- (6) The wide diameter and diameter variability of some graphs, International Conference on Graph Theory and its Applications, December 11 – 13, 2008, Amritha Viswa Vidya Peetham, Coimbatore, India.

Papers published / communicated

- (1) Manju K. Menon, A. Vijayakumar, The P_3 intersection graph, Util. Math. 75 (2008), 35 – 50.
- (2) Manju K. Menon, A. Vijayakumar, The Edge C_4 graph of a graph, Proceedings of the International Conference

- on Discrete Mathematics, Ramanujan Math. Soc. Lect. Notes Ser. 7 (2008), 245 – 248.
- (3) Manju K. Menon, A. Vijayakumar, Dynamics of the P_3 intersection graph (communicated).
- (4) Manju K. Menon, A. Vijayakumar, The Edge C_4 graph of some graph classes (communicated).
- (5) Manju K. Menon, A. Vijayakumar, Some domination parameters in $E_4(G)$ (communicated).
- (6) Manju K. Menon, Daniella Ferrero, A. Vijayakumar, The w -wide diameter and diameter variability of some graphs (in preparation).

Chapter 2

The P_3 intersection graph of a graph

This chapter deals with the graph operator known as ‘the P_3 intersection graph’. We study the conditions for the $P_3(G)$ to be connected, bipartite, tree, geodetic, block etc. The existence of a finite family of forbidden subgraphs for the $P_3(G)$ to be H -free, H being a finite graph, is proved and the forbidden subgraph

Some results of this chapter are included in the following papers.

1. Manju K. Menon, A. Vijayakumar, The P_3 intersection graph, Util. Math. 75 (2008), 35 - 50.
2. Manju K. Menon, A. Vijayakumar, Dynamics of the P_3 intersection graph, (communicated).

characterizations of G for which the $P_3(G)$ are complete, chordal etc are discussed. The relationship between the clique number, chromatic number, connectivity, independence number, domination number, radius and diameter of a graph and its $P_3(G)$ are also studied in detail.

2.1 The P_3 intersection graph of a graph

For any graph G which is the union of complete graphs, $P_3(G)$ is null graph. Hence in this chapter we do not consider such graphs. If G is a connected graph of order at most five then $P_3(G)$ is complete.

In general, the H -intersection graph of a connected graph need not be connected. But, in the case of $P_3(G)$, we have

Theorem 2.1.1. *$P_3(G)$ is connected if and only if G has exactly one component containing an induced P_3 .*

Proof. Suppose that G contains more than one component containing an induced P_3 . Let $a_1 - a_2 - a_3$ and $b_1 - b_2 - b_3$ be any

two induced 3-paths in G which lie in distinct components of G . Then by the definition of $P_3(G)$ the corresponding vertices $a_1a_2a_3$ and $b_1b_2b_3$ in $P_3(G)$ cannot be connected by a path and hence $P_3(G)$ is disconnected.

Let G have exactly one component containing an induced P_3 . Suppose that $x = a_1a_2a_3$ and $y = b_1b_2b_3$ are any two non-adjacent vertices in $P_3(G)$. If $a_i, i = 1, 2, 3$ and $b_j, j = 1, 2, 3$ are adjacent then $a_1a_2a_3, a_i b_j b_{j+1}$ or $a_i b_j b_{j-1}, b_1 b_2 b_3$ is a path connecting x and y . If a_i and b_j are not adjacent then let the shortest path connecting $a_i, i = 1, 2, 3$ and $b_j, j = 1, 2, 3$ be $a_i, c_1, c_2, \dots, c_n, b_j$. If $n = 1$, then $a_1a_2a_3, a_i c_1 b_j, b_1 b_2 b_3$ is a path connecting x and y . If $n \geq 2$, then $a_1a_2a_3, a_i c_1 c_2, \dots, c_{n-1} c_n b_j, b_1 b_2 b_3$ is a path connecting x and y in $P_3(G)$. Hence $P_3(G)$ is connected. \square

Theorem 2.1.2. *If G is k -connected; $k \geq 2$, then $P_3(G)$ is $(k - 1)$ -connected. Further $\kappa(P_3(G)) = \kappa(G) - 1$ if and only if G is $K_n - \{e\}$.*

Proof. Let G be k -connected. Then by Theorem 1.2.20, for any two vertices u and v , there exists at least k internally disjoint

$u - v$ paths.

Let $u_1u_2u_3$ and $v_1v_2v_3$ be any two distinct vertices in $P_3(G)$. In G , if u_i and v_j are connected by a path which contain at least one vertex other than these six vertices, then correspondingly there exists at least one path in $P_3(G)$ joining $u_1u_2u_3$ and $v_1v_2v_3$. So, it is enough to consider the paths in G involving one or more of these six vertices.

Case 1: Let u_i and v_j be non - adjacent for some $i, j \in \{1, 2, 3\}$.

Then corresponding to any path in G joining u_i and v_j , there exists a path in $P_3(G)$ joining $u_1u_2u_3$ and $v_1v_2v_3$. So $\kappa(P_3(G)) \geq \kappa(G)$.

Case 2: Let u_i be adjacent to v_j .

Case 2a: All u_i s and v_j s are distinct.

Then in between any u_i and v_j in G , there exists at most five internally disjoint $u_i - v_j$ paths involving those six vertices only. But there exists six internally disjoint paths joining $u_1u_2u_3$ and $v_1v_2v_3$ in $P_3(G)$ which are of the form $u_1u_2u_3, u_1v_ju_3, v_1v_2v_3$ and $u_1u_2u_3, v_1u_jv_3, v_1v_2v_3, j \in \{1, 2, 3\}$. Hence in this case also

$$\kappa(P_3(G)) \geq \kappa(G).$$

Case 2b: u_i s and v_j s share a common vertex.

In this case there exists more paths between $u_1u_2u_3$ and $v_1v_2v_3$ in $P_3(G)$ than the minimum number of paths between any u_i and v_j in G . Hence $\kappa(P_3(G)) \geq \kappa(G)$.

Case 2c: u_i s and v_j s share two common vertices.

In this case also, except when they form $K_4 - \{e\}$, the number of internally disjoint paths between $u_1u_2u_3$ and $v_1v_2v_3$ in $P_3(G)$ is greater than or equal to the minimum number of internally disjoint paths between any u_i and v_j in G . If there exists one more vertex, then $\kappa(P_3(G)) < \kappa(G)$ only when the newly adjoined vertex is adjacent to all the four vertices. If we adjoin more vertices to this graph also, $\kappa(P_3(G)) < \kappa(G)$ only when the adjoined vertices are adjacent to all the other existing vertices. Hence $\kappa(P_3(G)) < \kappa(G)$ only when G is $K_n - \{e\}$. For $K_n - \{e\}$, $\kappa(P_3(G)) = \kappa(G) - 1$. \square

As to the question whether every graph is the P_3 intersection graph of some graph, we have the following theorem:-

Theorem 2.1.3. *The following graphs G cannot be the P_3 intersection graph of any graph.*

1. G is a connected graph having at least three vertices and a pendant vertex .
2. There exists a vertex v in G with $d(v) = 2$ such that v is adjacent to any two non-adjacent vertices in G .
3. G is a connected triangle free graph having at least three vertices.

Proof. 1. Let G be a connected graph having at least three vertices. Let x be a pendant vertex of G and z be the unique vertex adjacent to x . If possible let there exist a graph H such that $P_3(H) = G$. Since there are at least three vertices, there exists a vertex adjacent to z and let it be y . Since x and y are two non-adjacent vertices in $G = P_3(H)$, we can assume that $x = a_1a_2a_3$ and $y = b_1b_2b_3$ where a_i s and b_j s are distinct vertices in H . Since z is adjacent to both x and y , z corresponds to a 3-path in H which must contain at least one a_i and b_j . So z must be of the form a_ib_jc or a_1cb_j or ca_ib_j .

Let $z = a_ib_jc$. If $i = 1$, then $a_2 - a_1 - b_j$ is a 3-path. But if this is an induced path, then x cannot remain a pendant vertex. So $a_2 - b_j$ is an edge in H . Then $a_3 - a_2 - b_j$ is

a 3-path. But if this is an induced path then x cannot remain a pendant vertex. So $a_3 - b_j$ is an edge in H . Then $a_1 - b_j - a_3$ is an induced 3-path. If the corresponding vertex $a_1 b_j a_3$ is different from z , then it is adjacent to x , a contradiction to the fact that x is a pendant vertex. If $a_1 b_j a_3 = z$, then we can show that there exists an induced 3-path with a_1 and two b_l s, $l = 1, 2, 3$ as its vertices. The corresponding vertex which is different from z is adjacent to x which will also lead to a contradiction. So G cannot be the P_3 -graph of any graph. The case is similar when $i = 2, 3$ also. The proof is similar when $z = a_i c b_j$ or $z = c a_i b_j$.

2. Suppose now that G has a vertex v with $d(v) = 2$ and let $G = P_3(H)$. Let v be adjacent to v_1 and v_2 where v_1 and v_2 are non-adjacent vertices. Let $v_1 = a_1 a_2 a_3$ and $v_2 = b_1 b_2 b_3$ where a_i s and b_j s are distinct vertices in H . Then v must be of the form $a_i b_j c$ or $a_i c b_j$ or $c a_i b_j$. So as in the proof given above, we can show that there exists a vertex adjacent to v which is different from both v_1 and v_2 , which is a contradiction to the fact that $d(v) = 2$.

3. Finally, let G be a connected triangle free graph. If possible assume that $G = P_3(H)$. Since G has at least three vertices it contains a vertex z which is adjacent to two non-adjacent vertices x and y . Let $x = a_1a_2a_3$ and $y = b_1b_2b_3$, where a_i s and b_j s are distinct vertices in H . Then z must be of the form a_ib_jc or a_2cb_j or ca_ib_j . Using the similar arguments as in the above proofs, we can show that there exists a vertex which is adjacent to both x and z , which is a contradiction to the fact that G is triangle free.

□

Lemma 2.1.4. *If G is a connected graph having at least five vertices, then $P_3(G)$ has at least three vertices.*

Proof. Let G be a connected graph having at least five vertices. Let x and y be two non-adjacent vertices of G . Let the shortest path connecting x and y be $x, v_1, v_2, \dots, v_n, y$. If $n \geq 3$ then $P_3(G)$ clearly contains at least three vertices. If $n = 2$ then since G is a connected graph having at least five vertices, the fifth vertex must be adjacent to at least one of x, v_1, v_2, y . Then there exists at least three induced 3-paths in G and hence $P_3(G)$ contains at least three vertices. If $n = 1$, there exists at least

two more vertices in G and they must be connected to x, v_1, y . In any case there exists at least three induced 3-paths in G and hence $P_3(G)$ contains at least 3 vertices. \square

Theorem 2.1.5. *Let G be a connected graph. Then $P_3(G)$ is bipartite if and only if G is $P_3, P_4, K_4 - \{e\}$ or paw.*

Proof. Let $P_3(G)$ be bipartite. Then $P_3(G)$ cannot contain triangles. So by Theorem 2.1.3 (3), the only bipartite P_3 intersection graphs are K_1 and K_2 . Again by Lemma 2.1.4, G can have at most four vertices. Since we are considering only connected graphs, the theorem follows. \square

Corollary 2.1.6. *For a connected graph G , $P_3(G)$ is a tree if and only if G is $P_3, P_4, K_4 - \{e\}$ or paw.*

Theorem 2.1.7. *For any connected graph G , $P_3(G)$ is a block.*

Proof. Suppose that $w = xyz$ is a cut vertex in $P_3(G)$. Then there exists two non adjacent vertices $u_1u_2u_3$ and $v_1v_2v_3$ such that the only path joining them is $u_1u_2u_3, w, v_1v_2v_3$. Then u_i s and v_j s are distinct and some $u_i, v_j = x, y$ or z . Thus we can find at least one more path joining $u_1u_2u_3$ and $v_1v_2v_3$, which is a contradiction to the fact that xyz is a cut vertex. \square

Theorem 2.1.8. *The only connected geodetic P_3 intersection graphs are the complete graphs.*

Proof. Let $P_3(G)$ be non-complete. Consider two non-adjacent vertices $u_1u_2u_3$ and $v_1v_2v_3$ in $P_3(G)$. Since G is connected, we may choose $v_1v_2v_3$ such that u_i is adjacent to some v_j for $i, j \in \{1, 2, 3\}$. Then there exists at least two disjoint paths of length two connecting $u_1u_2u_3$ and $v_1v_2v_3$ and hence $P_3(G)$ cannot be a geodetic graph. \square

2.2 Forbidden subgraph characterizations

In this section we prove that the P_3 intersection graphs have a forbidden subgraph characterization. Even though many well known classes of graphs have forbidden subgraph characterizations, the number of the forbidden subgraphs need not be finite. But, we prove that there exist only finitely many forbidden subgraphs for the P_3 intersection graph to be H - free where H is any finite graph. We also obtain forbidden subgraph characteri-

zations for the P_3 intersection graph to be chordal and complete.

Theorem 2.2.1. *If G is a P_3 intersection graph then $K_{1,4}$ is a forbidden subgraph for G .*

Proof. Suppose that $G = P_3(H)$ contains $K_{1,4}$ as an induced subgraph. Let v be the central vertex of $K_{1,4}$ and v_1, v_2, v_3, v_4 be its neighbors in G . Then v corresponds to an induced 3-path in H which intersects with all the four distinct 3-paths corresponding to v_1, v_2, v_3 and v_4 , which is not possible. Hence $K_{1,4}$ is a forbidden subgraph for the P_3 intersection graph. \square

Lemma 2.2.2. *Let $\varphi = \{G : P_3(G) \text{ is } H\text{-free}\}$ where H is any finite graph. Then the property $P, G \in \varphi$ is vertex hereditary.*

Proof. Let $G \in \varphi$. Suppose that $G - \{v\} \notin \varphi$. So $P_3(G - \{v\})$ contains H as an induced subgraph. Then this H will be induced in $P_3(G)$ also, which is a contradiction to the fact that $G \in \varphi$. \square

Theorem 2.2.3. *The collection φ has only a finite class of vertex minimal forbidden subgraphs.*

Proof. The property $G \in \varphi$ is vertex hereditary. So φ must have

vertex minimal forbidden subgraphs. Let F be the collection of all such vertex minimal forbidden subgraphs. Let $G_1 \in F$. Then $P_3(G_1)$ contains H as an induced subgraph. So, corresponding to a vertex in H there exists an induced 3-path in G_1 . This implies that the number of vertices in G_1 covered by these 3-paths cannot exceed $3n$ where n is the number of vertices in H . If G_1 contains more than $3n$ vertices, then there exists a vertex v in G_1 such that any induced 3-path containing v does not determine a vertex of H in $P_3(G_1)$. Then $G_1 - \{v\}$ is also forbidden for φ which is a contradiction to the vertex minimality of G_1 . Hence the number of vertices of G_1 is bounded by $3n$ and hence φ has only a finite class of vertex minimal forbidden subgraphs. \square

Corollary 2.2.4. *Let $\mathfrak{S} = \{G : P_3(G) \text{ is chordal}\}$. The collection \mathfrak{S} has an infinite class of vertex minimal forbidden subgraphs.*

Proof. Let $G \in \mathfrak{S}$. Then by Lemma 2.2.2, the property P , $G \in \mathfrak{S}$ is vertex hereditary. So \mathfrak{S} must have vertex minimal forbidden subgraphs. If G contains C_n , $n \geq 6$ as an induced subgraph, then $P_3(G)$ contains C_n , $n \geq 4$ and hence cannot be chordal.

Also $P_3(C_n - \{v\})$, $n \geq 6$ is chordal. So C_n , $n \geq 6$ are vertex minimal forbidden subgraphs for \mathfrak{S} . Thus there exists an infinite class of vertex minimal forbidden subgraphs for \mathfrak{S} . \square

Corollary 2.2.5. *Let $\Psi = \{G : P_3(G) \text{ is complete}\}$. Then any vertex minimal forbidden subgraph for Ψ has exactly six vertices.*

Proof. Let $G \in \Psi$. Then G is induced P_3 -free. So by Lemma 2.2.2, the property $G \in \Psi$ is vertex hereditary. So it has vertex minimal forbidden subgraphs. The $P_3(G)$ is complete for any graph having at most five vertices. So, a forbidden subgraph must have at least six vertices. Let G_1 be any vertex minimal forbidden subgraph for Ψ . Since G_1 is a forbidden subgraph for $P_3(G)$ being complete, it must have at least two disjoint 3-paths, $a_1 - a_2 - a_3$ and $b_1 - b_2 - b_3$. These six vertices are enough to induce a vertex minimal forbidden subgraph. \square

2.3 The chromatic number of $P_3(G)$

In this section we study the relationship between the chromatic number of G and that of $P_3(G)$.

