STOCHASTIC MODELLING USING MARKOV CHAINS

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Certificate

Certified that the thesis entitled '**Stochastic Modelling Using Markov Chains**' is a bonafide record of work done by Sri. Stephy Thomas under my guidance in the Department of Statistics, Cochin University of Science and Technology, Cochin-22, Kerala, India and that no part of it has been included anywhere previously for the award of any degree or title.

Cochin-22 Dr. V. K. Ramachandran Nair

4^{th} March 2011 (Supervising Guide)

Declaration

The thesis entitled '**Stochastic Modelling Using Markov Chains**' contains no material which has been accepted for the award of any Degree or Diploma in any University and to the best of my knowledge and belief, it contains no material previously published by any other person, except where due references are made in the text of the thesis.

Cochin-22 Stephy Thomas

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Chapter 1
Introduction

Modelling physical and natural phenomenon or the process of modelling has been a necessity for the human race because of many reasons. The task would have been simple if the phenomenon or the process was deterministic. Almost all physical equations related to motion assume that the hidden process is deterministic or they will tactically neglect the randomness that may arise because of the changes in acceleration due to gravity, friction in the air etc. Every modelling problem is thus simplified by neglecting the hidden random nature involved in the process.

With the Heisenberg's Uncertainty Principle randomness became an accepted fact and scientists were forced to model, by taking into consideration, the effects due to randomness. In order to incorporate randomness from single source, one dimensional random variables were introduced. However, later it was noticed that the randomness could be from more than one source. For example, the time taken by a mass to reach the ground depends on the distance as well as the air friction. Hence, the randomness is attributed not only to a single factor but also to two factors. Hence, in order to integrate randomness due to various factors or randomness from various sources multivariate random variables were introduced.

The problem was further worsened while dealing with random variables that change over time. Since time is not finite dimensional, time variables cannot be modelled by multivariate distributions. This paved the way for the introduction of stochastic processes that could be used to model the random quantities that vary over time or place. One of the basic characteristics of these variables is that they are related to each other. If these are independent, then they can be modelled using the one dimensional random variables. Stochastic processes can be classified based on the continuous or discrete nature of the random variable at each time and also that of the time scale.

A major class of stochastic processes is the one that satisfies the Markov property. Markov property states that the future depends on the present not on the past in statistical sense. Hence, by assuming that the processes follow Markov property it is assumed that the conditional distribution of a future value of the process is dependent only on the most recent available value. Once the initial distribution and the one-step transition probability matrix is known then the probabilistic properties of the whole process are tracked. The Markov process with its nice probabilistic properties and the dependence structure finds application in many areas of applied science including Physics, Chemistry and Genetics. In this study, problems in reliability, quality control and distribution have been modelled using the Markov process.

Chapter two is dedicated to the preparation of the theoretical background for introducing the various concepts in the study. The basic results regarding the continuous and discrete time Markov chains, transition probability matrices, infinitesimal generators etc. are discussed in the chapter. The various repair policies are also discussed in this chapter. Neuts (1975) introduced the phase type distribution from the continuous time Markov chains. This is a situation of modelling using the Markov chains. For an elaborate discussion on phase type distribution one may refer to Neuts (1981) and Neuts (1995). Quality control is an area where the Markov chain can be effectively exploited to model various strange situations. Start-up demonstration tests are the mechanism by which the quality of a product can be conveyed to the customer by the vendor. An elaborate discussion on the start-up demonstration tests and various concepts in the literature is dealt with in the chapter. The concept of runs and the various methods of making a record of the runs are also discussed in the chapter. Fu and Koutras (1994) have introduced Markov chain embedding as an effective procedure for deriving the distributions associated with runs. An introduction to the Markov chain embedding technique is provided in the chapter.

Most of the classical reliability modelling papers deal with two states namely, working and failure state systems. Here we model multi-state systems that allow the system to be in various intermediate states in between. The modelling of lifetime, after repair, has been confusing for those who are working with these situations. Earlier they were either assumed to be in perfect condition or were brought back to the situation just prior to the failure. Lam (1988) introduced geometric process model, which could satisfactorily model monotone random variables and this has been widely used to model the repair as well as the working time models as these can be considered to be monotone variables in the case of a two state system. There have been many attempts to model the multi-state systems involving repairs using extensions of the geometric process model (for details see Lam (2005) and the references therein). All these papers assume that the lifetimes and the repair times as a sequence of monotone random variables, which cannot be justified always in the case of a multistate system. Repair time for the critical failures will take more time than the minor failures even if it occurs prior to the second time. Hence, a state based modelling mechanism is essential to model these situations.

In the third chapter a multistate system with monotone lifetimes (repair times) is discussed. To generalize the model, the consecutive life times and repair times are considered to have Markov dependence. Neuts and Bhattacharjee (1981) proved that every distribution in the positive real domain can either be treated as a phase type random variable or approximated very well by the phase type distribution. Hence, the life times or the repair times in each state are assumed to follow phase type distribution. A regression model introduced by the Cox (1972) has been used to incorporate the impact of the repairs on the repair and the lifetimes assuming the repairs to be as the concomitant or the covariate variable. An expression for the long run cost for the assumed model is obtained. Further an algorithm which enables the evaluation of the best N (say) policy has been developed and the results are illustrated.

There is relatively little literature available in the discrete time reliability systems in which the life time is measured on a discrete scale, like number of copies by a printer, number of operations by a switch etc, in comparison to the usual continuous scale. The only attempt to model the repair or lifetimes in discrete time systems having impact of the repair on them has been done by Castro and Sajuan (2004) by the Power process. That can also be used only for modelling stochastically increasing or decreasing variables. But we made an attempt to model this in a more generalised frame work by assuming discrete phase type distribution (Kao (1997)) for the repair as well as the lifetimes and with the Cox regression models (Cox (1972)) to incorporate the impact of repair. Expression for the long run cost has been derived and the results are illustrated.

Any interruption while the repair is in progress may have high financial liabilities and such interruptions can be averted by incorporating appropriate protection procedures. The question of when the process should be protected has the natural answer of protection during the whole repair time. However, the problem becomes more complicated if the protection cost is very high compared to the repair cost. Such repair processes arise if the repairs have to be started from scrap, once interrupted. Interruption may be the immediate withdrawal of the repair person who has been working on repair or power failure etc. While repairing complicated machines an immediate withdrawal by the repairing personal may result in stating the repair from zero. Even though papers on interruption appear in the queuing scenario connecting with service breaks (See Krishnamoorthy et. al. (2009) and references therein), there are not sufficient work dealing with interruption in the reliability scenario. In chapter four, we try to model this situation assuming that the repair times follow Erlang distribution with n states. We will find the optimum value for the number of states to be protected.

Quality control is another area where probability and the Markov property can be effectively used. Classical quality control problems assume that the consecutive trials are independent. However, that may not be the case always. Generally they are Markov dependent. Start-up demonstration tests are the procedure of convincing the customer about the quality of certain products like heavy machines. Han and Gage (1983) introduced the concept of start-up demonstration tests. The results of the general theory of runs have been exploited widely in the context of start-up demonstration tests. Most of the researchers in start-up demonstration test describe it with the help of later or sooner waiting time problem of runs. For more discussions on the theory of start-up demonstration tests one may refer to Balakrishnan and Koutras (2002). Viveros and Balakrishnan (1993) introduced the Markov dependence structure in the start-up demonstration tests. The idea of repair or corrective action in the start-up scenario was introduced by Balakrishnan et. al. (1995).

Balakrishnan and Chan (1999) introduced two stage start-up demonstration tests in which the product is accepted if k_1 consecutive successes occur before l_1 failures and if the above event does not happen but k_2 consecutive successes occur before the next l_2 failures and the product is rejected if both the events do not happen. Smith and Griffith (2003) proposed a procedure having similarities with the Markov chain embedding technique introduced by Fu and Koutras (1994) in run scenario. A general procedure for finding various probabilities of interest in the start-up context was discussed by Aston and Martin (2005), start-up demonstration test was formulated as a special case of the competing patterns. Martin (2008) obtained a recursive formula for various variables of interest for different start-up demonstration tests.

In chapter five, we introduce two new start-up demonstration tests and derive the various characteristics involved. In the first model, we try to generalize the existing models in the start-up scenario. Here we assume that if consecutive failures occur for the product, it will be sent for repair and the rejection takes place when the number of random failure becomes large. In the second model, we reject the product based on the number of consecutive failures and the repair process will be triggered if the number of failures exceeds certain specified number. Consecutive failure occurs when the product is bad or has failed components and the random failures are due to the accidental causes. This is the motive for the introduction of the second model.

Runs have been finding applications in various fields like quality control, reliability, distributions, statistical testing procedures, genetics etc. Reliability of consecutive k out of n system can be computed by the application of runs. In many applications, along with the number of runs, it is of interest to find the number of occurrence of each outcome. In many practical situations, there involve more than two types of outcomes. Most of the earlier works dealing with the distributions of runs were dealing with the independent trails and they exploited combinatorics. For more applications on runs one may refer to Balakrishnan and Koutras (2002), Koutras (2003) and the references there in. Markov chain embedding has been widely used since Fu and Koutras (1994) in run scenario. Later Han and Aki (1999) further enhanced the Markov chain embedding technique by introducing multinomial and returnable type embeddings and they obtained the distribution of the runs in a sequence of multi- state trials. Fu and Lou (2003) discusses runs and patterns with Markov chain embedding in detail.

However, almost all works in this context were dealing with the runs alone. When we tried to model the multistate systems with an extension of the geometric process (Lam (2005)) with the Markov dependence between consecutive working and repair states, we were forced to derive the distribution of runs and the occurrence of each outcome. In chapter six, we derive the distribution of the runs and the occurrence of the events exploiting the Markov chain embedding methodology. Most interesting factor is that the various methods of counting viz. overlapping (Mood (1940) and Ling (1988)), non overlapping (Feller (1968)), partially overlapping (Aki and Hirano (2000)), runs of length greater than or equal to some specified number, Markov Binomial and Markov multinomial distributions comes as special cases of the what we discussed. The results obtained here can be easily extended to the l dependence (Aki and Hirano (2000)) case too. The study can be extended to the case of patterns also.

Geometric distributions and the negative binomial distributions are the well-known waiting time distributions. We also derive the expressions for obtaining the waiting time distributions associated with runs and the occurrence of each event too.

In the present study, we have discussed only some of the most important applications of the Markov chains, especially in the field of reliability, quality control and distributions. The above mentioned methods can be extended in framing nonparametric tests or in developing quality based sampling plans for multistate systems. Many problems in genetic modelling can be easily over ridden by the use of the Markov chains and it is still an area to be explored.

Chapter 2
Basic Concepts and the Literature Survey

2.1 Introduction

Ageing is an important characteristic prevailing in the universe and there has been evidence of ageing since the beginning of the universe. Darwin's evolution theory reiterates that dependence is attributed to ageing. In mechanical systems or the phenomenon having ageing, it is assumed to have either an increasing or a decreasing trend in the consecutive observations of the random variable. Hence there is dependence between the consecutive observations of the random phenomena under investigation. But most of the earlier papers assume that there is no dependence or in other words there is no ageing. In this chapter we will introduce some of the concepts that will be used in the thesis.

2.2 Stochastic Process

A stochastic process is a family $\{X(t), t \in T\}$ of random variables defined on the probability space (Ω, F, P). The set T is called the index set or the time set, and can assume values from any infinite subset of real numbers. $X(t)$ at each value of $t \in T$ is a random variable. Hence stochastic process is an infinite dimensional random vector indexed by a parameter. Depending on the index set being countable or uncountable a stochastic process is termed as discrete time or continuous time stochastic process. Normally a continuous time stochastic process is denoted by $\{X(t), t \in T\}$ and a discrete time stochastic process is denoted by $\{X_n, n \in N\}$, N being any subset of the set of integers. The set of possible values the random variable can assume is called the state space. Based on the state space being continuous or discrete, we can classify a stochastic process as discrete or continuous stochastic process. The function $X(t,\omega), \omega\in \Omega, t\in T$, is called the realizations or sample paths of the process $\{X(t), t\in T\}$.

2.2.1 Markov Process

Markov process is an important class of stochastic process. It finds application in almost every field, where there is dependence between the consecutive trials or attempts. Given the value of $X(t)$, the value of $X(u), u>t$, is independent of the value of $X(s), s<t$, then the stochastic process $\{X(t), t\in T\}$ is said to satisfy the Markov property**.** In simple terms, Markov property states that the future depends statistically on the present not on the past.

Hence a discrete time stochastic process $\{X_n, n\in N\}$ satisfies the Markov property if

$$P\left(X_{n+1}=x_{n+1}\middle|X_1=x_1, X_2=x_2, ..., X_n=x_n\right)=P\left(X_{n+1}=x_{n+1}\middle|X_n=x_n\right) \qquad (2.1)$$

Let us denote $P\left(X_{n+k}=j\middle|X_n=i\right)=P_{ij}^{n,n+k}$, then $P_{ij}^{n,n+k}$ is said to be k- step transition probability for transition from state i to state j through $n, n+k$. If $P_{ij}^{n,n+k}$ does not depend on n but on the length k in (2.1), then we say that the chain is homogeneous. For $k=1$, the transition probabilities are called one-step transition probabilities. A matrix whose elements are one-step transition probabilities is called the transition probability matrix (t.p.m). Any matrix whose elements are non-negative and each row sum is unity is called a transition probability matrix P. Hence $P=\left(\left(P_{ij}\right)\right)$ and let $P^{(n)}=\left(\left(P_{ij}^{(n)}\right)\right)$ be the n-step transition probability matrix. Then we can establish that $P^{(n)}$ is nothing other than P^n itself.

The initial probability vector, vector of probabilities that the process is in at each state at the start of the Markov chain, and the transition probability matrix of a homogeneous Markov process describes the process completely.

For a Markov chain $\{X_n, n \in N\}$, let f_{ii}^n denote the probability of return to state i, starting from state i for the first time in n steps, that is, $f_{ii}^n = P\left(X_n = i, X_r \neq i, r = 1, 2, ..., n-1 \mid X_0 = i\right)$. A state i is said to be recurrent if and only if $\sum_{n=1}^{\infty} f_{ii}^n = 1$, otherwise it is called transient. In other words, a state i is said to be recurrent if and only if the probability of ultimate return to state i is unity. The discrete time analogue of Chappman-Kolmogrov equation gives the method to finding the n-step transition probabilities, for $n > 1$, which is given by $p_{ij}^n = \sum_{k=1}^{\infty} p_{ik}^r \; p_{kj}^{n-r}$.

Consider a Markov process $\{X(t), t \in T\}$ with states $0, 1, 2, \ldots$. Let us assume that the usual transition probabilities are stationary that is the transition probability remains unaltered by the time at which transition occurs. Let $P_{ij}(t) = P\left(X(t+s) = j \middle| X(s) = i\right)$ be the probability of transition from state i to state j, during a duration of time t. Then for a discrete state, continuous time Markov process we have

$$\lim_{h \to 0+} \frac{1 - p_{ii}(h)}{h} = q_i \text{ and}$$

$$\lim_{h \to 0+} \frac{p_{ij}(h)}{h} = q_{ij} \text{ called the infinitesimal generators.}$$

Then the matrix $A = \begin{bmatrix} -q_0 & q_{01} & . & . \\ q_{10} & -q_1 & . & . \\ . & . & . & . \\ . & . & . & . \end{bmatrix}$ is called the matrix of infinitesimal generators. It can be seen that each row sum for the matrix is zero. q_{ij} is finite for every Markov process but q_i is finite for finite state Markov process and it can be infinite for infinite state Markov process.

The Chappman-Kolmgrov equation for the Markov process is given by $p_{ij}(t+s)=\sum_{k=1}^{\infty} p_{ik}(t)\, p_{kj}(s)$, for every s,t.

Kolmogrov's forward equation and backward differential equations (Ross,1997) are widely used in modelling problems using Markov processes. Kolmogrov's backward equations are given by

$p_{ij}'(t)=\sum_{k\neq i} q_{ik}\, p_{kj}(t)-q_j p_{ij}(t)$ for every i, j and $t \geq o$.

Kolmogrov's forward equations are given by, $p_{ij}'(t)=\sum_{k\neq i} q_{kj}\, p_{ij}(t)-q_j p_{ij}(t)$ for every i, j and $t \geq 0$.

2.2.3 Renewal Process

Renewal process is an important class of stochastic processes. Consider the situation of replacing a bulb immediately after failure. Let X_i, $i=1,2,3,\ldots$ denote the lifetime of the i^{th} bulb. Since the replacements are by a new one, lifetimes of each bulb can be assumed to be independent and identical. Hence it is natural to find the number of replacements by the time t or the time of failure of the n^{th} bulb. Renewal theory is associated with such problems.

A renewal process $\{N(t), t>0\}$ is a non negative integer valued stochastic process that registers the successive occurrence of an event during the time interval $(0,t]$, where the time durations between consecutive events are positive, independent, identically distributed random variables (Karlin and Taylor, 1975). Associated with a renewal process $\{N(t), t>0\}$ we can define another variable $\{S_n, n=0,1,2\ldots\}$, where S_n denotes the time of occurrence of the n^{th} event. Both $\{S_n, n=0,1,2\ldots\}$ and $\{N(t), t>0\}$ are invariably called the renewal processes. The time between the consecutive occurrence of the events, $X_i, i=1,2,3\ldots.$, are known to be as the inter-occurrence times. The relation connecting these variables is $N(t) \le n \Leftrightarrow S_n \ge t$. Further we have $S_n = \sum_{i=1}^{n} X_i$.

Renewal Process finds application in various fields including reliability theory and queuing theory. While modelling perfect repair models, models assuming that the system is as good as new, the consecutive lifetimes or the repair times can be assumed to be independent and identical. Hence renewal process can be used to model these problems. In queuing problems consecutive arrival times and the service times are assumed to be independent and identical. Hence the service times and the repair times can be assumed to be forming a renewal process. For an elaborate discussion on the renewal theory one may refer to classical books like Karlin and Taylor (1975) or Cox (1965) or any other books on Stochastic Processes.

Poisson process can be treated as a special case of the renewal process. A renewal process whose inter-occurrence times follow exponential distribution are known as the Poisson process. It can be proved that the $N(t)$ follows Poisson distribution in the case of a Poisson process. It can be seen that if the inter-occurrence times $X_i, i = 1, 2...$ follow exponential distribution with parameter λ, $N(t)$ follows Poisson distribution with parameter λt and S_n follows a gamma distribution with parameters (n, λ). Arrival process and the departure times of the classical Markovian queuing models are examples of the Poisson process.

One of the major advantages of the Poisson process is the lack of memory property of the exponential distribution. Lack of memory property says that the distribution of future value of the random variable given that the random variable has elapsed some time is independent of the time for which it has been working. In other words, the probabilistic statements about the residual value for the random quantity is the same as that of the quantity when observed from the beginning. Let X be a continuous random variable. Then X is said to satisfy the lack of memory property or the memoryless property if $P(X \ge s + t | X \ge s) = P(X \ge t)$,

s, t being real. The joint distribution of $S_i, i = 1, 2, ...n$ given that $N(t) = n$, in the case of a Poisson process, is the distribution of the order statistics from a sample of n observations taken from the uniform distribution on $[0, t]$. Also for $u < t$ and $k < n$, $P(N(u) = k \mid N(t) = n)$ follows Binomial distribution with parameters n and $\frac{u}{t}$.

Developing a formal definition for the random variables whose values tend to increase or decrease as the number of trials becomes large is given in this section. The amount of error while taking measurements will reduce considerably with the experience of the one who measures it. Given two random variables X and Y, X is said to be stochastically larger than Y or Y is stochastically smaller than X , if

$$\mathrm{P}(X > \alpha) \geq \mathrm{P}(Y > \alpha), \text{ for all real } \alpha \text{ (Lam 2005).}$$

This is denoted by $X \geq_{st} Y$ or $Y \leq_{st} X$. Furthermore, we say that a stochastic process $\{X_n, n \in N\}$ is stochastically decreasing if $X_n \geq_{st} X_{n+1}$ and stochastically increasing if $X_n \leq_{st} X_{n+1}$ for all n = 1, 2, ...

2.3 Phase Type Distribution

The first attempt of embedding a Makov process was carried out by Neuts(1975). He developed a distribution as the waiting time till absorption in a Markov Process with only one transient state. Since then Phase type (PH) distribution has been used widely, in modelling problems. Later the concept behind the development of the PH distribution paved way for a set of methods popularly known as Matrix Analytic Methods.

Consider a Markov process with states $\{1, 2, ..., m, m+1\}$ with infinitesimal generator $\begin{bmatrix} T & T^0 \\ 0 & 0 \end{bmatrix}$, where T is an $m \times m$ matrix and T^0 is a column vector of size m and the initial vector (α, α_{m+1}) where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_m)$. A probability distribution $F(.)$ on $[0, \infty)$ is a distribution of phase type (PH Distribution) (Neuts,1981) if and only if it is the time until absorption in the Finite Markov process. The pair (α, T) is called the parameters or the representation of $F(.)$.

For a phase type distribution, the distribution function $F(.)$ has a jump of height α_{m+1} at $x = 0$ and the density function is given by $f(x) = \alpha \exp(Tx) T^0$ and the $i^{th}, i \geq 0$ moments about zero is given by $\mu_i' = (-1)^i \, i!\left(\alpha T^{-1} e\right)$.

The Erlang distribution of order m with parameters $\lambda_1, \lambda_2, ..., \lambda_m$ has the representation $\alpha = (1, 0, 0, ...0)$ and $T = \begin{bmatrix} -\lambda_1 & \lambda_1 & & & \\ & -\lambda_2 & \lambda_2 & & \\ & & ... & & \\ & & & -\lambda_{m-1} & \lambda_{m-1} \\ & & & & -\lambda_m \end{bmatrix}$. When $\lambda_i = \lambda, \; i = 1, 2, ..., m$, Erlang distribution reduces to Gamma.

The Coxian distribution of order m is given as the Phase type distribution with parameters $\alpha = (1, 0, ..., 0)$ and $T = \begin{bmatrix} -\lambda_1 & p_1\lambda_1 & & & \\ & -\lambda_2 & p_2\lambda_2 & & \\ & & ... & & \\ & & & -\lambda_{m-1} & p_{m-1}\lambda_{m-1} \\ & & & & -\lambda_m \end{bmatrix}$

where $0 \leq p_i \leq 1$.

2.4 Concepts involved in the optimal replacement Policies

The time at which a system, that is subjected to repair, is to be replaced has been a puzzling question for all those who are working in area of reliability. The replacements are made so as to either maximize availability or minimize long run average cost. Optimal maintenance policies aim to provide optimum system reliability or availability and safety performances at the least possible maintenance cost.

2.4.1 Repair Policies

The repair policies can be broadly classified into two namely, corrective maintenance and preventive maintenance. Under corrective maintenance, repair facility is triggered at the time of system failure. Under preventive maintenance however, the repair is triggered at a specified time, even if the system is functioning, in order to keep the system in a predetermined state or condition. One of the major advantages of preventive maintenance over corrective maintenance is that it optimizes the availability of the system.

2.4.2 Bistate and Multistate Systems

The commonly used reliability systems are assumed to have two state viz. working and failure. A multistate model generalizes these models by admitting intermediate states between these extremes. A system can be assumed to have the states working, partially working, partially failed and failed. Multistate models are the most commonly used models for describing the development of longitudinal failure time data. A multistate model is defined as a discrete state stochastic process with the sojourn times in each state having some distribution. A change in state is called a transition. An excellent review on multistate models can be found in Houggard (1999). Griffith (1980) presents an axiomatic development of the multistate model.

A k *out of* $n:G$ system comprising of n units is in working condition if at least k components are in the working state, where as, a system which comprises of n units is said to be a k *out of* $n:F$ system if at least k components are in the failed or in non working condition. A *Consecutive* k *out of* $n:G$ system is the one which works if at least k consecutive components are in working condition. Similarly we can define *Consecutive* k *out of* $n:F$ system.

2.4.3 Maintenance Policies

Maintenance actions are classified according to the operating condition of the item restored by maintenance. At the initial stages, it is assumed that the failed systems after repair will be ‘as good as new’ and this maintenance policy is known as perfect repair. The replacement by a new component is assumed to be the perfect repair. Renewal theory is enough to model the perfect repair system. Barlow and Hunter (1960) introduced minimal repair model which restores the system to functioning condition, with the failure rate it had when it failed i.e., after the repair process, the system is ‘as bad as old’. However, in practice, most of the systems lie in between these extreme situations. Motivated by this Brown and Proschan (1983) introduced imperfect repair model in which the failed system will undergo a perfect repair with probability p and a minimal repair with probability (1-p). Clearly when p=1 it coincides with perfect repair and when p=0 it coincides with minimal repair. Block et. al. (1985) incorporated an inhomogenity to the above model by assuming the probability p to be age dependent. Kijima et. al. (1988) developed an imperfect repair model by using the idea of virtual age of the repairable system.

In most of the real life situations it has been observed that the lifetimes become shorter and the repair times become larger after each repair. Lam (1988a &1988b) introduced geometric process model, which could satisfactorily model monotone random variables. A sequence of non-negative independent random variables $\{X_n, n = 0,1,2,...\}$ is called a geometric process if for any $a > 0$ the distribution function of X_n is $F(a^n x)$ for $n = 0,1,2,...$ where $F(.)$ denote the distribution function of the random variable X_1. Later Wang & Pham (1996) introduced almost the same model in a different name called quasi renewal process. Various other imperfect models appeared in literature from time to time. An elaborate discussion on various imperfect models and their modelling methods can be read from Pham and Wang (1996), Wang and Pham (1996).

A sequence of non negative independent random variables $\{X_n, n \geq 0\}$ is called a power process (Castro and Sanujan,2004) with an associated sequence $\{\gamma_n, n = 0,1,2,...\}$ where $\gamma_n \in (0,\infty)$ and $\gamma_o = 1$, if the survival function of X_n is $\left[\overline{F}(x)\right]^{\gamma_n}, n = 0,1,...$ where $\overline{F}(.)$ is the survival function of X_0. Power process is widely accepted as discrete time analogue for the geometric processes.

2.4.2 Replacement Policies

The dominant replacement policies include N, T, (T, N) policies among others . In N policy, a system is replaced once the number of repairs exceeds some pre-assigned number N. In T policy, a replacement is carried out when the cumulative working period of the system exceeds some pre-assigned value T. In the case of (T, N) policy, replacement is started when the cumulative working period exceeds T or the number of repair exceeds N whichever occurs first.

2.5 Regression Models

There may exist heterogeneity due to repairs in the lifetime data or the repair time data. Covariates can be effectively exploited to model the heterogeneity present in the data. Such an approach has been used widely in the survival studies but rarely in the reliability scenario. In reliability context, the voltage level during the break down time of equipment is an example of covariate. Regression models are employed to understand and exploit the relationship between the main variables, sometime lifetime or the repair time of the components, and the covariates. The effect of covariates on lifetime variable may change over time and such covariates are referred to as time-dependent or time-varying covariates. The proportional hazards model introduced by Cox (1972) is the commonly employed regression model in survival analysis

2.5.1 Multiplicative Regression Models

The multiplicative regression model assumes that the covariates have a multiplicative effect on the hazard rate function of individuals. Here the infinitesimal generator for transition from state i to state j given the covariate to be as X is given by

$$q_{ij}(X) = \theta^X q_{ij}$$

where θ is the ageing factor, $q_{ij}(X)$ denote the infinitesimal generator of transitions from state i to state j under the covariate X and q_{ij} is the infinitesimal generator in the absence of any covariates. When $\theta > 1$, we will have stochastically decreasing random variable and when $\theta < 1$, we have stochastically increasing random variable. The much celebrated Proportional Hazards model(PH model) is an example for the multiplicative regression model.

