# STATISTICAL INFERENCE FOR SOME MEASURES OF SYSTEM AVAILABILITY

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## *Certificate*

Certified that the thesis entitled '**Statistical Inference for Some Measures of System Availability**' is a bonafide record of works done by Sri. Angel Mathew under my guidance in the Department of Statistics, Cochin University of Science and Technology, Cochin-22, Kerala, India and that no part of it has been included anywhere previously for the award of any degree or title.

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## *Declaration*

The thesis entitled '**Statistical Inference for Some Measures of System Availability**' contains no material which has been accepted for the award of any Degree or Diploma in any University and to the best of my knowledge and belief, it contains no material previously published by any other person, except where due references are made in the text of the thesis.

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**Angel Mathew**

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# Chapter 1

# Preliminaries

## 1.1 Introduction

In today's technological world, the consumers are more sensitive to the quality and performance of the products and hence the manufacturers have to make sure that their products meet the expected quality. Due to the increasing complexity of modern day equipments and general awareness about quality and especially safety issues, the assessment of system performance has a significant role in improving the quality of products. There are a number of measures that indicate the performance of a system. For non-repairable systems, reliability is an important performance measure.

*Reliability* is defined to be the probability that a unit or system can perform its intended function adequately over a specified period of time under stated operating conditions. In mathematical terms, the reliability of a component or a system at time '$t$' is defined as

$$R(t) = P(T > t),$$

where $T$ is the life length of the component. If $F(t)$ is the cumulative distribution function of the failure time and $f(t)$ is the corresponding probability density function, then the reliability function is given by,

$$R(t) = 1 - F(t) = \int_t^\infty f(u)du .$$

Reliability is an accepted measure of system performance if we consider non-repairable systems. However, if a system or its components are repairable, reliability is proved to be an incomplete measure of system performance because it does not consider the system maintenance. Maintainability is a measure that considers the maintenance of systems.

*Maintainability* is defined as the probability that a failed system can be made operable in a specified interval of downtime. The downtime consists of the time it takes to discover that a failure exists, identify the problem, acquire the appropriate tools and parts, and perform the necessary maintenance actions. Therefore, downtime is a function of the failure detection time, repair time, administrative time, and the logistics time connected with the repair cycle. Mathematically, the maintainability function of a system is given by,

$$H(t) = P[\tau \leq t],$$

where '$\tau$' is the total downtime. The maintainability function describes probabilistically how long a system remains in a failed state.

From the definitions of reliability and maintainability, it is clear that, reliability considers only the failure behaviours of the system and maintainability considers only the effects of maintenance actions. With increasing complexity and the resulting high operational and maintenance costs, greater emphasis has been placed on reducing system maintenance while improving reliability. So a measure that considers both the failure behaviours and the effects of maintenance actions is more appropriate for measuring the performance of a repairable system. In this regard, availability, which is a combined measure of reliability and maintainability, has received wide acceptance as a measure of performance of maintained systems.

*Availability* is defined as the probability that a system or component is performing its required function at a given point in time or over a stated period of time when operated and maintained in a prescribed manner (Ebeling, 1997). It is to be noted that the gain of a productive system is directly proportional to its availability. As a measure of performance criterion, the study of availability measures has a significant role in improving the effectiveness of repairable systems. The objective of this thesis is basically to discuss the statistical inference for various measures of system availability.

## 1.2 Repairable Systems

A system by definition can be said to be a collection of two or more components that have been assembled to perform one or more intended functions (Ascher and Feingold, 1984). It is obvious that with the passage of time, most of these systems may fail in the course of duty and will therefore need to be repaired to restore them to their intended functions.

A repairable system, as the name implies, is a system which can be restored to operating condition in the event of a failure by some maintenance action other than replacement of the entire system. The restoration can be done by any action including changing of parts, changes to adjustable settings, swapping of components etc. For example, a laptop computer not connected to an electrical power supply may fail to start if the battery is dead. In this case, replacing the battery with a new one may solve the problem. A television set is another example of a repairable system which upon failure may be restored to satisfactory condition by simply replacing either the failed resistor or transistor if that is the cause, or by adjusting the sweep or synchronization settings. Common examples of repairable systems include automobiles, computers, aircrafts, industrial machineries etc.

On the contrary, non-repairable systems are those that are discarded and replaced by new ones when they fail to perform the intended function. For example, a missile is a non-repairable system when it is launched. Other examples of non-repairable systems include electric bulbs, batteries, transistors etc. However, in the real world, it is obvious that most of the industrial machineries and consumer products are designed to be repaired rather than replaced upon failures. Therefore, the study on various techniques for analysing repairable systems has received a significant place in the current literature.

## 1.3 Maintenance and Renewal Theory

For repairable systems, maintenance plays a vital role in the performance of a system. Maintenance is defined as all actions which have an objective to retain an item in, or restore it to, a state in which it can perform the required function. The actions include the combination of all technical and corresponding administrative, managerial and supervision actions (Murthy et.al., 2008). Maintenance can significantly affect the quality of products after they have been produced. Maintenance actions performed on a repairable system can be categorized in two ways. It may be a corrective (unscheduled) maintenance or preventive (scheduled) maintenance.

Corrective maintenance actions are unscheduled actions intended to restore a system from a failed state into an operating state. The actions involve repair or replacement of all failed parts and components necessary for successful operation of the system. Since a component's lifetime is not known a priori, corrective maintenance is performed at unpredictable intervals. Its main objective is to restore the system to a satisfactory operating condition within the shortest possible time.

Preventive maintenance actions are scheduled actions carried out to improve equipment life and avoid any unplanned maintenance activity. It includes lubrication, testing, cleaning, adjusting, and minor component replacement to extend the life of equipment and facilities. Preventive maintenance is used to avoid costly effects of equipment breakdowns. The primary objective of preventive maintenance is to prevent the failure of equipment before it actually occurs. Improved system reliability, decreased cost of replacement and decreased system downtime are the benefits of preventive maintenance.

The study of repairable components and systems strongly depend on the model of repair or renewal involved in the maintenance process. For a repairable system, the life cycle can be described by a sequence of up and down states. Initially the system operates until the first failure occurs and then it is repaired and restored to its original operating state. It will fail again after some random time of operation, get repaired again, and this process of failure and repair will repeat. Now the sequences of failure and repair times can be considered as a sequence of independent and non-negative random variables constituting a renewal process. Each time a unit fails and is restored to operating condition, a renewal is said to have occurred. This type of renewal process is known as an *alternating renewal process* because the state of the component alternates between an operating state and a repair state. One of the main assumptions in renewal theory is that the failed components are replaced with new ones or repaired so they are '*as good as new*'.

### 1.4 Measures of System Availability

The definition of availability is somewhat flexible and there are various types of availability measures defined in the literature. A good survey and a systematic classification of availability measures are given in Lie et.al. (1977). Availability measures are classified by either the time interval of interest or the mechanisms for the system downtime. Based on the time interval, availability is classified into four categories: i) instantaneous or point availability, ii) limiting or steady state availability, iii) average availability, and iv) limiting average availability. If the mechanism for the system downtime is the primary concern, the availability definition is classified into three categories: i) inherent availability, ii) achieved availability, and iii) operational availability.

Consider a repairable system which can be in one of two states namely, '*up*' and '*down*'. By 'up' we mean that the system is still functioning and by 'down' we mean that the system is not functioning; in the latter case it is being repaired.

If we define,

$$\xi(t)=\begin{cases}1 & \text{if the system is functioning at time } t\\ 0 & \text{otherwise}\end{cases}, \tag{1.1}$$

then $\xi(t)$ represents the status of a repairable system at time '*t*'.

The availability measures that depend on the time interval are primarily based on the above binary function $\xi(t)$ and some of the important measures are:

*i*) The *instantaneous* or *point availability*, $A(t)$, is defined as the probability that the system is operational at any time, '*t*' and is given by:

$$A(t)=P[\xi(t)=1].$$

This is very similar to the reliability function, in that it gives a probability that a system will function at the given time, *t*. Unlike reliability, the instantaneous availability measure incorporates maintainability information. For systems which are required to perform a function at any random time, the point availability may be the most satisfactory measure.

*ii*) The *limiting* or *steady state availability*, *A*, is the limit of the instantaneous availability function as time approaches infinity and is given by,

$$A=\lim_{t\to\infty}A(t).$$

This quantity is the probability that the system will be available after it has been run for a long time, and is a satisfactory measure for systems which are to be operated continuously.

*iii*) The *average availability*, $A_{avg}(t)$, is the expected proportion of time in a specified interval $(0, t]$ that the system is available for use. It represents the mean value of the instantaneous availability function over the period $(0, t]$ and is expressed as:

$$A_{avg}(t)=\frac{1}{t}\int_{o}^{t}A(u)du. \tag{1.2}$$

*iv*) The *limiting average availability*, $A_{avg}$, is the average availability when $t \to \infty$ and is given by:

$$A_{avg} = \lim_{t \to \infty} A_{avg}(t) .$$

When it exists, limiting average availability is almost always equivalent to limiting availability.

There are different forms of the steady state availability depending on the definitions of uptime and downtime. Some of the important availability measures based on the mechanisms for the system downtime are:

*a*) *Inherent availability*: Inherent availability is defined as the probability that a system, when used under stated conditions, without considering any scheduling or preventive actions, in an ideal support environment, will operate satisfactorily at any point in time as required. It excludes ready preventive-maintenance downtime, logistic time, and administrative downtime and is expressed as:

$$A_i = \frac{MTBF}{MTBF + MTTR},$$

where *MTBF* is the mean time between failure and *MTTR* is the mean time to repair.

*b*) *Achieved availability*: Achieved availability is very similar to inherent availability with the exception that preventive maintenance downtimes are also included. Specifically, it is the steady state availability when considering corrective and preventive downtime of the system. It excludes logistic time and administrative downtime and it can be expressed as:

$$A_a = \frac{MTBM}{MTBM + \bar{M}},$$

where *MTBM* is the mean time between maintenance operations and $\bar{M}$ is the mean maintenance time resulting from both corrective and preventive maintenance actions.

c) *Operational availability*: It is the probability that a system, when used under stated conditions in an actual operational environment, will operate satisfactorily when called upon. It includes logistic time and administrative downtime and is expressed as:

$$A_O = \frac{MTBM}{MTBF + MDT},$$

where *MDT* is the mean maintenance downtime.

Emphasis in this thesis is centered on the availability measures based on the time interval of interest.

### 1.5 Inference on Availability Measures

Availability is a common metric to define guarantees among the vendor and the customer. Bergmann (1985) pointed out that during recent years buyers have realized the importance of good availability performance and they force the vendors to guarantee the availability performance. To be 'sure' that the guarantees are fulfilled, statistical techniques for estimating the availability measures have to be derived. One of the objectives of this thesis is to derive some estimators for measuring the availability characteristics and to study their statistical properties.

The properties of the availability measures are usually studied using the successive failure and repair times of the system. Consider a repairable system which is at any time either in operation or under repair after failure. Suppose that the system starts to operate at time $t = 0$. Let $\{X_n\}$ and $\{Y_n\}$ denote the sequences of failure and repair times, respectively. The first operating time and repair time, $X_1$ and $Y_1$, constitute the first cycle of the system. This behaviour is shown graphically in Figure 1.1.

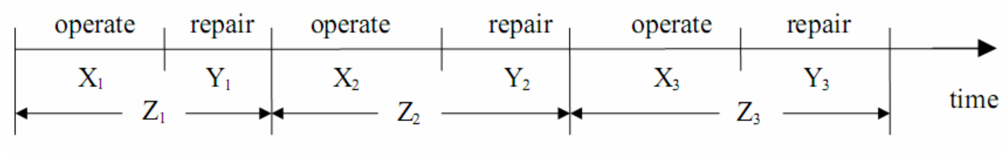


Figure 1.1 *Notional System Behaviour*

Assume that $\{X_n\}$ and $\{Y_n\}$ are independent sequences of independent and identically distributed (i.i.d.) non-negative random variables with common distribution functions $F_X(.)$ and $F_Y(.)$ respectively. Then the sequences of failure and repair times constitute an alternating renewal process. Let $M(t)$ be the renewal function associated with the sequence $\{Z_n\}$, where $Z_n = X_n + Y_n$ is the length of the $n^{\text{th}}$ cycle.

Now, the expression for the point availability $A(t)$ can be written as

$$A(t) = \overline{F}_X(t) + \overline{F}_X * M(t),$$

where $\overline{F}_X(.) = 1 - F_X(.)$ is the survival function of the failure time.

Assume that $F_X(.)$ and $F_Y(.)$ have positive mean $\mu_X$ and $\mu_Y$. Then using the theory of alternating renewal process, the expression for the limiting availability is given by (cf. Barlow and Proschan, 1975),

$$A = \lim_{t \to \infty} A(t) = \mu_X / (\mu_X + \mu_Y).$$

The estimation of these availability measures has been discussed extensively in the literature by several authors. The nonparametric point and interval estimation of the point availability has been discussed by Baxter and Li (1994) and Li (1999) in the case of complete and censored observations respectively. Ouhbi and Limnios (2003) constructed a nonparametric confidence interval for the point availability as a special case of Semi-Markov process. Since it is difficult to obtain closed form expressions for the point availability, except for few simple cases, in the literature more attention is being paid to the estimation of the limiting availability; see, for example, Mi (1995), Baxter and Li (1996), and Abraham and Balakrishna (2000). Parametric methods sometimes are not adequate for the analysis of repairable systems if the underlying distributional assumptions are not valid. So the inference procedures based on non-parametric methods are commonly used for the estimation of the availability measures due to their applicability and simplicity.

## 1.6 Censored Data

A unique feature of reliability data, especially failure time data, is that some of the data may be censored. Censored data arise when a component's life length is known to occur only in a certain period of time. In other words, a censored observation contains only partial information about the random variable of interest. In reliability context, the following types of censored data are of particular interest.

### *Right Censoring*

In both reliability and survival studies, right censoring is the most common form of censoring with lifetime data. In right censoring only lower bounds on lifetime are available for some individuals. Right censoring arises in certain situations because some individuals are still surviving at the time that the study is terminated. Type I and Type II censoring schemes are two different forms of right censoring.

### *Type I Censoring*

Censoring that occurs as a function of time is called Type I censoring. Type I censoring occurs if an experiment is started at a given time for a set of subjects or items, and the experiment is stopped at a predetermined time. For example, in Type I censoring, we put $n$ items on test and terminate the experiment at a pre-assigned time $t_0$. In this case, the data consists of the life times of items that failed before the time $t_0$, say, $x_{(1)}, x_{(2)}, ..., x_{(m)}$, assuming that $m$ items failed before $t_0$, and the fact that $(n-m)$ items have survived beyond $t_0$. Here $t_0$, the time of termination, is fixed, while $m$, the number of items that failed before $t_0$, is a random variable.

***Type II Censoring***

Type II censoring occurs when an experiment is continued until a predetermined number of subjects under study have failed. For example, in Type II censoring, we put $n$ items on test and terminate the experiment when a pre-assigned number of items, say, $r$ have failed. In this case the data consist of the life times of the r items that failed, $x_{(1)}, x_{(2)}, ..., x_{(r)}$ and the fact that $(n-r)$ items have survived beyond $x_{(r)}$. Here $r$, the number of items that failed, is fixed, while $x_{(r)}$, the time at which the experiment is terminated, is a random variable.

***Random Censoring***

A very simple random censoring process that is often realistic is the one in which each individual is assumed to have a lifetime $T$ and a censoring time $C$, where $T$ and $C$ are independent, continuous random variables. Suppose $n$ individuals are participated in a study. For each subject in the study, one observes the minimum of the survival time $T_i$ and the censoring time $C_i$ and knows whether one has observed the survival time $T_i$ or the censoring time $C_i$. Then, the observed variable will be $(Y_i, \delta_i)$, where $Y_i = \min(T_i, C_i)$ and $\delta_i = I(T_i \leq C_i)$. Thus, the data on $n$ individuals consist of the pairs $(Y_i, \delta_i)$, $i = 1, 2, ..., n$. This censoring mechanism is known as right random censorship.

## 1.7 Some Useful Definitions and Results

In this section we quote some useful definitions and results which are frequently used in our discussion.

**Definition 1.1 (Brownian Motion)**

A standard Brownian motion or Wiener process $W = \{W(t): t \in U\}$, such that $U \equiv (-\infty, \infty)$ or $[0, \infty)$, is a stochastic process satisfying:

($i$) $W(0) = 0$, and $E[W(t)] = 0$ for any $t$;

($ii$) $W$ has independent increments, therefore, $W(t) - W(u)$ is independent of $W(u)$ for any $0 \le u \le t$.

($iii$) $W(t)$ has variance $t$; and

($iv$) $W$ is a Gaussian process with continuous sample paths.

**Lemma 1.1**

*Let $h(x)$ be a real measurable function and W be a standard Brownian motion process. Then $\int_0^t h(x)dW(x)$ follows a normal distribution with mean 0 and variance $\sigma_h^2(t)$, where $\sigma_h^2(t) = \int_0^t [h(x)]^2 dx$.*

*Proof.* See Shorack and Wellner (1986, pp. 91).

**Definition 1.2 (Brownian Bridge)**

A standard Brownian bridge, $\{W^0(t): t \in S \equiv [0,1]\}$, is a stochastic process distributed as conditioned standard Brownian motion conditioned upon $W(1) = 0$. That is, a standard Brownian bridge is a Gaussian stochastic process such that

(i) $E[W^0(t)] = 0$, for all $t \in [0,1]$ and

(ii) $E[W^0(t)W^0(s)] = \min(t,s) - ts$, for all $t, s \in [0,1]$.

**Lemma 1.2**

*Suppose that the function* $h:[0,1]\rightarrow R$ *is square integrable. Then* $\int_0^1 h(x)dW^0(x)$ *follows a normal distribution with mean 0 and variance* $\sigma_h^2$, *where*

$$\sigma_h^2 = \int_0^1 [h(x)]^2 dx - \left[\int_0^1 h(x)dx\right]^2.$$

*Proof*. See Shorack and Wellner (1986, pp. 92-94).

**Lemma 1.3**

*Let* $\{f_n(t)\}$ *be a sequence of finite, nondecreasing functions on* $[0,\ a]$ *and let* $f(t)$ *be a continuous, finite function on* $[0,\ a]$ *such that* $f_n(t)$ *converges uniformly to* $f(t)$ *as* $n\rightarrow\infty$. *Further, let* $g_n:[0,a]\rightarrow R,\ n=1,2,3,\ldots$ *be a sequence of functions which converges uniformly to* $g:[0,a]\rightarrow R$ *as* $n\rightarrow\infty$. *Then* $\int_0^x g_n(t)df_n(t)\rightarrow\int_0^x g(t)df(t)$ *uniformly in* $x\in[0,a]$ *as* $n\rightarrow\infty$, *where all integrals are to be interpreted as Lebesgue integrals.*

*Proof*. See Baxter and Li (1994).

**Lemma 1.4**

*Suppose that* $Z_n=(Z_{n1},Z_{n2},\ldots,Z_{nk})$ *is asymptotically* $N(\boldsymbol{\mu},n^{-1}\Sigma)$, *with* $\Sigma$ *a covariance matrix. Let* $g(\boldsymbol{z}),\ \boldsymbol{z}=(z_1,z_2,\ldots,z_k)$ *be a real-valued function having a nonzero differential at* $\boldsymbol{z}=\boldsymbol{\mu}$. *Put* $T=\left[\left.\frac{\partial g}{\partial z_j}\right|_{z=\boldsymbol{\mu}}\right]_{1\times k}$. *Then,*

$$\sqrt{n}\left(g(Z_n)-g(\boldsymbol{\mu})\right)\xrightarrow{L}N\left(\mathbf{0},T\Sigma T'\right),$$

*where* $\xrightarrow{L}$ *denotes convergence in distribution.*

*Proof*. See Serfling (1981, pp.122).

**Lemma 1.5 (Slutsky's Theorem)**

*If* $\{X_n\}$, $\{Y_n\}$ *and* $\{Z_n\}$ *are three sequences of random variables with* $X_n \xrightarrow{L} X$, $Y_n \xrightarrow{p} a$ *and* $Z_n \xrightarrow{p} b$, *where* $a$, $b$ *are finite constants, then*

$$X_n Y_n + Z_n \xrightarrow{L} aX + b,$$

*where* $\xrightarrow{L}$ *and* $\xrightarrow{p}$ *denote the convergence in distribution and probability respectively.*

*Proof.* See Chow and Teicher, (1978, pp. 249).

**Definition 1.3 (*m*-Dependent Random Variables)**

A sequence $\{X_n\}$ of random variables is said to be m-dependent if $(X_1, X_2, ..., X_k)$ and $(X_{n+k}, X_{n+k+1}, ...)$ are independent for any $k$ whenever $n > m$.

**Lemma 1.6 (Central Limit Theorem for *m*-Dependent Sequence)**

*Let* $\{X_n\}$ *be a stationary m-dependent sequence with* $E(X_n) = 0$ *and* $E\left(X_n^2\right) < \infty$. *Then*

$$\sqrt{n}\bar{X}_n \xrightarrow{L} N\left(0, \sigma^2\right) \text{ as } n \to \infty,$$

*where* $\sigma^2 = E\left(X_0^2\right) + 2\sum_{j=1}^{\infty} E\left(X_0 X_j\right)$.

*Proof.* See Ibragimov and Linnik (1971, pp. 370).

**Definition 1.4 (Strong Mixing Sequences)**

A sequence $\{X_n\}$ of random variables is said to be strongly mixing if $\alpha(h) = \text{Sup}\left\{\left|P(A \cap B) - P(A)P(B)\right| : A \in \Im_1^k(X) \text{ and } B \in \Im_{k+h}^{\infty}(X)\right\} \to 0$, as $h \to \infty$ where $\Im_1^k(X) = \sigma\left(X_i;\ 1 \le i \le k\right)$ and $\Im_{k+h}^{\infty}(X) = \sigma\left(X_i;\ i \ge k+h\right)$.

**Lemma 1.7 (Central Limit Theorem for Strong Mixing Sequence)**

*Let* $\{X_n\}$ *be a stationary strong mixing sequence with mixing coefficient* $\alpha(h)$, *and let* $E(X_n)=0$ *and* $E\left|X_n\right|^{2+\delta}<\infty$ *for some* $\delta>0$. *If* $\sum_{h=1}^{\infty}\alpha^{\delta/(2+\delta)}(h)<\infty$, *then as* $n\to\infty$,

$$\sqrt{n}\bar{X}_n \xrightarrow{L} N\left(0,\, \sigma^2\right),$$

*where* $\sigma^2 = E\left(X_0^2\right)+2\sum_{j=1}^{\infty}E\left(X_0X_j\right)$.

*Proof*. See Ibragimov and Linnik (1971, pp. 346).

**Definition 1.5 (Uniformly Continuous in Probability)**

A sequence $\{Y_n\}$ of random variables is said to be uniformly continuous in probability (u.c.i.p) if for every $\varepsilon>0$ there is a $\delta>0$ for which $P\left\{\underset{0\le k\le n\delta}{Max}\left|Y_{n+k}-Y_n\right|\ge\varepsilon\right\}<\varepsilon$ for all $n\ge1$.

**Lemma 1.8** (**Anscombe's Theorem**)

*If* $\{Y_n\}$ *are uniformly continuous in probability and* $t_a$, $a>0$ *be an integer valued random variable for which* $t_a/a$ *converges to a finite positive constant* $c$ *in probability and* $N_a=[ac]$, *where* $[x]$ *denotes the greatest integer part of* $x$. *Then,* $Y_{t_a}-Y_{N_a}\to0$ *in probability as* $a\to\infty$. *If in addition* $Y_n$ *converges in distribution to a random variable* $Y$, *then* $Y_{t_a}\to Y$ *as* $a\to\infty$.

*Proof*. See Woodroofe (1982, pp.11).

### 1.8 Summary of the Thesis

The discussions in previous sections reveal that there has been much research on the estimation of the point availability and limiting availability. However, the estimation of the availability measures like average availability is not discussed much in the literature. Also there are several occasions, where the existing estimation procedures for system availability are inadequate when the successive observations on the failure and repair times are dependent. Motivated by this, we propose estimators for the availability measures and establish their statistical properties, which are completely distribution free for the analysis of repairable systems.

The thesis is organized into seven chapters, of which the first one is an introductory chapter, where we discuss the basic concepts, relevance and scope of the study along with a review of literature.

Average availability is a valuable measure of performance of a repairable system as it captures availability behavior over a finite period of time. However, the estimation of this quantity is not yet discussed in the literature. Motivated by this, in Chapter 2 we consider the nonparametric estimation of the average availability of a system over the interval $[0,\ t]$. The nonparametric estimation of the average availability under three different sampling schemes is discussed in this Chapter. First, we consider the estimation in the case of complete observations, in which the sample consists of the failure and repair times of '$n$' complete cycles of system operation. Next, we discuss the estimation when the observations on the failure and repair time are subject to right censorship. Finally, we study the estimation when the process is observed continuously over a fixed period $[0,T]$, in which the number of failures and number of repairs completed before the time '$T$' are random variables. In each case, the asymptotic properties of the estimators are studied and they are shown to be consistent and asymptotically normal. A simulation study is also conducted in order to assess the performance of the proposed estimators in each case. The simulation study shows that the proposed estimators perform well even for reasonable sample sizes.

In the context of repairable system, another important measure of performance of a system is the interval reliability. The interval reliability, $R(x,t)$, is defined as the probability that at a specified time '$t$', the system is operating and will continue to operate for an interval of duration '$x$'. In the literature we have not come across any work on the estimation of the interval reliability. So in Chapter 3, we discuss the nonparametric estimation of the interval reliability when (i) the data are complete, (ii) the data are subject to right censorship, and (iii) the process is observed up to a specified time '$T$'. In each case the proposed estimators of the interval reliability are proved to be consistent and asymptotically normal. A simulation study is carried out to assess the performance of the estimators and the proposed method is also applied for analysing a real life data.

As time '$t$' progresses, the interval reliability, $R(x,t)$, converges to a positive quantity called the limiting interval reliability. If we want to know the extent to which the system will survive an interval of duration after it has been run for a long time, the limiting interval reliability is a useful measure. In Chapter 4, we consider the nonparametric estimation of the limiting interval reliability when the failure times and repair times form a sequence of i.i.d. bivariate random variables. Asymptotic properties of the estimators under the three sampling schemes are studied and a simulation study is carried out to assess the performance of the estimators. A testing of hypothesis procedure for the limiting interval reliability is also discussed in this Chapter.

One of the major limitations of the existing approaches in the study of system availability is the assumption of independence among successive sequences of failure and repair times. When the system is operating in a random environment it is natural to observe some dependence among the successive sequence of failure and repair times. Several non-Gaussian time series models such as first order random coefficient autoregressive models are discussed in the literature for modeling life time data; see for example Lawrance and Lewis (1977), Gaver and Lewis (1980), and Sim (1992). Inspired by this, the availability behavior of some stationary dependent sequences is discussed in Chapter 5. We derive the expression for the point availability when the successive sequences of failure and repair times are generated by stationary dependent sequences. The availability behavior of repairable systems when the failure and repair times are generated by first order Exponential Moving Average (EMA1) process and first order Exponential Autoregressive (EAR(1)) process are also discussed in this Chapter.

In the case of repairable systems, estimation of the availability measures is not discussed much when the successive failure and repair times are generated by some stationary dependent sequences except those considered by Abraham and Balakrishna (2000). Motivated by this, the nonparametric estimation of the limiting interval reliability for stationary strong mixing sequences is discussed in Chapter 6. The proposed estimators of the limiting interval reliability are proved to be consistent and asymptotically normal when (i) the data are complete, (ii) the data are subject to right censorship and (iii) the process is observed over a fixed period. A simulation study is reported to assess the performance of the estimators and it shows that the assumption of independence among successive sequences of failure and repair times underestimates the variance of the estimators significantly if the true process is stationary dependent. Also we extend the estimation results to the case of a coherent system of '$k$' independent functioning components in order to consider complex systems.

Finally in Chapter 7, we discuss the sequential interval estimation of the limiting interval reliability when the failure and repair times of a system form a stationary strong mixing bivariate sequence of random vectors. It is shown that the confidence interval is asymptotically consistent and the proposed stopping rule is asymptotically efficient as the width of the interval approaches zero. The general theory is applied to a stationary first order bivariate exponential autoregressive (*BEAR*(1)) sequence and the resulting stopping rule is compared with the stopping rule under the i.i.d. set-up. It is observed that when the true model is *BEAR*(1), the assumption of an i.i.d. sequence underestimates the sample size and leads to poor coverage probability. A simulation study also confirmed the same result. Finally, certain open problems and plan for future study are presented.

# Chapter 2

# Nonparametric Estimation of the Average Availability

## 2.1 Introduction

The average availability of a repairable system is defined as the expected proportion of time that the system is operating in the interval $[0,\ t]$. It represents the mean value of the point availability function over the interval $[0,\ t]$ and hence it captures the availability behavior of a system over a finite period of time. So, in the context of repairable systems, average availability is a valuable measure of system performance. Even though, there are several works available on the estimation of the point and limiting availability, the estimation of the average availability is not yet discussed in the literature. Motivated by this, in the present chapter, we discuss the nonparametric estimation of the average availability.

Consider a one-unit repairable system which is at any time either in operation or under repair after failure. Suppose that the system starts to operate at time $t = 0$. Let $\{X_n\}$ and $\{Y_n\}$ be two independent sequences of independent and identically distributed (i.i.d.) non-negative random variables representing the failure and repair times of the system with common distribution functions $F_X(.)$ and $F_Y(.)$ respectively. Assume that $F_X(.)$ and $F_Y(.)$ have positive means $\mu_X$ and $\mu_Y$ and finite variances $\sigma_X^2$ and $\sigma_Y^2$ respectively. Define $Z_n = X_n + Y_n$, for $n = 1, 2, \ldots$. Let $F_Z(.)$ be the distribution function of the sequence $\{Z_n\}$ having mean $\mu_Z = \mu_X + \mu_Y$.

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The results in this chapter are published in the journal ***Communication in Statistics-Theory and Methods*** (See Balakrishna and Mathew, 2009).

Let $S_n = \sum_{i=1}^{n} Z_i$ and define $N(t) = Sup\{n : S_n \le t\}$. Then $N(t)$ counts the number of cycles completed in the interval $[0, t]$ and $M(t) = E[N(t)]$ is the renewal function associated with the sequence $\{Z_n\}$.

From the definition of the average availability (cf. Equation 1.2), it follows that $A_{avg}(t)$ is not a probability, but represents the expected proportion of "uptime" over the interval $[0, t]$ of system operation.

At any time '$t$', we have $S_{N(t)} \le t < S_{N(t)+1}$ and hence $M(t)\mu_Z \le t < (M(t)+1)\mu_Z$. Assuming that the system is operating at time $t$ = 0, $\bar{\alpha}(t)$, the average up time in the interval $[0, t]$ can be written as

$$\bar{\alpha}(t) = \begin{cases} t - M(t)\mu_Y & \text{if} \quad M(t)\mu_Z \le t < M(t)\mu_Z + \mu_X \\ (M(t)+1)\mu_X & \text{if} \quad M(t)\mu_Z + \mu_X \le t < (M(t)+1)\mu_Z \end{cases}.$$

If we define the indicator function, $\lambda(t) = I\{M(t)\mu_Z + \mu_X \le t\}$, we can write

$$\bar{\alpha}(t) = \lambda(t)\{(M(t)+1)\mu_X\} + (1-\lambda(t))\{t - M(t)\mu_Y\}.$$

Now the average availability can be expressed as:

$$A_{avg}(t) = \frac{1}{t}[\lambda(t)\{(M(t)+1)\mu_X\} + (1-\lambda(t))\{t - M(t)\mu_Y\}]. \qquad (2.1)$$

Thus the average availability can be written in terms of the renewal function $M(t)$ and the mean failure and repair times, $\mu_X$ and $\mu_Y$.

If the system is under repair at time $t$ = 0, then the expression for average up time takes the form:

$$\bar{\alpha}^*(t) = \eta(t)\{t - (M(t)+1)\mu_Y\} + \{1-\eta(t)\}M(t)\mu_X,$$

where $\eta(t) = I\{M(t)\mu_Z + \mu_Y \le t\}$ and hence the expression for the average availability will be

$$A^*_{avg}(t) = \frac{1}{t}[\eta(t)\{t - (M(t)+1)\mu_Y\} + \{1-\eta(t)\}M(t)\mu_X].$$

As $t \to \infty$, the estimators of $A_{avg}(t)$ and $A^*_{avg}(t)$ have similar asymptotic properties and their proofs are almost identical. Hence in this chapter, we present the asymptotic properties of the estimators of $A_{avg}(t)$ defined by (2.1).

The nonparametric estimation of the average availability under three different sampling schemes is discussed in this Chapter. In Section 2.2, we discuss the nonparametric estimation of $A_{avg}(t)$ based on complete observations. Section 2.3 discusses the estimation in the case of censored observations and in Section 2.4, we consider the estimation based on continuous observation over a fixed period. Some numerical illustrations are presented in Section 2.5. An application of the proposed method is illustrated using a compressor failure data in Section 2.6. Finally, Section 2.7 summarizes major conclusions of the study.

## 2.2 Estimation in the case of Complete Observations

Suppose that observations on the failure and repair times of '*n*' complete cycles of system operation, $(X_1,Y_1),(X_2,Y_2),...,(X_n,Y_n)$ are available.

Let $\hat{F}_X(t)$ and $\hat{F}_Y(t)$ denote the empirical distribution function of the random variables *X* and *Y* respectively. By definition,

$$\hat{F}_X(t) = \frac{1}{n}\sum_{i=1}^{n} I\{X_i \le t\} \text{ and } \hat{F}_Y(t) = \frac{1}{n}\sum_{i=1}^{n} I\{Y_i \le t\}.$$

Then a natural nonparametric estimators of $\mu_X$ and $\mu_Y$ are given by

$$\hat{\mu}_X = \int_0^\infty x d\hat{F}_X(x) = \bar{X} \text{ and } \hat{\mu}_Y = \int_0^\infty x d\hat{F}_Y(x) = \bar{Y} \text{ respectively.}$$

By definition, the renewal function associated with the sequence $\{Z_n\}$ is given by,

$$M(t) = \sum_{k=1}^{\infty} F_Z^{(k)}(t),$$

with $F_Z^{(k)}(t) = P[S_k \le t]$ is the *k*-fold convolution of $F_Z(t)$ and $F_Z(t) = F_X * F_Y(t)$, where '$*$' denotes the convolution operator.

Nonparametric estimation of the renewal function has been discussed by many authors; see, for example, Frees (1986), Grubel and Pitts (1993), Harel et al. (1995). For fixed *t*, Baxter and Li (1994) proposed a method for constructing nonparametric confidence intervals for the renewal function which is easier to compute than that of Frees (1986).

Thus, an estimator for $M(t)$ is given by,

$$\hat{M}_n(t) = \sum_{k=1}^{\infty} \hat{F}_Z^{(k)}(t), \tag{2.2}$$

where $\hat{F}_Z(t) = \hat{F}_X * \hat{F}_Y(t)$.

We propose an estimator for the average availability as

$$\hat{A}_{avg}(t) = \frac{\hat{\bar{\alpha}}_n(t)}{t}, \tag{2.3}$$

where $\hat{\bar{\alpha}}_n(t) = \hat{\lambda}_n(t)\{(\hat{M}_n(t)+1)\hat{\mu}_X\} + (1-\hat{\lambda}_n(t))\{t - \hat{M}_n(t)\hat{\mu}_Y\}$,

with $\hat{\lambda}_n(t) = I\{\hat{M}_n(t)\hat{\mu}_Z + \hat{\mu}_X \le t\}$ and $\hat{\mu}_Z = \hat{\mu}_X + \hat{\mu}_Y$.

The strong consistency of the proposed estimator is established in the following theorem.

**Theorem 2.1**

*As* $n \to \infty$, $\hat{A}_{avg}(t) \to A_{avg}(t)$ *almost surely (a.s.).*

*Proof.* Baxter and Li (1994) studied asymptotic properties of the estimator $\hat{M}_n(t)$ defined by (2.2) and shown that $\hat{M}_n(t) \to M(t)$ (a.s.) as $n \to \infty$.

By the strong law of large numbers, we have $\hat{\mu}_X \to \mu_X$, $\hat{\mu}_Y \to \mu_Y$ and $\hat{\mu}_Z \to \mu_Z$ (a.s.) as $n \to \infty$.

Using the fact that $\hat{M}_n(t)\hat{\mu}_Z + \hat{\mu}_X \to M(t)\mu_Z + \mu_X$ (a.s.), we can conclude that $\hat{\lambda}_n(t) \to \lambda(t)$ (a.s.) as $n \to \infty$.

Thus, $\hat{\bar{\alpha}}_n(t) \to \bar{\alpha}(t)$ (a.s.) and hence $\hat{A}_{avg}(t) \to A_{avg}(t)$ (a.s.) as $n \to \infty$.

In order to prove the weak convergence of $\hat{A}_{avg}(t)$ we use the following lemma.

**Lemma 2.1** *Let F be a distribution function with* $F(0)=0$*, and let* $X_1, X_2, ..., X_n$ *denote a random sample from F. Let* $Z \in D$ *be of bounded variation, where D denotes the set of right continuous functions with left-hand limits on* [0, 1]. *Then as* $n \to \infty$*, the process* $\{\sqrt{n} Z * (\hat{F} - F)(t)\}$ *converges weakly to* $\{Z * (W^0 \circ F)(t)\}$ *in D, where* $\{W^0(t), 0 \le t \le 1\}$ *denote a Brownian bridge and* $\circ$ *denote functional composition.*

*Proof.* See Harel et al. (1995).

For establishing the weak convergence of $\hat{A}_{avg}(t)$, let us define $\Delta\mu_X = \hat{\mu}_X - \mu_X$, $\Delta\mu_Y = \hat{\mu}_Y - \mu_Y$, $\Delta\mu_Z = \hat{\mu}_Z - \mu_Z$, $\Delta M(t) = \hat{M}_n(t) - M(t)$ and $\Delta\lambda(t) = \hat{\lambda}_n(t) - \lambda(t)$.

