Stochastic Modelling

STUDIES ON CLASSICAL AND RETRIAL INVENTORY WITH POSITIVE SERVICE TIME

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Certificate

This is to certify that the thesis entitled '**Studies On Classical And Retrial Inventory With Positive Service Time**' submitted to the Cochin University of Science and Technology by Mrs. Lalitha. K for the award of the degree of Doctor of Philosophy under the Faculty of Science, is a bonafide record of studies carried out by her under my supervision in the Department of Mathematics, Cochin University of Science and Technology. This report has not been submitted previously for considering the award of any degree, fellowship or similar titles elsewhere.

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Declaration

I, Lalitha. K hereby declare that this thesis entitled '**STUDIES ON CLASSICAL AND RETRIAL INVENTORY WITH POSITIVE SERVICE TIME**' contains no material which had been accepted for any other Degree, Diploma or similar titles in any university or institution and that to the best of my knowledge and belief, it contains no material previously published by any person except where due references are made in the text of the thesis.

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CHAPTER 1

Introduction

The purpose of this thesis is to combine several concepts from queuing theory and inventory and use them in modelling and analysis. Until 1947 it was assumed, while analyzing problems in queues with finite capacity, when the buffer is full any further arrival is lost. However this is not the case in reality. A customer who could not get admission into the system may keep trying until he succeeds or quits because a time reaches when he does not derive any benefit out of the service, whichever occurs first. This type of queueing problem was first analyzed by Kosten [**27**] in 1947 and such type of queues are referred to as retrial queues. Retrial queues arise in a natural way in communication systems, at enquiry counters attached with offices, in hospitals and so on. Multiserver retrial queues are complex compared to single server queue. Still more complex is the retrial multiserver queues where the servers are separated, which arises as follows. Suppose there are c servers who are separated so that neither a server nor an arriving customer knows the status of the rest of the $c - 1$ servers. Thus if the present arrival to a particular server finds that server busy then he has to retry to access even other servers. This type of situation arises in, for example, at reception counters where there are a few telephones with distinct numbers. This problem is analyzed in Mushkov, Jacob, Ramakrishnan, Krishnamoorthy and Dudin [**50**] in 2006.

Inventory system was formally investigated in the most simple situation by Harris in 1915 which was subsequently analysed independently by Wilson in 1918 and the famous Harris-Wilson EOQ formula was realized. Most of the initial work in inventory theory were on deterministic models. Realizing the importance of uncertainty of the demand process and of the lead time, probabilistic models started getting investigated. Nevertheless the basic assumption in all these was that the time required to serve the item(s) was negligible. So in case item is available at demand epoch it is instantly served. Else a queue gets formed, provided backlog is permitted. Krishnamoorthy and Raju in a series of papers [**39, 41**], analyzed inventory with local purchase during stock out period, whenever a demand occurs, to earn customer good-will. However these were also restricted to the case of negligible service time. In practice a positive duration of service, deterministic or random, is needed to serve the item(s). Thus Berman, Kim and Shimshack in 1993, came up with the notion of inventory with positive service time. Since then there are several developments in the analysis of such inventory models.

In this thesis we combine models in classical/retrial queues with inventory involving positive service time. In some cases we introduce local purchase during stock out period, to improve the reliability of the system. This local purchase is assumed to be instantly done so that customers are not lost on account of lack of availability of the item. We also introduce disaster that removes all inventoried items instantly.

Next we provide a brief account of queues and inventory. In the sequel we also provide a brief account of the matrix geometric solution. Then we proceed to provide a brief review of the work that were done in the direction of the problems discussed in this thesis.

1.1. Classical and Retrial Queues

Lining up for some form of service is a common phenomenon, be it visible or invisible, by human beings or by inanimate objects. It is more organized or, sometimes, is made to be so in the modern world and therefore a systematic study of a line up or equivalently a queueing process is instinctively more rewarding academically. A classical queueing system can be described as customers arriving for service, waiting for service if service is not immediate and if having waited for service, leaving the system after being served.

A queue is formed when either there is positive service time or there are no sufficient servers for the arriving customers. Some examples of a queue are customers arriving at a bank and aeroplanes waiting for their turn to land in busy airports.

Queueing systems in which arriving customers find all servers and waiting positions (if any) occupied, may retry for service after a period of time. Such queues are called retrial queues or queues with repeated attempts. One of the most obvious example is provided by a person who desires to make a phone call. If the line is busy, then he cannot queue up, but can try sometime later.

Retrial queues are a type of networking with reserving after blocking. The classical queueing models do not take into account the phenomenon of retrials and therefore cannot be applied in solving a number of practically important problems. Retrial queues have been introduced to solve this deficiency.

1.2. Inventory Systems

In all business firms the system must keep a minimum amount of inventory at the time of order placing of inventory for the smooth and efficient running of the firm. The importance of inventory management for the quality of service of today's service systems is generally accepted and optimization of systems in order to maximize quality of service is therefore an important topic.

There are several factors affecting the inventory. They are demand, life time of items stored, damage due to external disaster, production rate, the time lag between order and supply, availability of space in the store etc. If all these parameters are known before hand, then the inventory model is called deterministic inventory model. If some or all of these parameters are not known with certainty then we consider them as random variables with some probability distribution and the resulting inventory model is then called stochastic inventory model.

Efficient management of inventory system is done by finding out optimal values of the decision variables. The important decision variables in inventory system are maximum capacity of the inventory, reordering point and order quantity. Several policies may be used to control an inventory system. Of these, the most important policy is the (s, S) policy. An inventory system may be based on periodic review (e.g., ordering every week or every month), in which new orders are placed at the start of each period. Alternatively the system may be based on continuous review where a new order is placed when the inventory level drops to a certain level, called the reorder point. An example of periodic review occurs in gas stations where new deliveries arrive at the start of each week. Continuous review occurs in retail stores where items (such as cosmetics) are replenished only when their level on the shelf drops to the reorder point.

The time elapsed between an order and its physical materialization is termed as lead time. If the replenishment is instantaneous then lead time is zero, otherwise the system is said to have positive lead time.

Inventory models have a wide range of applications in the decision making of government military organization, industries, hospitals, banks, educational institutions etc. Study and research in this fast growing field of applied mathematics, taking models from practical situations, contributes significantly to the progress and development of human society.

In most of the analysis of inventory systems the decay and disaster factors are ignored. But in several practical situations these factors play an important role in decision making.Examples are electronic equipments stored and exhibited on a sales counter, perishable goods like food stuffs, chemicals, crops vulnerable to insects and natural calamities like earth quake, rains, storms etc.

1.2.1. Inventory with positive service time. In all works reported in inventory prior to 1993 it was assumed that the time required to serve the item to the customer is negligible. As a consequence if the item is available at a demand epoch, the customer need not have to wait; a queue can be formed only when the inventory level becomes zero and lead time is positive.

We come across several real life situations where the service time is not negligible. In this case a queue will be formed even when the item is available. Thus the problem in inventory with service time may appear as a problem in queue. Nevertheless, this is not the case. The server stays idle even when there are customers in the system in the absence of inventoried items for processing.

Shortages of inventory occur in systems with positive lead time, in systems with negative reordering points or in multi commodity inventory system in which an order is placed only when the inventory level of at least two commodities fall to or below than the reorder level. Shortage cost is the penalty incurred when we run out of stock. It includes potential loss of income and moreover subjective cost of loss in customer's goodwill. There are different methods to tackle the stock out periods of the inventory. One of the method is to consider the demands during dry periods as 'lost sales'. The other is partial or full backlogging of the demands.Lost sale causes a loss in the profit and back logging results in the increase in the waiting time of the customer. In order to avoid these two possibilities in this thesis we adopt the notion of local purchase. If a customer enters for service when the inventory level is zero we make a local purchase of the item at a higher cost. Thus we can decrease the waiting time of the customer and thereby holding cost of the customer. Local purchases are made to improve the good will of the customers with the system especially in a newly opened shop or where there is a competition between near by shops.

1.2.2. Quasi-Birth and Death process (QBD). Consider a continuous time Markov chain on the two-dimensional state space $\{(0, j),\, 1 \leq j \leq m'\} \cup \{(n, j), n \geq 1, 1 \leq j \leq m\}$. The first co-ordinate n is called the level and the second co-ordinate j is called the phase of the state (n, j). The Markov process is called a QBD if one-step transition from a state are restricted to states in the same level or in the two adjacent levels: it is possible to move in one step from (n, j) to (n', j') only if $n' = n, n + 1$ or $n - 1$ (in the last case $n \geq 1$). If the transition rate from (n, j) to $(n'j')$ does not depend on n and n', but only on $n' - n$ then the Markov process is called a Level Independent Quasi-Birth Death (LIQBD) process and the infinitesimal generator Q is given by

$$Q = \begin{bmatrix} B_1 & B_0 & 0 & 0 & \cdots & \cdots \\ B_2 & A_1 & A_0 & 0 & \cdots & \cdots \\ 0 & A_2 & A_1 & A_0 & \cdots & \cdots \\ 0 & 0 & A_2 & A_1 & \cdots & \cdots \\ \vdots & \vdots & & \vdots & & \vdots \end{bmatrix}$$

where B_1 is a square matrix of order m', B_0 is an $m' \times m$, B_2 is an $m \times m'$ and A_0, A_1 and A_2 are square matrices of order m.

If the transition rates depend on the level then the Markov Process is called a Level Dependent Quasi Birth Death (LDQBD) Process and the infinitesimal generator Q is then given by

$$Q = \begin{bmatrix} A_{10} & A_{00} & 0 & 0 & 0 & 0 & \cdots \\ A_{21} & A_{11} & A_{01} & 0 & 0 & 0 & \cdots \\ 0 & A_{22} & A_{12} & A_{02} & 0 & 0 & \cdots \\ 0 & 0 & A_{23} & A_{13} & A_{03} & 0 & \cdots \\ \vdots & \vdots & & \vdots & & \vdots & \end{bmatrix}. \tag{1.2.1}$$

All models discussed in this thesis are either LIQBD or LDQBD

1.2.3. Matrix analytic method. A matrix analytic approach to stochastic models was introduced by Neuts [**53**] to provide an algorithmic analysis for $M|G|1$ and $GI|M|1$ type of queueing models. Matrix analytic methods constitute a success story, illustrating the enrichment of science, applied probability by a technology, that of digital computers.

The following theorem gives a brief description of Matrix Analytic Method applied for solving Quasi-Birth Death Process (QBD).