2.6 Runs and its Various Counting Methods

A run is defined as an uninterrupted sequence of an outcome. Let the outcomes of a binary experiment be SSSFSSFFFFF. Then we have a success run of length three initially, then a failure run of length one, again a success run of length two and finally a failure run of length four. Runs find application in almost every field of human activity. Continuous Sampling Plans in the Statistical Quality Assurance is an example of runs. We will accept the project if c good items be produced consecutively, that is, if a run of c good items is produced. Runs find applications in the field of Statistical Inference where various test procedure for randomness is considered.

2.6.1 Various Counting Procedures

Different ways of counting the runs are available in related literature. Feller (1968) discussed the most classical way of counting the runs called the non-overlapping success runs of length k. Overlapping success runs of length k were introduced by Ling (1988). Later Aki and Hirano (2000) generalized the above two models by introducing l overlapping success runs.

The counting of the runs can be done in different ways. In the non-overlapping success runs (Feller, 1968) when k consecutive successes are observed it is said that a success run of length k has occurred and we start counting from scratch at the end of the completed run. But in the case of overlapping success runs (Ling, 1988 and Mood, 1940) we will start counting for the run starting from the second element of the preceding run. An uninterrupted success run of length $l > k$ preceded and followed by a failure accounts for $l - k + 1$ overlapping success runs of length k. In this case the first k accounts for the first success run of length , second runs starts from the second element in the run to the $(k+1)^{st}$ element in the run, third run is from third element to $(k+2)^{th}$ element etc. Aki and Hirano (2000) brought these counting procedures into a single platform by introducing μ-overlapping runs. In the μ-overlapping runs of length of k the last μ elements of the preceding run will be used in counting the second run. By assuming $\mu = 0$, it reduces to the non-overlapping case studied in Feller(1968) and when $\mu = k - 1$ it reduces to the overlapping counterpart as mentioned in Ling (1988).

Success runs of length exactly equal to k and greater than or equal to k were studied by Mood (1940). In runs of length greater than or equal to k, every run of length greater than or equal to k will account for a single unit where as in the case of runs of length exactly equal to k will only be accounted for. Consider the following sequence of 18 binary trials of Success (S) and Failure (F). FFSSSFSSSSSSFFFSFS. There are three non-overlapping success runs of length three, two non-overlapping failure runs of length two, whereas there are five overlapping success runs of length three of which three are contributed from the second series of successes and three overlapping failure runs of length two. Also there are three one-overlapping success runs of length three, only one success of run length exactly equal to three and two runs of length greater than or equal to three.

2.6.2 Waiting Time Distributions

Like the geometric distribution in the independent scenario we can have an associated waiting time distribution. But unlike there, here we can have two special class of waiting times, sooner and the later waiting times, in the case of runs. We will explain these in the case of binary trials which can be easily generalized to multistate trials. Let $X_1, X_2, ...$ be a sequence of binary trials and k_1 , k_2 be any two positive integers.

The waiting time for the occurrence of the k_1 consecutive successes or k_2 consecutive failure runs is known as the sooner waiting time. But the later waiting is defined as the time for the occurrence of both, k_1 consecutive successes and k_2 consecutive failure runs.

2.7 Literature Survey

Optimal replacement policies for various models under various assumptions have been studied and are being studied extensively. In literature Lam (1988) analyzed N policy and T policy using geometric process model and he proved that N policy is better than T policy under certain conditions. Zhang (1994) used bivariate (T ,N) policy and showed that bivariate policy is better than univariate N and T policies. Other works on the geometric process model in maintenance analysis include Stadje and Zuckerman (1990), Lam (1991), Lam (1995), Lam and Zhang (1996) and Zhang (1999) and Zhang et. al. (2001).

Zhang (2004) derived the optimal replacement policy for a system with two failure states and one working state. Zhang et. al. (2002) obtained the optimal replacement policy for a deteriorating multistate system with k failure state and one working state. Lam and Tse (2003) discussed optimal replacement policy for a multistate system with one failure state and k working state. Lam (2005) discussed optimal replacement policy for a monotone multistate system without any restriction on number of states. Zhang et. al. (2007) obtained the bivariate optimal replacement policy for a multistate repairable system with k failure states and one working state. All these papers use a geometric process based approach for modelling multistate systems.

Interruption and vacation are two concepts that are used in the queuing literature widely. By interruption we mean the shocks that cause the non-functioning of the service facility while the vacation points to the situation where the service facility went on vacation. For elaborate discussion on the queuing models dealing with interruptions one may refer to Pramod(2009) or Krishnamoorthy et. al. (2009). Protecting the repair facility so that the interruptions will not affect the repair process is dealt in this thesis.

A start-up demonstration is a mechanism by which the vendor demonstrates to the customer the reliability of equipment with regard to its working. While purchasing complex systems like power generators, chain saws, water pumps etc. reliability of the system is often assessed by its reliability with respect to starting. The results of the general theory of runs have been exploited widely in the context of start-up demonstration tests. Most of the researchers in start-up demonstration test describe it with the help of later and sooner waiting time problem of runs. In chapter five we propose two start-up demonstration tests, first preserving the interest of the buyer and the second preserving the interest of the seller.

The credit of introducing the concept of start-up demonstration test goes to Han and Gage (1983). They introduced a model in which the product is accepted when k consecutive successes occur. They considered the case in which each trial is independently and identically distributed. Viveros and Balakrishnan (1995) considered the same model of start-up demonstration tests and derived the expressions for the mean and variance of the number of trials required to complete the test. Inference procedures for the start-up demonstration tests were also discussed. These authors also discussed the problem when the consecutive start-ups are dependent in Markovian fashion. Balakrishnan et. al (1995) developed the probability generating function (pgf) of the random variable denoting the length of the test in Markovian dependent trials. The idea of repair or corrective action in the start-up scenario was introduced by Balakrishnan et. al.(1995). A sequential correction plan in which equipment is intervened with repair after each failure was studied by Balakrishnan et. al.(1997). Extensions of these start-up demonstration tests to m^{th} order Markov dependent series were studied by Aki et. al. (1996) and Balakrishnan et. al. (1997)

Balakrishnan and Chan (2000) introduced a start-up demonstration test in which the product is accepted if k consecutive successes occur prior to d random failures and the product is rejected if this event does not happen. Mean time for the termination of the experiment and various conditional expectations were also discussed. Explicit expressions for various variables of interest under Markovian dependence were studied by Martin (2004). Balakrishnan and Chan (1999) introduced two stage start-up demonstration tests in which the product is accepted if k_1 consecutive successes occur before l_1 failures and if the above event does not take place but k_2 consecutive successes occur before the next l_2 failures then also the product is accepted and we reject the product if both the events do not happen. Koutras and Balakrishnan (1999) discussed another start-up demonstration test based on the scan statistics.

Smith and Griffith (2003) proposed a procedure having similarities with the Markov chain embedding technique introduced by Fu and Koutras (1994) in run scenario and later refined by Koutras and Alexandrou (1995). Since its introduction the technique of Markov chain embedding has been exploited widely in the context of runs. Fu (1996) introduced forward and backward principle to cover the case of arbitrary patterns. Koutras (1997) obtained the distribution of waiting times associated with runs using the said method. Han and Aki (1999) further enhanced the Markov chain embedding technique by introducing multinomial and returnable type embeddings and they obtained the distribution of the runs in a sequence of multi- state trials. For an elaborate discussion on Markov chain embedding technique interested readers may refer to Koutras (2003). Many authors like Antzoulakos (2001) and Balasubramanian et. al. (1993) discussed the problem of sooner and later waiting time introduced by Ebneshahrashoob and Sobel(1990), which can be compared with start-up demonstration tests. A general procedure for finding various probabilities of interest in the start-up context was discussed by Aston and Martin (2005) formulating start-up demonstration test as a special case of the competing patterns. Martin (2008) obtained a recursive formula for various variables of interest for different start-up demonstration tests. For a comprehensive discussion on various start-up demonstration tests and their comparisons one may refer to Smith and Griffith (2008).

The idea of runs comes into picture in almost every applied area of Statistics involving experimental trials with two or more possible outcomes in each trial. It finds its application in theoretical scenario too by its involvement in the non-parametric statistical testing. For an elaborate discussion on the application of the runs, the interested readers may refer to Shwanger (1983) and Koutras (2003) and Balakrishnan and Koutras (2002) and references there in. In many applications, along with the number of runs, it is of interest to find the number of occurrence of each each outcome. In many practical situations, there involve more than two types of outcomes. For example, consider an application in quality control. Classical quality assurance problems consider sampling plans with two types of products, good and bad or in other words conforming and non-conforming. This however may not be the case always. There are cases in which we have more than two types of products viz., best, good and bad. In this case if we develop some sampling procedure the runs come into fray. Naturally in such case the number of occurrence of each item would also evolve interest. When it comes to the reliability scenario, one of the natural generalizations of the classical *consecutive* k *out of* n :G system will be *consecutive* $\left(k_1, k_2, ..., k_m\right)$ *out of* n: G system, which will work if consecutive k_i components are in state i for any $i \in \{1, 2, ...m\}$, possible states of the components. Then it is of interest to find the various reliability parameters and the number of components functioning.

Many papers addressing the distribution of the runs in trials with two possible outcomes appeared in the literature. Most of the earlier works studied the distribution concerning the runs in the independent trial case and later many papers appeared dealing with various dependencies. For more references one may go through Koutras (2003). Aki (1985) introduced the binomial distribution of order k based on the belief that success (failure) yields success (failure) with varying probability.

Doi and Yamamoto (1998) obtained the joint distribution of c kinds of success runs in $(c+1)$ state trial case. Shinde and Kotwal (2006) derived the joint distribution of runs in Multi state trials using the conditional pgfs. Koutras (1997) studied the waiting time distributions in trinary trials. The joint distribution of runs, success, failures and patterns and expressions for the distributional properties of waiting times of bistate trials was derived by Chadjiconstantinidis et. al.(2000). Distribution of number of failures and number of trials before the first occurrence of a success run of length k was studied by Aki and Hirano (1994). Even though there are many papers available in literature dealing with Binomial distribution of order k, its number shrinks sharply as we move on to the multinomial distribution. Wang and Yang (1995) defines a Markov multinomial distribution under some conditions regarding the transition probabilities.

Fu and Koutras (1994) proposed a unified and a simple method for finding the distribution of success runs of identical and non-identical Bernoulli trials using Markov Chain embedding technique. Till then most of the papers dealing with runs used combinatorial identities to obtain the distributions concerning runs. Koutras and Alexandrou (1995) refined the method of Markov chain embedding. Fu (1996) introduced forward and backward principle to cover the case of arbitrary patterns. Koutras (1997) obtained the distribution of waiting times associated with runs using the said method. Han and Aki (1999) further enhanced the Markov chain embedding technique by introducing multinomial and returnable type embeddings and they obtained the distribution of the runs in a sequence of multi-state trials.

In this chapter we considered the basic concepts that are involved with modelling problems to be dealt with in this thesis. A quick review of the available literature is also done in the chapter.

Chapter 3
Maintenance Policy for a Multi-State System[1]

3.1 Introduction

Consider a 3-out-of 4 G: System, which will work if three of the four units in the system work. Hence we can classify the working stages into two substates namely, all the four units are working and only three units are working. The system will be in failure if only two, one or no units are working condition. Also assume that the repair facility is started only if the system fails completely. Also once the repair is triggered the system need not be in the best working condition at the end of the repair. Hence in cases the system may be in four units working state or three or two units working state. In the conventional monotone lifetime models we assume that the lifetimes after each repair is less than the preceding ones. But this may not be the case over here. For example consider the case if the last repair system was brought back to three unit working state while the current repair brought it to the four unit working system. Naturally the lifetime after the current repair will be expected to be stochastically larger than the preceding one. Moreover the lifetime after the current repair cannot be expected to be at par with the new system. The repairs should have some impact on the lifetime and the lifetime depends on the state in which it is working. This cannot be achieved with the existing monotone lifetime models.

In this chapter we model similar situations using Phase-Type distribution. The Cox regression models are exploited to model the impact due to repairs.

As mentioned in section 2.7, one major characteristic of the studies dealing with multistate system is that the consecutive states are independent. But this is not the case always, for example, a car which had repair due to engine failure has high

[1] A part of this chapter is communicated in Nair and Thomas(2011)

probability of another engine failure. Also in all these models assume a monotone structure for the life times after repair. This cannot be justified always. For example consider a two component parallel system with the working state given by both components working and one component working. It is natural that the lifetime with two component working system is stochastically greater than the lifetime with one component working system even if the system had undergone repair. From the above considered system it is trivial that there are cases in which the life times are not monotone.

Hence in this chapter we will consider a multistate system in which consecutive states are Markov dependent. Also we will assume that the sojourn time in each state is a state dependent Phase type distribution. The remaining part of the chapter is organized as follows. In section 3.2 we formally introduce the model. Necessary notations used in this chapter are introduced in section 3.3. Expression for the long run reward and the optimal replacement policy algorithms are discussed in section 3.4. An algorithm which enables calculation of the optimal policy is given in section 3.5. A numerical illustration is performed in section 3.6.

3.2 The Model

Consider a multistate system with k'working and l' repair states respectively given by $W'=\{1,2,....,k'\}$ and $F'=\{k'+1,k'+2,...,k'+l'\}$and $\Omega'=W'\bigcup F'$.In this chapter, we will assume that the states are Markov dependent. Also we will incorporate the possibility of transition from a working state to another working state. Let us assume that u_n denote the time at which nth transitions takes place. Let $Z(t)$ denote the system state at time t. Then we have

$$P(Z(u_{n+1})=j\,|\,Z(u_n)=i)=p_{ij},\ i\in\Omega',j\in\Omega'-\{i\}.$$

Let $P=(p_{ij})$ denote the matrix of transition probabilities from state i to state j. Now we partition P as $P=\begin{bmatrix}P^{(1,1)} & P^{(1,0)}\\ P^{(0,1)} & P^{(0,0)}\end{bmatrix}$ where $P^{(1,1)}\left(P^{(0,0)}\right)$ denote the submatrix of transition probability between working (failure) states and the probability submatrix of transition from a working (failure) state to a failure (working) state is shown by $P^{(1,0)}\left(P^{(0,1)}\right)$.

We will also assume that the sojourn time in each state is phase type distributed with parameters depending on the state in which it occupy. Since the Phase type distribution is dense in the class of distribution functions of the random variables having non-negative range (Neuts and Bhattacharjee, 1981), our assumption is not restrictive. By the definition of the phase type distribution there exists phases associated with each state of which one phase is recurrent and all other phases are transient. Let k_i and l_j respectively denote the number of transient phases associated with i^{th} working state and j^{th} repair state. Let us denote $k=\sum_{i=1}^{k'}k_i$ and $l=\sum_{j=1}^{l'}l_j$. Also let the sojourn times associated with i^{th} working and j^{th} failure state, $i=1,2,..,k'$ and $j=1,2,...,l'$ given by $PH\left(\alpha_i,T_i\right)$ and $PH\left(\beta_j,S_j\right)$ respectively.

Let $\hat{\alpha}=diag\left(\alpha_1,\alpha_2,...,\alpha_{k'}\right)$ and $\hat{\beta}=diag\left(\beta_1,\beta_2,...,\beta_{l'}\right)$ are respectively $k\times k$ and $l\times l$ matrix with non-negative terms in the i^{th} row denote the probability distribution of the process coming to that state. Note that both $\hat{\alpha}$ and $\hat{\beta}$ are not diagonal matrices, but diagonal in the partitioned form. Similarly we denote $E^{(1)}=diag\left(e_{(1)},e_{(2)},...,e_{(k')}\right)$ and $E^{(2)}=diag\left(e_{(k'+1)},e_{(k'+2)},...,e_{(k'+l')}\right)$ with $e_{(i)}$ denoting the column vector of dimension equal to the number of substates in state i. Let $T_i^{(0)}$ denote the vector of transition intensity from the transient state to the recurrent state in the case of working state i ie., $T_i^{(0)}=T_ie_{(i)}$. Similarly, $S_j^{(0)}$ denote the column vector of transition intensities from the transient to recurrent state corresponding to the repair time in the j^{th} failure state.

Now we embed two separate Markov processes with the working times and the repair times. The totality of all the phases associated with each working state followed by that of failure states. Hence the state space is given by

$$\Omega = \left\{1_1, 1_2, ..., 1_{k_1}, ..., k_1^{'}, k_2^{'}, ..., k_{k'}^{'}, (k'+1)_1, (k'+1)_2, ..., (k'+1)_{l_1}, ..., (k'+l')_1, ..., (k'+l')_{l'}\right\}$$

Let $P^{(1)}$ and $P^{(2)}$ respectively denote the matrices of transition intensities of working time and repair time Markov processes. The repair (working) states are absorbing with respect to the working (repair) times. $P^{(1)} = \begin{bmatrix} T & T_0 \\ 0 & 0 \end{bmatrix}$, where $T = (t_{ij})$ is a $k \times k$ and $T_0 = (t_{ij}^0)$ a $k \times l$ matrix respectively denoting the infinitesimal generators of transition from a working state to a working state and working state to a failure state. Clearly we have

$$t_{ij} = \begin{cases} t_{ij}^{(k)}, & if\ i, j\ are\ phases\ in\ the\ state\ k \\ p_{lk}\alpha_{kj}t_i^0, & if\ i\ and\ j\ are\ respectively\ phses\ of\ state\ l\ and\ k \end{cases}$$ and

$$t_{ij}^0 = p_{ik}\beta_{kj}t_i^0, if\ j\ is\ a\ subset\ of\ the\ repair\ state\ k$$

where t_i^0 is the infinitesimal generator to the absorbing state in the phase type distribution corresponding to the i^{th} state that is, $t_i^0 = -\sum t_{ij}$, where summation is over all transient phases in the state i. If $\hat{T} = diag(T_1, T_2, ..., T_{k'})$ and $\hat{S} = diag(S_1, S_2, ..., S_{l'})$. Then under the matrix notations,

$$T = \hat{T} - \hat{T}E^{(1)}P^{(1,1)}\hat{\alpha} \text{ and } T_0 = -\hat{T}E^{(1)}P^{(1,0)}\hat{\beta},$$

Similarly if $S = (s_{ij})_{l \times l}$ denote the matrix of transition intensities of the transitions between the repair states i to j and $S_0 = (s_{ij}^0)_{l \times k}$ denote the transition intensity from repair state i to the working state j then $P^{(2)} = \begin{bmatrix} 0 & 0 \\ S_0 & S \end{bmatrix}$. Clearly we have

$$s_{ij} = \begin{cases} s_{ij}^{(k)}, & \text{if } i \text{ and } j \text{ are phases of the failure state } k \\ p_{lk}\beta_{kj}s_i^0, & \text{if } i \text{ and } j \text{ are phases of the repair states } l \text{ and } k \text{ respectively} \end{cases}$$

and

$$s_{ij}^0 = p_{lk}\alpha_{kj}s_i^0, \ \text{if } i \text{ and } j \text{ are the phases of the repair state } l \text{ and working state } k \text{ respectively}$$

where s_i^0 denote the transition intensity from state i to the corresponding absorbing state. Hence proceeding as in the case of working state we have

$$S = \hat{S} - \hat{S}E^{(2)}P^{(0,0)}\hat{\beta} \ \text{ and } \ S_0 = -\hat{S}E^{(2)}P^{(0,1)}\hat{\alpha}$$

We incorporate the effect of repair on the state sojourn times by using the regression models. In our study we will consider the number of repairs the system has undergone as the concomitant variable. Let $\lambda_{ij}^{(n)}$ denote the transition intensity for transitions from state i to state j after n repairs with $\lambda_{ij}^{(0)} = \lambda_{ij}$ denoting the initial transition intensity function. Then by the multiplicative assumption $\lambda_{ij}^{(n)} = \theta_i^{(n-1)}\lambda_{ij}$, where θ_i denote the ageing factor of the i^{th} state. Hence here we will assume that $t_{ij}^{(n)} = \delta_i^{(n-1)}t_{ij}, \ i, j \in W$ and

$t_{ij}^{(n,0)} = \dfrac{t_{ij}^{(0)}}{\sum_{l \in F} t_{il}^{(0)}} \delta_i^{(n-1)} \sum_{k \in W} t_{ik}, \ i \in W, j \in F$. Similarly for the repair times we have $s_{ij}^{(n)} = \delta_i^{(n-1)}s_{ij}, \ i, j \in W$. It can be seen that if $\theta_i > 1$ for every $i \in W$ then the sequence forms a stochastically increasing sequence. Also, $\theta_i < 1$ for every $i \in W$ forms a stochastically decreasing sequence.

3.3 Notations

Throughout this chapter we will use the following notations

e_j - Unit column vector of length j.

$e_{(j,i)}$ - Column vector of length j with all elements zeros except i^{th} element where we have unity.

X_n - r.v denoting the life time after $(n-1)^{th}$ repair.

Y_n - r.v denoting the repair time for n^{th} failure.

F_n, f_n - D.F and p.d.f of X_n.

G_n, g_n - D.F and p.d.f of Y_n.

$\underline{F}_n, \underline{f}_n$ - Vectorized D.F and p.d.f of X_n.

$\underline{G}_n, \underline{g}_n$ - Vectorized D.F and p.d.f of Y_n.

$\delta = (\delta_i)$ - Matrix of ageing factor of the working times

$\Gamma = (\gamma_i)$ - Matrix of ageing factor for the repair times.

$x(t)$ - Diagonal matrix of size k with the i^{th} diagonal term, $x^{(i)}(t)$, denote the probability that the system is in working state i at time t.

$x_0(t)$ - Diagonal matrix of size l with j^{th} diagonal term, $x_0^{(j)}(t)$, denote the probability that the system is in failure state j at time t.

$X(n)$ - Failure state at n^{th} failure

$Y(n)$ - Working state after the n^{th} repair, $i \in W$.

$$A(S,t) = [\exp(St) - I] S^{-1} S_0 = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,k} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,k} \\ \vdots & \vdots & \vdots & \vdots \\ a_{l,1} & a_{l,2} & \cdots & a_{l,k} \end{bmatrix}$$

$$B(T) = T^{-1} T_0 = \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,l} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,l} \\ \vdots & \vdots & \vdots & \vdots \\ b_{k,1} & b_{k,2} & \cdots & b_{k,l} \end{bmatrix}$$

$$D(S,t)=\frac{d}{dt}A(S,t)=\exp(St)S_0$$

3.4 Long run average cost

Under the above mentioned notations, by Kolmogrov forward equations, we have $x'(t)=x(t)T$ with $x(0)=\mu$.

$$\frac{d}{dt}\log x(t)=T\Rightarrow \log x(t)=Tt+c$$

$$\Rightarrow x(t)=\mu\exp(Tt)$$

Proceeding exactly in the same way

$$x_0^{(j)}(t+h)=x_0^{(j)}(t)P\left(Z(t+h)=j\middle|Z(t)=j\right)+\sum_{i\in W}x^{(i)}(t)P\left(Z(t+h)=j\middle|Z(t)=i\right)$$

But as far as the working states are concerned, failure states are all absorbing hence the probability on the first term will become zero. Hence dividing the above equation by h and taking the limit as $h\to 0$, it reduces to

$x_0^{(j)}{}'(t)=\sum_{i\in W}x^{(i)}(t)q_{ij}$, q_{ij} denote the infinitesimal generator from state i to state j.

Hence we have $x_0{}'(t)=x(t)T_0$

$$\Rightarrow x_0{}'(t)=\mu\exp(Tt)T_0.$$

On integrating we have $x_0(t)=\mu\exp(Tt)T^{-1}T_0+C$ with the initial condition $x_0(0)=\mathbf{0}$.

Hence $\mu T^{-1}T_0=C\Rightarrow x_0(t)=\mu\left[\exp(Tt)-I\right]T^{-1}T_0$.

Now $\underline{F}_1(t)=P(X_1<t)=x_0(t)=\mu\left[\exp(Tt)-I\right]T^{-1}T_0$.

Hence $F_1(t) = x_0(t)e_{(l)} = \mu\left[\exp(Tt) - I\right]T^{-1}T_0 e_l$. But the definition of infinitesimal generators we have $Te_{(k)} + T_0 e_{(l)} = \mathbf{0} \Rightarrow T^{-1}T_0 e_{(l)} = -e_k$.

Hence $F_1(t) = -\mu\left[\exp(Tt) - I\right]e_k = 1 - \mu\exp(Tt)e_k$.

Also $\underline{f}_1(t) = \dfrac{d}{dt}\underline{F}_1(t) = \mu\exp(Tt)TT^{-1}T_0 = \mu\exp(Tt)T_0$.

$$f_1(t) = \frac{d}{dt}F_1(t) = \mu\exp(Tt)T_0 e_l.$$

$$E(X_1) = -\mu T^{-1}e_k.$$

We will assume that the repair times depend on the last failure state. Hence the distribution of failure states becomes the initial distribution of the repair times.

Hence $\bar{G}_{(1)}(t) = P(\bar{Y} < t) = \sum_{i\in F}\int_0^\infty P\left(\bar{Y} < t \middle| X_1 = x, X(1) = i\right)P(X_1 = x, X(1) = i)\,dx$

$$= \sum_{i\in F}\int_0^\infty e'_{(l,i)}\left[\exp(St) - I\right]S^{-1}S_0\mu\exp(Tx)T_0 e_{(l,i)}dx$$

$$= \sum_{i\in F} e'_{(l,i)}\left[\exp(St) - I\right]S^{-1}S_0(-\mu)T^{-1}T_0 e_{(l,i)}$$

$$= -\mu B(T)A(S,t)$$

Also $\bar{g}_1(t) = -\sum_{i\in F} e'_{(l,i)}\exp(St)S_0\mu T^{-1}T_0 e_{(l,i)} = -\mu B(T)D(S,t)$.

$$\underline{G}_1(t) = \mu B(T)A(S,t)$$

$$E(Y_1) = \mu B(T)\int_0^\infty\left[\exp(St) - I\right]S^{-1}S_0 e_{(k)} = \mu B(T)S^{-1}e_l$$

$$P(X_2 < t) = \sum_{j\in W}\int_0^\infty P\left(X_2 < t \middle| Y_1 = y, Y(1) = j\right)dy$$

$$= \sum_{i\in W}\int_0^\infty e'_{(l,i)}\left[\exp(\delta Tt) - I\right](\delta T)^{-1}(\delta T_0)\mu B(T)D(S,y)\,dy$$

Then $\int_0^\infty D(S,t)\,dt = \int_0^\infty \exp(St)\,S_0\,dt$

$$= S^{-1}S_0 = B(S).$$

Hence $P(X_2 < t) = \sum_{i\in F} e'_{(l,i)} \left[\exp(\delta T t) - I\right](\delta T)^{-1}(\delta T_0)\,\mu B(T)B(S)e_{(l,i)}$

$$= -\mu B(T)B(S)A(\delta T,t).$$

$\underline{f}_2(t) = -\mu B(T)B(S)D(\delta T,t)$.