Now,

$$\begin{aligned}\hat{\bar{\alpha}}_n(t) - \bar{\alpha}(t) &= \hat{\lambda}_n(t)(\hat{M}_n(t)+1)\hat{\mu}_X + (1-\hat{\lambda}_n(t))\{t - \hat{M}_n(t)\hat{\mu}_Y\} \\ &\quad -[\lambda(t)(M(t)+1)\mu_X + (1-\lambda(t))\{t - M(t)\mu_Y\}] \\ &= \Delta[\lambda(t)M(t)\mu_X] + \Delta[\lambda(t)\mu_X] - \Delta[M(t)\mu_Y] \\ &\quad - t\Delta\lambda(t) + \Delta[\lambda(t)M(t)\mu_Y].\end{aligned}$$

Letting $\Delta[AB] = \hat{A}_n\hat{B}_n - AB = \Delta A\Delta B + A\Delta B + B\Delta A$, we can write

$$\begin{aligned}\hat{\bar{\alpha}}_n(t) - \bar{\alpha}(t) &= \lambda(t)[M(t)+1]\Delta\mu_X - M(t)[1-\lambda(t)]\Delta\mu_Y \\ &\quad + [\lambda(t)\mu_X - (1-\lambda(t))\mu_Y]\Delta M(t) + R_1 + R_2, \end{aligned} \tag{2.4}$$

where

$$R_1 = [(M(t)+1)\mu_X - (t - M(t)\mu_Y)]\Delta\lambda(t) \text{ and}$$

$$\begin{aligned}R_2 &= [M(t)+1]\Delta\mu_X\Delta\lambda(t) + \Delta M(t)\Delta[\lambda(t)\mu_X] \\ &\quad -[1-\lambda(t)]\Delta M(t)\Delta\mu_Y + \Delta\lambda(t)\Delta[M(t)\mu_Y].\end{aligned}$$

We have,

$$\begin{aligned}\Delta M(t) &= \hat{M}_n(t) - M(t)\\ &= \sum_{k=1}^{\infty}\left(\hat{F}_Z^{(k)} - F_Z^{(k)}\right)(t)\\ &= (\hat{F}_Z - F_Z) * \sum_{k=1}^{\infty}\left(\hat{F}_Z^{(k-1)} + \hat{F}_Z^{(k-2)} * F_Z + \cdots + \hat{F}_Z * F_Z^{(k-2)} + F_Z^{(k-1)}\right)(t)\\ &= \Delta F_Z * \hat{M}_n * M(t)\\ &= \Delta F_Z * \left(M + \hat{M}_n - M\right) * M(t)\\ &= \Delta F_Z * M * M(t) + \Delta F_Z * \Delta M * M(t)\\ &= \Delta F_Z * M * M(t) + \Delta F_Z * \Delta F_Z * \hat{M}_n * M(t). \qquad (2.5)\end{aligned}$$

But,

$$\begin{aligned}\Delta F_Z(t) &= \hat{F}_X * \hat{F}_Y(t) - F_X * F_Y(t)\\ &= \Delta F_X * \Delta F_Y + F_X * \Delta F_Y + F_Y * \Delta F_X .\end{aligned}$$

Thus,

$$\Delta M(t) = [\Delta F_X * \Delta F_Y + F_X * \Delta F_Y + F_Y * \Delta F_X] * M * M(t) + \Delta F_Z * \Delta F_Z * \hat{M}_n * M(t) .$$

Substituting for $\Delta M(t)$ in (2.4), we get,

$$\begin{aligned}\hat{\bar{\alpha}}_n(t) - \bar{\alpha}(t) \;=\; & \lambda(t)[M(t)+1]\Delta\mu_X + [\lambda(t)\mu_X - (1-\lambda(t))\mu_Y] F_Y * M * M * \Delta F_X(t)\\ & + [\lambda(t)\mu_X - (1-\lambda(t))\mu_Y] F_X * M * M * \Delta F_Y(t) - M(t)[1-\lambda(t)]\Delta\mu_Y\\ & + R_1 + R_2 + R_3 ,\end{aligned}$$

where

$$R_3 = [\lambda(t)\mu_X - (1-\lambda(t))\mu_Y][M * M * \Delta F_X * \Delta F_Y(t) - \hat{M}_n * M * \Delta F_Z * \Delta F_Z(t)] .$$

Introducing the notations,

$$K_1(t) = \frac{\lambda(t)[M(t)+1]}{t},\; K_2(t) = \frac{[\lambda(t)\mu_X - (1-\lambda(t))\mu_Y]}{t},\; K_3(t) = \frac{M(t)[1-\lambda(t)]}{t},$$

$J_X(t) = F_X * M * M(t)$, and $J_Y(t) = F_Y * M * M(t)$,

we can write

$$\sqrt{n}[\hat{A}_{avg}(t)(t) - A_{avg}(t)] = \sqrt{n}[K_1(t)\Delta\mu_X + K_2(t)J_Y * \Delta F_X(t)]$$

$$+\sqrt{n}[K_2(t)J_X * \Delta F_Y(t) - K_3(t)\Delta\mu_Y] + \frac{\sqrt{n}}{t}(R_1 + R_2 + R_3).$$

$$= \sqrt{n}(I_1 + I_2 + I_3),$$

where $I_1 = K_1(t)\Delta\mu_X + K_2(t)J_Y * \Delta F_X(t)$, $I_2 = K_2(t)J_X * \Delta F_Y(t) - K_3(t)\Delta\mu_Y$ and

$$I_3 = \frac{1}{t}(R_1 + R_2 + R_3). \tag{2.6}$$

For every $\varepsilon > 0$,

$$P[|\sqrt{n}\Delta\lambda(t)| > \varepsilon] = P[|\Delta\lambda(t)| > \varepsilon/\sqrt{n}]$$

$= P[|\Delta\lambda(t)| = 1]$, as $|\Delta\lambda(t)|$ can take only values 0 and 1.

$\to 0$ as $n \to \infty$, since $\hat{\lambda}_n(t) \to \lambda(t)$ (a.s.) as $n \to \infty$.

Thus $\sqrt{n}R_1 \to 0$ in probability.

Since $R_2$ contains only terms of the form $\Delta A \Delta B$, on the similar lines it can be shown that $\sqrt{n}R_2 \to 0$ in probability as $n \to \infty$.

Also every term in $R_3$ contains a convolution of two differences such as $\Delta A * \Delta B$. By writing $\sqrt{n}\Delta A * \Delta B = \sqrt{n}\Delta A * \hat{B}_n - \sqrt{n}\Delta A * B$, it is easy to see that the two terms on the right-hand side converge almost surely to the same limit by using Lemma 1.3 and hence $\sqrt{n}R_3 \to 0$ in probability. Thus $\sqrt{n}I_3 \to 0$ in probability as $n \to \infty$.

Consider,

$$\sqrt{n}I_1 = \sqrt{n}[K_1(t)\Delta\mu_X + K_2(t)J_Y * \Delta F_X(t)]$$

$$= \sqrt{n}[K_1(t)\int_0^\infty x d\Delta F_X(x) + K_2(t)\int_0^t J_Y(t-x) d\Delta F_X(x)]$$

$$= \sqrt{n}\int_0^\infty [K_1(t)x + K_2(t)J_Y(t-x)] d\Delta F_X(x)$$

$$\xrightarrow{L} \int_0^\infty [K_1(t)x + K_2(t)J_Y(t-x)] d(W_X^0 \circ F_X)(x), \text{ by Lemma 2.1.}$$

$$= \int_0^1 [K_1(t)F_X^{-1}(y) + K_2(t)J_Y(t - F_X^{-1}(y))]dW_X^0(y) \text{, by change of variable.}$$

As an application of Lemma 1.2, $\sqrt{n}I_1$ follows a normal distribution with mean 0 and variance $\sigma_1^2(t)$ as $n \to \infty$, where

$$\sigma_1^2(t) = \int_0^1 [K_1(t)F_X^{-1}(y) + K_2(t)J_Y(t - F_X^{-1}(y))]^2 dy$$
$$- \left[ \int_0^1 [K_1(t)F_X^{-1}(y) + K_2(t)J_Y(t - F_X^{-1}(y))]dy \right]^2$$
$$= \int_0^\infty [K_1(t)x + K_2(t)J_Y(t-x)]^2 dF_X(x)$$
$$- \left[ \int_0^\infty [K_1(t)x + K_2(t)J_Y(t-x)]dF_X(x) \right]^2$$
$$= K_1^2(t) \left[ \int_0^\infty x^2 dF_X(x) - \left( \int_0^\infty x dF_X(x) \right)^2 \right]$$
$$+ K_2^2(t) \left[ \int_0^t J_Y^2(t-x)dF_X(x) - \left( \int_0^t J_Y(t-x)dF_X(x) \right)^2 \right]$$
$$+ 2K_1(t)K_2(t) \left[ \int_0^t J_Y(t-x)x dF_X(x) - \int_0^\infty x dF_X(x) \int_0^t J_Y(t-x)dF_X(x) \right]$$

Thus,

$$\sigma_1^2(t) = K_1^2(t)\sigma_X^2 + K_2^2(t)\left[ J_Y^2 * F_X(t) - \left[ J_Y * F_X(t) \right]^2 \right]$$
$$+ 2K_1(t)K_2(t)\left[ J_Y * V_X(t) - \mu_X J_Y * F_X(t) \right], \quad (2.7)$$

where $V_X(t) = \int_0^t x dF_X(x)$.

Similarly, it can be shown that

$$\sqrt{n}I_2 \xrightarrow{L} N(0, \sigma_2^2(t)) \text{ as } n \to \infty,$$

where

$$\sigma_2^2(t) = K_2^2(t)\left[ J_X^2 * F_Y(t) - \left[ J_X * F_Y(t) \right]^2 \right] + K_3^2(t)\sigma_Y^2$$
$$-2K_2(t)K_3(t)\left[ J_X * V_Y(t) - \mu_Y J_X * F_Y(t) \right], \quad (2.8)$$

with $V_Y(t) = \int_0^t x dF_Y(x)$.

Since $\Delta F_X$ and $\Delta F_Y$ are independent, $I_1$ and $I_2$ are also independent. This leads to the following theorem.

**Theorem 2.2**

*For any fixed 't', as* $n \to \infty$, $\sqrt{n}[\hat{A}_{avg}(t) - A_{avg}(t)] \xrightarrow{L} N(0, \sigma^2(t))$, *where* $\xrightarrow{L}$ *denotes convergence in distribution and*

$$\sigma^2(t) = \sigma_1^2(t) + \sigma_2^2(t), \tag{2.9}$$

*with* $\sigma_1^2(t)$ *and* $\sigma_2^2(t)$ *are given by* (2.7) *and* (2.8) *respectively.*

Let $S_X^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$ and $S_Y^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$ be estimators of $\sigma_X^2$ and $\sigma_Y^2$ respectively. Then an estimator $\hat{\sigma}^2(t)$ of $\sigma^2(t)$ can be obtained on replacing $\mu_X$, $\mu_Y$, $\sigma_X^2$, $\sigma_Y^2$, $F_X(.)$, $F_Y(.)$, and $M(.)$ by $\bar{X}$, $\bar{Y}$, $S_X^2$, $S_Y^2$, $\hat{F}_X(.)$, $\hat{F}_Y(.)$, and $\hat{M}_n(.)$ respectively in (2.9). Using Lemma 1.3, it can be shown that $\hat{\sigma}^2(t) \to \sigma^2(t)$ almost surely as $n \to \infty$.

Thus, given a significance level $\alpha \in (0,1)$, an approximate large sample $100(1-\alpha)\%$ confidence interval for $A_{avg}(t)$ is

$$\hat{A}_{avg}(t) - z_{\alpha/2}\frac{\hat{\sigma}(t)}{\sqrt{n}} \le A_{avg}(t) \le \hat{A}_{avg}(t) + z_{\alpha/2}\frac{\hat{\sigma}(t)}{\sqrt{n}},$$

where $z_{\alpha/2}$ denotes the upper $\alpha/2$ quantile of the standard normal distribution.

## 2.3 Estimation in the case of Censored Observations

Suppose that observations on the failure and repair time are subject to right censorship. In practice, a censored failure time occurs when the system is removed before failure for some preventive maintenance and a censored repair time occurs when the repair work is terminated before the repair is completed due to some technical reason; for example, see Baxter and Li (1996) and Li (1999). Let $X_1, X_2, ..., X_n$ $(Y_1, Y_2, ..., Y_n)$ denote the failure (repair) times and $C_1, C_2, ..., C_n$ $(D_1, D_2, ..., D_n)$ denote the random censoring times associated with the failure (repair) times having distribution functions $F_X(F_Y)$ and $G_C(G_D)$ respectively. Suppose that the four random sequences $\{X_i\}$, $\{Y_i\}$, $\{C_i\}$ and $\{D_i\}$ are mutually independent and continuous. Under the censoring model, instead of observing $X_i$, we observe the pair $(T_i, \delta_i)$, $i = 1, 2, ..., n$, where $T_i = \min(X_i, C_i)$ and $\delta_i = I(X_i \le C_i)$. Let $H_X(t) = 1 - (1 - F_X(t))(1 - G_C(t))$ be the distribution function of $T_i$ and $\tau_X = \inf\{x : H_X(x) = 1\} \le \infty$ be the least upper bound for the support of $H_X(.)$.

With right-censored data, the most commonly used nonparametric estimator of $F_X(t)$ is the product limit estimator (PLE) (Kaplan and Meier, 1958)

$$\hat{F}_{X,c}(t) = 1 - \prod_{i=1}^{n} \left[ 1 - \frac{\delta_{(i)}}{n - i + 1} \right]^{I(T_{(i)} \le t)} \quad \text{for } t \le T_{(n)},$$

$$= 1 \qquad \text{for } t > T_{(n)},$$

where $T_{(1)} \le T_{(2)} \le ... \le T_{(n)}$ are the order statistics of $T_1, T_2, ..., T_n$ and $\delta_{(i)}$ denotes the concomitant associated with $T_{(i)}$.

Similarly, we can construct the product limit estimator $\hat{F}_{Y,c}(t)$ of $F_Y(t)$.
Let $H_Y(t) = 1 - (1 - F_Y(t))(1 - G_D(t))$ and $\tau_Y = \inf\{x : H_Y(x) = 1\}$.
Then, a natural nonparametric estimator of $\mu_X(\mu_Y)$ is

$$\hat{\mu}_{X,c} = \int_0^\infty \hat{\bar{F}}_{X,c}(t)dt \quad \left( \hat{\mu}_{Y,c} = \int_0^\infty \hat{\bar{F}}_{Y,c}(t)dt \right),$$

where $\bar{F}_X = 1 - F_X$ $(\bar{F}_Y = 1 - F_Y)$.

Let $\hat{M}_{c,n}(t)$ be an estimator of the renewal function $M(t)$ obtained by replacing $F_X(.)$ and $F_Y(.)$ with $\hat{F}_{X,c}(.)$ and $\hat{F}_{Y,c}(.)$ respectively.

Then,

$$\hat{M}_{c,n}(t) = \sum_{k=1}^{\infty} \hat{F}_{Z,c}^{(k)}(t), \tag{2.10}$$

where $\hat{F}_{Z,c}(t) = \hat{F}_{X,c} * \hat{F}_{Y,c}(t)$.

In this case a nonparametric estimator of $A_{avg}(t)$ is given by

$$\hat{A}_{avg,c}(t) = \frac{\hat{\bar{\alpha}}_{c,n}(t)}{t}, \tag{2.11}$$

where $\hat{\bar{\alpha}}_{c,n}(t) = \hat{\lambda}_c(t)\{(\hat{M}_{c,n}(t)+1)\hat{\mu}_{X,c}\} + (1-\hat{\lambda}_c(t))\{t - \hat{M}_{c,n}(t)\hat{\mu}_{Y,c}\}$,

with $\hat{\lambda}_{c,n}(t) = I\{\hat{M}_{c,n}(t)\hat{\mu}_{Z,c} + \hat{\mu}_{X,c} \le t\}$ and $\hat{\mu}_{Z,c} = \hat{\mu}_{X,c} + \hat{\mu}_{Y,c}$.

Before going to study the asymptotic properties of the estimator $\hat{A}_{avg,c}(t)$, we shall define $\mu_{X,c} = \int_0^{\tau_X} \bar{F}_X(t)dt$, $\mu_{Y,c} = \int_0^{\tau_Y} \bar{F}_Y(t)dt$, $\lambda_c(t) = I\{M(t)\mu_{Z,c} + \mu_{X,c} \le t\}$, $\mu_{Z,c} = \mu_{X,c} + \mu_{Y,c}$ and $\bar{\alpha}_c(t) = \lambda_c(t)\{(M(t)+1)\mu_{X,c}\} + (1-\lambda_c(t))\{t - M(t)\mu_{Y,c}\}$.

**Theorem 2.3**

*As $n \to \infty$, $\hat{A}_{avg,c}(t) \to A_{avg,c}(t)$ almost surely for $t < \tau$, where $\tau = \min(\tau_X, \tau_Y)$ and $A_{avg,c}(t) = \bar{\alpha}_c(t)/t$.*

*Proof*. Li (1999) discussed the nonparametric estimation of the renewal function defined by (2.10) with right censored data and proved that $\hat{M}_{c,n}(t) \to M(t)$ almost surely as $n \to \infty$.

Asymptotic properties of the mean survival time for right censored data have been discussed by Susarala and Van Ryzin (1980) and Stute and Wang (1994). Based on their results it is easy to see that $\hat{\mu}_{X,c} \to \mu_{X,c}$ (a.s.) as $n \to \infty$, where $\mu_{X,c}$ may not be equal to $\mu_X$ since the data $(L_i, \delta_i), i = 1, 2, ..., n$ provide no information about $F_X(.)$ beyond $\tau_X$. Similarly $\hat{\mu}_{Y,c} \to \mu_{Y,c}$ (a.s.) as $n \to \infty$.

Then, for $t < \tau$, it can be shown that $\hat{M}_{c,n}(t)\hat{\mu}_{Z,c} + \hat{\mu}_{X,c} \to M(t)\mu_{Z,c} + \mu_{X,c}$ (a.s.) and hence $\hat{\lambda}_{c,n}(t) \to \lambda_c(t)$ (a.s.) as $n \to \infty$.

Thus, $\hat{\bar{\alpha}}_{c,n}(t) \to \bar{\alpha}_c(t)$ (a.s.) leads to the conclusion that $\hat{A}_{avg,c}(t) \to A_{avg,c}(t)$ almost surely as $n \to \infty$.

**Remark:** *If $F_X$, $F_Y$, $G_C$ and $G_D$ have unbounded support, then $\tau_X = \tau_Y = \infty$ and hence $A_{avg,c}(t) = A_{avg}(t)$. Also if the least upper bound for the support of $F_X$ and $F_Y$ are less than or equal to $\tau_X$ and $\tau_Y$ respectively, even if they have bounded support, $A_{avg,c}(t) = A_{avg}(t)$, as $\mu_{X,c} = \mu_X$ $(\mu_{Y,c} = \mu_Y)$.*

For proving the weak convergence of $\hat{A}_{avg,c}(t)$ we use the following lemma.

**Lemma 2.2** *As $n \to \infty$, $\{\sqrt{n}\Delta_c F_X(t),\ t < \tau_X\}$ $\left(\{\sqrt{n}\Delta_c F_Y(t),\ t < \tau_Y\}\right)$ converges weakly to $\{\bar{F}_X(t)(W_X \circ U_X)(t),\ t < \tau_X\}$ $\left(\{\bar{F}_Y(t)(W_Y \circ U_Y)(t),\ t < \tau_Y\}\right)$, where $\{W_X(t),\ t \ge 0\}$ and $\{W_Y(t),\ t \ge 0\}$ are two independent standard Brownian motions and*

$$U_X(t) = \int_0^t \frac{dF_X(x)}{\bar{F}_X(x)\bar{H}_X(x)} \quad \left(U_Y(t) = \int_0^t \frac{dF_Y(x)}{\bar{F}_Y(x)\bar{H}_Y(x)}\right).$$

*Proof*. See Fleming and Harrington (1990, pp.235).

In order to establish the weak convergence of $\hat{A}_{avg,c}(t)$, we introduce the notations $\Delta_c F_X(t) = \hat{F}_{X,c}(t) - F_X(t)$, $\Delta_c F_Y(t) = \hat{F}_{Y,c}(t) - F_Y(t)$, $\Delta_c \mu_X = \hat{\mu}_{X,c} - \mu_{X,c}$, $\Delta_c \mu_Y = \hat{\mu}_{Y,c} - \mu_{Y,c}$, $\Delta_c M(t) = \hat{M}_{c,n}(t) - M(t)$ and $\Delta_c \lambda(t) = \hat{\lambda}_{c,n}(t) - \lambda(t)$.

Let us denote

$$K_{1,c}(t) = \frac{\lambda_c(t)[M(t)+1]}{t},\ K_{2,c}(t) = \frac{[\lambda_c(t)\mu_{X,c} - (1-\lambda_c(t))\mu_{Y,c}]}{t}$$

and $K_{3,c}(t) = \dfrac{M(t)[1-\lambda_c(t)]}{t}$.

By proceeding in the lines of the proof of Theorem 2.2, we can write

$$\sqrt{n}[\hat{A}_{avg,c}(t) - A_{avg,c}(t)] = \sqrt{n}(I_{1,c} + I_{2,c} + I_{3,c}),$$

where

$$I_{1,c} = K_{1,c}(t)\Delta_c\mu_X + K_{2,c}(t)J_Y * \Delta_c F_X(t),$$

$$I_{2,c} = K_{2,c}(t)J_X * \Delta_c F_Y(t) - K_{3,c}(t)\Delta_c\mu_Y,$$

and $I_{3,c}$ is obtained by replacing terms in $I_3$ defined by (2.6) with the corresponding terms of the censored versions defined in this section. Following the same arguments used in Theorem 2.2, it can be shown that $\sqrt{n}I_{3,c} \to 0$ in probability as $n \to \infty$.

Since $\Delta_c F_X$ and $\Delta_c F_Y$ are independent, $I_{1,c}$ and $I_{2,c}$ are also independent. Hence in order to establish the weak convergence of the estimator, it is sufficient to show that $\sqrt{n}I_{1,c} \xrightarrow{L} N(0, \sigma_{1,c}^2(t))$ and $\sqrt{n}I_{2,c} \xrightarrow{L} N(0, \sigma_{2,c}^2(t))$.

Consider

$$\begin{aligned}\sqrt{n}I_{1,c} &= \sqrt{n}[K_{1,c}(t)\Delta_c\mu_X + K_{2,c}(t)J_Y * \Delta_c F_X(t)] \\ &= \sqrt{n}[K_{1,c}(t)\int_0^{\tau_X} x\,d\Delta_c F_X(x) + K_{2,c}(t)J_Y * \Delta_c F_X(t)] \\ &= \sqrt{n}\int_0^{\tau_X} [K_{1,c}(t)x + K_{2,c}(t)J_Y(t-x)]\,d\Delta_c F_X(x).\end{aligned}$$

Using Lemma 1.3 and Lemma 2.2, it follows that,

$$\sqrt{n}I_{1,c} \xrightarrow{L} \int_0^{\tau_X} [K_{1,c}(t)x + K_{2,c}(t)J_Y(t-x)]\,d\{\bar{F}_X(x)(W_X \circ U_X)(x)\}$$

Consider,

$$\begin{aligned}&\int_0^{\tau_X} [K_{1,c}(t)x + K_{2,c}(t)J_Y(t-x)]\,d\{\bar{F}_X(x)(W_X \circ U_X)(x)\} \\ &\quad = \int_0^{\tau_X} [K_{1,c}(t)x + K_{2,c}(t)J_Y(t-x)]\bar{F}_X(x)\,d(W_X \circ U_X)(x) \\ &\quad\quad + \int_0^{\tau_X} [K_{1,c}(t)x + K_{2,c}(t)J_Y(t-x)](W_X \circ U_X)(x)\,d\bar{F}_X(x). \qquad (2.12)\end{aligned}$$

But we have,

$$\int_0^{\tau_X} [K_{1,c}(t)x + K_{2,c}(t)J_Y(t-x)](W_X \circ U_X)(x)d\bar{F}_X(x)$$

$$= \int_0^{\tau_X} [K_{1,c}(t)x + K_{2,c}(t)J_Y(t-x)]\int_0^{x} d(W_X \circ U_X)(y)d\bar{F}_X(x)$$

$$= \int_0^{\tau_X}\int_0^{x} [K_{1,c}(t)x + K_{2,c}(t)J_Y(t-x)]d(W_X \circ U_X)(y)d\bar{F}_X(x)$$

$$= \int_0^{\tau_X}\int_y^{\tau_X} [K_{1,c}(t)x + K_{2,c}(t)J_Y(t-x)]d\bar{F}_X(x)\, d(W_X \circ U_X)(y),$$

$$= \int_0^{\tau_X} R_X(t,y)\, d(W_X \circ U_X)(y),$$

where, $R_X(t,y) = \int_y^{\tau_X} [K_{1,c}(t)x + K_{2,c}(t)J_Y(t-x)]d\bar{F}_X(x)$.

Now, (2.12) becomes,

$$\int_0^{\tau_X} [K_{1,c}(t)x + K_{2,c}(t)J_Y(t-x)]d\{\bar{F}_X(x)(W_X \circ U_X)(x)\}$$

$$= \int_0^{\tau_X} \Big[[K_{1,c}(t)x + K_{2,c}(t)J_Y(t-x)]\bar{F}_X(x) + R_X(t,x)\Big] d(W_X \circ U_X)(x)$$

$$= \int_0^{U_X(\tau_X)} \Big[\{K_{1,c}(t)U_X^{-1}(y) + K_{2,c}(t)J_Y(t-U_X^{-1}(y))\}\bar{F}_X(U_X^{-1}(y)) + R_X(t,U_X^{-1}(y))\Big] dW_X(y).$$

As an application of Lemma 1.1, $\sqrt{n}I_{1,c}$ is asymptotically normally distributed with mean 0 and variance $\sigma_{1,c}^2(t)$, where

$$\sigma_{1,c}^2(t) = \int_0^{U_X(\tau_X)} \Big[\{K_{1,c}(t)U_X^{-1}(y) + K_{2,c}(t)J_Y(t-U_X^{-1}(y))\}\bar{F}_X(U_X^{-1}(y)) + R_X(t,U_X^{-1}(y))\Big]^2 dy$$

$$= \int_0^{\tau_X} \Big[[K_{1,c}(t)x + K_{2,c}(t)J_Y(t-x)]\bar{F}_X(x) + R_X(t,x)\Big]^2 dU_X(x). \qquad (2.13)$$

Similarly, it can be shown that,

$$\sqrt{n} I_2 \xrightarrow{L} N(0, \sigma^2_{2,c}(t)),$$

where

$$\sigma^2_{2,c}(t) = \int_0^{\tau_Y} \left[ K_{2,c}(t) J_X(t-x) - K_{3,c}(t) x ] \bar{F}_Y(x) + R_Y(t,x) \right]^2 dU_Y(x), \tag{2.14}$$

with $R_Y(t,x) = \int_x^{\tau_X} [K_{2,c}(t) J_X(t-y) - K_{3,c}(t) y] d\bar{F}_Y(y)$.

Thus the weak convergence of the estimator is established by the following theorem.

**Theorem 2.4**

*For any fixed 't', as* $n \to \infty$, $\sqrt{n}[\hat{A}_{avg,c}(t) - A_{avg,c}(t)] \xrightarrow{L} N(0, \sigma^2_c(t))$, *with*

$$\sigma^2_c(t) = \sigma^2_{1,c}(t) + \sigma^2_{2,c}(t), \tag{2.15}$$

*where* $\sigma^2_{1,c}(t)$ *and* $\sigma^2_{2,c}(t)$ *are given by* (2.13) *and* (2.14) *respectively.*

In order to construct a consistent estimator of $\sigma^2_c(t)$, we use a consistent estimator $\hat{U}_X(t)$ of $U_X(t)$ proposed by Miller (1981) and it is given by,

$$\hat{U}_X(t) = \sum_{i:T_{(i)} \le t} \frac{\delta_{(i)}}{(n-i)(n-i+1)},$$

where $\delta_{(i)}$ is the concomitant associated with $T_{(i)}$ as defined before.

Similarly, an estimator $\hat{U}_Y(t)$ of $U_Y(t)$ can be constructed. On replacing $\mu_{X,c}$, $\mu_{Y,c}$, $F_X(.)$, $F_Y(.)$, $U_X(.)$, $U_Y(.)$ and $M(.)$ by $\hat{\mu}_{X,c}$, $\hat{\mu}_{Y,c}$, $\hat{F}_{X,c}(.)$, $\hat{F}_{Y,c}(.)$, $\hat{U}_X(.)$, $\hat{U}_Y(.)$ and $\hat{M}_{c,n}(.)$ respectively in (2.15), an estimator $\hat{\sigma}^2_c(t)$ of $\sigma^2_c(t)$ is obtained, which by using Lemma 1.3, can be shown to be strongly consistent as $n \to \infty$. Thus, given a significance level $\alpha \in (0,1)$, an approximate large sample $100(1-\alpha)\%$ confidence interval for $A_{avg,c}(t)$ is

$$\hat{A}_{avg,c}(t) - z_{\alpha/2} \frac{\hat{\sigma}_c(t)}{\sqrt{n}} \le A_{avg,c}(t) \le \hat{A}_{avg,c}(t) + z_{\alpha/2} \frac{\hat{\sigma}_c(t)}{\sqrt{n}}.$$

### 2.4. Estimation in the case of Continuous Observation over a Fixed Period.

Suppose that the process is observed continuously over a fixed period $[0,T]$. Now, the number of failures and number of repairs completed before the time '$T$' are random variables. Let $N_X(T)$ and $N_Y(T)$ denote the number of completed failures and repairs up to time $T$. Then the empirical estimators for the distribution functions $F_X(t)$ and $F_Y(t)$ can be defined as

$$\hat{F}_{X,T}(t) = \frac{1}{N_X(T)} \sum_{i=1}^{N_X(T)} I\{X_i \le t\} \quad \text{and}$$

$$\hat{F}_{Y,T}(t) = \frac{1}{N_Y(T)} \sum_{i=1}^{N_Y(T)} I\{Y_i \le t\}.$$

In this case, natural nonparametric estimators for $\mu_X$ and $\mu_Y$ are given by

$$\hat{\mu}_X = \int_0^\infty x d\hat{F}_{X,T}(x) = \frac{1}{N_X(T)} \sum_{i=1}^{N_X(T)} X_i = \bar{X}_{N_X(T)} \quad \text{and}$$

$$\hat{\mu}_Y = \int_0^\infty x d\hat{F}_{Y,T}(x) = \frac{1}{N_Y(T)} \sum_{i=1}^{N_Y(T)} Y_i = \bar{Y}_{N_Y(T)} \quad \text{respectively.}$$

An estimator of the renewal function $M(t)$ in this case is given by,

$$\hat{M}_T(t) = \sum_{k=1}^{\infty} \hat{F}_{Z,T}^{(k)}(t),$$

where $\hat{F}_{Z,T}(t) = \hat{F}_{X,T} * \hat{F}_{Y,T}(t)$.

As a nonparametric estimator of $A_{avg}(t)$ we consider

$$\hat{A}_{avg,T}(t) = \frac{\hat{\hat{\alpha}}_T(t)}{t}, \tag{2.16}$$

where $\hat{\hat{\alpha}}_T(t) = \hat{\lambda}_T(t)\{(\hat{M}_T(t)+1)\hat{\mu}_{X,T}\} + (1-\hat{\lambda}_T(t))\{t - \hat{M}_T(t)\hat{\mu}_{Y,T}\}$

with $\hat{\lambda}_T(t) = I\{\hat{M}_T(t)\hat{\mu}_{Z,T} + \hat{\mu}_{X,T} \le t\}$ and $\hat{\mu}_{Z,T} = \hat{\mu}_{X,T} + \hat{\mu}_{Y,T}$.

The strong consistency of the proposed estimator is established in the following theorem.

**Theorem 2.5**

*As* $T \to \infty$, $\hat{A}_{avg,T}(t) \to A_{avg}(t)$ *almost surely.*

*Proof.* The almost sure convergence of $\hat{F}_{X,T}(t)$ and $\hat{F}_{Y,T}(t)$ follows from the fact that both $N_X(T)$ and $N_Y(T)$ tends to infinity as $T \to \infty$. Hence it is straight forward to verify that $\hat{M}_T(t) \to M(t)$ almost surely as $T \to \infty$.

By the strong law of large numbers for random sums, we have, $\hat{\mu}_{X,T} \to \mu_X$, $\hat{\mu}_{Y,T} \to \mu_Y$ and $\hat{\mu}_{Z,T} \to \mu_Z$ (a.s.). As $\hat{M}_T(t)\hat{\mu}_{Z,T} + \hat{\mu}_{X,T} \to M(t)\mu_Z + \mu_X$ (a.s.), $\hat{\lambda}_T(t) \to \lambda(t)$ (a.s.) as $T \to \infty$.

Thus, $\hat{\bar{\alpha}}_T(t) \to \bar{\alpha}(t)$ (a.s.) and hence $\hat{A}_{avg,T}(t) \to A_{avg}(t)$ almost surely as $T \to \infty$.

In order to study the weak convergence of $\hat{A}_{avg,T}(t)$ by introducing the notation $\Delta_T A = \hat{A}_T - A$, proceeding as in Theorem 2.2, we can write,

$$\sqrt{T}[\hat{A}_{avg,T}(t) - A_{avg}(t)] = \sqrt{T}(I_{1,T} + I_{2,T} + I_{3,T}),$$

where

$$I_{1,T} = K_1(t)\Delta_T\mu_X + K_2(t)J_Y * \Delta_T F_X(t),\ I_{2,T} = K_2(t)J_X * \Delta_T F_Y(t) - K_3(t)\Delta_T\mu_Y$$

and $I_{3,T}$ is obtained by replacing $\Delta$ by $\Delta_T$ in $I_3$ defined in (2.6).

Following the arguments in Theorem 2.2 and using the results stated in Ouhbi and Liminos (2003) it is straight forward to show that $\sqrt{T}I_{3,T} \to 0$ in probability as $T \to \infty$.

Consider,

$$\sqrt{T}I_1 = \sqrt{T}[K_1(t)\Delta_T\mu_X + K_2(t)J_Y * \Delta_T F_X(t)]$$

$$= \sqrt{T}\int_0^\infty [K_1(t)x + K_2(t)J_Y(t-x)]d\Delta F_T(x)$$

$$= \sqrt{\frac{T}{N_X(T)}}\sqrt{N_X(T)}\int_0^\infty [K_1(t)x + K_2(t)J_Y(t-x)]d\Delta F_T(x).$$

$$\xrightarrow{L} \sqrt{\mu_X}\int_0^\infty [K_1(t)x + K_2(t)J_Y(t-x)]d(W_X^0 \circ F_X)(x)\text{, by Lemma 2.1}$$

and using the fact that $\frac{N_X(T)}{T} \to \frac{1}{\mu_X}$ as $T \to \infty$.

$$= \int_0^1 \sqrt{\mu_X}[K_1(t)F_X^{-1}(y) + K_2(t)J_Y(t - F_X^{-1}(y))]dW_X^0(y)\text{, by change of}$$

variable.

Proceeding as in Theorem 2.2, the last integral follows a normal distribution with mean 0 and variance $\sigma_{1,T}^2(t)$, where

$$\sigma_{1,T}^2(t) = K_1^2(t)\mu_X\sigma_X^2 + K_2^2(t)\mu_X\left[J_Y^2 * F_X(t) - [J_Y * F_X(t)]^2\right]$$
$$+2K_1(t)K_2(t)\mu_X[J_Y * V_X(t) - \mu_X J_Y * F_X(t)]. \qquad (2.17)$$

Similarly, it can be shown that

$$\sqrt{T}I_T \xrightarrow{L} N(0, \sigma_{2,T}^2(t)),$$

where

$$\sigma_{2,T}^2(t) = K_2^2(t)\mu_Y\left[J_X^2 * F_Y(t) - [J_X * F_Y(t)]^2\right] + K_3^2(t)\mu_Y\sigma_Y^2$$
$$-2K_2(t)K_3(t)\mu_Y[J_X * V_Y(t) - \mu_Y J_X * F_Y(t)]. \qquad (2.18)$$

Thus, we have proved the following theorem.

**Theorem 2.6**

*For any fixed 't', as* $T \to \infty$, $\sqrt{T}[\hat{A}_{avg,T}(t) - A_{avg}(t)] \xrightarrow{L} N(0, \sigma_T^2(t))$, *where*

$$\sigma_T^2(t) = \sigma_{1,T}^2(t) + \sigma_{2,T}^2(t), \qquad (2.19)$$

*with* $\sigma_{1,T}^2(t)$ *and* $\sigma_{2,T}^2(t)$ *are given in* (2.17) *and* (2.18) *respectively.*

Let $S_{X,T}^2 = \frac{1}{N_X(T)}\sum_{i=1}^{N_X(T)}(X_i - \bar{X}_{N_X(T)})^2$ and $S_{Y,T}^2 = \frac{1}{N_Y(T)}\sum_{i=1}^{N_Y(T)}(Y_i - \bar{Y}_{N_Y(T)})^2$

be estimators of $\sigma_X^2$ and $\sigma_Y^2$ respectively.

On replacing $\mu_X$, $\mu_Y$, $\sigma_X^2$, $\sigma_Y^2$, $F_X(.)$, $F_Y(.)$ and $M(.)$ by $\overline{X}_{N_X(T)}$, $\overline{Y}_{N_Y(T)}$, $S_{X,T}^2$, $S_{Y,T}^2$, $\hat{F}_{X,T}(.)$, $\hat{F}_{Y,T}(.)$ and $\hat{M}_T(.)$ respectively in (2.19) we get a consistent estimator $\hat{\sigma}_T^2(t)$ of $\sigma^2(t)$.

Thus, given a significance level $\alpha \in (0,1)$, for large '$T$', an approximate $100(1-\alpha)\%$ confidence interval for $A_{avg}(t)$ is

$$\hat{A}_{avg,T}(t) - z_{\alpha/2}\frac{\hat{\sigma}_T(t)}{\sqrt{T}} \le A_{avg}(t) \le \hat{A}_{avg,T}(t) + z_{\alpha/2}\frac{\hat{\sigma}_T(t)}{\sqrt{T}}.$$

## 2.5 Simulation Study

In this section we present a simulation study in order to assess the performance of the proposed estimator in the case of i) complete observations, ii) censored observations and (iii) continuous observation over a fixed period. We use the algorithm proposed by Schneider et al. (1990) for computing the renewal function. Let $0 = t_0 < t_1 < ... < t_m = t$ be an equally spaced partition of $[0,\ t]$, where the choice of $m$ depends on $t$ and on the data. An algorithm for computing the estimates and the confidence interval for $A_{avg}(t)$ can be summarized as follows.

1. Compute $\hat{F}_X$, $\hat{F}_Y$, $\hat{\mu}_X$, $\hat{\mu}_Y$ and the standard deviations $\hat{\sigma}_X$ and $\hat{\sigma}_Y$.

2. Find $\hat{F}_Z(t_i) = \sum_{j=1}^{m} \hat{F}_X(t_i - t_j)[\hat{F}_Y(t_j) - \hat{F}_Y(t_{j-1})]$ for $i = 1, 2, ..., m.$

3. Evaluate $\hat{M}(t)$ using the recursive relationship

$$\hat{M}(t_i) = \hat{F}_Z(t_i) + \sum_{j=1}^{i} \hat{M}(t_i - t_j)[\hat{F}_Z(t_j) - \hat{F}_Z(t_{j-1})],\ \text{for}\ i = 1, 2, ..., m.$$

   and compute $\hat{A}_{avg}(t)$.

4. Compute $\hat{J}_X(t_i)$, $\hat{J}_Y(t_i)$, $\hat{V}_X(t_i)$ and $\hat{V}_Y(t_i)$ then $\hat{J}_X * \hat{F}_Y(t_i)$, $\hat{J}_X^2 * \hat{F}_Y(t_i)$, $\hat{J}_X * \hat{V}_Y(t_i)$, $\hat{J}_Y * \hat{F}_X(t_i)$, $\hat{J}_Y^2 * \hat{F}_X(t_i)$ and $\hat{J}_Y * \hat{V}_X(t_i)$ recursively for $i = 1, 2, ..., m.$

5. Substitute the values obtained in the above steps to evaluate $\hat{\sigma}^2(t)$.

The same algorithm can be used to compute the confidence interval for $A_{avg,c}(t)$ and $\hat{A}_{avg,T}(t)$ defined in (2.11) and (2.16) respectively after appropriate modifications.

Consider first the case of complete observations. Suppose that the distribution $F_X$ of the failure times is gamma with shape parameter 3 and scale parameter 2 and the repair times also follow a gamma distribution with shape parameter 1 and scale parameter 2. Three time points *t* = 2.5, *t* = 5 and *t* = 7.5 are considered for the simulation. The exact values of $A_{avg}(t)$ at these points are obtained using *Mathematica*. In Table 2.1, '*n*' denotes the number of observations of operating and repair times, $\bar{\hat{A}}_{avg}(t)$ denotes the average of $\hat{A}_{avg}(t)$ over 100 repetitions at '*t*', $\bar{\hat{\sigma}}(t)$ denotes the sample mean of the estimated standard error of the estimate and $A_{avg,L}(t)$ and $A_{avg,U}(t)$ denote the 95% lower and upper confidence limits for $A_{avg}(t)$ respectively. The values given in parenthesis represent the mean square error of the corresponding estimators.

In order to check the performance of the estimator under censoring we suppose that $F_X$ is a gamma distribution with shape parameter 3 and scale parameter 2, and that $F_Y$ is a gamma distribution with shape parameter 2 and scale parameter 1. Further assume that censoring distributions are exponential with $G_C(t) = 1 - e^{-0.05t}$ and $G_D(t) = 1 - e^{-0.1t}$. The results of the simulation study are presented in Table 2.2. Here *X%* and *Y%* denote the average percentage of censoring rate associated with the failure time and the repair time respectively.

Table 2.3 presents the result of the simulation study in the case of continuous observation over a fixed period [0, *T*] using the same distributions for generating the failure and repair times as in the case of complete observations. Here $\bar{N}(T)$ denotes the average number of cycles completed up to time '*T*'.

From the Tables 2.1, 2.2 and 2.3, it can be seen that even for moderate sample sizes, the standard deviation of the estimate is small and the width of the confidence interval is reasonably narrow.