Lemma 2.3.1. *For a connected graph G , $\omega(P_3(G)) \geq \omega(G) - 1$.*

Proof. Let $\omega(G) = k$. Since G is non-complete and connected, there exists a vertex u adjacent to at least one vertex of the k -clique in G . If u is joined to t vertices of this k -clique then there are $t(k - t)$ induced 3-paths in G where u is common to all these induced 3-paths. So $\omega(P_3(G)) \geq t(k - t)$. Now, if $t(k - t) < k - 1$ then $k < (t + 1)$ which is a contradiction to the fact that $\omega(G) = k$. So $\omega(P_3(G)) \geq k - 1$. \square

Theorem 2.3.2. *For a connected graph G , $\chi(P_3(G)) \geq \chi(G) - 1$. The equality holds if and only if G is either $K_n - \{e\}$ or a complete graph with a pendant vertex attached to it.*

Proof. Let $\chi(G) = k$. Then there exists a vertex v in G with color k such that its neighbors v_1, v_2, \dots, v_{k-1} have distinct colors $1, 2, \dots, k - 1$ respectively.

If these k vertices form a complete subgraph, then $\omega(G) \geq k$. So $\chi(P_3(G)) \geq \omega(P_3(G)) \geq k - 1$, by Lemma 2.3.1.

If these k vertices do not form a complete subgraph, then let 'm' be the size of maximal clique in the subgraph induced by these k vertices. Clearly v is a vertex in this m -clique. Then among the k vertices, there are $k - m$ vertices adjacent to v which are not in the m -clique. Let v_i be such a vertex. Then this v_i can be adjacent to at most $m - 1$ vertices of the m -clique. In any case we can find at least k distinct induced 3-paths having a common vertex. The corresponding k vertices in $P_3(G)$ will form a complete subgraph having k vertices and hence $\chi(P_3(G)) \geq k$.

Hence the equality holds only when there is a k -clique in G . Since G is connected and non-complete, there exists a vertex u_1 which is adjacent to some of the v_i s in the k -clique. If u_1 is adjacent to t vertices of the k -clique where $2 \leq t \leq k - 2$, then there exists at least k distinct induced 3-paths having a common vertex. Hence, in this case $\chi(P_3(G)) > k - 1$. So u_1 can be adjacent with either 1 or $k - 1$ vertices of the k -clique. If there exists one more vertex in G other than these $k + 1$ vertices, then also we can find at least k induced 3-paths having a common vertex and hence $\chi(P_3(G)) \geq k$. So when

$\chi((P_3(G))) = \chi(G) - 1$, there exists exactly $k + 1$ vertices such that u_1 is adjacent to 1 or $k - 1$ vertices of the k -clique. If u_1 is adjacent to only one vertex of the k -clique, then the graph is a complete graph with a pendant vertex attached to it and if u_1 is adjacent to $k - 1$ vertices of the k -clique, then the graph is $K_{k+1} - \{e\}$ and hence the result. \square

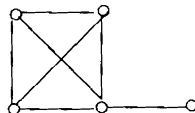
Theorem 2.3.3. *Given any two positive numbers a and b where $a > 1$ and $b \geq a - 1$, there exists a graph G such that $\chi(G) = a$ and $\chi(P_3(G)) = b$.*

Proof. Consider the following cases:

Case 1: $b = a - 1$

The graph G is obtained by attaching a pendant vertex to any one vertex of K_a .

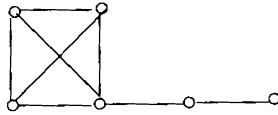
Illustration: When $a = 4; b = 3$,



Case 2: $b = a$

Consider the graph G in Case 1. Then attach a single vertex to the pendant vertex of G . This is the required graph.

Illustration: When $a = 4; b = 4$

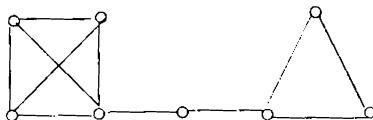


Case 3: $b > a$

Subcase 3a: $b \leq 2a - 1$

Consider the graph G in case 1. Any one vertex of K_{b-a+1} is joined to the pendant vertex of G . This is the required graph.

Illustration: When $a = 4; b = 6$

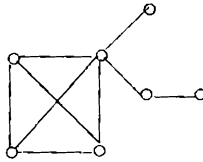


Subcase 3b: $b > 2a - 1$

Let k be the maximum integer satisfying the equation ${}^k C_2 + (a - 1)k = b$. Join k pendant vertices to the same vertex of K_a .

Replace any one of these pendant vertices by $K_{b - [{}^k C_2 + (a - 1)k]}$.

Illustration: When $a = 4; b = 9$



In all the above cases, $P_3(G) = K_b$ and hence $\chi(P_3(G)) = b$.

Since all possible cases have been covered, the result follows. \square

2.4 Some other graph parameters of

$P_3(G)$

In this section we study the relationship between the parameters such as domination number γ , independence number α , radius

and diameter of G and those of $P_3(G)$.

Theorem 2.4.1. *Given any two positive numbers a and b where $a > 1$ there exists a graph G such that $\gamma(G) = a$ and $\gamma(P_3(G)) = b$.*

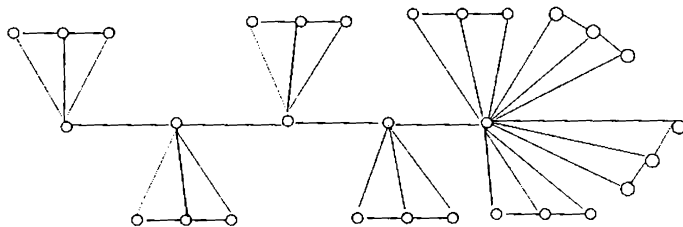
Proof. Consider the following cases.

Case 1: Suppose $a < b$.

Consider an induced $v_1 - v_a$ path. To each $v_i, i = 1, 2, \dots, a-1$, join an induced 3-path - $w_{i1} - w_{i2} - w_{i3}$. To v_a join $2(b - a + 1)$ disjoint induced 3-paths. This is the required graph G . Clearly $\gamma(G) = a$. Consider the $a - 1$ vertices in $P_3(G)$ which are of the form $w_{i1}v_iv_{i+1}, i = 1, 2, \dots, a - 1$. In $P_3(G)$, these vertices will dominate all the vertices except the vertices corresponding to the $2(b - a + 1)$ disjoint paths joined to v_a . These $2(b - a + 1)$ vertices can be dominated exactly by $b - a + 1$ vertices which are of the form $u_iv_au_j$ where u_i and u_j are vertices in any two of the disjoint induced P_3 s joined to v_a . The above described

collection of $a - 1$ vertices together with these $b - a + 1$ vertices will form a minimum dominating set for $P_3(G)$. Hence $\gamma(P_3(G)) = (b - a + 1) + (a - 1) = b$.

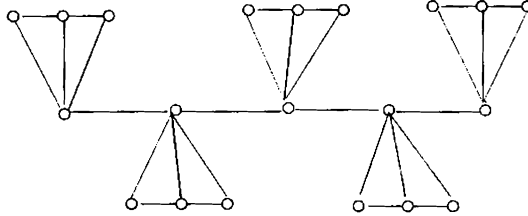
As an example, consider $a = 5; b = 6$. The corresponding graph G is,



Case 2: Suppose $a = b$.

Consider an induced $v_1 - v_a$ path. To each $v_i, i = 1, 2, \dots, a$ join an induced 3-path, $w_{i1}w_{i2}w_{i3}$. This is the required graph G . Clearly $\gamma(G) = a$. In $P_3(G)$, vertices of the form $w_{i1}v_iw_{i3}, i = 1, 2, \dots, a$ form a minimum dominating set. Hence $\gamma(P_3(G)) = a = b$.

As an example, consider $a = 5; b = 5$. The corresponding graph G is,

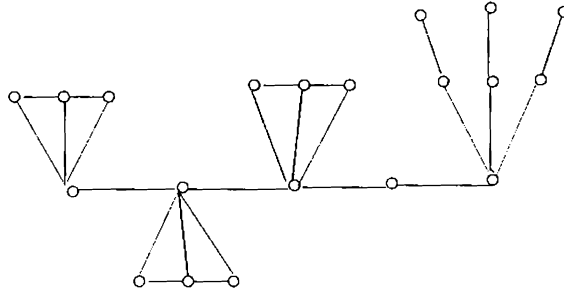


Case 3: Suppose $a > b$.

Consider an induced $v_1 - v_{b+1}$ path. To each $v_i, i = 1, 2, \dots, b-1$ join an induced 3-path, $w_{i1}w_{i2}w_{i3}$. To v_{b+1} , attach $a - b + 1$ disjoint K_2 s. This is the required graph G . Clearly $\gamma(G) = (b - 1) + (a - b + 1) = a$. In $P_3(G)$ the $(b - 1)$ vertices which are of the form $w_{i1}v_iw_{i3}, i = 1, 2, \dots, b - 1$ and $v_b c_1 c_2$ where c_1, c_2 are the vertices in any K_2 attached to v_b will dominate all the vertices. Clearly this is the minimum number of vertices in any dominating set of $P_3(G)$. Hence $\gamma(P_3(G)) = b$.

As an example, consider $a = 6; b = 4$. The corresponding

graph G is,



□

Theorem 2.4.2. *Given any two positive numbers a and b where $a > 1$, there exists a graph G such that $\alpha(G) = a$ and $\alpha(P_3(G)) = b$.*

Proof. Consider the following cases:

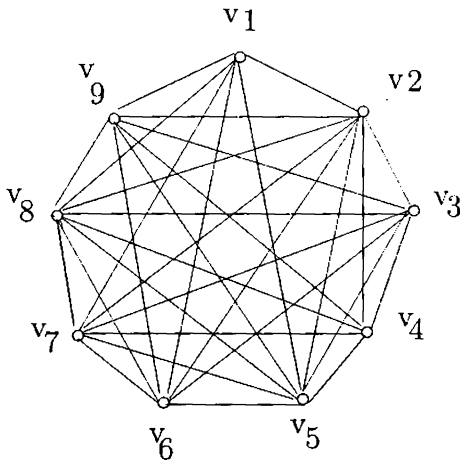
Case 1: Suppose $a < b$.

Consider a K_{3b} whose vertices are labelled as v_1, v_2, \dots, v_{3b} . From this graph, edges of the form $v_{3k-2} - v_{3k}$, $k = 1, 2, \dots, b$ and edges whose end vertices are both of the form v_{3k+1} , $k = 0, 1, \dots, a - 1$ are deleted. This is the required graph G . Clearly

$\alpha(G) = a$ where a maximum independent set is $\{v_1, v_4, \dots, v_{3a-2}\}$.

Also $\alpha(P_3(G)) = b$ where a maximum independent set is $\{v_{3k-2}, v_{3k-1}, v_{3k}; k = 1, 2, \dots, b\}$.

As an example, consider $a = 2; b = 3$. The corresponding graph G is,

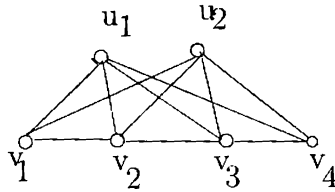


Case 2: Suppose $a = b$.

Consider $G = (K_a)^c \vee P_{2a}$. Clearly $\alpha(G) = a$. $\alpha(P_3(G)) = a$ where the maximum independent set is $\{v_i u_i v_{a+i}; i = 1, 2, \dots, a\}$

where v_i and v_{a+i} are vertices in P_{2a} and u_i is a vertex in $(K_a)^c$.

As an example, consider $a = 2; b = 2$. The corresponding graph G is,



Case 3: Suppose $a > b$.

Subcase 3a: Let $a \geq 2b$.

Consider $G = K_{a,b}$ with the partition $\{u_1, u_2, \dots, u_a\}$ and $\{v_1, v_2, \dots, v_b\}$. Clearly $\alpha(G) = a$. Since a maximum independent set in $P_3(G)$ is $\{u_i v_i u_{b+i}; i = 1, 2, \dots, b\}$, $\alpha(P_3(G)) = b$.

As an example, consider $a = 6; b = 2$. Then the graph G is $K_{6,2}$.

Subcase 3b: Let $a < 2b$.

Let $t = \lfloor a/2 \rfloor$. Let $G = K_{a,b} \vee P_{2(b-t)}$. Let the partition of $K_{a,b}$ be $\{u_1, u_2, \dots, u_a\}$ and $\{v_1, v_2, \dots, v_b\}$ and let the vertices in the path be $w_1, w_2, \dots, w_{2(b-t)}$. Then $\alpha(G) = a$. Consider the following independent set of vertices in $P_3(G)$, $\{u_i v_i u_{t+i}; i = 1, 2, \dots, t, w_j v_{t+j} w_{b-t+j}, j = 1, 2, \dots, b - k\}$. This is an independent set having maximum number of vertices in $P_3(G)$. Hence $\alpha(P_3(G)) = b$.

As an example, when $a = 5; b = 3$, the graph $G = K_{5,3} \vee P_2$. \square

Theorem 2.4.3. *For a connected graph G , $\text{rad}(P_3(G)) \leq \text{rad}(G) + 1$. The equality holds only when $\text{rad}(G) = 1$. Further if $\text{rad}(G) \geq 4$ then $\text{rad}(P_3(G)) < \text{rad}(G)$.*

Proof. Let u be a center of G . So $d(u, v) \leq \text{rad}(G)$ for all $v \in V(G)$. Since G is not a complete graph, there exists an induced 3-path having u as a vertex in it. Let the corresponding vertex in $P_3(G)$ be $a_1 a_2 a_3$ where u is some a_i . Let $b_1 b_2 b_3$ be any other vertex in $P_3(G)$. If $d(u, b_j) = 1$, then $a_1 a_2 a_3, u b_j b_{j+1}$ or

$ub_jb_{j-1}, b_1b_2b_3$ is a path connecting $a_1a_2a_3$ and $b_1b_2b_3$ and hence $d(a_1a_2a_3, b_1b_2b_3) \leq 2 = d(u, b_j) + 1$. Now, if $d(u, b_j) = k > 1$, let a shortest path connecting u and b_j be $u, c_1, c_2, \dots, c_{k-1}, b_j$. Then $a_1a_2a_3$ and $b_1b_2b_3$ are connected by a path $a_1a_2a_3, uc_1c_2, \dots, c_{k-2}c_{k-1}b_j, b_1b_2b_3$. So $d(a_1a_2a_3, b_1b_2b_3) \leq k = d(u, b_j)$.

This implies that $d(a_1a_2a_3, b_1b_2b_3) \leq d(u, b_j) + 1 \leq \text{rad}(G) + 1$, since $d(u, b_j) \leq \text{rad}(G)$. Hence $e(a_1a_2a_3) \leq \text{rad}(G) + 1$. Therefore $\text{rad}(P_3(G)) \leq \text{rad}(G) + 1$.

Now, let $\text{rad}(P_3(G)) = \text{rad}(G) + 1$. We have proved that if $d(u, b_j) > 1$, then $d(a_1a_2a_3, b_1b_2b_3) \leq d(u, b_j) \leq \text{rad}(G)$. So $e(a_1a_2a_3) \leq \text{rad}(G)$ and hence $\text{rad}(P_3(G)) \leq \text{rad}(G)$. So the equality does not hold when $\text{rad}(G) > 1$.

Consider the case when $\text{rad}(G) \geq 4$. Consider $a_1a_2a_3$ where u is some a_i and let $b_1b_2b_3$ be any other vertex in $P_3(G)$. Let $d(u, b_j) = k$ and $a_i, c_1, c_2, \dots, c_{k-1}, b_j$ be a shortest path connecting a_i and b_j . Then $a_1a_2a_3$ and $b_1b_2b_3$ are connected by a path $a_1a_2a_3, uc_1c_2, c_2c_3c_4, \dots, c_{k-2}c_{k-1}b_j, b_1b_2b_3$. So if $k \leq 3$, then $d(a_1a_2a_3, b_1b_2b_3) \leq 3$ and if $k \geq 4$, then $d(a_1a_2a_3, b_1b_2b_3) < k$.