THEOREM 1.2.1. *A continuous time QBD with infinitesimal generator Q of the form* (1.2.1) *is positive recurrent if and only if the minimal non-negative solution R to the matrix quadratic equation*

$$R^2A_2 + RA_1 + A_0 = 0 \tag{1.2.2}$$

has spectral radius less than 1 and the finite systems of equations

$$x_0A_{10} + x_1A_{21} = 0$$

$$x_{i-1}A_{0,i-1} + x_iA_{1i} + x_{i+1}A_{2,i+1} = 0 \quad (1 \leq i \leq N-2)$$

$$x_{N-2}A_{0,N-2} + x_{N-1}(A_{1,N-1} + RA_2) = 0$$

has a unique solution for $x_0, \ldots, x_{N-1}$. If the matrix $A = A_0+A_1+A_2$ where $A_{0i} = A_0$, $A_{1i} = A_1$ for $i \geq N$ is irreducible, then $sp(R) < 1$ if and only if $\pi A_0e < \pi A_2e$ where π is the stationary probability vector of the generator matrix A and satisfies the equation $\pi A = 0$ and $\pi e = 1$ where $e = (1, \ldots, 1)'$.

If $x = (x_0, x_1, \ldots)$ is the stationary probability vector of Q then x_i's $(i \geq N)$ are given by

$$x_{N+r-1} = x_{N-1}R^r \text{ for } r \geq 1.$$

To find the minimal solution of (1.2.2) we can use the iterative formula given by $R_{n+1} = -(R_n^2 A_2 + A_0)A_1^{-1}$, $n = 0, 1, 2, \ldots$ with $R_0 = 0$

1.3. Review of Related Work

1.3.1. Works on inventory. In 1915 Harris [**24**] started the mathematical modelling of inventory problems and derived the famous EOQ formula that was popularized by Wilson. A systematic analysis of the (s, S) inventory system using renewal theoretic arguments is provided in Arrow, Karlin and Scarf [**2**]. Hadley and Whitin [**23**] gave several applications of different inventory models. Gross and Harris [**21**] considered the inventory systems with state dependent lead times. Sivazlian [**63**] analyzed the continuous review (s, S) inventory system with general inter arrival times and unit demand in which he shows that the limiting distribution of the position inventory is uniform and independent of the inter arrival time distribution. Sahin [**60**] analyzed continuous review (s, S) inventory with continuous state space and constant lead time. Srinivasan [**64**] discussed an (s, S) inventory problem with arbitrarily distributed interarrival times and lead times.

Manoharan et.al. [**47**] discussed the case of non-identically distributed interarrival times. Krishnamoorthy and Lakshmi [**35**] analyzed problems with Markov dependent re-ordering levels and Markov dependent replenishment quantities. Krishnamoorthy and Manoharan [**46**] modelled an inventory system with varying reorder levels and random lead time. Krishnamoorthy and Varghese [**44**] considered a two commodity inventory problem with Markov shift in demand for the commodity. Krishnamoorthy and Raju [**39**] introduced N-policy to the (s, S) inventory system with positive lead time and local purchase when the inventory level is zero

Berman, Kim and Shimshack [**13**] introduced positive service time in inventory in which the service time is assumed to be constant. They determined optimal order quantity Q that minimizes the total cost rate using dynamic programing technique.

Subsequently, Berman and Kim [**12**] extended that model to random service time. Parthasarathy and Vijayalakshmi [**57**] discussed transient analysis of an $(S-1, S)$ inventory model with deteriorating items and obtained the solution using continued fraction.

Viswanath et.al [**66**] studied an (s, S) inventory policy with service time by considering vacation to server and correlated lead time. They considered quite general distribution for interarrival time, duration of service time and duration of a vacation. Server goes on vacation whenever there is either no customer left behind in the system at departure epoch or when the inventory level drops to zero or both occur simultaneously. Schwarz et.al [**61**] discussed $M|M|1$ queueing systems with inventory where the lead times are exponentially distributed. They analyzed the problem for both (r, Q) and (r, S) inventory policies and derived stationary distribution of joint queue length and inventory level in explicit product form. Also they discussed the problem of order placements any where on the set $\{0, 1, \ldots, s\}$ according to a given probability distribution. Krishnamoorthy et.al. [**38**] introduced the N-policy for commencement of service, once the server is switched off in the absence of customers in the system. Here the service time is positive and lead time is zero. They obtained analytical solution to this model. They establish a product form solution to the system state and thus produce a decomposition of the state space. Murthy and Ramanarayan [**49**] discussed (s, S) inventory system with defective items in the replenished items, where the lead time is positive with arbitrary distribution. Krishnamoorthy and Varghese [**43**] analyzed an inventory model where the items are damaged due to decay and disaster. They assumed that the lead time is zero and the service time is negligible. A detailed survey on inventory with positive service time is given in Krishnamoorthy et.al [**36**].

1.3.2. Works on retrial queue and retrial inventory. Retrial queues or queues with repeated attempts have been extensively investigated (See the survey papers by Yang and Templeton [**67**], Falin [**18**] and the book by Falin and Templeton [**19**]). Subsequent development on retrial queues can be found in Artalejo [**3**]. The latest addition to books on retrial queues is authored by Artalejo and Gomez-Correl [**6**]. In this they discussed the algorithmic approach. Artalejo, Krishnamoorthy and Lopez-Herrero [**9**] were the first to study inventory policies with positive lead time coupled with retrial of unsatisfied customers and their approach turns out to be algorithmic. Ushakumari [**65**] obtained analytical solution to the above problem in 2006. Krishnamoorthy and Mohammad Ekramol Islam [**31**] analyzed an (s, S) inventory system with retrial of customers. Here the lead time and inter-retrial times are assumed to be exponentially distributed.

Krishnamoorthy and Jose [**33**] compared three (s, S) inventory system with retrial of customers where the service time and lead time are positive. They investigated these systems to obtain performance measures and construct suitable cost functions for the three cases. In 2002 Artelajo et.al [**8**] discussed an $M|G|1$ retrial queue where the server goes for an orbital search, when he is free. Thus the system can decrease the idle time of the server as well as the waiting time of the customer. Neuts and Rao [**55**] discussed an $M|M|c$ retrial queue in which the model is LDQBD process and they suggested a truncation procedure, the idea is to make retrial rate to be constant when the number of customers in the orbit exceeds some level.

1.4. An Outline of the Work in this Thesis

This thesis is divided into six chapters including this introductory chapter. Second chapter contains investigation of two models. In the first model we consider a single item, continuous review (s, S) inventory model with one server. Arrival of customers form a Poisson process with rate λ and service times of customers are exponentially distributed random variables with parameter μ, one unit of item is needed for each customer. Lead time is assumed to be zero. An arriving customer, who finds the server busy, proceed to an orbit of infinite capacity and makes successive repeated attempts until it finds the server free. The inter retrial times have an exponential distribution with parameter $i\theta$ when there are i customers in the orbit. Here we get an analytical solution to the model. We construct a cost function and numerical examples are given. In the second model we consider a more general set up involving arbitrarily distributed service time. All other assumptions are same as that in the first model. We consider the number of customers in the orbit and the inventory level at the departure epoch of a customer. Thus we have an embedded Markov chain. Here also we analyze a cost function.

In chapter 3, we consider five distinct inventory models with positive service time and positive lead time.In all these it is assumed that customers arrive to a single server system according to a Poisson process with rate λ and service times are exponentially distributed random variables with parameter μ. Each customer require one unit of inventory. We follow an (s, S) inventory policy. When the inventory level depletes to s we place an order for $Q = S - s$ quantity of inventory. The distribution of lead time is exponential with parameter β. In model 1 customers do not join the system when the inventory level is zero. In model 2 customers join the system even when the inventory level is zero. In model 3 and 4 we make a local purchase of one and s units of items respectively, whenever a customer arrives to find the inventory level zero, at an extra cost. In model 5 under the same situation we make a local purchase of S units, thus cancelling the existing order for procurement of inventory as the maximum capacity of inventory is S. Numerical examples are given to compare performance of these models in terms of appropriate cost functions.

In chapter 4 we introduce retrial of unsatisfied customers into the models discussed in chapter 3, with the assumption that there is no waiting space for the customers at the service station other than to the one who is being served. An arriving customer who finds the server busy, proceeds to an orbit of infinite capacity and makes successive repeated attempts until it finds the server free. The inter retrial times can be modelled according to different disciplines depending on each particular application.In telephone systems the repeated attempts are made individually by each blocked customer following an exponential law of rate θ. This is the classical retrial policy where the rate is $i\theta$ when there are $i \geq 0$ customers in the orbit. Another retrial policy is the constant retrial policy in which the probability of repeated attempts is independent of the number of customers in the orbit. Here we assume that the inter retrial times have an exponential distribution with constant rate θ. Here also we compare the cost functions through numerical investigations.

In chapter 5 we consider (s, S) inventory systems with the possibility of destruction of inventoried items due to disasters.Here we discuss two models. Customers arrive to a single server system according to a Poisson process with parameter λ where service times are exponentially distributed random variables with parameter μ. We assume that disaster destroys all the inventoried items but not the customers. For example, in godowns food items are destroyed by natural calamities. Here we assume the inter disaster times to be exponentially distributed with parameter δ.It is assumed that lead time is also exponentially distributed random variables with parameter β. In Model I we assume further that customers do not join the system when the inventory level is zero. However in Model II it is assumed that customers join even when the inventory level is zero. Thus stability in Model II is affected by the lead time parameter. We compare the two models through numerical examples by constructing suitable cost functions.

In chapter 6 we consider a multi server queue coupled with an inventory following (s, S) policy and retrial of customers. Customers arrive to the system with c-servers according to Poisson process with rate λ. The service times are exponentially distributed with parameter μ. One item is needed for each customer. An arriving primary customer, who finds all servers busy, will go to an orbit of infinite capacity and tries again for the service. Inter retrial time follows exponential distribution with parameter θ. The lead time follows exponential distribution with rate β. We assume that customers do not join the system when the inventory level is zero. A cost function is constructed and numerically investigated.

CHAPTER 2

Inventory with Retrial and Service Time

2.1. Introduction

In classical queuing theory it is very often assumed that a customer who cannot get service immediately on arrival (as the server is busy) either joins the waiting line, and then is served according to some queue discipline, or leaves the system forever. However, as a matter of fact, the assumption about the loss of customers who opted to leave the system is just a first order approximation to a real situation. Usually such a customer after a random time returns to the system and tries to get service again. Such a queue is known as retrial queue (or queues with returning customers, repeated attempts etc.). In retrial queues an arriving customer, who finds the inventory level zero or server busy, proceeds to an orbit and repeats his attempts. Retrial queues have been used to model problems in telephone, computer and communication systems. For a detailed discussion of retrial queues one can refer to Falin [**18**], Falin and Templeton [**19**], Yang and Templeton [**67**] and Artalejo [**3**].

In most of the papers on inventory it is assumed that the service time is negligible.This means that at a demand epoch if the item is available,it is immediately served to the customer. However,in real life situations this assumption is too restrictive. The first attempt at analyzing inventory problems with positive service time was due to Berman et.al [**13**]. This was essentially a deterministic inventory model. Subsequently, Berman and Sapna [**14**], Arivaringan et.al [**1**], Krishnamoorthy et.al. [**38**] have discussed inventory with positive service time under various assumptions.