**Table 2.1** Simulation results in the case of complete observations

| $t$ | $A_{avg}(t)$ | $n$ | $\bar{\hat{A}}_{avg}(t)$ | $\bar{\sigma}(t)$ | $A_{avg,L}(t)$ | $A_{avg,U}(t)$ |
|---|---|---|---|---|---|---|
| | | 25 | 0.96171 | 0.00537 | 0.95118 | 0.97223 |
| | | | (0.0324) | (0.0048) | | |
| | | 50 | 0.95916 | 0.00407 | 0.95118 | 0.96714 |
| | | | (0.0267) | (0.0028) | | |
| 2.5 | 0.95852 | 75 | 0.95809 | 0.00346 | 0.95131 | 0.96486 |
| | | | (0.0242) | (0.0021) | | |
| | | 100 | 0.95872 | 0.00296 | 0.95292 | 0.96451 |
| | | | (0.0162) | (0.0012) | | |
| | | 150 | 0.95933 | 0.00236 | 0.95470 | 0.96396 |
| | | | (0.0131) | (0.0008) | | |
| | | 25 | 0.88572 | 0.01532 | 0.85570 | 0.91574 |
| | | | (0.0358) | (0.0056) | | |
| | | 50 | 0.88704 | 0.01146 | 0.86459 | 0.90950 |
| | | | (0.0272) | (0.0033) | | |
| 5 | 0.88641 | 75 | 0.88895 | 0.00895 | 0.87141 | 0.90650 |
| | | | (0.0209) | (0.0019) | | |
| | | 100 | 0.88650 | 0.00793 | 0.87095 | 0.90204 |
| | | | (0.0170) | (0.0015) | | |
| | | 150 | 0.88462 | 0.00674 | 0.87141 | 0.89783 |
| | | | (0.0151) | (0.0010) | | |
| | | 25 | 0.84120 | 0.02231 | 0.79746 | 0.88493 |
| | | | (0.0324) | (0.0059) | | |
| | | 50 | 0.84571 | 0.01518 | 0.81595 | 0.87547 |
| | | | (0.0223) | (0.0026) | | |
| 7.5 | 0.84232 | 75 | 0.84136 | 0.01297 | 0.81595 | 0.86677 |
| | | | (0.0199) | (0.0021) | | |
| | | 100 | 0.84458 | 0.01085 | 0.82332 | 0.86585 |
| | | | (0.0178) | (0.0016) | | |
| | | 150 | 0.84190 | 0.00913 | 0.82400 | 0.85980 |
| | | | (0.0138) | (0.0010) | | |

**Table 2.2** Simulation results in the case of censored observations

| $t$ | $A_{avg}(t)$ | $n$ | $\bar{\hat{A}}_{avg,c}(t)$ | $\bar{\hat{\sigma}}_c(t)$ | $X\,\%$ | $Y\,\%$ | $A_{avg,L}(t)$ | $A_{avg,U}(t)$ |
|---|---|---|---|---|---|---|---|---|
| | | 25 | 0.96102 | 0.00584 | 25.16 | 16.28 | 0.94956 | 0.97247 |
| | | | (0.0233) | (0.0044) | | | | |
| | | 50 | 0.96091 | 0.00425 | 25.76 | 16.98 | 0.95258 | 0.96924 |
| | | | (0.0157) | (0.0020) | | | | |
| 2.5 | 0.95852 | 75 | 0.96077 | 0.00351 | 24.25 | 17.57 | 0.95389 | 0.96765 |
| | | | (0.0136) | (0.0014) | | | | |
| | | 100 | 0.96109 | 0.00284 | 24.19 | 17.07 | 0.95552 | 0.96666 |
| | | | (0.0117) | (0.0010) | | | | |
| | | 150 | 0.96090 | 0.00243 | 24.57 | 17.80 | 0.95613 | 0.96567 |
| | | | (0.0085) | (0.0006) | | | | |
| | | 25 | 0.89173 | 0.01792 | 25.68 | 17.88 | 0.85661 | 0.92686 |
| | | | (0.0338) | (0.0189) | | | | |
| | | 50 | 0.88793 | 0.01196 | 24.50 | 17.16 | 0.86449 | 0.91136 |
| | | | (0.0227) | (0.0032) | | | | |
| 5 | 0.88641 | 75 | 0.88795 | 0.00979 | 24.35 | 17.85 | 0.86877 | 0.90713 |
| | | | (0.0190) | (0.0026) | | | | |
| | | 100 | 0.88843 | 0.00839 | 24.28 | 16.88 | 0.87199 | 0.90487 |
| | | | (0.0160) | (0.0017) | | | | |
| | | 150 | 0.88728 | 0.00709 | 24.71 | 17.61 | 0.87339 | 0.90117 |
| | | | (0.0130) | (0.0013) | | | | |
| | | 25 | 0.84946 | 0.02408 | 25.28 | 17.48 | 0.80227 | 0.89665 |
| | | | (0.0379) | (0.0221) | | | | |
| | | 50 | 0.84442 | 0.01586 | 25.18 | 16.72 | 0.81334 | 0.87551 |
| | | | (0.0242) | (0.0034) | | | | |
| 7.5 | 0.84232 | 75 | 0.84543 | 0.01325 | 24.88 | 17.56 | 0.81946 | 0.87140 |
| | | | (0.0193) | (0.0023) | | | | |
| | | 100 | 0.84333 | 0.01187 | 25.53 | 17.16 | 0.82007 | 0.86659 |
| | | | (0.0157) | (0.0018) | | | | |
| | | 150 | 0.84212 | 0.00982 | 24.33 | 17.29 | 0.82289 | 0.86136 |
| | | | (0.0157) | (0.0015) | | | | |

**Table 2.3** Simulation results in the case of continuous observation over a fixed period '*T*'

| $t$ | $A_{avg}(t)$ | $T$ | $\bar{\hat{A}}_{avg,T}(t)$ | $\bar{\hat{\sigma}}_T(t)$ | $\bar{N}(T)$ | $A_{avg,L}(t)$ | $A_{avg,U}(t)$ |
|---|---|---|---|---|---|---|---|
| 2.5 | 0.95852 | 250 | 0.94723 (0.0308) | 0.00343 (0.0022) | 31.05 | 0.94380 | 0.95066 |
| | | 500 | 0.96432 (0.0214) | 0.00160 (0.0009) | 61.93 | 0.96271 | 0.96592 |
| | | 1000 | 0.95839 (0.0199) | 0.00133 (0.0006) | 125.42 | 0.95707 | 0.95972 |
| 5 | 0.88641 | 250 | 0.88814 (0.0134) | 0.00627 (0.0015) | 31.36 | 0.88187 | 0.89440 |
| | | 500 | 0.88191 (0.0111) | 0.00558 (0.0007) | 63.61 | 0.87633 | 0.88749 |
| | | 1000 | 0.88599 (0.0131) | 0.00401 (0.0005) | 126.37 | 0.88198 | 0.89000 |
| 7.5 | 0.84232 | 250 | 0.84661 (0.0393) | 0.00856 (0.0019) | 32.32 | 0.83805 | 0.85518 |
| | | 500 | 0.84413 (0.0217) | 0.00632 (0.0011) | 60.43 | 0.83781 | 0.85046 |
| | | 1000 | 0.84164 (0.0114) | 0.00506 (0.0007) | 127.69 | 0.83658 | 0.84670 |

## 2.6 Data Analysis

We carry out a data analysis to illustrate an application of the proposed estimation procedure using compressor failure data given in Table 7.1 and Table 11.7 of Rausand and Høyland (2004). The data consists of the operating and repair times of 90 critical failures of a specific compressor at a Norwegian process plant in the time period from 1968 until 1989. In the given data set the failure times are measured in days and the repair times are measured in hours. For the meaningful computation purpose we convert the failure time data to hours and the data are summarized in Table 2.4. The average availability $A_{avg}(t)$ is estimated at various time points using the given data set. The 95% confidence intervals are also computed for the average availability at these time points and are summarized in Table 2.5.

**Table 2.4** Compressor failure data (Rausand and Høyland, 2004)

| Sl. No. | Failure Time | Repair Time | Sl. No. | Failure Time | Repair Time | Sl. No. | Failure Time | Repair Time |
|---|---|---|---|---|---|---|---|---|
| 1 | 24 | 1.25 | 31 | 2160 | 0.5 | 61 | 1104 | 0.25 |
| 2 | 72 | 135 | 32 | 2064 | 18 | 62 | 552 | 2.25 |
| 3 | 12 | 0.08 | 33 | 3168 | 2.5 | 63 | 1848 | 13.5 |
| 4 | 2100 | 5.33 | 34 | 4.8 | 0.33 | 64 | 1584 | 0.5 |
| 5 | 3840 | 154 | 35 | 4.8 | 0.5 | 65 | 12 | 0.25 |
| 6 | 600 | 0.5 | 36 | 4.8 | 2 | 66 | 60 | 0.17 |
| 7 | 12 | 1.25 | 37 | 4.8 | 0.33 | 67 | 1848 | 1.75 |
| 8 | 168 | 2.5 | 38 | 52.8 | 4 | 68 | 12 | 0.5 |
| 9 | 2148 | 15 | 39 | 168 | 20 | 69 | 132 | 1 |
| 10 | 1584 | 6 | 40 | 312 | 6 | 70 | 1680 | 2 |
| 11 | 96 | 4.5 | 41 | 768 | 6.3 | 71 | 1008 | 2 |
| 12 | 744 | 32.5 | 42 | 1416 | 15 | 72 | 4392 | 38 |
| 13 | 1464 | 9.5 | 43 | 4320 | 23 | 73 | 96 | 0.33 |
| 14 | 768 | 0.25 | 44 | 24 | 4 | 74 | 12672 | 2 |
| 15 | 4224 | 81 | 45 | 228 | 5 | 75 | 2976 | 40.5 |
| 16 | 3360 | 12 | 46 | 168 | 28 | 76 | 2376 | 4.28 |
| 17 | 480 | 0.25 | 47 | 15132 | 16 | 77 | 96 | 1.62 |
| 18 | 2724 | 1.66 | 48 | 504 | 11.5 | 78 | 3312 | 1.33 |
| 19 | 6492 | 5 | 49 | 1608 | 0.42 | 79 | 12 | 3 |
| 20 | 1176 | 7 | 50 | 5808 | 38.33 | 80 | 6492 | 5 |
| 21 | 24 | 39 | 51 | 12 | 10.5 | 81 | 3552 | 120 |
| 22 | 312 | 106 | 52 | 60 | 9.5 | 82 | 1008 | 0.5 |
| 23 | 1008 | 6 | 53 | 1248 | 8.5 | 83 | 4.8 | 3 |
| 24 | 456 | 5 | 54 | 1368 | 17 | 84 | 7.2 | 3 |
| 25 | 24 | 17 | 55 | 192 | 34 | 85 | 3012 | 11.58 |
| 26 | 24 | 5 | 56 | 24 | 0.17 | 86 | 6432 | 8.5 |
| 27 | 3168 | 2 | 57 | 1056 | 0.83 | 87 | 3312 | 13.5 |
| 28 | 12 | 2 | 58 | 5892 | 0.75 | 88 | 3096 | 29.5 |
| 29 | 684 | 0.33 | 59 | 156 | 1 | 89 | 12 | 29.5 |
| 30 | 24 | 0.17 | 60 | 504 | 0.25 | 90 | 2364 | 112 |

**Table 2.5** Average availability computation of compressor failure data

| $t$ | $\hat{A}_{avg}(t)$ | $\hat{\sigma}(t)$ | $A_{avg,L}(t)$ | $A_{avg,U}(t)$ |
|---|---|---|---|---|
| 7.5 | 0.92065 | 0.15394 | 0.88885 | 0.95245 |
| 10.0 | 0.91404 | 0.16677 | 0.87958 | 0.94849 |
| 12.5 | 0.87392 | 0.24460 | 0.82339 | 0.92446 |
| 15.0 | 0.85940 | 0.27276 | 0.80305 | 0.91576 |
| 17.5 | 0.87735 | 0.23793 | 0.82819 | 0.92651 |
| 20.0 | 0.88976 | 0.21385 | 0.84558 | 0.93394 |
| 25.0 | 0.88773 | 0.21772 | 0.84274 | 0.93271 |

## 2.7 Conclusion

We have discussed the nonparametric estimation of the average availability when the failure and repair times of a system are mutually independent sequences of i.i.d. random variables. The proposed estimators of the average availability are proved to be consistent and asymptotically normal when (i) the data are complete, (ii) the data are subject to right censorship, and (iii) the data are observed over a fixed time period. The simulation study shows that the proposed estimators perform well even for reasonable sample sizes. Finally, the estimation procedure corresponding to the complete sample is illustrated using a real life data.

# Chapter 3

# Nonparametric Estimation of the Interval Reliability

### 3.1 Introduction

An important measure of successful performance of a system in the context of repairable system is the interval reliability. The interval reliability, $R(x,t)$, of a repairable system is defined as the probability that the system is operating at a specified time '*t*' and will continue to operate for an interval of duration '*x*'. See for example, Barlow and Hunter (1961). If $\xi(t)$ represents the status of the system at time '*t*' as defined in (1.1), the interval reliability of the system is given by,

$$R(x,t) = P[\xi(s) = 1,\ t \le s \le t + x]. \tag{3.1}$$

From the definition, it is clear that, the interval reliability becomes reliability when $t = 0$ and point availability at time '*t*' as $x \to 0$. Thus, the interval reliability is one of the most important measures of system performance from the viewpoint of reliability and availability, and is useful in many practical situations. A typical example is the model of a standby generator, in which '*t*' is the time until the electric power stops and '*x*' is the required time until the electric power recovers again. In this case, the interval reliability represents the probability that a standby generator will be able to operate during the interruption of the electric power (cf., Nakagawa 2005).

Let $\{X_n\}$ and $\{Y_n\}$ be independent sequences of i.i.d. non-negative random variables with common distribution functions $F_X(.)$ and $F_Y(.)$

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Some results of this chapter have been communicated as entitled 'Nonparametric Estimation of the Interval Reliability' (See Balakrishna and Mathew, 2011a).

respectively. Define $Z_n = X_n + Y_n$ and let $F_Z(.) = F_X * F_Y(.)$ be its distribution function. Also let $S_n = \sum_{i=1}^{n} Z_i$ and $M(t) = \sum_{k=1}^{\infty} F_Z^{(k)}(t)$ be the renewal function associated with the sequence $\{Z_n\}$, where $F_Z^{(k)}(t) = P[S_k \le t]$.

Now, the interval reliability can be written as:

$$R(x,t) = \overline{F}_X(t+x) + \sum_{k=1}^{\infty} \int_0^t P[X_{n+1} > t + x - u \,|\, S_n = u] dF_Z^{(k)}(u) . \tag{3.2}$$

The first term in the interval reliability function reflects the probability that the first period of operation is of length $t+x$ or greater. The subsequent integral expressions reflect the probability that the $k-th$ failure occurs at time $u$ and the following period of operation is of length $t+x-u$ or greater.

Since $X_{n+1}$ is independent of $S_n$, (3.2) becomes

$$R(x,t) = \overline{F}_X(t+x) + \sum_{k=1}^{\infty} \int_0^t P[X_{n+1} > t + x - u] dF_Z^{(k)}(u)$$

$$= \overline{F}_X(t+x) + \sum_{k=1}^{\infty} \int_0^t \overline{F}_X(t+x-u) dF_Z^{(k)}(u)$$

$$= \overline{F}_X(t+x) + \int_0^t \overline{F}_X(t+x-u) dM(u) . \tag{3.3}$$

For example, when $F_X(t) = 1 - e^{-\lambda_1 t}$ and $F_Y(t) = 1 - e^{-\lambda_2 t}$, the interval reliability function is given by:

$$R(x,t) = \left[ \frac{\lambda_2}{\lambda_1 + \lambda_2} + \frac{\lambda_1}{\lambda_1 + \lambda_2} e^{-(\lambda_1 + \lambda_2)t} \right] e^{-\lambda_1 x}$$

$$= A(t)\overline{F}_X(x) .$$

Thus, when the sequences of failure and repair times are generated from two independent exponential distributions, the interval reliability function $R(x,t)$ is the product of the point availability function at time '$t$', $A(t)$ and the reliability function at time '$x$', $\overline{F}_X(x)$.

In general, the interval reliability may not be the product of point availability function and reliability function. However, as a combined measure of availability and reliability, the interval reliability has a significant role in the study of repairable system performance.

We consider the nonparametric estimation of the interval reliability in this chapter. The organization of this chapter is as follows: Section 3.2 discusses the nonparametric estimation of the interval reliability when the data on '*n*' complete cycles of system operation are available. Section 3.3 discusses the estimation when the data are subject to right censorship and in Section 3.4, we consider the estimation when the process is observed up to a specified time '*T*'. In Section 3.5, a simulation study is presented and an application of the proposed method is illustrated using a compressor failure data. Finally, Section 3.6 provides brief conclusions of the study.

### 3.2 Estimation in the case of Complete Observations

Suppose that observations on the failure and repair times of '*n*' complete cycles of system operation, $(X_1, Y_1), (X_2, Y_2), ..., (X_n, Y_n)$ are available. In this case, a nonparametric estimator of the interval reliability, $R(x,t)$, is given by:

$$\hat{R}(x,t) = \hat{\bar{F}}_X(t+x) + \int_0^t \hat{\bar{F}}_X(t+x-u) d\hat{M}_n(u), \tag{3.4}$$

where $\hat{\bar{F}}_X(.)$ and $\hat{M}_n(.)$ are the nonparametric estimators of $\bar{F}_X(.)$ and $M(.)$ defined as in Section 2.2.

Baxter and Li (1994) studied asymptotic properties of the estimator $\hat{M}_n(t)$ and shown that $\hat{M}_n(t) \to M(t)$ almost surely as $n \to \infty$. By Gilvenko-Cantelli theorem, $\hat{F}_X(u) \to F_X(u)$ uniformly in $u$ as $n \to \infty$ with probability one.

Hence, by Lemma 1.3, $\hat{R}(x,t) \to R(x,t)$ almost surely as $n \to \infty$ for any fixed $t$ and $x$.

In order to study the weak convergence of $\hat{R}(x,t)$, let us denote $\Delta F_X(t) = \hat{F}_X(t) - F_X(t)$, $\Delta F_Y(t) = \hat{F}_Y(t) - F_Y(t)$ and $\Delta M(t) = \hat{M}_n(t) - M(t)$.

We have,

$$\Delta R(x,t) = \hat{R}(x,t) - R(x,t)$$

$$= \hat{\overline{F}}_X(t+x) - \overline{F}_X(t+x) + \int_0^t \hat{\overline{F}}_X(t+x-u)d\hat{M}(u) - \int_0^t \overline{F}_X(t+x-u)dM(u)$$

$$= \Delta\overline{F}_X(t+x) + \Delta[\int_0^t \overline{F}_X(t+x-u)dM(u)]$$

$$= \Delta\overline{F}_X(t+x) + \Delta[\overline{F}_X(x)M(t) + \int_0^t M(u)dF_X(t+x-u)].$$

Now using the fact that $\Delta[AB] = \hat{A}\hat{B} - AB = \Delta A\Delta B + A\Delta B + B\Delta A$, we can write

$$\Delta R(x,t) = \Delta\overline{F}_X(t+x) + M(t)\Delta\overline{F}_X(x) + \overline{F}_X(x)\Delta M(t) + \int_0^t \Delta M(u)dF_X(t+x-u)$$
$$+\int_0^t M(u)d\Delta F_X(t+x-u) + Q_1, \tag{3.5}$$

where $Q_1 = \Delta\overline{F}_X(x)\Delta M(t) + \int_0^t \Delta M(u)d\Delta F_X(t+x-u)$.

Following the equation (2.5) derived in Chapter 2, we can write

$$\Delta M(t) = M * M * \Delta F_Z(t) + \Delta F_Z * \Delta F_Z * \hat{M} * M(t),$$

where $\Delta F_Z(t) = \hat{F}_X * \hat{F}_Y(t) - F_X * F_Y(t)$

$$= \Delta F_X * \Delta F_Y + F_X * \Delta F_Y + F_Y * \Delta F_X.$$

Thus,

$$\Delta M(t) = M * M * F_Y * \Delta F_X(t) + M * M * F_X * \Delta F_Y(t) + \Delta Q(t)$$
$$= J_Y * \Delta F_X(t) + J_X * \Delta F_Y(t) + \Delta Q(t),$$

where $J_X(t) = M * M * F_X(t)$, $J_Y(t) = M * M * F_Y(t)$ and

$$\Delta Q(t) = M * M * \Delta F_X * \Delta F_Y(t) + \Delta F_Z * \Delta F_Z * \hat{M} * M(t).$$

Substituting for $\Delta M(t)$ in (3.5), we get

$$\Delta R(x,t) = \Delta\overline{F}_X(t+x) + M(t)\Delta\overline{F}_X(x) + \overline{F}_X(x)J_Y * \Delta F_X(t) + \overline{F}_X(x)J_X * \Delta F_Y(t)$$

$$+\int_0^t J_Y * \Delta F_X(u)dF_X(t+x-u) + \int_0^t J_X * \Delta F_Y(u)dF_X(t+x-u)$$

$$+\int_0^t M(u)d\Delta F_X(t+x-u) + Q_1 + Q_2,$$

where $Q_2 = \Delta\overline{F}_X(x)\Delta Q(t) + \int_0^t \Delta Q(u)d\Delta F_X(t+x-u)$.

Now,

$$\sqrt{n}\Delta R(x,t) = \sqrt{n}I_1 + \sqrt{n}I_2 + \sqrt{n}(Q_1 + Q_2),$$

where $I_1 = \Delta\overline{F}_X(t+x) + M(t)\Delta\overline{F}_X(x) + \overline{F}_X(x)J_Y * \Delta F_X(t)$

$$+\int_0^t J_Y * \Delta F_X(u)dF_X(t+x-u) + \int_0^t M(u)d\Delta F_X(t+x-u)$$

and $I_2 = \overline{F}_X(x)J_X * \Delta F_Y(t) + +\int_0^t J_X * \Delta F_Y(u)dF_X(t+x-u)$.

Here $Q_1$ and $Q_2$ contain terms of the form $\Delta A\Delta B$ and $\int_0^t \Delta A(u)d\Delta B(t+x-u)$.

By writing,

$$\sqrt{n}\int_0^t \Delta A(u)d\Delta B(t+x-u) = \sqrt{n}\int_0^t \Delta A(u)d\hat{B}(t+x-u) - \sqrt{n}\int_0^t \Delta A(u)dB(t+x-u),$$

it is easy to see that the two terms on the right-hand side converge almost surely to the same limit by Lemma 1.3 and hence $\sqrt{n}(Q_1 + Q_2) \to 0$ in probability as $n \to \infty$.

Define, $V_1(x,t) = \int_0^t J_Y(t-u)dF_X(u+x)$.

Now,

$$\int_0^t J_Y * \Delta F_X(u)dF_X(t+x-u) = \int_0^t\int_0^u J_Y(u-y)d\Delta F_X(y)dF_X(t+x-u)$$

$$= \int_0^t \int_y^t J_Y(u-y)\,dF_X(t+x-u)\,d\Delta F_X(y)$$

$$= -\int_0^t V_1(x,t-y)\,d\Delta F_X(y).$$

Thus, $I_1$ can be written as

$$I_1 = -\int_0^{t+x} d\Delta F_X(u) - M(t)\int_0^x d\Delta F_X(u) + \overline{F}_X(x)\int_0^t J_Y(t-u)\,d\Delta F_X(u)$$

$$-\int_0^t V_1(x,t-u)\,d\Delta F_X(u) - \int_x^{t+x} M(t+x-u)\,d\Delta F_X(u).$$

If we define,

$$K_1(x,u) = I_{(0<u<t)}[\overline{F}_X(x)J_Y(t-u) - V_1(x,t-u)] - I_{(0<u<x)}[M(t)+1]$$

$$-I_{(x<u<t+x)}[M(t+x-u)+1],$$

we can write

$$\sqrt{n}I_1 = \sqrt{n}\int_0^\infty K_1(x,u)\,d\Delta F_X(u)$$

$$\xrightarrow{L} \int_0^\infty K_1(x,u)\,d(W_X^0 \circ F_X)(u), \text{ by Helly-Bray Theorem}$$

$$= \int_0^1 K_1(x,F_X^{-1}(y))\,dW_X^0(y), \text{ by change of variable.}$$

As an application of Lemma 1.2, $\sqrt{n}I_1$ follows a normal distribution with mean 0 and variance $\sigma_1^2(x,t)$ as $n \to \infty$, where

$$\sigma_1^2(x,t) = \int_0^1 \left[K_1(x,F_X^{-1}(y))\right]^2 dy - \left[\int_0^1 K_1(x,F_X^{-1}(y))\,dy\right]^2$$

$$= \int_0^\infty \left[K_1(x,u)\right]^2 dF_X(u) - \left[\int_0^\infty K_1(x,u)\,dF_X(u)\right]^2. \tag{3.6}$$

If we define, $V_2(x,t) = \int_0^t J_X(t-u)\,dF_Y(u+x)$, we can write

$$\sqrt{n}I_2 = \sqrt{n}\int_0^{\infty} K_2(x,u)d\Delta F_Y(u),$$

where $K_2(x,u) = I_{(0<u<t)}[\bar{F}_X(x)J_X(t-u) - V_2(x,t-u)]$.

Proceeding similarly as above, it can be shown that as $n \to \infty$,

$$\sqrt{n}I_2 \xrightarrow{L} N(0, \sigma_2^2(x,t)),$$

where

$$\sigma_2^2(x,t) = \int_0^{\infty} [K_2(x,u)]^2 \, dF_Y(u) - \left[\int_0^{\infty} K_2(x,u)dF_Y(u)\right]^2. \tag{3.7}$$

Since $\Delta F_X$ and $\Delta F_Y$ are independent, $I_1$ and $I_2$ are also independent. This leads to the following theorem.

**Theorem 3.1**

*For any fixed t and x, as* $n \to \infty$,

*(i)* $\hat{R}(x,t) \to R(x,t)$ *almost surely and*

*(ii)* $\sqrt{n}[\hat{R}(x,t) - R(x,t)] \xrightarrow{L} N(0, \sigma^2(x,t))$, *where*

$$\sigma^2(x,t) = \sigma_1^2(x,t) + \sigma_2^2(x,t), \tag{3.8}$$

*with* $\sigma_1^2(x,t)$ *and* $\sigma_2^2(x,t)$ *are given by* (3.6) *and* (3.7) *respectively.*

**Remark:** *If we choose x as 0, then the estimator of the interval reliability,* $\hat{R}(x,t)$, *reduces to the estimator of the point availability,* $\hat{A}(t) = \hat{R}(0,t)$ *and the asymptotic properties of* $\hat{A}(t)$ *follows immediately from Theorem 3.1 by choosing* $x = 0$.

An estimator $\hat{\sigma}^2(x,t)$ of $\sigma^2(x,t)$ can be obtained on replacing $F_X(.)$, $F_Y(.)$ and $M(.)$ by $\hat{F}_X(.)$, $\hat{F}_Y(.)$ and $\hat{M}_n(.)$ in (3.8) respectively. Using Lemma 1.3, it can be shown that $\hat{\sigma}^2(x,t) \to \sigma^2(x,t)$ almost surely as $n \to \infty$.

Thus, given a significance level $\alpha \in (0,1)$, an approximate large sample $100(1-\alpha)\%$ confidence interval for $R(x,t)$ is

$$\hat{R}(x,t) - z_{\alpha/2}\frac{\hat{\sigma}(x,t)}{\sqrt{n}} \le R(x,t) \le \hat{R}(x,t) + z_{\alpha/2}\frac{\hat{\sigma}(x,t)}{\sqrt{n}},$$

where $z_{\alpha/2}$ denotes the upper $\alpha/2$ quantile of the standard normal distribution.

### 3.3 Estimation in the case of Censored Observations

Suppose that observations on the failure and repair time are subject to right censorship. In order to avoid repetition, we follow the same notations defined in Section 2.3.

In this case a nonparametric estimator of $R(x,t)$ is given by

$$\hat{R}_c(x,t) = \hat{\bar{F}}_{X,c}(t+x) + \int_0^t \hat{\bar{F}}_{X,c}(t+x-u)d\hat{M}_{c,n}(u), \tag{3.9}$$

where $\hat{\bar{F}}_{X,c}(.)$ and $\hat{M}_{c,n}(.)$ are the nonparametric estimators of $\bar{F}_X(.)$ and $M(.)$ defined as in Section 2.3.

Li (1999) discussed the nonparametric estimation of the renewal function with right-censored data and proved that $\hat{M}_{c,n}(t) \to M(t)$ almost surely as $n \to \infty$. Hence, by the uniform convergence of $\hat{F}_{X,c}(.)$ and using Lemma 1.3, it is easy to show that, for any fixed $t+x<\tau$, $\hat{R}_c(x,t) \to R(x,t)$ almost surely as $n \to \infty$.

In order to establish the weak convergence of $\hat{R}_c(x,t)$, let us define, $\Delta_c F_X(t) = \hat{F}_{X,c}(t) - F_X(t)$, $\Delta_c F_Y(t) = \hat{F}_{Y,c}(t) - F_Y(t)$ and $\Delta_c M(t) = \hat{M}_c(t) - M(t)$.

By proceeding in the lines of the proof of Theorem 3.1, we can write

$$\sqrt{n}[\hat{R}_c(x,t) - R(x,t)] = \sqrt{n}I_{1,c} + \sqrt{n}I_{2,c} + \sqrt{n}(Q_{1c} + Q_{2,c}),$$

where $I_{1,c}$, $I_{2,c}$, $Q_{1c}$ and $Q_{2,c}$ are obtained by replacing $\Delta F_X(t)$ and $\Delta F_Y(t)$ with $\Delta_c F_X(t)$ and $\Delta_c F_Y(t)$ respectively in $I_1$, $I_2$, $Q_1$ and $Q_2$ defined in Section 3.2. Following the same arguments used in Theorem 3.1, it can be shown that $\sqrt{n}(Q_{1c}+Q_{2,c}) \to 0$ in probability as $n \to \infty$.

Since $\Delta_c F_X(.)$ and $\Delta_c F_Y(.)$ are independent, $I_{1,c}$ and $I_{2,c}$ are also independent. Hence in order to establish the weak convergence of the estimator, $\hat{R}_c(x,t)$, it is sufficient to show that $\sqrt{n} I_{1,c} \xrightarrow{L} N(0, \sigma_{1,c}^2(x,t))$ and $\sqrt{n} I_{2,c} \xrightarrow{L} N(0, \sigma_{2,c}^2(x,t))$.

Consider,

$$\sqrt{n} I_{1,c} = \sqrt{n} \int_0^{t+x} K_1(x,u) d\Delta_c F_X(u)$$

$$\xrightarrow{L} \int_0^{t+x} K_1(x,u) d\{\overline{F}_X(u)(W_X \circ U_X)(u)\}, \text{ by Lemma 2.2.}$$

$$= \int_0^{t+x} K_1(x,u)\overline{F}_X(u) d(W_X \circ U_X)(u) + \int_0^{t+w} K_1(x,u)(W_X \circ U_X)(u) d\overline{F}_X(u).$$

Now,

$$\int_0^{t+x} K_1(x,u)(W_X \circ U_X)(u) d\overline{F}_X(u) = \int_0^{t+x} K_1(x,u) \int_0^u d(W_X \circ U_X)(z) d\overline{F}_X(u)$$

$$= \int_0^{t+x} \int_z^{t+x} K_1(x,u) d\overline{F}_X(u) d(W_X \circ U_X)(z)$$

$$= \int_0^{t+x} P_1(x,z) d(W_X \circ U_X)(z),$$

where $P_1(x,z) = \int_z^{t+x} K_1(x,u) d\overline{F}_X(u)$.

Thus,

$$\sqrt{n} I_{1,c} \xrightarrow{L} \int_0^{t+x} [K_1(x,u)\overline{F}_X(u) + P_1(x,u)] d(W_X \circ U_X)(u)$$

$$= \int_0^{U_X(t+x)} [K_1(x,U_X^{-1}(y))\bar{F}_X(U_X^{-1}(y)) + P_1(x,U_X^{-1}(y))]dW_X(y).$$

As an application of Lemma 1.1, $\sqrt{n}I_{1,c}$ follows a normal distribution with mean 0 and variance $\sigma_{1,c}^2(x,t)$, as $n \to \infty$, where

$$\sigma_{1,c}^2(x,t) = \int_0^{U_X(t+x)} \left[K_1(x,U_X^{-1}(y))\bar{F}_X(U_X^{-1}(y)) + P_1(x,U_X^{-1}(y))\right]^2 d(y)$$

$$= \int_0^{t+x} \left[K_1(x,u)\bar{F}_X(u) + P_1(x,u)\right]^2 dU_X(u), \tag{3.10}$$

with $U_X(t) = \int_0^t \frac{dF_X(u)}{\bar{F}_X(u)\bar{H}_X(u)}$.

Similarly, it can be shown that

$$\sqrt{n}I_{2,c} \xrightarrow{L} N(0,\sigma_{2,c}^2(x,t)),$$

where

$$\sigma_{2,c}^2(x,t) = \int_0^t \left[K_2(x,u)\bar{F}_Y(u) + P_2(x,u)\right]^2 dU_Y(u), \tag{3.11}$$

with $P_2(x,u) = \int_u^t K_2(x,y)d\bar{F}_Y(y)$ and $U_Y(t) = \int_0^t \frac{dF_Y(u)}{\bar{F}_Y(u)\bar{H}_Y(u)}$.

Thus, we have proved the following theorem.

**Theorem 3.2**

*For any fixed t and x, as* $n \to \infty$,

*(i)* $\hat{R}_c(x,t) \to R(x,t)$ *almost surely for* $t + x < \tau$, *where* $\tau = \min(\tau_X, \tau_Y)$

*(ii)* $\sqrt{n}[\hat{R}_c(x,t) - R(x,t)] \xrightarrow{L} N(0,\sigma_c^2(x,t))$ *with*

$$\sigma_c^2(x,t) = \sigma_{1,c}^2(x,t) + \sigma_{2,c}^2(x,t), \tag{3.12}$$

*where* $\sigma_{1,c}^2(x,t)$ *and* $\sigma_{2,c}^2(x,t)$ *are given by* (3.10) *and* (3.11) *respectively.*

On replacing $F_X(.)$, $F_Y(.)$, $U_X(.)$, $U_Y(.)$ by their corresponding consistent estimators in (3.12), a consistent estimator $\hat{\sigma}_c^2(x,t)$ of $\sigma_c^2(x,t)$ is obtained. Thus, given a significance level $\alpha \in (0,1)$, an approximate large sample $100(1-\alpha)\%$ confidence interval for $R(x,t)$ is

$$\hat{R}_c(x,t) - z_{\alpha/2}\frac{\hat{\sigma}_c(x,t)}{\sqrt{n}} \le R(x,t) \le \hat{R}_c(x,t) + z_{\alpha/2}\frac{\hat{\sigma}_c(x,t)}{\sqrt{n}}.$$

**3.4 Estimation in the case of Continuous Observation over a Fixed Period.**

Suppose that the process is observed continuously over a fixed period $[0,T]$. In this case, as a nonparametric estimator of $R(x,t)$ we consider

$$\hat{R}_T(x,t) = \hat{\bar{F}}_{X,T}(t+x) + \int_0^t \hat{\bar{F}}_{X,T}(t+x-u)d\hat{M}_T(u), \tag{3.13}$$

where $\hat{\bar{F}}_{X,T}(.)$ and $\hat{M}_T(.)$ are the nonparametric estimators of $\bar{F}_X(.)$ and $M(.)$ defined as in Section 2.4.

The almost sure convergence of $\hat{F}_{X,T}(t)$ and $\hat{F}_{Y,T}(t)$ follows from the fact that both $N_X(T)$ and $N_Y(T)$ tend to infinity as $T \to \infty$. Thus it is straightforward to verify that $\hat{M}_T(t) \to M(t)$ almost surely and hence $\hat{R}_T(x,t) \to R(x,t)$ almost surely as $T \to \infty$.

Introducing the notation $\Delta_T A = \hat{A}_T - A$ and proceeding similar to Section 3.2, we can write,

$$\sqrt{T}[\hat{R}_T(x,t) - R(x,t)] = \sqrt{T} I_{1,T} + \sqrt{T} I_{2,T} + \sqrt{T}(Q_{1,T} + Q_{2,T}),$$

where $I_{1,T}$, $I_{2,T}$, $Q_{1,T}$ and $Q_{2,T}$ are obtained by replacing $\Delta$ by $\Delta_T$ in $I_1$, $I_2$, $Q_1$ and $Q_2$ respectively.

Following the arguments in Theorem 3.1 and using the results applied in Section 2.4, it is straightforward to show that $\sqrt{T}(Q_{1,T} + Q_{2,T}) \to 0$ in probability as $T \to \infty$.

Writing $\sqrt{T} I_{1,T} = \sqrt{\frac{T}{N_X(T)}} \sqrt{N_X(T)}\; I_{1,T}$ and using the fact that $N_X(T)/T \to 1/\mu_X$ as $T \to \infty$, proceeding in the lines of the proof of Theorem 3.1, we can show that $\sqrt{T} I_{1,T} \xrightarrow{L} N(0, \mu_X \sigma_1^2(x,t))$ and $\sqrt{T} I_{2,T} \xrightarrow{L} N(0, \mu_Y \sigma_2^2(x,t))$ as $T \to \infty$, where $\sigma_1^2(x,t)$ and $\sigma_2^2(x,t)$ are given in (3.6) and (3.7) respectively.

This leads to the following theorem.

**Theorem 3.3**

*For any fixed t and x, as* $T \to \infty$,

*(i)* $\hat{R}_T(x,t) \to R(x,t)$ *almost surely and*

*(ii)* $\sqrt{T}[\hat{R}_T(x,t) - R(x,t)] \xrightarrow{L} N\left(0, \sigma_T^2(x,t)\right)$, *where*

$$\sigma_T^2(x,t) = \mu_X \sigma_1^2(x,t) + \mu_Y \sigma_2^2(x,t)\,. \tag{3.14}$$

On replacing $\mu_X$, $\mu_Y$, $F_X(.)$, $F_Y(.)$ and $M(.)$ by $\overline{X}_{N_X(T)}$, $\overline{Y}_{N_Y(T)}$, $\hat{F}_{X,T}(.)$, $\hat{F}_{Y,T}(.)$ and $\hat{M}_T(.)$ respectively in (3.14) we get a consistent estimator $\hat{\sigma}_T^2(x,t)$ of $\sigma^2(x,t)$. Thus, given $\alpha \in (0,1)$, for large '$T$', an approximate $100(1-\alpha)\%$ confidence interval for $R(x,t)$ is

$$\hat{R}_T(x,t) - z_{\alpha/2} \frac{\hat{\sigma}_T(x,t)}{\sqrt{T}} \le R(x,t) \le \hat{R}_T(x,t) + z_{\alpha/2} \frac{\hat{\sigma}_T(x,t)}{\sqrt{T}}\,.$$

## 3.5 Numerical Study

In this section we carry out a simulation study to assess the finite sample performance of the proposed estimators in the case of i) complete observations, ii) censored observations and (iii) continuous observation over a fixed period.

For the case of complete observations, we assume that the distribution of the failure times is gamma with shape parameter 3 and scale parameter 2 and the repair times also follow a gamma distribution with shape parameter 1 and scale parameter 2. The time points $t$ = 2.5, 5 and $x$ = 0, 0.25, 0.5 are considered for the simulation study. The exact values of $R(x,t)$ at these points are obtained using *Mathematica*. The results of the simulation study are summarized in Table 3.1. Here '*n*' denotes the number of completed cycles of the failure and repair times, $\hat{\hat{R}}(x,t)$ and $\hat{\hat{\sigma}}(x,t)$ denote the average of $\hat{R}(x,t)$ and $\hat{\sigma}(x,t)$ over 100 repetitions, and $R_L(x,t)$ and $R_U(x,t)$ denote the 95% lower and upper confidence limits for $R(x,t)$ respectively. The values given in parenthesis represent the mean square error (MSE) of the corresponding estimators.

In order to check the performance of the estimator under censoring scheme we use the same distribution for the failure and repair times as in the case of complete observations. Further we assume that censoring distributions are exponential with cumulative distribution functions $G_C(t) = 1 - e^{-0.05t}$ and $G_D(t) = 1 - e^{-0.1t}$ respectively. The results of the simulation study are presented in Table 3.2. The average percentage of censoring rate associated with the failure time and the repair time are denoted by *X*% and *Y*% respectively.