So $e(a_1a_2a_3) < k \leq \text{rad}(G)$. Hence $\text{rad}(P_3(G)) < \text{rad}(G)$. \square

Remark 2.4.1. The condition $\text{rad}(G) = 1$ is not sufficient for the equality $\text{rad}(P_3(G)) = \text{rad}(G) + 1$. For eg:- if $G = K_{1,n}$, $n \geq 3$, then $\text{rad}(G) = \text{rad}(P_3(G)) = 1$.

Theorem 2.4.4. For a connected graph G , $\text{diam}(P_3(G)) \leq \text{diam}(G)$. Further, if $\text{diam}(G) \geq 4$ then $\text{diam}(P_3(G)) < \text{diam}(G)$.

Proof. Since G is not a complete graph $\text{diam}(G) > 1$. By the arguments similar to those in the above proof, we can prove that for any two vertices $a_1a_2a_3$ and $b_1b_2b_3$ in $P_3(G)$, $d(a_1a_2a_3, b_1b_2b_3) \leq d(a_i, b_j) \leq \text{diam}(G)$. So $\text{diam}(P_3(G)) \leq \text{diam}(G)$.

Let $\text{diam}(G) \geq 4$. Let $a_1a_2a_3$ and $b_1b_2b_3$ be any two vertices in $P_3(G)$ such that $d(a_1a_2a_3, b_1b_2b_3) = \text{diam}(P_3(G))$. Using the similar arguments as in the above proof, we can show that $d(a_1a_2a_3, b_1b_2b_3) < d(a_i, b_j) \leq \text{diam}(G)$. Hence $\text{diam}(P_3(G)) < \text{diam}(G)$. \square

Note: Let $G = K_2 \vee 4 K_1$. Then $\text{rad}(G) = 1$, $\text{rad}(P_3(G)) = 2$, $\text{diam}(G) = 2$, $\text{diam}(P_3(G)) = 2$. Hence the bounds in Theorems

2.4.3 and 2.4.4 are strict.

Chapter 3

The edge C_4 graph of a graph

In this chapter, we study another graph operator - the edge C_4 graph of a graph. This operator is also called the edge graph in [66], as mentioned earlier. We construct infinitely many pairs

Some results of this chapter are included in the following papers.

1. Manju K. Menon, A. Vijayakumar, The edge C_4 graph of a graph, Proceedings of the International Conference on Discrete Mathematics, Ramanujan Math. Soc. Lect. Notes Ser. 7 (2008), 245 - 248.
2. Manju K. Menon, A. Vijayakumar, The edge C_4 graph of some graph classes (communicated).
3. Manju K. Menon, A. Vijayakumar, Some domination parameters in $E_4(G)$ (communicated).

of non isomorphic graphs G and H such that $E_4(G) = E_4(H)$. We also prove that $E_4(G)$ has no forbidden subgraph characterization. We include in this chapter, the relationship between different types of domination numbers in G and those in $E_4(G)$. We also study the conditions for $E_4(G)$ to be a special class of graphs such as the threshold graphs, cographs, block graphs, geodetic graphs, weakly geodetic graphs etc.

3.1 The edge C_4 graph of a graph

For any graph G , $E_4(G)$ is a supergraph of $L(G)$. So $E_4(G)$ is connected if and only if exactly one component of G contains edges. It is well known [79] that the only pair of non-isomorphic graphs having the same line graph is $K_{1,3}$ and K_3 . But we observe that in the case of the edge C_4 graphs, there are infinitely many pairs of non-isomorphic graphs having isomorphic edge C_4 graphs. However we are yet to obtain such pairs of same order.

Theorem 3.1.1. *There exist infinitely many pairs of non isomorphic graphs whose edge C_4 graphs are isomorphic.*

Proof. Let $G = K_{1,n}$. If $n = 2k - 1$, then take $H = K_2 \vee (k-1)K_1$ and if $n = 2k$, then take $H = 2K_1 \vee kK_1$. Clearly G and H are non isomorphic graphs. But $E_4(G) = E_4(H) = K_n$. \square

In [8], Beineke proved the existence of nine forbidden subgraphs for a graph to be a line graph. But, we prove that there is no forbidden subgraph characterization for $E_4(G)$.

Theorem 3.1.2. *There is no forbidden subgraph characterization for $E_4(G)$.*

Proof. We shall prove that given any graph G , we can find a graph H such that G is an induced subgraph of $E_4(H)$. For any graph G , let $H = G \times K_2$. Then in $E_4(H)$, all the vertices of the form uu' where u is a vertex in G and u' is the corresponding vertex in the copy of G used in the construction of $G \times K_2$, will induce G . For, if u and v are any two adjacent vertices in G , uu' and vv' correspond to adjacent vertices in $E_4(H)$ as $uu'v'v$ forms a C_4 in H . If u and v are any two non adjacent vertices in G then uu' and vv' are non adjacent vertices in $E_4(H)$. \square

Theorem 3.1.3. *For a tree T , $E_4(T) \cong E_4(T^c)$ if and only if T is K_1 , P_4 or $K_{1,3}$.*

Proof. Let $E_4(T) \cong E_4(T^c)$. If T is a tree having n vertices then T has $n - 1$ edges. But T^c has $n(n - 1)/2 - (n - 1)$ edges. Since $E_4(T) \cong E_4(T^c)$, both T and T^c must have the same number of edges.

So $n(n - 1)/2 - (n - 1) = n - 1$ and hence $n = 1$ or 4 . If $n = 1$, $T = K_1$ and if $n = 4$, $T = K_{1,3}$ or P_4 .

Converse is trivially true. □

3.2 Diameter, Radius and Center

In this section, we study the relationships between the diameter and radius of G and those of $E_4(G)$. In [50], it has been shown that for any graph G without isolated vertices, there is a graph H such that $C(H) = G$ and $C(L(H)) = L(G)$. We prove a similar result for $E_4(G)$ also.

Theorem 3.2.1. *For a connected graph G , $\text{diam}(G) - 1 \leq \text{diam}(E_4(G)) \leq \text{diam}(G) + 1$.*

Proof. We shall first prove the inequality on the right.

Let $\text{diam}(E_4(G)) = k$. Suppose that $\text{diam}(G) < k - 1$. Then

for any two vertices v_1 and v_2 in G , $d_G(v_1, v_2) < k - 1$. Consider any two edges $e_1 = v_1 - v'_1$ and $e_2 = v_2 - v'_2$ in G . But $d_G(v_1, v_2) < k - 1$. Hence in $E_4(G)$, $d_{E_4(G)}(e_1, e_2) \leq d_G(v_1, v_2) + 1 < k$, which is a contradiction to the fact that $\text{diam}(E_4(G)) = k$. Thus $\text{diam}(E_4(G)) \leq \text{diam}(G) + 1$.

Next let $\text{diam}(G) = k$. Suppose that $\text{diam}(E_4(G)) < k - 1$. Let u and v be any two vertices in G and let $u - u'$, $v - v'$ be any two edges incident with u and v respectively. But $d_{E_4(G)}(uu', vv') < k - 1$. So $d_G(u, v) \leq d_{E_4(G)}(uu', vv') + 1 < k$, which is a contradiction to the fact that $\text{diam}(G) = k$. \square

Theorem 3.2.2. *For a connected graph G , $\text{rad}(G) - 1 \leq \text{rad}(E_4(G)) \leq \text{rad}(G) + 1$.*

Proof. Let $\text{rad}(E_4(G)) = k$. If possible, let $\text{rad}(G) < k - 1$. Then there exists a vertex ' a ' in G so that $d_G(a, b) < k - 1$ for any vertex b in G . Consider an edge adjacent to a , say e_a . Let $e = v_1 - v_2$ be any edge in G .

$$d_{E_4(G)}(e_a, e) \leq \max\{d_G(a, v_1), d_G(a, v_2)\} + 1$$

$< k - 1 + 1 = k$ and hence $e(e_a) < k$. Thus $\text{rad}(E_4(G)) \leq \text{rad}(G) + 1$.

Finally let $rad(G) = k$. Suppose that $rad(E_4(G)) < k - 1$. Then there exists a vertex uu' in $E_4(G)$ such that $e(uu') < k - 1$. Consider the vertex u in G . Let v be any vertex in G . Let vv' be any edge incident with v . Then $d_G(u, v) \leq d_{E_4(G)}(uu', vv') + 1 < k$, and hence $e(u) < k$, which is a contradiction to the fact that $rad(G) = k$. \square

Note: The bounds in Theorems 3.2.1 and 3.2.2 are strict.

If G is a bow, then $diam(G) = 2$, $diam(E_4(G)) = 3$, $rad(G) = 1$ and $rad(E_4(G)) = 2$.

If G is C_4 , then $diam(G) = 2$, $diam(E_4(G)) = 1$ and $rad(G) = 2$, $rad(E_4(G)) = 1$.

Theorem 3.2.3. *For any graph G without isolated vertices, there exists a supergraph H such that $C(H) = G$ and $C(E_4(H)) = E_4(G)$.*

Proof. Consider $G \vee 2K_2$. Let the K_2 s be $a - a'$ and $b - b'$. Attach $a'' - a'''$ to $a - a'$ such that a' is adjacent to a'' and a is adjacent to a''' . Similarly attach $b'' - b'''$ to $b - b'$ such that b is

adjacent to b'' and b' is adjacent to b''' . The graph so obtained is H .

We have,

$$e(u) = 2, \text{ if } u \in V(G).$$

$$= 3, \text{ if } u \in \{a, a', b, b'\}.$$

$$= 4, \text{ if } u \in \{a'', a''', b'', b'''\}.$$

Hence $C(H) = G$.

Now, let u_1, u_2, \dots, u_m be the vertices in G and x be any vertex in $E_4(H)$. Then,

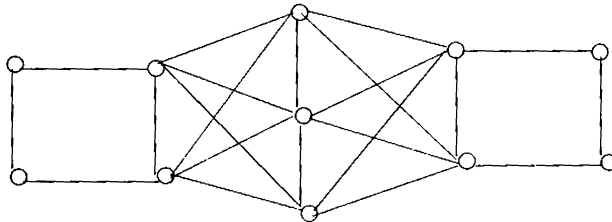
$$e(x) = 2, \text{ if } x \in \{u_i u_j / u_i \text{ is adjacent to } u_j \text{ in } G, i, j = 1, 2, \dots, m, i \neq j\}.$$

$$= 3, \text{ if } x \in \{a u_i, a' u_i, b u_i, b' u_i\}, i = 1, 2, \dots, m.$$

$$= 4, \text{ if } x \in \{a' a'', a a''', b' b'', b b''', a'' a''', b'' b'''\}.$$

Hence $C(E_4(H)) = E_4(G)$.

Illustration: If $G = P_3$, then H is



□

3.3 Some domination parameters in

$$E_4(G)$$

In this section, we study the relationship between different types of dominations in G and those in $E_4(G)$.

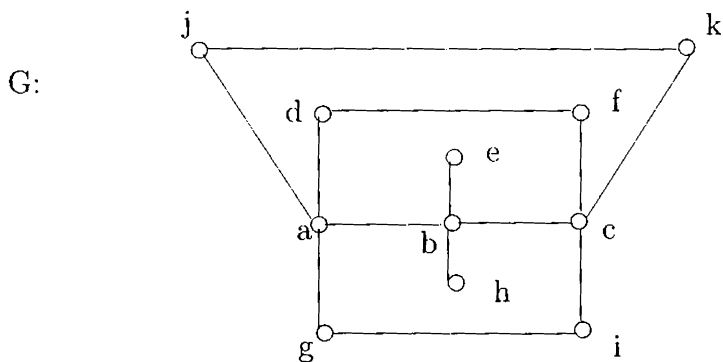
Theorem 3.3.1. *For any graph G with out isolated vertices, there exists a dominating set in G corresponding to any dominating set in $E_4(G)$. Further for such a graph G , $\gamma(G) \leq 2\gamma(E_4(G))$.*

Proof. Let G be any graph having no isolated vertices. Then corresponding to any component in G , there exists a component in $E_4(G)$ and vice versa.

Let $\{e_1 = v_1v'_1, e_2 = v_2v'_2, \dots, e_b = v_bv'_b\}$ be a dominating set in $E_4(G)$. Consider $S = \{v_1, v'_1, v_2, v'_2, \dots, v_b, v'_b\}$. Then $S \subseteq V(G)$. Let w be any vertex in $V(G)$. Since G is a connected graph, w must be the end vertex of an edge $w - w'$. But the vertex ww' in $E_4(G)$ is dominated and hence is adjacent to at least one of the b vertices. Let e_i be adjacent to ww' in $E_4(G)$.

Then in G , either e_i is incident with $w - w'$ or e_i and $w - w'$ are the opposite edges of some C_4 . In both the cases, w is dominated by v_i or v'_i . Thus S is a dominating set of G and hence $\gamma(G) \leq 2\gamma(E_4(G))$. \square

Note: Corresponding to any dominating set in G , there need not be a dominating set in $E_4(G)$. As an example, consider the following graph:



Corresponding to the dominating set $\{a, b, c\}$ in this graph, we cannot find a dominating set in its $E_4(G)$.

Corollary 3.3.2. *For any graph G , $\gamma_{pr}(G) \leq 2\gamma(E_4(G))$*

Proof. Let $\gamma(E_4(G)) = k$. Let a minimum dominating set of $E_4(G)$ be $\{e_1 = v_1v'_1, e_2 = v_2v'_2, \dots, e_k = v_kv'_k\}$. Let $S = \{v_1, v'_1, v_2, v'_2, \dots, v_k, v'_k\}$. Then S is a dominating set in G by Theorem 3.3.1. Further, S allows a perfect matching and hence the result. \square

Corollary 3.3.3. *For a connected graph G , $\gamma_c(G) \leq 2\gamma_c(E_4(G))$.*

Proof. Let $S' = \{e_1 = v_1v'_1, e_2 = v_2v'_2, \dots, e_k = v_kv'_k\}$ be a minimal connected dominating set in $E_4(G)$. Consider $S = \{v_1, v'_1, v_2, v'_2, \dots, v_k, v'_k\}$ in G , which is a dominating set in G by Theorem 3.3.1. By the definition of $E_4(G)$, S is connected and hence the result. \square

Corollary 3.3.4. *For a connected graph G , $\gamma_w(G) \leq 2i_w(E_4(G))$.*

Proof. Let $i_w(E_4(G)) = k$ and $S' = \{e_1 = v_1v'_1, e_2 = v_2v'_2, \dots, e_k = v_kv'_k\}$ be a minimal weakly connected independent dominating set in $E_4(G)$. Since S' is independent, no two e_i s are adjacent. Further since $\langle S' \rangle_w$ is connected, for any e_i in S' , $e(e_i) \leq 2$ in $\langle S' \rangle_w$. So in $\langle S' \rangle_w$, e_i and e_j are either adjacent or there exists an $e_{i,j}$ which is adjacent to both e_i and e_j . Then in G , the edge $e_{i,j}$ is either incident with e_i, e_j or is opposite to e_i, e_j in

some C_4 . Consider $S = \{v_1, v'_1, v_2, v'_2, \dots, v_k, v'_k\}$. In both these cases, $\langle S \rangle_w$ is weakly connected. Also S' is a dominating set by Theorem 3.3.1. Hence the result. \square

Corollary 3.3.5. *For any connected graph G , $\gamma_t(G) \leq 2\gamma_t(E_4(G))$.*

Proof. Let S be a minimal total dominating set in $E_4(G)$. Let S' be the set of all end vertices of the corresponding edges in G . Then $N(S') = V$. For, consider any vertex $v \in S'$. It is clearly dominated by the other vertex v' in S' such that $vv' \in S$. Also by Theorem 3.3.1, S' forms a dominating set for G . Thus $N(S') = V(G)$ and hence the result. \square

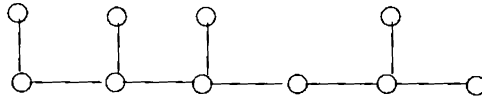
Theorem 3.3.6. *Given any two integers a and b , there exists a graph G such that $\gamma(G) = a$ and $\gamma(E_4(G)) = b$. Further, if $a \leq 2b$, there exists a connected graph G such that $\gamma(G) = a$ and $\gamma(E_4(G)) = b$.*

Proof. Case 1 : $b \leq a \leq 2b$.