In this chapter we consider two models of inventory with positive service time and retrial of customers. The difference between these two models is that, in the first we assume the service times are exponentially distributed with parameter μ and in the second model, service times have general distribution with distribution function $G(\cdot)$. The first one is analyzed as a continuous time Markov chain whereas the second using the embedded Markov chain technique. The inventory control is governed by the (s, S) policy. We assume that the lead time is zero. There is no waiting space for customers at the service station, except for the one undergoing service. If at an epoch at which a customer joins for service and if the inventory level turns out to be s, an order is instantly placed for $Q = S - s$ units which is received immediately. Each demand is exactly for one item. The system is manned by one server. If an arriving customer finds the server busy it proceeds to an orbit of infinite capacity and makes repeated attempts until it finds the server free. Primary customers arrive according to a Poisson process with rate λ. The inter-retrial times follow exponential distribution with linear rate $i\theta$ when there are i customers in the orbit.

2.2. The Mathematical Model and Analysis of Model I

We consider a single item, continuous review (s, S) inventory model. Arrival of customers form a Poisson process with rate λ. Service times of customers are independent and identically distributed exponential random variables with parameter μ. Arrival and service process are independent of each other. Service times of customers are mutually independent. Order is placed and immediately delivered at epoch at which customers join for service, with the inventory level equal to $s\ (\geq 0)$. That is, lead time is assumed to be zero. Further shortage cost is assumed to be infinity. An arriving customer who finds the server busy, proceeds to an orbit of infinite capacity and makes successive repeated attempts until he finds the server free. The inter-retrial times have an exponential distribution with parameter $i\theta$ when there are i customers in the orbit.

Let $N(t)$ be the number of customers in the orbit and $I(t)$ is the corresponding inventory level at time t. Define

$$C(t) = \begin{cases} 0 & \text{if the server is idle at time } t \\ 1 & \text{if the server is busy at time } t. \end{cases}$$

Now $X(t) = \{(N(t), C(t), I(t));\ t \geq 0\}$ is a Continuous Time Markov Chain (CTMC) with state space $S' = \cup_{i=0}^{\infty} l(i)$ where $l(i) = \{(i, 0, j), s \leq j \leq S-1\} \cup$

$\{(i, 1, j), s + 1 \leq j \leq S\}$. Since the demand is exactly for one unit and only one customer is served at a time, the level (number of customers in the orbit) increases or decreases by one unit. Therefore it is skip free to the left as well as to the right. Further the phase representing the inventory level decreases by 1 unit up to s and then goes back to S. Thus the model is a LDQBD (Level Dependent Quasi-Birth-Death process). The infinitesimal generator $\bar{Q}$ of the process has the block tridiagonal:

$$\bar{Q} = \begin{bmatrix} A_{10} & A_0 & 0 & 0 & 0 & \cdots \\ A_{21} & A_{11} & A_0 & 0 & 0 & \cdots \\ 0 & A_{22} & A_{12} & A_0 & 0 & \cdots \\ 0 & 0 & A_{23} & A_{13} & A_0 & \cdots \\ \vdots & & & \vdots & & \end{bmatrix}$$

where A_0, A_{1i} $(i \geq 0)$ and A_{2i} $(i \geq 1)$ are square matrices of the same order $2(S - s)$ and they are given by

$$A_{1i} = \begin{bmatrix} -(\lambda + i\theta) I_{S-s} & \lambda E \\ \mu I_{S-s} & -(\lambda + \mu) I_{S-s} \end{bmatrix}, \quad A_{2i} = \begin{bmatrix} 0 & i\theta E \\ 0 & 0 \end{bmatrix}$$

$$A_0 = \begin{bmatrix} 0 & 0 \\ 0 & \lambda I_{S-s} \end{bmatrix}, \quad \text{where } E = \begin{bmatrix} 0 & 1 \\ I_{S-s-1} & 0 \end{bmatrix}$$

and is of order $(S - s) \times (S - s)$. Next we investigate the condition for stability of the system.

2.2.1. System stability. When the number of customers in the orbit is sufficiently large, majority of the customers fail to access the server and do not result in significant change in the number of customers in the orbit. Under this condition, we can find a sufficiently large N such that the retrial rates $N\theta$ and $(N + 1)\theta$ do not differ significantly. In other words we can find N sufficiently large such that A_{1i}, A_{2i} can be approximated by $A_{1i} = A_1$, $A_{2i} = A_2$, respectively whenever $i \geq N$. This results in the difference between equilibrium probabilities corresponding to $\bar{Q}$ and $\hat{Q}$ (given below) turning out to be minimal. If the number of customers is restricted to an approximately chosen number N, then the change on the equilibrium probability vector is minimal. This truncation (see Neuts and Rao [**55**]) modifies the infinitesimal generator $\bar{Q}$ to the following form where $A_{1i} = A_1$ and $A_{2i} = A_2$ for $i \geq N$.

$$\hat{Q} = \begin{bmatrix} A_{10} & A_0 & & & & & & \\ A_{21} & A_{11} & A_0 & & & & & \\ & A_{22} & A_{12} & A_0 & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & A_{2N-1} & A_{1N-1} & A_0 & & \\ & & & & A_2 & A_1 & A_0 & \\ & & & & & A_2 & A_1 & A_0 \\ & & & & & & \ddots & \ddots & \ddots \end{bmatrix}.$$

Define the generator A as $A = A_0 + A_1 + A_2$. Then

$$A = \begin{bmatrix} -(\lambda + N\theta)I_{S-s} & (\lambda + N\theta)E \\ \mu I_{S-s} & -\mu I_{S-s} \end{bmatrix}$$

. Let $\boldsymbol{\pi}$ be the steady state probability vector of the generator matrix A. That is $\boldsymbol{\pi} A = 0$ and $\boldsymbol{\pi}\boldsymbol{e} = 1$. The vector $\boldsymbol{\pi}$ can be partitioned as $\boldsymbol{\pi} = (\boldsymbol{\pi}', \boldsymbol{\pi}'')$, where $\boldsymbol{\pi}' = (\pi_1, \pi_2, \ldots, \pi_{S-s})$ and $\boldsymbol{\pi}'' = (\pi_{S-s+1}, \pi_{S-s+2}, \ldots, \pi_{2(S-s)})$. It is easily seen that the solution to $\pi A = 0$ with $\pi e = 1$ is given by

$$\begin{aligned} \boldsymbol{\pi}' &= \frac{\mu}{\lambda + N\theta + \mu}\left(\frac{1}{S-s}, \frac{1}{S-s}, \ldots, \frac{1}{S-s}\right) \\ \boldsymbol{\pi}'' &= \frac{\lambda + N\theta}{\lambda + N\theta + \mu}\left(\frac{1}{S-s}, \frac{1}{S-s}, ..., \frac{1}{S-s}\right) \end{aligned}$$

This leads to the following

THEOREM 2.2.1. *The system is stable if and only if $\lambda < \mu$.*

PROOF. We have from the well known result (see Neuts [**53**]) for positive recurrence of $\hat{Q}$, the rate of drift to the left (in terms of level) has to be higher than that to the right; i.e., $\boldsymbol{\pi} A_0 \mathbf{e} < \boldsymbol{\pi} A_2 \mathbf{e}$ for stability of the system and vice versa. After some algebra this reduces to

$$\frac{\lambda + N\theta}{\lambda + N\theta + \mu}\lambda < \frac{\mu}{\lambda + N\theta + \mu}N\theta$$

which reduces to $\lambda < \mu$ as $N \to \infty$. $\square$

2.3. The Steady State Probability Vector of $\hat{Q}$

To get a complete picture of the system it is essential to compute the long run system state probability vector whenever it exists That is we have to calculate the steady-state probability vector of $\hat{Q}$ under the stability condition. Let the steady-state probability vector **x** of $\hat{Q}$ be partitioned according to the level as $\mathbf{x} = (x(0), x(1), x(2), \ldots)$ where the subvectors $x(i)$, $i \geq 0$, contains $2(S-s)$ elements. These subvectors satisfy the equations

$$x(0)A_{10} + x(1)A_{21} = 0 \tag{2.3.1}$$

$$x(i-1)A_0 + x(i)A_{1i} + x(i+1)A_{2,i+1} = 0; \; i \geq 1 \tag{2.3.2}$$

Again partition the subvector $x(i)$, $i \geq 0$ as

$$x(i) = (x(i,0), x(i,1)) \text{ where the subvectors } x(i,j), \; j = 0, 1$$

contain $S - s$ elements each. That is, $x(i,0) = (y_{i\,0\,s}, y_{i,0,s+1} \cdots \; y_{i,0,S-1})$ and $x(i,1) = (y_{i,1,s+1}, y_{i,1,s+2} \ldots y_{i1S})$. Equations (2.3.1) and (2.3.2) give rise to the following relations:

$$-\lambda x(0,0)I_{S-s} + \mu x(0,1)I_{S-s} = 0 \tag{2.3.3}$$

$$[\lambda x(0,0) + \theta x(1,0)]E - (\lambda + \mu)x(0,1)I_{S-s} = 0 \tag{2.3.4}$$

$$-(\lambda + i\theta)x(i,0) + \mu x(i,1) = 0 \tag{2.3.5}$$

$$[\lambda x(i-1,1) - (\lambda + \mu)x(i,1)]I_{S-s} + [\lambda x(i,0) + (i+1)\theta x(i+1,0)]E = 0 \tag{2.3.6}$$

From equation (2.3.3) we have

$$x(0,1) = \rho x(0,0) \text{ where } \rho = \frac{\lambda}{\mu} \tag{2.3.7}$$

Let $x(0,0) = \eta(1,1,\ldots,1)$. Then equation (2.3.7) gives $x(0,1) = \rho\eta(1,1,\ldots,\ 1)$. From equation (2.3.4) we have,

$$x(1,0) = \rho\frac{\lambda}{\theta}\eta(1,1,\ldots,1).$$

Equation (2.3.5) gives $x(i,1) = \frac{\lambda+i\theta}{\mu}x(i,0)$ for $i \geq 0$. Finally, (2.3.6) gives

$$x(i,0) = \left[\frac{\rho^i}{i!\theta^i}\prod_{k=0}^{i-1}(\lambda + k\theta)\right]\eta(1,1,\ldots,1)\text{for } i \geq 0.$$

Thus

$$x(i,1) = \left[\frac{\rho^{i+1}}{i!\theta^i}\prod_{k=1}^{i}(\lambda + k\theta)\right]\eta(1,1,\ldots,1) \text{ for } i \geq 0.$$

Now to find η we use the normalizing condition $\sum_{i=0}^{\infty} x(i)\mathbf{e} = 1$. Then we get $\eta = \frac{1}{S-s}(1-\rho)^{\frac{\lambda}{\theta}+1}$. Hence

$$x(i,0) = \left[\frac{1}{S-s}\frac{\rho^i}{i!\theta^i}(1-\rho)^{\frac{\lambda}{\theta}+1}\prod_{k=0}^{i-1}(\lambda + k\theta)\right](1,1,\ldots,1)$$

That is,

$$y_{i0j} = (\frac{1}{S-s})\left[\frac{\rho^i}{i!\theta^i}(1-\rho)^{\frac{\lambda}{\theta}+1}\prod_{k=0}^{i-1}(\lambda + k\theta)\right] \text{ for } s \leq j \leq S-1.$$

Hence $y_{i0j} = P[N = i, C = 0, I = j] = P[N = i, C = 0]P[I = j]$.