Table 3.3 summarizes the result of the simulation study in the case of continuous observation over a fixed period [0, *T*] using the same distributions for generating the failure and repair times as in the case of complete observations. Here $\bar{N}(T)$ denotes the average number of cycles completed up to time '*T*'.

From the simulation study, we see that even for moderate sample sizes, the proposed estimators perform well and the width of the confidence interval is reasonably narrow.

We also carry out a data analysis to illustrate an application of the proposed estimation procedure using compressor failure data given in Table 2.4 of Chapter 2. The interval reliability $R(x,t)$ is computed with $t$ = 15, 20, 25 and $x$ = 0, 2.5, 5, 7.5 hours. The 95% confidence intervals are also computed for the interval reliability at these time points and are summarized in Table 3.4.

**Table 3.1** Simulation results in the case of complete observations

| $t$ | $x$ | $n$ | $R(x,t)$ | $\hat{\hat{R}}(x,t)$ | $\hat{\hat{\sigma}}(x,t)$ | $R_L(x,t)$ | $R_U(x,t)$ |
|---|---|---|---|---|---|---|---|
| 2.5 | 0 | 25 | 0.84728 | 0.83935 (0.0645) | 0.39748 (0.0635) | 0.68354 | 0.99516 |
| | | 50 | | 0.84549 (0.0564) | 0.40419 (0.0527) | 0.73346 | 0.95753 |
| | | 100 | | 0.84908 (0.0341) | 0.40733 (0.0294) | 0.76924 | 0.92891 |
| | 0.25 | 25 | 0.81613 | 0.81467 (0.0766) | 0.40872 (0.0539) | 0.65445 | 0.97489 |
| | | 50 | | 0.81078 (0.0558) | 0.42052 (0.0345) | 0.69422 | 0.92734 |
| | | 100 | | 0.81683 (0.0334) | 0.42569 (0.0223) | 0.73339 | 0.90026 |
| | 0.5 | 25 | 0.78454 | 0.77501 (0.0818) | 0.43977 (0.0543) | 0.60262 | 0.94739 |
| | | 50 | | 0.78299 (0.0723) | 0.43798 (0.0473) | 0.66159 | 0.90440 |
| | | 100 | | 0.78022 (0.0343) | 0.44630 (0.0255) | 0.69275 | 0.86770 |
| 5 | 0 | 25 | 0.75778 | 0.74911 (0.0720) | 0.51735 (0.0219) | 0.54631 | 0.95191 |
| | | 50 | | 0.75393 (0.0450) | 0.52114 (0.0140) | 0.60948 | 0.89838 |
| | | 100 | | 0.75921 (0.0478) | 0.52129 (0.0100) | 0.65704 | 0.86138 |
| | 0.25 | 25 | 0.72567 | 0.73218 (0.0856) | 0.51854 (0.0254) | 0.52891 | 0.93545 |
| | | 50 | | 0.72249 (0.0567) | 0.52121 (0.0134) | 0.57802 | 0.86696 |
| | | 100 | | 0.72111 (0.0442) | 0.52819 (0.0105) | 0.61759 | 0.82464 |
| | 0.5 | 25 | 0.69385 | 0.69972 (0.0852) | 0.52903 (0.0233) | 0.49234 | 0.90710 |
| | | 50 | | 0.69184 (0.0555) | 0.52957 (0.0173) | 0.54505 | 0.83863 |
| | | 100 | | 0.69323 (0.0414) | 0.53257 (0.0109) | 0.58885 | 0.79762 |
| 7.5 | 0 | 25 | 0.74775 | 0.75098 (0.0872) | 0.60868 (0.0365) | 0.51238 | 0.98958 |
| | | 50 | | 0.74835 (0.0675) | 0.61096 (0.0197) | 0.57900 | 0.91770 |
| | | 100 | | 0.74861 (0.0520) | 0.60723 (0.0163) | 0.62959 | 0.86763 |
| | 0.25 | 25 | 0.71640 | 0.71810 (0.0896) | 0.61437 (0.0301) | 0.47727 | 0.95893 |
| | | 50 | | 0.71631 (0.0613) | 0.61921 (0.0223) | 0.54467 | 0.88795 |
| | | 100 | | 0.71551 (0.0505) | 0.61631 (0.0141) | 0.59471 | 0.83631 |
| | 0.5 | 25 | 0.68529 | 0.68428 (0.0772) | 0.62085 (0.0316) | 0.44091 | 0.92765 |
| | | 50 | | 0.68724 (0.0601) | 0.62112 (0.0216) | 0.51507 | 0.85941 |
| | | 100 | | 0.68606 (0.0517) | 0.62053 (0.0148) | 0.56444 | 0.80768 |

**Table 3.2** Simulation results in the case of censored observations

| $t$ | $x$ | $n$ | $R(x,t)$ | $\hat{\hat{R}}_c(x,t)$ | $\hat{\hat{\sigma}}_c(x,t)$ | X% | Y% | $R_L(x,t)$ | $R_U(x,t)$ |
|---|---|---|---|---|---|---|---|---|---|
| 2.5 | 0 | 25 | 0.84728 | 0.83986 (0.0765) | 0.39047 (0.0587) | 22.88 | 16.16 | 0.68680 | 0.99292 |
| | | 50 | | 0.85047 (0.0445) | 0.39299 (0.0403) | 26.08 | 17.84 | 0.74154 | 0.95940 |
| | | 100 | | 0.84564 (0.0276) | 0.41556 (0.0293) | 24.72 | 17.52 | 0.76419 | 0.92709 |
| | 0.25 | 25 | 0.81613 | 0.82063 (0.0877) | 0.39935 (0.0781) | 25.44 | 19.04 | 0.66409 | 0.97718 |
| | | 50 | | 0.81019 (0.0396) | 0.43771 (0.0310) | 22.40 | 17.20 | 0.68887 | 0.93152 |
| | | 100 | | 0.81966 (0.0410) | 0.42795 (0.0358) | 22.84 | 17.16 | 0.73578 | 0.90354 |
| | 0.5 | 25 | 0.78454 | 0.79149 (0.0631) | 0.42736 (0.0561) | 23.04 | 17.44 | 0.62396 | 0.95901 |
| | | 50 | | 0.77585 (0.0473) | 0.45014 (0.0331) | 24.48 | 18.16 | 0.65108 | 0.90063 |
| | | 100 | | 0.78033 (0.0315) | 0.45793 (0.0220) | 22.52 | 17.72 | 0.69058 | 0.87009 |
| 5 | 0 | 25 | 0.75778 | 0.74990 (0.0726) | 0.52929 (0.0215) | 22.56 | 19.68 | 0.54242 | 0.95738 |
| | | 50 | | 0.76730 (0.0664) | 0.53135 (0.0159) | 25.36 | 17.84 | 0.62002 | 0.91459 |
| | | 100 | | 0.75363 (0.0410) | 0.54908 (0.0119) | 23.40 | 17.60 | 0.64601 | 0.86125 |
| | 0.25 | 25 | 0.72567 | 0.73263 (0.0733) | 0.53007 (0.0222) | 23.04 | 17.28 | 0.52484 | 0.94042 |
| | | 50 | | 0.72778 (0.0603) | 0.51482 (0.0209) | 25.28 | 16.88 | 0.58508 | 0.87048 |
| | | 100 | | 0.72619 (0.0367) | 0.55210 (0.0137) | 25.32 | 16.96 | 0.61798 | 0.83440 |
| | 0.5 | 25 | 0.69385 | 0.69539 (0.0858) | 0.54801 (0.0494) | 24.96 | 16.96 | 0.48057 | 0.91021 |
| | | 50 | | 0.69273 (0.0452) | 0.53414 (0.0194) | 24.72 | 18.16 | 0.54468 | 0.84079 |
| | | 100 | | 0.69655 (0.0379) | 0.55862 (0.0160) | 25.04 | 17.40 | 0.58706 | 0.80604 |
| 7.5 | 0 | 25 | 0.74775 | 0.74551 (0.0702) | 0.61146 (0.0275) | 24.16 | 19.18 | 0.50582 | 0.98520 |
| | | 50 | | 0.74957 (0.0573) | 0.60993 (0.0246) | 23.84 | 16.76 | 0.58051 | 0.91863 |
| | | 100 | | 0.74879 (0.0396) | 0.60854 (0.0135) | 23.50 | 17.88 | 0.62952 | 0.86806 |
| | 0.25 | 25 | 0.71640 | 0.71425 (0.0637) | 0.61803 (0.0286) | 25.04 | 16.48 | 0.47198 | 0.95652 |
| | | 50 | | 0.71785 (0.0495) | 0.61995 (0.0218) | 24.20 | 17.72 | 0.54601 | 0.88969 |
| | | 100 | | 0.71549 (0.0310) | 0.61908 (0.0127) | 24.00 | 17.70 | 0.59415 | 0.83683 |
| | 0.5 | 25 | 0.68529 | 0.69205 (0.0786) | 0.62094 (0.0261) | 25.20 | 19.04 | 0.44864 | 0.93546 |
| | | 50 | | 0.68404 (0.0415) | 0.62017 (0.0191) | 24.36 | 17.20 | 0.51214 | 0.85594 |
| | | 100 | | 0.68591 (0.0334) | 0.62138 (0.0147) | 24.50 | 18.10 | 0.56412 | 0.80770 |

**Table 3.3** Simulation results in the case of continuous observation over a fixed period '*T*'

| $t$ | $x$ | $T$ | $R(x,t)$ | $\hat{\hat{R}}_T(x,t)$ | $\hat{\hat{\sigma}}_T(x,t)$ | $\bar{N}(T)$ | $R_L(x,t)$ | $R_U(x,t)$ |
|---|---|---|---|---|---|---|---|---|
| 2.5 | 0 | 250 | 0.84728 | 0.84345 | 0.95755 | 31.44 | 0.72475 | 0.96215 |
| | | | | (0.0805) | (0.1047) | | | |
| | | 500 | | 0.84780 | 0.98334 | 60.72 | 0.76161 | 0.93400 |
| | | | | (0.0562) | (0.0983) | | | |
| | | 1000 | | 0.84643 | 0.98311 | 125.32 | 0.78550 | 0.90737 |
| | | | | (0.0288) | (0.0584) | | | |
| | 0.25 | 250 | 0.81613 | 0.82279 | 1.04782 | 32.16 | 0.69290 | 0.95268 |
| | | | | (0.0612) | (0.0974) | | | |
| | | 500 | | 0.81274 | 1.02838 | 64.00 | 0.72260 | 0.90288 |
| | | | | (0.0531) | (0.0628) | | | |
| | | 1000 | | 0.81556 | 1.04327 | 123.72 | 0.75089 | 0.88022 |
| | | | | (0.0212) | (0.0387) | | | |
| | 0.5 | 250 | 0.78454 | 0.78330 | 1.06905 | 30.40 | 0.65078 | 0.91583 |
| | | | | (0.0574) | (0.0848) | | | |
| | | 500 | | 0.79001 | 1.08108 | 62.00 | 0.69525 | 0.88477 |
| | | | | (0.0460) | (0.0627) | | | |
| | | 1000 | | 0.78617 | 1.08486 | 124.92 | 0.71893 | 0.85341 |
| | | | | (0.0394) | (0.0459) | | | |
| 5 | 0 | 250 | 0.75778 | 0.76089 | 1.25714 | 31.84 | 0.60505 | 0.91673 |
| | | | | (0.0711) | (0.0732) | | | |
| | | 500 | | 0.75936 | 1.25272 | 63.36 | 0.64955 | 0.86916 |
| | | | | (0.0640) | (0.0594) | | | |
| | | 1000 | | 0.75889 | 1.28293 | 123.28 | 0.67937 | 0.83841 |
| | | | | (0.0338) | (0.0304) | | | |
| | 0.25 | 250 | 0.72567 | 0.73020 | 1.28595 | 30.44 | 0.57079 | 0.88961 |
| | | | | (0.0823) | (0.0875) | | | |
| | | 500 | | 0.72102 | 1.27204 | 63.52 | 0.60952 | 0.83252 |
| | | | | (0.0557) | (0.0645) | | | |
| | | 1000 | | 0.72740 | 1.29428 | 123.36 | 0.64718 | 0.80762 |
| | | | | (0.0441) | (0.0466) | | | |
| | 0.5 | 250 | 0.69385 | 0.68660 | 1.28439 | 31.56 | 0.52739 | 0.84582 |
| | | | | (0.0865) | (0.1125) | | | |
| | | 500 | | 0.70012 | 1.28668 | 61.64 | 0.58734 | 0.81290 |
| | | | | (0.0566) | (0.0619) | | | |
| | | 1000 | | 0.69146 | 1.30958 | 124.00 | 0.61029 | 0.77263 |
| | | | | (0.0336) | (0.0294) | | | |
| 7.5 | 0 | 250 | 0.74775 | 0.74905 | 1.47665 | 30.68 | 0.56600 | 0.93210 |
| | | | | (0.0851) | (0.0754) | | | |
| | | 500 | | 0.74889 | 1.47736 | 62.32 | 0.61939 | 0.87839 |
| | | | | (0.0545) | (0.0483) | | | |
| | | 1000 | | 0.74733 | 1.48785 | 123.94 | 0.65511 | 0.83955 |
| | | | | (0.0477) | (0.0447) | | | |
| | 0.25 | 250 | 0.71640 | 0.71815 | 1.49230 | 30.26 | 0.53316 | 0.90314 |
| | | | | (0.0812) | (0.0831) | | | |
| | | 500 | | 0.71706 | 1.50387 | 62.36 | 0.58524 | 0.84888 |
| | | | | (0.0565) | (0.0586) | | | |
| | | 1000 | | 0.71655 | 1.48486 | 124.72 | 0.62452 | 0.80858 |
| | | | | (0.0432) | (0.0364) | | | |
| | 0.5 | 250 | 0.68529 | 0.68430 | 1.52581 | 31.22 | 0.49516 | 0.87344 |
| | | | | (0.0789) | (0.0986) | | | |
| | | 500 | | 0.68327 | 1.54502 | 62.30 | 0.54784 | 0.81870 |
| | | | | (0.0544) | (0.0606) | | | |
| | | 1000 | | 0.68637 | 1.53015 | 124.82 | 0.59153 | 0.78121 |
| | | | | (0.0374) | (0.0427) | | | |

**Table 3.4** Interval reliability computation of compressor failure data

| $t$ | $x$ | $\hat{\hat{R}}(x,t)$ | $\hat{\sigma}(x,t)$ | $R_L(x,t)$ | $R_U(x,t)$ |
|---|---|---|---|---|---|
| 15 | 0 | 0.97716 | 0.36676 | 0.90139 | 1.00000 |
| | 2.5 | 0.97138 | 0.38039 | 0.89279 | 1.00000 |
| | 5 | 0.96710 | 0.39054 | 0.88641 | 1.00000 |
| | 7.5 | 0.96564 | 0.39399 | 0.88424 | 1.00000 |
| 20 | 0 | 0.97330 | 0.38119 | 0.89454 | 1.00000 |
| | 2.5 | 0.97166 | 0.38531 | 0.89206 | 1.00000 |
| | 5 | 0.89060 | 0.44750 | 0.79814 | 0.98305 |
| | 7.5 | 0.88750 | 0.45354 | 0.79380 | 0.98120 |
| 25 | 0 | 0.92945 | 0.43896 | 0.83876 | 1.00000 |
| | 2.5 | 0.92560 | 0.44657 | 0.83334 | 1.00000 |
| | 5 | 0.92123 | 0.45485 | 0.82726 | 1.00000 |
| | 7.5 | 0.91884 | 0.45956 | 0.82390 | 1.00000 |

In Table 3.4, the upper limit of the confidence interval for the interval reliability, $R_U(x,t)$, is greater than 1 at several time points because the estimated value of the interval reliability at these time points is near to its maximum value 1. So, if $R_U(x,t)$ is greater than 1, it is replaced by 1 in the Table to make meaningful interpretation.

### 3.6 Conclusion

In this Chapter, we considered the nonparametric estimation of the interval reliability when the failure and repair times of a system are mutually independent sequences of i.i.d. random variables. The proposed estimators of the interval reliability are proved to be consistent and asymptotically normal when (i) the data are complete, (ii) the data are subject to right censorship and (iii) the data are observed over a fixed period. The simulation study confirmed the performance of the proposed estimators for reasonable sample sizes. Finally, the estimation procedure corresponding to the complete sample is illustrated using a real life data.

# Chapter 4

# Nonparametric Estimation of the Limiting Interval Reliability

## 4.1 Introduction

In many practical situations, one may be interested in knowing the extent to which the system will survive an interval of duration after it has been run for a long time. In such situation, the limiting interval reliability may be used as an appropriate measure for system effectiveness. Also, since it is difficult to obtain closed form expressions for the interval reliability, except for few simple cases, in the literature more attention is being paid to its limiting measure. The limiting interval reliability is the limiting value of the interval reliability, $R(x,t)$, as $t \to \infty$. In this chapter, we consider the nonparametric estimation of the limiting interval reliability when the failure and repair times form a bivariate sequence of i.i.d. random variables.

Let $\{(X_n, Y_n),\ n \geq 1\}$ be a bivariate sequence of independent and identically distributed (i.i.d) non-negative random variables with marginal distribution functions $F_X(.)$ and $F_Y(.)$ respectively. Assume that $F_X(.)$ and $F_Y(.)$ have positive means $\mu_X$ and $\mu_Y$ and finite variances $\sigma_X^2$ and $\sigma_Y^2$ respectively. Under this set-up it is proved by using the theory of renewal processes that

$$R(x,t) \to R(x) = \frac{\upsilon(x)}{\mu_X + \mu_Y} \text{ as } t \to \infty,$$

where $\upsilon(x) = \int_x^\infty \overline{F}_X(u)du = \int_0^\infty (u-x) I_{(u>x)} dF_X(u)$ (See Barlow and Hunter 1961).

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Some results of this chapter have been communicated as entitled 'Nonparametric Estimation of the Interval Reliability' (See Balakrishna and Mathew, 2011a).

If we define $\psi(.) = \overline{F}_X(.)/\mu_X$ as the density of the asymptotic recurrence time of a renewal process governed by the distribution function $F_X(.)$, the limiting interval reliability, $R(x)$ can be written in the form

$$R(x) = \frac{\mu_X}{\mu_X + \mu_Y}[1 - \Psi(x)],$$

where $\Psi(.)$ is the distribution function with density $\psi(.)$.

Thus, the limiting interval reliability, $R(x)$ is the product of the limiting probability that the system is available at some point and the limiting probability that it survives an interval of duration at least '$x$' (Baxter, 1981).

For example, when $F_X(t) = 1 - e^{-\lambda_1 t}$ and $F_Y(t) = 1 - e^{-\lambda_2 t}$, the limiting interval reliability function is given by:

$$R(x) = \frac{\lambda_2}{\lambda_1 + \lambda_2} e^{-\lambda_1 x}$$

$$= A \cdot \overline{F}_X(x),$$

where $A = \dfrac{\lambda_2}{\lambda_1 + \lambda_2}$ is the limiting availability.

Thus, when the sequences of failure and repair times are generated from two independent exponential distributions, the limiting interval reliability function, $R(x)$, is the product of the limiting availability, $A$ and the reliability function at time '$x$', $\overline{F}_X(x)$.

The chapter is organized as follows. In Section 4.2, we discuss the nonparametric estimation of the limiting interval reliability when the observations on failure and repair times of '$n$' complete cycles are available. Section 4.3 discusses the estimation in the case of censored observations. In Section 4.4, we consider the estimation of $R(x)$ when the process is observed up to a specified time '$T$'. Some numerical illustrations are presented in Section 4.5 to assess the performance of the proposed estimators and finally a conclusion of the study is provided in Section 4.6.

### 4.2 Estimation in the case of Complete Observations

Suppose that observations on the failure and repair times of '$n$' complete cycles $(X_1,Y_1),(X_2,Y_2),...,(X_n,Y_n)$ are available. Let $\hat{F}_X(t)$ and $\hat{F}_Y(t)$ denote the empirical distribution functions of the random variables $X$ and $Y$ respectively. By definition,

$$\hat{F}_X(t)=\frac{1}{n}\sum_{i=1}^{n}I_{(X_i\le t)} \text{ and } \hat{F}_Y(t)=\frac{1}{n}\sum_{i=1}^{n}I_{(Y_i\le t)}.$$

Then, natural nonparametric estimators for $\mu_X$ and $\mu_Y$ are given by

$$\hat{\mu}_X=\int_0^\infty x d\hat{F}_X(x)=\bar{X} \text{ and } \hat{\mu}_Y=\int_0^\infty x d\hat{F}_Y(x)=\bar{Y} \text{ respectively.}$$

Since $\upsilon(x)=\int_x^\infty \bar{F}_X(u)du=\int_0^\infty (u-x)I_{(u>x)}dF_X(u)$, a natural nonparametric estimator of $\upsilon(x)$ is

$$\hat{\upsilon}(x)=\int_0^\infty (u-x)I_{(u>x)}d\hat{F}_X(u)=\bar{U},$$

where $\bar{U}=\frac{1}{n}\sum_{i=1}^{n}U_i$ with $U_i=(X_i-x)I_{(X_i>x)}$.

Thus, a natural nonparametric estimator of $R(x)$ is given by,

$$R_n(x)=\frac{\bar{U}}{\bar{X}+\bar{Y}}. \tag{4.1}$$

By the strong law of large numbers, we have as $n\to\infty$, $\bar{X}\to\mu_X$, $\bar{Y}\to\mu_Y$ and $\bar{U}\to\upsilon(x)$ almost surely and hence we conclude that $R_n(x)\to R(x)$ almost surely as $n\to\infty$.

Since $\{(X_n,Y_n),\ n\ge 1\}$ is a bivariate sequence of i.i.d. random variables, by central limit theorem we have as $n\to\infty$

$$\sqrt{n}\left(\bar{X}-\mu_X,\bar{Y}-\mu_Y\right)\xrightarrow{L}N_2(\mathbf{0},\Sigma_2),$$

where $N_2(0,\Sigma_2)$ is a 2-variate normal vector with mean $\mathbf{0'}=(0,0)$ and dispersion matrix

$$\Sigma_2 = \begin{pmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{pmatrix}.$$

Now, by the Cramer-Wold device (Billingsley, 1968, pp.49) we have as $n \to \infty$

$$\sqrt{n}\left(\bar{X}-\mu_X, \bar{Y}-\mu_Y, \bar{U}-\upsilon(x)\right) \xrightarrow{L} N_3(\mathbf{0},\Sigma_3),$$

where $N_3(0,\Sigma_3)$ is a 3-variate normal vector with mean $\mathbf{0'}=(0,0,0)$ and dispersion matrix

$$\Sigma_3 = \begin{pmatrix} \sigma_X^2 & \sigma_{XY} & \sigma_{XU} \\ \sigma_{XY} & \sigma_Y^2 & \sigma_{YU} \\ \sigma_{XU} & \sigma_{YU} & \sigma_U^2 \end{pmatrix},$$

with $\sigma_U^2 = \mathrm{var}(U_i)$, $\sigma_{XY} = \mathrm{cov}(X_i,Y_i)$, $\sigma_{XU} = \mathrm{cov}(X_i,U_i)$ and $\sigma_{YU} = \mathrm{cov}(Y_i,U_i)$.

If we define, $f(x,y,z) = z/(x+y)$, then $f(\bar{X},\bar{Y},\bar{U}) = R_n(x)$.

Now, the partial derivatives of $f(.)$ are

$$\left.\frac{\partial f}{\partial x}\right|_{(\mu_X,\mu_Y,\upsilon(x))} = -\frac{\upsilon(x)}{(\mu_X+\mu_X)^2},$$

$$\left.\frac{\partial f}{\partial y}\right|_{(\mu_X,\mu_Y,\upsilon(x))} = -\frac{\upsilon(x)}{(\mu_X+\mu_X)^2}, \text{ and}$$

$$\left.\frac{\partial f}{\partial z}\right|_{(\mu_X,\mu_Y,\upsilon(x))} = \frac{1}{(\mu_X+\mu_X)^2}.$$

Hence by using Lemma 1.4, it can be verified that,

$$\sqrt{n}\left(R_n(x)-R(x)\right) \xrightarrow{L} N\left(0,\tau^2(x)\right) \text{ as } n \to \infty,$$

where

$$\tau^2(x) = \frac{\upsilon^2(x)(\sigma_X^2+2\sigma_{XY}+\sigma_Y^2)}{(\mu_X+\mu_Y)^4} - \frac{2\upsilon(x)(\sigma_{XU}+\sigma_{YU})}{(\mu_X+\mu_Y)^3} + \frac{\sigma_U^2}{(\mu_X+\mu_Y)^2}. \qquad (4.2)$$

Thus we have the following theorem.

**Theorem 4.1**

*For any fixed 'x', as* $n \to \infty$,

(i) $R_n(x) \to R(x)$ *almost surely and*

(ii) $\sqrt{n}\left(R_n(x) - R(x)\right) \xrightarrow{L} N\left(0, \tau^2(x)\right)$,

*where* $\tau^2(x)$ *is given in* (4.2) .

Let $s_X^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$, $s_Y^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$, $s_U^2 = \frac{1}{n-1}\sum_{i=1}^{n}(U_i - \bar{U})^2$, $s_{XY} = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})$, $s_{XU} = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})(U_i - \bar{U})$, and $s_{YU} = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})(U_i - \bar{U})$ be estimators of $\sigma_X^2$, $\sigma_Y^2$, $\sigma_U^2$, $\sigma_{XY}$, $\sigma_{XU}$ and $\sigma_{YU}$ respectively. Then an estimator $\hat{\tau}(x)$ of $\tau(x)$ can be obtained on replacing $\mu_X$, $\mu_Y$, $\upsilon(x)$, $\sigma_X^2$, $\sigma_Y^2$, $\sigma_U^2$, $\sigma_{XY}$, $\sigma_{XU}$ and $\sigma_{YZ}$ by $\bar{X}$, $\bar{Y}$, $\bar{U}$, $s_X^2$, $s_Y^2$, $s_U^2$, $s_{XU}$, $s_{XY}$, and $s_{YU}$ respectively. Now it is straightforward to verify that $\hat{\tau}(x) \to \tau(x)$ almost surely as $n \to \infty$.

Thus, given a significance level $\alpha \in (0,1)$, an approximate large sample $100(1-\alpha)\%$ confidence interval for $R(x)$ is

$$R_n(x) - z_{\alpha/2}\frac{\hat{\tau}(x)}{\sqrt{n}} \le R(x) \le R_n(x) + z_{\alpha/2}\frac{\hat{\tau}(x)}{\sqrt{n}}.$$

If we define, $Z_i = X_i + Y_i$, then $Z_i$ represents the length of the *i*-th cycle of system operation and $\{Z_i\}$ have mean $\mu_Z = \mu_X + \mu_Y$ and variance $\sigma_Z^2 = \sigma_X^2 + 2\sigma_{XY} + \sigma_Y^2$.

Now, the asymptotic variance of the estimator $R_n(x)$ given in (4.2) can be written as

$$\tau^2(x) = \frac{\upsilon^2(x)}{\mu_Z^4}\sigma_Z^2 - \frac{2\upsilon(x)}{\mu_Z^3}\sigma_{ZU} + \frac{\sigma_U^2}{\mu_Z^2}.$$

$$= R_n^2(x)\frac{\sigma_Z^2}{\mu_Z^2} - 2R_n^2(x)\frac{\sigma_{ZU}}{\mu_Z \upsilon(x)} + R_n^2(x)\frac{\sigma_U^2}{\upsilon^2(x)}$$

$$= R_n^2(x)\left[c_Z^2 - 2c_{ZU} + c_U^2\right], \qquad (4.3)$$

where $c_Z = \dfrac{\sigma_Z}{\mu_Z}$, and $c_U = \dfrac{\sigma_U}{\upsilon(x)}$ are respectively the coefficients of variation of $Z$ and $U$, and $c_{ZU} = \dfrac{\sigma_{ZU}}{\mu_Z \upsilon(x)}$.

Thus, the asymptotic variance of $R_n(x)$ is functionally dependent on $R(x)$. This aspect pose some difficulty in testing a hypothesis about $R(x)$ by using $R_n(x)$, the rejection region would thus depends upon $R(x)$. However by a suitable transformation $h(R_n(x))$, we can achieve the feature that $h(R_n(x))$ is asymptotically $N\left(h(R(x)), \kappa^2(x)\right)$, where $\kappa^2(x)$ does not depend on $R(x)$. This technique is called a variance stabilizing transformation. In order to obtain $h(R_n(x))$ is asymptotically $N\left(h(R(x)), \kappa^2(x)\right)$, we choose $h$ to be the solution of the differential equation

$$\frac{dh}{dR(x)} = \frac{c}{R(x)}, \text{ where } c \text{ is a constant.}$$

Solving the above equation for $h$ we get

$$h(R(x)) = -\log\left(R(x)\right) = \log\left(1/R(x)\right), \text{ by choosing } c = -1.$$

Obviously,

$$\sqrt{n}\left(h(R_n(x)) - h(R(x))\right) \xrightarrow{L} N(0, \kappa^2(x)), \text{ as } n \to \infty,$$

where $\kappa^2(x) = c_Z^2 - 2c_{ZU} + c_U^2$.

As a consequence, an approximate large sample $100(1-\alpha)\%$ confidence interval for $h(R(x))$ is

$$h(R_n(x)) - z_{\alpha/2}\frac{\hat{\kappa}(x)}{\sqrt{n}} \le h(R(x)) \le h(R_n(x)) + z_{\alpha/2}\frac{\hat{\kappa}(x)}{\sqrt{n}},$$

where $\hat{\kappa}(x)$ is a consistent estimator of $\kappa(x)$ obtained by replacing $c_Z$, $c_U$ and $c_{ZU}$ by their corresponding consistent estimators.

Now, by rewriting the confidence interval in a convenient form we get;

$$R_n(x)\exp\left\{-z_{\alpha/2}\frac{\hat{\kappa}(x)}{\sqrt{n}}\right\} \le R(x) \le R_n(x)\exp\left\{z_{\alpha/2}\frac{\hat{\kappa}(x)}{\sqrt{n}}\right\}.$$

Next, we consider hypothesis testing about the interval reliability for a system with distribution-free failure and repair time.

Suppose we want to test the hypothesis

$$H_0 : R(x) \le R_0 \quad \text{versus} \quad H_a : R(x) > R_0,$$

where $R_0$ is a constant level of interval reliability.

Since $h\left(R(x)\right) = -\log R(x)$ is a monotonic decreasing function of $R(x)$, $R(x) > R_0$ if and only if $h\left(R(x)\right) < h\left(R_0\right)$.

Therefore, an equivalent test is

$$H_0 : h\left(R(x)\right) \ge h\left(R_0\right) \quad \text{versus} \quad H_a : h\left(R(x)\right) < h\left(R_0\right)$$

Thus, the decision rule of the test is:

Reject $H_0$ if $h\left(R_n(x)\right) < C_\alpha$,

where $\alpha$ is the pre-specified significance level of the test and $C_\alpha$ is the critical value satisfying

$$\alpha = P\left(h\left(R_n(x)\right) < C_\alpha \mid R(x) = R_0\right)$$

$$= P\left(\frac{\sqrt{n}\left(h(R_n(x)) - h(R_0)\right)}{\hat{\kappa}(x)} < \frac{\sqrt{n}\left(C_\alpha - h(R_0)\right)}{\hat{\kappa}(x)}\right).$$

Since $\dfrac{\sqrt{n}\left(h(R_n(x)) - h(R_0)\right)}{\hat{\kappa}(x)} \xrightarrow{L} N(0,1)$, $C_\alpha$ can be determined by

$$\frac{\sqrt{n}\left(C_\alpha - h(R_0)\right)}{\hat{\kappa}(x)} = -z_\alpha,$$

where $-z_\alpha$ is the lower $\alpha$-th quantile of the standard normal distribution.

That is,

$$C_\alpha = h(R_0) - \frac{z_\alpha \hat{\kappa}(x)}{\sqrt{n}}.$$

Thus, the decision rule of the test is

Reject $H_0$ if

$$h\left(R_n(x)\right) < h(R_0) - \frac{z_\alpha \hat{\kappa}(x)}{\sqrt{n}} \tag{4.4}$$

$$\text{or, } -\log R_n(x) < -\log R_0 - \frac{z_\alpha \hat{\kappa}(x)}{\sqrt{n}}$$

$$\text{or, } R_n(x) > R_0 \exp\left(\frac{z_\alpha \hat{\kappa}(x)}{\sqrt{n}}\right).$$

Using this decision rule, the power function of the test is given by,

$$\begin{aligned}
\beta(R) &= P\left( h\left(R_n(x)\right) < h(R_0) - \frac{z_\alpha \hat{\kappa}(x)}{\sqrt{n}} \middle| R(x) = R \right) \\
&= P\left( \frac{\sqrt{n}\left(h\left(R_n(x)\right) - h\left(R\right)\right)}{\hat{\kappa}(x)} < \frac{\sqrt{n}\left(h(R_0) - h(R)\right)}{\hat{\kappa}(x)} - z_\alpha \right) \\
&= \Phi\left( \frac{\sqrt{n}\left(h(R_0) - h(R)\right)}{\hat{\kappa}(x)} - z_\alpha \right) \\
&= \Phi\left( \frac{\sqrt{n}\log(R_0 / R)}{\hat{\kappa}(x)} - z_\alpha \right),
\end{aligned}$$

where $\Phi(.)$ is the cumulative standard normal distribution function.

### 4.3 Estimation in the case of Censored Observations

Suppose that observations on the failure and repair times $(X_1, Y_1), (X_2, Y_2), ..., (X_n, Y_n)$ are subject to right censorship. Let $(C_1, D_1), (C_2, D_2), ..., (C_n, D_n)$ denote the random censoring times associated with the failure and repair times having marginal distribution functions $G_C(.)$ and $G_D(.)$ respectively. We assume that the sequences $\{(X_i, Y_i)\}$ and $\{(C_i, D_i)\}$ are mutually independent. Under the censoring model, instead of observing $X_i$ and $Y_i$, we observe the pairs $(T_i, \delta_i)$ and $(V_i, \eta_i)$, $i = 1, 2, ..., n$, where $T_i = \min(X_i, C_i)$, $V_i = \min(Y_i, D_i)$, $\delta_i = I_{(X_i \le C_i)}$ and $\eta_i = I_{(Y_i \le D_i)}$. Let $\hat{F}_{X,c}(t)$ and $\hat{F}_{Y,c}(t)$ be the product limit estimator of $F_X(t)$ and $F_Y(t)$ respectively as defined in Section 2.3.

Let $H_X(.)$ and $H_Y(.)$ be the distribution function of $T_i$ and $V_i$ respectively. Also let $\tau_X$ and $\tau_Y$ be the least upper bound for the support of $H_X(.)$ and $H_Y(.)$ respectively.

Let $T_{(1)} \le T_{(2)} \le ... \le T_{(n)}$ and $V_{(1)} \le V_{(2)} \le ... \le V_{(m)}$ be the order statistics of $T_1, T_2, ..., T_n$ and $V_1, V_2, ..., V_m$ respectively and let $\delta_{(i)}$ and $\eta_{(i)}$ denote the concomitant associated with $T_{(i)}$ and $V_{(i)}$ respectively.

Using the Kaplan-Meier integrals (Stute and Wang, 1994), the nonparametric estimators of $\mu_X$, $\mu_Y$ and $\upsilon(x)$ are given by,

$$\hat{\mu}_{X,c} = \int_0^\infty u d\hat{F}_{X,c}(u) = \sum_{i=1}^{n} W_{X,i} T_{(i)},$$

$$\hat{\mu}_{Y,c} = \int_0^\infty u d\hat{F}_{Y,c}(u) = \sum_{i=1}^{n} W_{Y,i} V_{(i)} \text{ and}$$

$$\hat{\upsilon}_c(x) = \int_0^\infty (u-x) I_{(u>x)} d\hat{F}_{X,c}(x) = \sum_{i=1}^{n} W_{X,i} (T_{(i)} - x) I_{(T_{(i)}>x)},$$

where

$$W_{X,i} = \frac{\delta_{(i)}}{n-i+1} \prod_{j=1}^{i-1} \left[ \frac{n-j}{n-j+1} \right]^{\delta_{(j)}} \text{ and}$$

$$W_{Y,i} = \frac{\eta_{(i)}}{n-i+1} \prod_{j=1}^{i-1} \left[ \frac{n-j}{n-j+1} \right]^{\eta_{(j)}}.$$

Thus, a nonparametric estimator of $R(x)$ is given by

$$\hat{R}_c(x) = \frac{\hat{\upsilon}_c(x)}{\hat{\mu}_{X,c} + \hat{\mu}_{Y,c}}. \tag{4.5}$$

In order to study the asymptotic properties of the estimator $\hat{R}_c(x)$ we need the following two lemmas due to Stute and Wang (1994) and Stute (1995).

Before stating the Lemmas, let us define

$$\gamma_0(x) = \frac{1}{1-G(x)}$$

$$\gamma_1(x) = \frac{1}{1-H(x)}\int_x^{\tau}\varphi(t)dF(t)$$

$$\gamma_2(x) = \int_{-\infty}^{x}\frac{\gamma_1(t)}{1-G(t)}dG(t)$$

where $F$, $G$, $H$ and $\tau$ are as defined in this section ignoring the suffices.

**Lemma 4.1** *Let $\varphi$ be any F-integrable function. Assume that F and G are continuous. If $\hat{F}$ is the Kaplan-Meier product limit estimator of $F$, then*

$$\lim_{n\to\infty}\int_0^{\infty}\varphi d\hat{F} = \int_0^{\tau}\varphi dF \text{ with probability 1.}$$

*Proof*: See Stute and Wang (1994).

**Lemma 4.2** *Assume the conditions*

*(i)* $\int_0^{\tau}\frac{\varphi^2(x)}{\bar{G}(x)}dF(x) < \infty$ *and*

*(ii)* $\int_0^{\tau}|\varphi(x)|\sqrt{C(x)}dF(x) < \infty$, *where* $C(x) = \int_0^{x}\frac{dG(t)}{\bar{H}(t)\bar{G}(t)}$.

*Then,* $\int\varphi d(\hat{F}-F) = n^{-1}\sum_{i=1}^{n}B_i + o_P(n^{-1/2})$,

*where* $B_i = \varphi(T_i)\gamma_0(T_i)\delta_i + \gamma_1(T_i)(1-\delta_i) - \gamma_2(T_i)$ *are independent and identically distributed with mean 0 and variance* $\sigma_c^2$. *Also*

$$\sqrt{n}\left[\int_0^{\infty}\varphi d\hat{F} - \int_0^{\tau}\varphi dF\right] \xrightarrow{L} N(0,\sigma_c^2) \text{ as } n\to\infty.$$

*Proof*: See Stute (1995).

Introduce the notations,

$$\mu_{X,c} = \int_0^{\tau_X} u dF_{X,c}(u), \quad \mu_{Y,c} = \int_0^{\tau_Y} u dF_{Y,c}(u), \quad \upsilon_c(x) = \int_0^{\tau_X} (u-x) I(u>x) dF_{X,c}(u)$$

and $R_c(x) = \dfrac{\upsilon_c(x)}{\mu_{X,c} + \mu_{Y,c}}$.

Using the strong law of Kaplan-Meier integrals stated in Lemma 4.1, it is easy to verify that $\hat{\mu}_{X,c} \to \mu_{X,c}$, $\hat{\mu}_{Y,c} \to \mu_{Y,c}$, and $\hat{\upsilon}_c(x) \to \upsilon_c(x)$ almost surely as $n \to \infty$ and hence we conclude that $\hat{R}_c(x) \to R_c(x)$ almost surely as $n \to \infty$.