$E(X_2) = -\mu B(T)B(S)(\delta T)^{-1}e_k$.

Similarly we have

$$\underline{G}_2(T) = -\mu B(T)B(S)B(\delta T)A(\gamma S,t).$$

$$\underline{g}_2(t) = -\mu B(T)B(S)B(\delta T)D(\gamma S,t)$$

$$E(Y_2) = \mu B(T)B(S)B(\delta T)(\gamma S)^{-1}e_l.$$

Proceeding exactly in the same way we obtain the following

$$\underline{F}_n(t) = -\mu B(T)B(S)B(\delta T)B(\gamma S)\ldots B\left(\delta^{(n-2)}T\right)B\left(\gamma^{(n-2)}S\right)A\left(\delta^{(n-1)}T,t\right)$$

$$\underline{f}_n(t) = -\mu B(T)B(S)B(\delta T)B(\gamma S)\ldots B\left(\delta^{(n-2)}T\right)B\left(\gamma^{(n-2)}S\right)D\left(\delta^{(n-1)}T,t\right)$$

$$E(X_n) = -\mu B(T)B(S)B(\delta T)B(\gamma S)\ldots B\left(\delta^{(n-2)}T\right)B\left(\gamma^{(n-2)}S\right)\left(\delta^{(n-1)}T\right)^{-1}e_k \qquad (3.1)$$

Also $\underline{G}_n(T) = -\mu B(T)B(S)B(\delta T)B(\gamma T)\ldots B\left(\delta^{(n-1)}T\right)A\left(\gamma^{(n-1)}S,t\right)$.

$$\underline{g}_n(T) = -\mu B(T)B(S)B(\delta T)B(\gamma T)\ldots B\left(\delta^{(n-1)}T\right)D\left(\gamma^{(n-1)}S,t\right)$$

$$E(Y_n) = \mu B(T)B(S)B(\delta T)B(\gamma T)\ldots B\left(\delta^{(n-1)}T\right)\left(\gamma^{(n-1)}S\right)^{-1}e_l. \qquad (3.2)$$

Let $D_n = \left(T^{-1}T_0S^{-1}S_0\right)^n$ and $E = \left(T^{-1}T_0\right)S^{-1}$.

Then $E(X_n) = -\mu\left(T^{-1}T_0S^{-1}S_0\right)^{(n-1)} T^{-1}\delta^{-(n-1)}e_k = -\mu D_{n-1}T^{-1}\delta^{-(n-1)}e_k$ (3.3)

and

$$E(Y_n) = \mu\left(T^{-1}T_0S^{-1}S_0\right)^{(n-1)}\left(T^{-1}T_0\right)S^{-1}\gamma^{-(n-1)}e_l = \mu D_{n-1}E\gamma^{-(n-1)}e_l \tag{3.4}$$

Now we calculate the long run average reward. Our objective is to find the value of N which optimizes the long run average reward. Let $L(N)$ denote the long run average reward at N^{th} failure. It is clear that the replacement time sequences $\{J_n, n = 1,2,3,....\}$ forms a renewal process. The time between consecutive replacements forms the replacement cycle. The long run expected cost is given by $\frac{E(r)}{E(T)}$ where $E(r)$ is the expected reward in a replacement cycle and $E(T)$ is the expected length of the replacement cycle.

$$L(N) = \frac{\sum_{i=1}^{N} E(R_i) - \sum_{i=1}^{N-1} E(C_i) - C_r}{\sum_{i=1}^{N} E(X_n) + \sum_{i=1}^{N-1} E(Y_n) + \tau}$$

$$= \frac{-A_1\sum_{n=1}^{N}\mu D_{n-1}T^{-1}a^{-(n-1)}e_k - A_0\sum_{n=1}^{N-1}\mu D_{n-1}Eb^{-(n-1)}e_l - c_r}{-\sum_{n=1}^{N}\mu D_{n-1}T^{-1}a^{-(n-1)}e_k + \sum_{n=1}^{N-1}\mu D_{n-1}Eb^{-(n-1)}e_l}. \tag{3.5}$$

By simplification it can be seen that

$$L(N+1) - L(N) = \frac{(A_0 + A_1)\left\{\sum_{n=1}^{N-1}\mu D_{n-1}Eb^{-(n-1)}e_l\mu D_N T^{-1}a^{-N}e_k - \sum_{n=1}^{N}\mu D_{n-1}T^{-1}a^{-(n-1)}e_k\mu D_{N-1}Eb^{-(N-1)}e_l\right\}}{\left\{-\sum_{n=1}^{N}\mu D_{n-1}T^{-1}a^{-(n-1)}e_k - \sum_{n=1}^{N-1}\mu D_{n-1}Eb^{-(n-1)}e_l\right\}}$$

$$\frac{-c_r\left\{\mu D_N T^{-1}a^{-N}e_k - \mu D_{N-1}Eb^{-(N-1)}e_l\right\}}{\left\{-\sum_{n=1}^{N+1}\mu D_{n-1}T^{-1}a^{-(n-1)}e_k - \sum_{n=1}^{N}\mu D_{n-1}Eb^{-(n-1)}e_l\right\}}$$

For a convex function in N the cost function $L(N)$ increases with N initially and then start decreasing. Hence the optimum number of repairs is the value of N for which the $L(N+1)-L(N)$ becomes negative for the first time ie, $\underset{N}{Inf}\left[L(N+1)-L(N)\le 0\right]$. This is not easy to find the roots making use of the equation given above. Hence we develop an algorithm which serves a great deal in finding the optimal solution to the problem.

3.5 Numerical Implementation

For the sake of developing the algorithm, on proceeding similar lines with Castro and Perez-Ocon,2006, we assume that the system is monotone and we define the following functions.

$$B(0)=\frac{-\mu T^{-1}e_k\mu Ee_l}{\mu Ee_l-\mu D_1T^{-1}\delta^{-1}e_k} \tag{3.6}$$

$$B(N)=\frac{\mu D_N T^{-1}a^{-N}e_k\sum_{n=1}^{N-1}\mu D_{n-1}Eb^{-(n-1)}e_l-\mu D_{N-1}Eb^{-(N-1)}e_l\sum_{n=1}^{N}\mu D_{n-1}T^{-1}a^{-(n-1)}e_k}{\mu D_{N-1}Eb^{-(N-1)}e_l-\mu D_N T^{-1}a^{-N}e_k} \tag{3.7}$$

It can be easily seen that the term in the denominator is the sum of the mean of two positive random variables and numerator is the product of two positive terms. Hence $B(0)$ is positive.

$$B(N+1)-B(N)=\left[\mu D_{N+1}T^{-1}a^{-(N+1)}e_k\mu D_{N-1}Eb^{-(N-1)}e_l-\mu D_N T^{-1}a^{-N}e_k\mu D_N Eb^{-N}e_l\right]$$

$$\frac{\left[\sum_{n=1}^{N}\mu D_{n-1}Eb^{-(n-1)}e_l-\sum_{n=1}^{N+1}\mu D_{n-1}T^{-1}a^{-(n-1)}e_k\right]}{\left[\mu D_{N-1}Eb^{-(N-1)}e_l-\mu D_N T^{-1}a^{-N}e_k\right]\left[\mu D_N Eb^{-N}e_l-\mu D_{N+1}T^{-1}a^{-(N+1)}e_k\right]}$$

.

It can be seen that the denominator is the product of two positive terms. Now the second term in the numerator is again greater than zero, in order to prove that $B(N)$ is increasing it is remaining to prove that first term in numerator is greater than zero. $\mu D_N T^{-1} a^{-N} e_k$ correspond to the negative of the mean working time of the system or component after n^{th} repair. Since we are assuming that the working (repair) times are stochastically decreasing (increasing), expected working (repair) times forms a decreasing (increasing) sequence. That is,

$-\mu D_{N+1} T^{-1} a^{-(N+1)} e_k < -\mu D_N T^{-1} a^{-N} e_k$ and $\mu D_{N-1} E b^{-(N-1)} e_l < \mu D_N E b^{-N} e_l$.

$\Rightarrow \mu D_{N+1} T^{-1} a^{-(N+1)} e_k \mu D_{N-1} E b^{-(N-1)} e_l > \mu D_N T^{-1} a^{-N} e_k \mu D_N E b^{-N} e_l$

Hence we have $B(N+1) - B(N) > 0 \ \ \forall N$. So $B(N)$ forms an increasing function of N .Now we will prove that $B(N)$ is bounded above. We have

$$B(N) = \frac{\mu D_N T^{-1} a^{-N} e_k \sum_{n=1}^{N-1} \mu D_{n-1} E b^{-(n-1)} e_l - \mu D_{N-1} E b^{-(N-1)} e_l \sum_{n=1}^{N} \mu D_{n-1} T^{-1} a^{-(n-1)} e_k}{\mu D_{N-1} E b^{-(N-1)} e_l - \mu D_N T^{-1} a^{-N} e_k}$$

Since the denominator is positive and $B(N)$ is positive for all N, second term in the numerator is greater than the first term, else $B(N)$ would have been negative. Hence we have

$$B(N) \leq \frac{-\mu D_{N-1} E b^{-(N-1)} e_l \sum_{n=1}^{N} \mu D_{n-1} T^{-1} a^{-(n-1)} e_k}{\mu D_{N-1} E b^{-(N-1)} e_l - \mu D_N T^{-1} a^{-N} e_k}$$

$$B(N) \leq \frac{-\mu D_{N-1} E b^{-(N-1)} e_l \sum_{n=1}^{N} \mu D_{n-1} T^{-1} a^{-(n-1)} e_k}{\mu D_{N-1} E b^{-(N-1)} e_l}$$

$$\leq -\sum_{n=1}^{N} \mu D_{n-1} T^{-1} a^{-(n-1)} e_k \ . \qquad (3.8)$$

Hence we had proved that $B(N)$ is a monotone increasing sequence which is bounded above, for stochastically increasing working times and stochastically decreasing lifetime. Hence it is bounded deteriorating systems.

Our objective is to find that value of N that maximizes the expected reward. It reduces to find minimum N such that

$$L(N+1)-L(N)<0 \Rightarrow \frac{c_r}{A_1+A_0}<B(N).$$

Hence the maximum value for the long run reward $L(N)$ is obtained when $N=N_{opt}$ where $N_{opt}=\min\limits_{N\geq 0}\left\{\frac{c_r}{A_1+A_0}<B(N)\right\}$, $B(N)$ being in the form given in equations (3.6) and (3.7). If the minimum value of N is such way that $\frac{c_r}{A_1+A_0}=B(N)$, then N and $N+1$ maximize the long run reward $L(N)$.

As special cases if $\frac{c_r}{A_1+A_0}<B(0)$, then $N_{opt}=0$ i.e., it is better to replace the system by a new component rather than going for repair. Also if $\frac{c_r}{A_1+A_0}>\lim\limits_{N\to\infty}B(N)=-\mu DT^{-1}[I-a]^{-1}e_{(k)}$, then $N_{opt}=\infty$ that is , it is better to repair systems than going for replacement.

Since deriving an analytic optimal solution that maximizes the long run average reward per unit time is extremely complicated, we will try to find an algorithm for numerical identification of nonnegative minimum value of N that satisfies the condition $\frac{c_r}{A_1+A_0}<B(N)$. In other words, it is the root of the equation $G(N)=\frac{c_r}{A_1+A_0}-B(N)$. In general the solution to the above equation need not be integers, in that case we will take the largest integer less than the zero of $G(N)$. The problem is equivalent to finding the zeros of $R(N)$ given by

$$R(1)=c_r\left(\mu Ee_l-\mu D_1T^{-1}\delta^{-1}e_k\right)+\left(A_0+A_1\right)\mu T^{-1}e_k\mu Ee_l$$

and for $N \geq 2$

$$R(N) = c_r\left(\mu D_{N-1} E b^{-(N-1)} e_l - \mu D_N T^{-1} a^{-N} e_k\right) - \left(A_0 + A_1\right)$$

$$\left(\mu D_N T^{-1} a^{-N} e_k \sum_{n=1}^{N-1} \mu D_{n-1} E b^{-(n-1)} e_l - \mu D_{N-1} E b^{-(N-1)} e_l \sum_{n=1}^{N} \mu D_{n-1} T^{-1} a^{-(n-1)} e_k\right).$$

Now we formulate an algorithm to find the solution to the above problem. Our algorithm can be described as discrete analogue of the Halley's Rational method (Ortega and Rheinboldt,2000) for finding the root of the equation.

Algorithm

Step One. Check whether $\dfrac{c_r}{A_1 + A_0} > -\mu D T^{-1}\left[I - a^{-1}\right]^{-1} e_{(k)}$. If it hold, then $N_{opt} = \infty$ and the search ends.

Step Two. If the inequality $\dfrac{c_r}{A_1 + A_0} < B(0)$ is satisfied then $N_{opt} = 0$ and here also no further exploration is needed.

Step Three. Now fix the following values $N_0 = 0$, $N_1 = 1$ and $N_2 = 2$.

Step Four. Check whether $sign\left[G(N_1)\right] \neq sign\left[G(N_2)\right]$ and if it satisfies $N_{opt} = 1$, the algorithm terminates.

Step Five. Proceeding in line with rational Halley's method for finding the zeros of an polynomial, calculate $d = G(N_2) - G(N_1)$ and $\hat{d} = G(N_2) - 2G(N_1) + G(N_0)$.

Step Six. $\Delta = \dfrac{G(N_1)}{d}\left[1 - \dfrac{G(N_1)\hat{d}}{d^2}\right]^{-1}$.

Step Seven. Now we obtain the new values of $N_i\,'s$ as

$$N_0 = N_1,\ \ N_1 = \max_{N \in N}\{N < N_1 + \Delta\} \text{ and } N_2 = N_1 + 1 .$$

Step Eight. Now go to ***Step Four***.

3.6 Numerical Illustration

Through out this chapter we were assuming that the lifetimes and the repair times are phase type distributed. In the numerical illustration we will assume that our system consist of two working states and three failure states. Special cases of the Phase type distribution mamely, Erlang distribution and the mixture exponential distribution are used for modelling the lifetimes and the repair rimes. Specific reason for going for models is their mathematical tractability. Another reason for selecting Erlang and mixture exponential distribution is that the coefficient of variation of the first is less than one and the latter is greater than zero. Weibull distribution is widely used in the reliability scenario. We employ an approximation of the Weibull distribution to model the sojourn time in the third repair state.

We will assume transition probability between various states to be as follows

$$P = \begin{bmatrix} 0 & 0.2 & 0.3 & 0.3 & 0.2 \\ 0.1 & 0 & 0.3 & 0.3 & 0.3 \\ 0.4 & 0.4 & 0 & 0.1 & 0.1 \\ 0.4 & 0.3 & 0.2 & 0 & 0.1 \\ 0.3 & 0.2 & 0.4 & 0.1 & 0 \end{bmatrix}.$$

The sojourn times in first working state and the first failure state (third state) are distributed according to a Erlangian distribution with four, five substates respectively and the infinitesimal generator partitioned for the recurrent states and the initial distribution for each are respectively given by

$$\begin{bmatrix} -0.75 & 0.75 & 0 \\ 0 & -0.75 & 0.75 \\ 0 & 0 & -0.75 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} -6 & 6 & 0 & 0 \\ 0 & -6 & 6 & 0 \\ 0 & 0 & -6 & 6 \\ 0 & 0 & 0 & -6 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}.$$

The sojourn times in the second working state and the second failure state (fourth state) are distributed according to mixture exponential distribution with three substates and the infinitesimal generators of the recurrent states and the initial distribution for each state are respectively given by $diag(\text{-}1/3,\text{-}1/2,\text{-}1/4)$, $\begin{bmatrix} 0.3 & 0.2 & 0.5 \end{bmatrix}$ and $diag\begin{bmatrix} \text{-}1.2 & \text{-}2 & \text{-}0.5 \end{bmatrix}$, $\begin{bmatrix} 0.75 & 0.15 & 0.1 \end{bmatrix}$.

For modelling the repair time in the third state we will use the phase type approximation of the Weibull distribution discussed in Kao (2002). The infinitesimal generator for the recurrent states and the initial distribution are respectively given by

$$\begin{bmatrix} -5.5997 & 5.5997 & 0 & 0 & 0 & 0 \\ 0 & -5.5997 & 5.5997 & 0 & 0 & 0 \\ 0 & 0 & -5.5997 & 4.1667 & 0 & 0 \\ 0 & 0 & 0 & -5.5997 & 4.5853 & 0 \\ 0 & 0 & 0 & 0 & -5.5997 & 5.5997 \\ 0 & 0 & 0 & 0 & 0 & -5.5997 \end{bmatrix} \quad \text{and}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

For $a = \begin{bmatrix} 1.05 & 1.07 \end{bmatrix}$ and $b = \begin{bmatrix} 0.95 & 0.97 & 0.93 \end{bmatrix}$ with replacement cost $c_r = 180$, uptime reward rate $A_1 = 25$ and the downtime cost rate $A_0 = 5$ the long run average reward per unit time is shown in figure 3.1 for different values of number of repairs.

Figure 3.1 Long run average reward for various values of number of repairs.

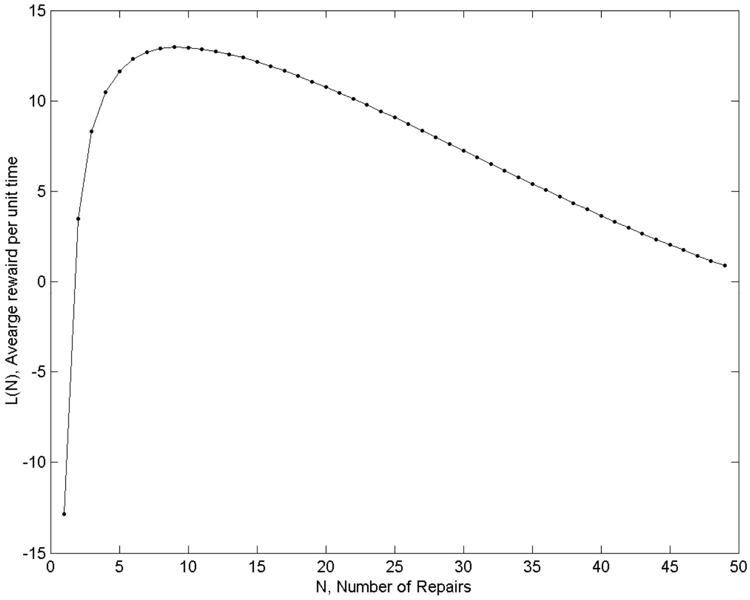
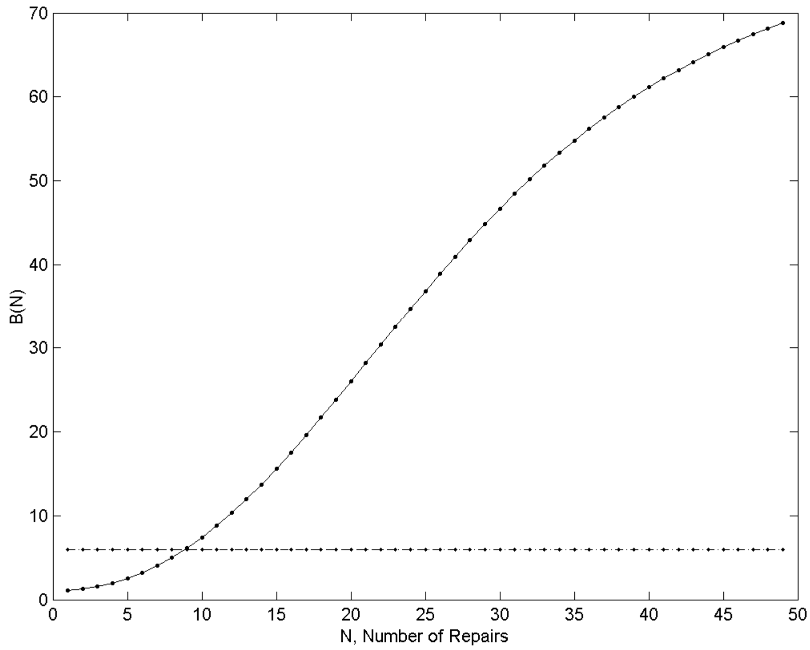


Figure 3.2 Plot with $B(N)$ *and* $\frac{c_r}{A_1 + A_0}$ *for different values of number of repairs,* N.



It can be seen from the figure 3.1 that the maximum reward per unit time is 12.955 and is attained when the number of repairs is $N_{opt} = 9$. Values $B(N)$ for various values of number of repairs and $\frac{c_r}{A_1 + A_0}$ are plotted in figure 3.2. The optimal value of N is obtained at the largest integer where these two lines intersect. It can be seen that the lines intersect at $N = 9$. Hence the discussions above are verified using the numerical example.

3.7 Conclusion

In this chapter we considered a general multistate system in which the lifetime and the repair times are not monotone and the consecutive states are Markov dependent. Long run average cost for the system is developed. An algorithm is developed keeping in mind that the system is monotone increasing.

Chapter 4
Optimal Protection Policy for Repair Facility with Interruption

4.1 Introduction

Vacation is a concept that is used widely in queuing literature. This chapter is motivated by the concept of vacation. Consider the case of software development. Assume that the development is done by an individual. One of the major threats of employing a single individual for the execution of the project is that the project may need to be dealt from scrap if the concerned person leaves the company without notice. This is because the approach and logic applied varies from person to person. Hence the company should keep alternatives open to tackle such situations. But keeping a standby from the beginning is not admissible from the cost perspective. It may be economical to restart if the concerned person leaves abruptly than keeping a standby from the beginning. Furthermore when the project enters some subsequent stages it might be desirable to have a standby than starting from scrap. Hence the most admissible choice will be to go for maintaining a standby resource once the development enters some critical stage. The time at which the standby person should be hired is a serious decision making problem.

In this chapter we deal with these types of problems. We assume that the spans of various stages are exponentially distributed with parameters depending on the stage.

4.2 Model

Consider a multi-state repair facility with n states. The repair facility can be interrupted because of many reasons like shut down at the repair facility, attrition of the service personal etc. Here we assume that the interruption will result in restarting the repair facility when the repair process is in certain class of states. But since restarting the repair facility from the scrap cannot be financially viable always, we introduce the concept of protecting some of the repair states from interruption. When the repairs are in these states, the shocks or the interruption will not have any impact of the repairs, that is, the repairs will continue uninterrupted. Hence it will be desirable to protect every step of the repair. This incurs a heavy financial commitment as the cost of protecting a state be too high. Hence it will be interesting to find the number of states to be protected.

Suppose there are n states in the repair procedure. Assume that the initial k are unprotected. Hence the number of protected states will be $n-k$. Let U and P respectively denote the set of unprotected and protected states. Once the process enters a protected state, it is protected till the completion of the repair facility. Hence transitions are assumed to be from unprotected state to same class or unprotected to protected and within the protected class. Transitions are not allowed from a protected state to unprotected state.

Let us assume that the repair times are distributed as an Erlang random variable with parameters (n,λ). Hence the state at which the process is in at time t is, one plus the number of renewals by the time t. Hence the probability that the process is in state $m+1$ at time t is given by $\frac{\exp(-\lambda t)(\lambda t)^m}{m!}, m=0,1,2,...,n-1$.

Then $$P(X(t)\in U)=\sum_{m=0}^{k-1}\frac{\exp(-\lambda t)(\lambda t)^m}{m!} \tag{4.1}$$

$$P(X(t)\in P)=\sum_{m=k}^{n-1}\frac{\exp(-\lambda t)(\lambda t)^m}{m!} \tag{4.2}$$

$$P(\text{Repair is completed by time } t)=\sum_{m=n}^{\infty}\frac{\exp(-\lambda t)(\lambda t)^m}{m!} \tag{4.3}$$

Let X, X_u and X_p respectively be the random variable denoting repair time, the time spent in the unprotected states and the time spent in protected state till the completion of the repair. Let $X(t)$ denote the state occupied by the repair facility at time t. Let random variable Y denote the time for the occurrence of the shock or the interruption. We will assume that the shocks occur according to an exponential distribution with parameter δ. Let c denote the cost incurred per unit time when the system is under repair. Let c_p denote the protection cost per unit time. Our objective is to find k so that the expected cost will be minimized during the procedure. Let $L(k)$ denote the expected cost with k unprotected states.

We can broadly classify the whole time interval into three distinct cases, assuming that the shock has occurred at time t;

(i) repair process is still in unprotected states at time t and the cost will be ct,

(ii) repair process is in protected state at time t and

(iii) repair process has completed by the time t. Let $L(k\mid t)$ denote the expected cost with k unprotected states when the occurrence of the shock is given to be at t. Then under situation (i), the cost will be

$$\left(ct+L(k)\right)P\left(X\left(t\right)\in U\right) \tag{4.4}$$

Now under (ii), assume that the repair process is in state i, $i\in P$ at time t, then expected time for the completion of repair after the i^{th} state is given by $\frac{(n-i)}{\lambda}$. Also since the sojourn time in state i is assumed to be exponentially distributed, the expected time in the state is independent the time for which it had been working in state i which implies that the expected time of stay in state i after time t is given by $\frac{1}{\lambda}$. Hence the cost incurred for repair, if $X\left(t\right)\in i$, is given by

$$c\left(t+\frac{n-i+1}{\lambda}\right) \tag{4.5}$$

Now in (iii), the repair process has completed by the time t. Hence if we consider the transitions from each state as a renewal in a Poisson Process, repair has completed by the time t imply that the number of renewals for the corresponding renewal process is $n,\ n+1,\ \ldots$. Let the process is in at state i at time t that is $(i-1)$ renewals had occurred by the time t.

Let $Y_1, Y_2, \ldots$ denote the interoccurence times and $N(t)$ denote the number of renewals by the time t for the renewal process. Let $S_k = \sum_{i=1}^{k} Y_i$. Then it is known that the distribution of S_k given that $N(t) = i$ is the distribution of the k^{th} order statistic of i uniform random variables over the range $(0,t)$.