Also we have

$$x(i,1) = \left[\frac{1}{S-s}\frac{\rho^{i+1}}{i!\theta^i}(1-\rho)^{\frac{\lambda}{\theta}+1}\prod_{k=1}^{i}(\lambda + k\theta)\right](1,1,\ldots,1)$$

from which we get

$$y_{i1j} = (\frac{1}{S-s})\left[\frac{\rho^{i+1}}{i!\theta^i}(1-\rho)^{\frac{\lambda}{\theta}+1}\prod_{k=1}^{i}(\lambda+k\theta)\right] \text{ for } s+1 \leq j \leq S.$$

This tells us that

$$y_{i1j} = P[N=i, C=1, I=j] = P[N=i, C=1]P[I=j].$$

We sum up these results in the following.

THEOREM 2.3.1. *The steady state probability vector* **x** *of* $\hat{Q}$ *be partitioned as* **x** $= (x(0), x(1), x(2), \ldots)$ *where each* $x(i)$ *is again partitioned as* $x(i) = (x(i,0), x(i,1))$, $i \geq 0$. *Then*

$$x(0,0) = \eta(1,1,\ldots,1)$$

$$x(i,0) = \left[\frac{\rho^i}{i!\theta^i}\prod_{k=0}^{i-1}(\lambda+k\theta)\right]\eta(1,1,\ldots,1),\ i \geq 0$$

$$x(i,1) = \left[\frac{\rho^{i+1}}{i!\theta^i}\prod_{k=1}^{i}(\lambda+k\theta)\right]\eta(1,1,\ldots,1),\ i \geq 0$$

$$\textit{where} \quad \eta = \frac{(1-\rho)^{\frac{\lambda}{\theta}+1}}{S-s} \textit{ and } \rho = \frac{\lambda}{\mu}$$

Thus we arrive at a product form solution for the system state distribution. This naturally leads to the decomposition of the joint generating function.

2.4. System Performance Measures

Let $\mathbf{x} = (x(0), x(1), x(2), \ldots)$ be the steady-state probability vector of $\hat{Q}$. Each $x(i)$, $i \geq 0$ is partitioned as $x(i) = (x(i,0), x(i,1))$ where $x(i,0) = (y_{i,0,s}, y_{i,0,s+1}, \ldots, y_{i,0,S-1})$ and $x(i,1) = (y_{i,1,s+1}, y_{i,1,s+2} \ldots y_{i,1,S})$. Then we have the following expressions for the performance measures:

a. Expected number of customers, EC in the orbit is given by

$$\text{EC} = \sum_{i=1}^{\infty} ix(i)\mathbf{e} = \frac{\rho(\lambda+\rho\theta)}{(1-\rho)\theta}$$

b. Expected inventory level, EI is given by

$$\text{EI} = \sum_{i=0}^{\infty}\sum_{j=s}^{S-1} jy_{i0j} + \sum_{i=0}^{\infty}\sum_{j=s+1}^{S} jy_{i1j} = \frac{S+s-1}{2} + \rho.$$

c. Expected re-order rate, ER is given by

$$\text{ER} = \lambda\sum_{i=0}^{\infty} y_{i0s} + \theta\sum_{i=1}^{\infty} iy_{i0s} = \frac{\lambda}{S-s}.$$

d. Expected rate of departures, ED after completing service is given by

$$\text{ED} = \mu\sum_{i=0}^{\infty}\sum_{j=s+1}^{S} y_{i1j} = \lambda$$

e. Probability that the server is busy

$$= \sum_{i=0}^{\infty}\sum_{j=s+1}^{S} y_{i1j} = \rho$$

f. Over all retrial rate, ORR is given by

$$\text{ORR} = \theta\sum_{i=1}^{\infty} ix_i\mathbf{e} = \frac{\rho(\lambda+\rho\theta)}{1-\rho}$$

g. Successful retrial rate, SRR is given by

$$\text{SRR} = \theta\sum_{i=0}^{\infty} i\sum_{j=s}^{S-1} y_{i0j} = \rho\lambda$$

h. Probability of the number of customers in the orbit exceeding a given number, say R is

$$P[N>R] = (1-\rho)^{\frac{\lambda}{\theta}+1}\sum_{i>R}\left\{\frac{\rho^i}{i!\theta^i}[\prod_{k=0}^{i-1}(\lambda+k\theta) + \rho\prod_{k=1}^{i}(\lambda+k\theta)]\right\}$$

This measure is of great significance since systems are designed so as to minimize the expected waiting time of customers.

i. Since there is no queue formed in the orbit where the queue discipline is not first in first out, it is not easy to compute the waiting time distribution. So we proceed to compute the expected waiting time of a tagged customer.

Expected waiting time, EWT (excluding service time) of such a customer

$$= \frac{\rho}{1-\rho}\left(\frac{1}{\mu}+\frac{1}{\theta}\right) \quad \text{(see [\textbf{11}])}$$

j. Stochastic decomposition.

We have $E[N] = E[N_\infty] + \frac{E[N_0]}{1-\rho}$, where

$E[N]$ = Expected number of customers in the orbit $= \frac{\rho(\lambda+\rho\theta)}{\theta(1-\rho)}$

$E[N_\infty]$ = Expected number of customers in the queue excluding the customer receiving service,if any in the standard queue$= \rho^2/1-\rho$.

$E[N_0]$ =Expected number of customers in the orbit when the server is idle

= $\sum_{i=0}^{\infty} ix(i,0)$= $\frac{\rho\lambda}{\theta}$

2.5. Cost Function

To construct the cost function we define the following costs as

C = fixed ordering cost

c_1 = procurement cost/unit

c_2 = holding cost of inventory/unit/unit time

In terms of these we define the expected total cost function as

$$\text{ETC} = F(s,Q) = [C + Qc_1]\,\text{ER} + c_2\text{EI}$$

That is

$$F(s,Q) = [C + Qc_1]\frac{\lambda}{Q} + c_2[\frac{Q+2s-1}{2} + \rho].$$

Then $F(s,Q)$ is a separable and convex function of s and Q namely $c_1\lambda + c_2(s+\rho-\frac{1}{2})$ and $\frac{C\lambda}{Q} + \frac{c_2 Q}{2}$. We note that F is linear in s. Since no shortage is permitted, the optimal value of s is zero. Again we notice that the optimal value of Q is given by $\sqrt{\frac{2C\lambda}{c_2}}$. Hence the optimal value of S is also $\sqrt{\frac{2C\lambda}{c_2}}$. Thus the expected minimum cost of the system is

$$\sqrt{2Cc_2}\lambda + \frac{c_2}{2}(2\rho - 1) + c_1\lambda.$$

2.6. The Mathematical Model and Analysis of Model II

We consider a single server queueing system to which primary customers arrive according to a Poisson process with rate λ. If an arriving customer finds the server busy, it leaves the service area and joins the orbit to repeat its attempts from there. The inter retrial time follows an exponential distribution with linear rate $i\theta$ when there are i customers in the orbit. We follow an (s, S) inventory policy. The lead time is assumed to be zero. Service times are independently and identically distributed with distribution function $G(\cdot)$. Let $\beta(z) = \int_0^\infty e^{-zt} dG(t)$ be Laplace-Stieltjes transform of $G(t)$. $\beta_k = (-1)^k \beta^{(k)}(0)$ be the k^{th} row moment of the service time, $\rho = \lambda\beta_1$ is the system load due to primary calls. The inter arrival times, the interval between repeated attempts and service times are assumed to be mutually independent.

Let $N(t)$ be the number of customers in the orbit and $I(t)$ be the inventory level at time t. Let t_i be the time at which the i^{th} service completion occurs and
$N_i = N(t_i+) =$ Number of customers in the orbit immediately after the i^{th} departure and I_i be the corresponding inventory level. Thus $\{(N_i, I_i), i \geq 1\}$ forms a Markov chain on the state space $S' = \cup_{n=0} l(n)$ where $l(n) = \{(n, s), (n, s+1), \ldots, (n, S-1)\}$, $n \geq 0$.

Let $\gamma_i =$ Number of primary customers which arrive to the system during the service time of the i^{th} customer and

$$k_n = P(\gamma_i = n) = \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^n}{n!} dG(t), \quad n = 0, 1, \ldots$$

whose generating function $K(z) = \sum_{n=0}^\infty k_n z^n = \beta(\lambda - \lambda z)$. Its mean value $E(\gamma_i) = \sum_{n=0}^\infty n k_n = \rho$.

We have

$$N_i = N_{i-1} - B_i + \gamma_i \tag{2.6.1}$$

where

$$B_i = 1 \quad \text{if the } i^{\text{th}} \text{ customer is from the orbit}$$
$$= 0 \quad \text{if the } i^{\text{th}} \text{ customer is a primary customer.}$$

Then the one step transition probabilities of the Markov chain $r_{mn} = P\{N_i = n | N_{i-1} = m\}$ are given by the formula

$$r_{mn} = \frac{\lambda}{\lambda + m\theta} k_{n-m} + \frac{m\theta}{\lambda + m\theta} k_{n-m+1}, \; m, n = 0, 1, 2, \ldots$$

and $r_{mn} \neq 0$ only for $m = 0, 1, \ldots, n+1$

The transition probability matrix associated with the Markov chain is given by

$$P = \begin{bmatrix} A_{00} & A_{01} & A_{02} & \cdots \\ A_{10} & A_{11} & A_{12} & \cdots \\ 0 & A_{21} & A_{22} & \cdots \\ 0 & 0 & A_{31} & \cdots \\ \cdots & \cdots & \cdots & \end{bmatrix}$$

where

$$A_{mn} = \begin{array}{c} \\ (m,s) \\ (m,s+1) \\ \vdots \\ \vdots \\ (m,S-1) \end{array} \begin{array}{c} \begin{array}{ccccc} (n,s) & (n,s+1) & \cdots & & (n,S-1) \end{array} \\ \begin{pmatrix} 0 & 0 & \cdots & 0 & \Delta \\ \Delta & 0 & \cdots & 0 & 0 \\ 0 & \Delta & \cdots & 0 & 0 \\ & & & & \\ 0 & 0 & \cdots & \Delta & 0 \end{pmatrix} \end{array}$$

and $\Delta = \frac{\lambda}{\lambda + m\theta} k_{n-m} + \frac{m\theta}{\lambda + m\theta} k_{n-m+1}$

2.6.1. Ergodicity of $\{(N_i, I_i)\}$.

THEOREM 2.6.1. *The embedded Markov chain* $\{(N_i, I_i)\}$ *is ergodic if and only if* $\rho < 1$.