Consider $P_{2b} = \{v_1, v_2, \dots, v_{2b}\}$. Attach a pendant vertex to each of $v_{2i-1}, i = 1, 2, \dots, b$. Then to each $v_{2i}, i = 1, 2, \dots, a - b$, attach a pendant vertex. This is the required graph G . Then $\gamma(G) = b + a - b = a$. Also, $\gamma(E_4(G)) = b$ since the set of vertices

$\{v_{2i-1}v_{2i}; i = 1, 2, \dots, b\}$ is a dominating set of minimum cardinality in $E_4(G)$.

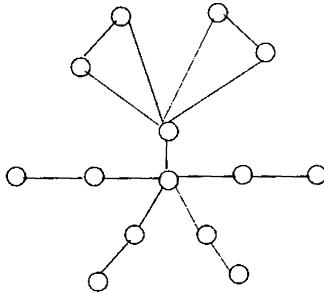
For example, when $a = 4; b = 3$, G is



Case 2 : $a < b$.

Consider $K_{1,a}$. Replace a pendant vertex of $K_{1,a}$ by $K_1 \vee (b-a+1)K_2$. To all other pendant vertices of $K_{1,a}$, attach a pendant vertex. Let u_1 be the central vertex of $K_{1,a}$ and $\{v_1, v_2, \dots, v_a\}$ be the vertices attached to u_1 . Suppose that v'_2, v'_3, \dots, v'_a are respectively the pendant vertices attached to v_2, v_3, \dots, v_a . Let $w_1 - w'_1, w_2 - w'_2, \dots, w_{b-a+1} - w'_{b-a+1}$ be the $b-a+1$ K_2 s joined to v_1 . This is the required graph G . Then $\{v_1, v_2, \dots, v_a\}$ forms a minimum dominating set in G and hence $\gamma(G) = a$. In $E_4(G)$, $\{w_i w'_i; i = 1, 2, \dots, b-a, v_1 w_{b-a+1}, v_i v'_i; i = 2, 3, \dots, a\}$ is a dominating set of minimum cardinality and hence $\gamma(E_4(G)) = b-a+1+a-1 = b$.

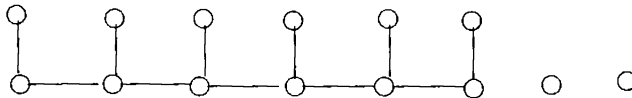
For example, when $a = 5; b = 6$, G is



Case 3: $a > 2b$.

G is $P_{2b} \circ K_1$ together with $a - 2b$ isolated vertices.

For example, when $a = 8; b = 3$, G is



□

Lemma 3.3.7. For any connected graph G , $\alpha(E_4(G)) \leq \beta(G)$.

Proof. Let $\alpha(E_4(G)) = k$ and $S = \{e_1, e_2, \dots, e_k\}$ be the maximum vertex independent set in $E_4(G)$. Consider the corresponding edges in G . Since e_i and e_j are not adjacent in S , the corresponding edges in G are also independent. \square

Theorem 3.3.8. *For any connected graph G , $\text{diam}(G) - 2 \leq \gamma_c(E_4(G)) \leq 2\beta(G) - 1$.*

Proof. By Theorem 1.2.16, $\text{diam}(E_4(G)) - 1 \leq \gamma_c(E_4(G))$. Using Theorem 3.2.1, we get the left inequality. By Theorem 1.2.17, $\gamma_c(E_4(G)) \leq 2\alpha(E_4(G)) - 1$. Then by applying Lemma 3.3.7, we get the right inequality. \square

Note: In Theorem 3.3.8, the left bound is strict for P_5 and the right bound is strict for $K_{1,3}$.

Theorem 3.3.9. *For any connected graph G , $\gamma(E_4(G)) \leq \gamma'(G)$. Further, equality holds if the edge domination of G is the efficient edge domination.*

Proof. Let S form a minimal edge dominating set in G so that $\gamma'(G) = |S|$. In $E_4(G)$, let S' be the set of vertices corresponding to the edges in S . Clearly S' dominates all the vertices in

$E_4(G)$. Hence $\gamma(E_4(G)) \leq \gamma'(G)$.

Let $S = \{e_1, e_2, \dots, e_k\}$ be an efficient edge dominating set in G so that $\gamma'_e(G) = |S|$. Let S' be any dominating set in $E_4(G)$.

Claim: $|S'| \geq |S|$.

Suppose that $e_i \in S'$ dominates e_l and e_m in $E_4(G)$ where $l, m \in \{1, 2, \dots, k\}$. Since S is an efficient edge dominating set in G , both e_l and e_m cannot be adjacent with e_i . So the only possibility is that e_l and e_m are opposite to e_i in some C_4 s in G . But e_i in G must be dominated by an edge $e_k \in S$. Even then, S is not an efficient edge dominating set. Thus any dominating set in $E_4(G)$ must have $|S|$ elements. Hence the equality. \square

Note: Even if $\gamma(E_4(G)) = \gamma'(G)$, the edge domination need not be efficient. As an example, $G = K_{1,n}$.

Corollary 3.3.10. *For any graph $G = (n, m)$, $\gamma(E_4(G)) < n/2$.*

Proof. Using Theorem 1.2.12 and Theorem 3.3.9, we get $\gamma(E_4(G)) \leq \lfloor n/2 \rfloor \leq n/2$. But $\gamma'(G) = n/2$ is possible only when n is even. But, by Theorem 1.2.13, for even n , $\gamma'(G) = n/2$ if and only if $G = K_n$ or $K_{n/2, n/2}$. When $G = K_n$ or $K_{n/2, n/2}$, $E_4(G)$ is complete and hence $\gamma(E_4(G)) = 1$. \square

Notation 3.3.1. The graph $C_{3,k}$ denotes the graph obtained from a C_3 and k copies of K_2 by joining one end of each K_2 with a fixed vertex of C_3 . The graph obtained from C_4 by joining a vertex of C_4 with the center of $S(K_{1,k})$ is denoted by $C_{4,k}$.

Corollary 3.3.11. *If a connected graph $G = (n, m)$ is a tree or a unicyclic graph then $\gamma(E_4(G)) = \lfloor n/2 \rfloor$ if and only if G is one of the following.*

(1) subdivision of a star, (2) C_5 , (3) C_7 , (4) $C_{3,k}$.

Proof. By Corollary 3.3.10, $\gamma(E_4(G)) < n/2$. So $\gamma(E_4(G)) = \lfloor n/2 \rfloor$ is possible only when n is odd. By Theorem 1.2.14 for any tree of order $n \neq 2$, $\gamma'(G) = (n-1)/2$ if and only if G is isomorphic to the subdivision of a star. If G is $S(K_{1,k})$, $\gamma(E_4(G)) = k$. By Theorem 1.2.15, for a connected unicyclic graph $G = (n, m)$, $\gamma'(G) = \lfloor n/2 \rfloor$ if and only if G is isomorphic to either C_4 , C_5 , C_7 , $C_{3,k}$ or $C_{4,k}$. We have $\gamma(E_4(C_4)) = 1$, $\gamma(E_4(C_5)) = 2$, $\gamma(E_4(C_7)) = 3$, $\gamma(E_4(C_{3,k})) = k + 1$, $\gamma(E_4(C_{4,k})) = k + 1$. Thus the result follows. \square

Theorem 3.3.12. *Let G be a clique dominated graph with a dominating clique S . If every vertex $v \notin S$ is adjacent to at least two vertices of S , then $E_4(G)$ is also a clique dominated*

graph.

Proof. Let $S = \{v_1, v_2, \dots, v_k\}$ be a dominating clique of a clique dominated graph G . Let $\{e_1, e_2, \dots, e_n\}$ be the set of all edges in the graph induced by S . Then the set of vertices $S' = \{e_1, e_2, \dots, e_n\}$ in $E_4(G)$ induces a complete subgraph by the definition of $E_4(G)$. Also, S' forms a dominating clique.

Let $e_x = v_x v'_x \notin S'$ be a vertex in $E_4(G)$. If x or $x' \in \{1, 2, \dots, k\}$, then e_x in $E_4(G)$ is dominated by S' . So let both v_x and $v'_x \notin S$. But, both v_x and v'_x are adjacent to at least two vertices in S . Let v_x be adjacent to v_i in S and let v'_x be adjacent to $v_j, i \neq j$ in S . Then $v_i - v_j$ and $v_x - v'_x$ are opposite edges of some C_4 in G and hence the vertex e_x is dominated by S' . So S' is a dominating clique in $E_4(G)$ and hence it is a clique dominated graph. \square

Note: The converse of Theorem 3.3.12 need not be true, as in the case of $G = K_n \circ K_1$.

3.4 $E_4(G)$ and some graph classes

Theorem 3.4.1. *For a connected graph G , $E_4(G)$ is complete if and only if G is a complete multipartite graph .*

Proof. Let G be a connected graph such that $E_4(G)$ is complete. We shall first show that G is a cograph and is paw-free. If G contains an induced P_4 then the first and the third edges in P_4 correspond to two non adjacent vertices in $E_4(G)$, and $E_4(G)$ is not complete. Further if G contains a paw as an induced sub-graph then the pendant edge and the edge in the triangle of the paw to which the pendant edge is not adjacent correspond to non adjacent vertices in $E_4(G)$. Hence G is also paw-free. Thus G is a paw-free cograph.

Claim: G is a complete multipartite graph.

If not, \bar{G} is not a union of complete graphs. Then \bar{G} contains an induced P_3 . But, \bar{G} is disconnected as G is a connected graph. Hence \bar{G} is a disconnected graph containing an induced P_3 and so G has a paw, giving a contradiction. This proves the claim.

Conversely suppose that G is a complete multipartite graph. Let e_1 and e_2 be any two edges in G . If they are not adjacent, they are opposite edges of some C_4 in G since G is a complete multipartite graph. Hence $E_4(G)$ is a complete graph. \square

Theorem 3.4.2. *For a connected graph G , $E_4(G)$ is bipartite if and only if G is either an even cycle of length greater than five or a path.*

Proof. If G is either a path or an even cycle of length greater than five, then $E_4(G)$ is bipartite.

Let $E_4(G)$ be a bipartite graph. Then it cannot contain odd cycles. But if G contains a $K_{1,3}$ or a K_3 then $E_4(G)$ contains a K_3 . $E_4(C_4)$ is K_4 which is not bipartite. Since $E_4(G)$ is fixed for C_n ; $n \neq 4$, G cannot contain odd cycles. Hence $E_4(G)$ is bipartite only for the even cycles of length greater than five or paths. \square

Corollary 3.4.3. *For a connected graph G , $E_4(G)$ is a tree if and only if G is a path.*

Theorem 3.4.4. *Let G be a connected graph such that $E_4(G)$ is a threshold graph. Then $\gamma(G) \leq 2$.*

Proof. We know by Lemma 1.2.5 that $E_4(G)$ is a threshold graph if and only if $dilw(E_4(G)) = 1$. Also $dilw(E_4(G)) \geq \gamma(E_4(G))$ by Lemma 1.2.4. Then the theorem follows from Theorem 3.3.1.

□

Notation 3.4.1. The graph obtained from K_4 by attaching two pendant vertices to the same vertex of K_4 is denoted by H in the following Theorem.

Theorem 3.4.5. *If G is a threshold graph then $E_4(G)$ is a threshold graph if and only if G is $\{\text{moth}, H\}$ -free.*

Proof. Let G be a threshold graph. If G contains a moth graph or H as an induced subgraph, then $E_4(G)$ contains a $2K_2$ and hence cannot be threshold.

Conversely, suppose that G is a $\{\text{moth}, H\}$ -free threshold graph. Since G is threshold, $dilw(G) = 1$ and hence $\gamma(G) = 1$. So G must have a universal vertex u .

If at most two vertices in $N(u)$ are of degree greater than one, then $E_4(G)$ cannot contain an induced $2K_2$, C_4 or P_4 .

Now, let $k \geq 3$ vertices in $N(u)$ be of degree greater than one.

Claim: There exist three vertices $u_1, u_2, u_3 \in N(u)$ such that the vertex u_2 is adjacent to u_1 and u_3 .

If $k = 3$, this claim holds true. If $k > 3$, let u_1, u_2, u_3 and u_4 be four vertices of degree greater than one in $N(u)$ such that u_1 is adjacent to u_2 and u_3 is adjacent to u_4 . Since G is threshold, it cannot contain an induced $2K_2$ and hence u_3 or u_4 must be adjacent to u_1 or u_2 . Let u_3 be adjacent to u_1 . Then u_2, u_1, u_3, u_4 forms an induced P_4 which is not possible since G is threshold. In this case, if u_4 is adjacent to u_2 , then G contains an induced C_4 which is again not possible. Hence the claim.

Further, if u_1 and u_3 are adjacent, the vertex u can have at most one more neighbor since G is H -free. In this case also $E_4(G)$ is threshold since it is $\{2K_2, C_4, P_4\}$ -free. On the other hand if u_1 and u_3 are not adjacent, then since G is moth-free, the vertex u can have at most one more neighbor. In this case also $E_4(G)$ is threshold. □

Note: If G is a connected graph such that $E_4(G)$ is a cograph,

then $\gamma(G) \leq 4$. This follows from Lemma 1.3.3 and Theorem 3.3.1.

Theorem 3.4.6. *Let G be a connected graph. Then $E_4(G)$ is a weakly geodetic graph if and only if G is {paw, 4-pan}-free.*

Proof. If G contains a paw in which $C_3 = u_1, u_2, u_3$ and a is a pendant vertex attached to u_1 , then in $E_4(G)$, $d(au_1, u_2u_3) = 2$, but they have two common neighbors u_1u_2 and u_1u_3 . Similarly if G contains a 4-pan in which $C_4 = u_1, u_2, u_3, u_4$ and a is a pendant vertex attached to u_1 , then in $E_4(G)$, $d(au_1, u_3u_4) = 2$, but they have two neighbors u_1u_2 and u_1u_4 .

Conversely, suppose that G is a {paw, 4-pan} free graph. If G is an acyclic graph, there exists a unique shortest path joining any two vertices in $E_4(G)$. Thus $E_4(G)$ is weakly geodetic.

Next, suppose that G contains cycles.

If $g(G) = 3$ then G contains a C_3 with vertices u_1, u_2, u_3 .

Claim: G is a cograph.

Suppose that G contains an induced $P_4 = v_1, v_2, v_3, v_4$. Let $u_1 \neq v_1$. Consider a shortest path $u_1, a_1, a_2, \dots, a_k, v_1$ joining u_1 and

v_1 . Since G is paw free, a_1 must be adjacent to at least one more u_i , $i = 2, 3$. Proceeding like this, v_1 and then v_2 must be adjacent to at least two u_i s. This implies that v_1 and v_2 must have a common neighbor among the u_i s. Let it be u_1 . Then v_1, u_1, v_2 form a C_3 . Since G is paw-free, v_3 must be adjacent to at least one of v_1 and u_1 . But, since v_1, v_2, v_3, v_4 is an induced P_4 , v_3 must be adjacent to u_1 . Then v_1, u_1, v_3 will form a C_3 in G . Again, since G is paw-free, v_4 must be adjacent to u_1 . Now, consider v_1, u_1, v_2 with the edge $u_1 - v_4$. Since G is paw-free, v_4 must be adjacent to v_1 or v_2 , which is a contradiction. Thus G is a paw-free cograph. Thus by Theorem 3.4.1, $E_4(G)$ is complete and hence is weakly geodetic.

If $g(G) = 4$, then G contains a $C_4 = u_1, u_2, u_3, u_4$. If $G = C_4$, then $E_4(G) = K_4$. If there exists a vertex v_1 in G which is adjacent to u_1 , v_1 must be adjacent to u_3 also since G is 4-paw-free. Similarly if there exists a vertex v_2 which is adjacent to u_2 , v_2 must be adjacent to u_4 . If there exists a vertex v'_1 which is adjacent to v_1 , it must be adjacent to both u_2 and u_4 . Hence G is a complete bipartite graph. Since $g(G) = 4$, G is paw-free. Again by Theorem 3.4.1, $E_4(G)$ is complete, and hence weakly geodetic.