By applying the central limit theorem for Kaplan-Meier integrals stated in Lemma 4.2 and using the Cramer-Wold device (Billingsley, 1968, pp.49) it can be shown that as $n \to \infty$

$$\sqrt{n}\left(\hat{\mu}_{X,c} - \mu_{X,c},\, \hat{\mu}_{X,c} - \mu_{X,c},\, \hat{\upsilon}_c(x) - \upsilon_c(x)\right) \xrightarrow{L} N_3(\mathbf{0}, \Sigma_c),$$

where $N_3(0, \Sigma_c)$ is a 3-variate normal vector with mean $\mathbf{0}' = (0,0,0)$ and dispersion matrix

$$\Sigma_c = \begin{pmatrix} \sigma^2_{X,c} & \sigma_{XY,c} & \sigma_{XU,c} \\ \sigma_{XY,c} & \sigma^2_{Y,c} & \sigma_{YU,c} \\ \sigma_{XU,c} & \sigma_{YU,c} & \sigma^2_{U,c} \end{pmatrix},$$

with $\sigma^2_{X,c} = \mathrm{var}(B_{X,i})$, $\sigma^2_{Y,c} = \mathrm{var}(B_{Y,i})$, $\sigma^2_{U,c} = \mathrm{var}(B_{U,i})$, $\sigma_{XY,c} = \mathrm{cov}(B_{X,i}, B_{Y,i})$, $\sigma_{XU,c} = \mathrm{cov}(B_{X,i}, B_{U,i})$ and $\sigma_{YU,c} = \mathrm{cov}(B_{Y,i}, B_{U,i})$, in which $B_{X,i}$, $B_{Y,i}$ and $B_{U,i}$ are as defined in Lemma 4.2 by choosing $\varphi(t) = t$ for $B_{X,i}$ and $B_{Y,i}$ and $\varphi(t) = (t-x) I_{(t>x)}$ for $B_{U,i}$.

Note that,

$$f(\hat{\mu}_{X,c}, \hat{\mu}_{Y,c}, \hat{\upsilon}_c(x)) = \hat{R}_c(x).$$

Now, by Lemma 1.4, we can show that

$$\sqrt{n}\left(\hat{R}_c(x) - R_c(x)\right) \xrightarrow{L} N\left(0, \tau^2_c(x)\right) \text{ as } n \to \infty,$$

where

$$\tau_c^2(x) = \frac{\upsilon_c^2(x)(\sigma_{X,c}^2 + 2\sigma_{XY,c} + \sigma_{Y,c}^2)}{(\mu_{X,c} + \mu_{Y,c})^4} - \frac{2\upsilon_c(x)(\sigma_{XU,c} + \sigma_{YU,c})}{(\mu_{X,c} + \mu_{Y,c})^3} + \frac{\sigma_{U,c}^2}{(\mu_{X,c} + \mu_{Y,c})^2}. \quad (4.6)$$

This leads to the following theorem.

**Theorem 4.2**

*For fixed 'x', as* $n \to \infty$,

*(i)* $\hat{R}_c(x) \to R_c(x)$ *almost surely and*

*(ii)* $\sqrt{n}\left(\hat{R}_c(x) - R_c(x)\right) \xrightarrow{L} N\left(0, \tau_c^2(x)\right)$,

*where* $\tau_c^2(x)$ *is given in* (4.6).

In order to construct a confidence interval for $R(x)$, a consistent estimator $\hat{\tau}_c(x)$ of $\tau_c(x)$ can be obtained by using the Jackknife estimate proposed by Stute (1996). The Jackknife estimators of $\sigma_{X,c}^2$ and $\sigma_{XY,c}$ are given by,

$$\hat{\sigma}_{X,c}^2 = (n-1)\sum_{k=1}^{n}\left(S_{X,n}^{(k)} - \overline{S}_{X,n}\right)^2,$$

$$\sigma_{XY,c} = (n-1)\sum_{k=1}^{n}\left(S_{X,n}^{(k)} - \overline{S}_{X,n}\right)\left(S_{Y,n}^{(k)} - \overline{S}_{Y,n}\right)$$

where

$$S_{X,n}^{(k)} = \sum_{i=1}^{k}\frac{T_{(i)}\delta_{(i)}}{n-i}\prod_{j=1}^{i-1}\left[\frac{n-j-1}{n-j}\right]^{\delta_{(j)}} + \sum_{i=k+1}^{n}\frac{T_{(i)}\delta_{(i)}}{n-i+1}\prod_{j=1}^{k-1}\left[\frac{n-j-1}{n-j}\right]^{\delta_{(j)}}\prod_{j=k+1}^{i-1}\left[\frac{n-j}{n-j+1}\right]^{\delta_{(j)}},$$

$$S_{X,n}^{(k)} = \sum_{i=1}^{k}\frac{V_{(i)}\eta_{(i)}}{n-i}\prod_{j=1}^{i-1}\left[\frac{n-j-1}{n-j}\right]^{\eta_{(j)}} + \sum_{i=k+1}^{n}\frac{V_{(i)}\eta_{(i)}}{n-i+1}\prod_{j=1}^{k-1}\left[\frac{n-j-1}{n-j}\right]^{\eta_{(j)}}\prod_{j=k+1}^{i-1}\left[\frac{n-j}{n-j+1}\right]^{\eta_{(j)}},$$

$$\bar{S}_{X,n} = \frac{1}{n}\sum_{k=1}^{n} S_{X,n}^{(k)} \text{ and } \bar{S}_{Y,n} = \frac{1}{n}\sum_{k=1}^{n} S_{Y,n}^{(k)}.$$

The Jackknife estimators for $\sigma_{Y,c}^2$, $\sigma_{U,c}^2$, $\sigma_{XU,c}$, and $\sigma_{YU,c}$ may be constructed in a similar manner. Using these estimators, it is easier to construct an asymptotic confidence interval for the limiting interval reliability.

### 4.4 Estimation in the case of Continuous Observation over a Fixed Period

Suppose that the process is observed over a fixed period $[0,T]$. Let $N(T)$ denote the number of cycles completed in the interval $[0, T]$. Then, $N(T) = \text{Sup}\{n : S_n \le T\}$, where $S_n = \sum_{i=1}^{n} Z_i$ with $Z_i = X_i + Y_i$.

Let us assume that, for some $\delta > 0$, $E\left(X_1^{2+\delta}\right) < \infty$ and $E\left(Y_1^{2+\delta}\right) < \infty$.

Define,

$$\alpha_T(x) = \lambda(T)\sum_{j=1}^{N(T)+1} U_j + \left(1-\lambda(T)\right)\left(\sum_{j=1}^{N(T)} U_j + (T - S_{N(T)} - x) I_{(T-S_{N(T)}>x)}\right),$$

where $\lambda(T) = I_{(S_{N(T)}+X_{N(T)+1}\le T < S_{N(T)+1})}$.

Then, an estimator of the limiting interval reliability, $R(x)$, is given by,

$$R_T(x) = \frac{\alpha_T(x)}{T}. \tag{4.7}$$

We can write (4.7) as

$$R_T(x) = \frac{1}{T}\sum_{j=1}^{N(T)+1} U_j + \frac{1-\lambda(T)}{T}\left((T - S_{N(T)} - x) I_{(T-S_{N(T)}>x)} - U_{N(T)+1}\right). \tag{4.8}$$

Consider,

$$\frac{1}{T}\left((T - S_{N(T)} - x) I_{(T-S_{N(T)}>x)} - U_{N(T)+1}\right) \le \frac{1}{T}\left((T - S_{N(T)}) + U_{N(T)+1}\right)$$
$$\le \frac{2}{T} Z_{N(T)+1} \to 0 \text{ a.s. as } T \to \infty.$$

Clearly as $T \to \infty$, $N(T) \to \infty$ almost surely and hence by the strong law of large numbers for random sum of i.i.d. random variables,

$$\frac{S_{N(T)}}{N(T)} \to (\mu_X + \mu_Y) = \mu_Z \text{ as } T \to \infty .$$

Now, from the following inequality

$$\frac{S_{N(T)}}{N(T)} \le \frac{T}{N(T)} < \frac{S_{N(T)+1}}{N(T)}$$

it follows that, $\dfrac{N(T)}{T} \to \dfrac{1}{\mu_Z}$ almost surely as $T \to \infty$.

Therefore,

$$\frac{1}{T}\sum_{j=1}^{N(T)+1} U_j = \frac{N(T)+1}{T}\frac{1}{N(T)+1}\sum_{j=1}^{N(T)+1} U_j$$

$$\to \frac{\upsilon(x)}{\mu_Z} = R(x) \text{ almost surely as } T \to \infty .$$

Hence $R_T(x)$ is strongly consistent for $R(x)$.

Define,

$$\xi_j = \mu_Z U_j - \upsilon(x) Z_j,\ j = 1,2,...$$

Then, $\xi_j$'s are i.i.d. with mean 0 and common variance

$$\gamma^2(x) = \mu_Z^2 \sigma_U^2 + \upsilon^2(x)\sigma_Z^2 - 2\upsilon(x)\mu_Z\sigma_{UZ} . \tag{4.9}$$

Thus, by the Central limit theorem, we have

$$\sqrt{n}\left(\mu_Z \bar{U} - \upsilon(x)\bar{Z}\right) \xrightarrow{L} N\left(0, \gamma^2(x)\right).$$

Now, using the central limit theorem for a random sum of i.i.d. random variables, we have

$$\frac{1}{\sqrt{N(T)+1}}\sum_{j=1}^{N(T)+1} \xi_j = \frac{1}{\sqrt{N(T)+1}}\sum_{j=1}^{N(T)+1} [\mu_Z U_j - \upsilon(x) Z_j]$$

$$\xrightarrow{L} N\left(0, \gamma^2(x)\right) . \tag{4.10}$$

Let us write,

$$\mu_Z \sum_{j=1}^{N(T)+1} U_j - \upsilon(x)T = \sum_{j=1}^{N(T)+1} [\mu_Z U_j - \upsilon(x) Z_j] + \left( \sum_{j=1}^{N(T)+1} Z_j - T \right) \upsilon(x)$$

$$= \sum_{j=1}^{N(T)+1} \xi_j + \left( S_{N(T)+1} - T \right) \upsilon(x) . \tag{4.11}$$

For $\varepsilon > 0$, consider

$$P\left[ \frac{Z_n}{\sqrt{n}} > \varepsilon \right] \le P\left[ X_n > \varepsilon\sqrt{n}/2 \right] + P\left[ Y_n > \varepsilon\sqrt{n}/2 \right]$$

$$\le \frac{E\left( X_1^{2+\delta} \right) + E\left( Y_1^{2+\delta} \right)}{n^{1+\delta/2}} \text{ for some } \delta > 0, \text{ by Markov inequality.}$$

As the expectation on the right hand side are finite, it follows that

$$\sum_{n=1}^{\infty} P\left[ \frac{Z_n}{\sqrt{n}} > \varepsilon \right] < \infty \Rightarrow \frac{Z_n}{\sqrt{n}} \to 0 \text{ a.s. as } n \to \infty,$$

and hence $\frac{Z_{N(T)}}{\sqrt{N(T)}} \to 0$ a.s. as $T \to \infty$.

Thus,

$$0 \le \frac{S_{N(T)+1} - T}{\sqrt{N(T)+1}} \le \frac{Z_{N(T)+1}}{\sqrt{N(T)+1}} \to 0 \text{ a.s. as } T \to \infty .$$

Hence, from (4.10) and (4.11) we get as $T \to \infty$

$$\frac{\mu_Z \sum_{j=1}^{N(T)+1} U_j - \upsilon(x)T}{\sqrt{N(T)+1}} \xrightarrow{L} N\left(0, \gamma^2(x)\right) \tag{4.12}$$

Also,

$$\frac{1}{\sqrt{N(t)+1}} \left( (T - S_{N(T)} - x) I_{(T - S_{N(T)} > x)} - U_{N(T)+1} \right) \le \frac{1}{\sqrt{N(t)+1}} \left( (T - S_{N(T)}) + U_{N(T)+1} \right)$$

$$\le \frac{2}{\sqrt{N(T)+1}} Z_{N(T)+1} \to 0 \text{ a.s. as } T \to \infty .$$

Thus, in view of (4.8) we can write,

$$\frac{R_T(x) - \frac{1}{T}\sum_{j=1}^{N(T)+1} U_j}{\sqrt{N(T)+1}\big/T} \to 0 \text{ almost surely as } T \to \infty \tag{4.13}$$

Now,

$$\frac{R_T(x) - R(x)}{\mu_Z^{-1}\left(\sqrt{N(T)+1}\big/T\right)} = \frac{R_T(x) - \frac{1}{T}\sum_{j=1}^{N(T)+1} U_j}{\mu_Z^{-1}\left(\sqrt{N(T)+1}\big/T\right)} + \frac{\frac{1}{T}\sum_{j=1}^{N(T)+1} U_j - R(x)}{\mu_Z^{-1}\left(\sqrt{N(T)+1}\big/T\right)}.$$

By (4.12) and (4.13) we get,

$$\frac{R_T(x) - R(x)}{\mu_Z^{-1}\left(\sqrt{N(T)+1}\big/T\right)} \xrightarrow{L} N\left(0, \gamma^2(x)\right) \text{ as } T \to \infty$$

Finally considering $\frac{N(T)+1}{T} \to \frac{1}{\mu_Z}$, it follows that

$$\sqrt{T}\left(R_T(x) - R(x)\right) \xrightarrow{L} N\left(0, \gamma^2(x)\big/\mu_Z^3\right) \text{ as } T \to \infty .$$

Thus we proved the following theorem.

**Theorem 4.3**

*For fixed* $x$, *as* $T \to \infty$,

*(i)* $R_T(x) \to R(x)$ *almost surely and*

*(ii)* $\sqrt{T}\left(R_T(x) - R(x)\right) \xrightarrow{L} N\left(0, \gamma^2(x)\big/\mu_Z^3\right)$,

*where* $\gamma^2(x)$ *is given in* (4.9).

Define,

$$\bar{U}_{N(T)} = \frac{1}{N(T)}\sum_{i=1}^{N(T)} U_i \,,\ \bar{Z}_{N(T)} = \frac{1}{N(T)}\sum_{i=1}^{N(T)} Z_i \,,\ S_{U,T}^2 = \frac{1}{N(T)-1}\sum_{i=1}^{N(T)}\left(U_i - \bar{U}_{N(T)}\right)^2,$$

$$S_{Z,T}^2 = \frac{1}{N(T)-1}\sum_{i=1}^{N(T)}\left(Z_i - \bar{Z}_{N(T)}\right)^2 \text{ and } S_{UZ,T} = \frac{1}{N(T)-1}\sum_{i=1}^{N(T)}(U_i - \bar{U}_{N(T)})(Z_i - \bar{Z}_{N(T)}) .$$

Then a consistent estimator $\hat{\gamma}(x)$ of $\gamma(x)$ can be obtained on replacing $\mu_Z$, $\upsilon(x)$, $\sigma_U^2$, $\sigma_Z^2$ and $\sigma_{UZ}$ by $\bar{U}_{N(T)}$, $\bar{Z}_{N(T)}$, $S_{U,T}^2$, $S_{Z,T}^2$ and $S_{UZ,T}$ respectively in

(4.9). In this case, for large '*T*', an approximate $100(1-\alpha)\%$ confidence interval for $R(x)$ is given by

$$R_T(x) - z_{\alpha/2}\frac{\hat{\gamma}(x)}{\bar{U}^3\sqrt{T}} \le R(x) \le R_T(x) + z_{\alpha/2}\frac{\hat{\gamma}(x)}{\bar{U}^3\sqrt{T}}.$$

### 4.5 Numerical Study

In this section we carry out an extensive simulation study to assess the performance of the proposed estimator for the limiting interval reliability when (i) data on '*n*' complete cycles of system operation are available, (ii) data are subject to right censorship and (iii) the process is observed up to a specified time '*T*'.

Consider first the case of complete observations. Suppose that the joint distribution of failure and repair times follows a Gumbel's bivariate exponential distribution having bivariate survivor function $S(x,y) = exp\left(-\lambda_1 x - \lambda_2 y - \lambda xy\right)$ with $\lambda_1 = 1/6$, $\lambda_2 = 1/2$ and $\lambda = 0.75$. The time points $x$ = 0, 0.1, 0.25 and 0.5 are considered for the simulation and the corresponding $R(x)$ are also calculated. In Table 4.1, *n* denotes the number of observations of failure and repair times, $\bar{R}_n(x)$ and $\hat{\bar{\tau}}_n(x)$ denote the averages of $R_n(x)$ and $\hat{\tau}(x)$ over 500 repetitions and $R_L(x)$ and $R_U(x)$ are respectively the 95% lower and upper confidence limits for $R(x)$. The mean square errors of the estimators are written within the parenthesis.

In order to assess the performance of the proposed estimator under censoring scheme we use the same Gumbel's bivariate exponential distribution for the failure and repair times as in the case of complete observations. Further we assume that censoring distributions are also generated from a bivariate Gumbel's exponential distribution with bivariate survivor function $S(x,y) = exp\left(-0.05x - 0.10y - 0.5xy\right)$. The results of the simulation study are presented in Table 4.2. Here *X%* and *Y%* denote the average percentage of censoring rate associated with the failure time and the repair time respectively.

**Table 4.1** Simulation results in the case of complete observations

| $x$ | $R(x)$ | $n$ | $\overline{R}_n(x)$ | $\overline{\tau}_n(x)$ | $R_L(x)$ | $R_U(x)$ |
|---|---|---|---|---|---|---|
| 0.00 | 0.75000 | 50 | 0.74794 | 0.09428 | 0.72180 | 0.77407 |
| | | | (0.0451) | (0.0283) | | |
| | | 100 | 0.74864 | 0.09393 | 0.73023 | 0.76705 |
| | | | (0.0317) | (0.0199) | | |
| | | 250 | 0.74946 | 0.09481 | 0.73770 | 0.76121 |
| | | | (0.0195) | (0.0125) | | |
| | | 500 | 0.74986 | 0.09518 | 0.74152 | 0.75821 |
| | | | (0.0141) | (0.0094) | | |
| 0.10 | 0.73760 | 50 | 0.73546 | 0.09643 | 0.70873 | 0.76218 |
| | | | (0.0427) | (0.0266) | | |
| | | 100 | 0.73697 | 0.09734 | 0.71790 | 0.75605 |
| | | | (0.0312) | (0.0207) | | |
| | | 250 | 0.73626 | 0.09809 | 0.72410 | 0.74841 |
| | | | (0.0192) | (0.0129) | | |
| | | 500 | 0.73801 | 0.09758 | 0.72946 | 0.74657 |
| | | | (0.0142) | (0.0093) | | |
| 0.25 | 0.71939 | 50 | 0.71711 | 0.10271 | 0.68864 | 0.74558 |
| | | | (0.0449) | (0.0297) | | |
| | | 100 | 0.71724 | 0.10243 | 0.69716 | 0.73731 |
| | | | (0.0320) | (0.0207) | | |
| | | 250 | 0.71904 | 0.10219 | 0.70637 | 0.73171 |
| | | | (0.0201) | (0.0130) | | |
| | | 500 | 0.71877 | 0.10285 | 0.70975 | 0.72778 |
| | | | (0.0144) | (0.0097) | | |
| 0.50 | 0.69003 | 50 | 0.68754 | 0.11086 | 0.65682 | 0.71827 |
| | | | (0.0478) | (0.0301) | | |
| | | 100 | 0.69043 | 0.10892 | 0.66908 | 0.71178 |
| | | | (0.0332) | (0.0217) | | |
| | | 250 | 0.68811 | 0.11037 | 0.67443 | 0.70179 |
| | | | (0.0216) | (0.0141) | | |
| | | 500 | 0.68906 | 0.11002 | 0.67942 | 0.69871 |
| | | | (0.0147) | (0.0097) | | |

**Table 4.2** Simulation results in the case of censored observations

| $x$ | $R(x)$ | $n$ | $\overline{R}_{c,n}(x)$ | $\overline{\tau}_c(x)$ | $X\%$ | $Y\%$ | $R_L(x)$ | $R_U(x)$ |
|---|---|---|---|---|---|---|---|---|
| 0.00 | 0.75000 | 50 | 0.75239 (0.0445) | 0.13546 (0.0362) | 22.60 | 15.60 | 0.71484 | 0.78994 |
| | | 100 | 0.74853 (0.0385) | 0.13752 (0.0288) | 21.90 | 17.40 | 0.72157 | 0.77548 |
| | | 250 | 0.74964 (0.0199) | 0.13516 (0.0197) | 23.40 | 16.66 | 0.73289 | 0.76640 |
| | | 500 | 0.74692 (0.0130) | 0.13846 (0.0144) | 22.74 | 16.54 | 0.73478 | 0.75905 |
| 0.10 | 0.73760 | 50 | 0.73771 (0.0397) | 0.14490 (0.0376) | 22.00 | 16.60 | 0.69755 | 0.77788 |
| | | 100 | 0.72195 (0.0325) | 0.14336 (0.0280) | 22.30 | 17.10 | 0.69385 | 0.75005 |
| | | 250 | 0.73522 (0.0229) | 0.14461 (0.0184) | 23.32 | 16.58 | 0.71729 | 0.75314 |
| | | 500 | 0.73481 (0.0178) | 0.14538 (0.0103) | 23.13 | 16.82 | 0.72207 | 0.74755 |
| 0.25 | 0.71939 | 50 | 0.71894 (0.0410) | 0.16271 (0.0315) | 25.00 | 16.20 | 0.67384 | 0.76404 |
| | | 100 | 0.70698 (0.0399) | 0.16243 (0.0217) | 23.26 | 16.44 | 0.67514 | 0.73882 |
| | | 250 | 0.71677 (0.0231) | 0.16305 (0.0147) | 22.74 | 16.69 | 0.69656 | 0.73698 |
| | | 500 | 0.72126 (0.0196) | 0.16254 (0.0099) | 23.17 | 16.54 | 0.70702 | 0.73551 |
| 0.50 | 0.69003 | 50 | 0.69822 (0.0499) | 0.18681 (0.0404) | 24.20 | 16.20 | 0.64644 | 0.75000 |
| | | 100 | 0.67765 (0.0405) | 0.18520 (0.0287) | 22.92 | 17.24 | 0.64135 | 0.71395 |
| | | 250 | 0.68612 (0.0244) | 0.18372 (0.0160) | 23.22 | 16.58 | 0.66335 | 0.70889 |
| | | 500 | 0.68902 (0.0175) | 0.18685 (0.0115) | 23.42 | 16.82 | 0.67264 | 0.70540 |

**Table 4.3** Simulation results in the case of continuous observation over a fixed period '*T*'

| $x$ | $R(x)$ | $T$ | $\bar{R}_T(x)$ | $\bar{\gamma}_T(x)$ | $\bar{N}(T)$ | $R_L(x)$ | $R_U(x)$ |
|---|---|---|---|---|---|---|---|
| 0.00 | 0.75000 | 200 | 0.75235<br>(0.0540) | 0.82831<br>(0.2426) | 23.90 | 0.63755 | 0.86714 |
| | | 500 | 0.75062<br>(0.0439) | 0.83930<br>(0.1977) | 45.92 | 0.67705 | 0.82418 |
| | | 750 | 0.75186<br>(0.0318) | 0.85160<br>(0.1246) | 74.18 | 0.69091 | 0.81281 |
| | | 1000 | 0.75134<br>(0.0212) | 0.85915<br>(0.0914) | 95.88 | 0.69809 | 0.80459 |
| 0.10 | 0.73760 | 200 | 0.73546<br>(0.0427) | 0.94643<br>(0.2628) | 23.90 | 0.60429 | 0.86662 |
| | | 500 | 0.73697<br>(0.0312) | 0.95355<br>(0.1768) | 48.76 | 0.65339 | 0.82056 |
| | | 750 | 0.73626<br>(0.0192) | 0.95917<br>(0.1487) | 72.12 | 0.66761 | 0.80490 |
| | | 1000 | 0.73801<br>(0.0142) | 0.96085<br>(0.1040) | 96.98 | 0.67846 | 0.79757 |
| 0.25 | 0.71939 | 200 | 0.71711<br>(0.0449) | 1.01111<br>(0.2895) | 23.90 | 0.57698 | 0.85724 |
| | | 500 | 0.71724<br>(0.0320) | 1.02024<br>(0.2037) | 47.20 | 0.62781 | 0.80666 |
| | | 750 | 0.71904<br>(0.0201) | 1.02989<br>(0.1739) | 70.08 | 0.64533 | 0.79275 |
| | | 1000 | 0.71877<br>(0.0144) | 1.03154<br>(0.1146) | 95.54 | 0.65483 | 0.78270 |
| 0.50 | 0.69003 | 200 | 0.68754<br>(0.0478) | 1.10757<br>(0.3014) | 24.32 | 0.53404 | 0.84104 |
| | | 500 | 0.69043<br>(0.0332) | 1.10892<br>(0.0242) | 47.76 | 0.59323 | 0.78763 |
| | | 750 | 0.68811<br>(0.0216) | 1.10972<br>(0.1841) | 72.62 | 0.60869 | 0.76753 |
| | | 1000 | 0.68906<br>(0.0147) | 1.11017<br>(0.1197) | 95.86 | 0.62025 | 0.75787 |

Table 4.3 summarizes the result of the simulation study in the case of continuous observation over a fixed period [0, *T*] using the same Gumbel's bivariate exponential distribution for generating the failure and repair times as in the case of complete observations. In Table 4.3, the average number of cycles completed up to time '*T*' is denoted by $\overline{N}(T)$. The results of the simulation study confirmed the performance of the proposed estimators for reasonable sample sizes.

A data analysis is also carried out using the compressor failure data given in Table 2.4 to illustrate the application of the proposed estimation procedure in the case of complete observation. The limiting interval reliability $R(x)$ is computed with $x = 0,\ 2.5,\ 5,\ 7.5$ hours and the corresponding 95% confidence intervals are computed and summarized in Table 4.4.

**Table 4.4** Limiting interval reliability computation of compressor failure data

| $x$ | $\hat{R}(x)$ | $\hat{\tau}(x)$ | $R_L(x)$ | $R_U(x)$ |
|---|---|---|---|---|
| 0 | 0.99088 | 0.02216 | 0.98630 | 0.99546 |
| 2.5 | 0.98941 | 0.02354 | 0.98454 | 0.99427 |
| 5 | 0.98794 | 0.02503 | 0.98277 | 0.99311 |
| 7.5 | 0.98655 | 0.02649 | 0.98108 | 0.99202 |

## 4.6. Conclusion

In this chapter, we discussed the nonparametric estimation of the limiting interval reliability when the failure and repair times form a sequence of i.i.d. bivariate random variables. A testing of hypothesis procedure for the limiting interval reliability is also discussed. The performance of the proposed estimators is verified using a simulation study. We illustrated the application of the proposed estimation procedure with a real life data set.

# Chapter 5

# Availability Behavior of Some Stationary Dependent Sequences

## 5.1 Introduction

One of the major limitations in the study of system performance of repairable systems is the assumption of independence among the successive sequence of failure and repair times of the system. It is natural to expect some sort of dependence among the successive sequence of failure and repair times when the system is working in a random environment. So, it is important to consider suitable models for repairable systems that can incorporate the dependence structure. Motivated by this idea, Kijima and Sumita (1986) discussed the point process models for the reliability of repairable systems when the survival times are generated by some stationary dependent sequences. Several non-Gaussian time series models such as first order random coefficient autoregressive models are discussed in the literature for modeling life time data. See for example Lawrance and Lewis (1977), Gaver and Lewis (1980), and Sim (1992). However, the properties of the availability measures are not discussed much when the successive failure and repair times are generated by stationary dependent sequence of random variables. Motivated by this, in this chapter, we consider the availability behavior of some stationary dependent sequences.

In Section 5.2, we derive the expression for the point availability when the successive sequences of failure and repair times are generated by stationary dependent sequence of random variables. Section 5.3 and 5.4 discuss the availability behavior of a one-unit system when the sequences of failure and repair

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Some results of this chapter have been communicated as entitled 'Availability Behavior of some Stationary Dependent Sequences' (See Balakrishna and Mathew, 2011b).

times are generated by two independent first order Exponential Moving Average (EMA1) processes and two independent first order Exponential Autoregressive (EAR(1)) processes respectively. Finally, Section 5.5 summarizes major conclusions of the study.

### 5.2 Point Availability of Stationary Dependent Sequences

Let $\{X_n\}$ and $\{Y_n\}$ be two independent sequences of stationary dependent non-negative random variables with distribution functions $F_X(.)$ and $F_Y(.)$ respectively. Suppose $U_n$ denotes the sum of first $n$ operating intervals such that $U_n = \sum_{i=1}^{n} X_i$ and let $F_X^{(n)}(.)$ be the distribution function of $U_n$. Let $V_n = \sum_{i=1}^{n} Y_i$ be the sum of first $n$ repair intervals and $F_Y^{(n)}(.)$ be the distribution function of $V_n$.

To capture the behavior of cycles that contain one operation interval and one repair interval it is useful to define $Z_n = X_n + Y_n$ as the length of the $n$-th such cycle. Let $S_n$ be the total elapsed time of the first $n$ cycles such that $S_n = \sum_{i=1}^{n} Z_i = U_n + V_n$, and let $F_Z^{(n)}(t) = P[S_n \le t]$ be the distribution function of $S_n$. Then, $F_Z^{(n)}(t) = F_X^{(n)} * F_Y^{(n)}(t)$.

Now, the point availability of the repairable system is given by,

$$A(t) = P[\xi(t) = 1]$$

$$= \overline{F}_X(t) + \sum_{n=1}^{\infty} \int_0^t P[X_{n+1} > t-u \mid S_n = u] dF_Z^{(n)}(u) . \qquad (5.1)$$

The first term in the point availability function (5.1) reflects the probability that the first period of operation is of length $t$ or greater. The subsequent integral expressions reflect the probability that the $n$-th failure occurs at time $u$ and the following period of operation is of length $(t-u)$ or greater.

Consider,

$$\int_0^t P[X_{n+1} > t-u \,|\, S_n = u]\,dF_Z^{(n)}(u)$$

$$= \int_0^t dF_Z^{(n)}(u) - \int_0^t P[X_{n+1} \le t-u \,|\, S_n = u]\,dF_Z^{(n)}(u)$$

$$= F_Z^{(n)}(t) - P[X_{n+1} + S_n \le t]$$

$$= F_Z^{(n)}(t) - P[U_{n+1} + V_n \le t]$$

$$= F_X^{(n)} * F_Y^{(n)}(t) - F_X^{(n+1)} * F_Y^{(n)}(t)\,.$$

Thus,

$$A(t) = \overline{F}_X(t) + \sum_{i=1}^{n} F_X^{(n)} * F_Y^{(n)}(t) - \sum_{i=1}^{n} F_X^{(n+1)} * F_Y^{(n)}(t)$$

$$= \overline{F}_X(t) + \sum_{i=1}^{n} \left( F_X^{(n)} - F_X^{(n+1)} \right) * F_Y^{(n)}(t)\,. \tag{5.2}$$

If we assume $\{X_n\}$ and $\{Y_n\}$ are two independent sequences of i.i.d. non-negative random variables then $F_X^{(n+1)}(t) = F_X * F_X^{(n)}(t)$ and (5.2) reduces to

$$A(t) = \overline{F}_X(t) + \sum_{i=1}^{n} \overline{F}_X * F_X^{(n)} * F_Y^{(n)}(t)$$

$$= \overline{F}_X(t) + \overline{F}_X * \left( \sum_{i=1}^{n} F_X^{(n)} * F_Y^{(n)}(t) \right)$$

$$= \overline{F}_X(t) + \overline{F}_X * M(t)\,,$$

where $M(t) = \sum_{i=1}^{n} F_X^{(n)} * F_Y^{(n)}(t)$ is the renewal function associated with the sequence $\{Z_n\}$.

Let $F_X^{(n)*}(s)$ and $F_Y^{(n)*}(s)$ denote the Laplace transforms of $F_X^{(n)}(t)$ and $F_Y^{(n)}(t)$ respectively.

Now, from (5.2), the Laplace transform of $A(t)$ is given by

$$A^*(s) = \overline{F}_X^*(s) + \sum_{n=1}^{\infty} \left( F_X^{(n)*}(s) - F_X^{(n+1)*}(s) \right) F_Y^{(n)*}(s)\,. \tag{5.3}$$

So, if the Laplace transforms $F_X^{(n)*}(s)$ and $F_Y^{(n)*}(s)$ are known, then the point availability function can be obtained by inverting the Laplace transform $A^*(s)$ given in (5.3). In the next two Sections, we consider the availability behavior of two stationary dependent time series models.

### 5.3 Availability Behavior of Exponential Moving-Average Processes

Assume that the sequences of failure and repair times, $\{X_n\}$ and $\{Y_n\}$, are generated by two independent first-order Exponential Moving Average (EMA1) processes (Lawrence and Lewis, 1977) defined as

$$X_n = \begin{cases} \beta_1\varepsilon_n & \text{with probability } \beta_1, \\ \beta_1\varepsilon_n + \varepsilon_{n+1} & \text{with probability } 1 - \beta_1. \end{cases}$$

and

$$Y_n = \begin{cases} \beta_2\eta_n & \text{with probability } \beta_2, \\ \beta_2\eta_n + \eta_{n+1} & \text{with probability } 1 - \beta_2. \end{cases}, \ (0 \le \beta_1, \beta_2 \le 1, n = 1, 2, 3, \ldots),$$

where $\{\varepsilon_n\}$ and $\{\eta_n\}$ are two independent i.i.d. exponential random sequences with parameters $\lambda_1$ and $\lambda_2$ respectively. The simplest aspects of the EMA1 model are the exponential marginal distribution of the intervals and the non-markovian dependence among the adjacent members of the sequence.

In the case of EMA1 process the Laplace transforms of $F_X^{(n)}(t)$ and $F_Y^{(n)}(t)$ are given by

$$F_X^{(n)*}(s) = \frac{\lambda_1}{\lambda_1 + s}\left[\frac{\lambda_1(\lambda_1 + 2\beta_1 s)}{(\lambda_1 + \beta_1 s)\{\lambda_1 + (1+\beta_1)s}\right]^{n-1} \text{ and}$$

$$F_Y^{(n)*}(s) = \frac{\lambda_2}{\lambda_2 + s}\left[\frac{\lambda_2(\lambda_2 + 2\beta_2 s)}{(\lambda_2 + \beta_2 s)\{\lambda_2 + (1+\beta_2)s}\right]^{n-1}, \ n \ge 1.$$

Inserting $F_X^{(n)*}(s)$ and $F_Y^{(n)*}(s)$ in (5.3) we get

$$A^*(s) = \frac{1}{s + \lambda_1} + \frac{\lambda_1\lambda_2\{s\beta_1(1-\beta_1) + \lambda_1\}(s\beta_2 + \lambda_2)(s + s\beta_2 + \lambda_2)}{k(s)}, \tag{5.4}$$

where

$$k(s) = s(s+\lambda_1)(s+\lambda_2)\Big[ s\beta_2(1+\beta_2)(s\beta_1+\lambda_1)(s+s\beta_1+\lambda_1) + \big(s\beta_1(1+\beta_1)+\lambda_1\big)\lambda_2^2$$
$$+\big(s^2\beta_1(1+\beta_1)(1+2\beta_2) + s(1+2\beta_1+2\beta_2)\lambda_1 + \lambda_1^2\big)\lambda_2\Big].$$

On inverting the Laplace transform (5.4) we get the point availability function $A(t)$ for the EMA1 process. Obviously, for values of $(\beta_1, \beta_2) = (0,0)$ or $(1,1)$, $A(t)$ becomes the point availability function when the sequences of failure and repair times are generated by two independent i.i.d. exponential random variables. Figure 5.1 gives the graph of $A(t)$ for different values of $\beta_1$ and $\beta_2$ by choosing $\lambda_1 = 1$ and $\lambda_2 = 5$.

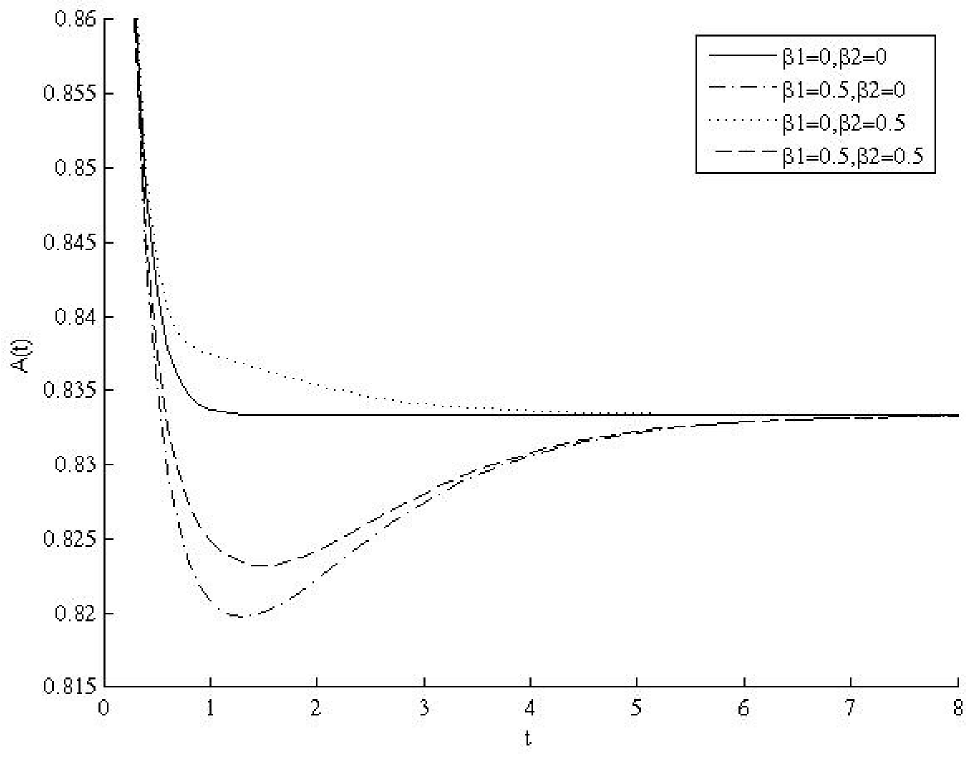


**Figure 5.1**

The availability function for EMA1 process. The function is plotted for values $(\beta_1, \beta_2) = (0,0), (0.5,0), (0,0.5)$ and $(0.5,0.5)$ using $\lambda_1 = 1$ and $\lambda_2 = 5$.

From the figure, it can be seen that the point availability function in the i.i.d. set up overestimate or underestimate $A(t)$ in the presence of dependence among successive observations. Also, it is seen that the point availability function converge to a common limit, called the limiting availability, as $t \to \infty$ irrespective of the values of $\beta_1$ and $\beta_2$.

Because the study of the point availability function $A(t)$ is too hard, due to the complexity of the exact expression, in the literature more attention is being paid to the limiting behavior of $A(t)$. The limiting availability, *A*, can be obtained from (5.4) using the fact that

$$\lim_{t\to\infty} A(t) = \lim_{s\to 1} sA^*(s).$$

Applying this result we get,

$$A = \lim_{t\to\infty} A(t) = \frac{\lambda_2}{\lambda_1 + \lambda_2} = \frac{1/\lambda_1}{1/\lambda_1 + 1/\lambda_2},$$

which is same as the limiting availability in the i.i.d. exponential case.

Even though, the limiting availability remains the same for both set-ups, the properties of its estimators may be different in both cases. In order to study the effect of dependence among successive sequences of failure and repair times in the estimation of the limiting availability, we consider the asymptotic properties of two commonly used estimators for the limiting availability.

First, we consider the estimator in the case of complete observations. Suppose that observations on the failure times and the repair times of '*n*' complete cycles $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ are available. In this case a natural estimator of the limiting availability *A* is

$$\hat{A}_n = \frac{\bar{X}_n}{\bar{X}_n + \bar{Y}_n}, \tag{5.5}$$

where $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $\bar{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i$.

As $\{X_n\}$ and $\{Y_n\}$ are stationary and ergodic, we have $\overline{X}_n \to 1/\lambda_1$ and $\overline{Y}_n \to 1/\lambda_2$ almost surely as $n \to \infty$ and hence we conclude that $\hat{A}_n \to A$ almost surely as $n \to \infty$.

Since the EMA1 process is a 1-dependent process, using the central limit theorem for *m*-dependent process (Lemma 1.6) we have as $n \to \infty$,

$\sqrt{n}\left(\overline{X}_n - 1/\lambda_1\right) \xrightarrow{L} N(\mathbf{0}, \sigma_X^2)$ and

$\sqrt{n}\left(\overline{Y}_n - 1/\lambda_2\right) \xrightarrow{L} N(\mathbf{0}, \sigma_Y^2)$,

where $\sigma_X^2 = \{1 + 2\beta_1(1-\beta_1)\}/\lambda_1^2$ and $\sigma_Y^2 = \{1 + 2\beta_2(1-\beta_2)\}/\lambda_2^2$.