PROOF. To investigate the positive recurrence of the Markov chain we shall use Foster's criterion which states that an irreducible and aperiodic Markov Chain is positive recurrent if there exists a non-negative function $f(s')$, $s' = (n, j) \in S'$, $n \geq 0$, $s \leq j \leq S - 1$, and $\epsilon > 0$ such that the mean drift

$$\eta_{s'} = E[f(N_{i+1}, I_{i+1}) - f(N_i, I_i)|(N_i, I_i) = (n, j)]$$

is finite and $\eta_{s'} \leq -\epsilon$ for all $s' \in S'$ except perhaps a finite number.

Let $f(N_i, I_i) = N_i$. Then

$$\begin{aligned} \eta_{s'} &= E(N_{i+1} - N_i|N_i = n) \\ &= E[-B_{i+1} + \gamma_{i+1}|N_i = n], \quad \text{from (2.6.1).} \\ &= \frac{-n\theta}{\lambda + n\theta} + \rho \end{aligned}$$

Allowing $n \to \infty$ we get $\lim_{n\to\infty} \eta_{(n,j)} = -1 + \rho$.

The limit is negative if and only if $\rho < 1$. Thus $\rho < 1$ is sufficient condition for the positive recurrence of the Markov Chain.

To analyze the non ergodicity we use the Theorem 1 in Sennott et.al. [**62**]. The Markov chain $\{(N_i, I_i)\}$ is non ergodic if the mean drift is bounded below, $\eta_{s'} < \infty$ for all $s' \in S'$ and there exist an index n_0 such that $\eta_{s'} \geq 0$ for $n \geq n_0$. If $\rho \geq 1$ it is clear that $\eta_{s'} \geq 0$ for $n \geq 1$. Further more, in this model the mean down drift is bounded below since $N_{i+1} - N_i \geq -1$. Hence the proof. $\square$

THEOREM 2.6.2. *The system state distribution has a product form solution given by*

$$
\begin{aligned}
y_{nj} &= \lim_{i\to\infty} P(N_i = n, I_i = j) \\
&= y_n \frac{1}{Q},\ n \geq 0,\ s \leq j \leq S-1, Q = S - s
\end{aligned}
$$

where N_i = *Number of customers in the orbit immediately after the* i^{th} *service completion and* I_i *is the corresponding inventory level.* y_n *is the stationary probability that there are* n *customers in the* $M|G|1$ *retrial queue.*

PROOF. Let $\mathbf{x} = (x(0), x(1), \ldots)$ be the stationary probability vector associated with the Markov chain where $x(n) = (y_{ns}, y_{n,s+1}, \ldots, y_{n,S-1})$, $n \geq 0$. The stationary probabilities are given by the unique solution to $\mathbf{x} = \mathbf{x}P$ and $\mathbf{x}e = 1$ where e is the column vector with all entries equal to 1. That is

$$
\begin{aligned}
x(0)A_{00} + x(1)A_{10} &= x(0) \\
x(0)A_{01} + x(1)A_{11} + x(2)A_{21} &= x(1) \\
x(0)A_{02} + x(1)A_{12} + x(2)A_{22} + x(3)A_{32} &= x(2) \\
\ldots\ldots
\end{aligned}
\tag{2.6.2}
$$

Substituting
$$
\begin{aligned}
x(n) &= (y_{ns}, y_{n,s+1}, \ldots, y_{n,S-1}),\ n \geq 0, \\
&= y_n\left(\frac{1}{Q}, \frac{1}{Q}, \ldots, \frac{1}{Q}\right) \\
&= y_n \frac{1}{Q}(1, \ldots, 1) \\
&= y_n \frac{1}{Q} e \text{ where } e = (1, \ldots, 1) \text{ in (2.6.2)}
\end{aligned}
$$

we get the solution which turns out to be unique due to normalizing condition. Here y_n, $n \geq 0$ is the stationary probabilities that there are n customers at a departure epoch and hence at arbitrary epoch in an $M|G|1$ retrial queue. Then the stationary probabilities of the system at departure epoch is given by $y_{nj} = y_n \frac{1}{Q}$ for $n \geq 0$, $s \leq j \leq S-1$. $\square$

THEOREM 2.6.3. *For the $M|G|1$ retrial queue distribution of the number of customers in the orbit at departure epoch is same as that of the number of customers at arbitrary epoch. Hence we have* $\lim_{t\to\infty} P(N(t) = n, I(t) = j) = y_n \frac{1}{Q}$, $n \geq 0$, $s \leq j \leq S-1$.

2.6.2. Generating function. Let $\phi(z,x)$ be the generating function of y_{nj} defined by

$$\begin{aligned} \phi(z,x) &= \sum_{j=s}^{S-1} \sum_{n=0}^{\infty} z^n x^j y_{nj} \\ &= \frac{1}{Q} \sum_{j=s}^{S-1} x^j \phi(z) \end{aligned}$$

where $\phi(z)$ is the generating function of the stationary distribution y_n of the $M|G|1$ retrial queue.

2.7. System Performance Measures

(1) Average inventory size EI is given by

$$\mathrm{EI} = \sum_{n=0}^{\infty} \sum_{j=s}^{S-1} j y_{nj} = \frac{S+s-1}{2}$$

(2) Expected number of customers EC in the orbit is given by

$$\begin{aligned} \mathrm{EC} &= \sum_{n=1}^{\infty} n x(n) e = \sum_{n=1}^{\infty} n y_n \frac{1}{Q} e \\ &= \frac{1}{Q} \sum_{n=1}^{\infty} n y_n = \frac{1}{Q} \text{ (Expected number of customers} \\ &\qquad\qquad \text{in the } M|G|1 \text{ retrial queue).} \end{aligned}$$

(3) Expected cycle length from replenishment to replenishment EG is given by= E[time for Q services] $+$ E [duration of time the server is idle in between Q services]

$$= \beta_1 Q + [\frac{1}{\lambda + (\mathrm{EC})\theta}]Q.$$

2.8. Cost Function

To construct the cost function we define the costs as follows:

Let c_1 = procurement cost/unit

c_2 = holding cost of inventory/unit/unit time.

Then expected total cost function $F(s, Q)$ is

$$F(s,Q) = \frac{C + c_1 Q}{\mathrm{EG}} + c_2 \mathrm{EI}$$
$$= \frac{C + c_1 Q}{[\beta_1 + \frac{1}{\lambda + (\mathrm{EC})\theta}]Q} + c_2 \mathrm{EI}$$

2.9. Numerical Illustration of Model I

The following tables show the effect of parameters on some performance measures.

Variations in arrival rate λ

λ	ORR	SRR	EWT
2.0	5.333333	1.333333	2.666667
2.1	6.533333	1.470000	3.111111
2.2	8.066667	1.613333	3.666667
2.3	10.076190	1.763333	4.380952
2.4	12.800000	1.920000	5.333333
2.5	16.666667	2.083333	6.666667
2.6	22.533333	2.253333	8.666667
2.7	32.400000	2.430000	12.000000
2.8	52.266667	2.613333	18.666667

TABLE 2.1. $\mu = 3$, $\theta = 1$

Variations in service rate μ

μ	ORR	SRR	EWT
3.0	5.333333	1.333333	2.666667
3.1	4.809384	1.290323	2.404692
3.2	4.375000	1.250000	2.187500
3.3	4.009324	1.212121	2.004662
3.4	3.697479	1.176471	1.848739
3.5	3.428571	1.142857	1.714286
3.6	3.194444	1.111111	1.597222
3.7	2.988871	1.081081	1.494436
3.8	2.807018	1.052632	1.403509
3.9	2.645074	1.025641	1.322537

TABLE 2.2. $\lambda = 2, \theta = 1$

Variations in retrial rate θ

θ	ORR	EWT
1.5	6.000000	2.000000
1.6	6.133333	1.916667
1.7	6.266667	1.843137
1.8	6.400000	1.777778
1.9	6.533333	1.719298
2.0	6.666667	1.666667
2.1	6.800000	1.619048
2.2	6.933333	1.575758
2.3	7.066667	1.536232
2.4	7.200000	1.500000

TABLE 2.3. $\lambda = 2, \mu = 3$

2.9.1. Interpretations of the numerical results in the tables. In table 2.1, as the arrival rate λ increases the number of customers in the orbit becomes larger so that the overall retrial rate, successful retrial rate and expected waiting time increase. As the service rate μ increases the customers will be served more rapidly so that the number of customers in the orbit gets decreased and as a consequence the overall retrial rate, successful retrial rate and expected waiting time will decrease (see table 2.2). Table 2.3 indicates that as the retrial rate increases the overall retrial rate increases and the expected waiting time decreases.