Finally, Let $g(G) = k$, $k > 4$. Let $u_1, u_2, u_3, \dots, u_k$ be a C_k in G . Then $E_4(G)$ also contains a C_k . This C_k is not a part of any complete subgraph in $E_4(G)$ and hence $b(E_4(G)) \leq k$. Since G does not contain any C_4 , two vertices in $E_4(G)$ are adjacent if and only if the corresponding edges in G are adjacent. Thus $E_4(G)$ cannot contain a b -cycle of length less than k and so $b(E_4(G)) = k$ where $k > 4$. By Lemma 1.2.8, G is weakly geodetic if and only if $b(G) \geq 5$. Thus $E_4(G)$ is a weakly geodetic graph. \square

Theorem 3.4.7. *For a connected graph G , $E_4(G)$ is a geodetic graph if and only if G is $\{C_{2n}, n > 2, 4\text{-pan}, (2n-1)\text{-pan}; n > 1\}$ -free.*

Proof. Let G be a geodetic graph. If G contains a 4-pan, there exists more than one shortest path joining two vertices in $E_4(G)$ as proved earlier. If G contains a $C_{2n} = u_1, u_2, \dots, u_{2n}$, then u_1u_2 and $u_{n+1}u_{n+2}$ in $E_4(G)$ are connected by more than one shortest paths and hence $E_4(G)$ is not geodetic. If G contains a $(2n-1)$ -pan in which $C_{2n-1} = u_1, u_2, \dots, u_{2n-1}$ and a is a pendant vertex attached to u_1 , then au_1 and u_nu_{n+1} in $E_4(G)$ are connected by more than one shortest path and hence $E_4(G)$ is not geodetic.

Conversely, assume that G is $\{4\text{-pan}, C_{2n}, (2n - 1)\text{-pan}\}$ -free. If G is an acyclic graph there exists a unique shortest path joining any two vertices in $E_4(G)$ and hence is geodetic. So consider the graphs G containing cycles.

Let $g(G) = 3$. Since G is paw-free, $E_4(G)$ is complete and hence is geodetic. If $g(G) = 4$, $E_4(G)$ is complete since G is 4-pan-free and thus geodetic. If $g(G) = 2n - 1, n > 2$, then G contains a $C_{2n-1} = u_1, u_2, \dots, u_{2n-1}$. If $G = C_{2n-1}$, then $E_4(G) = C_{2n-1}$ and hence geodetic. If a is a vertex attached to u_1 , since G is $(2n - 1)$ -pan-free, a must be adjacent to at least one more u_i . But this is impossible since $g(G) = 2n - 1$. Since G is C_{2n} -free, $g(G) \neq 2n, n > 2$. Hence in all the cases, it follows that G is geodetic. \square

Theorem 3.4.8. *If G is a connected graph, then $E_4(G)$ is a block graph if and only if G is $\{\text{paw}, 4\text{-pan}, C_n; n \geq 5\}$ -free.*

Proof. Let G be a block graph. If G contains a paw in which $C_3 = u_1, u_2, u_3$ and a is the pendant vertex adjacent to u_1 , then $E_4(G)$ contains a $C_4 = au_1, u_1u_2, u_2u_3, u_3u_1$ which is not a part of any complete subgraph. Thus $b(E_4(G)) \leq 4$. Similarly if

G contains a 4-pan, in which $C_4 = u_1, u_2, u_3, u_4$ and a is a pendant vertex adjacent to u_1 , then $E_4(G)$ contains a $C_4 = au_1, u_1u_2, u_3u_4, u_4u_1$ which is not a part of any complete subgraph and hence $b(E_4(G)) \leq 4$. If G contains a C_n , $n > 4$, then $E_4(G)$ also contains a C_n , $n > 4$. This C_n forms a b -cycle and hence $b(E_4(G)) \leq n$ and hence $E_4(G)$ is not a block graph.

Conversely, suppose that G is {paw, 4-pan, C_n ; $n > 4$ }-free. If G is an acyclic graph, then $E_4(G)$ cannot contain a b -cycle and hence is a block graph. Now, consider the graphs G containing cycles. Since G is C_n , $n \geq 5$ -free, $g(G) = 3$ or 4. But since G is {paw, 4-pan}-free, $E_4(G)$ is complete as proved earlier and thus is a block graph. \square

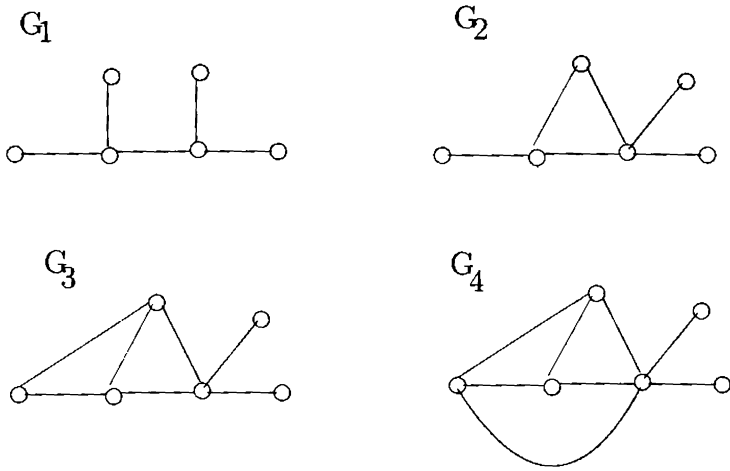


Figure 3.4.1

Theorem 3.4.9. *If G is a connected split graph, then $E_4(G)$ is a split graph if and only if G does not contain any of the graphs shown in Figure 3.4.1 as induced subgraphs.*

Proof. Let both G and $E_4(G)$ be split graphs. Then by Lemma 1.2.9, both G and $E_4(G)$ cannot contain induced $\{2K_2, C_4\}$ and no C_5 . If G contains any one of the graphs shown in Figure 3.4.1, then $E_4(G)$ contains a $2K_2$ which is a contradiction to the fact that $E_4(G)$ is a split graph.

Conversely, suppose that G is a split graph which does not contain any of the graphs shown in Figure 3.4.1 as induced sub-

graphs.

Let G be a tree. Since G is a split graph, G cannot contain $P_n, n \geq 5$ as an induced subgraph (Since $2K_2 \subset$ induced P_5). If $G \cong P_n, n < 5$, then $E_4(G)$ is a split graph. If G is a tree other than P_4 , and if there are more vertices in G , then the additional vertices can be attached only to the mid vertices of P_4 . If vertices are attached only to a mid vertex of P_4 , then $E_4(G)$ is a split graph. Since G is G_1 -free, vertices cannot be attached to both the mid vertices of P_4 . If a split graph G is any tree other than G_1 in Figure 3.4.1, then $E_4(G)$ is a split graph.

Let G be not a tree. Since G is a split graph, G is $\{C_4, C_5\}$ -free. Further, since G is $2K_2$ -free, G is $C_n, n > 6$ -free. Hence $g(G) = 3$ and $c(G) = 3$. Let $u_1u_2u_3$ be any C_3 in G . If $G \cong C_3$, then $E_4(G) \cong C_3$ and hence is a split graph. So G contains more than these three vertices. If more vertices are attached to a unique vertex of the C_3 , then $E_4(G)$ is a split graph. If a vertex each is attached to any two vertices of the C_3 , then also $E_4(G)$ is a split graph. Since G is $\{G_2, G_3, G_4\}$ -free, the only case remaining is attaching a single vertex to each of the u_i s. Then also $E_4(G)$ is a split graph. Hence the proof. \square

Chapter 4

Dynamics of $P_3(G)$ and

$E_4(G)$

Graph dynamics deals with the study of convergence, divergence, fixedness, periodicity etc of graph operators [66]. The dynamics of both $P_3(G)$ and $E_4(G)$ are discussed in this chapter. We have also included some results on their touching numbers.

Some results of this chapter are included in the following papers.

1. Manju K. Menon, A. Vijayakumar, Dynamics of the P_3 intersection graph, (communicated).
2. Manju K. Menon, A. Vijayakumar, The edge C_4 graph of a graph, Proceedings of the International Conference on Discrete Mathematics, Ramanujan Math. Soc. Lect. Notes Ser. 7 (2008), 245 - 248.

4.1 Dynamics of $P_3(G)$

If G is the union of complete graphs, then $P_3(G) = \phi$, the null graph. So we do not consider such graphs in this section.

Lemma 4.1.1. *Let G be a connected graph. If $P_3(G)$ is not complete, then $\omega(P_3(G)) \geq \omega(G)$. Equality holds if and only if G is one of the following.*

1. G is a complete graph with two pendant vertices attached to any two of its distinct vertices.
2. The graph G is as in figure 2(a) or 2(b):

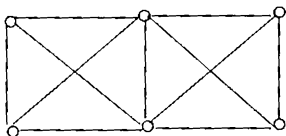


Figure : 2(a)

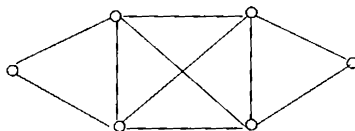


Figure : 2(b)

Proof. Let $\omega(G) = k$. So consider a $K_k \subseteq G$. Let the vertices of the K_k be $\{u_1, u_2, \dots, u_k\}$. Since G is not complete, $K_k \subset G$. Let $v \in V(G) - V(K)$ be a vertex which is adjacent to t vertices u_1, u_2, \dots, u_t of K_k . Then the vertices of the form vu_iu_j , $1 \leq i \leq t$, $t+1 \leq j \leq k$ form a $K_{t(k-t)}$ in $P_3(G)$. But $t(k-t)$

$\geq k - 1$ as proved in Lemma 2.3.1. If $t(k - t) = k - 1$, then $t = 1$ or $k - 1$.

Case1: $t = k - 1$.

Let v be adjacent to u_1, u_2, \dots, u_{k-1} . Since $P_3(G)$ is not complete, there must exist more vertices in G . Consider the case when there exists a vertex w adjacent to some u_i , $i = 1, 2, \dots, k - 1$, but not adjacent to u_k . If w is not adjacent to v , then vu_iu_k ; $i = 1, 2, \dots, k - 1$, wu_iv and wu_iu_k will form a K_{k+1} in $P_3(G)$. If w is adjacent to v , then since v is adjacent to $k - 1$ vertices of K_k , w can be adjacent to at most $k - 2$ vertices of K_k . If w is not adjacent to u_{k-1} , then vu_iu_k ; $i = 1, 2, \dots, k - 1$, wvu_{k-1} and wu_iu_k , $i = 1, 2, \dots, k - 2$ will form at least a K_{k+1} in $P_3(G)$. Similarly if w is adjacent to u_k , we can find at least a K_{k+1} in $P_3(G)$.

Therefore consider the case when such a w does not exist. Since $P_3(G)$ is not complete, the vertex v must have a neighbor x having an induced P_3 which is independent of the u_i s and v . Then vu_iu_k ; $i = 1, 2, \dots, k - 1$ and xvu_i ; $i = 1, 2, \dots, k - 1$ will form at least a K_{k+1} in $P_3(G)$. Thus in this case, $\omega(P_3(G)) \geq \omega(G) + 1$.

Case2: $t = 1$.

Let v be adjacent to u_1 alone. Since $P_3(G)$ is not complete, there must exist an induced P_3 which is independent of u_1 and v . Let there exist w adjacent to some u_i s. If w is also adjacent to u_1 , then there exists at least a K_{k+1} in $P_3(G)$. If w is adjacent to more than one vertex of K_k , then also at least a K_{k+1} is contained in $P_3(G)$. If w is adjacent to exactly one u_i , $i \neq 1$, then $\omega(P_3(G)) = k$ except when $k = 3$ [But when $k = 3$, $P_3(G)$ is complete]. This is the graph mentioned in 1 of the statement. Now, if v or w has a neighbor, then $\omega(P_3(G)) > k$. If more than two vertices are adjacent to the u_i s, then also $\omega(P_3(G)) > k$.

So consider the case when u_i s have no neighbor other than v . Since $P_3(G)$ is not complete, v must have a neighbor x having an induced P_3 consisting of x but none of v or the u_i s. In this case also we can find at least a K_{k+1} in $P_3(G)$.

If $t(k - t) = k$, then we get $k = 4$ and $t = 2$. So let v be adjacent to u_1 and u_2 . Since $P_3(G)$ is not complete, it must contain more vertices. If there exists a vertex w which is adjacent to v and if w is not adjacent to u_1 or u_2 , then $\omega(P_3(G)) > 4$. So consider the case when w is adjacent with v , u_1 , u_2 (w can

be adjacent only to these two vertices of K_4]). Then the graph is the graph shown in Figure : 2(a) of the Lemma.

For the graph G in Figure : 2(a), $\omega(P_3(G)) = \omega(G) = 4$. If we join one more vertex to this graph, then $\omega(P_3(G)) > \omega(G)$. Next consider the case when w is not adjacent to v . Then $\omega(P_3(G)) > \omega(G)$ except when w is adjacent to both u_3 and u_4 . Then the graph is the graph shown in Figure : 2(b) of the Lemma.

For the graph G in Figure : 2(b), $\omega(P_3(G)) = \omega(G)$. As in the above case, if this graph contains more vertices, then $\omega(P_3(G)) > \omega(G)$. Hence the lemma is proved. \square

Theorem 4.1.2. *There are no P_3 -periodic graphs.*

Proof. If a graph G is P_3 -fixed, then $\omega(P_3(G)) = \omega(G)$. From Lemma 4.1.1, $\omega(P_3(G)) = \omega(G)$ only for the above three types of graphs. But, none of them are fixed under P_3 . Thus there does not exist any graph with period one. Again, from Lemma 4.1.1, $\omega(P_3^2(G)) > \omega(G)$. Hence $P_3^n(G) \neq P_3(G)$ for any $n > 1$. \square

Remark 4.1.1. Since 1 - periodic graphs are fixed, it follows

that there does not exist any P_3 - fixed graphs.

Theorem 4.1.3. *If a graph G is P_3 -convergent , then it is $P_3^n(G)$ -complete for some $n \geq 1$.*

Proof. Let G be a P_3 -convergent graph. If G is none of the three graphs mentioned in Lemma 4.1.1, then $\omega(P_3(G)) > \omega(G)$. Thus the clique size of the iterated graphs goes on increasing. So if G converges, it converges to some complete graph.

If G is one among the three graphs mentioned in Lemma 4.1.1, then by Theorem 4.1.2 $P_3(G) \neq G$ and $\omega(P_3^2(G)) > \omega(G)$. This indicates that in both the cases, the clique size goes on increasing. Also we know that if $P_3^k(G)$ is a complete graph, then $P_3^{k+1}(G) = \phi$. Hence if G converges, it converges to some complete graph. □

Note: By the Theorem 4.1.3, it follows that all the P_3 -convergent graphs are P_3 -mortal graphs .

The following seems to be an interesting open problem.

Problem: Are there any P_3 -divergent graphs?

4.2 Dynamics of $E_4(G)$

In this section we study the dynamical properties of the edge C_4 graph of a graph.

Theorem 4.2.1. *A connected graph G is E_4 -convergent if and only if G is P_n , $K_{1,3}$ or C_n ($n \neq 4$).*

Proof. The paths P_n converge to ϕ since $E_4^n(P_n) = \phi$. For $K_{1,3}$, $E_4(K_{1,3}) = K_3$, $E_4^n(K_{1,3}) = K_3$, for $n > 1$ and hence $K_{1,3}$ converges to K_3 . All cycles except C_4 are E_4 -fixed.

If G contains a vertex of degree > 3 , then $E_4(G)$ contains K_4 . Then $K_6 \subseteq E_4^2(G)$, $K_{15} \subseteq E_4^3(G)$, $K_{105} \subseteq E_4^4(G)$ and so on. Thus in the subsequent iterations the clique size goes on increasing and hence G diverges. So if G is E_4 -convergent, then $\Delta(G) \leq 3$.