Hence

$$f_X\left(x \mid N(t) = i-1\right) = \frac{(i-1)!}{(n-1)!(i-n-1)!}\left(\frac{x}{t}\right)^{n-1}\left(1-\frac{x}{t}\right)^{i-n-1}\frac{1}{t}.$$

Hence

$$E\left(X \mid N(t) = i-1\right) = \int_0^t x\frac{(i-1)!}{(k-1)!(i-k-1)!}\left(\frac{x}{t}\right)^{n-1}\left(1-\frac{x}{t}\right)^{i-n-1}\frac{1}{t}dx.$$

Putting $y = \frac{x}{t}$ the integral reduces to a beta integral and on simplification we get

$$E\left(X \mid N(t) = i-1\right) = \frac{(i-1)!}{(n-1)!(i-n-1)!}t\frac{n!(i-k-1)!}{i!} = \frac{nt}{i}.$$

Hence $E\left(X \mid T = t\right) = \sum_{i=n+1}^{\infty}\frac{nt}{i}\frac{exp(-\lambda t)(\lambda t)^{i-1}}{(i-1)!}$

$$= \frac{exp(-\lambda t)n}{\lambda}\sum_{i=n+1}^{\infty}\frac{(\lambda t)^i}{(i)!}$$

Now it is remaining to find out the protection cost under the condition (ii). Let us assume that $X(t) \in i,\ i \in P$. Then the time for which protection is given by time t is $t - X_u$. Then the distribution of X_u is the k^{th} order statistic of a random sample of size $i-1$ taken from a uniform distribution over the interval $(0,t)$. Hence

$$f_{X_u}\left(x \mid X(t)=i\right)=\frac{(i-1)!}{(k-1)!(i-k-1)!}\left(\frac{x}{t}\right)^{k-1}\left(1-\frac{x}{t}\right)^{i-k-1}\frac{1}{t}$$

Hence the expected time in protected states till time t when the process is in state $i, i\in P$ at time t is $I=\int_0^t (t-x)\frac{(i-1)!}{(k-1)!(i-k-1)!}\left(\frac{x}{t}\right)^{k-1}\left(1-\frac{x}{t}\right)^{i-k-1}\frac{1}{t}dx$, Putting $y=\frac{x}{t}$ we have

$$\begin{aligned} I &= \frac{(i-1)!}{(k-1)!(i-k-1)!}\int_0^1 t(1-y)\,y^{k-1}(1-y)^{i-k-1}\,dy \\ &= \frac{t(i-k)}{i} \end{aligned} \tag{4.6}$$

Following similar argument in condition (ii), expected time to the completion of the repair after time t and $X(t)\in i$ is $\frac{n-i+1}{\lambda}$.

Hence

$$\begin{aligned} L(k \mid t) &= \left(ct+L(k)\right)\sum_{i=0}^{k-1}P\left(X(t)=i\right)+\sum_{i=k}^{n-1}c\left(t+\frac{n-i+1}{\lambda}\right)P\left(X(t)=i\right) \\ &\quad c_p\sum_{i=k}^{n-1}\left(\frac{t(i-k)}{i}+\frac{n-i+1}{\lambda}\right)P\left(X(t)=i\right)+\frac{cnt}{\lambda}\sum_{i=n+1}^{\infty}P\left(X(t)=i\right) \\ &\quad +\frac{c_p t(n-k)}{\lambda}\sum_{i=n+1}^{\infty}P\left(X(t)=i\right) \end{aligned}$$

$$\begin{aligned} &= ct\sum_{i=0}^{n-1}P\left(X(t)=i\right)+L(k)\sum_{i=0}^{k-1}P\left(X(t)=i\right)+\frac{\left(c+c_p\right)}{\lambda}\sum_{i=k}^{n-1}(n-i+1)P\left(X(t)=i\right) \\ &\quad +c_p\sum_{i=k}^{n-1}\frac{t(i-k)}{i}P\left(X(t)=i\right)+\left(\frac{\left(c+c_p\right)nt}{\lambda}-\frac{c_p kt}{\lambda}\right)\sum_{i=n+1}^{\infty}P\left(X(t)=i\right) \end{aligned}$$

Integrating over the range of t, we have

$$\begin{aligned} L(k) &= \int_0^\infty L(k \mid t)\,\delta\exp(-\delta t)\,dt \\ &= cI_1+L(k)I_2+\left(c+c_p\right)I_3+c_pI_4+\left(\frac{\left(c+c_p\right)n}{\lambda}-\frac{c_pk}{\lambda}\right)I_5 \end{aligned} \tag{4.7}$$

where

$$I_1 = \sum_{i=0}^{n-1} \int_0^{\infty} t \frac{\exp(-\lambda t)(\lambda t)^i}{i!} \delta \exp(-\delta t)\, dt$$
$$= \sum_{i=0}^{n-1} \frac{\lambda^i \delta}{i!} \int_0^{\infty} \exp\left(-(\lambda+\delta)t\right) t^{i+1} dt$$
$$= \sum_{i=0}^{n-1} \frac{(i+1)\delta\lambda^i}{(\lambda+\delta)^{i+2}}$$

Hence $$I_1 = \frac{1}{\delta} - \frac{\left((n+1)\delta+\lambda\right)\left(\frac{\lambda}{\lambda+\delta}\right)^n}{\delta(\lambda+\delta)} \qquad (4.8)$$

$$I_2 = \sum_{i=0}^{k-1} \int_0^{\infty} \frac{\exp(-\lambda t)(\lambda t)^i}{i!} \delta \exp(-\delta t)\, dt$$
$$= \sum_{i=0}^{k-1} \frac{\lambda^i \delta}{i!} \int_0^{\infty} \exp\left(-(\lambda+\delta)t\right) t^i dt$$
$$= \sum_{i=0}^{k-1} \frac{\delta\lambda^i}{(\lambda+\delta)^{i+1}}$$

Hence $$I_2 = 1 - \left(\frac{\lambda}{\delta+\lambda}\right)^k \qquad (4.9)$$

$$I_3 = \sum_{i=k+1}^{n} \frac{n-i+1}{\lambda} \int_0^{\infty} \frac{\exp(-\lambda t)(\lambda t)^{i-1}}{(i-1)!} \delta \exp(-\delta t)\, dt$$
$$= \sum_{i=0}^{n-1} \frac{n-i+1}{\lambda} \frac{\lambda^{i-1}\delta}{(i-1)!} \int_0^{\infty} \exp\left(-(\lambda+\delta)t\right) t^{i-1} dt$$
$$= \sum_{i=0}^{n-1} \frac{(n-i+1)\delta\lambda^{i-2}}{(\lambda+\delta)^i}$$

Hence $$I_3 = \frac{\lambda^{k-1}\left((n-k)\delta - \lambda + \lambda\left(\frac{\lambda}{\lambda+\delta}\right)^{k-n}\right)}{\delta(\lambda+\delta)^k} \qquad (4.10)$$

Now

$$I_4=\sum_{i=k+1}^{n}\frac{i-k}{i}\int_0^{\infty}t\frac{\exp(-\lambda t)(\lambda t)^{i-1}}{(i-1)!}\delta\exp(-\delta t)\,dt$$

$$=\sum_{i=0}^{n-1}\frac{i-k}{i}\frac{\lambda^{i-1}\delta}{(i-1)!}\int_0^{\infty}\exp\left(-(\lambda+\delta)t\right)t^{i}dt$$

$$=\sum_{i=0}^{n-1}\frac{(i-k)}{i}\frac{i\delta\lambda^{i-1}}{(\lambda+\delta)^{i+1}}=\sum_{i=0}^{n-1}\frac{(i-k)\delta\lambda^{i-1}}{(\lambda+\delta)^{i+1}}$$

$$I_4=\frac{\lambda^{k}(\delta+\lambda)^{n+1}-\lambda^{n}(\delta+\lambda)^{k}\left[(1-k+n)\delta+\lambda\right]}{\delta(\delta+\lambda)^{n+k}} \tag{4.11}$$

$$I_5=\int_0^{\infty}\frac{exp(-\lambda t)n}{\lambda}\sum_{i=n+1}^{\infty}\frac{(\lambda t)^{i}}{(i)!}\delta\exp(-\delta t)\,dt$$

$$=\frac{n\delta}{\lambda}\sum_{i=n+1}^{\infty}\frac{\lambda^{i}}{i!}\int_0^{\infty}\exp\left(-(\lambda+\delta)t\right)t^{i}dt$$

$$=\frac{n\delta}{\lambda}\sum_{i=n+1}^{\infty}\frac{\lambda^{i}}{i!}\frac{i!}{(\lambda+\delta)^{i}}$$

$$=\frac{n\delta}{\lambda}\sum_{i=n+1}^{\infty}\frac{\lambda^{i}}{(\lambda+\delta)^{i}}$$

On Simplification we get

$$I_5=\frac{\lambda^{n+1}}{(\delta+\lambda)^{n}\delta} \tag{4.12}$$

Now making use of equations (4.8) to (4.12) in equation (4.7), we will have the expression for $L(k),\ k=1,2,...,n$.

For $k=0$, that is the case when all the states are protected, the expected cost will be independent of the time at which shock occurs and the expected time to complete the repair process is given by $\frac{n}{\lambda}$. Hence the expected cost in this will be $\left(c+c_p\right)\frac{n}{\lambda}$.

4.3 Numerical Illustration

The results obtained in this chapter are illustrated with the help of a numerical example. The illustrations are performed with the help of MATHEMATICA®. The results are validated with various values of the parameters λ, δ, c and c_p, the values of the number of states are assumed to be constant at $n = 20$. The results are summarized in table4.1. The figures in the boldface letter indicated the optimal values.

Table 4.1 Long run cost for various values of $(\lambda, \delta, c, c_p)$ *and constant at* $n = 20$

k \ $(\lambda, \delta, c, c_p)$	**10,2,50,80**	**10,2,30,80**	**10,2,10,80**	**10,6,10,80**
1	284.8659	239.6491	194.4323	172.0237
2	**283.1871**	236.5269	189.8667	164.6362
3	283.522	234.7298	185.9376	158.2151
4	286.2235	**234.4728**	182.7222	153.3396
5	291.7047	236.0039	180.3031	**150.9361**
6	300.4494	239.6085	178.7675	152.4862
7	313.0238	245.6147	**178.2056**	160.3594
8	330.09	254.3991	178.7081	178.3454
9	352.4218	266.3926	180.3635	212.5053
10	380.9226	282.0876	183.2527	272.5327
11	416.6469	302.0449	187.443	373.931
12	460.824	326.9017	192.9793	541.4956
13	514.8861	357.3793	199.8725	814.8825
14	580.5	394.2919	208.0837	1257.515
15	659.6041	438.5543	217.5045	1970.83
16	754.4498	491.1901	227.9303	3117.056
17	867.6497	553.338	239.0262	4955.655
18	1002.231	626.2574	250.2833	7901.593
19	1161.7	711.3311	260.9622	12618.54
20	1350.107	810.064	270.0213	20167.93

4.4 Conclusion

We introduced the concept of protection during the repair process in this chapter. An optimal modelling assuming that the sojourn times in each state as exponential distribution is also done. Since the most of the lifetime distribution can be approximated by Coxian distribution, the results can be applied to the single state systems, splitting the total stay as to be combination of exponential random variables.

Chapter 5
Two-Phase Start-up Demonstration Tests[2]

5.1 Introduction

Business people demonstrates or convinces customer about the reliability of heavy machine by start-ups and such a procedure is normally termed as start-up demonstration test. Most of the earlier studies in the start-up demonstration tests were assuming that the consecutive attempts are independent. But this may not be situation as success yields success with higher probability. For example a team winning a game has higher chance for winning the next match as the win in a match will increase the morale of the team as a whole. Also vendor is always motivated to sell his product. Hence he will make attempts to convince the customer about the quality of the product with necessary steps like looking for new or alternative product if the initial attempts make to have positive impact on sthe customer. But at the same time as the customer is vigilant he might be having certain conditions for the rejection of the product. Hence in this chapter we will introduce two test procedures which a vendor can be applied for selling his product.

5.2 Models

The proposed models consist of two phases. We will incorporate repair action in phase one. Let us consider the first model. The product is accepted in the first phase if k_1 consecutive successes occur prior to c_1 consecutive failure and d_1 failed attempts and we reject it in the phase one if we observe d_1 failures ahead of k_1 consecutive successes and c_1 consecutive failure. If c_1 consecutive failures occur prior to k_1 consecutive successes and d_1 failures, we will take product for repair and the repaired product, in second phase, is accepted if k_2 consecutive

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[2] A part of this chapter is communicated in Nair and Thomas(2011)

successes occur before we observe d_2 failures and we reject the product otherwise. If we take $c_1 = d_1 + 1$ our model reduces to the case with rejection of units upon observing d failures proposed by Balakrishnan and Chan (2000) and later studied by Martin (2004) and Smith and Griffith (2005). When $c_1 = 1$ our model reduces to single corrective model discussed by Balakrishnan et. al (1995,1997).

In the second model we will sent the product for repair if accidental causes exceeds specified limits and once a permanent failure observed for the product, leading to consecutive failure, we will reject the product. In phase one, we will accept if k_1 consecutive successes prior to c_1 consecutive failures and d_1 random failures and if c_1 consecutive failures occur before k_1 consecutive successes and d_1 random failures, we reject the product. The product is sent for repair if d_1 random failures come ahead of the other two events. Now in order to make sure that the test terminates with probability one, in the second phase we reject the product if d_2 random failures occur ahead of the k_2 consecutive successes. On the contrary we accept the product.

It is natural to think that success (failure) leads to success (failure) with high probability. Hence throughout this paper we will assume that the probability of success (failure) depends on the number of just preceding consecutive success (failure), discussed as l dependent sequences by Aki and Hirano (2000). Independent and identical and Markov dependent trials comes as special cases to our proposed model. So let $p_{ss}(x)$ denotes the probability for a success following a success run of length x and $p_{fs}(x)$ denotes the probability for a success following a failure run of length x. Similarly $p_{ff}(x)$ $\left(p_{sf}(x)\right)$ denote probability for a failure following a failure (success) run of length x.

For both the models of start-up demonstration tests, we proceed by embedding a Markov chain with each phase of the test. The states of the embedded

Markov chain, in both phases, are given by the triplet (x, y, z) where x denotes the length of the current run, y denotes the type of the current run i.e. s if a success run is going on and f otherwise. z corresponds to the number of failures occurred till now. Also throughout this paper we will assume that $\underline{1}'$ unit vector of appropriate dimension so that the matrix multiplication is conformable.

Let $\left\{A_n^{(1)}, n \geq 1\right\}$ be the Markov chain associated with the first phase of the test. Then the transition probabilities of the Markov chain $\left\{A_n^{(1)}, n \geq 1\right\}$ are given by

$$P\left(A_{n+1}^{(1)} = (x+1, s, z) \middle| A_n^{(1)} = (x, s, z)\right) = p_{ss}(x) \quad if \quad 0 \leq z \leq d_1 - 1, \; 0 \leq x \leq k_1 - 1$$
$$P\left(A_{n+1}^{(1)} = (x+1, f, z+1) \middle| A_n^{(1)} = (x, f, z)\right) = p_{ff}(x) \quad if \quad 0 \leq z \leq d_1 - 1, \; 0 \leq x \leq c - 1$$
$$P\left(A_{n+1}^{(1)} = (1, f, z+1) \middle| A_n^{(1)} = (x, s, z)\right) = p_{sf}(x) \quad if \quad 0 \leq z \leq d_1 - 1, \; 0 \leq x \leq k_1 - 1$$
$$P\left(A_{n+1}^{(1)} = (1, s, z) \middle| A_n^{(1)} = (x, f, z)\right) = p_{fs}(x) \quad if \quad 0 \leq z \leq d_1 - 1, \; 0 \leq x \leq c - 1$$

and

$$P\left(A_{n+1}^{(1)} = (x, f, d_1) \middle| A_n^{(1)} = (x, f, d_1)\right) = 1 \quad 0 \leq x \leq c - 1$$
$$P\left(A_{n+1}^{(1)} = (k_1, s, z) \middle| A_n^{(1)} = (k_1, s, z)\right) = 1 \quad 0 \leq z \leq d_1 - 1$$
$$P\left(A_{n+1}^{(1)} = (c, f, z) \middle| A_n^{(1)} = (c, f, z)\right) = 1 \quad 0 \leq z \leq d_1$$

Based on the decision whether we accept, reject or repair we can divide the states of the Markov chain into four classes viz., Non-decisive, accepting, rejecting and repair classes. We will continue with making attempts as long as the Markov chain is in non-decisive class. Once the Markov chain enters the accepting, rejecting or repair class, we will accept, reject or repair the product. Hence the states in the accepting, rejecting and repair classes are all absorbing. Clearly the last three equations correspond to the absorbing states. Now we can partition the transition probability matrix corresponding to the first phase of the start-up demonstration test as shown

$$P_1 = \begin{bmatrix} R^{(1)} & Q_a^{(1)} & Q_t^{(1)} & Q_r^{(1)} \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

where $R^{(1)}$ denote the matrix of transition probabilities between the non-decisive states or non-absorbing states. $Q_a^{(1)}, Q_t^{(1)}$ and $Q_r^{(1)}$ respectively denote the transition probabilities from non-decisive states to the accepting, repair/corrective and rejecting states

Let $\tau_a^{(1)}, \tau_t^{(1)}$ and $\tau_r^{(1)}$ denote the number of transition required either to accept or to repair or to reject the product in the first phase. Then by using the Chappman-Kolmogrov equation and the first phase t.p.m we have

$$\begin{aligned} &P\left(\tau_a^{(1)} = n\right) = \alpha^{(1)}\left(R^{(1)}\right)^{n-1} Q_a^{(1)} \underline{1}', \quad P\left(\tau_t^{(1)} = n\right) = \alpha^{(1)}\left(R^{(1)}\right)^{n-1} Q_t^{(1)} \underline{1}' \\ &P\left(\tau_r^{(1)} = n\right) = \alpha^{(1)}\left(R^{(1)}\right)^{n-1} Q_r^{(1)} \underline{1}' \end{aligned} \tag{5.1}$$

where $\underline{1}'$ denote the unit vector of appropriate dimension so that the matrix multiplication is conformable and $\alpha^{(1)}$ denote the initial probability distribution.

Then the

$$\begin{aligned} P(\text{Acceptance of the product in phase 1}) &= \sum_{n=1}^{\infty} \alpha^{(1)}\left(R^{(1)}\right)^{n-1} Q_a^{(1)} \underline{1}' = \alpha^{(1)}\left[I - R^{(1)}\right]^{-1} Q_a^{(1)} \underline{1}' \\ P(\text{Rejection of the product in phase 1}) &= \sum_{n=1}^{\infty} \alpha^{(1)}\left(R^{(1)}\right)^{n-1} Q_r^{(1)} \underline{1}' = \alpha^{(1)}\left[I - R^{(1)}\right]^{-1} Q_r^{(1)} \underline{1}' \\ P(\text{Repair product }) &= \sum_{n=1}^{\infty} \alpha^{(1)}\left(R^{(1)}\right)^{n-1} Q_t^{(1)} \underline{1}' = \alpha^{(1)}\left[I - R^{(1)}\right]^{-1} Q_t^{(1)} \underline{1}' \end{aligned} \tag{5.2}$$

As in phase one we can associate a Markov chain in the second phase also. Let $\left\{A_n^{(2)}, n \geq 1\right\}$ be the Markov chain associated with the second phase of the test. Here again we will denote the states of the Markov chain by triplet as denoted in the phase one. Transition probabilities in the second phase is given by

$$P\left(A_{n+1}^{(2)}=(x+1,s,z)\middle|A_n^{(2)}=(x,s,z)\right)=p_{ss}(x) \quad if \quad 0\le z\le d_2-1,\ \ 0\le x\le k_2-1$$
$$P\left(A_{n+1}^{(2)}=(x+1,f,z+1)\middle|A_n^{(2)}=(x,f,z)\right)=p_{ff}(x) \quad if \quad 0\le x,z\le d_2-1$$
$$P\left(A_{n+1}^{(2)}=(1,f,z+1)\middle|A_n^{(2)}=(x,s,z)\right)=p_{sf}(x) \quad if \quad 0\le z\le d_2-1,\ \ 0\le x\le k_2-1$$
$$P\left(A_{n+1}^{(2)}=(1,s,z)\middle|A_n^{(2)}=(x,f,z)\right)=p_{fs}(x) \quad if \quad 0\le x,z\le d_1-1$$

and

$$P\left(A_{n+1}^{(2)}=(x,f,d_2)\middle|A_n^{(2)}=(x,f,d_2)\right)=1 \quad 0\le x\le d_2$$
$$P\left(A_{n+1}^{(2)}=(k_2,s,z)\middle|A_n^{(2)}=(k_2,s,z)\right)=1 \quad 0\le z\le d_2-1$$

Since there is no scope for repair in the second phase, we can partition the states of the Markov chain into subclasses (i) Non-decisive class (ii) accepting class and (iii) rejecting class. On achieving any states in the class leads to the acceptance or rejection of the product then such a class is respectively called as accepting and rejecting class. As in phase one the transition probability matrix associated with the embedded Markov chain is given by

$$P_2=\begin{bmatrix} R^{(2)} & Q_a^{(2)} & Q_r^{(2)} \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

with $R^{(2)}$ denoting the matrix of transition probabilities between the non-decisive states, $Q_a^{(2)}$ and $Q_r^{(2)}$ respectively denote the matrix of transition probability from a non-decisive state to an accepting, rejecting state. As the test is stopped once we reach any of the accepting or rejecting states, those states are all absorbing states. Also let the number of trials required either to accept or to reject the product in the second phase be denoted respectively by $\tau_a^{(2)}$ and $\tau_r^{(2)}$. Then by exploiting Chappman-Kolmogrov (Bhat 2002, Feller 1968) equation and the second phase t.p.m we have

$$P\left(\tau_a^{(2)}=n\right)=\alpha^{(2)}\left(R^{(2)}\right)^{n-1}Q_a^{(2)}\underline{1}', \quad P\left(\tau_r^{(2)}=n\right)=\alpha^{(2)}\left(R^{(2)}\right)^{n-1}Q_r^{(2)}\underline{1}' \qquad (5.3)$$

$$P(\text{Acceptance of the product in phase } 2)=\sum_{n=1}^{\infty}\alpha^{(2)}\left(R^{(2)}\right)^{n-1}Q_a^{(2)}\underline{1}'=\alpha^{(2)}\left[I-R^{(2)}\right]^{-1}Q_a^{(2)}\underline{1}'$$

$$P(\text{Rejection of the product in phase } 2)=\sum_{n=1}^{\infty}\alpha^{(2)}\left(R^{(2)}\right)^{n-1}Q_r^{(2)}\underline{1}'=\alpha^{(2)}\left[I-R^{(2)}\right]^{-1}Q_r^{(2)}\underline{1}' \tag{5.4}$$

Hence from equations (5.2) and (5.4) we have

$$P(\text{Acceptance of the product })=\alpha^{(1)}\left[I-R^{(1)}\right]^{-1}\left[Q_a^{(1)}\underline{1}'+Q_t^{(1)}\underline{1}'\alpha^{(2)}\left[I-R^{(2)}\right]^{-1}Q_a^{(2)}\underline{1}'\right]$$

$$P(\text{Rejection of the product })=\alpha^{(1)}\left[I-R^{(1)}\right]^{-1}\left[Q_r^{(1)}\underline{1}'+Q_t^{(1)}\underline{1}'\alpha^{(2)}\left[I-R^{(2)}\right]^{-1}Q_r^{(2)}\underline{1}'\right] \tag{5.5}$$

Now we will consider the states in the different classes in both phases of the test. Since both tests vary only in the first phase and Markov chain $\left\{A_n^{(2)}, n\geq 1\right\}$ and their partitions are the same for both phases of the test. They differ only in their partition of the states of the Markov chain $\left\{A_n^{(1)}, n\geq 1\right\}$. First we shall consider the case for the first model. States in the non-decisive class, which forms a $2+(k_1-1)d_1+(d_1-c_1+1)(c_1-1)+(c_1-1)(c_1-2)/2$ dimensional vector, can be further subdivided into three sub classes, namely.,

(i) $\{(0,s,0),(x,s,z),1\leq x\leq k_1-1,\ \ 0\leq z\leq d_1-1\}$ with $1+(k_1-1)d_1$ states

(ii) class with $1+(d_1-c_1+1)(c_1-1)$ states given by $\{(0,f,0),(1,f,1),...(c_1-1,f,c_1-1),(1,f,2),...,(c_1-1,f,c_1),....,$ $(1,f,d_1-c_1),...,(c_1-1,f,d-1)\}$.

(iii) Class given by $\{(1,f,d_1-c_1+2),...,(c_1-2,f,d_1-1),(1,f,d_1-c_1+3),$ $...,(c_1-3,f,d_1-1),...,(1,f,d_1-1)\}$, with dimension $(c_1-1)(c_1-2)/2$.

Accepting class, a row vector with dimension d_1 is given by $\{(k_1,s,0),(k_1,s,1),...,(k_1,s,d_1-1)\}$. Rejecting class given by $\{(1,f,d_1),(2,f,d_1),$

$\left(3,f,d_1\right),...,\left(c,f,d_1\right)\}$ has c_1 states. A $\left(d_1-c_1\right)$ states class, with elements given by $\left\{\left(c_1,f,c_1\right),\left(c_1,f,c_1+1\right),...,\left(c_1,f,d_1-1\right)\right\}$ forms the repair class.

For the second model, the rejecting class and the repair class got replaced each other. Hence the class of states $\left\{\left(1,f,d_1\right),\left(2,f,d_1\right),\left(3,f,d_1\right),...,\left(c,f,d_1\right)\right\}$ forms the rejection class, corresponds to the rejection of the products. Clearly there are c_1 states in the class. Once the Markov chain enters any of the states in the repair class, with $\left(d_1-c_1\right)$ states, given by $\left\{\left(c_1,f,c_1\right),\left(c_1,f,c_1+1\right),...,\left(c_1,f,d_1-1\right)\right\}$ then the product is sent for repair/corrective action. The non-decisive class and the accepting class remains to be the same.