If we define $f(x, y) = x/(x+y)$, then $f(\overline{X}_n, \overline{Y}_n) = \hat{A}_n$ and hence by using Lemma 1.4, we can show that

$$\sqrt{n}\left(\hat{A}_n - A\right) \xrightarrow{L} N(0, \tau^2),$$

where $\tau^2$ is given by

$$\begin{aligned}
\tau^2 &= \frac{2\lambda_1^2\lambda_2^2}{(\lambda_1+\lambda_1)^4}\{1+\beta_1(1-\beta_1)+\beta_2(1-\beta_2)\} \\
&= 2A^2(1-A)^2\{1+\beta_1(1-\beta_1)+\beta_2(1-\beta_2)\} \\
&= 2A^2(1-A)^2(1+\rho_1+\rho_2), \qquad (5.6)
\end{aligned}$$

with $\rho_1 = \operatorname{corr}(X_i, X_{i+1}) = \beta_1(1-\beta_1)$ and $\rho_2 = \operatorname{corr}(Y_i, Y_{i+1}) = \beta_2(1-\beta_2)$ are the lag 1 autocorrelations of $\{X_n\}$ and $\{Y_n\}$ respectively.

Next, we consider the properties of the estimator in the case of continuous observations over a fixed period. Suppose that the process is observed over a fixed period $[0,\ T]$. In this case a natural estimator of the limiting availability is given by,

$$\overline{A}(T) = \frac{\alpha(T)}{T}, \qquad (5.7)$$

where $\alpha(T)$ is the total operating time in the interval $[0, T]$.

Let $Z_n = X_n + Y_n$ and $S_n = Z_1 + Z_2 + ... + Z_n$, $n = 1, 2....$.

If we define $N(T) = \sup\{n : S_n \le T\}$, then $N(T)$ counts the number of cycles completed in the interval $[0,T]$ and hence $\alpha(T)$ can be represented as

$$\alpha(T) = \lambda(T)\sum_{j=1}^{N(T)+1} X_j + \left(1-\lambda(T)\right)\left\{\sum_{j=1}^{N(T)} X_j + T - S_{N(T)}\right\},$$

where $\lambda(T) = I(S_{N(T)} + X_{N(T)+1} \le T < S_{N(T)+1})$.

Now, we can write (5.7) as

$$\overline{A}(T) = \frac{1}{T}\sum_{j=1}^{N(T)+1} X_j + \frac{\left(1-\lambda(T)\right)}{T}\left\{T - S_{N(T)} - X_{N(T)+1}\right\}. \tag{5.8}$$

For the stationary *m*-dependent sequence it is proved that (Janson, 1983) $N(T)/T \to 1/\mu_Z$ and $S_{N(T)}/N(T) \to \mu_Z$ almost surely as $T \to \infty$, where $\mu_Z$ is the mean of the sequence $\{Z_n\}$.

Using this result we have,

$$\frac{N(T)}{T} \to \frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}.$$

and hence

$$\frac{1}{T}\sum_{j=1}^{N(T)+1} X_j = \frac{N(T)+1}{T}\frac{1}{N(T)+1}\sum_{j=1}^{N(T)+1} X_j \to \frac{\lambda_2}{\lambda_1+\lambda_2} = A$$

Also,

$$\left|\frac{\left(T - S_{N(T)} - X_{N(T)+1}\right)}{T}\right| \le \frac{Z_{N(T)+1}}{T} \to 0 \text{ (a.s.) as } T \to \infty.$$

Applying these results in (5.8), we can show that

$$\overline{A}(T) \to A \quad \text{almost surely as } T \to \infty.$$

If we define, $W_j = \dfrac{X_j}{\lambda_2} - \dfrac{Y_j}{\lambda_1}$, $j = 1, 2, 3, ...$, then the sequence $\{W_j\}$ is also stationary and 1-dependent with $E(W_j) = 0$.

Now, using the central limit theorem for *m*-dependent sequence (Lemma 1.6) we have as $n \to \infty$,

$$\sqrt{n}\bar{W}_n = \sqrt{n}\left(\bar{X}_n/\lambda_2 - \bar{Y}_n/\lambda_1\right) \xrightarrow{L} N(0, \nu^2),$$

where $\nu^2 = \frac{2}{\lambda_1^2 \lambda_2^2}\{1 + \beta_1(1-\beta_1) + \beta_2(1-\beta_2)\}$.

Using the results of Janson (1983) we have

$$\frac{1}{\sqrt{N(T)+1}} \sum_{j=1}^{N(T)+1} \left(X_j/\lambda_2 - Y_j/\lambda_1\right) \xrightarrow{L} N(0, \nu^2) \text{ as } T \to \infty. \tag{5.9}$$

Let us write

$$\frac{\lambda_1 + \lambda_2}{\lambda_1 \lambda_2} \sum_{j=1}^{N(T)+1} X_j - \frac{T}{\lambda_1} = \sum_{j=1}^{N(T)+1} \left(X_j/\lambda_2 - Y_j/\lambda_1\right) + \left(S_{N(T)+1} - T\right)/\lambda_1 .$$

For $\varepsilon > 0$, we have

$$P\left[\frac{Z_n}{\sqrt{n}} > \varepsilon\right] = P[Z_n \le \varepsilon\sqrt{n}]$$

$$\le \frac{E\left[Z_1^{2+\delta}\right]}{\varepsilon^{2+\delta} n^{1+\delta/2}}, \text{ for some } \delta > 0 \text{, by Markov Inequality.}$$

As the expectation on the right hand side is finite, it follows that

$$\sum_{n=1}^{\infty} P\left[\frac{Z_n}{\sqrt{n}} > \varepsilon\right] \le \frac{E\left[Z_1^{2+\delta}\right]}{\varepsilon^{2+\delta}} \sum_{n=1}^{\infty} \frac{1}{n^{1+\delta/2}} < \infty, \text{ for any } \delta > 0.$$

and consequently $\frac{Z_n}{\sqrt{n}} \to 0$ (a.s.) as $n \to \infty$.

Since $N(T) \to \infty$ as $T \to \infty$, we obtain

$$\frac{Z_{N(T)}}{\sqrt{N(T)}} \to 0 \text{ (a.s.) as } T \to \infty.$$

Thus, we have

$$0 \le \frac{S_{N(T)+1} - T}{\sqrt{N(T)+1}} \le \frac{Z_{N(T)+1}}{\sqrt{N(T)+1}} \to 0 \text{ (a.s.) as } T \to \infty. \tag{5.10}$$

Hence from (5.9) and (5.10) we get as $T \to \infty$,

$$\left( \frac{\lambda_1 + \lambda_2}{\lambda_1 \lambda_2} \sum_{j=1}^{N(T)+1} X_j - \frac{T}{\lambda_1} \right) \Big/ \sqrt{N(T)+1} \xrightarrow{L} N(0, \nu^2)$$

or
$$\frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2} \left( \frac{1}{T} \sum_{j=1}^{N(T)+1} X_j - A \right) \Big/ \left( \sqrt{N(T)+1} \big/ T \right) \xrightarrow{L} N(0, \nu^2). \tag{5.11}$$

In view of (5.8) and (5.10) we can write,

$$\left( \bar{A}(T) - \frac{1}{T} \sum_{j=1}^{N(T)+1} X_j \right) \Big/ \left( \sqrt{N(T)+1} \big/ T \right) \to 0 \text{ (a.s.) as } T \to \infty \tag{5.12}$$

By writing

$$\frac{\bar{A}(T) - A}{\sqrt{N(T)+1} \big/ T} = \frac{\bar{A}(T) - \frac{1}{T} \sum_{j=1}^{N(T)+1} X_j}{\sqrt{N(T)+1} \big/ T} + \frac{\frac{1}{T} \sum_{j=1}^{N(T)+1} X_j - A}{\sqrt{N(T)+1} \big/ T}$$

and applying the results (5.11) and (5.12) it can be shown that as $T \to \infty$,

$$\frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2} \frac{\bar{A}(T) - A}{\sqrt{N(T)+1} \big/ T} \xrightarrow{L} N(0, \nu^2).$$

Finally considering $\frac{N(T)+1}{T} \to \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2}$, it follows that

$$\sqrt{T} \left( \bar{A}(T) - A \right) \xrightarrow{L} N\left(0, \kappa^2\right),$$

where
$$\begin{aligned} \kappa^2 &= \left( \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2} \right)^3 \nu^2 \\ &= \frac{2\lambda_1 \lambda_2}{(\lambda_1 + \lambda_2)^3} \{1 + \beta_1(1-\beta_1) + \beta_2(1-\beta_2)\} \\ &= \frac{2A(1-A)}{(\lambda_1 + \lambda_2)} \{1 + \rho_1 + \rho_2\}. \end{aligned} \tag{5.13}$$

In order to study the effect of the dependence among successive observations of $\{X_n\}$ and $\{Y_n\}$ in the asymptotic variance of the estimators suggested for $A$ we consider the following ratio

$$E = \frac{\text{asymptotic variance under the EMA1 model}}{\text{asymptotic variance under i.i.d. set-up}}$$

The asymptotic variance of $\hat{A}_n$ and $\overline{A}(T)$ under the EMA1 model are given in (5.6) and (5.13) respectively. Let $\tau_*^2$ and $\kappa_*^2$ denote the asymptotic variance of $\hat{A}_n$ and $\overline{A}(T)$ when $\{X_n\}$ and $\{Y_n\}$ are i.i.d. exponential sequence of random variables. Then we have

$$\tau_*^2 = 2A^2(1-A)^2 \text{ and } \kappa_*^2 = \frac{2A(1-A)}{(\lambda_1+\lambda_2)}.$$

On substitution we get

$$E = \frac{\tau^2}{\tau_*^2} = \frac{\kappa^2}{\kappa_*^2}$$

$$= 1 + \beta_1(1-\beta_1) + \beta_2(1-\beta_2).$$

$$= 1 + \rho_1 + \rho_2.$$

The graph of the ratio *E* for different values of $\beta_1$ and $\beta_2$ is shown in figure 5.2.

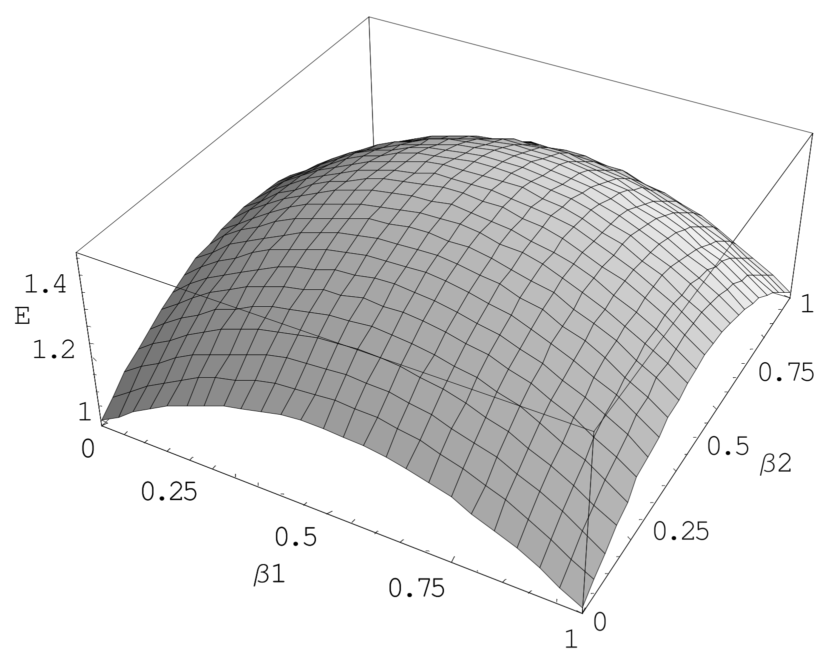


**Figure 5.2** Graph of the ratio *E* for various values of $\beta_1$ and $\beta_2$.

From the above figure, it is clear that the ratio $E$ is always greater than unity in the presence of autocorrelation among the successive sequences of failure and repair times. For example, when the marginal autocorrelations $\rho_1 = 0.15$ and $\rho_2 = 0.10$, then the ratio $E = 1.25$. This indicates that the assumption of independence among the successive sequences of failure and repair times underestimates the variance of the estimators by 25% if the true process is generated by EMA1 model. This may lead to erroneous conclusions in the inference procedure for the limiting availability.

### 5.4 Availability Behavior of Exponential Autoregressive Processes

Suppose that the sequences of failure and repair times, $\{X_n\}$ and $\{Y_n\}$, are modeled by two independent first-order Exponential Autoregressive (EAR(1)) process (Gaver and Lewis, 1980) defined as

$$X_n = \begin{cases} \rho_1 X_{n-1} & \text{with probability } \rho_1, \\ \rho_1 X_{n-1} + \varepsilon_n & \text{with probability } 1 - \rho_1. \end{cases} \quad \text{and}$$

$$Y_n = \begin{cases} \rho_2 Y_{n-1} & \text{with probability } \rho_2, \\ \rho_2 Y_{n-1} + \eta_n & \text{with probability } 1 - \rho_2. \end{cases}, \quad (0 \le \rho_1, \rho_2 < 1, n = 1, 2, 3, ...)$$

where $\{\varepsilon_n\}$ and $\{\eta_n\}$ are two independent sequences of i.i.d. exponential random variables with parameters $\lambda_1$ and $\lambda_2$ respectively. Obviously when $\rho_1 = \rho_2 = 0$, $\{X_n\}$ and $\{Y_n\}$ reduce to the i.i.d. sequences of exponential random variables. The exponential marginal distribution of the intervals and the Markovian dependence among adjacent members of the sequence are the simplest aspects of the EAR(1) model.

The Laplace transforms of $F_X^{(n)}(t)$ and $F_Y^{(n)}(t)$ in the EAR(1) model are given by,

$$F_X^{(n)*}(s) = \frac{\lambda_1}{\lambda_1 s + s^2\left(\dfrac{1-\rho_1^n}{1-\rho_1}\right)} \prod_{r=1}^{n-1} \left( \frac{\lambda_1 + s\rho_1\left(\dfrac{1-\rho_1^{n-r}}{1-\rho_1}\right)}{\lambda_1 + s\left(\dfrac{1-\rho_1^{n-r}}{1-\rho_1}\right)} \right)$$

and

$$F_Y^{(n)*}(s) = \frac{\lambda_2}{\lambda_2 s + s^2\left(\frac{1-\rho_2^n}{1-\rho_2}\right)} \prod_{r=1}^{n-1} \left( \frac{\lambda_2 + s\rho_2\left(\frac{1-\rho_2^{n-r}}{1-\rho_2}\right)}{\lambda_2 + s\left(\frac{1-\rho_2^{n-r}}{1-\rho_2}\right)} \right), \quad n \geq 1.$$

Substituting $F_X^{(n)*}(s)$ and $F_Y^{(n)*}(s)$ in (5.3) we get $A^*(s)$ and the point availability function $A(t)$ can be found by inverting the Laplace transform $A^*(s)$. Due to complex analytical form of $A^*(s)$ we omit the expression for $A^*(s)$. However, the limiting availability $A$ in this case is obtained as

$$A = \lim_{t\to\infty} A(t) = \lim_{s\to 1} sA^*(s) = \frac{\lambda_2}{\lambda_1 + \lambda_2}.$$

Thus, the limiting availability in the EAR(1) model is same as the limiting availability in the case of i.i.d. exponential random variables irrespective of the values of $\rho_1$ and $\rho_2$.

In order to compare the asymptotic properties of the estimators of the limiting availability in the case of EAR(1) model with the i.i.d. exponential model we consider the two estimators discussed in Section 5.3.

Based on complete observations on the failure and repair times of '*n*' complete cycles the proposed estimator for $A$ is

$$\hat{A}_n = \frac{\bar{X}_n}{\bar{X}_n + \bar{Y}_n}.$$

The consistency of the estimator is obvious as the sequences $\{X_n\}$ and $\{Y_n\}$ are stationary and ergodic. The asymptotic normality of the estimator follows from the fact that the EAR(1) process is strong mixing (Chernick, 1977). By the central limit theorem for strong mixing sequence (Lemma 1.7) we have as $n \to \infty$,

$$\sqrt{n}\left(\bar{X}_n - 1/\lambda_1\right) \xrightarrow{L} N(\mathbf{0}, \sigma_X^2) \text{ and}$$

$$\sqrt{n}\left(\bar{Y}_n - 1/\lambda_2\right) \xrightarrow{L} N(\mathbf{0}, \sigma_Y^2),$$

where

$$\sigma_X^2 = \mathrm{Var}(X_1) + 2\sum_{h=2}^{\infty} \mathrm{cov}(X_1, X_h)$$

$$= \frac{1}{\lambda_1^2}\left(1 + 2\sum_{h=2}^{\infty} \rho_1^{h-1}\right) = \frac{1}{\lambda_1^2}\left(\frac{1+\rho_1}{1-\rho_1}\right)$$

and

$$\sigma_Y^2 = \mathrm{Var}(Y_1) + 2\sum_{h=2}^{\infty} \mathrm{cov}(Y_1, Y_h) = \frac{1}{\lambda_2^2}\left(\frac{1+\rho_2}{1-\rho_2}\right).$$

Now, proceeding as in Section 5.3 it is easy to show that

$$\sqrt{n}\left(\hat{A}_n - A\right) \xrightarrow{L} N(0, \tau^2),$$

where $\tau^2$ is given by

$$\tau^2 = \frac{\lambda_1^2 \lambda_2^2}{(\lambda_1 + \lambda_2)^4}\left(\frac{1+\rho_1}{1-\rho_1} + \frac{1+\rho_2}{1-\rho_2}\right)$$

$$= A^2(1-A)^2\left(\frac{1+\rho_1}{1-\rho_1} + \frac{1+\rho_2}{1-\rho_2}\right). \tag{5.14}$$

In the case of continuous observation over a fixed period $[0, T]$ the proposed estimator of $A$ is

$$\bar{A}(T) = \frac{\alpha(T)}{T}.$$

The asymptotic properties of the estimator $\bar{A}(T)$ in the EAR(1) process can be proved similarly as in the EMA1 model discussed in Section 5.3 by applying the central limit theorem for strong mixing sequence (Lemma 1.7). So we omit the proof of consistency and asymptotic normality of the estimator.

As $T \to \infty$ it can be shown that

$$\sqrt{T}\left(\bar{A}(T) - A\right) \xrightarrow{L} N\left(0, \kappa^2\right),$$

where

$$\kappa^2 = \frac{A(1-A)}{(\lambda_1 + \lambda_2)}\left(\frac{1+\rho_1}{1-\rho_1} + \frac{1+\rho_2}{1-\rho_2}\right). \tag{5.15}$$

In order to see how sensitive the asymptotic variance of the estimators given in (5.14) and (5.15) are for different values of the marginal autocorrelations of $\{X_n\}$ and $\{Y_n\}$ we consider the ratio

$$E = \frac{\text{asymptotic variance under the EAR(1) model}}{\text{asymptotic variance under i.i.d. set-up}}$$

$$= \frac{1}{2}\left(\frac{1+\rho_1}{1-\rho_1} + \frac{1+\rho_2}{1-\rho_2}\right).$$

Figure 5.3 shows the graph of *E* for different possible values of $\rho_1$ and $\rho_2$.

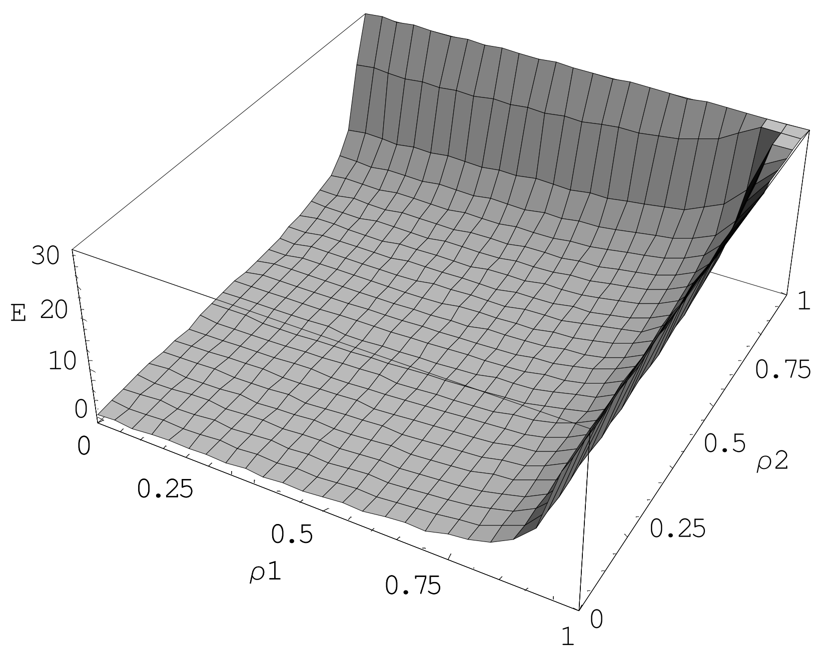


**Figure 5.3** Graph of the ratio *E* for various values of $\rho_1$ and $\rho_2$.

From the figure, it is obvious that the ratio *E* is always greater than unity in the presence of autocorrelation and increases rapidly as $\rho_1$ and $\rho_2$ increases. This means that the assumption of an i.i.d. sequence underestimates the variance of the estimators significantly if the true process is EAR(1). For example, when the marginal autocorrelations $\rho_1 = 0.50$ and $\rho_2 = 0.20$, then the ratio $E = 2.25$. This indicates that the assumption of independence among the successive sequences of failure and repair times underestimates the variance of the estimators by 125% if the true process is generated by EAR(1) model.

### 5.5 Conclusion

In this chapter, we discussed the availability behavior of some stationary dependent sequence of random variables. The general expression for point availability was derived when the failure and repair times are generated by stationary dependent sequence of random variables. The availability behavior of two time series models, first order Exponential moving average process (EMA1 model) and first order Exponential autoregressive process (EAR(1) model) were studied. The asymptotic properties of the estimators of the limiting availability for both time series models were studied and the efficiency of the asymptotic variance of the estimators were compared with the corresponding estimators in the i.i.d. exponential case. The comparison showed that the ignorance of autocorrelation present in the data underestimate the asymptotic variance of the estimators significantly.

# Chapter 6

# Estimation of the Limiting Interval Reliability for Stationary Dependent Sequences

## 6.1 Introduction

In the case of repairable systems, the estimation of the availability measures is not discussed much when the successive sequences of failure and repair times are generated by some stationary dependent sequences except those considered by Abraham and Balakrishna (2000). Motivated by this, in the present chapter we consider the nonparametric estimation of the limiting interval reliability, when the sequences of failure times $\{X_n\}$ and repair times $\{Y_n\}$ are generated by stationary strong mixing sequences of random variables. It is assumed that $\{X_n\}$ and $\{Y_n\}$ are two independent stationary strong mixing sequences of random variables with mixing coefficients $\alpha_X(h)$ and $\alpha_Y(h)$ respectively (See Definition 1.4). The estimation of the limiting interval reliability is studied under the three sampling schemes discussed in previous chapters.

The chapter is organized as follows. In Section 6.2, we consider the nonparametric estimation of the limiting interval reliability in the case of complete observations. Section 6.3 discusses the estimation in the case of censored observations. In Section 6.4, we consider the estimation of $R(x)$ when the process is observed up to a specified time '$T$'. An extension of the estimation results to complex systems is discussed in Section 6.5 and some numerical illustrations are presented in Section 6.6 to assess the performance of the proposed estimators. We

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The results in this chapter have been communicated as entitled 'Nonparametric Estimation of the Limiting Interval Reliability for Stationary Dependent Sequences' (See Balakrishna and Mathew, 2011c).

conclude the chapter in Section 6.7.

## 6.2 Estimation in the case of Complete Observations

Suppose that observations on $n$ failure times $X_1, X_2, ..., X_n$ and $m$ repair times $Y_1, Y_2, ..., Y_m$ are recorded for a repairable one-unit system. In practice, usually we use equal sample sizes for the failure and repair time observations based on the fact that a repair is done for each failure. However, if a failed component is replaced by a new component instead of repairing it, then the number of failures is greater than the number of repairs. On the other hand, due to missing of records or some other reasons, a failure time may not be recorded but the corresponding repair time is recorded. In such cases, the number of repair times is greater than the number of failure times.

Assume that the failure times $X_1, X_2, ..., X_n$ and repair times $Y_1, Y_2, ..., Y_m$ are generated from stationary strong mixing sequences of random variables $\{X_n\}$ and $\{Y_n\}$ having continuous cumulative distribution functions $F_X(.)$ and $F_Y(.)$ respectively. If we define $U_i = (X_i - x)I_{(X_i > x)}$, then nonparametric estimators for $\mu_X$, $\mu_Y$ and $\upsilon(x)$ are given by,

$$\hat{\mu}_X = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}_n\,,\ \hat{\mu}_Y = \frac{1}{m}\sum_{i=1}^{m} Y_i = \bar{Y}_m \text{ and } \hat{\upsilon}(x) = \frac{1}{n}\sum_{i=1}^{n} U_i = \bar{U}_n\,.$$

Thus, a nonparametric estimator of the limiting interval reliability $R(x)$ is

$$\hat{R}(x) = \frac{\bar{U}_n}{\bar{X}_n + \bar{Y}_m}\,. \tag{6.1}$$

Since $\{X_n\}$ and $\{Y_n\}$ are strictly stationary, it follows that $\bar{X}_n \to \mu_X$ and $\bar{U}_n \to \upsilon(x)$ almost surely as $n \to \infty$ and $\bar{Y}_m \to \mu_Y$ almost surely as $m \to \infty$. Hence we may conclude that $\hat{R}(x) \to R(x)$ almost surely as $m, n \to \infty$.

If we define $f(x_1, x_2, x_3) = \frac{x_3}{x_1 + x_2}$, then $f(\bar{X}_n, \bar{Y}_m, \bar{U}_n) = \hat{R}(x)$ and $f(\mu_X, \mu_Y, \upsilon(x)) = R(x)$.

Employing a Taylor series expansion of $f$ about the point $(a_1, a_2, a_3)$ to the first order approximation we have,

$$f(x_1, x_2, x_3) \cong f(a_1, a_2, a_3) + \sum_{i=1}^{3} \left. \frac{\partial f}{\partial x_i} \right|_{\underline{x}=\underline{a}} (x_i - a_i),$$

where $\underline{x} = (x_1, x_2, x_3)$ and $\underline{a} = (a_1, a_2, a_3)$.

Since $\frac{\partial f}{\partial x_1} = -\frac{x_3}{(x_1 + x_2)^2}$, $\frac{\partial f}{\partial x_2} = -\frac{x_3}{(x_1 + x_2)^2}$ and $\frac{\partial f}{\partial x_3} = \frac{1}{x_1 + x_2}$, we can write,

$$\begin{aligned} \hat{R}(x) &= f(\bar{X}_n, \bar{Y}_m, \bar{U}_n) \\ &\cong R(x) - \frac{\upsilon(x)}{(\mu_X + \mu_Y)^2} (\bar{X}_n - \mu_X) - \frac{\upsilon(x)}{(\mu_X + \mu_Y)^2} (\bar{Y}_m - \mu_Y) \\ &\quad + \frac{1}{\mu_X + \mu_Y} (\bar{U}_n - \upsilon(x)). \end{aligned} \tag{6.2}$$

Let $B = \left( \frac{-\upsilon(x)}{(\mu_X + \mu_Y)^2}, \frac{-\upsilon(x)}{(\mu_X + \mu_Y)^2}, \frac{1}{\mu_X + \mu_Y} \right)$ be a row vector and $V_n = \sqrt{n} \left( \bar{X}_n - \mu_X, \bar{Y}_m - \mu_Y, \bar{U}_n - \upsilon(x) \right)'$.

Now, from the asymptotic expansion (6.2) we can write,

$$\sqrt{n} \left( \hat{R}(x) - R(x) \right) \cong BV_n .$$

Assume that, as $n \to \infty$ and $m \to \infty$, $(n/m) \to \theta$, where $0 < \theta < \infty$. Now, the remainder terms in the Taylor series expansion of $\hat{R}(x)$ multiplied by $\sqrt{n}$ converges to zero almost surely as $n \to \infty$.

By the central limit theorem for strong mixing sequence (Lemma 1.7) we have as $n \to \infty$,

$$\sqrt{n}\left(\bar{Y}_m - \mu_Y\right) \xrightarrow{D} N\left(0,\ \theta\sigma_{YY}\right) \text{ and}$$

$$\sqrt{n}\left(\bar{X}_n - \mu_X,\ \bar{U}_n - \upsilon(x)\right) \xrightarrow{D} N_2\left(0,\ \Sigma_2\right),$$

where $N_2(0, \Sigma_2)$ is a 2-variate normal vector with mean $\mathbf{0} = (0,0)'$ and dispersion matrix

$$\Sigma_2 = \begin{pmatrix} \sigma_{XX} & \sigma_{XU} \\ \sigma_{XU} & \sigma_{UU} \end{pmatrix},$$

with

$$\sigma_{XX} = \text{var}(X_1) + 2\sum_{h=2}^{\infty} \text{cov}(X_1, X_h),$$

$$\sigma_{YY} = \text{var}(Y_1) + 2\sum_{h=2}^{\infty} \text{cov}(Y_1, Y_h),$$

$$\sigma_{UU} = \text{var}(U_1) + 2\sum_{h=2}^{\infty} \text{cov}(U_1, U_h) \text{ and}$$

$$\sigma_{XU} = \text{cov}(X_1, U_1) + \sum_{h=2}^{\infty} \text{cov}(X_1, U_h) + \sum_{h=2}^{\infty} \text{cov}(X_h, U_1).$$

Now, by the Cramer-Wold device (Billingsley, 1968, pp.49), we have as $n \to \infty$,

$$\sqrt{n}\left(\bar{X}_n - \mu_X,\ \bar{Y}_m - \mu_Y,\ \bar{U}_n - \upsilon(x)\right) \xrightarrow{L} N_3(\mathbf{0}, \Sigma_3),$$

where $N_3(0, \Sigma_3)$ is a 3-variate normal vector with mean $\mathbf{0} = (0,0,0)'$ and dispersion matrix

$$\Sigma_3 = \begin{pmatrix} \sigma_{XX} & 0 & \sigma_{XU} \\ 0 & \theta\sigma_{YY} & 0 \\ \sigma_{XU} & 0 & \sigma_{UU} \end{pmatrix}.$$

Thus,

$$\sqrt{n}\left(\hat{R}(x) - R(x)\right) \cong BV_n$$

$$\xrightarrow{D} N\left(0,\ \tau^2(x)\right),$$

where

$$\tau^2(x) = B\,\Sigma_3\,B'.$$

Now, $\tau^2(x)$

$$= \left( \frac{-\upsilon(x)}{(\mu_X+\mu_Y)^2}, \frac{-\upsilon(x)}{(\mu_X+\mu_Y)^2}, \frac{1}{\mu_X+\mu_Y} \right) \begin{pmatrix} \sigma_{XX} & 0 & \sigma_{XU} \\ 0 & \theta\sigma_{YY} & 0 \\ \sigma_{XU} & 0 & \sigma_{UU} \end{pmatrix} \begin{pmatrix} \dfrac{-\upsilon(x)}{(\mu_X+\mu_Y)^2} \\ \dfrac{-\upsilon(x)}{(\mu_X+\mu_Y)^2} \\ \dfrac{1}{\mu_X+\mu_Y} \end{pmatrix}$$

$$= \frac{\upsilon^2(x)}{(\mu_X+\mu_Y)^4}(\sigma_{XX}+\theta\sigma_{YY}) - \frac{2\upsilon(x)}{(\mu_X+\mu_Y)^3}\sigma_{XU} + \frac{1}{(\mu_X+\mu_Y)^2}\sigma_{UU}. \qquad (6.3)$$

Thus we have proved the following theorem.

**Theorem 6.1**

*If $\{X_n\}$ and $\{Y_n\}$ are two mutually independent strictly stationary and strong mixing sequence of non-negative random variables such that for some $\delta > 0$, $E(X_1^{2+\delta}) < \infty$, $E(Y_1^{2+\delta}) < \infty$, $\sum_{h=1}^{\infty} \alpha_X^{\delta/(2+\delta)}(h) < \infty$ and $\sum_{h=1}^{\infty} \alpha_Y^{\delta/(2+\delta)}(h) < \infty$, then for any fixed 'x' as $n \to \infty$,*

*(i) $\hat{R}(x) \to R(x)$ almost surely and*

*(ii) $\sqrt{n}\left(\hat{R}(x) - R(x)\right) \xrightarrow{L} N\left(0, \tau^2(x)\right)$,*

*where $\tau^2(x)$ is given in* (6.3).

A consistent estimator $\hat{\tau}^2(x)$ of $\tau^2(x)$ can be obtained by replacing $\mu_X$, $\mu_Y$, $\upsilon(x)$, $\sigma_{XX}$, $\sigma_{YY}$, $\sigma_{UU}$ and $\sigma_{XU}$ with their corresponding consistent estimators in (6.3). Obviously $\bar{X}_n$, $\bar{Y}_m$, $\bar{U}_n$ and $(n/m)$ are the consistent estimators for $\mu_X$, $\mu_Y$, $\upsilon(x)$ and $\theta$ respectively. In order to construct consistent estimators for $\sigma_{XX}$, $\sigma_{YY}$, $\sigma_{UU}$ and $\sigma_{XU}$, we use the moving-block jackknife method for variance estimation with dependent data (Kunsch, 1989). The moving-block jackknife estimators for $\sigma_{XX}$, $\sigma_{YY}$, $\sigma_{UU}$ and $\sigma_{XU}$ respectively are

$$\hat{\sigma}^2_{XX,l_1} = \frac{l_1}{n-l_1+1}\sum_{i=1}^{n-l_1+1}\left(\bar{X}_i^{(l_1)} - (n+l_1-1)^{-1}\sum_{j=1}^{n-l_1+1}\bar{X}_j^{(l_1)}\right)^2,$$

$$\hat{\sigma}^2_{ZZ,l_2} = \frac{l_2}{m-l_2+1}\sum_{i=1}^{m-l_2+1}\left(\bar{Y}_i^{(l_2)} - (m+l_2-1)^{-1}\sum_{j=1}^{m-l_2+1}\bar{Y}_j^{(l_2)}\right)^2,$$

$$\hat{\sigma}^2_{UU,l_1} = \frac{l_1}{n-l_1+1}\sum_{i=1}^{n-l_1+1}\left(\bar{U}_i^{(l_1)} - (n+l_1-1)^{-1}\sum_{j=1}^{n-l_1+1}\bar{U}_j^{(l_1)}\right)^2, \text{ and}$$

$$\hat{\sigma}_{XU,l_1} = \frac{l_1}{n-l_1+1}\sum_{i=1}^{n-l_1+1}\left(\bar{X}_i^{(l_1)} - (n+l_1-1)^{-1}\sum_{j=1}^{n-l_1+1}\bar{X}_j^{(l_1)}\right)\left(\bar{U}_i^{(l_1)} - (n+l_1-1)^{-1}\sum_{j=1}^{n-l_1+1}\bar{U}_j^{(l_1)}\right),$$

where $\bar{X}_i^{(l_1)} = l_1^{-1}\sum_{j=i}^{i+l_1-1} X_j$, $\bar{Y}_i^{(l_2)} = l_2^{-1}\sum_{j=i}^{i+l_2-1} Y_j$, $\bar{U}_i^{(l_1)} = l_1^{-1}\sum_{j=i}^{i+l_1-1} U_j$ and $l_1 = l(n)$ and $l_2 = l(m)$ are the block sizes.

If we assume that for some $\delta > 0$, $E\left[\left|X_1\right|^{6+\delta}\right] < \infty$, $E\left[\left|Y_1\right|^{6+\delta}\right] < \infty$, $\sum k^2\alpha_1(k)^{\delta/(6+\delta)} < \infty$ and $\sum k^2\alpha_2(k)^{\delta/(6+\delta)} < \infty$, then the estimators $\hat{\sigma}^2_{XX,l_1}$, $\hat{\sigma}^2_{YY,l_2}$, $\hat{\sigma}^2_{UU,l_1}$ and $\hat{\sigma}_{XU,l_1}$ converge almost surely to $\sigma_{XX}$, $\sigma_{YY}$, $\sigma_{UU}$ and $\sigma_{XU}$ respectively if $l_1 = o(n)$, $l_2 = o(m)$ and $l_1, l_2 \to \infty$ (Kunsch, 1989).

Then, it is easy to see that

$\hat{\tau}^2(x) \to \tau^2(x)$ almost surely as $n \to \infty$.

Thus, given a significance level $\alpha \in (0,1)$, an approximate large sample $100(1-\alpha)\%$ confidence interval for the limiting interval reliability $R(x)$ is

$$\hat{R}(x) - z_{\alpha/2}\frac{\hat{\tau}(x)}{\sqrt{n}} \le R(x) \le \hat{R}(x) + z_{\alpha/2}\frac{\hat{\tau}(x)}{\sqrt{n}}.$$

### 6.3 Estimation in the case of Censored Observations

Suppose that observations on the failure times $X_1, X_2, ..., X_n$ and the repair times $Y_1, Y_2, ..., Y_m$ are subject to right censorship. Let $C_1, C_2, ..., C_n$ and $D_1, D_2, ..., D_m$ denote the censoring times associated with the failure and repair times respectively. Assume that the censoring times $C_1, C_2, ..., C_n$ and $D_1, D_2, ..., D_m$ are mutually independent i.i.d. random variables having continuous cumulative distribution functions $G_C(.)$ and $G_D(.)$ respectively. Under the censoring model, instead of observing $X_i$ and $Y_i$, we observe the pairs $(T_i, \delta_i)$, $i = 1, 2, ..., n$ and $(V_i, \eta_i)$, $i = 1, 2, ..., m$, where $T_i = \min(X_i, C_i)$, $V_i = \min(Y_i, D_i)$, $\delta_i = I_{(X_i \le C_i)}$ and $\eta_i = I_{(Y_i \le D_i)}$. Let $H_X(.)$ and $H_Y(.)$ be the distribution function of $T_i$ and $V_i$ respectively. Also let $\tau_X$ and $\tau_Y$ be the least upper bound for the support of $H_X(.)$ and $H_Y(.)$ respectively.

Let $T_{(1)} \le T_{(2)} \le ... \le T_{(n)}$ and $V_{(1)} \le V_{(2)} \le ... \le V_{(m)}$ be the order statistics of $T_1, T_2, ..., T_n$ and $V_1, V_2, ..., V_m$ respectively and let $\delta_{(i)}$ and $\eta_{(i)}$ denote the concomitant associated with $T_{(i)}$ and $V_{(i)}$ respectively. Now, the Kaplan-Meier estimators of $F_X(t)$ and $G_C(t)$ are given by,

$$\hat{F}_{X,c}(t) = 1 - \prod_{T_{(i)} \le t} \left( \frac{n-i}{n-i+1} \right)^{\delta_{(i)}} \text{ and}$$

$$\hat{G}_C(t) = 1 - \prod_{T_{(i)} \le t} \left( \frac{n-i}{n-i+1} \right)^{1-\delta_{(i)}}.$$

Similarly, we can construct the Kaplan-Meier estimators $\hat{F}_{Y,c}(t)$ and $\hat{G}_D(t)$ for $F_Y(t)$ and $G_D(t)$.

Now, the nonparametric estimators for $\mu_X$, $\mu_Y$ and $\upsilon(x)$ are given by,

$$\hat{\mu}_{X,c} = \int_0^\infty u d\hat{F}_{X,c}(u), \quad \hat{\mu}_{Y,c} = \int_0^\infty u d\hat{F}_{Y,c}(u) \text{ and } \hat{\upsilon}_c(x) = \int_0^\infty (u-x) I_{(u>x)} d\hat{F}_{X,c}(x).$$

Thus, a nonparametric estimator of $R(x)$ is given by,

$$\hat{R}_c(x) = \frac{\hat{\upsilon}_c(x)}{\hat{\mu}_{X,c} + \hat{\mu}_{Y,c}}. \tag{6.4}$$

Let us define,

$$\gamma_0(x) = \frac{1}{1-G(x)}$$

$$\gamma_1(x) = \frac{1}{1-H(x)} \int_x^{\tau} \varphi(t) dF(t)$$

$$\gamma_2(x) = \int_{-\infty}^{x} \frac{\gamma_1(t)}{1-G(t)} dG(t) ,$$

where $\varphi$ is an $F$-integrable function and $F$, $G$, $H$ and $\tau$ are as defined in this section ignoring their suffices.