If G is a tree which is neither P_n nor $K_{1,3}$, then K_4 is contained in at least the third iterated graph and hence G cannot converge. Next, consider the graphs which are not trees. If G is not a cycle, then G contains a cycle with a pendant edge as a subgraph (need not be induced). Then K_4 is a subgraph at least in the second

iteration and hence in the subsequent iterations the clique size will go on increasing and hence cannot converge. Also, C_4 is not convergent since $E_4(C_4) = K_4$ and in the subsequent iterations, the clique size goes on increasing. \square

Corollary 4.2.2. *A connected graph G is E_4 -periodic if and only if $G \cong C_n, n \neq 4$. The cycles $C_n, n \neq 4$ have period one.*

Proof. A graph G is convergent if and only if G is either periodic or there is some positive integer n with $E_4^n(G)$ periodic. But from Theorem 4.2.1, the only E_4 - convergent graphs are $P_n, K_{1,3}, C_n; n \neq 4$. But P_n converges to the null graph and $K_{1,3}$ converges to K_3 . So it is easy to verify that the only periodic graphs are the cycles $C_n, n \neq 4$ and they have period one. \square

Corollary 4.2.3. *The transition number $t(P_n) = n, t(K_{1,3}) = 1$ and for $n \neq 4, t(C_n) = 0$.*

Proof. The transition with respect to E_4 of a graph G is zero if G is periodic and the smallest number n such that $E_4^n(G)$ is periodic. $E_4(P_n) = P_{n-1}, E_4^n(P_n) = \Phi$ and hence $t(P_n) = n$. $E_4(K_{1,3}) = K_3, E_4^2(K_{1,3}) = E_4(K_3) = K_3$. Thus $t(K_{1,3}) = 1$. Since $C_n, n \neq 4$ are periodic, for $n \neq 4, t(C_n) = 0$. \square

Corollary 4.2.4. *For a connected graph G , $E_4(G)$ is a mortal graph if and only if G is a path.*

Theorem 4.2.5. *If G is a tree then for $E_4(G)$, the only semibasins are the paths and the only basins are $P_n, n \leq 4$.*

Proof. Let G be a tree. If G contains a $K_{1,3}$, then $E_4(G)$ contains a K_3 and hence $E_4(G) \not\subseteq G$. Also for any P_n , $E_4(P_n) = P_{n-1}$. Hence, among the trees the only semi basins are the paths.

If $G \cong P_n, n \geq 5$, then $\Delta(G^c) = n - 2$. If $P_n = v_1, v_2, \dots, v_n$, then in $E_4(P_n^c)$, $d(v_1 v_n)$ is at least $(n - 3) + (n - 3) = 2n - 6$. For $n \geq 5$, $2n - 6 > n - 2$. So $E_4(P_n^c) \not\subseteq P_n^c$. If $G \cong P_n; n < 5$, then $E_4(P_n^c) \subseteq P_n^c$. \square

4.3 The touching number

In this section, we consider those graphs G for which the touching number is defined.

Theorem 4.3.1. *For any graph G , $t_k(P_3(G)) \geq 8 vt_k(G)$, $k \geq 4$ and $t_3(P_3(G)) \geq 9 vt_k(G)$ for any $k > 3$.*

Proof. Let $x_1, x_2, \dots, x_k, x_1$ be an induced C_k in G . Let y be a touching vertex of this cycle which is adjacent to x_i alone. Then $x_1x_2x_3, x_2x_3x_4, \dots, x_{k-1}x_kx_1, x_kx_1, x_2$ forms a C_k in $P_3(G)$. Also yx_ix_{i+1} and yx_ix_{i-1} are two vertices in $P_3(G)$ such that yx_ix_{i+1} is adjacent to $x_ix_{i+1}x_{i+2}, x_{i-1}x_ix_{i+1}, x_{i-2}x_{i-1}x_i, x_{i+1}x_{i+2}x_{i+3}$ and yx_ix_{i-1} are all adjacent to $x_ix_{i+1}x_{i+2}, x_{i-1}x_ix_{i+1}, x_{i-2}x_{i-1}x_i$ and $x_{i-3}x_{i-2}x_{i-1}$. All these eight edges are touching edges to the C_k in $P_3(G)$. Hence $t_k(P_3(G)) \geq 8 vt_k(G)$, $k \geq 4$.

Again, let $C_k = x_1, x_2, \dots, x_k, x_1$ be an induced C_k in G . If y is a touching vertex of the cycle which touches x_i alone, then $yx_ix_{i+1}, x_{i-1}x_ix_{i+1}$ and yx_ix_{i-1} induce a C_3 in $P_3(G)$. Then $x_ix_{i+1}x_{i+2}, x_{i-2}x_{i-1}x_i$ and $x_{i+1}x_{i+2}x_{i+3}$ are all adjacent to all the three vertices of the C_3 and hence the result. \square

Note: The bounds in the above theorem are strict.

If G is 4 - pan, then, $P_3(G) = K_6$. But, $vt_4(G) = 1$; $t_4(P_3(G)) = 8$; $t_3(P_3(G)) = 9$.

Theorem 4.3.2. For any graph G , $t_n(E_4(G)) \geq 2 t_n(G)$. Further if G contains C_4 as a subgraph where either an edge or two

consecutive edges of C_4 are the edges of the C_n which determines the touching number, then $t_n(E_4(G)) > 2 t_n(G)$.

Proof. Let a C_n in G be $x_1, x_2, \dots, x_n, x_1$. Then $x_1x_2, x_2x_3, \dots, x_nx_1$ is a C_n in $E_4(G)$. If yx_i is a touching edge in G , then $E_4(G)$ contains two touching edges say $yx_i - x_ix_{i+1}$ and $yx_i - x_{i-1}x_i$. Thus $t_n(E_4(G)) \geq 2 t_n(G)$.

Let $C_4 = a_1a_2a_3a_4$ be a subgraph of G . Further, suppose that the edge a_3a_4 is a touching edge in G . Then a_1a_4, a_2a_3, a_1a_2 are touching edges in $E_4(G)$. Hence $t_n(E_4(G)) > 2 t_n(G)$. The proof is similar for the case when any two consecutive edges of C_4 are the edges of the C_n mentioned above. \square

Note: The bound in the above theorem is strict.

For example, if G is k - pan, $k \neq 4$. Then $t_k(G) = 1$ and $t_k(E_4(G)) = 2$.

Chapter 5

The wide diameter and diameter variability

An interconnection network connects the processors of a parallel and distributed system. This can always be represented by a graph, where each vertex represents a processor and each edge represents a vertex to vertex communication link. For routing problems in interconnection networks it is important to find short containers between any two vertices, since the w -wide di-

Some results of this chapter are included in the following paper.
Manju K. Meunon, Daniela Ferrero, A. Vijayakumar, The wide diameter and diameter variability of some graphs (in preparation).

ameter tells us the maximum communication delay when there are up to $w - 1$ faulty nodes in a network modelled by a graph G . This is a consequence of Menger's Theorem. In fact, the maximum integer w such that there exists a non empty container of width w between every pair of distinct vertices is the vertex connectivity $\kappa(G)$. Indeed it is only interesting to study $1 \leq w \leq \kappa(G)$. In networks, communication is the critical issue and the diameter of the graph is a measure of the transmission. In fact the diameter of a graph can be affected by adding or deleting edges. In [33], Graham and Harary studied whether the diameter of hypercubes changed or not on increasing or decreasing the number of edges. The diameter variability arising from the change of edges of a graph G is defined in [77].

5.1 The w -wide diameter

In this section we study the w -wide diameter of some graph operators such as $P_3(G)$, $E_4(G)$ and $L(G)$. We also include results on the w -wide diameter of the join of two graphs.

Lemma 5.1.1. *If there exists a container of width w between*

any two vertices in G , then there exists a container of width w between any two vertices in $P_3(G)$.

Proof. Let there exists a container of width w between any two vertices in G . Let $u_1u_2u_3$ and $v_1v_2v_3$ be any two vertices in $P_3(G)$. Then in G , between any u_i and v_j , $i, j \in \{1, 2, 3\}$, there exists w internally vertex disjoint paths.

Consider u_i and v_j , $i, j \in \{1, 2, 3\}$ where u_i and v_j are non adjacent vertices in G .

Claim: Corresponding to the vertex disjoint paths in $C_w(u_i, v_j)$, there exist vertex disjoint paths connecting $u_1u_2u_3$ and $v_1v_2v_3$ in $P_3(G)$.

For, if $u_i - v_{j+1} - v_j \in C_w(u_i, v_j)$, then $u_1u_2u_3 - u_iv_{j+1}v_j - v_1v_2v_3$ is a path joining $u_1u_2u_3$ and $v_1v_2v_3$ in $P_3(G)$. If $u_i - u_{i+1} - v_j \in C_w(u_i, v_j)$, then $u_1u_2u_3 - u_iu_{i+1}v_j - v_1v_2v_3$ is a path joining $u_1u_2u_3$ and $v_1v_2v_3$ in $P_3(G)$. Let $P = u_i - a_1 - a_2 - a_3 - \dots - a_{k-1} - v_j \in C_w(u_i, v_j)$. If P is an induced path then $u_1u_2u_3 - u_ia_1a_2 - \dots - a_{k-2}a_{k-1}v_j - v_1v_2v_3$ is a path joining $u_1u_2u_3$ and $v_1v_2v_3$ in $P_3(G)$. If P is not an induced path, then consider the induced path $u_i, b_1, b_2, \dots, b_m, v_j$ joining u_i and v_j where $b_1, b_2, \dots, b_m \in \{a_1, a_2, \dots, a_{k-1}\}$. Then $u_1u_2u_3, u_ib_1b_2,$

$b_1b_2b_3, \dots, b_{m-1}b_mv_j, v_1v_2v_3 \in C_w(u_1u_2u_3, v_1v_2v_3)$. Since we are considering internally vertex disjoint paths between u_i and v_j in $C_w(u_i, v_j)$, all the corresponding paths joining $u_1u_2u_3$ and $v_1v_2v_3$ in $P_3(G)$ are internally vertex disjoint.

Now, consider the case when each u_i is adjacent to each v_j . Then as in the above argument, we can find vertex disjoint paths joining $u_1u_2u_3$ and $v_1v_2v_3$ in $P_3(G)$ corresponding to the vertex disjoint paths in $C_w(u_i, v_j)$. \square

Theorem 5.1.2. *For a connected graph G with $w > 1$, $D_w(P_3(G)) \leq [(D_w(G) + 1)/2]$ if $D_w(G)$ is even, and $D_w(P_3(G)) \leq [D_w(G)/2] + 1$ if $D_w(G)$ is odd.*

Proof. Let $D_w(G) = l$. Then there exists a container of width w between any two vertices u and v in G which is of length l . Let $u_1u_2u_3$ and $v_1v_2v_3$ be any two vertices in $P_3(G)$. Then by Lemma 5.1.1, corresponding to any $C_w(u_i, v_j)$; $i, j \in \{1, 2, 3\}$ there exists a $C_w(u_1u_2u_3, v_1v_2v_3)$. Consider the $C_w(u_i, v_j)$ of length l . Let $u_i, a_1, a_2, \dots, a_{l-1}, v_j$ be a path of length l in $C_w(u_i, v_j)$. This corresponds to a path of maximum length in $C_w(u_1u_2u_3, v_1v_2v_3)$. Clearly the maximum length occurs when $u_i, a_1, a_2, \dots, a_{l-1}, v_j$ in $C_w(u_i, v_j)$ is an induced path. Then $u_1u_2u_3, u_i a_1 a_2, a_2 a_3 a_4,$

$a_4a_5a_6, \dots, a_{k-2}a_{k-1}v_j, v_1v_2v_3$ in $C_w(u_1u_2u_3, v_1v_2v_3)$ is of length $\lceil (l+1)/2 \rceil$ if l is even and $\lceil l/2 \rceil + 1$ if l is odd. Hence the result. \square

Lemma 5.1.3. *If there exists a container of width w between any two vertices in G , then there exists a container of width w between any two vertices in $E_4(G)$.*

Proof. Let there exists a container of width w between any two vertices in G . Let $e_1 = u_1v_1$ and $e_2 = u_2v_2$ be any two vertices in $E_4(G)$ and let $u_1 \neq u_2$. Since u_1 and u_2 are any two vertices in G , there exists a container of width w between them. Consider that container $C_w(u_1, u_2)$. If $u_1 - u_2$ is a member of that container, then e_1, u_1u_2, e_2 is a path joining e_1 and e_2 in $E_4(G)$. If $u_1 - v_1 - u_2 \in C_w(u_1, u_2)$, then e_1, v_1u_2, e_2 is a path joining e_1 and e_2 in $E_4(G)$. If $u_1 - a_1 - a_2 - \dots - a_k - v_2 \in C_w(u_1, u_2)$, then $e_1, u_1a_1, a_1a_2, \dots, a_kv_2, e_2$ is a path joining e_1 and e_2 in $E_4(G)$. Thus corresponding to the w internally disjoint paths in $C_w(u_1, u_2)$, there exist at least w internally vertex disjoint paths between e_1 and e_2 . Hence the result. \square

Theorem 5.1.4. *For a connected graph G with $w > 1$, $D_w(E_4(G)) \leq D_w(G) + 1$.*

Proof. Let $D_w(G) = k$. Then there exists a $C_w(u, v)$ of length k between any two vertices $u, v \in V(G)$. Let $e_1 = u_1v_1$ and $e_2 = u_2v_2$ be any two vertices in $E_4(G)$ and let $u_1 \neq u_2$. Then by the Lemma 5.1.3, corresponding to the container $C_w(u_1, u_2)$ in G , there exists $C_w(e_1, e_2)$ in $E_4(G)$. Consider the path of length k in $C_w(u_1, u_2)$. Let it be $u_1 - a_1 - a_2 - \dots - a_{k-1} - u_2$. Then correspondingly there exists $e_1, u_1a_1, a_1a_2, \dots, a_{k-1}u_2, e_2$ in $C_w(e_1, e_2)$ which is a path of length $k + 1$. \square

Corollary 5.1.5. *For a connected graph G with $w > 1$, $D_w(L(G)) \leq D_w(G) + 1$.*

Theorem 5.1.6. *If $G_1 = (n_1, m_1)$ and $G_2 = (n_2, m_2)$ are any two connected graphs having containers of width w_1 and w_2 respectively between any two of its vertices, then there exists containers of width $1 \leq w \leq \text{Minimum}\{w_1 + n_2, w_2 + n_1, \delta(G_1) + \delta(G_2)\}$ between any two vertices of $G_1 \vee G_2$.*

Proof. Let u and v be any two vertices in G_1 . Then the container of width w_1 between u and v in G_1 together with the $u-v$ paths of the form $u - w_i - v$, where $w_i \in G_2$ give $w_1 + n_2$ internally disjoint paths between u and v in $G_1 \vee G_2$. Similarly, if u and v are any two vertices in G_2 then there exists $w_2 + n_1$ internally

disjoint paths between u and v in $G_1 \vee G_2$. Now let $u \in G_1$ and $v \in G_2$. Then in $G_1 \vee G_2$, there exist paths of the form $u - u' - v$, where u' is a neighbor of u in G_1 and $u - v' - v$, where v' is a neighbor of v in G_2 . Thus in this case there exists at least $\delta(G_1) + \delta(G_2)$ paths between u and v . Hence the proof. \square

Remark 5.1.1. For any two connected graphs G_1 and G_2 with at least two vertices, $D_2(G_1 \vee G_2) = 2$.

5.2 The diameter variability of some graph operators

In this section, the diameter variability of $P_3(G)$, $E_4(G)$ and $L(G)$ are studied.

Theorem 5.2.1. *For a connected graph G , $\delta(P_3(G)) \geq \delta(G)$. The equality is attained if and only if G is $K_k - \{\text{two independent edges}\}$ for some $k > 3$.*

Proof. Let $u_1u_2u_3$ be any vertex in $P_3(G)$. Then $u_1u_2u_3$ will have the minimum degree when all the three u_i 's in G have minimum

degree $\delta(G)$ and further more the neighbors of u_1 and u_3 are exactly the same and all the neighbors of u_2 are from the neighbors of u_1 and u_3 . Let the neighbors of u_1 and u_3 be $w_1, w_2, \dots, w_{\delta(G)-1}$ and let the neighbors of u_2 be $w_1, w_2, \dots, w_{\delta(G)-2}$. Further the induced 3-paths in G will be the minimum if and only if $w_1, w_2, \dots, w_{\delta(G)-1}$ form a complete graph. Then, in $P_3(G)$, the only vertices adjacent to $u_1u_2u_3$ are $w_{\delta(G)-1}u_1u_2, w_{\delta(G)-1}u_3u_2, w_{\delta(G)-1}w_iu_2; i = 1, 2, \dots, \delta(G) - 2$. Thus $\delta(P_3(G)) \geq \delta(G)$.