Now let us consider the partition of states in the second phase. Since the conditions of acceptance and rejection remain same in both the models, the partition is same for both the models. The class $\left(k_2,s,z\right), 0\le z\le d_2-1$ with d_2 states forms the accepting class, where as the class of d_2 states given by $\left(x,f,d_2\right)$, $1\le x\le d_2$. Non-decisive class consists of r states , where $r=2+\left(k_2-1\right)d_2+d_2\left(d_2-1\right)/2$, states and is given by

$$\left\{(0,s,0),(1,s,0),...,\left(k_2-1,s,0\right),(1,s,1),...,\left(k_2-1,s,d_2-1\right),(0,f,0),(1,f,1)...,\right.$$
$$\left.\left(d_2-1,f,d_2-1\right),(1,f,2),...,\left(d_2-2,f,d_2-1\right),...,\left(1,f,d_2-1\right)\right\}.$$

5.3 Results

In this section we will derive the expressions for the probability generating functions (pgfs) of the random variable the number of trials under different scenario. Let N and $N^{(i)}$, $i=1,2$ denote the number of trials in the test and in the i^{th} state respectively. Then we have

Theorem 5.1*:* If $\phi_N(z), \phi_{N^{(i)}}(z),\ i=1,\ 2$ denote the pgf of $N, N^{(i)},\ i=1,\ 2$ .Then

i. $$\phi_N(z) = [z-1]\alpha^{(1)}\left[I - zR^{(1)}\right]^{-1}\left\{\underline{1}' + zQ_t^{(1)}1'\alpha^{(2)}\left[I - zR^{(2)}\right]^{-1}\underline{1}'\right\} \tag{5.6}$$

ii. $$\phi_{N^{(i)}}(z) = [z-1]\alpha^{(i)}\left[I - zR^{(i)}\right]^{-1}1' + 1, \quad i = 1,2$$

Proof:

We have $\phi_N(z) = E(z^n)$

$$= \sum_{n=1}^{\infty}\left\{z^n\left[\alpha^{(1)}\left(R^{(1)}\right)^{n-1}\left(Q_a^{(1)}\underline{1}' + Q_r^{(1)}\underline{1}'\right) + \sum_{\mu=1}^{n-1}\alpha^{(1)}\left(R^{(1)}\right)^{\mu-1}Q_t^{(1)}\underline{1}'\alpha^{(2)}\left(R^{(2)}\right)^{n-\mu-1}\right.\right.$$

$$\left.\left.\left[Q_a^{(2)}\underline{1}' + Q_r^{(2)}\underline{1}'\right]\right]\right\} \tag{5.7}$$

$$\sum_{n=1}^{\infty} z^n\alpha^{(1)}\left(R^{(1)}\right)^{n-1}\left(Q_a^{(1)}1' + Q_r^{(1)}\underline{1}'\right) = \sum_{n=1}^{\infty} z^n\alpha^{(1)}\left(R^{(1)}\right)^{n-1}\left(\underline{1}' - R^{(1)}\underline{1}' - Q_t^{(1)}\underline{1}'\right)$$

$$= z\sum_{n=1}^{\infty}\alpha^{(1)}\left(zR^{(1)}\right)^{n-1}\underline{1}' - \sum_{n=0}^{\infty}\alpha^{(1)}\left(zR^{(1)}\right)^{n}\underline{1}' + \alpha^{(1)}\underline{1}' - \sum_{n=1}^{\infty} z^n\alpha^{(1)}\left(R^{(1)}\right)^{n-1}Q_t^{(1)}\underline{1}'$$

$$= [z-1]\alpha^{(1)}\left[I - zR^{(1)}\right]^{-1}\underline{1}' + 1 - \sum_{n=1}^{\infty} z^n\alpha^{(1)}\left(R^{(1)}\right)^{n-1}Q_t^{(1)}\underline{1}' \tag{5.8}$$

$$\sum_{n=1}^{\infty} z^n\sum_{\mu=1}^{n-1}\alpha^{(1)}\left(R^{(1)}\right)^{\mu-1}Q_t^{(1)}\underline{1}'\alpha^{(2)}\left(R^{(2)}\right)^{n-\mu-1}\left\{Q_a^{(2)}\underline{1}' + Q_r^{(2)}\underline{1}'\right\}$$

$$= \sum_{n=1}^{\infty} z^n\sum_{\mu=1}^{n-1}\alpha^{(1)}\left(R^{(1)}\right)^{\mu-1}Q_t^{(1)}\underline{1}'\alpha^{(2)}\left(R^{(2)}\right)^{n-\mu-1}\left\{1' - R^{(2)}\underline{1}'\right\}$$

Interchanging the order of summation, we have

$$= \sum_{\mu=1}^{\infty}\alpha^{(1)}\left(R^{(1)}\right)^{(\mu-1)}Q_t^{(1)}\underline{1}'\alpha^{(2)}\sum_{n=\mu+1}^{\infty} z^n\left(R^{(2)}\right)^{(n-\mu-1)}\left[\underline{1}' - R^{(2)}\underline{1}'\right]$$

$$\sum_{n=\mu+1}^{\infty} z^n\left(R^{(2)}\right)^{(n-\mu-1)}\left[1' - R^{(2)}\underline{1}'\right] = z^{\mu}\sum_{n=\mu+1}^{\infty} z^{(n-\mu)}\left(R^{(2)}\right)^{(n-\mu-1)}\left[\underline{1}' - R^{(2)}\underline{1}'\right]$$

$$= z^{\mu}\left\{z\sum_{n'=1}^{\infty}\left(zR^{(2)}\right)^{(n'-1)}\underline{1}' - \sum_{n'=1}^{\infty}\left(zR^{(2)}\right)^{(n')}\underline{1}'\right\}$$

$$= z^{\mu}\left[z\sum_{n'=1}^{\infty}\left(R^{(2)}\right)^{(n'-1)}\underline{1}' - \sum_{n'=0}^{\infty}\left(R^{(2)}\right)^{(n')}\underline{1}' + \underline{1}'\right]$$

$$= z^{\mu}\left\{(z-1)\left[I - zR^{(2)}\right]^{-1}\underline{1}' + \underline{1}'\right\} \tag{5.9}$$

$$\sum_{n=1}^{\infty} z^{n}\sum_{\mu=1}^{n-1}\alpha^{(1)}\left(R^{(1)}\right)^{\mu-1}Q_t^{(1)}\underline{1}'\alpha^{(2)}\left(R^{(2)}\right)^{n-\mu-1}\left\{Q_a^{(2)}\underline{1}' + Q_r^{(2)}1'\right\}\underline{1}'$$

$$= \sum_{\mu=1}^{\infty}\alpha^{(1)}\left(R^{(1)}\right)^{(\mu-1)}Q_t^{(1)}\underline{1}'\alpha^{(2)}z^{\mu}\left\{(z-1)\left[I - zR^{(2)}\right]^{-1}\underline{1}' + \underline{1}'\right\}$$

Using (5.9)

$$= \sum_{\mu=1}^{\infty}\alpha^{(1)}z\left(zR^{(1)}\right)^{(\mu-1)}Q_t^{(1)}\underline{1}'\alpha^{(2)}(z-1)\left[I - zR^{(2)}\right]^{-1}\underline{1}' + \sum_{\mu=1}^{\infty}\alpha^{(1)}z^{\mu}\left(R^{(1)}\right)^{(\mu-1)}Q_t^{(1)}\underline{1}' \tag{5.10}$$

Now using (5.8) and (5.10) in (5.7), we have

$$\phi_N(z) = [z-1]\alpha^{(1)}\left[I - zR^{(1)}\right]^{-1}\underline{1}' + 1 - \sum_{n=1}^{\infty} z^{n}\alpha^{(1)}\left(R^{(1)}\right)^{n-1}Q_t^{(1)}\underline{1}' +$$

$$\sum_{\mu=1}^{\infty}\alpha^{(1)}z\left(zR^{(1)}\right)^{(\mu-1)}Q_t^{(1)}\underline{1}'\alpha^{(2)}(z-1)\left[I - zR^{(2)}\right]^{-1}\underline{1}' + \sum_{\mu=1}^{\infty}\alpha^{(1)}z^{\mu}\left(R^{(1)}\right)^{(\mu-1)}Q_t^{(1)}\underline{1}'$$

$$\phi_N(z) = [z-1]\alpha^{(1)}\left[I - zR^{(1)}\right]^{-1}\left\{\underline{1}' + zQ_t^{(1)}\underline{1}'\alpha^{(2)}\left[I - zR^{(2)}\right]^{-1}\underline{1}'\right\}$$

(ii) Also we have,

$$P\left(N^{(1)} = n\right) = P\left(\tau_a^{(1)} = n\right) + P\left(\tau_r^{(1)} = n\right) + P\left(\tau_t^{(1)} = n\right)$$

$$= \alpha^{(1)}\left(R^{(1)}\right)^{n-1}\left(Q_a^{(1)}\underline{1}' + Q_r^{(1)}\underline{1}' + Q_t^{(1)}1'\right)\underline{1}' = \alpha^{(1)}\left(R^{(1)}\right)^{n-1}\left(\underline{1}' - R^{(1)}\underline{1}'\right)$$

$$\phi_{N^{(1)}}(z)=\sum_{n=1}^{\infty} z^n P\left(N^{(1)}=n\right)=\sum_{n=1}^{\infty} z^n \alpha^{(1)}\left(R^{(1)}\right)^{(n-1)}\left(\underline{1}'-R^{(1)}\underline{1}'\right)$$

$$=z\sum_{n=1}^{\infty} z^{(n-1)}\alpha^{(1)}\left(R^{(1)}\right)^{(n-1)}\underline{1}'-\sum_{n=1}^{\infty} z^n\alpha^{(1)}\left(R^{(1)}\right)^{n}\underline{1}'$$

$$=z\sum_{n=1}^{\infty}\alpha^{(1)}\left(zR^{(1)}\right)^{(n-1)}\underline{1}'-\sum_{n=0}^{\infty}\alpha^{(1)}\left(zR^{(1)}\right)^{n}\underline{1}'+\alpha^{(1)}\underline{1}'$$

$$=[z-1]\sum_{n=1}^{\infty}\alpha^{(1)}\left(zR^{(1)}\right)^{(n-1)}\underline{1}'+1=[z-1]\alpha^{(1)}\left[I-zR^{(1)}\right]^{-1}\underline{1}'+1$$

Working on the same line as in (ii) we can prove (iii)

$$\phi_{N^{(2)}}(z)=[z-1]\alpha^{(2)}\left[I-zR^{(2)}\right]^{-1}\underline{1}'+1$$

Remark 5.1: We can find the expected number of trials in the test and in each phase of the test by taking the derivative of the pgfs with respect to s and setting $s=1$. Hence we have from the above generating function

1. $E(N)=\alpha^{(1)}\left[I-R^{(1)}\right]^{-1}\underline{1}'+\alpha^{(1)}\left[I-R^{(1)}\right]^{-1}Q_t^{(1)}\underline{1}'\alpha^{(2)}\left[I-R^{(2)}\right]^{-1}\underline{1}'$
2. $E\left(N^{(1)}\right)=\alpha^{(1)}\left[I-R^{(1)}\right]^{-1}\underline{1}'$
3. $E\left(N^{(2)}\right)=\alpha^{(2)}\left[I-R^{(2)}\right]^{-1}\underline{1}'$

It will be interesting to find the expected number of trials before we accept or reject the product.

Theorem 5.2*:* Expected number trials given that accept or reject the product is

$$E\left(N\middle|Accept\right)=\alpha^{(1)}\left[I-R^{(1)}\right]^{-2}Q_a^{(1)}\underline{1}'+\alpha^{(1)}\left[I-R^{(1)}\right]^{-1}\left\{Q_t^{(1)}\underline{1}'\alpha^{(2)}\left[I-R^{(2)}\right]^{-1}\right.$$

$$\left.+\left[I-R^{(1)}\right]^{-1}Q_t^{(1)}\underline{1}'\alpha^{(2)}\right\}\left[I-R^{(2)}\right]^{-1}Q_a^{(2)}\underline{1}'$$

$$E\left(N\middle|\text{Reject}\right)=\alpha^{(1)}\left[I-R^{(1)}\right]^{-2}Q_r^{(1)}\underline{1}'+\alpha^{(1)}\left[I-R^{(1)}\right]^{-1}\left\{Q_t^{(1)}\underline{1}'\alpha^{(2)}\left[I-R^{(2)}\right]^{-1}\right.$$

$$\left.+\left[I-R^{(1)}\right]^{-1}Q_t^{(1)}\underline{1}'\alpha^{(2)}\right\}\left[I-R^{(2)}\right]^{-1}Q_r^{(2)}\underline{1}'$$

Proof :

We have

$$E\left(N\middle|Accept\right)=\sum_{n=1}^{\infty}n\alpha^{(1)}\left(R^{(1)}\right)^{(n-1)}Q_a^{(1)}\underline{1}'+\sum_{n=1}^{\infty}n\sum_{\mu=1}^{n-1}\alpha^{(1)}\left(R^{(1)}\right)^{\mu-1}Q_t^{(1)}\underline{1}'\alpha^{(2)}\left(R^{(2)}\right)^{n-\mu-1}Q_a^{(2)}\underline{1}'$$

First let us consider

$$\sum_{n=1}^{\infty}n\alpha^{(1)}\left(R^{(1)}\right)^{(n-1)}Q_a^{(1)}\underline{1}'=\alpha^{(1)}\left[I-R^{(1)}\right]^{-2}Q_a^{(1)}\underline{1}'$$

Also we have

$$\sum_{n=1}^{\infty}n\sum_{\mu=1}^{n-1}\alpha^{(1)}\left(R^{(1)}\right)^{\mu-1}Q_t^{(1)}\underline{1}'\alpha^{(2)}\left(R^{(2)}\right)^{n-\mu-1}Q_a^{(2)}\underline{1}'$$

$$=\sum_{\mu=1}^{\infty}\alpha^{(1)}\left(R^{(1)}\right)^{(\mu-1)}Q_t^{(1)}\underline{1}'\alpha^{(2)}\sum_{n=\mu+1}^{\infty}n\left(R^{(2)}\right)^{(n-\mu-1)}Q_r^{(1)},$$

Interchanging the order of summations

$$=\sum_{\mu=1}^{\infty}\alpha^{(1)}\left(R^{(1)}\right)^{(\mu-1)}Q_t^{(1)}\underline{1}'\alpha^{(2)}\left\{\sum_{n=\mu+1}^{\infty}(n-\mu)\left(R^{(2)}\right)^{(n-\mu-1)}Q_a^{(2)}\underline{1}'+\right.$$

$$\left.\mu\sum_{n=\mu+1}^{\infty}\left(R^{(2)}\right)^{(n-\mu-1)}Q_a^{(2)}\underline{1}'\right\}$$

$$=\sum_{\mu=1}^{\infty}\alpha^{(1)}\left(R^{(1)}\right)^{(\mu-1)}Q_t^{(1)}\underline{1}'\alpha^{(2)}\left\{\left[I-R^{(2)}\right]^{-2}Q_a^{(2)}\underline{1}'+\mu\left[I-R^{(2)}\right]^{-1}Q_a^{(2)}\underline{1}'\right\}$$

$$=\alpha^{(1)}\left[I-R^{(1)}\right]^{-1}\left\{Q_t^{(1)}\underline{1}'\alpha^{(2)}\left[I-R^{(2)}\right]^{-1}+\right.$$

$$\left.\left[I-R^{(1)}\right]^{-1}Q_t^{(1)}\underline{1}'\right\}\alpha^{(2)}\left[I-R^{(2)}\right]^{-1}Q_a^{(2)}\underline{1}'$$

$$E\left(N\middle|Accept\right)=\alpha^{(1)}\left[I-R^{(1)}\right]^{-2}Q_a^{(1)}\underline{1}'+\alpha^{(1)}\left[I-R^{(1)}\right]^{-1}\left\{Q_t^{(1)}\underline{1}'\alpha^{(2)}\left[I-R^{(2)}\right]^{-1}\right.$$

$$\left.+\left[I-R^{(1)}\right]^{-1}Q_t^{(1)}\underline{1}'\alpha^{(2)}\right\}\left[I-R^{(2)}\right]^{-1}Q_a^{(2)}\underline{1}'$$

Proceeding as in (i) we can prove (ii).

5.4 Numerical Illustrations

In this section we will consider examples for each model to validate the above-discussed results. Authors had developed a MATLAB® program that generates, for given values of k_1, d_1, c_1, k_2 and d_2, the involved sub matrices and computes various probabilities and the expected number of trials under different conditions. Here we will consider a l dependent trials discussed by Aki and Hirano (1999). The constants under the given conditions are given by

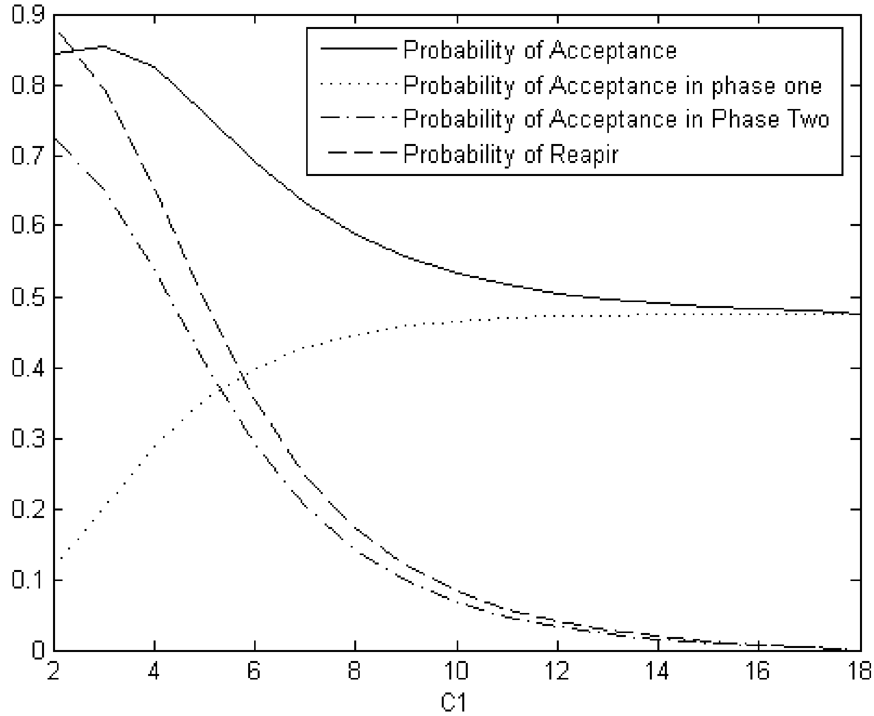
$$k_1=12;\, p_{ss}(t)=1\text{-}1/(t+1);\, p_{ff}(t)=1\text{-}1/(t^{(1/2)}+1);$$
$$k_2=5;\, d_2=8;\, p_{ss}(t)=1\text{-}(1/(t+1))^2;\, p_{ff}(t)=1\text{-}(t^{(1/3)}/(t+2));$$

with the initial distribution, in both phases, as equally likely in the states $(0,s,0)$ and $(0,f,0)$.

5.4.1 Model 1

First we consider the first model and obtain different probabilities for values of c_1 less than $d_1=18$. Figure 1 probability of acceptance in the test, probability of acceptance in phase one, phase two are depicted. Probabilities of repair for the product for various values of c_1 are also given.

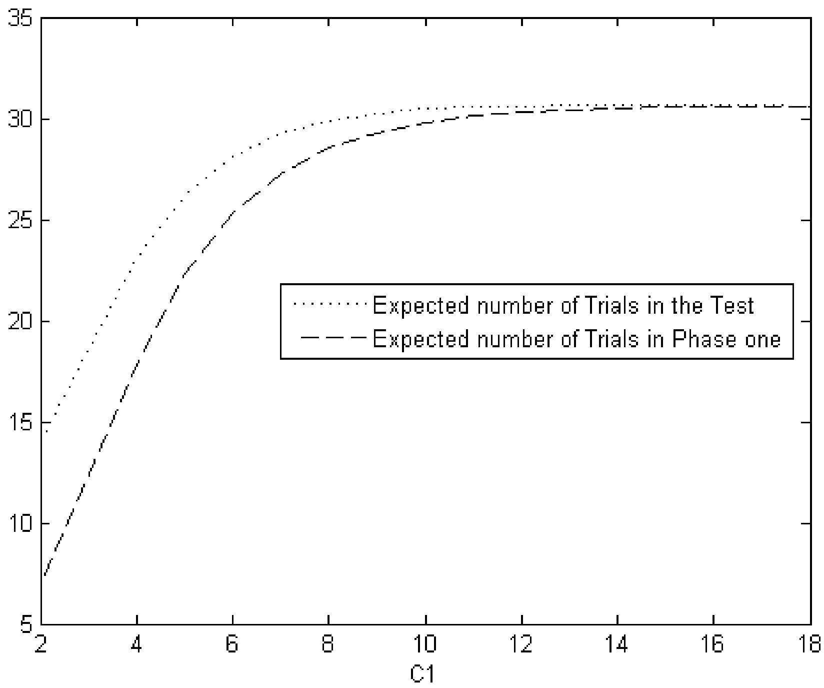
Figure 5.1 Values of different Probabilities for various values of c_1 *in the case of model 1*



It can be seen from the figure5.1 that as c_1 increases the probability of repair converges to zero and the Probability of acceptance of the product in the test and in phase one stabilize to the same limit.

In figure 5.2 expected number of trials required to terminate the test and in phase one are depicted for various values of c_1 .

Figure 5.2 Values of different Expected value measures for various values of c_1 in the case of Model 1



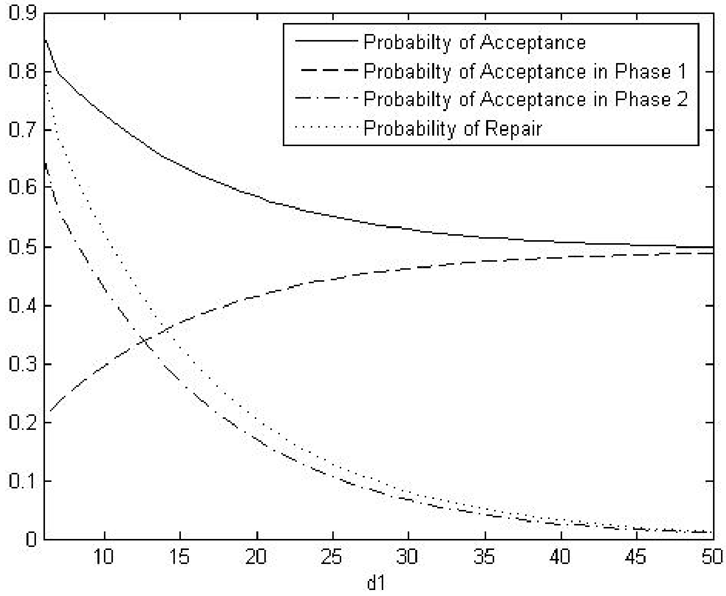
The expected number of trials in the test and in phase one, even though distant apart for small values of c will converge to the same limit as c increases. Also it can be seen that the expected number of trials will converge to a limit as the values of c_1 increases.

5.4.2 Model 2

Here we will obtain different probabilities for values of d_1 ranging from six to 50. For the given constant value of $c = 6$.

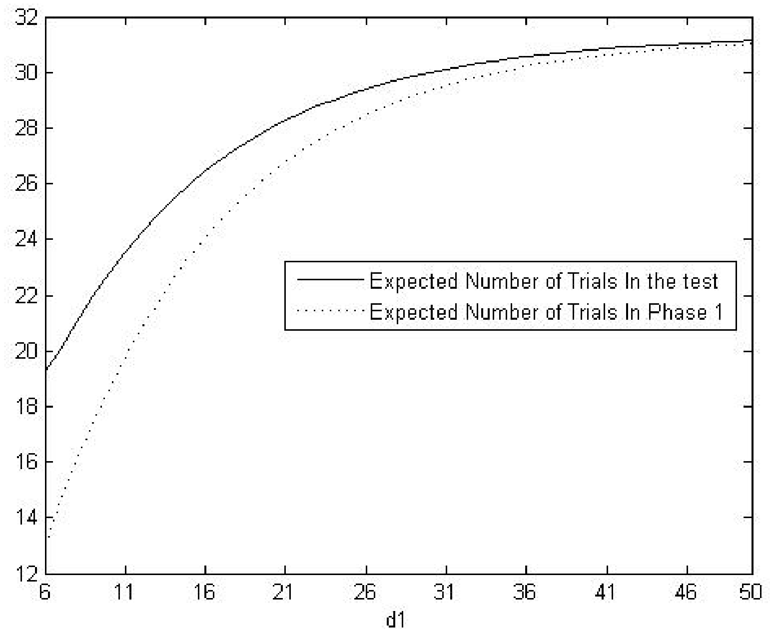
In figure 5.3 as in figure 1, plots of various probabilities against corresponding d_1 value is shown. The probability of interest include probability of acceptance, probability of acceptance in phase one, phase two and the probability of repair.

Figure 5.3 Values of different Probabilities for various values of c_1 in the case of model 2



As the value of d_1 increase the probability of repair as well as probability of acceptance in phase two converges to zero. Also Probability of acceptance in the test and in phase one converges to the same limit.

Figure 5.4 Values of different Expected value measures for various values of c_1 in the case of Model 2



It can be seen that as the value of d_1 increases the expected number of trials becomes convergent. It asserts the logical results that the number of trials in phases one becomes equal to the number of trials in the test.

5.5 Conclusion

In this chapter we proposed two new models of start-up demonstration having two phases with the condition for the corrective action in the first model being specified number of consecutive failures and in the second model being specified number of random failures.

Chapter 6
Distribution of Runs and Occurrence of each outcome[3]

6.1 Introduction

Runs find application in almost every field of human activity. In most of the run related problems we are interested in finding the distribution of runs alone. But in some of the problems we may need to find the joint distribution of runs as well as the number of occurrences of each type of outcome. For example consider a system with k causes for failure denoted by $1,2,...,k$. A probable question that may arise is whether the consecutive causes for failures are dependent or not. Let n_{ij} denote the number of failures with current cause of failure as j and the previous cause as i and n_i denote the number of failures due to cause i, $i, j = 1,2,...k$. Now it is interesting to derive the joint distribution of the variables $\left(n_{ij}, n_i, i, j = 1,2,...k\right)$. In this chapter we deal with these classes of problems.

6.2 Distribution of Runs and occurrence of each outcome

Let $Z_1, Z_2, ...$ be a sequence of outcomes in a multistate trial with m outcomes. Let $X_n = \left(X_n^{(1)}, X_n^{(2)}, ..., X_n^{(m)}\right)$ be the random vector denoting the number of runs in a sequence of n trials where $X_n^{(i)}$ denote the number of runs of i^{th} type in n trials. Also let $Y_n = \left(Y_n^{(1)}, Y_n^{(2)}, ..., Y_n^{(m)}\right)$, with $Y_n^{(i)}$ denote the occurrence of i^{th} type outcome in n trials, denote the random vector denoting the number of occurrence of each

[3] A part of this chapter is communicated in Nair and Thomas (2011)

outcomes. Our objective is to find the Joint distribution of (X_n, Y_n). We have the following definition.

Definition 1 (Han and Aki, 1999):

The random vector X_n is called a Markov chain embeddable vector of multinomial type (MVM), if

(1). there exists a Markov chain $\{\alpha_t, t \geq 0\}$ defined on a state space Ω,

(2). there exists a partition $\{C_x, x \geq 0\}$ on the state space Ω,

(3). for every x, $\Pr(X_n = x) = \Pr(\alpha_n \in C_x)$ and

(4). $\Pr(\alpha_t \in C_{x+x^*} \mid \alpha_{t-1} \in C_x) = 0$ if $x^* \neq 0$ or $x^* \neq e_k$, $k = 1, 2, \ldots m$.