In order to study the asymptotic properties of the estimator defined in (6.4) we use the following Lemma due to Ghouch et.al. (2010).

**Lemma 6.1** *Let $\hat{F}$ be the Kaplan-Meier product limit estimator of the cumulative distribution function F corresponding to a stationary strong mixing sequence of random variables with mixing coefficient $\alpha(.)$. If there exist $v > 3$ such that $\alpha(h) = O(h^{-v})$ and $\int |\varphi(t)|^{v} dF(t) < \infty$ for any F-integrable function $\varphi$, then*

$$\lim_{n\to\infty} \int_0^{\infty} \varphi d\hat{F} = \int_0^{\tau} \varphi dF \text{ with probability 1.}$$

*Also,* $\int \varphi d(\hat{F} - F) = n^{-1} \sum_{i=1}^{n} B_i + o_P(n^{-1/2})$,

*where* $B_i = \varphi(T_i)\gamma_0(T_i)\delta_i + \gamma_1(T_i)(1-\delta_i) - \gamma_2(T_i)$ *are strong mixing sequences of random variables with mixing coefficient* $\alpha(h)$ *and*

$$\sqrt{n}\left[ \int_0^{\infty} \varphi d\hat{F} - \int_0^{\tau} \varphi dF \right] \xrightarrow{L} N(0, \sigma_c^2) \text{ as } n \to \infty.$$

*Proof*: See Ghouch et.al. (2010).

Define,

$$\mu_{X,c} = \int_0^{\tau_X} u dF_{X,c}(u) ,\ \mu_{Y,c} = \int_0^{\tau_Y} u dF_{Y,c}(u) ,\ \upsilon_c(x) = \int_0^{\tau_X} (u-x) I(u > x) dF_{X,c}(u) ,$$

and $R_c(x) = \dfrac{\upsilon_c(x)}{\mu_{X,c} + \mu_{Y,c}}$.

Now, it can be verified that $\hat{\mu}_{X,c} \to \mu_{X,c}$ and $\hat{\upsilon}_c(x) \to \upsilon_c(x)$ almost surely as $n \to \infty$ by choosing $\varphi(u) = u$ and $\varphi(u) = (u - x)I_{(u>x)}$ respectively in Lemma 6.1. Also, by choosing $\varphi(u) = u$, $\hat{\mu}_{Y,c} \to \mu_{Y,c}$ almost surely as $m \to \infty$ and hence we conclude that $\hat{R}_c(x) \to R_c(x)$ almost surely as $m, n \to \infty$.

Again, by applying Lemma 6.1 and using the Cramer-Wold device (Billingsley, 1968, pp.49) it can be shown that as $n \to \infty$,

$$\sqrt{n}\left(\hat{\mu}_{X,c} - \mu_{X,c},\, \hat{\mu}_{X,c} - \mu_{X,c},\, \hat{\upsilon}_c(x) - \upsilon_c(x)\right) \xrightarrow{L} N_3(\mathbf{0}, \Sigma_c),$$

where $N_3(0, \Sigma_c)$ is a 3-variate normal vector with mean $\mathbf{0} = (0,0,0)$ and dispersion matrix

$$\Sigma_c = \begin{pmatrix} \sigma_{XX,c} & 0 & \sigma_{XU,c} \\ \sigma_{XY,c} & \theta\sigma_{YY,c} & 0 \\ \sigma_{XU,c} & 0 & \sigma_{UU,c} \end{pmatrix},$$

with $\sigma_{XX,c} = \text{var}(B_{X_1}) + 2\sum_{h=2}^{\infty} \text{cov}(B_{X_1}, B_{X_h})$,

$$\sigma_{YY,c} = \text{var}(B_{Y_1}) + 2\sum_{h=2}^{\infty} \text{cov}(B_{Y_1}, B_{Y_h}),$$

$$\sigma_{UU,c} = \text{var}(B_{U_1}) + 2\sum_{h=2}^{\infty} \text{cov}(B_{U_1}, B_{U_h}) \text{ and}$$

$$\sigma_{XU,c} = \text{cov}(B_{X_1}, B_{U_1}) + \sum_{h=2}^{\infty} \text{cov}(B_{X_1}, B_{U_h}) + \sum_{h=2}^{\infty} \text{cov}(B_{X_h}, B_{U_1}),$$

in which $B_{X,i}$, $B_{Y,i}$ and $B_{U,i}$ are as defined in Lemma 6.1 by choosing $\varphi(u) = u$ for $B_{X,i}$ and $B_{Y,i}$ and $\varphi(u) = (u - x)I_{(u>x)}$ for $B_{U,i}$.

Since $f(\hat{\mu}_{X,c}, \hat{\mu}_{Y,c}, \hat{\upsilon}_c(x)) = \hat{R}_c(x)$, proceeding as in Section 6.1, it can be shown that,

$$\sqrt{n}\left(\hat{R}_c(x)-R_c(x)\right)\xrightarrow{L} N\left(0,\tau_c^2(x)\right) \text{ as } n\to\infty,$$

where

$$\tau_c^2(x)=\frac{\upsilon_c^2(x)(\sigma_{XX,c}+\theta\sigma_{YY,c})}{(\mu_{X,c}+\mu_{Y,c})^4}-\frac{2\upsilon_c(x)\sigma_{XU,c}}{(\mu_{X,c}+\mu_{Y,c})^3}+\frac{\sigma_{UU,c}}{(\mu_{X,c}+\mu_{Y,c})^2}. \tag{6.5}$$

This leads to the following theorem.

**Theorem 6.2**

*If $\{X_n\}$ and $\{Y_n\}$ are two mutually independent strictly stationary and strong mixing sequence of non-negative random variables such that for some $v>3$ $E(X_1^v)<\infty$, $E(Y_1^v)<\infty$, $\alpha_X(h)=O(h^{-v})$ and $\alpha_Y(h)=O(h^{-v})$, then for any fixed 'x', as $n\to\infty$,*

*(i) $\hat{R}_c(x)\to R_c(x)$ almost surely and*

*(ii) $\sqrt{n}\left(\hat{R}_c(x)-R_c(x)\right)\xrightarrow{L} N\left(0,\tau_c^2(x)\right)$,*

*where $\tau_c^2(x)$ is given in* (6.5).

A consistent estimator $\hat{\tau}_c^2(x)$ of $\tau^2(x)$ can be obtained by replacing $\mu_{X,c}$, $\mu_{Y,c}$, $\upsilon_c(x)$, $\theta$, $\sigma_{XX,c}$, $\sigma_{YY,c}$, $\sigma_{UU,c}$ and $\sigma_{XU,c}$ with their corresponding consistent estimators in (6.5). The consistent estimators for $\mu_X$, $\mu_Y$, $\upsilon(x)$ and $\theta$ are $\hat{\mu}_{X,c}$, $\hat{\mu}_{Y,c}$, $\hat{\upsilon}_c(x)$ and $(n/m)$ respectively. In order to construct consistent estimators for $\sigma_{XX,c}$, $\sigma_{YY,c}$, $\sigma_{UU,c}$ and $\sigma_{XU,c}$, we apply the moving-block jackknife method for variance estimation with dependent data discussed in Section 6.1 to the stationary strong mixing sequence of random variables $\{\hat{B}_{X,i}\}$, $\{\hat{B}_{Y,i}\}$ and $\{\hat{B}_{U,i}\}$ obtained by plugging in $\hat{F}$ and $\hat{G}$ instead of $F$ and $G$ in $B_{X,i}$, $B_{Y,i}$ and $B_{U,i}$. Now, using $\hat{\tau}_c^2(x)$ we can construct an asymptotic confidence interval for the limiting interval reliability.

### 6.4 Estimation in the case of Continuous Observation over a Fixed Period

Suppose that the process is observed over a fixed period $[0,\ T]$. In this section, we assume that a repair is done for each failure. Define $Z_n = X_n + Y_n$ and $S_n = Z_1 + Z_2 + ... + Z_n,\ n = 1, 2....$ Let $\mu_Z = \mu_X + \mu_Y$ and $\sigma_{ZZ}$ be the mean and variance of the sequence $\{Z_n\}$. Also let $N(T) = \text{Sup}\{n : S_n \leq T\}$.

Define,

$$\alpha_T(x) = \lambda(T) \sum_{j=1}^{N(T)+1} U_j + \left(1 - \lambda(T)\right)\left( \sum_{j=1}^{N(T)} U_j + (T - S_{N(T)} - x) I(T - S_{N(T)} > x) \right)$$

where $\lambda(T) = I(S_{N(T)} + X_{N(T)+1} \leq T < S_{N(T)+1})$ and $U_j = (X_j - x) I_{(X_j > x)}$.

Then, a nonparametric estimator of $R(x)$ is given by

$$R_T(x) = \frac{\alpha_T(x)}{T}. \tag{6.6}$$

Now, by applying the central limit theorem for strong mixing sequence of random variables (Lemma 1.7) to the stationary strong mixing sequence $\xi_j = \mu_Z U_j - \upsilon(x) Z_j,\ j = 1, 2, ...$ and proceeding in the similar lines as in Section 4.4, we can prove the consistency and asymptotic normality of the estimator $R_T(x)$ defined in (6.6). In order to avoid repetition we omit the proof and the results are stated in the following theorem.

**Theorem 6.3**

*If $\{X_n\}$ and $\{Y_n\}$ are two mutually independent strictly stationary and strong mixing sequence of non-negative random variables such that for some $\delta > 0$, $E(X_1^{2+\delta}) < \infty$, $E(Y_1^{2+\delta}) < \infty$, $\sum_{h=1}^{\infty} \alpha_X^{\delta/(2+\delta)}(h) < \infty$ and $\sum_{h=1}^{\infty} \alpha_Y^{\delta/(2+\delta)}(h) < \infty$, then for any fixed $x$, as $T \to \infty$,*

*(i) $R_T(x) \to R(x)$ almost surely and*

*(ii) $\sqrt{T}\left(R_T(x) - R(x)\right) \xrightarrow{L} N\left(0, \gamma^2(x)/\mu_Z^3\right)$,*

*where $\gamma^2(x) = \mu_Z^2 \sigma_{UU} + \upsilon^2(x) \sigma_{ZZ} - 2\upsilon(x) \mu_Z \sigma_{UZ}$.*

In the next section, we extend the results of the previous sections to a coherent system of $k$ independently functioning components.

### 6.5 Limiting Interval Reliability of a Coherent System

Consider a system consisting of $k$ components. Suppose the state of the $i^{th}$ component is denoted by the binary variable $x_i$ given by,

$$x_i = \begin{cases} 1 & \text{if component } 'i' \text{ is functioning,} \\ 0 & \text{if component } 'i' \text{ is failed.} \end{cases}, \text{ for } i = 1, 2, ..., k .$$

Now, the state of the system can be described by a binary function $\phi = \phi(x_1, x_2, ..., x_k)$ defined by,

$$\phi = \begin{cases} 1 & \text{if the system is functioning,} \\ 0 & \text{if the system is failed.} \end{cases} .$$

We assume that the state of the system is determined completely by the states of the components and the function $\phi = \phi(x_1, x_2, ..., x_k)$ is called the structure function of the system.

For example, the structure function of a series system of $k$ components can be written as

$$\phi(x_1, x_2, ..., x_k) = x_1 x_2 ... x_k .$$

For a parallel system of $k$ components, the structure function will be of the form

$$\phi(x_1, x_2, ..., x_k) = 1 - (1 - x_1)(1 - x_2)...(1 - x_k) .$$

The reliability function of the system is defined by

$$h = P[\phi(x_1, x_2, ..., x_k) = 1] = E[\phi(x_1, x_2, ..., x_k)] .$$

Under the assumption of independent components we may represent the system reliability as a function of component reliabilities:

$$h = h(p_1, p_2, ..., p_k) ,$$

where $p_i = P[x_i = 1]$ denotes the reliability of the $i$-th component.

A system of components is said to be coherent if (a) its structure function $\phi$ is increasing and (b) each component is relevant. A detailed discussion of coherent systems can be found in Barlow and Proschan (1975). Series structure, parallel structure and *k*-out-of-*n* structure are all trivial examples of coherent systems.

Suppose we have a coherent system of *k* independent components. Let $R_i(x)$ denote the limiting interval reliability of the *i*-th component. Then the limiting interval reliability of the system is given by,

$$R_S(x) = h\left(R_1(x), R_2(x), ..., R_k(x)\right),$$

where $h(.)$ is the reliability function of the system.

For example, the limiting interval reliability of a series structure of *k* independent components is

$$R_S(x) = R_1(x)R_2(x)...R_k(x).$$

Also, in the case of a parallel structure of k independent components the limiting interval reliability is given by,

$$R_S(x) = 1-\left(1-R_1(x)\right)\left(1-R_2(x)\right)...\left(1-R_k(x)\right).$$

Assume that $n_i$ failure times $X_{i1}, X_{i2}, ..., X_{in_i}$ and $m_i$ repair times $Y_{i1}, Y_{i2}, ..., Y_{im_i}$ are observed for component $'i'$, $i = 1, 2, ..., k$.

A nonparametric estimator for the limiting interval reliability of the *i*-th component is given by,

$$\hat{R}_i(x) = \frac{\bar{U}_{in_i}}{\bar{X}_{in_i} + \bar{Y}_{im_i}} \text{ for } i = 1, 2, ..., k,$$

where

$$\bar{X}_{in_i} = \frac{1}{n_i}\sum_{j=1}^{n_i} X_{ij}, \ \bar{Y}_{im_i} = \frac{1}{m_i}\sum_{j=1}^{m_i} Y_{ij} \text{ and } \bar{U}_{in_i} = \frac{1}{n_i}\sum_{j=1}^{n_i} (X_{ij}-x) I_{(X_{ij}>x)}.$$

If we assume that $(n_i / m_i) \to \theta_i$, where $0 < \theta_i < \infty$, then under the assumptions stated in Theorem 6.1, we can show that as $n_i \to \infty$,

$\hat{R}_i(x) \to R_i(x)$ almost surely and

$\sqrt{n_i}\left(\hat{R}_i(x) - R_i(x)\right) \xrightarrow{L} N\left(0,\ \tau_i^2(x)\right)$,

where

$$\tau_i^2(x) = \frac{\upsilon_i^2(x)(\sigma_{XX,i} + \theta_i \sigma_{YY,i})}{(\mu_{X,i} + \mu_{Y,i})^4} - \frac{2\upsilon_i(x)\sigma_{XU,i}}{(\mu_{X,i} + \mu_{Y,i})^3} + \frac{\sigma_{UU,i}}{(\mu_{X,i} + \mu_{Y,i})^2}. \tag{6.7}$$

Now, a nonparametric estimator of the limiting interval reliability of the system is given by,

$$\hat{R}_S(x) = h\left(\hat{R}_1(x), \hat{R}_2(x), \ldots, \hat{R}_k(x)\right). \tag{6.8}$$

Since $\hat{R}_i(x)$ is a consistent estimator of $R_i(x)$ for $i = 1, 2, \ldots, k$, the consistency of the estimator $\hat{R}_S(x)$ is obvious.

In order to establish the asymptotic normality of the estimator $\hat{R}_S(x)$, without loss of generality, assume that $n_1 \le n_2 \le \ldots \le n_k$ and $(n_1 / n_i) \to \omega_i$, where $0 < \omega_i \le 1$ for $i = 1, 2, \ldots, k$.

By the delta method (Rao 1973, pp. 388), we have, approximately,

$$\hat{R}_S(x) - R_S(x) = \sum_{i=1}^{k} \frac{\partial h}{\partial R_i(x)}\left(\hat{R}_i(x) - R_i(x)\right).$$

Then, approximately,

$$\begin{aligned}\sqrt{n_1}\left(\hat{R}_S(x) - R_S(x)\right) &= \sum_{i=1}^{k} \frac{\partial h}{\partial R_i(x)} \sqrt{n_1}\left(\hat{R}_i(x) - R_i(x)\right) \\ &= \sum_{i=1}^{k} \frac{\partial h}{\partial R_i(x)} \sqrt{\omega_i}\sqrt{n_i}\left(\hat{R}_i(x) - R_i(x)\right) \\ &\xrightarrow{L} \sum_{i=1}^{k} \frac{\partial h}{\partial R_i(x)} \sqrt{\omega_i} Q_i \text{ as } n_1 \to \infty,\end{aligned}$$

where $Q_i$'s are independent normal variates with mean 0 and variance $\tau_i^2(x)$ defined by (6.7).

Thus, as $n_1 \to \infty$,

$$\sqrt{n_1}\left(\hat{R}_S(x) - R_S(x)\right) \xrightarrow{L} N\left(0,\, \tau_S^2(x)\right),$$

where $\tau_S^2(x) = \sum_{i=1}^{k}\left(\frac{\partial h}{\partial R_i(x)}\right)^2 \omega_i \tau_i^2(x)$. (6.9)

An estimator $\hat{\tau}_S^2(x)$ of $\tau_S^2(x)$ can be obtained by replacing $\frac{\partial h}{\partial R_i(x)}$, $\omega_i$ and $\tau_i^2(x)$ by $\left.\frac{\partial h}{\partial R_i(x)}\right|_{R_i=\hat{R}_i}$, $(n_1/n_i)$ and $\hat{\tau}_i^2(x)$ respectively in (6.9), where $\hat{\tau}_i^2(x)$ is obtained similarly to Section 6.2. It is easy to verify that $\hat{\tau}_S^2(x) \to \tau_S^2(x)$ almost surely as $n_1 \to \infty$. Thus, given a significance level $\alpha \in (0,1)$, an approximate large sample $100(1-\alpha)\%$ confidence interval for the limiting interval reliability $R_S(x)$ is

$$\hat{R}_S(x) - z_{\alpha/2}\frac{\hat{\tau}_S(x)}{\sqrt{n_1}} \le R_S(x) \le \hat{R}_S(x) + z_{\alpha/2}\frac{\hat{\tau}_S(x)}{\sqrt{n_1}}.$$

The estimation of the limiting interval reliability $R_S(x)$ of a coherent system in the case of censored observations and continuous observation over a fixed period can be carried out in a similar manner.

### 6.6 Simulation Study

A simulation study is conducted in this section to assess the performance of the proposed estimators and to compare their efficiencies with corresponding estimators in the i.i.d. set-up. Here, we assume that the failure and repair times are generated from two independent EAR(1) models given by,

$$X_n = \begin{cases} 0.5X_{n-1} & \text{with probability } 0.5, \\ 0.5X_{n-1} + \varepsilon_n & \text{with probability } 0.5. \end{cases} \text{ and}$$

$$Y_n = \begin{cases} 0.25Y_{n-1} & \text{with probability } 0.25, \\ 0.25Y_{n-1} + \eta_n & \text{with probability } 0.75. \end{cases},\ n = 1, 2, 3, \ldots,$$

where $\{\varepsilon_n\}$ and $\{\eta_n\}$ are two independent i.i.d. exponential sequences with parameters $\lambda_1 = 1/6$ and $\lambda_2 = 1/2$ respectively. Thus $\{X_n\}$ and $\{Y_n\}$ have exponential marginal distributions with mean failure time $\mu_X = 6$ and mean repair time $\mu_Y = 2$ respectively.

We consider the limiting interval reliability $R(x)$ at time points $x = 0,\ 0.25,\ 0.50,\ 0.75$ and 1.00 for the simulation study. In order to compare the performance of the estimators of $R(x)$ in the stationary dependent case (EAR(1) model) with that of the i.i.d. exponential case, we compute the empirical coverage probabilities in the case of EAR(1) model and the i.i.d exponential model respectively.

The results of the simulation study in the case of complete observations are summarized in Table 6.1. Here *n* and *m* denote the number of failure and repair time observations respectively. The notations $\bar{\hat{R}}(x)$, $\bar{\hat{\tau}}^2(x)$ and *CP* denote the average of the estimated value of $R(x)$, its asymptotic variance $\tau^2(x)$ and the empirical coverage probability of 95% confidence interval for $R(x)$ over 750 repetitions in the stationary dependent case. The same quantities are also computed by assuming the stationary dependent failure and repair times as i.i.d. exponential observations ignoring the autocorrelations present in the data. Let $\bar{\hat{\tau}}_*^2(x)$ and $CP^*$ denote the average of the asymptotic variance and the empirical coverage probabilities in the i.i.d. case. Note that the estimated value of $R(x)$ is the same for both the stationary dependent and i.i.d. case. The values within the parenthesis represent the *MSE* of the estimators.

In the case of censored observations, the censored failure and repair times are generated from two independent exponential distribution with cumulative distribution functions $G_C(t) = 1 - e^{-0.05t}$ and $G_D(t) = 1 - e^{-0.1t}$ respectively and the results are summarized in Table 6.2. Here *X%* and *Y%* denote the average percentage of censoring rate associated with the failure and repair times respectively. The simulation results in the case of continuous observations are given in Table 6.3.

**Table 6.1** Simulation results in the case of complete observations

| $x$ | $R(x)$ | $n$ | $m$ | $\bar{\hat{R}}(x)$ | EAR(1) Model | | i.i.d. Case | |
|---|---|---|---|---|---|---|---|---|
| | | | | | $\bar{\hat{\tau}}^2(x)$ | $CP$ | $\bar{\hat{\tau}}_*^2(x)$ | $CP^*$ |
| 0.00 | 0.75000 | 25 | 20 | 0.75098 (0.0459) | 0.17619 (0.0218) | 0.9367 | 0.08013 (0.0283) | 0.6179 |
| | | 75 | 70 | 0.74986 (0.0361) | 0.16823 (0.0156) | 0.9392 | 0.07382 (0.0179) | 0.6243 |
| | | 150 | 145 | 0.75013 (0.0213) | 0.16672 (0.0124) | 0.9413 | 0.07259 (0.0131) | 0.6338 |
| 0.25 | 0.71939 | 25 | 20 | 0.71816 (0.0437) | 0.19433 (0.0269) | 0.9326 | 0.08512 (0.0318) | 0.6020 |
| | | 75 | 70 | 0.72017 (0.0319) | 0.18996 (0.0183) | 0.9341 | 0.07962 (0.0243) | 0.6215 |
| | | 150 | 145 | 0.71998 (0.0198) | 0.18854 (0.0112) | 0.9392 | 0.07946 (0.0167) | 0.6298 |
| 0.50 | 0.69003 | 25 | 20 | 0.70143 (0.0447) | 0.22198 (0.0328) | 0.9284 | 0.09117 (0.0329) | 0.5919 |
| | | 75 | 70 | 0.70019 (0.0306) | 0.21316 (0.0235) | 0.9331 | 0.08494 (0.0228) | 0.6012 |
| | | 150 | 145 | 0.68918 (0.0184) | 0.21194 (0.0192) | 0.9403 | 0.08322 (0.0173) | 0.6194 |
| 0.75 | 0.66187 | 25 | 20 | 0.65902 (0.0491) | 0.24129 (0.0311) | 0.9421 | 0.09514 (0.0287) | 0.6226 |
| | | 75 | 70 | 0.66867 (0.0384) | 0.23417 (0.0204) | 0.9439 | 0.09192 (0.0187) | 0.6354 |
| | | 150 | 145 | 0.66091 (0.0176) | 0.23273 (0.0156) | 0.9503 | 0.09015 (0.0133) | 0.6546 |
| 1.00 | 0.63486 | 25 | 20 | 0.63918 (0.0421) | 0.26518 (0.0291) | 0.9399 | 0.10398 (0.0288) | 0.6188 |
| | | 75 | 70 | 0.63217 (0.0348) | 0.25631 (0.0167) | 0.9403 | 0.09894 (0.0183) | 0.6202 |
| | | 150 | 145 | 0.63772 (0.0182) | 0.25536 (0.0139) | 0.9468 | 0.09711 (0.0146) | 0.6332 |

**Table 6.2** Simulation results in the case of censored observations

| $x$ | $R(x)$ | $n$ | $m$ | $\bar{\hat{R}}_c(x)$ | $X\%$ | $Y\%$ | EAR(1) Model | | i.i.d. Case | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | $\bar{\hat{\tau}}_c^2(x)$ | $CP$ | $\bar{\hat{\tau}}_*^2(x)$ | $CP^*$ |
| | | 25 | 20 | 0.75165 | 23.12 | 15.60 | 0.18464 | 0.9248 | 0.08127 | 0.6319 |
| | | | | (0.0534) | | | (0.0247) | | (0.0304) | |
| 0.00 | 0.75000 | 75 | 70 | 0.75201 | 22.90 | 17.40 | 0.17823 | 0.9368 | 0.07613 | 0.6346 |
| | | | | (0.0381) | | | (0.0168) | | (0.0213) | |
| | | 150 | 140 | 0.75128 | 23.60 | 16.62 | 0.17652 | 0.9448 | 0.07419 | 0.6423 |
| | | | | (0.0287) | | | (0.0133) | | (0.0194) | |
| | | 25 | 20 | 0.72102 | 22.96 | 17.24 | 0.2143 | 0.9332 | 0.08553 | 0.6290 |
| | | | | (0.0567) | | | (0.0298) | | (0.0318) | |
| 0.25 | 0.71939 | 75 | 70 | 0.71878 | 23.22 | 17.58 | 0.20963 | 0.9429 | 0.07957 | 0.6471 |
| | | | | (0.0341) | | | (0.0203) | | (0.0243) | |
| | | 150 | 140 | 0.72093 | 23.44 | 16.80 | 0.20449 | 0.9461 | 0.07824 | 0.6568 |
| | | | | (0.0213) | | | (0.0167) | | (0.0167) | |
| | | 25 | 20 | 0.70098 | 23.32 | 17.10 | 0.22482 | 0.9430 | 0.09051 | 0.6326 |
| | | | | (0.0527) | | | (0.0362) | | (0.0366) | |
| 0.50 | 0.69003 | 75 | 70 | 0.71982 | 24.24 | 18.58 | 0.21541 | 0.9503 | 0.08646 | 0.6389 |
| | | | | (0.0353) | | | (0.0246) | | (0.0249) | |
| | | 150 | 140 | 0.69994 | 23.18 | 17.82 | 0.21267 | 0.9521 | 0.08538 | 0.6460 |
| | | | | (0.0199) | | | (0.0189) | | (0.0146) | |
| | | 25 | 20 | 0.66302 | 24.80 | 17.00 | 0.24933 | 0.9324 | 0.10143 | 0.6402 |
| | | | | (0.0514) | | | (0.0344) | | (0.0321) | |
| 0.75 | 0.66187 | 75 | 70 | 0.67067 | 23.92 | 16.94 | 0.24418 | 0.9468 | 0.09642 | 0.6498 |
| | | | | (0.0335) | | | (0.0239) | | (0.0198) | |
| | | 150 | 140 | 0.66932 | 23.68 | 17.62 | 0.24174 | 0.9512 | 0.09454 | 0.6509 |
| | | | | (0.0188) | | | (0.0191) | | (0.0124) | |
| | | 25 | 20 | 0.63345 | 23.98 | 15.86 | 0.27216 | 0.9401 | 0.10467 | 0.6357 |
| | | | | (0.0538) | | | (0.0371) | | (0.0304) | |
| 1.00 | 0.63486 | 75 | 70 | 0.63418 | 22.00 | 16.92 | 0.26032 | 0.9453 | 0.10298 | 0.6415 |
| | | | | (0.0417) | | | (0.0289) | | (0.0172) | |
| | | 150 | 140 | 0.63612 | 25.08 | 16.40 | 0.25998 | 0.9506 | 0.10026 | 0.6469 |
| | | | | (0.0223) | | | (0.0175) | | (0.0138) | |

**Table 6.3** Simulation results in the case of continuous observations

| $x$ | $R(x)$ | $T$ | $\bar{\hat{R}}_T(x)$ | $\bar{N}(T)$ | EAR(1) Model | | i.i.d. Case | |
|---|---|---|---|---|---|---|---|---|
| | | | | | $\bar{\hat{\tau}}^2(x)$ | $CP$ | $\bar{\hat{\tau}}_*^2(x)$ | $CP^*$ |
| 0.00 | 0.75000 | 250 | 0.75124 (0.0523) | 29.92 | 1.31087 (0.2316) | 0.9270 | 0.55466 (0.0379) | 0.5876 |
| | | 500 | 0.75231 (0.0409) | 45.68 | 1.32192 (0.1772) | 0.9432 | 0.55312 (0.0286) | 0.5957 |
| | | 750 | 0.75093 (0.0322) | 74.24 | 1.31246 (0.1298) | 0.9443 | 0.56253 (0.0205) | 0.6012 |
| 0.25 | 0.71939 | 250 | 0.72046 (0.0496) | 28.80 | 1.49156 (0.2508) | 0.9259 | 0.61457 (0.0396) | 0.5948 |
| | | 500 | 0.72133 (0.0372) | 48.76 | 1.49351 (0.1878) | 0.9313 | 0.6089 (0.0292) | 0.6023 |
| | | 750 | 0.72021 (0.0288) | 72.84 | 1.48816 (0.1349) | 0.9381 | 0.61107 (0.0213) | 0.6144 |
| 0.50 | 0.69003 | 250 | 0.68544 (0.0517) | 29.78 | 1.6689 (0.2409) | 0.9292 | 0.67892 (0.0413) | 0.5995 |
| | | 500 | 0.70127 (0.0402) | 47.58 | 1.65612 (0.1947) | 0.9367 | 0.66785 (0.0329) | 0.6037 |
| | | 750 | 0.68945 (0.0325) | 71.46 | 1.68876 (0.1381) | 0.9498 | 0.66206 (0.0287) | 0.6126 |
| 0.75 | 0.66187 | 250 | 0.66978 (0.0496) | 30.32 | 1.83566 (0.2603) | 0.9354 | 0.72454 (0.0422) | 0.6123 |
| | | 500 | 0.66013 (0.0363) | 47.36 | 1.84689 (0.1898) | 0.9476 | 0.72167 (0.0385) | 0.6212 |
| | | 750 | 0.66276 (0.0234) | 72.12 | 1.85063 (0.1434) | 0.9512 | 0.71423 (0.0318) | 0.6345 |
| 1.00 | 0.63486 | 250 | 0.63939 (0.0479) | 29.40 | 2.02394 (0.2789) | 0.9346 | 0.74678 (0.0427) | 0.6141 |
| | | 500 | 0.63443 (0.0317) | 48.34 | 2.01652 (0.1992) | 0.9501 | 0.75779 (0.0379) | 0.6177 |
| | | 750 | 0.63644 (0.0192) | 73.66 | 2.03081 (0.1528) | 0.9523 | 0.76651 (0.0322) | 0.6240 |

From the Tables, we can see that the estimated asymptotic variance of the estimators in the stationary dependent case is approximately twice of that in the i.i.d. case and hence the confidence interval of the estimators in the i.i.d. case is shorter than that in the stationary dependent case. Also, the empirical coverage probabilities of the estimators corresponding to 95% confidence interval in the i.i.d. set-up is around 0.60 – 0.65 and that in the case of stationary dependent model is around 0.90 – 0.95. This suggests that when the failure and repair times are some stationary dependent, the ignorance of autocorrelation present in the data will lead to poor coverage probabilities and this may lead to wrong interpretations in the inference procedures.

Finally, we consider the case of a coherent system of three independent components, in which components 2 and 3 are in parallel and component 1 is in series with components 2 and 3 as shown in Figure 6.1. Here the system functions if and only if component 1 works and at least one of components 2 and 3 works.

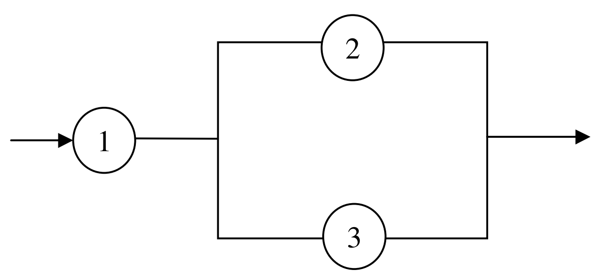


Figure 6.1

Then, the limiting interval reliability of the system is given by,

$$R_S(x) = R_1(x)\left[R_2(x) + R_3(x) - R_2(x)R_3(x)\right],$$

where $R_i(x)$ is the limiting interval reliability of the *i*-th component, $i = 1, 2, 3$.

To simplify the presentation of the table, we assume that $n_1 = n_2 = n_3 = n$ and $m_1 = m_2 = m_3 = m$.

Now, a nonparametric estimator of $R_S(x)$ is given by

$$\hat{R}_S(x) = \hat{R}_1(x)\left[\hat{R}_2(x) + \hat{R}_3(x) - \hat{R}_2(x)\hat{R}_3(x)\right],$$

where

$$\hat{R}_i(x) = \frac{\bar{U}_{in}}{\bar{X}_{in} + \bar{Y}_{im}} \text{ for } i = 1,2,3.$$

The asymptotic variance of the estimator $\hat{R}_S(x)$ in this case is given by,

$$\tau_S^2(x) = \left[R_2(x) + R_3(x) - R_2(x)R_3(x)\right]\tau_1^2(x) + R_1(x)\left[1 - R_3(x)\right]\tau_2^2(x)$$
$$+ R_1(x)\left[1 - R_2(x)\right]\tau_3^2(x).$$

For the simulation, we assume that the failure and repair times of component 1 are generated using the same EAR(1) models defined in this Section. We further assume that the failure and repair times of component 2 and 3 are identically distributed and are generated from two independent EMA1 models given by,

$$X_n = \begin{cases} 0.75\varepsilon_n & \text{with probability } 0.75, \\ 0.75\varepsilon_n + \varepsilon_{n+1} & \text{with probability } 0.25. \end{cases}$$

and

$$Y_n = \begin{cases} 0.5\eta_n & \text{with probability } 0.50, \\ 0.5\eta_n + \eta_{n+1} & \text{with probability } 0.50. \end{cases}, \ (n = 1,2,3,...),$$

where $\{\varepsilon_n\}$ and $\{\eta_n\}$ are two independent i.i.d. exponential random sequences with parameters $\lambda_1 = 0.1$ and $\lambda_2 = 0.25$ respectively. The results of the simulation study are shown in Table 6.4. In the Table, LCL and UCL represent the lower and upper limits of the 95% confidence interval for $\hat{R}_S(x)$. From the Table, it can be seen that the proposed estimator for the interval reliability of the given coherent system performs well and shows consistent performance even for small sample sizes.

**Table 6.4** Simulation results in the case of coherent systems

| $x$ | $R_S(x)$ | $n$ | $m$ | $\bar{\hat{R}}_S(x)$ | $\bar{\hat{\tau}}_S^2(x)$ | $LCL$ | $UCL$ |
|---|---|---|---|---|---|---|---|
| 0.00 | 0.68878 | 25 | 20 | 0.69023 (0.0413) | 0.22343 (0.0327) | 0.50494 | 0.87552 |
| | | 75 | 70 | 0.68996 (0.0346) | 0.20822 (0.0174) | 0.58669 | 0.79323 |
| | | 150 | 145 | 0.68707 (0.0278) | 0.20501 (0.0132) | 0.61461 | 0.75953 |
| 0.25 | 0.65319 | 25 | 20 | 0.65251 (0.0428) | 0.24231 (0.0346) | 0.45955 | 0.84547 |
| | | 75 | 70 | 0.65416 (0.0332) | 0.22967 (0.0169) | 0.54570 | 0.76262 |
| | | 150 | 145 | 0.65409 (0.0241) | 0.22613 (0.0127) | 0.57799 | 0.73019 |
| 0.50 | 0.61913 | 25 | 20 | 0.62172 (0.0489) | 0.26012 (0.0359) | 0.42179 | 0.82165 |
| | | 75 | 70 | 0.61965 (0.0366) | 0.24791 (0.0234) | 0.50696 | 0.73234 |
| | | 150 | 145 | 0.61788 (0.0271) | 0.24568 (0.0166) | 0.53856 | 0.69720 |
| 0.75 | 0.58656 | 25 | 20 | 0.59234 (0.0454) | 0.28156 (0.0353) | 0.38434 | 0.80034 |
| | | 75 | 70 | 0.57839 (0.0317) | 0.26678 (0.0228) | 0.46149 | 0.69529 |
| | | 150 | 145 | 0.58823 (0.0239) | 0.26513 (0.0172) | 0.50583 | 0.67063 |
| 1.00 | 0.55544 | 25 | 20 | 0.56065 (0.0411) | 0.29987 (0.0332) | 0.34599 | 0.77531 |
| | | 75 | 70 | 0.55787 (0.0335) | 0.28712 (0.0255) | 0.43660 | 0.67914 |
| | | 150 | 145 | 0.55689 (0.0197) | 0.28564 (0.0187) | 0.47136 | 0.64242 |

## 6.7. Conclusion

In this chapter, we discussed the nonparametric estimation of the limiting interval reliability when the failure and repair times are generated by two mutually independent strictly stationary dependent sequences of random variables. The proposed estimators were shown to be consistent and asymptotically normal under three different sampling schemes. A simulation study was conducted to assess the performance of the estimators in the stationary dependent case to the corresponding estimators in the i.i.d. set-up. The simulation study showed that if the true process is generated from stationary dependent sequences of random variables, the ignorance of autocorrelation among successive observations leads to poor coverage probabilities in the estimation procedure. Finally, we extended the estimation procedures to a coherent system of $k$ independent components and illustrated the computation of confidence interval based on simulated samples.

# Chapter 7

# Sequential Interval Estimation of the Limiting Interval Reliability

## 7.1 Introduction

In this chapter, we consider the sequential interval estimation of the limiting interval reliability of a repairable system when the sequences of failure and repair times are generated by a bivariate stationary dependent sequence. In Section 7.2, we discuss the estimation of the limiting interval reliability $R(x)$ for a stationary strong mixing bivariate sequence of failure and repair times. Section 7.3 considers the sequential interval estimation of $R(x)$. In section 7.4, we consider the sequential interval estimation in the case of a bivariate exponential autoregressive (BEAR) model. A numerical study is also performed in Section 7.5 to assess the performance of the proposed sequential decision rule. Finally, Section 7.6 provides brief conclusions of the study.

## 7.2 Estimation of the Limiting Interval Reliability

Suppose that $\{(X_n, Y_n),\ n \geq 1\}$ is strictly stationary and strong mixing in the sense that as $h \to \infty$,

$$\alpha(h) = \text{Sup}\left\{\left|P(A \cap B) - P(A)P(B)\right| : A \in \mathfrak{I}_1^k(X,Y) \text{ and } B \in \mathfrak{I}_{k+h}^{\infty}(X,Y)\right\} \to 0,$$

where

$$\mathfrak{I}_1^k(X,Y) = \sigma\left(X_i, Y_i;\ 1 \leq i \leq k\right) \text{ and } \mathfrak{I}_{k+h}^{\infty}(X,Y) = \sigma\left(X_i, Y_i;\ i \geq k+h\right).$$

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The results in this chapter have been accepted for publication as entitled 'Sequential Interval Estimation of the Limiting Interval Availability for a Bivariate Stationary Dependent Sequence' in the journal ***Statistics*** (See Balakrishna and Mathew, 2010).

When the observations on the failure and repair times of '*n*' complete cycles, $(X_1,Y_1),(X_2,Y_2),...,(X_n,Y_n)$, are available, a natural estimator for the limiting interval reliability $R(x)$ is

$$\hat{R}_n(x) = \frac{\bar{U}_n}{\bar{X}_n + \bar{Y}_n}, \tag{7.1}$$

where $\bar{X}_n = \sum_{i=1}^{n} X_i \Big/ n$, $\bar{Y}_n = \sum_{i=1}^{n} Y_i \Big/ n$ and $\bar{U}_n = \sum_{i=1}^{n} U_i \Big/ n$ with $U_i = (X_i - x) I_{(X_i > x)}$.

Since $\{(X_n,Y_n),\ n \ge 1\}$ is strictly stationary it follows that $\bar{X}_n \to \mu_X$, $\bar{Y}_n \to \mu_Y$ and $\bar{U}_n \to \upsilon(x)$ almost surely as $n \to \infty$ and hence we conclude that $\hat{R}_n(x) \to R(x)$ almost surely as $n \to \infty$.

In order to establish the asymptotic normality of the estimator $\hat{R}_n(x)$, we assume that for some $\delta > 0$, $E(X_1^{2+\delta}) < \infty$, $E(Y_1^{2+\delta}) < \infty$ and $\sum_{h=1}^{\infty} \alpha^{\delta/(2+\delta)}(h) < \infty$.