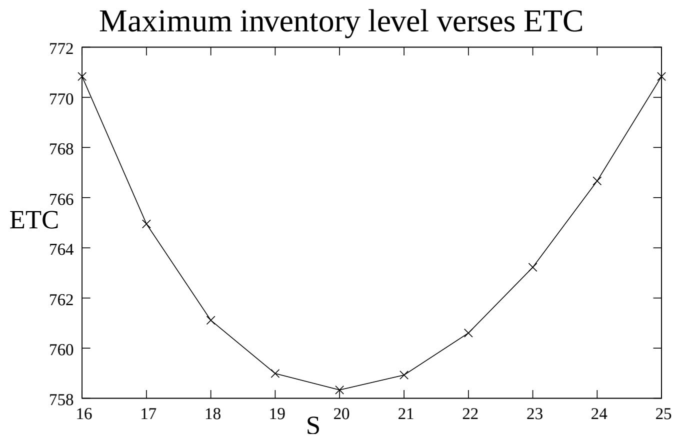


FIGURE 2.1. $\lambda = 5$, $\mu = 6$, $C = 1000$, $c_1 = 50$, $c_2 = 25$

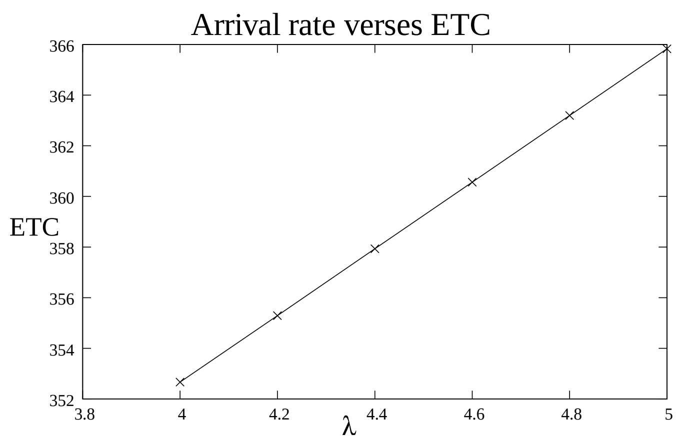


FIGURE 2.2. $S = 25$, $C = 1000$, $c_1 = 50$, $\mu = 6$, $c_2 = 25$

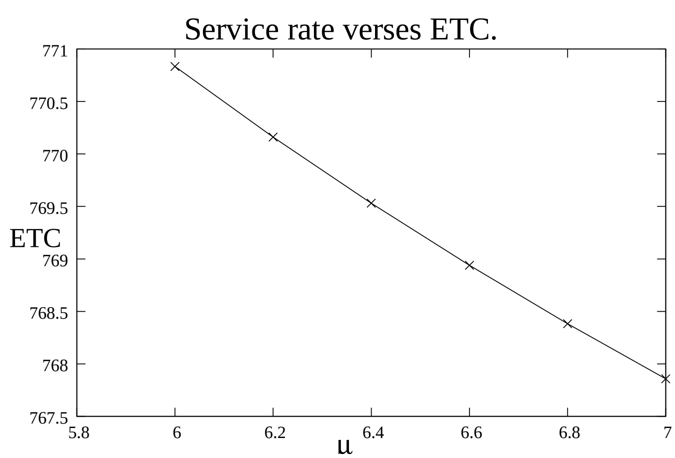


FIGURE 2.3. $S = 25$, $C = 1000$, $c_1 = 50$, $\lambda = 5$, $c_2 = 25$

2.9.2. Interpretation of the Graphs. The average cost per unit time, ETC is shown in the figure 2.1 for various values of S and for the given input parameters. The cost decreases with increasing values of S, attains a minimum and then increases. Figure 2.2 shows that as the arrival rate λ increases the cost also increases. From figure 2.3 we conclude that as the service rate μ increases the cost decreases.

CHAPTER 3

Comparison of Some Inventory Models Involving Positive Service Time

3.1. Introduction

In the previous chapter we discussed two retrial inventory systems with positive service time and zero lead time. In this chapter we propose to compare a few classical queueing models with inventory where the service time and lead time are positive. This is done by introducing what we call 'local purchase' at a demand epoch while stock is out. In an inventory system if the lead time is positive shortages of item may occur. At that time the newly arriving customer may or may not join the system. If he joins his waiting time will increase which increases the holding cost of the customer. If he leaves it is a loss to the system. In order to minimize the loss we adopt the method of local purchase at a higher cost, if a customer arrives when the inventory is zero. Krishnamoorthy and Raju [**39**] introduced N-policy to the (s, S) inventory system with positive lead time and local purchase when the inventory level is zero. They assumed that the service time is negligible.

The assumptions of this chapter are as follows: Arrival of customers to a single server system form a Poisson process with rate λ and service times are exponentially distributed with parameter μ. Each customer demands one unit of commodity. When the inventory level depletes to s we place an order for fixed quantity $Q = S - s$. The lead time follows an exponential distribution with parameter β. In Model I, we assume that customers do not join the system when the inventory level is zero. In Model II, customers are assumed to join the system even when the inventory level is zero. In the following models we make local purchase of the commodity, if a customer arrives when the inventory level is zero in order to cut short the waiting time of customers. Local purchases are made at a higher cost. Local purchase is assumed to be instantaneous. In models III and 1V local purchases are assumed to be made for one unit and s units of inventory, respectively, if a customer enters for service while the inventory level is zero. Under the same situation in model V we assume that a local purchase of S units is made resulting in cancellation of the existing order as the maximum capacity of inventory is S.

3.2. Mathematical Modelling of Model I

Customers arrive to the single server system according to a Poisson process of rate λ. Service times are exponentially distributed with parameter μ. We follow an (s, S) inventory system. The lead time is exponentially distributed with parameter β. Customers do not join the system when the inventory level is zero. Let $N(t)$ be the number of customers in the system and $I(t)$ be the corresponding inventory level at time t. Then $\{(N(t), I(t)), t \geq 0\}$ is a LIQBD process with the state space $\{(i, j), 0 \leq j \leq S : i \geq 0\}$. The infinitesimal generator $\overline{Q}$ of the process has the following form.

$$\overline{Q} = \begin{bmatrix} A_{00} & A_0 & 0 & 0 & \cdots & \cdots \\ A_2 & A_1 & A_0 & 0 & \cdots & \cdots \\ 0 & A_2 & A_1 & A_0 & \cdots & \cdots \\ 0 & 0 & A_2 & A_1 & A_0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \tag{3.2.1}$$

where A_{00}, A_2, A_1, A_0 are square matrices of order $(S + 1)$ and they are given by

$$A_0 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 0 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ & \cdots & & & \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \mu & 0 & \cdots & 0 & 0 \\ 0 & \mu & \cdots & 0 & 0 \\ \cdots & & \cdots & & \\ 0 & 0 & \cdots & \mu & 0 \end{bmatrix}$$

$$
A_{00} = \begin{array}{cc}
 & \begin{array}{ccccccccc} 0 & 1 & \cdots & s & s+1 & \cdots & S-s & \ldots & S \end{array} \\
\begin{array}{c} 0 \\ 1 \\ \vdots \\ s \\ s+1 \\ \vdots \\ \vdots \\ S \end{array} &
\begin{pmatrix}
-\beta & & & & & & \beta & & \\
 & -(\lambda+\beta) & \cdots & & & & & & \\
 & & & & & & & \ddots & \\
 & & \cdots & -(\lambda+\beta) & & & & & \beta \\
 & & & & -\lambda & & & & \\
 & & & & & & \ddots & & \\
 & & & & & & & \ddots & \\
 & & & & & & & & -\lambda
\end{pmatrix}
\end{array}
$$

$$
A_{1} = \begin{array}{cc}
 & \begin{array}{ccccccccc} 0 & 1 & \cdots & s & s+1 & \cdots & S-s & \ldots & S \end{array} \\
\begin{array}{c} 0 \\ 1 \\ \vdots \\ s \\ s+1 \\ \vdots \\ \vdots \\ S \end{array} &
\begin{pmatrix}
-\beta & & & & & & \beta & & \\
 & -\omega & \cdots & & & & & & \\
 & & & & & & & \ddots & \\
 & & \cdots & -\omega & & & & & \beta \\
 & & & & -\Omega & & & & \\
 & & & & & & \ddots & & \\
 & & & & & & & \ddots & \\
 & & & & & & & & -\Omega
\end{pmatrix}
\end{array}
$$

where $\omega = \lambda + \beta + \mu$, $\Omega = \lambda + \mu$.

3.3. Mathematical Modelling of Model II

The only difference of this model from the first is that customers join the system even when the inventory level is zero. Here also $\{(N(t), I(t)), t \geq 0\}$ is a LIQBD process on the state space $\{(i, j), 0 \leq j \leq S, i \geq 0\}$. Then the generator has the form (3.2.1) where the blocks A_{00}, A_2, A_1, A_0 are square matrices of the same order $(S+1)$ and they are given by

$$
A_{00} = \begin{bmatrix} -(\lambda+\beta)I_{s+1} & E_1 \\ 0 & -\lambda I_{S-s} \end{bmatrix} \text{ where } E_1 = [0 \ \beta I_{s+1}]_{(s+1)\times(S-s)}
$$

$$A_0 = \lambda I_{S+1},\, A_2 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \mu & 0 & \cdots & 0 & 0 \\ 0 & \mu & \cdots & 0 & 0 \\ \cdots & & \cdots & & \cdots \\ \cdots & & \cdots & & \cdots \\ 0 & 0 & \cdots & \mu & 0 \end{bmatrix}$$

$$A_1 = \begin{array}{cc} & \begin{array}{ccccccccc} 0 & 1 & \cdots & s & s+1 & \cdots & S-s & \ldots & S \end{array} \\ \begin{array}{c} 0 \\ 1 \\ \vdots \\ s \\ s+1 \\ \vdots \\ \vdots \\ S \end{array} & \begin{pmatrix} -\Delta & & & & & & \beta & & \\ & -\omega & \cdots & & & & & & \\ & & & & & & & \ddots & \\ & & \cdots & -\omega & & & & & \beta \\ & & & & -\Omega & & & & \\ & & & & & & \ddots & & \\ & & & & & & & \ddots & \\ & & & & & & & & -\Omega \end{pmatrix} \end{array}$$

with $\omega = \lambda + \beta + \mu$, $\Omega = \lambda + \mu$, $\Delta = \lambda + \beta$.

3.4. Analysis of Models I and II

3.4.1. System Stability. Define the generator matrix A (for each model) as $A = A_0 + A_1 + A_2$. Let $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots, \pi_S)$ be the stationary probability vector associated with the matrix A where $\boldsymbol{\pi}A = 0$ and $\boldsymbol{\pi}e = 1$. Solving $\boldsymbol{\pi}A = 0$ we get

$$\pi_k = \begin{cases} \left(\frac{\mu+\beta}{\mu}\right)^{k-1} \frac{\beta}{\mu}\pi_0, & k = 1, \ldots, s \\ \left(\frac{\mu+\beta}{\mu}\right)^{s} \frac{\beta}{\mu}\pi_0, & k = s+1, \ldots, S-s \end{cases}$$

$$\pi_{S-s+k} = \frac{\beta}{\mu}\left[\left(\frac{\mu+\beta}{\mu}\right)^{s} - \left(\frac{\mu+\beta}{\mu}\right)^{k-1}\right]\pi_0, \quad k = 1, \ldots, s$$

π_0 can be evaluated from

$$\boldsymbol{\pi}e = 1 \text{ and } \pi_0 = \left[1 + (S-s)\left(\frac{\mu+\beta}{\mu}\right)^{s} \frac{\beta}{\mu}\right]^{-1} \tag{3.4.1}$$

THEOREM 3.4.1. *The system is stable if and only if* $\lambda < \mu$ *for model I and* $\lambda < \mu(1-\pi_0)$ *for model II, where* (3.4.1) *gives* π_0.

PROOF. From the well-known result (Neuts [**53**]) on positive recurrence of $\overline{Q}$, which states that $\boldsymbol{\pi} A_0 e < \pi A_2 e$, the result follows. □

3.4.2. Computation of the steady-state probability vector of $\overline{Q}$.

Let $X = (x(0), x(1), \ldots)$ be the stationary probability vector associated with $\overline{Q}$, where $x(i)$ is the probability vector associated with level i. Then $X\bar{Q} = 0$ and $Xe = 1$.