Instead of the partition C_x, here we will partition as $C_{x,y}$. We assume that the cardinality of $C_{x,y}$ is β ie., $\beta = \left|C_{x,y}\right|$. Then we denote

$$C_{x,y} = \left\{C_{x,y;1}, C_{x,y;2}, \ldots, C_{x,y;\beta}\right\}.$$

Now associated with the above Markov chain we can associate two matrices given by

$$A_t^{(k)}(x, y) = \left(\Pr(\alpha_t \in C_{x,y+e_k;j} \mid \alpha_{t-1} \in C_{x,y;i})\right)_{\beta\times\beta} \quad k = 1, 2, \ldots m.$$

$$B_t^{(k,l)}(x, y) = \left(\Pr(\alpha_t \in C_{x+e_k,y+e_l;j} \mid \alpha_{t-1} \in C_{x,y;i})\right)_{\beta\times\beta} \quad k, l = 1, 2, \ldots m.$$

The matrix $A_t^{(k)}(x, y)$ corresponds to an increase in the number of occurrence of k^{th} type outcome where as $B_t^{(l,k)}(x, y)$ corresponds to an increase in both l^{th} type runs and k^{th} type outcome. Now we have the probability vector of the t^{th} step of the Markov chain as

$$f_t(x, y) = \left(\Pr\left(\alpha_t \in C_{x,y;1}\right), \Pr\left(\alpha_t \in C_{x,y;2}\right), \ldots \Pr\left(\alpha_t \in C_{x,y;\beta}\right)\right), 0 \leq x \leq l_t, 0 \leq y \leq n_t,$$

$$t = 0, 1, \ldots n.$$

where $l_n = \left(l_n^{(1)}, l_n^{(2)}, ..., l_n^{(m)}\right)$, $l_n^{(i)} = \max\left\{x^{(i)} : \Pr(X_n^{(i)} = x^{(i)}) > 0\right\}$,

$n_t = \left(n_t^{(1)}, n_t^{(2)}, ..., n_t^{(m)}\right)$, $n_t^{(i)} = \max\left\{y^{(i)} : \Pr(y_n^{(i)} = y^{(i)}) > 0\right\}$

Theorem 6.1: The sequence of probability vectors
$f_t(x, y), 0 \le x \le l_t, 0 \le y \le n_t, t = 0,1,...n$ satisfies

$$f_t(x, y) = \sum_{k=1}^{m} f_{t-1}(x, y - e_k) \underset{t}{A}^{(k)}(x, y - e_k) I(y - e_k \ge 0) +$$

$$\sum_{k,l=1}^{m} f_{t-1}(x - e_k, y - e_k) \underset{t}{B}^{(l,k)}(x - e_l, y - e_k) I(x - e_l \ge 0, y - e_k \ge 0). \tag{6.1}$$

where $I(P) = \begin{cases} 1 & \textit{if } P \textit{ is } \text{true} \\ 0 & \textit{otherwise} \end{cases}$

then the probability distribution function is given by $P(X_n = x, Y_n = y) = f_n(x, y)\underline{1}^{'}$.

Proof:

We can prove the above theorem as proceeding in the same line as Han and Aki (1999) done for the MVM.

The recurrence relation follows immediately as a consequence of the Chapman-Kolmogrov equations and from the form of the matrices $\underset{t}{A}^{(k)}(x, y)$ and $\underset{t}{B}^{(l,k)}(x, y)$, $k, l = 1, 2, ...m$.

Also $\Pr(X_n = x, Y_n = y) = \Pr(\alpha_n \in C_{x,y}) = \sum_{i=1}^{\beta} \Pr(\alpha_n \in U_{x,y;i}) = f_n(x, y)\underline{\mathbf{1}}^{'}$.
Hence the theorem.

Now we consider the case when $\underset{t}{A}^{(k)}(x, y)$ and $\underset{t}{B}^{(l,k)}(x, y)$ are independent of (x, y).

Let $z=(z_1,z_2,...,z_m)$, $z^x=z_1^{x_1}z_2^{x_2}...z_m^{x_m}$, $s=(s_1,s_2,...,s_m)$ and $s^y=s_1^{y_1}s_2^{y_2}...s_m^{y_m}$. Consider the probability generating function of the random vectors X_n and Y_n

$$\phi_n(z_1,z_2,...,z_m,s_1,....s_m)=\sum_{0\le x\le l_t}\sum_{0\le y\le n_t}\Pr(X_n=x,Y_n=y)z_1^{x_1}z_2^{x_2}...z_m^{x_m}s_1^{y_1}s_2^{y_2}....s_m^{y_m}$$

.We introduce $\chi_{(\delta)}^{(k)}=\left\{x:x^{(k)}=\delta;0\le x^{(j)}\le l_t^{(j)},\ j=1,2,...,k-1,k+1,...,m\right\}$

$$\eta_{(\delta)}^{(k)}=\left\{y:y^{(k)}=\delta;0\le y^{(j)}\le n,\ j=1,2,...,k-1,k+1,...,m\right\}\qquad k=1,2,...m.$$

Theorem 6.2: If $A_t^{(k)}(x,y)$ and $B_t^{(l,k)}(x,y)$ does not depend on (x,y), that is $A_t^{(k)}(x,y)=A_t^{(k)}$ and $B_t^{(l,k)}(x,y)=B_t^{(l,k)}$, $k=1,2,...m$ for all (x,y), we have

$$\phi_t(z,s)=\phi_{t-1}(z,s)\left(\sum_{k=1}^{m}s_k\left(A_t^{(k)}+\sum_{l=1}^{m}z_lB_t^{(l,k)}\right)\right)\tag{6.2}$$

Proof:

We have $\phi_t(z,s)=\sum_{0\le x\le l_t}\sum_{0\le y\le n_t}f_t(x,y)z^xs^y$. Using (6.1) we will have

$$\phi_t(z,s)=\sum_{0\le x\le l_t}\sum_{0\le y\le n_t}\sum_{k=1}^{m}\Big(f_{t-1}(x,y-e_k)A_t^{(k)}I(y-e_k\ge 0)+$$

$$\sum_{k=1}^{m}f_{t-1}(x-e_l,y-e_k)B_t^{(l,k)}I(x-e_l\ge 0,y-e_k\ge 0)\Big)z^xs^y$$

$$=\sum_{k=1}^{m}s_k\sum_{0\le x\le l_t}\sum_{0\le y\le n_t}f_{t-1}(x,y-e_k)A_t^{(k)}I(y-e_k\ge 0)z^xs^{y-e_k}+$$

$$\sum_{k,l=1}^{m}z_ls_k\sum_{0\le x\le l_t}\sum_{0\le y\le n_t}f_{t-1}(x-e_l,y-e_k)B_t^{(l,k)}I(x-e_l\ge 0,y-e_k\ge 0)z^{x-e_l}s^{y-e_k}$$

$$= \sum_{k=1}^{m} s_k \sum_{0\le x\le l_t} \sum_{0\le y\le n_t - e_k} f_{t-1}(x,y) A_t^{(k)} z^x s^y +$$

$$\sum_{k,l=1}^{m} z_l s_k \sum_{0\le x\le l_t - e_l} \sum_{0\le y\le n_t - e_k} f_{t-1}(x,y) B_t^{(l,k)} z^x s^y$$

Note that $l_t - l_{t-1}$ and $n_t - n_{t-1}$ is zero or a finite sum of different e_k's, $k = 1,2,...m$. Hence we have two cases viz., (1) $l_t - l_{t-1} = \mathbf{0}$ and $n_t - n_{t-1} = \sum_{j=1}^{r} e_{i_j}$

and (2) $l_t - l_{t-1} = \sum_{j=r_1}^{R_1} e_{i_j}$ and $n_t - n_{t-1} = \sum_{j=r_2}^{R_2} e_{i_j}$.

Case 1:

$$l_t - l_{t-1} = \mathbf{0} \text{ and } n_t - n_{t-1} = \sum_{j=1}^{r} e_{i_j}$$

$$\phi_t(z,s) = \sum_{k=1}^{m} s_k \sum_{0\le x\le l_{t-1}} \sum_{0\le y\le n_{t-1}+\sum_{j=1}^{r} e_{i_j} - e_k} f_{t-1}(x,y) A_t^{(k)} z^x s^y + \sum_{k,l=1}^{m} z_k s_k \sum_{0\le x\le l_{t-1}-e_l} \sum_{0\le y\le n_{t-1}\sum_{j=1}^{r} e_{i_j} - e_k} f_{t-1}(x,y) B_t^{(l,k)} z^x s^y \qquad (6.3)$$

$$\sum_{k=1}^{m} s_k \sum_{0\le x\le l_{t-1}} \sum_{0\le y\le n_{t-1}\sum_{j=1}^{r} e_{i_j} - e_k} f_{t-1}(x,y) A_t^{(k)} z^x s^y =$$

$$\left(\sum_{\rho=1}^{r} + \sum_{\rho=r+1}^{m}\right) s_{i_\rho} \sum_{0\le x\le l_{t-1}} \sum_{0\le y\le n_{t-1}+\sum_{j=1}^{r} e_{i_j} - e_{i_\rho}} f_{t-1}(x,y) A_t^{(i_\rho)} z^x s^y$$

$$=\sum_{\rho=1}^{r} s_{i_\rho} \sum_{0\le x\le l_{t-1}} \sum_{0\le y\le n_{t-1}+\sum_{j=1,j\ne\rho}^{r} e_{i_j}} f_{t-1}(x,y) \underset{t}{A}^{(i_\rho)} z^x s^y +$$

$$\sum_{\rho=r+1}^{m} s_{i_\rho} \sum_{0\le x\le l_{t-1}} \sum_{0\le y\le n_{t-1}+\sum_{j=1}^{r} e_{i_j}-e_k} f_{t-1}(x,y) \underset{t}{A}^{(i_\rho)} z^x s^y$$

$$=\sum_{\rho=1}^{r} s_{i_\rho} \sum_{0\le x\le l_{t-1}} \left[\sum_{0\le y\le n_{t-1}} f_{t-1}(x,y) \underset{t}{A}^{(i_\rho)} z^x s^y + \sum_{j=1,j\ne\rho}^{r} \sum_{y\in \eta^{i_j}_{\left(n^{i_\rho}_{t-1}+1\right)}} f_{t-1}(x,y) \underset{t}{A}^{(i_\rho)} z^x s^y \right]$$

$$+\sum_{\rho=r+1}^{m} s_{i_\rho} \sum_{0\le x\le l_{t-1}} \left[\sum_{0\le y\le n_{t-1}-e_{i_\rho}} f_{t-1}(x,y) \underset{t}{A}^{(i_\rho)} z^x s^y + \sum_{j=1}^{r} \sum_{y\in \eta^{i_j}_{\left(n^{i_\rho}_{t-1}+1\right)}} f_{t-1}(x,y) \underset{t}{A}^{(i_\rho)} z^x s^y \right] \tag{6.4}$$

But,

$$\sum_{0\le y\le n_{t-1}-e_{i_\rho}} f_{t-1}(x,y) \underset{t}{A}^{(i_\rho)} z^x s^y = \sum_{0\le y\le n_{t-1}} f_{t-1}(x,y) \underset{t}{A}^{(i_\rho)} z^x s^y - \sum_{y\in \eta^{i_\rho}_{\left(n^{i_\rho}_{t-1}\right)}} f_{t-1}(x,y) \underset{t}{A}^{(i_\rho)} z^x s^y \tag{6.5}$$

Now $\displaystyle\sum_{k,l=1}^{m} z_l s_k \sum_{0\le x\le l_t-e_l} \sum_{0\le y\le n_t-e_k} f_{t-1}(x,y) \underset{t}{B}^{(l,k)} z^x s^y$

$$=\sum_{l=1}^{m} \left((\sum_{\rho=1}^{r} + \sum_{\rho=r+1}^{m}) z_l s_{i_\rho} (\sum_{0\le x\le l_{t-1}} - \sum_{x\in \chi^{i_\rho}_{\left(l^{i_\rho}_{t-1}\right)}}) \sum_{0\le y\le n_t+\sum_{j=1}^{r} e_{i_j}-e_k} f_{t-1}(x,y) \underset{t}{B}^{(l,k)} z^x s^y \right)$$

$$=\sum_{l=1}^{m} z_l \left\{ \sum_{\rho=1}^{r} s_{i_\rho} \sum_{0\le x\le l_{t-1}} \left[\sum_{0\le y\le n_{t-1}} f_{t-1}(x,y) \underset{t}{B}^{(l,i_\rho)} z^x s^y + \sum_{y\in \eta^{i_\rho}_{\left(n^{i_\rho}_{t-1}+1\right)}} f_{t-1}(x,y) \underset{t}{B}^{(l,i_\rho)} z^x s^y \right] \right.$$

$$-\sum_{\rho=1}^{r} s_{i_\rho} \sum_{x\in \chi^{i_\rho}_{\left(l^{i_\rho}_{t-1}\right)}} \left[\sum_{0\le y\le n_{t-1}} f_{t-1}(x,y) \underset{t}{B}^{(l,i_\rho)} z^x s^y + \sum_{y\in \eta^{i_\rho}_{\left(n^{i_\rho}_{t-1}+1\right)}} f_{t-1}(x,y) \underset{t}{B}^{(l,i_\rho)} z^x s^y \right]$$

$$+\sum_{\rho=r+1}^{m} s_{i_\rho} \sum_{0\le x\le l_{t-1}} \left[\sum_{0\le y\le n_{t-1}} f_{t-1}(x,y) B_t^{(l,i_\rho)} z^x s^y + \sum_{y\in \eta^{i_\rho}_{\left(n^{i_\rho}_{t-1}+1\right)}} f_{t-1}(x,y) B_t^{(l,i_\rho)} z^x s^y \right]$$

$$\left. -\sum_{\rho=r+1}^{m} s_{i_\rho} \sum_{x\in \chi^{i_\rho}_{\left(l^{i_\rho}_{t-1}\right)}} \left[\sum_{0\le y\le n_{t-1}} f_{t-1}(x,y) B_t^{(l,i_\rho)} z^x s^y + \sum_{y\in \eta^{i_\rho}_{\left(n^{i_\rho}_{t-1}+1\right)}} f_{t-1}(x,y) B_t^{(l,i_\rho)} z^x s^y \right] \right\} \tag{6.6}$$

But $f_{t-1}(l_{t-1}, y) B_t^{(l,k)} = 0$ for every $l = 1,2,...m$.

$$f_{t-1}(x, n_{t-1}+e_k) A_t^{(k)} = 0, f_{t-1}(x, n_{t-1}+e_k) B_t^{(l,k)} = 0 \text{ for every } l,k = 1,2,..,m$$

Multiplying (6.3) by $\underline{\mathbf{1}}'$ and applying (6.4), (6.5) and (6.6) and putting s_k, $z_k = 1$, $k = 1,2,...m$ we have the theorem.

Case 2:

$$l_t - l_{t-1} = \sum_{j=r_1}^{R_1} e_{i_j} \text{ and } n_t - n_{t-1} = \sum_{j=r_2}^{R_2} e_{i_j}$$

$$\phi_t(z,s) = \sum_{k=1}^{m} s_k \sum_{0\le x\le l_t} \sum_{0\le y\le n_{t-1}+\sum_{j=1}^{r} e_{i_j} - e_k} f_{t-1}(x,y) A_t^{(k)} z^x s^y +$$

$$\sum_{k,l=1}^{m} z_l s_k \sum_{0\le x\le l_{t-1}+\sum_{j=1}^{r} e_{i_j} - e_l} \sum_{0\le y\le n_{t-1}\sum_{j=1}^{r} e_{i_j} - e_k} f_{t-1}(x,y) B_t^{(l,k)} z^x s^y \tag{6.7}$$

Consider the first term in equation (6.7),

$$\sum_{k=1}^{m} s_k \sum_{0\le x\le l_{t-1}+\sum_{j=1}^{r} e_{i_j}} \sum_{0\le y\le n_{t-1}+\sum_{j=1}^{r} e_{i_j} - e_k} f_{t-1}(x,y) A_t^{(k)} z^x s^y$$

$$= (\sum_{\rho=1}^{r_1} + \sum_{\rho=r_1+1}^{m}) s_{i_\rho} (\sum_{0\le x\le l_{t-1}} + \sum_{x\in \chi^{i_\rho}_{\left(l^{i_\rho}_{t-1}+1\right)}}) \sum_{0\le y\le n_{t-1}+\sum_{j=1}^{r} e_{i_j} - e_{i_\rho}} f_{t-1}(x,y) A_t^{(i_\rho)} z^x s^y$$

$$=\sum_{\rho=1}^{r_1} s_{i_\rho}\Big(\sum_{0\le x\le l_{t-1}} + \sum_{x\in\chi^{i_\rho}_{\left(l^{i_\rho}_{t-1}+1\right)}}\Big) \sum_{0\le y\le n_{t-1}+\sum_{j=1,j\neq\rho}^{r} e_{i_j}} f_{t-1}(x,y)\,A_t^{(i_\rho)} z^x s^y +$$

$$\sum_{\rho=r_1+1}^{m} s_{i_\rho}\Big(\sum_{0\le x\le l_{t-1}} + \sum_{x\in\chi^{i_\rho}_{\left(l^{i_\rho}_{t-1}+1\right)}}\Big) \sum_{0\le y\le n_{t-1}+\sum_{j=1}^{r} e_{i_j}-e_k} f_{t-1}(x,y)\,A_t^{(i_\rho)} z^x s^y$$

$$=\sum_{\rho=1}^{r_1} s_{i_\rho}\Big(\sum_{0\le x\le l_{t-1}} + \sum_{x\in\chi^{i_\rho}_{\left(l^{i_\rho}_{t-1}+1\right)}}\Big) \sum_{0\le y\le n_{t-1}} f_{t-1}(x,y)\,A_t^{(i_\rho)} z^x s^y +$$

$$\sum_{\rho=1}^{r_1} s_{i_\rho}\Big(\sum_{0\le x\le l_{t-1}} + \sum_{x\in\chi^{i_\rho}_{\left(l^{i_\rho}_{t-1}+1\right)}}\Big) \sum_{j=1,j\neq\rho}^{r_1} \sum_{y\in\chi^{(i_j)}_{2(n_{t-1}+1)}} f_{t-1}(x,y)\,A_t^{(i_\rho)} z^x s^y +$$

$$\sum_{\rho=r_1+1}^{m} s_{i_\rho}\Big(\sum_{0\le x\le l_{t-1}} + \sum_{x\in\chi^{i_\rho}_{\left(l^{i_\rho}_{t-1}+1\right)}}\Big) \sum_{0\le y\le n_{t-1}-e_{i_\rho}} f_{t-1}(x,y)\,A_t^{(i_\rho)} z^x s^y +$$

$$\sum_{\rho=r_1+1}^{m} s_{i_\rho}\Big(\sum_{0\le x\le l_{t-1}} + \sum_{x\in\chi^{i_\rho}_{\left(l^{i_\rho}_{t-1}+1\right)}}\Big) \sum_{j=1,j\neq\rho}^{r} \sum_{y\in\chi^{(i_j)}_{2(n_{t-1}+1)}} f_{t-1}(x,y)\,A_t^{(i_\rho)} z^x s^y$$

Since $f_{t-1}(x, n_{t-1}+e_k)\,A_t^{(k)}=0$, second and the fourth term in the above summation becomes identically equal to zero. Then we have

$$\sum_{k=1}^{m} s_k \sum_{0\le x\le l_{t-1}+\sum_{j=1}^{r} e_{i_j}} \;\sum_{0\le y\le n_{t-1}+\sum_{j=1}^{r} e_{i_j}-e_k} f_{t-1}(x,y)\,A_t^{(k)} z^x s^y$$

$$=\sum_{\rho=1}^{r_1} s_{i_\rho}\Big(\sum_{0\le x\le l_{t-1}} + \sum_{x\in\chi^{i_\rho}_{\left(l^{i_\rho}_{t-1}+1\right)}}\Big) \sum_{0\le y\le n_{t-1}} f_{t-1}(x,y)\,A_t^{(i_\rho)} z^x s^y +$$

$$\sum_{\rho=r_1+1}^{m} s_{i_\rho}\Big(\sum_{0\le x\le l_{t-1}} + \sum_{x\in\chi^{i_\rho}_{\left(l^{i_\rho}_{t-1}+1\right)}}\Big) \sum_{0\le y\le n_{t-1}-e_{i_\rho}} f_{t-1}(x,y)\,A_t^{(i_\rho)} z^x s^y \qquad (6.8)$$

Now let us consider the second term in equation (6.7)

$$
\sum_{k,l=1}^{m} z_l s_k \sum_{0\le x\le l_t-e_l}\ \sum_{0\le y\le n_t-e_k} f_{t-1}(x,y)\,B_t^{(l,k)} z^x s^y
$$

$$
=\sum_{l=1}^{m}\left(\Big(\sum_{k=1}^{r_1}+\sum_{k=r_1+1}^{m}\Big) z_l s_k \sum_{0\le x\le l_{t-1}+\sum_{p=1}^{r_2} e_{i_p}-e_l}\ \sum_{0\le y\le n_{t-1}+\sum_{j=1}^{r_1} e_{i_j}-e_k} f_{t-1}(x,y)\,B_t^{(l,k)} \right) z^x s^y
$$

$$
=\sum_{l=1}^{m}\left\{ \sum_{\rho=1}^{r} s_{i_\rho} \sum_{0\le x\le l_{t-1}+\sum_{j=1,j\ne\rho}^{r} e_{i_j}}\ \sum_{0\le y\le n_{t-1}+\sum_{j=1,j\ne\rho}^{r} e_{i_j}} f_{t-1}(x,y)\,B_t^{(l,i_\rho)} z^x s^y + \right.
$$

$$
\left. \sum_{\rho=r+1}^{m} s_{i_\rho} \sum_{0\le x\le l_{t-1}+\sum_{j=1}^{r} e_{i_j}-e_{i_\rho}}\ \sum_{0\le y\le n_{t-1}+\sum_{j=1}^{r} e_{i_j}-e_{i_\rho}} f_{t-1}(x,y)\,B_t^{(l,i_\rho)} z^x s^y \right\}.
$$

$$
=\sum_{l=1}^{m} z_l \left\{ \sum_{\rho=1}^{r} s_{i_\rho} \sum_{0\le x\le l_{t-1}}\ \sum_{0\le y\le n_{t-1}} f_{t-1}(x,y)\,B_t^{(l,i_\rho)} z^x s^y + \sum_{\rho=1}^{r} s_{i_\rho} \sum_{0\le x\le l_{t-1}}\ \sum_{j=1,j\ne\rho}^{r}\ \sum_{y\in\eta^{(i_j)}_{(n_{t-1}+1)}} f_{t-1}(x,y)\,B_t^{(l,i_\rho)} z^x s^y + \right.
$$

$$
\sum_{\rho=1}^{r} s_{i_\rho} \sum_{0\le y\le n_{t-1}}\ \sum_{j=1,j\ne\rho}^{r}\ \sum_{y\in\chi^{(i_j)}_{(n_{t-1}+1)}} f_{t-1}(x,y)\,B_t^{(l,i_\rho)} z^x s^y - \sum_{\rho=1}^{r} s_{i_\rho} \sum_{j=1,j\ne\rho}^{r}\ \sum_{x\in\chi^{i_j}_{\left(l^{i_j}_{t-1}+1\right)}}\ \sum_{y\in\eta^{(i_j)}_{\left(n^{i_j}_{t-1}+1\right)}} f_{t-1}(x,y)\,B_t^{(l,i_\rho)} z^x s^y +
$$

$$
\sum_{\rho=r+1}^{m} s_{i_\rho} \sum_{0\le x\le l_{t-1}-e_{i_\rho}}\ \sum_{0\le y\le n_{t-1}-e_{i_\rho}} f_{t-1}(x,y)\,B_t^{(l,i_\rho)} z^x s^y +
$$

$$
\sum_{\rho=r+1}^{m} s_{i_\rho} \sum_{0\le x\le l_{t-1}}\ \sum_{j=1,j\ne\rho}^{r}\ \sum_{y\in\eta^{(i_j)}_{(n_{t-1}+1)}} f_{t-1}(x,y)\,B_t^{(l,i_\rho)} z^x s^y +
$$

$$
\sum_{\rho=r+1}^{m} s_{i_\rho} \sum_{0\le y\le n_{t-1}}\ \sum_{j=1,j\ne\rho}^{r}\ \sum_{y\in\chi^{(i_j)}_{(n_{t-1}+1)}} f_{t-1}(x,y)\,B_t^{(l,i_\rho)} z^x s^y -
$$

$$
\left. \sum_{\rho=r+1}^{m} s_{i_\rho} \sum_{j=1,j\ne\rho}^{r}\ \sum_{x\in\chi^{i_j}_{\left(l^{i_j}_{t-1}+1\right)}}\ \sum_{y\in\eta^{(i_j)}_{\left(n^{i_j}_{t-1}+1\right)}} f_{t-1}(x,y)\,B_t^{(l,i_\rho)} z^x s^y \right\} \tag{6.9}
$$

But,

$$\sum_{\rho=r+1}^{m} s_{i_\rho} \sum_{0 \le x \le l_{t-1}-e_{i_\rho}} \sum_{0 \le y \le n_{t-1}-e_{i_\rho}} f_{t-1}(x,y) B_t^{(i_\rho)} z^x s^y = \sum_{\rho=r+1}^{m} s_{i_\rho} \sum_{0 \le x \le l_{t-1}} \sum_{0 \le y \le n_{t-1}} f_{t-1}(x,y) B_t^{(i_\rho)} z^x s^y -$$

$$\sum_{\rho=r+1}^{m} s_{i_\rho} \sum_{0 \le x \le l_{t-1}} \sum_{j=1, j \ne \rho}^{r} \sum_{y \in \eta_{(n_{t-1})}^{(i_j)}} f_{t-1}(x,y) B_t^{(i_\rho)} z^x s^y - \sum_{\rho=r+1}^{m} s_{i_\rho} \sum_{0 \le y \le n_{t-1}} \sum_{j=1, j \ne \rho}^{r} \sum_{y \in \chi_{(l_{t-1})}^{(i_j)}} f_{t-1}(x,y) B_t^{(i_\rho)} z^x s^y +$$

$$\sum_{\rho=r+1}^{m} s_{i_\rho} \sum_{j=1, j \ne \rho}^{r} \sum_{x \in \chi^{i_j}_{\left(l_{t-1}^{i_j}\right)}} \sum_{y \in \eta^{(i_j)}_{\left(n_{t-1}^{i_j}\right)}} f_{t-1}(x,y) B_t^{(i_\rho)} z^x s^y \qquad (6.10)$$

Now, it is worthwhile to note that $f_{t-1}(l_{t-1}+e_l, y) B_t^{(l,k)} = 0$, and split the summation based on l into two regions as above for the case of k.

Multiplying (6.7) by $\underline{\mathbf{1}}'$ and substituting (6.8), (6.9) and (6.10), s_k, $z_k = 1$, $k = 1, 2, \ldots m$, we have the theorem.

Hence in both the cases theorem holds.

From the theorem, when the transition probability matrices are independent of (x, y), we have

$$\phi_n(z,s) = \phi_0(z,s) \prod_{t=1}^{n} \left(\sum_{k=1}^{m} s_k A_t^{(k)} + \sum_{k=1}^{m} s_k z_k B_t^{(k)} \right) \underline{1}',$$

where $\phi_0(z,s) = \sum_{0 \le x \le l_0} \sum_{0 \le y \le n_0} f_0(x,y) z^x s^y$.

Corollary 6.1: Han and Aki (1999) derived the distribution of runs in the multi state trial. The same formula can be obtained by putting $s_k = 1$, $k = 1, 2, \ldots, m$ and let $\sum_{k=1}^{m} A_t^{(k)} = A_t$.

Corollary 6.2: Now put $z_k = 1,\ k = 1,2,...,m$ we will get the generating function corresponding to the multinomial distribution. In such cases the matrices $B^{(l,k)} = 0$, for every $k,l = 1,2,...m$. Then the generating function becomes $\phi_n(1,s) = \phi_0(1,s)\prod_{t=1}^{n}\left(\sum_{k=1}^{m} s_k A_t^{(k)}\right)\underline{1}'$.

Now we will consider the case when the transition rates are homogeneous. Throughout this paper we will use the following notations.