Since $\{(X_n,Y_n),\ n \ge 1\}$ is strictly stationary and strong mixing with mixing coefficient $\alpha(h)$, under the above assumptions, by the central limit theorem for such sequences (Lemma 1.7) we have as $n \to \infty$,

$$\sqrt{n}\left(\bar{X}_n - \mu_X, \bar{Y}_n - \mu_Y\right) \xrightarrow{L} N_2(\mathbf{0}, \Sigma_2),$$

where $N_2(0,\Sigma_2)$ is a bivariate normal vector with mean $\mathbf{0} = (0,0)'$ and dispersion matrix

$$\Sigma_2 = \begin{pmatrix} \sigma_{XX} & \sigma_{XY} \\ \sigma_{XY} & \sigma_{YY} \end{pmatrix},$$

with $\sigma_{XX} = \mathrm{var}(X_1) + 2\sum_{h=2}^{\infty} \mathrm{cov}(X_1, X_h)$, $\sigma_{YY} = \mathrm{var}(Y_1) + 2\sum_{h=2}^{\infty} \mathrm{cov}(Y_1, Y_h)$

and $\sigma_{XY} = \mathrm{cov}(X_1, Y_1) + \sum_{h=2}^{\infty} \mathrm{cov}(X_1, Y_h) + \sum_{h=2}^{\infty} \mathrm{cov}(X_h, Y_1)$.

If we define, $\bar{Z}_n = \sum_{i=1}^{n} Z_i \Big/ n$, with $Z_i = X_i + Y_i$, then it follows that as $n \to \infty$

$$\sqrt{n}\left(\bar{Z}_n - \mu_Z\right) \xrightarrow{L} N(0, \sigma_{ZZ}),$$

where $\mu_Z = \mu_X + \mu_Y$ and $\sigma_{ZZ} = \sigma_{XX} + 2\sigma_{XY} + \sigma_{YY}$.

It is to be noted that $\{(U_n, Z_n),\ n \geq 1\}$ is also strictly stationary and by the Cramer-Wold device (Billingsley, 1968, pp.49), it may be verified that as $n \to \infty$

$$\sqrt{n}\left(\bar{U}_n - \upsilon(x), \bar{Z}_n - \mu_Z\right) \xrightarrow{L} N_2(\mathbf{0}, \Sigma_2^*), \tag{7.2}$$

where $\Sigma_2^* = \begin{pmatrix} \sigma_{UU} & \sigma_{UZ} \\ \sigma_{UZ} & \sigma_{ZZ} \end{pmatrix}$,

with $\sigma_{UU} = \text{var}(U_1) + 2\sum_{h=2}^{\infty} \text{cov}(U_1, U_h)$ and

$$\sigma_{UZ} = \text{cov}(U_1, Z_1) + \sum_{h=2}^{\infty} \text{cov}(U_1, Z_h) + \sum_{h=2}^{\infty} \text{cov}(U_h, Z_1).$$

If we define $g(x, y) = x / y$, then $g\left(\bar{U}_n, \bar{Z}_n\right) = \hat{R}_n(x)$.

Now, the partial derivatives of $g(.)$ are

$$\left.\frac{\partial g}{\partial x}\right|_{(\upsilon(x), \mu_Z)} = \frac{1}{\mu_Z}, \text{ and}$$

$$\left.\frac{\partial g}{\partial y}\right|_{(\upsilon(x), \mu_Z)} = -\frac{\upsilon(x)}{\mu_Z^2}.$$

Hence, by using Lemma 1.4, we can show that as $n \to \infty$,

$$\sqrt{n}\left(\hat{R}_n(x) - R(x)\right) \xrightarrow{L} N\left(0, \tau^2(x)\right),$$

where

$$\tau^2(x) = \frac{\sigma_{UU}}{\mu_Z^2} - 2\frac{\mu(w)\sigma_{UZ}}{\mu_Z^3} + \frac{\mu^2(w)\sigma_{ZZ}}{\mu_Z^4}. \tag{7.3}$$

Thus, we proved the following theorem.

**Theorem 7.1**

If $\{(X_n, Y_n),\ n \geq 1\}$ is a strictly stationary and strong mixing sequence of bivariate random vectors on $R_2^+ = \{(x, y) : 0 \leq x < \infty, 0 \leq y < \infty\}$ such that for some $\delta > 0$, $E(X_1^{2+\delta}) < \infty,\ E(Y_1^{2+\delta}) < \infty$ and $\sum_{h=1}^{\infty} \alpha^{\delta/(2+\delta)}(h) < \infty$, then $\hat{R}_n(x)$ is a consistent and asymptotically normal (CAN) estimator for $R(x)$.

Thus if $\tau^2(x)$ is known, for a given significance level $\alpha \in (0,1)$, a $100(1-\alpha)\%$ confidence interval for $R(x)$ is

$$\hat{R}_n(x) - z_{\alpha/2}\frac{\tau(x)}{\sqrt{n}} \leq R(x) \leq \hat{R}_n(x) + z_{\alpha/2}\frac{\tau(x)}{\sqrt{n}}.$$

**Remark 7.1** *When* $x = 0$*, the estimator* $\hat{R}_n(x)$ *reduces to* $\hat{R}_n(0) = \frac{\overline{X}_n}{\overline{X}_n + \overline{Y}_n} = \hat{A}_n$*, which is a CAN estimator for the limiting availability* $A = \mu_X / (\mu_X + \mu_Y)$*. Also it is straight forward to see that as* $n \to \infty$

$$\sqrt{n}\left(\hat{A}_n - A\right) \xrightarrow{L} N\left(0, \gamma^2\right),$$

*where* $\gamma^2 = \{\mu_Y^2 \sigma_{XX} + \mu_X^2 \sigma_{YY} - 2\mu_X \mu_Y \sigma_{XY}\} / (\mu_X + \mu_Y)^4$.

In the next section, we discuss the sequential confidence interval estimation for the limiting interval reliability.

## 7.3 Sequential Interval Estimation

In sequential Interval estimation our prime objective is to locate an interval, say $I_n$, based on the observations, $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$, such that

*i*) $P[R(x) \in I_n] \geq 1 - \alpha$ and

*ii*) width of $I_n \leq 2d$,

where $\alpha$ $(0 < \alpha < 1)$ and $d$ $(d > 0)$ are preassigned numbers.

Theorem 7.1 ensures that for large *n*,

$$P\left[\sqrt{n}\left|\hat{R}_n(x)-R(x)\right| \le z_{\alpha/2}\tau(x)\right] \ge 1-\alpha, \tag{7.4}$$

where $z_{\alpha/2}$ denotes the upper $\alpha/2$ quantile of the standard normal distribution.

Define, $I_n = \left[\hat{R}_n(x)-d,\ \hat{R}_n(x)+d\right]$.

If $\tau^2(x)$ is known, we could take $n_d = \min\left\{n : z_{\alpha/2}^2\tau^2(x)d^{-2} \le n\right\}$ as the number of observations. Then $I_{n_d}$ is the fixed accuracy confidence interval for the limiting interval reliability $R(x)$ of fixed width $2d$ with coverage probability $P[R(x)\in I_{n_d}] = P\left[\sqrt{n_d}\left|\hat{R}_{n_d}(x)-R(x)\right| \le d\sqrt{n_d}\right]$, which converges to $1-\alpha$ as $d\to 0$ due to (7.4) and the fact that $\lim_{d\to 0}\dfrac{d^2 n_d}{z_{\alpha/2}^2\tau^2(x)} = 1$.

However, $\tau^2(x)$ is unknown in practice, so we should replace it by a consistent estimator. A consistent estimator $\hat{\tau}_n^2(x)$ of $\tau^2(x)$ can be obtained by replacing $\upsilon(x)$, $\mu_Z$, $\sigma_{UU}$, $\sigma_{ZZ}$ and $\sigma_{UZ}$ with their corresponding consistent estimators in (7.3). Obviously $\bar{U}_n$ and $\bar{Z}_n$ are the consistent estimators for $\upsilon(x)$ and $\mu_Z$ respectively. In order to construct consistent estimators for $\sigma_{UU}$, $\sigma_{ZZ}$ and $\sigma_{UZ}$, we use the moving-block jackknife method for variance estimation with dependent data (Kunch, 1989). The moving-block jackknife estimators for $\sigma_{UU}$, $\sigma_{ZZ}$ and $\sigma_{UZ}$, respectively, are

$$\hat{\sigma}_{UU,l}^2 = \frac{l}{n-l+1}\sum_{i=1}^{n-l+1}\left(\bar{U}_i^{(l)} - (n+l-1)^{-1}\sum_{j=1}^{n-l+1}\bar{U}_j^{(l)}\right)^2,$$

$$\hat{\sigma}_{ZZ,l}^2 = \frac{l}{n-l+1}\sum_{i=1}^{n-l+1}\left(\bar{Z}_i^{(l)} - (n+l-1)^{-1}\sum_{j=1}^{n-l+1}\bar{Z}_j^{(l)}\right)^2 \text{ and}$$

$$\hat{\sigma}_{UZ,l} = \frac{l}{n-l+1}\sum_{i=1}^{n-l+1}\left(\bar{U}_i^{(l)} - (n+l-1)^{-1}\sum_{j=1}^{n-l+1}\bar{U}_j^{(l)}\right)\left(\bar{Z}_i^{(l)} - (n+l-1)^{-1}\sum_{j=1}^{n-l+1}\bar{Z}_j^{(l)}\right),$$

where $\bar{U}_i^{(l)} = l^{-1}\sum_{j=i}^{i+l-1} U_j$ , $\bar{Z}_i^{(l)} = l^{-1}\sum_{j=i}^{i+l-1} Z_j$ and $l = l(n)$ is the block size.

To establish the optimal properties of the sequential procedure we make the following assumption:

$A_1$: *For some* $\delta > 0$ , $E\left[\left|X_1\right|^{6+\delta}\right] < \infty,\ E\left[\left|Y_1\right|^{6+\delta}\right] < \infty$ *and* $\sum k^2 \alpha(k)^{\delta/(6+\delta)} < \infty$ .

Under the above assumption the estimators $\hat{\sigma}^2_{UU,l}$, $\hat{\sigma}^2_{ZZ,l}$ and $\hat{\sigma}_{UZ,l}$ converge almost surely to $\sigma_{UU}$ , $\sigma_{ZZ}$ and $\sigma_{UZ}$ respectively if $l = o(n)$ and $l \to \infty$ (Kunch, 1989).

Then, it is easy to see that

$$\hat{\tau}_n^2(x) \to \tau^2(x) \text{ almost surely as } n \to \infty . \tag{7.5}$$

Now, consider the stopping rule

$$N_d = \inf\left\{n \geq m : nd^2 \geq z^2_{\alpha/2}\hat{\tau}_n^2(x)\right\}, \tag{7.6}$$

where $m$ is the initial sample size.

The bounded length confidence interval is then

$$I_{N_d} = \left[\hat{R}_{N_d}(x) - d,\ \hat{R}_{N_d}(x) + d\right].$$

The various steps involved in the construction of sequential confidence interval for the limiting interval reliability are summarized below:

1) Take a preliminary sample of appropriate size $m$ , $(X_i, Y_i),\ i = 1, 2, ..., m$ and transform the data into $(U_i, Z_i),\ i = 1, 2, ..., m$ , where $U_i = (X_i - x)I_{(X_i > x)}$ and $Z_i = X_i + Y_i$ .

2) Estimate the unknown parameter $\tau^2(x)$ by

$$\hat{\tau}_n^2(x) = \frac{\hat{\sigma}^2_{UU,l}}{\bar{Z}_n^2} - 2\frac{\bar{U}_n\hat{\sigma}_{UZ,l}}{\bar{Z}_n^3} + \frac{\bar{U}_n^2\hat{\sigma}^2_{ZZ,l}}{\bar{Z}_n^4} .$$

3) For preassigned $d, (0 < d \leq 0.5)$ , calculate the stopping number $N_d$ defined by (7.6).

4) Take $N_d - m$ additional samples $(X_i, Y_i)$, $i = m+1, m+2, ..., N_d$. Then with the total sample of size $N_d$ construct the confidence interval

$$I_{N_d} = \left[\hat{R}_{N_d}(x) - d,\ \hat{R}_{N_d}(x) + d\right].$$

The desirable asymptotic properties of the stopping rule defined by (7.6) are given in the following theorem.

**Theorem 7.2**

*Under the assumption* $A_1$, *as* $d \to 0$,

(i) $\dfrac{N_d}{n_d} \to 1$ *almost surely*

(ii) $P[R(x) \in I_{N_d}] \to 1 - \alpha$ (*asymptotic consistency*)

(iii) $E\left(\dfrac{N_d}{n_d}\right) \to 1$ (*asymptotic efficiency*).

*Proof*

In order to prove (*i*) note that

$$d^{-2} z_{\alpha/2}^2 \left(\hat{\tau}_{N_d}^2(x) + N_d^{-h}\right) \le N_d \le (m-1) I_{(N_d = m)} + d^{-2} z_{\alpha/2}^2 \left(\hat{\tau}_{N_d - 1}^2(x) + (N_d - 1)^{-h}\right) + 1$$

$$\le m + d^{-2} z_{\alpha/2}^2 \left(\hat{\tau}_{N_d - 1}^2(x) + (N_d - 1)^{-h}\right).$$

Hence,

$$\frac{d^{-2} z_{\alpha/2}^2 \left(\hat{\tau}_{N_d}^2(x) + N_d^{-h}\right)}{n_d} \le \frac{N_d}{n_d} \le \frac{m}{n_d} + \frac{d^{-2} z_{\alpha/2}^2 \left(\hat{\tau}_{N_d - 1}^2(x) + (N_d - 1)^{-h}\right)}{n_d}.$$

Now using the fact that $\lim\limits_{d \to 0} \dfrac{d^2 n_d}{z_{\alpha/2}^2} = \tau^2(x)$ and from (7.5) it follows that as $d \to 0$,

$$\frac{N_d}{n_d} \to 1 \text{ almost surely.} \tag{7.7}$$

If we define,

$$\xi_j = \mu_Z U_j - \upsilon(x) Z_j,\ j = 1, 2, ...,$$

then $\xi_j$'s are also strictly stationary and it follows from (7.2) that as $n \to \infty$

$$\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\xi_j \xrightarrow{L} N\left(0, \gamma^2(x)\right),$$

where

$$\gamma^2(x) = \mu_Z^2\sigma_{UU} + \upsilon^2(x)\sigma_{ZZ} - 2\upsilon(x)\mu_Z\sigma_{UZ}. \tag{7.8}$$

To establish the asymptotic consistency property we use Anscombe's theorem (Lemma 1.8), which requires $\left\{\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\xi_j\right\}$ to be uniformly continuous in probability (u.c.i.p). (See Definition 1.5).

Letting $Q_n = \sum_{j=1}^{n}\xi_j$ and following Woodroofe (1982, pp.11), we can write

$$\left|\frac{Q_{n+k}}{\sqrt{n+k}} - \frac{Q_n}{\sqrt{n}}\right| \le \sqrt{n}\left|Q_{n+k} - Q_n\right| + \left[1 - \sqrt{\frac{n}{n+k}}\right]\left|\frac{Q_n}{\sqrt{n}}\right| \quad \text{for } k, n \ge 1.$$

If $\varepsilon, \delta > 0$ and $k \ge n\delta$, then the second term on the right is bounded by $C(\delta)\left|Q_n/\sqrt{n}\right|$, where $C(\delta) = 1 - (1+\delta)^{-1/2}$ and

$$P\left[\underset{0<k\le n\delta}{Max}\left|\left[1 - \sqrt{\frac{n}{n+k}}\right]\left|\frac{Q_n}{\sqrt{n}}\right|\right| > \frac{\varepsilon}{2}\right] \le P\left[\left|\frac{Q_n}{\sqrt{n}}\right| > \frac{\varepsilon}{2C(\delta)}\right],$$

which tends to zero as $\delta \to 0$ uniformly in $n \ge 1$, since $\left\{\left|Q_n/\sqrt{n}\right|, n \ge 1\right\}$ are stochastically bounded.

Since $\{\xi_j\}$ is a strong mixing sequence of random variables, by the maximal inequality for such random variables (Rio, 1995), we have,

$$P\left[\underset{0<k\le n\delta}{Max}\left|Q_{n+k} - Q_n\right| > \frac{\varepsilon\sqrt{n}}{2}\right] \le \frac{64}{n\varepsilon^2}Var\left(\sum_{j=n+1}^{n+n\delta}\xi_j\right)$$

$$\le \frac{64}{\varepsilon^2}\delta\gamma^2(w),$$

which is independent of $n \ge 1$ and tends to zero as $\delta \to 0$.

Thus, $\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\xi_j$ , $n \ge 1$ is u.c.i.p.

Now, by Anscombe's theorem (Lemma 1.8), we have as $d \to 0$

$$\frac{1}{\sqrt{N_d}}\sum_{j=1}^{N_d}\xi_j = \frac{1}{\sqrt{N_d}}\sum_{j=1}^{N_d}[\mu_Z U_j - \upsilon(x)Z_j] \xrightarrow{L} N\left(0, \gamma^2(x)\right).$$

Note that

$$\sqrt{N_d}\left(\hat{R}_{N_d}(x) - R(x)\right) = \sqrt{N_d}\left(\frac{\bar{U}_{N_d}}{\bar{Z}_{N_d}} - \frac{\upsilon(x)}{\mu_Z}\right)$$

$$= \frac{\frac{1}{\sqrt{N_d}}\sum_{j=1}^{N_d}[\mu_Z U_j - \upsilon(x)Z_j]}{\mu_Z \bar{Z}_{N_d}}.$$

Since $\bar{Z}_{N_d} \xrightarrow{P} \mu_Z$ as $d \to 0$ it follows from Slutsky's theorem (Lemma 1.5) that

$$\sqrt{N_d}\left(\hat{R}_{N_d}(x) - R(x)\right) \xrightarrow{L} N\left(0, \tau^2(x)\right).$$

Now,

$$P[R(x) \in I_{N_d}] = P\left[\left|\hat{R}_{N_d}(x) - R(x)\right| \le d\right]$$

$$= P\left[\frac{\sqrt{N_d}\left|\hat{R}_{N_d}(x) - R(x)\right|}{\tau(x)} \le \frac{d\sqrt{n_d}}{\tau(x)}\sqrt{\frac{N_d}{n_d}}\right],$$

which converges to $1-\alpha$ as $d \to 0$ due to (7.7) and the fact that $\lim_{d \to 0}\frac{d^2 n_d}{z_{\alpha/2}^2 \tau^2(x)} = 1$.

Let $0 < \varepsilon < 1$ be given, and define $a = (1-\varepsilon)n_d$ and $b = (1+\varepsilon)n_d$.

Note that,

$$E\left(N_d\right) = \sum_{n=m}^{\infty} nP[N_d = n] \ge aP[N_d \ge a]$$

and hence

$$E\left(\frac{N_d}{n_d}\right) \ge (1-\varepsilon)P[N_d \ge a].$$

Now, using (7.7)

$$\liminf_{d\to 0} E\left(\frac{N_d}{n_d}\right) \geq (1-\varepsilon) \tag{7.9}$$

Also note that,

$$E(N_d) = \sum_{n=m}^{\infty} nP[N_d = n]$$

$$\leq bP[N_d \leq b] + \sum_{n=b+1}^{\infty} nP[N_d = n]$$

$$= b + T(b),$$

where $T(b) = \sum_{n=b}^{\infty} P[N_d > n]$.

Now,

$$E\left(\frac{N_d}{n_d}\right) \leq (1+\varepsilon) + \frac{T(b)}{n_d}. \tag{7.10}$$

Consider,

$$T(b) = \sum_{n=b}^{\infty} P[N_d > n] \leq \sum_{n=b}^{\infty} P\left[n < c\left(\hat{\tau}_n^2(x) + n^{-h}\right)\right], \text{ where } c = d^{-2} z_{\alpha/2}^2$$

$$\leq \sum_{n=b}^{\infty} P[\hat{\tau}_n^2(x) > c^{-1}n - n^{-h}]$$

$$\leq \sum_{n=b}^{\infty} P[\hat{\tau}_n^2(x) > c^{-1}b - b^{-h}]$$

$$\leq \sum_{n=b}^{\infty} P[\hat{\tau}_n^2(x) - \tau^2(x) > c^{-1}(b - n_d) - b^{-h}]$$

$$\leq \sum_{n=b}^{\infty} P[\hat{\tau}_n^2(x) - \tau^2(x) > \varepsilon\tau^2(x) - \left\{d^2 / [z_{\alpha/2}^2(1+\varepsilon)\tau^2(x)]\right\}^h]$$

If we choose $d$ small enough so that

$$\varepsilon\tau^2(x) - \left\{d^2 / [z_{\alpha/2}^2(1+\varepsilon)\tau^2(x)]\right\}^h > \frac{1}{2}\varepsilon\tau^2(x),$$

then

$$T(b) \leq \sum_{n=b}^{\infty} P\left[\left|\hat{\tau}_n^2(x) - \tau^2(x)\right| > \frac{1}{2}\varepsilon\tau^2(x)\right] < \infty.$$

It is clear that for sufficiently small $d$, since $T(b) < \infty$, (7.5) together with (7.10) imply that,

$$\limsup_{d \to 0} E\left(\frac{N_d}{n_d}\right) \le (1+\varepsilon).$$

Combining this with (7.9) we get (*iii*). This completes the proof.

In the next section we discuss the sequential estimation for a specific bivariate sequence.

**7.4 Sequential Interval Estimation for a BEAR(1) Process**

In this section we discuss the application of the results obtained in Section 7.2 and 7.3 for a *BEAR(1)* model.

Let $(N_1, N_2)$ be a bivariate geometric random vector with support $S = \{(i, j): i, j \ge 1\}$ defined by Block *et. al.* (1988) with probability mass function

$$P[N_1 = n_1, N_2 = n_2] = \begin{cases} p_{01} p_{11}^{n_1 - 1} (p_{01} + p_{11})^{n_2 - n_1 - 1} \left(1 - (p_{01} + p_{11})\right); & n_1 < n_2 \\ p_{10} p_{11}^{n_2 - 1} (p_{10} + p_{11})^{n_1 - n_2 - 1} \left(1 - (p_{10} + p_{11})\right); & n_1 > n_2 \\ p_{11}^{n_1 - 1} p_{00}; & n_1 = n_2 \end{cases}, \tag{7.11}$$

where $0 \le p_{ij} \le 1$, $i, j = 0,1$ such that $p_{00} + p_{10} + p_{01} + p_{11} = 1$, $0 < p_{01} + p_{11} < 1$ and $0 < p_{10} + p_{11} < 1$.

Let $\{(I_1(n), I_2(n))\}$ be a sequence *i.i.d.* bivariate Bernoulli random vectors with $P[I_1(n) = i, I_2(n) = j] = p_{ij}$, $i, j = 0,1$, where $p_{ij}$'s are as in (7.11).

Suppose $\{(E_{1n}, E_{2n}), n = 0, \pm 1, \pm 2, \ldots\}$ is a sequence of *i.i.d.* bivariate exponential random vector denoted by $BVE(\lambda_1, \lambda_2, \rho)$ with mean $(\lambda_1^{-1}, \lambda_2^{-1})$ and correlation coefficient $\rho$ such that the sequences $(E_1, E_2)$, $(I_1, I_2)$ and $(N_1, N_2)$ are mutually independent.

Define,

$$X(n) = \begin{pmatrix} X_n \\ Y_n \end{pmatrix} = \begin{pmatrix} I_1(n) X_{n-1} + \pi_1 E_{1n} \\ I_2(n) Y_{n-1} + \pi_2 E_{2n} \end{pmatrix}, \; n = 2, 3, ... \tag{7.12}$$

where

$$X(1) = (X_1, Y_1)' = \left( \pi_1 \sum_{j=1}^{N_1} E_{1,-j}, \; \pi_2 \sum_{j=1}^{N_2} E_{2,-j} \right)',$$

with $\pi_1 = p_{01} + p_{00}$ and $\pi_2 = p_{10} + p_{00}$.

The sequence $\{X(n),\ n \geq 1\}$ defined by (7.12) is referred to as a Bivariate exponential autoregressive process of order 1 (*BEAR*(1)) process. For each $n \geq 1$, $X(n)$ has $BVE(\lambda_1, \lambda_2, \rho)$ distribution. It is shown in Abraham and Balakrishna (2000) that the *BEAR*(1) sequence $\{X(n)\}$ is stationary, ergodic and strong mixing with mixing parameter $\alpha(h) = (p_{10} + p_{11})^{h-1} + (p_{01} + p_{11})^{h-1}$, $h = 1, 2, \ldots$.

In particular, if $(E_1, E_2)$ has a Marshall-Olkin bivariate exponential distribution with survival function (Marshall and Olkin, 1967)

$$\overline{F}(x, y) = \exp\left[-b_1 x - b_2 y - b_{12} \max(x, y)\right], \quad x, y \geq 0,$$

where $b_1$, $b_2$ and $b_{12}$ are non-negative real numbers such that $\lambda_1 = b_1 + b_{12}$, $\lambda_2 = b_2 + b_{12}$ and $\rho = b_{12} / (b_1 + b_2 + b_{12})$ and if we choose $0 < \theta < (\lambda_1 + \lambda_2)^{-1}$, $\pi_1 = \lambda_1 \theta$, $\pi_2 = \lambda_2 \theta$, $p_{00} = \theta b_{12}$, $p_{01} = \theta b_1$, $p_{10} = \theta b_2$ and $p_{11} = 1 - \theta(b_1 + b_2 + b_{12})$, then the resulting *BEAR(1)* sequence $\{X(n), n \geq 1\}$ is stationary and strong mixing with mixing parameter $\alpha(h) = (1 - \pi_1)^{h-1} + (1 - \pi_2)^{h-1}$ and each $X(n)$ has a Marshall-Olkin bivariate exponential distribution for $n \geq 1$ (See Block *et. al.,* 1988).

If we define $V(n) = (X_n,\ Y_n,\ U_n)'$, the autocovariance matrix $\Gamma_V(k)$ of $\{V(n)\}$ becomes

$\Gamma_V(k) = Cov\big(V(n), V(n+k)\big)$

$$= \begin{pmatrix} (1-\pi_1)^k \lambda_1^{-2} & (1-\pi_2)^k \rho\lambda_1^{-1}\lambda_2^{-1} & (1-\pi_1)^k \lambda_1^{-2}(1+\lambda_1 x)e^{-\lambda_1 x} \\ (1-\pi_1)^k \rho\lambda_1^{-1}\lambda_2^{-1} & (1-\pi_2)^k \lambda_2^{-2} & (1-\pi_1)^k \rho(x)\lambda_1^{-1}\lambda_2^{-1}e^{-\lambda_1 x} \\ (1-\pi_1)^k \lambda_1^{-2}(1+\lambda_1 x)e^{-\lambda_1 x} & (1-\pi_2)^k \rho(x)\lambda_1^{-1}\lambda_2^{-1}e^{-\lambda_1 x} & (1-\pi_1)^k \lambda_1^{-2}e^{-\lambda_1 x}(2-e^{-\lambda_1 x}) \end{pmatrix}$$

where $\rho(x)$ is the correlation coefficient between $U_n$ and $Y_n$.

For the *BEAR(1)* sequence all the moments of $X_n$ and $Y_n$ are finite and hence those of $U_n$ and $Z_n$. In this case $\upsilon(x) = \lambda_1^{-1}e^{-\lambda_1 x}$ and $\mu_Z = (\lambda_1+\lambda_2)\lambda_1^{-1}\lambda_2^{-1}$. Also, it can be verified that $\sum_{h=1}^{\infty}\alpha^{\delta/(\delta+2)}(h) < \infty$. Thus it follows that

$$\sqrt{n}\left(\bar{U}_n - \mu(w),\ \bar{Z}_n - \mu_Z\right) \xrightarrow{L} N(0, \Sigma_2^*),$$

where $\Sigma_2^* = \begin{pmatrix} \sigma_{UU} & \sigma_{UZ} \\ \sigma_{UZ} & \sigma_{ZZ} \end{pmatrix}$,

with $\sigma_{UU} = \dfrac{2-\pi_1}{\pi_1\lambda_1^2}e^{-\lambda_1 x}(2-e^{-\lambda_1 x})$,

$$\sigma_{UU} = \frac{2-\pi_1}{\pi_1\lambda_1^2}e^{-\lambda_1 x}(1+\lambda_1 x) + \frac{\rho(x)e^{-\lambda_1 x}}{\lambda_1\lambda_2}\left(\frac{\pi_1+\pi_2-\pi_1\pi_2}{\pi_1\pi_2}\right) \text{ and}$$

$$\sigma_{ZZ} = \frac{2-\pi_1}{\pi_1\lambda_1^2} + \frac{2\rho}{\lambda_1\lambda_2}\left(\frac{\pi_1+\pi_2-\pi_1\pi_2}{\pi_1\pi_2}\right) + \frac{2-\pi_2}{\pi_2\lambda_2^2}.$$

Hence by applying the results of Theorem 7.1, we get

$$\sqrt{n}\left(\hat{R}_n(x) - R(x)\right) \xrightarrow{L} N\left(0, \tau^2(x)\right), \tag{7.13}$$

where
$$\tau^2(x) = \frac{\lambda_2^2 e^{-2\lambda_1 x}}{(\lambda_1+\lambda_2)^2}\left\{\frac{2-\pi_1}{\pi_1}\left(\frac{\lambda_2^2}{(\lambda_1+\lambda_2)^2} - \frac{2\lambda_2(1+\lambda_1 x)}{\lambda_1+\lambda_2} + 2e^{\lambda_1 x} - 1\right)\right.$$
$$\left. + \frac{2\lambda_1(\pi_1+\pi_2-\pi_1\pi_2)}{\pi_1\pi_2(\lambda_1+\lambda_2)}\left(\frac{\rho\lambda_2}{\lambda_1+\lambda_2} - \rho(x)\right) + \frac{2-\pi_2}{\pi_2}\frac{\lambda_1^2}{(\lambda_1+\lambda_2)^2}\right\}.$$

This in turn implies that, for a *BEAR*(1) sequence the estimator $\hat{R}_n(x)$ is CAN for the limiting interval availability $R(x) = \lambda_2 e^{-\lambda_1 x}/(\lambda_1+\lambda_2)$.

In order to establish the importance of $BEAR(1)$ model compared to the *i.i.d.* $BVE(\lambda_1, \lambda_2, \rho)$ sequence, without loss of generality we consider the case of $x = 0$. In this case, $R(x)$ reduces to the limiting availability $A = \lambda_2 / (\lambda_1 + \lambda_2)$ and hence (7.13) becomes

$$\sqrt{n}\left(\hat{R}_n(0) - A\right) \xrightarrow{L} N\left(0, \tau^2\right),$$

where

$$\tau^2 = \frac{2(1-\rho)(\lambda_1\lambda_2)^2(\pi_1 + \pi_2 - \pi_1\pi_2)}{\pi_1\pi_2(\lambda_1 + \lambda_2)^4}.$$

Note that the $BEAR(1)$ sequence reduces to the *i.i.d.* $BVE(\lambda_1, \lambda_2, \rho)$ sequence when $\pi_1 = \pi_2 = 1$. Let $\tau_*^2$ be the asymptotic variance of $\hat{R}_n(0)$ in the case of *i.i.d.* $BVE(\lambda_1, \lambda_2, \rho)$ case.

Then,

$$\tau_*^2 = \frac{2(1-\rho)(\lambda_1\lambda_2)^2}{(\lambda_1 + \lambda_2)^4}.$$

Let $n_d$ and $n_d^*$ denote the number of observations required to construct sequential confidence interval for the limiting availability *A*, of fixed width 2*d* and coverage probability $1-\alpha$ in the case of $BEAR(1)$ sequence and *i.i.d.* $BVE(\lambda_1, \lambda_2, \rho)$ sequence of failure and repair times. Then assuming the asymptotic variance $\tau^2$ and $\tau_*^2$ are known,

$$n_d = \min\{n : n \geq d^{-2} z_{\alpha/2}^2 \tau^2\} \text{ and}$$

$$n_d^* = \min\{n : n \geq d^{-2} z_{\alpha/2}^2 \tau_*^2\}.$$

Consider the ratio,

$$\frac{n_d}{n_d^*} \approx \frac{\tau^2}{\tau_*^2} = \frac{\pi_1 + \pi_2 - \pi_1\pi_2}{\pi_1\pi_2}$$

$$= \frac{1}{1-\rho_1} + \frac{1}{1-\rho_2} - 1,$$

where $\rho_1 = 1 - \pi_1$ and $\rho_2 = 1 - \pi_2$ are the marginal lag 1 autocorrelations of the sequences $\{X_n\}$ and $\{Y_n\}$ respectively. The following table gives the values of the ratio $n_d / n_d^*$ for a few values of $\rho_1$ and $\rho_2$.

| $\rho_2$ \ $\rho_1$ | 0.2 | 0.5 | 0.7 | 0.9 |
|---|---|---|---|---|
| 0.2 | 1.50 | 2.25 | 3.58 | 10.25 |
| 0.5 | 2.25 | 3.00 | 4.33 | 11.00 |
| 0.7 | 3.58 | 4.33 | 5.67 | 12.33 |
| 0.9 | 10.25 | 11.00 | 12.33 | 19.00 |

Note that the ratio $n_d / n_d^*$ is always greater than unity and increases as the marginal autocorrelations $\rho_1$ and $\rho_2$ increase. This indicates that under the assumption of independence the sample size is significantly underestimated if the true process is *BEAR*(1). For example, even when the autocorrelation is small $(\rho_1 = 0.2,\ \rho_2 = 0.2)$ the ratio $n_d / n_d^*$ is approximately equal to 1.50, indicating underestimation of 50%. Thus, when the successive sequences of failure and repair times are dependent, the assumption of independence make erroneous conclusions.

### 7.5. Numerical Study

In order to compare the performance of the sequential decision rule defined by (7.6) in the case of bivariate stationary dependent sequence with that of i.i.d. sequence, a simulation study is performed in this section. A sequence of failure and repair times are generated by a *BEAR*(1) model having bivariate Marshall-Olkin distribution with parameters $\lambda_1 = 0.06,\ \lambda_2 = 0.36$ and $\lambda_{12} = 0.14$. So the bivariate random vector has $BVE(0.2, 0.5, 0.25)$ distribution with mean (5, 2) and correlation coefficient $\rho = 0.25$. Here we assume that $p_{00} = 0.14$, $p_{01} = 0.06$, $p_{10} = 0.36$ and $p_{11} = 0.44$ so that the marginal autocorrelations of the failure and repair times are $\rho_1 = 0.8$ and $\rho_2 = 0.5$ respectively.

The 95% sequential confidence interval for $R(x)$ for several values of '$x$' and '$d$' are constructed for the $BEAR(1)$ process. We also construct such confidence intervals for $R(x)$ by treating the above data are as generated by an i.i.d. $BVE(0.2, 0.5, 0.25)$ distribution. We repeat this experiment 5000 times and then compute the empirical coverage probabilities in both cases. The results of the simulation study are summarized in Table 7.1, where $n_d$ represents the actual sample size required to construct a sequential confidence interval for $R(x)$ of width $2d$. The notations $\bar{N}_d$, $\hat{R}_{\bar{N}_d}(x)$, $CP$ and $\bar{N}_d^*$, $\hat{R}_{\bar{N}_d^*}(x)$, $CP^*$ denote the average sample size required, average value of the estimated $R(x)$, empirical coverage probability for the sequential confidence interval in the case of $BEAR(1)$ process and i.i.d. bivariate exponential model respectively. The initial sample size $m$ is taken as 10 in the simulation study.

**Table 7.1** Simulated coverage probabilities for limiting interval reliability

| $x$ | $R(x)$ | $d$ | $n_d$ | *BEAR(1) Case* | | | *Bivariate i.i.d. Case* | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | $\bar{N}_d$ | $\hat{R}_{\bar{N}_d}(x)$ | $CP$ | $\bar{N}_d^*$ | $\hat{R}_{\bar{N}_d^*}(x)$ | $CP^*$ |
| 0.0 | 0.71429 | 0.050 | 576 | 562.71 | 0.71333 | 0.9182 | 97.87 | 0.70895 | 0.6020 |
| | | 0.075 | 256 | 247.75 | 0.71241 | 0.9298 | 49.79 | 0.70132 | 0.6152 |
| | | 0.100 | 144 | 137.92 | 0.70965 | 0.9384 | 30.70 | 0.68527 | 0.6274 |
| | | 0.125 | 93 | 89.44 | 0.71029 | 0.9402 | 20.30 | 0.68758 | 0.6706 |
| | | 0.150 | 64 | 63.59 | 0.70934 | 0.9496 | 15.12 | 0.67991 | 0.6827 |
| 0.5 | 0.64631 | 0.050 | 866 | 843.05 | 0.64495 | 0.9187 | 139.83 | 0.63916 | 0.6118 |
| | | 0.075 | 385 | 379.28 | 0.64791 | 0.9221 | 61.15 | 0.63544 | 0.6191 |
| | | 0.100 | 217 | 212.04 | 0.64349 | 0.9274 | 34.60 | 0.62721 | 0.6242 |
| | | 0.125 | 139 | 126.94 | 0.63794 | 0.9343 | 23.15 | 0.61183 | 0.6605 |
| | | 0.150 | 97 | 89.53 | 0.63317 | 0.9424 | 16.73 | 0.58345 | 0.6785 |
| 1.0 | 0.58481 | 0.050 | 1142 | 1095.61 | 0.58212 | 0.9103 | 155.08 | 0.57982 | 0.6194 |
| | | 0.075 | 508 | 485.37 | 0.58964 | 0.9186 | 69.81 | 0.56392 | 0.6256 |
| | | 0.100 | 286 | 252.81 | 0.57565 | 0.9287 | 40.07 | 0.55152 | 0.6350 |
| | | 0.125 | 183 | 175.25 | 0.57921 | 0.9298 | 25.87 | 0.53395 | 0.6455 |
| | | 0.150 | 127 | 113.85 | 0.57539 | 0.9389 | 18.15 | 0.52437 | 0.6625 |

Table 7.1 reveals that the coverage probabilities of $R(x)$ under the assumption of i.i.d. model are significantly smaller than those under the $BEAR(1)$ case. This also indicates that the ignorance of autocorrelations present in the sequence will significantly under estimates the sample size.

### 7.6 Conclusion

In this chapter we have discussed the sequential confidence interval estimation of the limiting interval availability when the failure and repair times of a system form a stationary strong mixing sequence of bivariate random vectors. It is shown that the confidence interval is asymptotically consistent and the proposed stopping rule is asymptotically efficient as the width of the interval approaches zero. The general theory is applied to a stationary $BEAR(1)$ sequence and the resulting stopping rule is compared with the stopping rule under the i.i.d. set-up. It is observed that when the true model is $BEAR(1)$, the assumption of an i.i.d. sequence underestimates the sample size and leads to poor coverage probability. A simulation study also confirmed the same result.

### 7.7 Plan for Future Work

In Chapter 2 and 3, we consider the nonparametric estimation of the average availability and the interval reliability under three different sampling schemes. The estimation was carried out by assuming that the sequences of failure and repair times are two independent sequences of i.i.d. random variables. However, this assumption need not hold good in many situations. The repair times may depend on the previous failure time due to the influence of the operating environment on the system. When the failure and repair times form a bivariate i.i.d. sequence, the estimation of the availability measures; point availability, average availability and interval reliability, is an interesting research problem which is to be addressed.

The availability behavior and the estimation of the limiting interval reliability when the sequences of failure and repair times are generated by stationary dependent sequences of random variables were discussed in Chapter 5 and 6. When the system is working in a random environment, it is natural to observe dependence among successive sequences of failure times. The inference procedures for estimating various quantities in the survival analysis are discussed by several authors in this set-up. See, for example, Ying and Wei (1994), Cai and Roussas (1998), Cai (2001). However, the estimation of the availability measures; point availability, average availability and interval reliability, is not discussed in the literature when the sequences of failure and repair times are generated by some stationary mixing sequences of dependent random variables, except the case of limiting measures.

Throughout this thesis, we use the empirical distribution function and the Kaplan-Meier product limit estimator as a nonparametric estimator of the cumulative distribution function in the case of complete and censored observations respectively. These estimators can only give a step function as the estimates. There are several works available in the literature dealing with the estimation of smooth distribution functions using kernel type estimators. See, Reiss (1981) and Ghorai and Susarla (1990). The nonparametric estimators of the availability measures using smoothly estimated distribution functions may reduce the mean square errors of the estimators significantly. This can be considered as a future work in this direction.

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