It is well known that

$$x(i) = x(1)R^{i-1} \quad \text{for } i \geq 2 \tag{3.4.2}$$

where R is the minimal non-negative solution of the matrix equation $A_0 + RA_1 + R^2 A_2 = 0$.

$X\bar{Q} = 0$ gives

$$x(0)A_{00} + x(1)A_2 = 0 \tag{3.4.3}$$

$$x(0)A_0 + x(1)(A_1 + RA_2) = 0 \tag{3.4.4}$$

The vectors $x(0)$ and $x(1)$ can be obtained by solving the above equations subject to the normalizing condition $Xe = 1$.Then $x(i)$, $i \geq 2$ can be obtained from (3.4.2).

3.5. System Performance Measures

Let $X = (x(0), x(1), \ldots)$ be the steady-state probability vector of $\overline{Q}$ (for each model) and $x(i)$, $i \geq 0$ is partitioned as $x(i) = (y_{i0}, y_{i1}, \ldots, y_{iS})$. Then we have the following expressions for the performance measures.

(1) Expected number of customers, EC in the system is given by

$$\text{EC} = \sum_{i=1}^{\infty} i x(i) e$$

(2) Expected inventory level EI is given by

$$\text{EI} = \sum_{i=0}^{\infty} \sum_{j=1}^{S} j y_{ij}$$

(3) Expected re-order rate ER is given by

$$\text{ER} = \mu \sum_{i=1}^{\infty} y_{i,s+1}$$

(4) Expected rate of departure ED after completing service is given by

$$\text{ED} = \mu \sum_{i=0}^{\infty} \sum_{j=1}^{S} y_{ij}$$

Model I

(5) Expected waiting time in the system EW is given by

$$\text{EW} = \frac{\text{EC}}{\lambda}.$$

Model II

(6) Expected waiting time in the system EW is given by

$$\text{EW} = \frac{\text{EC}}{\lambda[1 - \sum_{i=0}^{\infty} y_{i0}]}.$$

Model I

(7) Expected number of customers EJ not joining the system when the inventory level is zero is given by

$$\text{EJ} = \lambda \sum_{i=0}^{\infty} y_{i0}.$$

3.6. Cost Function

To construct the cost function we define the following costs :

C = fixed ordering cost

C_1 = procurement cost/unit

C_2 = holding cost of inventory/unit/unit time

$C_3 =$ holding cost of customer/unit/unit time

$C_4 =$ cost due to loss of customer/unit/unit time

In terms of these we define the expected total cost function ETC for each model as follows.

Model I

$\text{ETC} = [C + QC_1]\text{ER} + C_2\text{EI} + C_3\text{EC} + C_4\text{EJ}$

Model II

$\text{ETC} = [C + QC_1]\text{ER} + C_2\text{EI} + C_3\text{EC}$

3.7. Mathematical Modelling of Model III

In addition to the assumption in second model, here it is assumed that a local purchase of one unit of item is made at a higher cost, if a customer enters for service when the inventory is zero. Let $N(t)$ be the number of customers in the system and $I(t)$ be the corresponding inventory level at time t. Then $\{(N(t), I(t)), t \geq 0\}$ is a LIQBD process with the state space $\{(0, j), 0 \leq j \leq S\} \cup \{(i, j), 1 \leq j \leq S, i \geq 1\}$. The infinitesimal generator $\overline{Q}$ of the process has the following form

$$\overline{Q} = \begin{bmatrix} A_{00} & A_{01} & 0 & 0 & \cdots \\ A_{10} & A_1 & A_0 & 0 & \cdots \\ 0 & A_2 & A_1 & A_0 & \cdots \\ \vdots & & \vdots & & \end{bmatrix} \tag{3.7.1}$$

where A_{00} is a square matrix of order $(S+1)$. A_{01} is of order $(S+1) \times S$, A_{10} of order $S \times (S+1)$. A_0, A_1, A_2 are square matrices of order S and they are given by

$$A_{00} = \begin{bmatrix} -(\lambda+\beta)I_{s+1} & E_1 \\ 0 & -\lambda I_{S-s} \end{bmatrix} \text{ where } E_1 = [0 \quad \beta I_{s+1}]_{(s+1)\times(S-s)}$$

$$A_{01} = \begin{bmatrix} \lambda e_1 \\ \lambda I_S \end{bmatrix} \text{ where } e_j \text{ is a row vector with ‘1’ in the } j\text{th place and zeros elsewhere.}$$

$$A_{10} = [\mu I_S \quad 0]; A_1 = \begin{bmatrix} -(\lambda+\beta+\mu)I_s & E_2 \\ 0 & -(\lambda+\mu)I_{S-s} \end{bmatrix}$$

where $E_2 = [0 \quad \beta I_s]_{s\times(S-s)}$; $A_0 = \lambda I_S$ and

$$A_2 = \begin{bmatrix} \mu & 0 & \dots & 0 & 0 \\ \mu & 0 & \dots & 0 & 0 \\ 0 & \mu & \dots & 0 & 0 \\ \dots & & \dots & & \dots \\ \dots & & \dots & & \dots \\ 0 & 0 & \dots & \mu & 0 \end{bmatrix}.$$

3.8. Mathematical Modelling of Model IV

Apart from the third model here we make a local purchase of s units as a new customer arrives for service when the inventory is zero. $\{(N(t), I(t)); t \geq 0\}$ is a LIQBD process on the state space $\{(0,j), 0 \leq j \leq S\} \cup \{(i,j), 1 \leq j \leq S, i \geq 1\}$.

The infinitesimal generator $\overline{Q}$ of the process has the form (3.7.1) where, A_{00}, A_{10}, A_1 ,A_0 are the same as in Model III, and A_{01} and A_2 are given by $A_{01} = \begin{bmatrix} \lambda e_s \\ \lambda I_S \end{bmatrix}$ where e_j is a row vector with '1' in the jth place and zeros elsewhere, and

$$A_2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ S \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & \cdots & s & \cdots & S-1 & S \end{array} \\ \begin{pmatrix} 0 & 0 & \cdots & \mu & \cdots & 0 & 0 \\ \mu & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & \mu & \cdots & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & & \cdots & & & \\ 0 & 0 & \cdots & 0 & \cdots & \mu & 0 \end{pmatrix} \end{array}.$$

3.9. Mathematical Modelling of Model V

The main difference of this model from those indicated in III, IV is that, in the present one we make a local purchase to bring the level back to S, whenever a customer arrives to an idle server with no inventory or at an epoch of departure of a customer resulting in zero inventory, but one or more customers in the queue. The existing replenishment order is cancelled. This is done so as to ensure that replenishment does not take place until the inventory on hand again goes down to s, as otherwise the on hand inventory may exceed the maximum permissible. Here also $\{(N(t), I(t)), t \geq 0\}$ is a Level Independent Quasi-Birth Death process on the state space $\{(0, j), 0 \leq j \leq S\} \cup \{(i, j), 1 \leq j \leq S, i \geq 1\}$. The infinitesimal generator $\overline{Q}$ of the process has the form of (3.7.1) where all the matrices except A_{01} and A_2 have the same form as in Model III.

$A_{01} = \begin{bmatrix} \lambda e_S \\ \lambda I_S \end{bmatrix}$ where e_j is a row vector with '1' in the jth place and zeros else-

where, and $A_2 = \begin{bmatrix} 0 & 0 & \cdots & 0 & \mu \\ \mu & 0 & \cdots & 0 & 0 \\ 0 & \mu & \cdots & 0 & 0 \\ \cdots & & \cdots & & \cdots \\ \cdots & & \cdots & & \cdots \\ 0 & 0 & \cdots & \mu & 0 \end{bmatrix}_{S \times S}$.

3.10. Mathematical Analysis of Models III, IV and V

3.10.1. System Stability. Define the generator matrix (for each model) A as $A = A_0 + A_1 + A_2$, where A_0, A_1, A_2 are the corresponding matrices of each model. Let $\pi = (\pi_1, \pi_2, \ldots, \pi_S)$ be the stationary probability vector associated with A. Then we have $\pi A = 0$ and $\pi e = 1$. Solving $\pi A = 0$ we get the following values for each model.

Model III.

$$\pi_k = \begin{cases} \left(\frac{\mu+\beta}{\mu}\right)^{k-2} \frac{\beta}{\mu}\pi_1, & k = 2, \ldots, s \\ \left(\frac{\mu+\beta}{\mu}\right)^{s-1} \frac{\beta}{\mu}\pi_1, & k = s+1, \ldots, Q+1 \end{cases}$$

$$\pi_{Q+k} = \frac{\beta}{\mu}\left[\left(\frac{\mu+\beta}{\mu}\right)^{s-1} - \left(\frac{\mu+\beta}{\mu}\right)^{k-2}\right]\pi_1,\ k = 2, 3, \ldots, s$$

Model IV.

$$\pi_k = \begin{cases} \left(\frac{\mu+\beta}{\mu}\right)^{k-1} \pi_1, & k = 1, 2, \ldots, s \\ \left[\left(\frac{\mu+\beta}{\mu}\right)^{s} - 1\right] \pi_1, & k = s+1, \ldots, Q+1 \end{cases}$$

$$\pi_{Q+k} = \left[\left(\frac{\mu+\beta}{\mu}\right)^{s} - \left(\frac{\mu+\beta}{\mu}\right)^{k-1}\right] \pi_1,\ k = 2, 3, \ldots, s$$

Model V.

$$\pi_k = \begin{cases} \left(\frac{\mu+\beta}{\mu}\right)^{k-1} \pi_1, & k = 1, 2, \ldots, s \\ \left(\frac{\mu+\beta}{\mu}\right)^{s} \pi_1, & k = s+1, \ldots, Q+1 \end{cases}$$

$$\pi_{Q+k} = \left[\left(\frac{\mu+\beta}{\mu}\right)^{s} - \frac{\beta}{\mu}\sum_{m=0}^{k-2}\left(\frac{\mu+\beta}{\mu}\right)^{m}\right] \pi_1,\ k = 2, \ldots, s$$

Using the noramlising condition $\pi e = 1$ we get π_1 and hence π_2, $\pi_3 \ldots, \pi_S$. Here $Q = S - s$

THEOREM 3.10.1. *The system in each model is stable if and only if $\lambda < \mu$.*

PROOF. For the positive recurrence of $\overline{Q}$ we have the well known results of Neuts (see [**53**]) which states that $\pi A_0 e < \pi A_2 e$; simplifying we get $\lambda < \mu$. □

3.10.2. Computation of the steady-state probability vectors of $\overline{Q}$.

Let $X = (x(0), x(1), \ldots)$ be the stationary probability vector associated with $\overline{Q}$ where $x(i)$ is the probability vector associated with level i. Then $X\overline{Q} = 0$ and $Xe = 1$. It is well known that

$$x(i) = x(1)R^{i-1} \text{ for } i \geq 2 \tag{3.10.1}$$

where R is the minimal non-negative solution of the matrix equation $A_0 + RA_1 + R^2A_2 = 0$. $x(0)$ and $x(1)$ are calculated from the equations

$$x(0)A_{00} + x(1)A_{10} = 0 \tag{3.10.2}$$

$$x(0)A_{01} + x(1)(A_1 + RA_2) = 0 \tag{3.10.3}$$

with the normalizing condition $Xe = 1$.