Further $\delta(P_3(G)) = \delta(G)$ only if the vertices of the graph are $u_1, u_2, u_3, w_1, w_2, \dots, w_{\delta(G)-1}$ as explained earlier. Then the graph is clearly $K_k - \{\text{two independent edges}\}$ for some $k > 3$. \square

Lemma 5.2.2. *For any connected graph G , $\kappa'(P_3(G)) \leq \delta(P_3(G)) \leq \Delta(P_3(G)) \leq 3\Delta^2(G) - 7\Delta(G) + 4 + 3(\Delta(G) - 1)! - (\Delta(G) - 2)!$.*

Proof. Both the inequalities on the left are obvious.

Let $u_1u_2u_3$ be a vertex in $P_3(G)$. Then $u_1u_2u_3$ has the maximum degree in $P_3(G)$ when u_1, u_2, u_3 and all their neighbors in G have degree $\Delta(G)$. Then $d(u_1u_2u_3) \leq (\Delta(G) - 1)(\Delta(G) - 1) + (\Delta(G) - 1)! + (\Delta(G) - 2)(\Delta(G) - 2)! + (\Delta(G) - 2)(\Delta(G) - 1) + (\Delta(G) - 1)(\Delta(G) - 1) + (\Delta(G) - 1)!$
 $\leq 2(\Delta(G) - 1)^2 + 2(\Delta(G) - 1)! + (\Delta(G) - 2)(\Delta(G) - 2)! + (\Delta(G) -$

$$\begin{aligned}
 & 2)(\Delta(G) - 1) \\
 & \leq 3\Delta^2(G) - 7\Delta(G) + 4 + 3(\Delta(G) - 1)! - (\Delta(G) - 2)!. \quad \square
 \end{aligned}$$

Note: If $G = C_7$, the bounds in the above lemma are strict and the common value is four.

Theorem 5.2.3. *For any connected graph G , $D^{+i}(P_3(G)) \leq 3\Delta^2(G) - 7\Delta(G) + 4 + 3(\Delta(G) - 1)! - (\Delta(G) - 2)!$, for any i .*

Proof. The proof follows from Lemmas 1.2.10 and 5.2.2. □

Lemma 5.2.4. *For any graph G , $2\delta(G) - 2 \leq \delta(E_4(G)) \leq \Delta(E_4(G)) \leq \Delta^2(G) - 1$.*

Proof. A vertex $e = uv$ in $E_4(G)$ has minimum degree when both u and v in G have minimum degree $\delta(G)$. Then, the $\delta(G) - 1$ edges incident on u and the $\delta(G) - 1$ edges incident on v form $2\delta(G) - 2$ neighbors of e in $E_4(G)$, and hence the left inequality holds.

Now, the maximum degree of a vertex $e = uv$ in $E_4(G)$ occurs when both u and v in G have the maximum degree $\Delta(G)$ and

any edge incident on u and any edge incident on v are opposite edges of some C_4 in G . The $\Delta(G) - 1$ edges incident on u [among the $\Delta(G)$ neighbors, $u - v$ is omitted] and the $\Delta(G) - 1$ edges incident on v forms $2\Delta(G) - 2$ neighbors of e in $E_4(G)$. Now, the maximum number of edges opposite to e in some C_4 of G is $(\Delta(G) - 1)^2$. Thus $\Delta(E_4(G)) \leq 2\Delta(G) - 2 + (\Delta(G) - 1)^2 = \Delta^2(G) - 1$.

□

Theorem 5.2.5. *For any connected graph G , $D^{+i}(E_4(G)) \leq \Delta^2(G) - 1$ for any i .*

Proof. For any connected graph G , $D^{+i}(G) \leq \kappa'(G) \leq \Delta(G)$. Hence the proof follows from Lemmas 1.2.10 and 5.2.3. □

Theorem 5.2.6. *For any connected graph G , $D^{+i}(L(G)) \leq 2(\Delta(G) - 1)$.*

Proof. For a connected graph G , $\Delta(L(G)) \leq 2(\Delta(G) - 1)$. Hence the proof follows from Lemma 1.2.10. □

5.3 The diameter variability of some graph operations

In this section, the diameter variability of the graph operations, 'join' and 'corona' of two graphs are studied.

Theorem 5.3.1. *Let $G_1 = (n_1, m_1)$ and $G_2 = (n_2, m_2)$ be two connected graphs such that at least one of them is not a complete graph. Then $D^{-1}(G_1 \vee G_2) = n_1(n_1 - 1)/2 - m_1 + n_2(n_2 - 1)/2 - m_2$.*

Proof. If at least one of G_1, G_2 is not complete, then $\text{diam}(G_1 \vee G_2) = 2$. So to decrease $\text{diam}(G_1 \vee G_2)$ by one, we have to add edges till both G_1 and G_2 become complete graphs. So to make G_1 a complete graph, we have to add $n_1(n_1 - 1)/2 - m_1$ edges and to make G_2 a complete graph, we have to add $n_2(n_2 - 1)/2 - m_2$ edges. □

Theorem 5.3.2. *If at least one of the two connected graphs $G_1 = (n_1, m_1)$ and $G_2 = (n_2, m_2)$ is not a complete graph, $D^{+0}(G_1 \vee G_2) \geq m_1 + m_2$.*

Proof. If at least one of G_1 and G_2 is not a complete graph, then

$\text{diam}(G_1 \vee G_2) = 2$. Even if we delete all the edges of G_1 and G_2 in $G_1 \vee G_2$, the diameter of $G_1 \vee G_2$ remain unchanged and thus the result. \square

Theorem 5.3.3. *If both $G_1 = (n_1, m_1)$ and $G_2 = (n_2, m_2)$ are graphs having diameter more than two, then $D^{+1}(G_1 \vee G_2) \leq \text{minimum}\{n_1, n_2\}$. If $\text{diam}(G_1) \leq 2$ and $\text{diam}(G_2) \geq 2$, then $D^{+1}(G_1 \vee G_2) \leq n_1$. If $\text{diam}(G_1) \leq 2$ and $\text{diam}(G_2) = 2$, then $D^{+1}(G_1 \vee G_2) \leq n_1(\Delta(G_2) + 1)$. Finally if both G_1 and G_2 are complete graphs, then $D^{+1}(G_1 \vee G_2) = 1$.*

Proof. If both G_1 and G_2 are not complete graphs, then $\text{diam}(G_1 \vee G_2) = 2$.

Let both $G_1 = (n_1, m_1)$ and $G_2 = (n_2, m_2)$ have diameter greater than two. Let $\text{minimum}\{n_1, n_2\} = n_1$. Consider a vertex u in G_2 such that there exists a vertex v in G_2 with $d_{G_2}(u, v) > 2$. In $G_1 \vee G_2$, if we delete all the edges of the form $u - u'$, $u' \in V(G_1)$ then $d_{G_1 \vee G_2}(u, v)$ is three and hence the diameter of $G_1 \vee G_2$ is increased by at least one.

Now let $\text{diam}(G_1) \leq 2$ and $\text{diam}(G_2) \geq 2$. Consider the vertices u and v in G_2 such that $d_{G_2}(u, v) > 2$. In $G_1 \vee G_2$, if we delete all the n_1 edges of the form $u - u'$ where $u' \in V(G_1)$, then $d_{G_1 \vee G_2}(u, v) = 3$ and hence the diameter of $G_1 \vee G_2$ will be increased by at least one.

Let $\text{diam}(G_1) \leq 2$ and $\text{diam}(G_2) = 2$. Choose a vertex u in G_2 having the minimum degree and with $e(u) = 2$. Let v be the vertex in G_2 with $d_{G_2}(u, v) = 2$. Let the neighbors of u in G_2 be w_1, w_2, \dots, w_k . In $G_1 \vee G_2$ delete all the edges of the form $x - w_i$ and $x - u$ where $x \in G_1$. Then $d_{G_1 \vee G_2}(u, x) = 3$ and hence the diameter of $G_1 \vee G_2$ will be increased by at least one. Since the degree of u is at most $\Delta(G_2)$, the result follows.

Finally assume that both G_1 and G_2 are complete graphs. Then $\text{diam}(G_1 \vee G_2) = 1$. So if we delete any one edge in $G_1 \vee G_2$, the diameter will be increased by one and hence the result. \square

Theorem 5.3.4. *For any two connected graphs $G_1 = (n_1, m_1)$ and $G_2 = (n_2, m_2)$ with $n_1, n_2 > 1$,*

1. $D^{-1}(G_1 \circ G_2) \leq n_1 n_2 (n_1 n_2 - 1) / 2 - n_1 m_2$

2. $D^{+0}(G_1 \circ G_2) \geq n_1 m_2$.

3. $D^{+1}(G_1 \circ G_2) = 1$.

Proof. For any two graph $G_1 = (n_1, m_1)$ and $G_2 = (n_2, m_2)$,
 $G_1 \circ G_2 = \text{diam}(G_1) + 2$.

1. In $G_1 \circ G_2$, if we make all the copies of G_2 a complete graph then the diameter of $G_1 \circ G_2$ will decrease by one. For that we have to add $n_1 n_2 (n_1 n_2 - 1) / 2 - n_1 m_2$ edges.

2. In $G_1 \circ G_2$, even if we delete all the edges in all the n_1 copies of G_2 , the diameter of $G_1 \circ G_2$ remains unchanged and hence the result.

3. Let u_i and u_j be the i^{th} and j^{th} vertices of G_1 with $\text{diam}(G_1) = d(u_i, u_j)$. Suppose that u_{i1} and u_{j1} are two vertices in the i^{th} and j^{th} copies of G_2 respectively. In $G_1 \circ G_2$, if we delete an edge $u_i - u_{i1}$ then $d(u_{i1}, u_{j1}) = \text{diam}(G_1) + 3$ and hence the result. □

Theorem 5.3.5. *If both G_1 and G_2 are complete graphs then*
$$D^{-2}(G_1 \circ G_2) = (n_1 + n_1 n_2)(n_1 + n_1 n_2 - 1)/2 - m_1 - n_1 m_2 - n_1 n_2.$$

Proof. If both G_1 and G_2 are complete graphs then $\text{diam}(G_1 \circ G_2) = 3$.

So to decrease diameter by at least two, any two vertices in $G_1 \circ G_2$ must be adjacent. There are $n_1 + n_1 n_2$ vertices in $G_1 \circ G_2$ and there are $m_1 + n_1 m_2 + n_1 n_2$ edges in $G_1 \circ G_2$. Thus the theorem. □

List of problems

1. Characterize P_3 intersection graphs.
2. Characterize non-isomorphic graphs having isomorphic P_3 intersection graphs.
3. If G is a P_3 intersection graph, explain a method to find H so that $P_3(H) = G$.
4. Characterize all self complementary P_3 intersection graphs.
5. Characterize graphs for which the P_3 intersection graph belongs to some special classes of graphs such as planar graphs, perfect graphs, distance hereditary graphs, ptolemaic graphs, split graphs, cographs etc.
6. Characterize P_3 convergent graphs.
7. Are there any P_3 divergent graphs?
8. Characterize $E_4(G)$.
9. Characterize non-isomorphic graphs having isomorphic edge C_4 graphs.

10. Find non isomorphic graphs of the same order having isomorphic edge C_4 graphs.
11. Characterize the graphs G for which $diam(E_4(G)) = diam(G) - 1$, $diam(E_4(G)) = diam(G)$ and $diam(E_4(G)) = diam(G) + 1$. Also characterize the graphs G for which $rad(E_4(G)) = rad(G) - 1$, $rad(E_4(G)) = rad(G)$, and $rad(E_4(G)) = rad(G) + 1$.
12. For any graph G , find a super graph H such that $C(H) = G$ and $C(E_4^i(H)) = E_4^i(G)$, $i \geq 2$.
13. Relationships between the k -path graph and the P_3 intersection graph.
14. The w - wide diameter and the diameter variability of other graph operators and graph operations.
15. The wide diameter $D_w(P_3(G))$ for $\kappa(G) < w \leq \kappa(P_3(G))$.

List of symbols

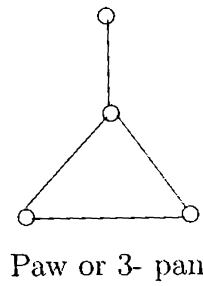
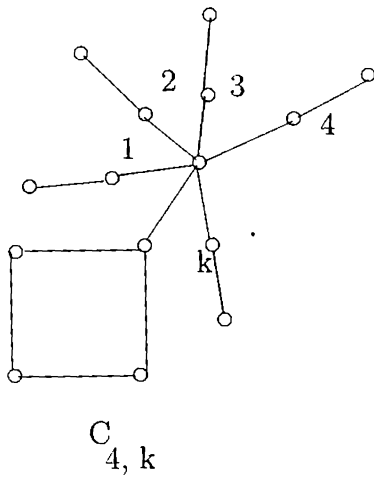
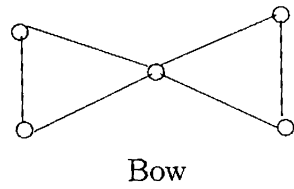
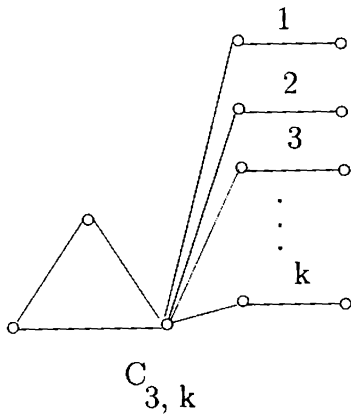
- $\lceil x \rceil$ - Smallest integer $\geq x$
 $\lfloor x \rfloor$ - Greatest integer $\leq x$
 $b(G)$ - Bulge of G
 $c(G)$ - Circumference of G
 $C(G)$ - Center of G
 C_n - Cycle of length n
 $C_w(u, v)$ - Container of width w between u and v
 $d(v)$ - Degree of v
 $\text{diam}(G)$ - Diameter of G
 $\text{dil}w(G)$ - Dilworth number of G
 $d(u, v)$ or $d_G(u, v)$ - Distance between u and v in G
 $D_w(G)$ - w -wide diameter of G
 $D^{-k}(G)$ - The least number of edges
 whose addition to G decreases
 the diameter by (at least) k
 $D^{+0}(G)$ - The maximum number of edges
 whose deletion from G
 does not change the diameter

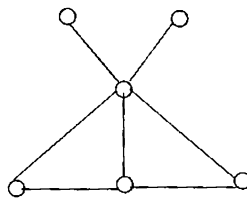
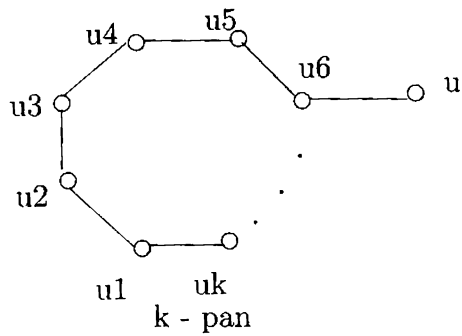
- $D^{+k}(G)$ - The least number of edges whose deletion from G increases the diameter by (at least) k
- E or $E(G)$ - Edge set of G
- $e(u)$ - Eccentricity of u
- $g(G)$ - Girth of G
- G^c - Complement of G
- $G \cong H$ - G is isomorphic to H
- $G \times H$ - Cartesian product of G and H
- $G \vee H$ - Join of G and H
- $G \cup H$ - Union of G and H
- $G \circ H$ - Corona of G and H
- $i_w(G)$ - Weakly connected independent domination number
- $K_{m,n}$ - Complete bipartite graph where m and n are the cardinalities of the partitions
- K_{n_1, n_2, \dots, n_k} - Complete multipartite graph with partite sets of cardinalities n_1, n_2, \dots, n_k
- K_n - Complete graph on n vertices
- $L(G)$ - Line graph of G
- m or $m(G)$ - Number of edges of G

$N[v]$	-	Closed neighborhood of v
$N(v)$	-	Open neighborhood of v
nG	-	n disjoint copies of G
n or $n(G)$	-	Number of vertices of G
P_n	-	Path on n vertices
$rad(G)$	-	Radius of G
$t(G)$	-	Transition number of G
$t_n(G)$	-	Touching number of G
V or $V(G)$	-	Vertex set of G
$\langle V \rangle$	-	Graph induced by V
$vt_k(G)$	-	Vertex touching number of G
$\alpha(G)$	-	Independence number of G
$\alpha_0(G)$	-	Vertex covering number of G
$\beta(G)$	-	Matching number of G
$\gamma(G)$	-	Domination number of G
$\gamma_i(G)$	-	Independence domination number of G
$\gamma_t(G)$	-	Total domination number of G
$\gamma_c(G)$	-	Connected domination number of G
$\gamma_{pr}(G)$	-	Paired domination number of G
$\gamma_{cl}(G)$	-	Clique domination number of G

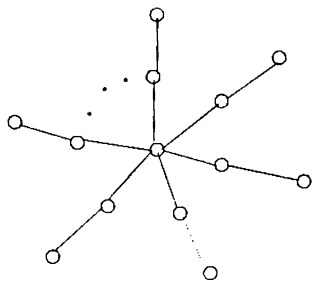
- $\gamma_w(G)$ - Weakly connected domination
number of G
- $\gamma'(G)$ - Edge domination number of G
- $\gamma'_e(G)$ - Efficient edge domination
number of G
- $\delta(G)$ - Minimum degree of vertices in G
- $\Delta(G)$ - Maximum degree of vertices in G
- $\kappa(G)$ - Vertex connectivity of G
- $\kappa'(G)$ - Edge connectivity of G
- $\chi(G)$ - Chromatic number of G
- $\omega(G)$ - Clique number of G
- $\Phi^n(G)$ - n^{th} iterated graph of G under Φ

List of Graphs

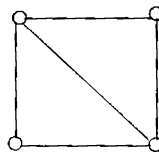




Moth



S(star)

 $K_4 - e$

Bibliography

- [1] B. D. Acharya, M. N. Vartak, A notion of convergence in graph theory, Graph Theory Newsletter 4 (5) (1975).
- [2] J. Akiyama, V. Chvátal, Packing paths perfectly, Discrete Math., 85 (1990), 247-256.
- [3] Aparna Lakshmanan S., Studies on some graph classes, Ph.D Thesis, Cochin University of Science and Technology, India (2008).
- [4] S. Arumugam, S. Velammal, Edge domination in graphs, Taiwanese J. Math., Vol 2 (1998), 173 - 179.
- [5] R. Balakrishnan, K. Ranganathan, A text book of graph theory, Springer (1999).