$D_1(z,s) = \sum_{k=1}^{m}\left(s_k\left(A^{(k)} + \sum_{l=1}^{m} z_l B^{(l,k)}\right)\right)$, $D = \sum_{k=1}^{m}\left(A^{(k)} + \sum_{l=1}^{m} B^{(l,k)}\right), B^{(\bullet,j)} = \sum_{l=1}^{m} B^{(l,j)}$ and $B^{(i,\bullet)} = \sum_{j=1}^{m} B^{(i,j)}$.

Now it will be interesting to find how the number of occurrence of each event and the number of runs are related. Let us define $Q_{X^{(i)}Y^{(j)}}(w) = \sum_{n=1}^{\infty} E\left(X_n^{(i)} Y_n^{j}\right) w^n$.

Theorem 6.3: If $A_t^{(k)}(x,y) = A^{(k)}$, $B_t^{(l,k)}(x,y) = B^{(l,k)}$ and $n_0 = 0, l_0 = 0$, we have

$$E\left(X_n^{(i)} Y_n^{(j)}\right) = \phi_0(1,1)\sum_{r=1}^{n}\left\{\sum_{p=1}^{r-1} D^{p-1} B^{(i,\bullet)} D^{r-p-1}\left(A^{(j)} + B^{(\bullet,j)}\right)\underline{1}' + \right.$$

$$\left.\sum_{q=1}^{n-r} D^{r-1}\left(A^{(j)} + B^{(\bullet,j)}\right) D^{q-1} B^{(i,\bullet)} 1' + D^{(r-1)} B^{(i,j)}\underline{1}'\right\}.$$

$$Q_{\left(X^{(i)},Y^{(i)}\right)}(w) = \frac{\phi_0(1,1)\left[I - wD\right]^{-1}}{(1-w)}\left\{B^{(i,\bullet)}\left[I - wD\right]^{-1}\left(A^{(i)} + B^{(\bullet,i)}\right) + \right.$$

$$\left. w^2\left(A^{(i)} + B^{(\bullet,i)}\right)\left[I - wD\right]^{-1} B^{(i,\bullet)} + wI\right\}\underline{1}'.$$

Proof:

We have

$$E\left(X_n^{(i)}Y_n^{(j)}\right)=\left.\frac{\partial^2\phi_n(z,s)}{\partial z_i\partial s_j}\right| z=1,s=1$$

$$=\left\{\frac{\partial^2\phi_0(z,s)}{\partial z_i\partial s_j}D_1^n(z,s)1'+\frac{\partial}{\partial z_i}\phi_0(z,s)\sum_{r=1}^{n}D_1^{r-1}(z,s)\left(A^{(j)}+z_jB^{(\bullet,j)}\right)D_1^{n-r}(z,s)\underline{1}'+\right.$$

$$\frac{\partial}{\partial s_j}\phi_0(z,s)\sum_{r=1}^{n}D_1^{r-1}(z,s)s_iB^{(i,\bullet)}D_1^{n-r}(z,s)+\phi_0(z,s)\sum_{r=1}^{n}\sum_{p=1}^{r-1}D_1^{p-1}(z,s)\sum_{k=1}^{m}\left(s_kB^{(i,k)}\right)D_1^{r-p-1}(z,s)$$

$$\left(A^{(j)}+\sum_{l=1}^{m}z_lB^{(l,j)}\right)D_1^{n-r}(z,s)\underline{1}'+\phi_0(z,s)\sum_{r=1}^{n}D_1^{r-1}(z,s)\left(A^{(j)}+\sum_{l=1}^{m}z_lB^{(l,j)}\right)\sum_{q=1}^{n-r}D_1^{q-1}(z,s)$$

$$\left.\left.\sum_{k=1}^{m}\left(s_kB^{(i,k)}\right)D^{n-r-q}(z,s)\underline{1}'+\phi_0(z,s)D_1^{r-1}(z,s)B^{(i,j)}D_1^{n-r}(z,s)\underline{1}'\right\}\right| z=1,s=1$$

Now when $n_0=0,\ \ l_0=0$, derivatives of $\phi_0(z,s)$ vanishes to zero.

Hence

$$E\left(X_n^{(i)}Y_n^{(j)}\right)=\phi_0(1,1)\sum_{r=1}^{n}\left\{\sum_{p=1}^{r-1}D^{p-1}B^{(i,\bullet)}D^{r-p-1}\left(A^{(j)}+B^{(\bullet,j)}\right)\underline{1}'+\right.$$

$$\left.\sum_{q=1}^{n-r}D^{r-1}\left(A^{(j)}+B^{(\bullet,j)}\right)D^{q-1}B^{(i,\bullet)}1'+D^{r-1}B^{(i,j)}\underline{1}'\right\} \tag{6.11}$$

Now consider $Q_{X^{(i)}Y^{(j)}}(w)=\sum_{n=1}^{\infty}E\left(X_n^{(i)}Y_n^{j}\right)w^n$

(6.12)

But

$$\sum_{n=1}^{\infty}\sum_{r=1}^{n}\sum_{p=1}^{r-1}D^{p-1}B^{(i,\bullet)}D^{r-p-1}\left(A^{(j)}+B^{(\bullet,j)}\right)w^n\underline{1}'=\sum_{p=1}^{\infty}\sum_{r=p+1}^{\infty}\sum_{n=r}^{\infty}D^{p-1}B^{(i,\bullet)}D^{r-p-1}\left(A^{(j)}+B^{(\bullet,j)}\right)w^n\underline{1}'$$

$$=\sum_{p=1}^{\infty}\sum_{r=p+1}^{\infty} D^{p-1}B^{(i,\bullet)}D^{r-p-1}\left(A^{(j)}+B^{(\bullet,j)}\right)\frac{w^{r}}{(1-w)}\underline{1}'$$

$$=\frac{[I-wD]^{-1}B^{(i,\bullet)}[I-wD]^{-1}\left(A^{(j)}+B^{(\bullet,j)}\right)\underline{1}'}{(1-w)} \tag{6.13}$$

and

$$\sum_{n=1}^{\infty}\sum_{r=1}^{n}\sum_{q=1}^{n-r} D^{r-1}\left(A^{(j)}+B^{(\bullet,j)}\right)D^{q-1}B^{(i,\bullet)}w^{n}\,\underline{1}'=\sum_{r=1}^{\infty}\sum_{q=1}^{\infty}\sum_{q=1}^{n-r} D^{r-1}\left(A^{(j)}+B^{(\bullet,j)}\right)D^{q-1}w^{n}B^{(i,\bullet)}\underline{1}'$$

$$=\frac{w^{2}[I-wD]^{-1}\left(A^{(j)}+B^{(\bullet,j)}\right)[I-wD]^{-1}B^{(i,\bullet)}\underline{1}'}{(1-w)} \tag{6.14}$$

Using equations (6.11), (6.13), (6.14) in (6.12), we have the result.

Remark 6.1: By changing the order of differentiation, we have a different looking but the same results.

$$E\left(X_{n}^{(i)}Y_{n}^{(j)}\right)=\phi_{0}(1,1)\sum_{r=1}^{n}\left\{\sum_{p=1}^{r-1} D^{p-1}\left(A^{(j)}+B^{(\bullet,j)}\right)D^{r-p-1}B^{(i,\bullet)}\underline{1}'+\right.$$

$$\left.\sum_{q=1}^{n-r} D^{r-1}B^{(i,\bullet)}D^{q-1}\left(A^{(j)}+B^{(\bullet,j)}\right)\underline{1}'+D^{(r-1)}B^{(i,j)}\underline{1}'\right\},$$

$$Q_{\left(X^{(i)},Y^{(i)}\right)}(w)=\frac{\phi_{0}(1,1)[I-wD]^{-1}}{(1-w)}\left\{\left(A^{(i)}+B^{(\bullet,i)}\right)[I-wD]^{-1}B^{(i,\bullet)}+\right.$$

$$\left.w^{2}B^{(i,\bullet)}[I-wD]^{-1}\left(A^{(i)}+B^{(\bullet,i)}\right)+wI\right\}\underline{1}'.$$

Theorem 6.4 : If $A_t^{(k)}(x,y) = A^{(k)}$, $B_t^{(l,k)}(x,y) = B^{(l,k)}$ and when $n_0 = 0$, we have

$$E\left(Y_n^{(j)}\right) = \phi_0(1,1)\sum_{r=1}^{n} D^{r-1}\left(A^{(j)} + B^{(\bullet,j)}\right)\underline{1}',$$

$$E\left(Y_n^{(i)}Y_n^{(j)}\right) = \phi_0(1,1)\sum_{r=1}^{n}\left\{\sum_{p=1}^{r-1} D^{p-1}\left(A^{(i)} + B^{(\bullet,i)}\right)D^{r-q-1}\left(A^{(j)} + B^{(\bullet,j)}\right)\underline{1}' + \sum_{q=1}^{n-r} D^{r-1}\left(A^{(j)} + B^{(\bullet,j)}\right)D^{q-1}\left(A^{(i)} + B^{(\bullet,i)}\right)\underline{1}'\right\}$$

Also

$$Q_{Y^{(i)}}(w) = \frac{w}{1-w}\phi_0(1,1)[I - wD]^{-1}\left(A^{(i)} + B^{(\bullet,j)}\right)\underline{1}'$$

$$Q_{Y^{(i)}Y^{(j)}}(w) = \frac{w}{1-w}\phi_0(1,1)[I - wD]^{-1}\left\{\left(A^{(i)} + B^{(\bullet,i)}\right)[I - wD]^{-1}\left(A^{(j)} + B^{(\bullet,j)}\right) + \left(A^{(j)} + B^{(\bullet,j)}\right)[I - wD]^{-1}\left(A^{(i)} + B^{(\bullet,i)}\right)\right\}\underline{1}'$$

Proof :

We have, $E\left(Y_n^{(j)}\right) = \frac{\partial}{\partial s_j}\phi_n(1,s)\Big|\, s = 1$

$$= \frac{\partial}{\partial s_j}\left[\phi_0(1,s)\left(\sum_{k=1}^{m} s_k\left(A^{(k)} + B^{(\bullet,k)}\right)\right)^n \underline{1}'\right]\Bigg|\, s = 1$$

$$= \frac{\partial}{\partial s_j}\phi_0(1,s)\left(\sum_{k=1}^{m} s_k\left(A^{(k)} + B^{(\bullet,k)}\right)\right)^n \underline{1}' + \phi_0(1,s)\sum_{r=1}^{n}\left[\left(\sum_{k=1}^{m} s_k\left(A^{(k)} + B^{(\bullet,k)}\right)\right)^{r-1}\left(A^{(j)} + B^{(\bullet,j)}\right)\left(\sum_{k=1}^{m} s_k\left(A^{(k)} + B^{(\bullet,k)}\right)\right)^{n-r}\underline{1}'\right]\Bigg|\, s = 1$$

But when $n_0 = 0$, $\frac{\partial}{\partial s_j}\phi_0(1,s) = 0$

Hence $E\left(Y_n^{(j)}\right) = \phi_0(1,1)\sum_{r=1}^{n} D^{r-1}\left(A^{(j)} + B^{(\bullet,j)}\right)\underline{1}'$.

$$E\left(Y_n^{(i)}Y_n^{(j)}\right) = \frac{\partial^2}{\partial s_i \partial s_j}\left[\phi_o(1,s)\left(\sum_{k=1}^{m} s_k\left(A^{(k)} + B^{(\bullet,k)}\right)\right)^n \underline{1}'\right]\Bigg| s = 1$$

$$= \frac{\partial^2}{\partial s_i \partial s_j}\left[\phi_0(1,s)\right] D_1^n(1,s)\underline{1}' + \frac{\partial \phi_0(1,s)}{\partial s_j}\sum_{r=1}^{n} D_1^{r-1}(1,s)\left(A^{(i)} + B^{(\bullet,i)}\right) D_1^{n-r}(1,s)\underline{1}' +$$

$$\frac{\partial \phi_0(1,s)}{\partial s_i}\sum_{r=1}^{n} D_1^{r-1}(1,s)\left(A^{(j)} + B^{(\bullet,j)}\right) D_1^{n-r}(1,s)\underline{1}' + \phi_0(1,s)\sum_{r=1}^{n}\left\{\sum_{p=1}^{r-1} D_1^{p-1}(1,s)\right.$$
$$\left(A^{(i)} + B^{(\bullet,i)}\right) D_1^{r-p-1}(1,s)\left(A^{(j)} + B^{(\bullet,j)}\right) D_1^{n-r}(1,s) + D_1^{r-1}(1,s)\left(A^{(j)} + B^{(\bullet,j)}\right)$$

$$\left.\sum_{q=1}^{n-r} D_1^{q-1}(1,s)\left(A^{(j)} + B^{(\bullet,j)}\right) D_1^{q-1}(1,s)\right\}\underline{1}'\Bigg| s = 1.$$

Under the assumption $n_0 = 0$, $\frac{\partial}{\partial s_j}\phi_0(1,s) = 0$ hence $\frac{\partial^2}{\partial s_j \partial s_i}\phi_0(1,s) = 0$

Hence

$$E\left(Y_n^{(i)}Y_n^{(j)}\right) = \phi_0(1,1)\sum_{r=1}^{n}\left\{\sum_{p=1}^{r-1} D^{p-1}\left(A^{(i)} + B^{(\bullet,i)}\right) D^{r-p-1}\left(A^{(j)} + B^{(\bullet,j)}\right)\underline{1}' +\right.$$

$$\left.\sum_{q=1}^{n-r} D^{r-1}\left(A^{(j)} + B^{(\bullet,j)}\right) D^{q-1}\left(A^{(i)} + B^{(\bullet,i)}\right)\underline{1}'\right\}.$$

Proceeding analogues to the second part of the theorem 6.3 we can prove the remaining results.

Hence we have the theorem.

Remark 6.2: By taking $s_k = 1,\ k = 1,2,...,m$ and proceeding as in the above theorem we will obtain the distributional properties of joint distribution of runs as derived by Han and Aki (1999).

Theorem 6.5: If $A_t^{(k)}(x,y) = A^{(k)}$ and $B_t^{(i,j)}(x,y) = B^{(x,y)},\ i,j,k = 1,2,...,m$ for all (x,y) and $t \ge 0$, then $\phi(z,s;w) = \phi_0(z,s)\left\{I - \left[\sum_{k=1}^{m} s_k A^{(k)} - \sum_{i=1}^{m}\sum_{j=1}^{m} s_i z_j B^{(i,j)}\right] w\right\}^{-1} \underline{1}'$

Proof: Under the conditions of the theorem,

$$\phi_n(z,s) = \phi_0(z,s)\left[\sum_{k=1}^{m} s_k A^{(k)} - \sum_{i=1}^{m}\sum_{j=1}^{m} s_i z_j B^{(i,j)}\right]^n \underline{1}'$$

Hence

$$\phi(z,s;w) = \sum_{n=1}^{\infty}\left\{\phi_0(z,s)\left[\sum_{k=1}^{m} s_k A^{(k)} - \sum_{i=1}^{m}\sum_{j=1}^{m} s_i z_j B^{(i,j)}\right]^n \underline{1}' w^n\right\}$$

$$= \phi_0(z,s)\sum_{n=1}^{\infty}\left\{\left[\sum_{k=1}^{m} s_k A^{(k)} - \sum_{i=1}^{m}\sum_{j=1}^{m} s_i z_j B^{(i,j)}\right]^n w^n \underline{1}'\right\}$$

$$= \phi_0(z,s)\left\{I - \left[\sum_{k=1}^{m} s_k A^{(k)} - \sum_{i=1}^{m}\sum_{j=1}^{m} s_i z_j B^{(i,j)}\right] w\right\}^{-1} \underline{1}'$$

Hence the theorem.

6.4 Distributions Associated with Waiting Times

Let $T_r^{(1)}, r = \left(r^{(1)}, r^{(2)},...,r^{(m)}\right)$ denote the sooner waiting times associated with the runs i.e. $T_r^{(1)}$ denote the number of trials required for the occurrence of i^{th} run $r^{(i)}$

times for any $i=1,2,...,m$. Also let $T_r^{(2)},\ r=\left(r^{(1)},r^{(2)},...,r^{(m)}\right)$ denote the later waiting times associated with the runs i.e., the minimum number of trials required for the occurrence of i^{th} run at least $r^{(i)}\ \forall\ i=1,2,...,m$ times and let $T_r^{(3)},\ r=\left(r^{(1)},r^{(2)},...,r^{(m)}\right)$ denote the waiting time till the occurrence of each run exactly $r^{(i)}, i=1,2,...,m$ times. Now let

$$h_r^{(i)}(n,y)=P\left(T_r^{(i)}=n,Y_n=y\right),\qquad y\geq 0\qquad i=1,2,3.$$

Theorem 6.6: The joint probability mass function $\left(T_r^{(i)},Y_{T_r^{(i)}}\right),\ i=1,2,3$ can be expressed as

i. $$h_r^{(1)}(n,y)=\sum_{i,j=1}^{m}\sum_{x^i\in\Psi^i}\sum_{k=1}^{\beta}\mu_k^{(i,j)}(x,y)f_{n-1}\left(x-e_i,y-e_j\right)e_k^{'}$$

ii. $$h_r^{(2)}(n,y)=\sum_{i,j=1}^{m}\sum_{x^i\in\Upsilon^i}\sum_{k=1}^{\beta}\mu_k^{(i,j)}(x,y)f_{n-1}\left(x-e_i,y-e_j\right)e_k^{'}.$$

iii. $$h_r^{(3)}(n,y)=\sum_{i,j=1}^{m}\sum_{k=1}^{\beta}\mu_k^{(i,j)}(r,y)f_{n-1}\left(r-e_i,y-e_j\right)e_k^{'}.$$

where , $\Psi^i=\left\{x:x^{(j)}<r^{(j)},j\neq i,x^{(i)}=r^i\right\}$, $\Upsilon^i=\left\{x:x^{(j)}>r^{(j)},x^{(i)}=r^i,j\neq i\right\}$ and

$\mu_k^{(i,j)}(x,y)=e_kB_n^{(i,j)}\left(x-e_i,y-e_j\right)1'$.

Proof:

To prove (i), we have

$$h_r^{(1)}(n,y)=P\left(T_r^{(1)}=n,Y_n=y\right)$$

Let us assume that the runs attain the value r for the first time with the occurrence of i^{th} type outcome. Also let $T_{r,i}^{(1)}$ denotes the waiting time for such an outcome to occur. Then

$$h_r^{(1)}(n,y)=\sum_{i=1}^{m}P\left(T_{r,i}^{(1)}=n,Y_n=y\right).$$

Let $x^i=\left(x^{(1)},x^{(2)},...,r^{(i)},...,x^{(m)}\right),\ i=1,2,...,m$ denote the outcome that attains the number of occurrence of runs equal to r only at i^{th} position i.e., $x^{(j)}\neq r^{(j)},\ x^{(i)}=r^{(i)}\ and\ j\neq i$. Then

$$h_r^{(1)}(n,y)=\sum_{i=1}^{m}\sum_{x^i\in\Psi^i}P\left(X_n=x^i,X_{n-1}=x^i-e_i,Y_n=y\right)$$

$$=\sum_{i=1}^{m}\sum_{x^i\in\Psi^i}\sum_{j=1}^{m}P\left(X_n=x^i,X_{n-1}=x^i-e_i,Y_n=y,Y_{n-1}=y-e_j\right)$$

Also let $J=\left[X_n=x^i,Y_n=y\right]$,

$$J_{i,j}=\left[X_{n-1}=x^i-e_i,Y_{n-1}=y-e_j\right]=\bigcup_{k=1}^{\beta}\left(\alpha_{n-1}\in C_{x-e_i,y-e_j,k}\right),$$

$$P\left(J\,\middle|\,J_{i,j,k}\right)=e_kB_n^{(i,j)}\left(x-e_i,y-e_j\right)1'=\mu_k^{(i,j)}(x,y)\text{ and}$$

$$P\left(J_{i,j,k}\right)=f_{n-1}\left(x-e_i,y-e_j\right)e_k'$$

Then $h_r^{(1)}(n,y)=\sum_{i=1}^{m}\sum_{x^i\in\Psi^i}\sum_{j=1}^{m}P\left(J\bigcap J_{i,j,n}\right)$

$$=\sum_{i,j=1}^{m}\sum_{x^i\in\Psi^i}\sum_{k=1}^{\beta}\mu_k^{(i,j)}(x,y)f_{n-1}\left(x-e_i,y-e_j\right)e_k'$$

Thus we get (i).

Proceeding as in (i) we can easily prove (ii) and (iii).

Now let us consider the double generating functions corresponding to the above waiting time distributions. Let us denote

$$H^{(i)}(z,s;w)=\sum_{r=1}^{\infty}\sum_{n=0}^{\infty}\sum_{y=0}^{n}h_r^{(i)}(n,y)\,z^n s^y w^r \qquad i=1,3$$

Theorem 6.7: If $A_t^{(k)}(x,y)=A^{(k)}$ and $B_t^{(i,j)}(x,y)=B^{(i,j)}$, $i,j,k=1,2,...,m$ for all (x,y). Then the following results hold

(i) $H^{(1)}(z,s;w)=zw\phi_0(w,s)\sum_{i,j=1}^{m}\dfrac{\mu^{(i,j)}w_i s_j}{\prod_{l=1,l\neq i}^{m}(1-w_l)}\sum_{k=1}^{\beta}\left\{\left[I-D_1(w,s)z\right]^{-1}\right\}e_k^{'}$

(ii) $H^{(3)}(z,s;w)=z\phi_0(w,s)\sum_{i,j=1}^{m}\mu^{(i,j)}w_i s_j\sum_{k=1}^{\beta}\left\{\left[I-D_1(w,s)z\right]^{-1}\right\}e_k^{'}$ where $D_1(w,s)$ is defined as above.

Proof:

(i) Under the conditions of the theorem we have $\mu_k^{(i,j)}(x-e_i,y-e_j)=\mu^{(i,j)}$.

$$H^{(1)}(z,s;w)=\sum_{r=1}^{\infty}\sum_{n=0}^{\infty}\sum_{y=0}^{n}h_r^{(1)}(n,y)\,z^n s^y w^r$$

$$=\sum_{r=1}^{\infty}\sum_{n=0}^{\infty}\sum_{y=0}^{n}\sum_{i,j=1}^{m}\sum_{x^i\in\Psi^i}\sum_{k=1}^{\beta}\mu_k^{(i,j)}(x-e_i,y-e_j)f_{n-1}(x-e_i,y-e_j)e_k^{'}z^n s^y w^r$$

$$=\sum_{i,j=1}^{m}\sum_{k=1}^{\beta}\mu_k^{(i,j)}\sum_{n=0}^{\infty}\sum_{y=0}^{n}\sum_{r=1}^{\infty}\sum_{x^i\in\Psi^i}f_{n-1}(x-e_i,y-e_j)e_k^{'}z^n s^y w^r$$

$$= \sum_{i,j=1}^{m} \sum_{k=1}^{\beta} \mu_k^{(i,j)} w_i s_j \sum_{n=0}^{\infty} \frac{\phi_{n-1}(w,s)}{\prod_{l=1, l\neq i}^{m} (1-w_l)} z^n e_k^{'} .$$

But $\phi_n(z,s) = \phi_0(z,s) \left[\sum_{k=1}^{m} s_k \left(A^{(k)} + \sum_{j=1}^{m} z_j B^{(j,k)} \right) \right]^n = \phi_0(z,s) D_1^n(z,s)$

$$H^{(1)}(z,s;w) = \sum_{i,j=1}^{m} \mu^{(i,j)} w_i s_j \sum_{k=1}^{\beta} \sum_{n=0}^{\infty} \frac{\phi_0(w,s) D_1^{n-1}(s,z)}{\prod_{l=1, l\neq i}^{m} (1-w_l)} z^n e_k^{'}$$

Thus we get,

$$H^{(1)}(z,s;w) = z\phi_0(w,s) \sum_{i,j=1}^{m} \frac{\mu^{(i,j)} w_i s_j}{\prod_{l=1, l\neq i}^{m} (1-w_l)} \sum_{k=1}^{\beta} \left\{ \left[I - D_1(s,z) z \right]^{-1} 1' \right\} e_k^{'}$$

(ii) can be proved similarly.

Now it is interesting to find the waiting time for the occurrence of specified number of particular outcome. Let $V_p^{(1)}, \, p = \left(p^{(1)}, p^{(2)}, ..., p^{(m)} \right)$ denote the sooner waiting times associated with the occurrence of outcomes that is $V_p^{(1)}$ denote the number of trials required for the occurrence of any of the outcome $p^{(i)}, i = 1, 2, ..., m$ times. Also let $V_p^{(2)}, p = \left(p^{(1)}, p^{(2)}, ..., p^{(m)} \right)$ denote the later waiting times associated with the occurrence of outcomes i.e., the minimum number of trials required for the occurrence of each outcome at least $p^{(i)}, i = 1, 2, ..., m$ times and let $V_p^{(3)}, p = \left(p^{(1)}, p^{(2)}, ..., p^{(m)} \right)$ denote the waiting time till the occurrence of each outcome exactly $p^{(i)}, i = 1, 2, ..., m$ times. Now let

$$g_p^{(i)}(x,n) = P\left(X_{V_p^{(i)}} = x, V_p^{(i)} = n \right), \qquad x \geq 0 \qquad i = 1, 2, 3$$

Theorem 6.8: The joint probability mass function $\left(X_{V_p^{(i)}}, V_p^{(i)}\right), i=1,2,3$ can be expressed as

1. $$g_p^{(1)}(x,n)=\sum_{i,j=1}^{m}\sum_{x^i\in\Omega^i}\sum_{k=1}^{\beta}\pi_k^{(j,i)}(x,y)f_{n-1}\left(x-e_j, y-e_i\right)e_k^{'}.$$

2. $$g_p^{(2)}(x,n)=\sum_{i,j=1}^{m}\sum_{x^i\in\Theta^i}\sum_{k=1}^{\beta}\pi_k^{(j,i)}(x,y)f_{n-1}\left(x-e_i, y-e_j\right)e_k^{'}.$$

3. $$g_p^{(3)}(x,n)=\sum_{i,j=1}^{m}\sum_{k=1}^{\beta}\pi_k^{(j,i)}(x,y)f_{n-1}\left(x-e_i, y-e_j\right)e_k^{'}.$$

where $\Omega^i=\left\{y: y^{(i,j)}<p^{(j)}, j\neq i, y^{(i,i)}=p^{(i)}\right\}$,

$\Theta^i=\left\{y: y^{(i,j)}\geq p^{(j)}, j\neq i, y^{(i,i)}=p^{(i)}\right\}$ and

$\pi_k^{(j,i)}(x,y)=e_k B_n^{(j,i)}\left(x-e_j, y-e_i\right)1'+e_k A_n^{(i)}(x, y-e_i)1'$.

Proof:

(i) We have $g_p^{(1)}(x,n)=P\left(X_{V_p^{(1)}}=x, V_p^{(1)}=n\right)$

Let us assume that the occurrence of i^{th} outcome attains the value p for the first time i.e., the number of occurrence of j^{th} type outcome is less than $p^{(j)}$ $\forall j\neq i$. Also let $V_{p,i}^{(1)}$ denotes the waiting time for such an outcome to occur. Then $g_p^{(1)}(x,n)=\sum_{i=1}^{m}P\left(X_n=x, V_{p,i}^{(1)}=n\right)$.