- [6] P. Baldi, P. Frasconi, P. Smyth , Modeling the internet and the web, John Wiley (2003).
- [7] H. J. Bandelt, M. Farber, P. Hell, Absolute reflexive retracts and absolute bipartite retracts, Discrete Appl. Math. 44 (1993), 9-20.
- [8] L. W. Beineke, On derived graphs and digraphs, in Beiträge zur Graphentheorie (1968), 17-23.
- [9] C. Berge, The theory of graphs, Methuen (1962).
- [10] N. L. Biggs, E. K. Lloyd, R. J. Wilson, Graph theory 1736 - 1936, Oxford University Press (1976).
- [11] B. Bollobás, Modern graph theory, Springer (1991).
- [12] A. Bouabdallah, C. Delorme, S. Djelloul, Edge deletion preserving the diameter of the hyper cube, Discrete Appl. Math. 63 (1995), 91 - 95.
- [13] A. Brandstädt, V. B. Le, J. P. Spinrad, Graph classes, SIAM (1999).
- [14] F. Buckley, F. Harary, Distance in graphs, Addison- Wesley (1990).

- [15] G. Chartrand, S. Schuster, On the independence number of complementary graphs, *Trans. New York Acad. Sci., Series II* 36 (3) (1974), 247 - 251.
- [16] V. Chvátal, P. L. Hammer, Aggregation of inequalities in integer programming, *Ann. Discrete Math.*, 1(1997), 145 - 162.
- [17] J. Clark, D. A. Holton, *A first look at graph theory*, Allied Publishers Limited (1991).
- [18] C. J. Colbourn, L. K. Stewart, Permutation graphs: connected domination and Steiner trees, *Discrete Math.* 86 (1990), 179-189.
- [19] D. G. Corneil, Y. Perl, I. K. Stewart, A linear recognition algorithm for cographs, *SIAM J. Comput.*, 14 (1985), 926 - 934.
- [20] M. B. Cozzens, L. L. Kelleher, Dominating cliques in graphs, *Discrete Math.* 86 (1990), 101 - 116.
- [21] G. B. Dantzig, D. R. Fulkerson, On the max-flow, min-cut theorem of networks in linear inequalities and related systems, *Ann. of Math. Stud.* 38, Princeton University Press (1956)

- [22] R. Diestel, Graph theory, Springer (1991).
- [23] P. Duchet, H. Meyniel, On Hadwiger's number and stability number, *Ann. Discrete Math.*, 13 (1982), 71 - 74.
- [24] E. Dunbar, J. W. Grossman, J. H. Hattingh, S. T. Hedetniemi, A. A. McRae, On weakly connected domination in graphs, *Discrete Math.* 167/168 (1997), 261 - 269.
- [25] F. Escalante, ber iterierte Cliquen - Graphen, *Abh. Math. Sem. Univ. Hamburg* 39 (1973), 58 - 68.
- [26] L. Euler, Solutio problematis ad geometriam situs pertinentis, *Comment. Academiae. Sci. I. Petropolitanae*, 8 (1736), 128 - 140
- [27] D. Ferrero, Connectivity of path graphs, *Acta. Math. Univ. Comeniana* LXXII, 1 (2003), 59 - 66.
- [28] D. Ferrero, C. Padro, Disjoint paths of bounded length in large generalized cycles, *Discrete Math.* 197/198 (1999), 285 - 298.
- [29] S. Földes, P. L. Hammer, The Dilworth number of a graph, *Ann. Discrete Math.*, 2 (1978), 211 - 219.

- [30] S. Földes, P. L. Hammer, Split graphs, *Congres. Numer.* 19 (1977), 311 - 315.
- [31] P. A. Fowler, The Königsberg bridges - 250 years later, *Amer. Math. Monthly*, 95 (1988), 42 - 43.
- [32] M. C. Golumbic, *Algorithmic graph theory and perfect graphs*, Academic Press, New York (1980).
- [33] N. Graham, F. Harary, Changing and unchanging the diameter of a hypercube, *Discrete Appl. Math.* 37/38 (1992), 265 - 274.
- [34] J. L. Gross, J. Yellen, *Graph theory and its applications*, CRC Press (1998).
- [35] F. Harary, *Graph theory*, Narosa (2000).
- [36] F. Harary, R. Z. Norman, Some propereties of line digraphs, *Rend. Circ. Mat. Palermo* 9 (1960), 161 - 168.
- [37] F. Harary, A characterization of block graphs, *Canad. Math. Bull.* 6 (1963), 1 - 6.
- [38] F. Harary, I. C. Ross, The square of a tree, *Bell System Tech. J.* 39 (1960), 641 - 647.

- [39] T. W. Haynes, S. T. Hedetniemi, P. J. Slater, *Fundamentals of domination in graphs*, Marcel Dekker, Inc. (1998).
- [40] T. W. Haynes, P. J. Slater, Paired domination and paired domatic number, *Congr. Numer.* 109 (1995), 67 - 72.
- [41] S. T. Hedetniemi, R. C. Laskar, *Connected domination in graphs*, *Graph theory and Combinatorics*, Academic Press, London, 209 - 218.
- [42] R. L. Hemminger, Digraphs with periodic line digraphs, *Stud. Sci. Math. Hung.* 9 (1974), 27 - 31.
- [43] X. Hou, T. Wang, On generalized wide diameter of graphs, *Taiwanese J. Math.*, Vol.7, No.2 (2003), 339 - 345.
- [44] E. Howorka, On metric properties of certain clique graphs, *J. Combin. Theory Ser.*, 27 (1979), 67 - 74.
- [45] D. F. Hsu, On container width and length in graphs, groups, and networks, *IEICE Trans. Fund. Electron. Commun. Comput. Sci.* E77-A (1994), 668 - 680.
- [46] L. H. Hsu, C. K. Lin, *Graph theory and interconnection networks*, CRC Press (2009).

-
- [47] S. R. Jayaram, Line domination in graphs, *Graphs Combin.*, 3 (1987), 357 - 363.
- [48] D. C. Kay, G. Chartrand, A characterization of certain Ptolemaic graphs, *Canad. J. Math*, 17 (1965), 342 - 346.
- [49] D. König, *Theorie der Endlichen und Unendlichen Graphen*, Leipzig (1936).
- [50] M. Knor, L. Niepel, L. Soltés, Centers in line graphs, *Math. Slovaca*, 43 (1993), 11 - 20.
- [51] M. Knor, L. Niepel, Graphs isomorphic to their path graphs, *Math. Bohem.* 127 (2002), 473 - 480.
- [52] J. Krausz, Démonstration nouvelle d'un théorème de Whitney sur les réseaux, *Mat. Fiz. Lapok* 50 (1943), 75 - 85.
- [53] S. C. Liaw, G. J. Chang, Wide diameter of butterfly networks, *Taiwanese J. Math.*, Vol.3, No.1 (1999), 83 - 88.
- [54] B. Liu, X. Zhang, Wide diameter and diameter of networks, *Ars Combin.* 89 (2008).
- [55] Manju K. Menon, A. Vijayakumar, Dynamics of the P_3 intersection graph, (communicated).

- [56] Manju K. Menon, A. Vijayakumar, Some domination parameters in $E_4(G)$, (communicated).
- [57] Manju K. Menon, A. Vijayakumar, The edge C_4 graph of a graph, International Conference on Discrete Mathematics Proceedings (ICDM 2006), Ramanujan Math. Soc. Lect. Notes Ser. 7 (2008), 245 - 248.
- [58] Manju K. Menon, A. Vijayakumar, The edge C_4 graph of some graph classes, (communicated).
- [59] Manju K. Menon, A. Vijayakumar, The P_3 intersection graph, Util. Math. 75 (2008), 35 - 50.
- [60] Manju K. Menon, Daniella Ferrero, A. Vijayakumar, The wide diameter and diameter variability of some graphs, (in preparation).
- [61] C. W. Marshall, Applied graph theory, Wiley (1971).
- [62] K. Menger, Zur allgemeinen Kurventheorie, Fund. Math. 10 (1927), 95 - 115
- [63] S. Mitchell, S. T. Hedetniemi, Edge domination in trees, Congr. Numer. 19 (1977), 489 - 509.

- [64] M. E. J. Newman, The structure and function of complex networks, *SIAM Rev* 45 (2003), 167 - 256.
- [65] O. Ore, *Theory of graphs*, Amer. Math. Soc. Coll. Publ. 38, Providence (R.I.)(1962).
- [66] E. Prisner, *Graph dynamics*, Longman (1995).
- [67] E. Prisner, A common generalization of line graphs and clique graphs, *J.Graph Theory* 18 (1994), 301 - 313.
- [68] S. B. Rao, Aparna Lakshmanan S., A. Vijayakumar, Co-graphic and global cographic domination number of a graph, *Ars Combin.*, (to appear)
- [69] S. B. Rao, A. Vijayakumar, On the median and the anti-median of a cograph, *Int. J. Pure Appl. Math.*, 46, 5 (2008), 703 - 710.
- [70] R. C. Read, On the number of self complimentary graphs and digraphs, *J. London Math. Soc.* 38 (1963), 99 - 104.
- [71] F. S. Roberts, *Discrete mathematical models with applications to social, biological and environmental problems*, Prentice - Hall (1976).

- [72] A. C. M. van Rooji, H. S. Wilf, The interchange graph of a finite graph, *Acta. Math. Acad. Sci. Hungar* 16 (1965), 263 - 269.
- [73] B. L. Schwartz, Infinite self interchange graphs, *Pac. J. Math.* 31 (1969), 497 - 504.
- [74] J. G. Stemple, M.E.Watkins, On planar geodetic graphs, *J. Combin. Theory* 4 (1968), 101-117.
- [75] J. L. Szwarcfiter, A survey on clique graphs, *Recent Advances in Algorithms and Combinatorics* (2003), 109-136.
- [76] L. E. Trotter. Jr., Line perfect graphs, *Math.Program.* 12 (1977), 255-259.
- [77] J. J. Wang, T. Y. Ho, D. Ferrero, T. Y. Sung, Diameter variability of cycles and tori, *Information Sciences* 178 (2008), 2960 - 2967.
- [78] D. B. West, *Introduction to Graph Theory*, Prentice - Hall (1999).
- [79] H. Whitney, Congruent graphs and connectivity of graphs, *Amer. J. Math.* 54 (1932), 150 - 168.

Index

- acyclic, 9
- adjacent
 - edges, 8
 - vertices, 8
- b*-cycle, 11, 92, 94
- basin, 19
- bipartite graph, 12, 87
- block, 14, 49
 - graph, 21, 25, 93
- bow, 23
- bulge, 11
- cartesian product, 23
- center, 10, 74
- chord, 10
- chordal, 10, 52
- chromatic number, 11, 53
- circumference, 9
- clique, 10
 - dominated graph, 15, 84
 - domination number, 15
 - number, 11
- closed neighborhood, 11
- cograph, 21, 24, 25, 89
- comparable, 14
- complement, 12
- complete
 - bipartite graph, 12
 - graph, 10, 53, 86
 - multipartite graph, 12, 86
- component, 10
- connected, 10
 - domination number, 15
- container, 110, 113

- convergent, 19
- corona, 23, 119
- cut vertex, 14
- cycle, 9
- degree, 8
- diameter, 10, 59, 72
 - variability, 22, 115
- Dilworth number, 14
- disconnected, 10
- distance, 10
- distance hereditary graph, 6
- divergent, 19
- dominating set, 14, 76
- domination number, 14, 58
- E_4
 - convergent, 103
 - periodic, 104
- eccentricity, 10
- edge
 - domination number, 17, 82
 - edge C_4 graph, 28, 70
 - edge connectivity, 14
- efficient
 - edge domination number, 17, 83
- end vertex, 7
- Eulerian, 9, 25, 35
- 1-factor, 9
- finite graph, 7
- fixed, 19
- fixed graph, 101
- forbidden subgraph, 11, 50, 71
- geodetic
 - graph, 21, 50, 92
- girth, 9
- graph, 7
- H - intersection graph, 29
- H -free, 9, 51, 88
- incident, 8
- incomparable, 14

-
- independence number, 13, 58
 - independent, 13
 - domination number, 14
 - induced subgraph, 9
 - internal vertex, 8
 - intersection graph, 17
 - isolated vertex, 8
 - isomorphic, 12, 70
 - iterated graph, 19
 - join, 23, 119
 - k -clique, 11
 - k -vertex cut, 13
 - k -regular, 8
 - length, 9
 - length of the container, 21
 - line graph, 17, 24
 - matching
 - number, 13
 - Menger's theorem, 110
 - mortal, 19, 105
 - moth, 23, 88
 - n -connected, 13, 27
 - null graph, 8
 - open neighborhood, 11
 - order, 8
 - origin, 8
 - P_3 intersection graph, 27, 42, 111
 - P_3
 - convergent, 102
 - divergent, 102
 - mortal, 102
 - periodic, 101
 - k - path graph, 18
 - paired
 - domination number, 15
 - path, 9
 - paw, 10
 - pendant edge, 8
 - pendant vertex, 8

- perfect
 - graph, 6, 20
 - matching, 9
- period, 19
- periodic, 19
- proper vertex coloring, 11

- radius, 10, 58, 72

- self complementary, 12
- semibasin, 19, 105
- size, 8
- spanning subgraph, 9
- split graph, 20, 25
- star, 12
- subdivision, 23
- subgraph, 9

- terminus, 8
- threshold graph, 20, 25, 88
- total domination number, 15,
79
- totally disconnected, 12

- touching number, 19, 105
- trail, 9
- transition number, 19, 104
- tree, 10, 49, 87
- triangle, 10
- trivial, 8

- $u - v$ cut, 14
- $u - v$ path, 9
- unicyclic, 9, 26, 84
- union, 23
- universal vertex, 8

- vertex
 - coloring, 11
 - connectivity, 13, 110
 - cover, 13
 - covering number, 13
 - hereditary, 11, 51
 - touching number, 19

- w -container, 21
- walk, 8

weakly

connected domination num-
ber, 16

connected independent dom-
ination number, 16

geodetic, 21, 25, 90

wide diameter, 21, 110

width, 21, 110, 113