Let $y^i=\left(y^{(1)}, y^{(2)},..., p^{(i)},..., y^{(m)}\right), i=1,2,...,m$ denote the outcome that attains the value equal to p only at i^{th} position that is,

$$y^{(i,j)} \neq p^{(j)},\ \ j \neq i\ \ and\ \ \mathrm{y}^{(i,i)} = p^{(i)}.$$

Then

$$g_p^{(1)}(x,n) = \sum_{i=1}^{m} \sum_{y^i \in \Omega^i} P\left(X_n = x, Y_{n-1} = y^i - e_i, Y_n = y^i\right)$$

$$= \sum_{i=1}^{m} \sum_{y^i \in \Omega^i} \left\{ \sum_{j=1}^{m} P\left(X_n = x, X_{n-1} = x - e_j, Y_n = y^i, Y_{n-1} = y^i - e_i\right) + \right.$$

$$\left. P\left(X_n = x, X_{n-1} = x, Y_n = y^i, Y_{n-1} = y^i - e_i\right)\right\}$$

Let $K = \left[X_n = x, Y_n = y^i\right]$,

$K_{j,i} = \left[X_{n-1} = x - e_j, Y_{n-1} = y^i - e_i\right] = \bigcup_{k=1}^{\beta}\left(\alpha_{n-1} \in C_{x-e_j, y-e_i, k}\right)$,

$K_{j,i,k} = \left[\alpha_{n-1} \in C_{x-e_j, y-e_i, k}\right], \qquad j = 0,1,...m$ with $e_0 = 0$

Also $P\left(K \middle| K_{j,i,k}\right) = \begin{cases} e_k B_n^{(j,i)}\left(x - e_j, y - e_i\right)\underline{1}', & j = 1,2,...m \\ e_k A_n^{(i)}\left(x, y - e_i\right)\underline{1}' & j = 0 \end{cases}$ and

$$P\left(K_{j,i,k}\right) = f_{n-1}\left(x - e_j, y - e_i\right)e_k^{'}$$

$$g_p^{(1)}(n,y) = \sum_{i=1}^{m} \sum_{x^i \in \Psi^i} \left\{ \sum_{j=1}^{m} \sum_{k=1}^{\beta} P\left(K \bigcap K_{j,i,k}\right) + P\left(K \bigcap K_{0,i,k}\right)\right\}$$

$= \sum_{i=1}^{m} \sum_{x^i \in \Psi^i} \sum_{k=1}^{\beta} \left\{ \sum_{j=1}^{m} e_k B_n^{(j,i)}\left(x - e_j, y - e_i\right)\underline{1}' + e_k A_n^{(i)}\left(x, y - e_i\right)\underline{1}' \right\} f_{n-1}\left(x - e_i, y - e_j\right)e_k^{'}$. Let us denote $\pi_k^{(j,i)}(x,y) = e_k B_n^{(j,i)}\left(x - e_j, y - e_i\right)\underline{1}' + e_k A_n^{(i)}\left(x, y - e_i\right)\underline{1}'$ then the above equation reduces to $g_p^{(1)}(x,n) = \sum_{i,j=1}^{m} \sum_{x^i \in \Omega^i} \sum_{k=1}^{\beta} \pi_k^{(j,i)}(x,y) f_{n-1}\left(x - e_j, y - e_i\right)e_k^{'}$.

Similarly we can prove the other two cases.

Now we define the double generating function corresponding to the above variables as

$$G^{(i)}(z,s;w)=\sum_{p=1}^{\infty}\sum_{n=0}^{\infty}\sum_{x=0}^{\infty}g_p^{(i)}(x,n)z^n s^x w^p \qquad i=1,3$$

Theorem 6.9: If $A_t^{(k)}(x,y)=A^{(k)}$ and $B_t^{(i,j)}(x,y)=B^{(i,j)}$, $i,j,k=1,2,...,m$ for all (x,y). Then the following results hold

(i) $$G^{(1)}(z,s;w)=z\phi_0(w,s)\sum_{i,j=1}^{m}\frac{\mu^{(i,j)}w_i s_j}{\prod_{l=1,l\neq i}^{m}(1-w_l)}\sum_{k=1}^{\beta}\left\{\left[I-\left(\sum_{k=1}^{m}s_k A^{(k)}-\sum_{i=1}^{m}\sum_{j=1}^{m}s_i z_j B^{(i,j)}\right)z\right]^{-1}\right\}e_k^{'}$$

(ii) $$G^{(3)}(z,s;w)=z\phi_0(w,s)\sum_{i,j=1}^{m}\mu^{(i,j)}w_i s_j\sum_{k=1}^{\beta}\left\{\left[I-\left(\sum_{k=1}^{m}s_k A^{(k)}-\sum_{i=1}^{m}\sum_{j=1}^{m}s_i z_j B^{(i,j)}\right)z\right]^{-1}\right\}e_k^{'}$$

Proof:

(i) Under the conditions of the theorem we have $\mu_k^{(i,j)}(x-e_i,y-e_j)=\mu_k^{(i,j)}$

$$G^{(1)}(z,s;w)=\sum_{p=1}^{\infty}\sum_{n=0}^{\infty}\sum_{x=0}^{\infty}g_p^{(1)}(x,n)z^n s^x w^p$$

$$=\sum_{p=1}^{\infty}\sum_{n=0}^{\infty}\sum_{x=0}^{n}\sum_{i=1}^{m}\sum_{x^i\in\Psi^i}\sum_{k=1}^{\beta}\left\{\sum_{j=1}^{m}e_k B^{(j,i)}\underline{1}'f_{n-1}(x-e_j,y-e_i)+e_k A^{(i)}\underline{1}'f_{n-1}(x,y-e_i)\right\}e_k^{'}z^n s^x w^p$$

$$=\sum_{i,j=1}^{m}\mu^{(i,j)}\sum_{n=0}^{\infty}\sum_{y=0}^{n}\sum_{r=1}^{\infty}\sum_{x^i\in\Psi^i}f_{n-1}(x-e_i,y-e_j)e_k^{'}z^n s^y w^r$$

$$=\sum_{i,j=1}^{m}\sum_{k=1}^{\beta}\mu_k^{(i,j)}w_i s_j\sum_{n=0}^{\infty}\frac{\phi_{n-1}(w,s)}{\prod_{l=1,l\neq i}^{m}(1-w_l)}z^n e_k^{'}\,.$$

But $\phi_n(z,s)=\phi_0(z,s)\left[\sum_{k=1}^{m}s_kA^{(k)}+\sum_{i=1}^{m}\sum_{j=1}^{m}s_iz_jB^{(i,j)}\right]^n=\phi_0(z,s)D_1^n(z,s)$.

$$G^{(1)}(z,s;w)=\sum_{i,j=1}^{m}\mu^{(i,j)}w_is_j\sum_{k=1}^{\beta}\sum_{n=0}^{\infty}\frac{\phi_0(w,s)D_1^{n-1}(z,s)}{\prod_{l=1,l\neq i}^{m}(1-w_l)}z^ne_k^{'}$$

$$=z\phi_0(w,s)\sum_{i,j=1}^{m}\frac{\mu^{(i,j)}w_is_j}{\prod_{l=1,l\neq i}^{m}(1-w_l)}\sum_{k=1}^{\beta}\left\{\left[I-\left(\sum_{k=1}^{m}s_kA^{(k)}-\sum_{i=1}^{m}\sum_{j=1}^{m}s_iz_jB^{(i,j)}\right)z\right]^{-1}\underline{1}'\right\}e_k^{'}$$

Proceeding on the same line as (i) we can prove (ii).

6.5 Illustrations

Here, as an illustration to the above mentioned method, we obtain the transition matrices for the distribution of runs and occurrence of each outcome under various ways of counting. Commonly used counting schemes of runs and the corresponding random variables with respect to usual Bernoulli trials are as follows.

- $X_{n,k,\mu}$ denote the number of μ overlapping success runs of length k in n Markov dependent Bernoulli trials(Aki and Hirano 2000).
- $G_{n,k}$ denote the number of success runs of length greater than or equal to k in n Markov dependent Bernoulli trials.
- $E_{n,k}$ denote the number of success runs of length exactly equal to k in n Markov dependent Bernoulli trials (Mood,1940).

6.4.1 *Joint distribution of* $\mu=\left(\mu^{(1)},\mu^{(2)},...,\mu^{(m)}\right)$ *overlapping runs of length* $k=(k_1,k_2,...,k_m)$ *(Aki and Hirano Sense) and the number of occurrence of each outcome.*

Let $X_{n,k,\mu}^{(i)}$ denote the number of μ^i overlapping i^{th} type run of length k_i and $Y_n^{(i)}$ be the number of occurrence of the i^{th} type outcome in n Markov dependent multistate trials. Clearly the upper end points of $X_{n,k,\mu}^{(i)}$ is given by $l^{(i)}{}_n = \left[(n-\mu_i)/(k_i-\mu_i)\right]$ and the random vector $\underline{X}_{n,k,\mu}$ can be treated as a Markov chain Embeddable Variable under the below transformations. Let $\{\alpha_t, t \ge 0\}$ is Markov chain defined on the state space Ω.

Now we partition the state space Ω i.e., $\Omega = \bigcup_{\underline{x}_n, \underline{y}_n \ge 0} C_{\underline{x}_n, \underline{y}_n}$ where

$$C_{\underline{x}_n} = \left\{ \bigcup_{i=1}^{m} (\underline{x}_n, \underline{y}_n; u_i, i) : 0 \le u_i \le k_i - 1 \right\} \qquad \underline{x}_n \ge 0$$

where $\underline{x}_n = \left\{ x_n^{(1)}, x_n^{(2)}, \ldots x_n^{(m)} \right\}$ denote the realizations $\underline{X}_{n,k,\mu}$ and $\underline{y}_n = \left\{ y_n^{(1)}, y_n^{(2)}, \ldots, y_n^{(m)} \right\}$ be a realization of the vector Y_n. Let b denote number of trailing identical symbols in the observed sequence and $u_i = b \bmod (k_i - \mu_i)\ \forall i \in W$.

Now the within state transition probabilities are given by

$$P\left(\alpha_{t+1} \in C_{\underline{x}, \underline{y}+e_j; u_j+1, j} \mid \alpha_t \in C_{\underline{x}, \underline{y}; u_j, j}\right) = p_{jj} \quad 0 \le u_j \le k_i - 1 \ \ , i, j \in W$$

$$P\left(\alpha_{t+1} \in C_{\underline{x}, \underline{y}+e_i; 1, i} \mid \alpha_t \in C_{\underline{x}, \underline{y}; u_j, j}\right) = p_{ji} \quad 0 \le u_j \le k_j - 1, i \ne j, i, j \in W$$

And the between state transition probabilities are given by

$$P\left(\alpha_t \in C_{\underline{x}+e_i, \underline{y}+e_i; l_i, i} \mid \alpha_t \in C_{\underline{x}, \underline{y}; k_i-1, i}\right) = p_{ii} \quad i \in W$$

where e_i denotes a m dimensional row vector i^{th} with element unity and all other elements as zeros.

$A_t^{(k)}$ is a matrix whose k^{th} column is given by

$$\left(\nu_{1,k}(t), \nu_{2,k}(t), \ldots, \nu_{k-1,k}(t), \theta_k(t), \nu_{k+1,k}(t), \ldots, \nu_{m,k}(t)\right)$$

where $\theta_i(t) = \begin{array}{c} (0,i) \\ (1,i) \\ . \\ (k_i-2,i) \\ (k_i-1,i) \end{array} \begin{bmatrix} 0 & p_{ii}(t) & 0 & . & 0 \\ 0 & 0 & p_{ii}(t) & . & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & . & p_{ii}(t) \\ 0 & 0 & 0 & . & 0 \end{bmatrix}_{k_i \times k_i}$

and

$$v_{ij}(t) = \begin{bmatrix} 0 & p_{ij}(t) & 0 & . & 0 \\ 0 & p_{ij}(t) & 0 & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & p_{ij}(t) & 0 & . & 0 \end{bmatrix}_{k_i \times k_j} \quad \forall \; i \neq j \qquad i, j \in W$$

The between state transition probability matrices $B_t^{(i,i)}(\underline{x}) = B_t^{(i,i)}$ will be a $\sum_{i=1}^{m} k_i \times \sum_{i=1}^{m} k_i$ have all their entries zero except at $\left(\sum_{j=1}^{i} k_j, \sum_{j=1}^{i-1} k_j + \mu_i + 1\right)$ where we have $p_{ii}(t)$.

Remark 6.3 : $N_{n,k}$ denote the number of non-overlapping (Feller's counting) success runs of length equal to k in n Markov dependent Bernoulli trials. $M_{n,k}$ denote the number of overlapping (Ling's counting) success runs of length equal to k in n Markov dependent Bernoulli trials. It can be seen that both these counting comes as special cases of the above discussed counting scheme. $\mu = \left(\mu^{(1)}, \mu^{(2)}, ..., \mu^{(m)}\right)$ overlapping runs of length $k = (k_1, k_2, ..., k_m)$ reduces to the non-overlapping counting scheme if $\mu^{(i)} = 0 \;\forall\; i = 1, 2, ..., m$ and it reduces to overlapping counting scheme if $\mu^{(i)} = k_i - 1 \;\forall\, i = 1, 2, ..., m$. We can easily obtain the transition probability matrices corresponding these cases by making respective substitution for $\mu^{(i)} \;\forall\; i = 1, 2, ..., m$ in the above transition matrices discussed for Aki and Hirano counting Scheme.

6.4.2 *Joint Distribution of number of runs of length at least* $k=(k_1,k_2,...,k_m)$ *and the number of occurrence of each outcome*

Let $G_{n,k}^{(i)}, Y_n^{(i)}$ denote the number of i^{th} type runs of length k_i and occurrence of i^{th} outcome respectively for $i=1,2,...,m$ in n trials The upper bounds for the number of runs of length at least k_i is given by $l_{n,k}^{(i)}=\left[(n+1)/\left(k_i+1\right)\right]$ $,i=1,2,...,m$. Now proceeding exactly as in the above two cases we will define a new Markov chain whose state space Ω is given by $\Omega=\bigcup_{\underline{x}_n,\underline{y}_n\geq 0} C_{\underline{x}_n,\underline{y}_n}$ where

$$C_{\underline{x}_n,\underline{y}_n}=\{\bigcup_{i\in W}\left(\underline{x}_n,\underline{y}_n;u_i,i\right):1\leq u_i\leq k_i\} \quad \underline{x},\underline{y}\geq 0$$

and $\underline{x}_n=\left\{x^{(1)}{}_n,x^{(2)}{}_n...x^{(m)}{}_n\right\}$ denote the realizations of

$G_{n,k}=\left(G_{n,k}^{(1)},G_{n,k}^{(2)},...,G_{n,k}^{(m)}\right)$ and $\underline{y}_n=\left\{y_n^{(1)},y_n^{(2)},...,y_n^{(m)}\right\}$ be a realization of the vector $Y_n=\left(Y_n^{(1)},Y_n^{(2)},...,Y_n^{(m)}\right)$.

$$u_i=\begin{cases} m & if \quad m\leq k_i-1, i\in W \\ k_i & if \quad m\geq k_i \ , i\in W \end{cases}$$

Now the within state transition probabilities are given by

$$P\left(\alpha_{t+1}\in C_{\underline{x},\underline{y}+e_i;u_i+1,i} \mid \alpha_t\in C_{\underline{x},\underline{y};u_i,i}\right)=p_{ii}\left(t\right) \quad if \quad 0\leq u_i\leq k_i-1, \ \ i\in W$$

$$P\left(\alpha_{t+1}\in C_{\underline{x},\underline{y}+e_j;1,j} \mid \alpha_t\in C_{\underline{x},\underline{y};u_i,j}\right)=p_{ij}\left(t\right) \quad if \ \ 0\leq u_i\leq k_i-1, \ \ i,j\in W$$

$$P\left(\alpha_{t+1}\in C_{\underline{x},\underline{y}+e_j;k_j,j} \mid \alpha_t\in C_{\underline{x},\underline{y};k_j,j}\right)=p_{jj}\left(t\right), \ \ j\in W$$

Also the between state transition probabilities are given by

$$P\left(\alpha_{t+1}\in C_{\underline{x}+e_i,\underline{y}+e_j;1,j} \mid \alpha_t\in C_{\underline{x},\underline{y};k,i}\right)=p_{ij}\left(t\right), i\neq j \ \ , \ \ i,j\in W$$

$A_t^{(k)}$ is a matrix whose k^{th} column is given by

$$\left(v_{1,k}(t), v_{2,k}(t), \ldots, v_{k-1,k}(t), \theta_k(t), v_{k+1,k}(t), \ldots, v_{m,k}(t)\right)$$

$$\text{where } \theta_i(t) = \begin{array}{c} (0,i) \\ (1,i) \\ . \\ (k_i-1,i) \\ (k_i,i) \end{array} \begin{bmatrix} 0 & p_{ii}(t) & 0 & . & 0 \\ 0 & 0 & p_{ii}(t) & . & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & . & p_{ii}(t) \\ 0 & 0 & 0 & . & p_{ii}(t) \end{bmatrix}_{k_i+1 \times k_i+1}$$

$$v_{ij}(t) = \begin{bmatrix} 0 & p_{ii}(t) & 0 & . & 0 \\ 0 & p_{ij}(t) & 0 & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & 0 & . & 0 \end{bmatrix}_{k_i+1 \times k_j+1} \qquad \forall i \neq j \quad i, j \in W$$

The between state transition probability matrices $B_t^{(i,i)}(\underline{x}) = B_t^{(i,i)}$ have all their entries zero except at $\left(\sum_{j=1}^{i}(k_j+1), \sum_{j=1}^{n}(k_j+2)\right), n = 0,1 \ldots m, n \neq i$ where we have p_{1n}.

6.4.3 *Joint Distribution of number of runs of length exactly equal to* $k = (k_1, k_2, \ldots, k_m)$ *and the number of occurrence of each outcome*

Denote the number of i^{th} type runs of length exactly equal to k_i and occurrence of i^{th} outcome in n trials respectively by $E_{n,k}^{(i)}, Y_n^{(i)}$ for $i = 1, 2, \ldots, m$. We can define a Markov chain corresponding to this case as discussed above. Then corresponding within state transition probabilities are given by

$$P\left(\alpha_t \in C_{\underline{x}, \underline{y}+e_i; u_i+1, i} \mid \alpha_t \in C_{\underline{x}, \underline{y}; u_i, i}\right) = p_{ii}(t) \;\; if \;\; 0 \leq u_i \leq k_i - 1, \;\; i \in W$$

$$P\left(\alpha_{t+1} \in C_{\underline{x},\underline{y}+e_j;1,j} \mid \alpha_t \in C_{\underline{x},\underline{y};u_i,i}\right) = p_{ij}(t) \quad if \quad j \neq i, 0 \leq u_i \leq k_i - 1, \quad i,j \in W$$

$$P\left(\alpha_t \in C_{\underline{x},\underline{y}+e_j;-1,j} \mid \alpha_t \in C_{\underline{x},\underline{y};-1,j}\right) = p_{jj} \quad j \in W$$

and the between state transition probabilities are as follows

$$P\left(\alpha_{t+1} \in C_{\underline{x}+e_i,\underline{y}+e_j;1,j} \mid \alpha_t \in C_{\underline{x},\underline{y};k_i,i}\right) = p_{ij} \quad if \quad i \neq j \quad , \quad i,j \in W$$

$A_t^{(k)}$ is a matrix whose k^{th} column is given by

$\left(\nu_{1,k}(t), \nu_{2,k}(t), ..., \nu_{k-1,k}(t), \theta_k(t), \nu_{k+1,k}(t), ..., \nu_{m,k}(t)\right)$

$$\text{where } \theta_i(t) = \begin{array}{c} (0,i) \\ (1,i) \\ . \\ (k_i,i) \\ (-1,i) \end{array} \begin{bmatrix} 0 & p_{ii}(t) & 0 & . & 0 \\ 0 & 0 & p_{ii}(t) & . & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & . & p_{ii}(t) \\ 0 & 0 & 0 & . & p_{ii}(t) \end{bmatrix}_{(k_i+2)\times(k_i+2)}$$

$$\nu_{ij}(t) = \begin{array}{c} (0,i) \\ (1,i) \\ . \\ (k,i) \\ (-1,i) \end{array} \begin{bmatrix} 0 & p_{ij}(t) & 0 & . & 0 \\ 0 & p_{ij}(t) & 0 & . & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & . & 0 \\ 0 & p_{ij}(t) & 0 & . & 0 \end{bmatrix}_{(k_i+2)\times(k_j+2)} \qquad i \neq j \quad i,j \in W$$

The between state transition probability matrices $B_t^{(i,i)}(\underline{x}) = B_t^{(i,i)}$ have all their entries zero except at $\left(\sum_{j=1}^{i} k_j + i, \sum_{j=1}^{n} k_j + n + 3\right), n = 1,2,...m \quad n \neq i$ and we will have p_{1n} there.

6.4.4 *Markov Multinomial distribution*

As mentioned earlier one can obtain the Markov multinomial distribution. By taking each $B^{(l,k)} = 0 \ \forall \ l, k = 1, 2, ...m$. For a Markov multinomial distribution

$$P\left(\alpha_{t+1} \in C_{\underline{x}+e_k;1,k} \mid \alpha_t \in C_{\underline{x};u,k}\right) = p_{kk} \quad u = 0,1$$

$$P\left(\alpha_{t+1} \in C_{\underline{x};1,i} \mid \alpha_t \in C_{\underline{x};u,j}\right) = p_{ji} \quad u = 0,1, i, j \in W$$

Then the k^{th} column of the matrix $A_t^{(k)}$ is given by

$$\left(\nu_{1,k}(t), \nu_{2,k}(t), ..., \nu_{k-1,k}(t), \theta_k(t), \nu_{k+1,k}(t), ..., \nu_{m,k}(t)\right)$$

where $\nu_{ij}(t) = \begin{bmatrix} 0 & p_{ij}(t) \\ 0 & p_{ij}(t) \end{bmatrix}_{2\times 2}$ for every $i \neq j$ and

$$\theta_i(t) = \begin{array}{c} (0,i) \\ (1,i) \end{array} \begin{bmatrix} 0 & p_{ii}(t) \\ 0 & p_{ii}(t) \end{bmatrix}_{2\times 2} \quad \text{for every } \ i \in W$$

6.1 Conclusions

In this chapter we derived a recurrence for the evaluation of the joint distribution of the runs and the outcomes of each type. The moment generating functions and the various expected values are also derived. The expression for the interdependence between the both characters also studied. The waiting time distributions under the sooner and the later scenario are also derived. The methodology is illustrated in the case of runs formed under different ways of counting.

Chapter 7
Conclusion and Future Works

Every real life problem has a solution, that has been the motivating stone for this thesis. The thesis deals with modelling situations that involve dependence between the random quantities of interest. Most of the natural phenomena in real life involve dependence between the consecutive observations. The assumption of dependence between the variables becomes impossible when the phenomenon or the characteristic under study involves some repetitions. Most of the modelling problems neglect the dependence that is redundant in them. Here we tried to model some of the scenarios where there is dependence.

In chapter three, we considered a multistate system with Markov dependence between the states occupied by the system. We assumed that the lifetimes and the repair times in each state follow Phase type distribution with the parameters depending on the state. We also assumed that the repairs will have an impact on the lifetime or the repair time. We modelled the impact of the repairs on the lifetimes or the repair times by the Cox multiplicative model. The number of repairs on the system is assumed to be the concomitant variable and the regression factor is assumed to be depending on the state. An expression for long run cost per cycle for the system is derived. An algorithm that simplified the procedures for finding the optimal number of repair, minimizing the expected cost per cycle, was also developed. Finally the algorithm was illustrated with a numerical example.

Eventhough we were able to derive an expression for the long run expected cost per unit time in the case of general multistate systems which do not assume any stochastic ordering between the lifetimes or repair times, the development of the algorithm was based on the assumption that the lifetime or the repair times are stochastically monotone random variables. Developing a simple method or algorithm for finding the optimal policy under the given condition is still interesting. We exploited the Phase type distribution to model the lifetimes and the repair times in each state. Exploiting general situation involving any distribution can also be dealt with in the future. Even in the case of approximating the lifetimes or the repair times by a Phase type distribution, estimating the number of recurrent states to be used is an open problem for almost a decade. Deriving the statistical inference procedures for the model is also an area to be explored. We assumed that the ageing factor depends only on the state which it is occupying. But systems in which the ageing factors depend on the state occupied till now is an interesting problem. Optimization based on the time dependent factors like availability is also a problem to be tackled in the future.

In chapter four, we introduced the concept of protection into the reliability modelling scenario. Till now we did not consider problems with the shocks or the failures that may cause the malfunctioning or even the nonfunctioning of the repair facility. It ha been assumed that there will not be any reliability concerns regarding the repair facility. We considered the situation where the repair had to be restarted from the scratch when an interruption happens to the repair facility. Long run cost for the completion of the repair has been developed assuming that the k states are unprotected while the remaining $n-k$ are protected states, shocks do not have any impact on the repair facility when the repair is in these states. Since high cost is involved with protecting the states, an optimal policy regarding the time of introduction of the repair facility is considered in the chapter. The results are illustrated with the help of a numerical example.

It has been assumed that the repair time in each state is an exponentially distributed random variable. Generalizing this into the general frame work is interesting and this is to be opened upon. Also we assume a sequential transition between the states. This is also restrictive. We also did not consider situations where we take away the protection after some time if the system is performing reasonably well.

In chapter five, we proposed two new models of start-up demonstration having two phases with the condition for the corrective action in the first model, being specified number of consecutive failures, and in the second model, being specified number of random failures. Expressions for various measures of interest like the probability of acceptance of the product, probability of rejection of the product, expected number of start-ups etc were developed. We assumed a Markov dependence between the consecutive trials. We considered a numerical example. The probability of acceptance of the product can be used to derive an optimal policy in both cases.

One of the possible generalizations is one with more than two phases and the results can be obtained in a similar fashion as we had done for the two-phase case. The models can be generalized to m Markov dependent case by proceeding as in Aston and Martin (2005).

In chapter six we obtained the joint distribution of runs and occurrence of outcomes there for multi-state outcomes, there by generalizing Han and Aki (1999) and Chadjiconstantinidis et. al (2000).Probability generating functions for the joint distribution of occurrence of events and the runs was also derived. Probability generating functions of the distribution of occurrence of the events and the runs were also derived. The results were in tune with Han and Aki (1999). Expected values of the joint variables and the marginal variables are also derived. We derived the expression for the waiting times also. The recurrence relation, we derived, can be used to evaluate the probabilities in the case of various types of counting as illustrated. A more general Markov Multinomial distribution was also derived Throughout the development of the chapter we assumed that the consecutive trials are Markov dependent.

All the results discussed in the chapter can be extended to the case of l dependent variables discussed by Aki and Hirano (2000). Patterns and scans are area of interest, generalizing the results to the case of patterns and the scans can also be considered in future.

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