That is, $x(0)e + x(1)(1 - R)^{-1} = 1$. Then $x(i)$, $(i \geq 2)$ can be found from (3.10.1)

3.11. System Performance Measures

Let $X = (x(0), x(1), \ldots)$ be the steady-state probability vector of $\overline{Q}$ (for each model) and $x(i)$, $i \geq 0$, be partitioned as $x(i) = (y_{i0}, y_{i1}, \ldots, y_{iS})$. Then we have the following expressions for the performance measures.

(1) Expected number of customers, EC in the system is given by

$$\text{EC} = \sum_{i=1}^{\infty} i x(i) e$$

(2) Expected inventory level EI is given by

$$\text{EI} = \sum_{i=0}^{\infty} \sum_{j=1}^{S} j y_{ij}$$

(3) Expected re-order rate ER is given by

$$\text{ER} = \mu \sum_{i=1}^{\infty} y_{i,s+1}$$

(4) Expected rate of departure ED after completing service is given by

$$\text{ED} = \mu \sum_{i=0}^{\infty} \sum_{j=1}^{S} y_{ij}$$

(5) Expected waiting time in the system EW is given by

$$\text{EW} = \frac{\text{EC}}{\lambda}.$$

(6) Expected rate of local purchase EL is given by

$$\text{EL} = \lambda y_{00} + \mu \sum_{i=2}^{\infty} y_{i1}.$$

In expressions under (1) to (6) above, it may be noted that the actual values differ for the three models.

3.12. Cost Function and Numerical Examples

To construct the cost function we define the following costs :

C = fixed ordering cost

C_1 = procurement cost/unit

C_2 = holding cost of inventory/unit/unit time

C_3 = holding cost of customer/unit/unit time

$(1 + k)C_1 zEL$=total local purchase cost of z units of inventory with a hike of k times C_1/unit.

In model V as we make a local purchase of S units and thus cancelling the existing order,the system losses the ordering cost already paid and $(ER - EL)$=the remaining rate of ordering inventory.

In terms of these we define the expected total cost function ETC for each model as follows.

Model III

$\text{ETC} = [C + QC_1]\text{ER} + C_2\text{EI} + C_3\text{EC} + (1 + l)C_1 EL$

Model IV

$\text{ETC} = [C + QC_1]\text{ER} + C_2\text{EI} + C_3\text{EC} + (1 + m)s \times C_1 EL$

Model V

$\text{ETC} = C\ \text{ER} + QC_1[\text{ER} - \text{EL}] + C_2\text{EI} + C_3\text{EC} + (1 + n)S \times C_1 \times EL$ where l, m, n are proper fractions and $l > m > n > 0$, as we know that when we make local purchase in large quantities, the hike in price decreases.

S	EC		EI		EW		ER	
	Models							
	I	II	I	II	I	II	I	II
25	9.0	19.5869	14.4906	13.1847	2.0	4.5294	0.3189	0.2918
27	9.0	17.3797	15.5136	14.4835	2.0	4.0023	0.2827	0.2653
29	9.0	15.9311	16.5310	15.6790	2.0	3.6558	0.2539	0.2419
31	9.0	14.9078	17.5442	16.8153	2.0	3.4108	0.2304	0.2218
33	9.0	14.1473	18.5545	17.9147	2.0	3.2288	0.2108	0.2045
35	9.0	13.5599	19.5624	18.9896	2.0	3.0881	0.1944	0.1895
37	9.0	13.0926	20.5685	20.0476	2.0	2.9763	0.1803	0.1765
39	9.0	12.7120	21.5732	21.0934	2.0	2.8852	0.1681	0.1650

S	EC			EI			ER		
	Models								
	III	IV	V	III	IV	V	III	IV	V
25	9.0	9.0	9.0	12.1954	12.6673	13.1607	0.2847	0.2709	0.2757
27	9.0	9.0	9.0	13.1151	13.5736	14.0745	0.2527	0.2417	0.2456
29	9.0	9.0	9.0	14.0299	14.4779	14.9856	0.2270	0.2181	0.2215
31	9.0	9.0	9.0	14.9412	15.3806	15.8948	0.2061	0.1988	0.2016
33	9.0	9.0	9.0	15.8498	16.2821	16.8024	0.1887	0.1825	0.1850
35	9.0	9.0	9.0	16.7563	17.1826	17.7088	0.1740	0.1688	0.1709
37	9.0	9.0	9.0	17.6591	18.0825	18.6144	0.1615	0.1569	0.1589
39	9.0	9.0	9.0	18.5628	18.9817	19.5192	0.1506	0.1467	0.1484

TABLE 3.1. Variations in maximum inventory level S. $\lambda = 4.5$, $\mu = 5$, $\beta = 1$, $s = 10$

s	EC		EI		EW		ER	
	Models							
	I	II	I	II	I	II	I	II
5	9.0	16.8534	25.1432	23.6914	2.0	3.8814	0.1067	0.1019
7	9.0	13.7409	25.8429	25.0709	2.0	3.1328	0.1128	0.1104
9	9.0	12.0304	26.6504	26.1997	2.0	2.7224	0.1192	0.1178
11	9.0	11.0033	27.5284	27.2480	2.0	2.4767	0.1260	0.1251
13	9.0	10.3537	28.4517	28.2692	2.0	2.3217	0.1333	0.1327
15	9.0	9.9293	29.4039	29.2809	2.0	2.2206	0.1413	0.1409
17	9.0	9.6456	30.3741	30.2889	2.0	2.1532	0.1501	0.1498
19	9.0	9.4529	31.3555	31.2950	2.0	2.1074	0.1600	0.1598

s	EI			EL			ER		
	Models								
	III	IV	V	III	IV	V	III	IV	V
5	21.8936	22.3451	23.4259	0.1886	0.0532	0.0346	0.0953	0.0936	0.0962
7	22.4611	22.9327	23.7708	0.1338	0.0316	0.0243	0.1010	0.0990	0.1007
9	23.1466	23.5784	24.2234	0.0948	0.0202	0.0171	0.1069	0.1048	0.1060
11	23.9098	24.2756	24.7741	0.0671	0.0135	0.0121	0.1131	0.1110	0.1120
13	24.7234	25.0170	25.4082	0.0476	0.0092	0.0086	0.1197	0.1178	0.1186
15	25.5693	25.7953	26.1102	0.0338	0.0064	0.0061	0.1269	0.1252	0.1260
17	26.4354	26.6033	26.8656	0.0240	0.0045	0.0043	0.1349	0.1334	0.1341
19	27.3138	27.4348	27.6617	0.0172	0.0032	0.0032	0.1439	0.1425	0.1433

TABLE 3.2. Variations in reorder level s. $\lambda = 4.5$, $\mu = 5$, $\beta = 1$, $S = 50$

β	EC I	EC II	EI I	EI II	EW I	EW II	EJ I
	Models						
1.6	9.0	14.4953	13.8410	13.3544	2.0	3.3229	0.1335
1.8	9.0	12.6046	14.0933	13.8111	2.0	2.8681	0.1017
2.0	9.0	11.5167	14.3020	14.1217	2.0	2.6063	0.0787
2.2	9.0	10.8326	14.4773	14.3543	2.0	2.4416	0.0617
2.4	9.0	10.3754	14.6266	14.5387	2.0	2.3316	0.0490
2.6	9.0	10.0565	14.7551	14.6901	2.0	2.2548	0.0393
2.8	9.0	9.8266	14.8670	14.8175	2.0	2.1995	0.0318
3.0	9.0	9.6567	14.9650	14.9267	2.0	2.1585	0.0260

β	EI III	EI IV	EI V	EL III	EL IV	EL V	EW III	EW IV	EW V
	Models								
1.6	11.6532	11.9495	12.4131	0.1744	0.0570	0.0454	2.0	2.0	2.0
1.8	11.8616	12.1112	12.4865	0.1371	0.0470	0.0389	2.0	2.0	2.0
2.0	12.0350	12.2468	12.5610	0.1094	0.0392	0.0334	2.0	2.0	2.0
2.2	12.1815	12.3624	12.6277	0.0883	0.0330	0.0288	2.0	2.0	2.0
2.4	12.3069	12.4623	12.6898	0.0721	0.0280	0.0249	2.0	2.0	2.0
2.6	12.4154	12.5495	12.7476	0.0594	0.0239	0.0216	2.0	2.0	2.0
2.8	12.5102	12.6267	12.8014	0.0494	0.0205	0.0189	2.0	2.0	2.0
3.0	12.5937	12.6952	12.8514	0.0414	0.0178	0.0165	2.0	2.0	2.0

TABLE 3.3. Variations in replenishment rate β. $\lambda = 4.5$, $\mu = 5$, $s = 5$, $S = 25$

μ	EC I	EC II	EW I	EW II	EI I	EI II	ER I	ER II
	Models							
5.5	4.5000	6.3838	1.0000	1.4731	13.7573	13.6536	0.2655	0.2662
6.0	3.0000	4.0148	0.6667	0.9264	13.7578	13.6630	0.2896	0.2916
6.5	2.2500	2.9435	0.5000	0.6791	13.7578	13.6612	0.3138	0.3167
7.0	1.8000	2.3330	0.4000	0.5382	13.7578	13.6590	0.3379	0.3418
7.5	1.5000	1.9385	0.3333	0.4472	13.7578	13.6567	0.3621	0.3669
8.0	1.2857	1.6628	0.2857	0.3835	13.7578	13.6543	0.3863	0.3921
8.5	1.1250	1.4592	0.2500	0.3365	13.7578	13.6520	0.4104	0.4172
9.0	1.0000	1.3027	0.2222	0.3004	13.7578	13.6498	0.4345	0.4423

μ	EC III	EC IV	EC V	EL III	EL IV	EL V	EW III	EW IV	EW V
	Models								
5.5	4.5000	4.5000	4.5000	0.1949	0.0622	0.0482	1.0000	1.0000	1.0000
6.0	3.0000	3.0000	3.0000	0.1916	0.0611	0.0474	0.6667	0.6667	0.6667
6.5	2.2500	2.2500	2.2500	0.1888	0.0603	0.0467	0.5000	0.5000	0.5000
7.0	1.8000	1.8000	1.8000	0.1864	0.0595	0.0461	0.4000	0.4000	0.4000
7.5	1.5000	1.5000	1.5000	0.1843	0.0589	0.0456	0.3333	0.3333	0.3333
8.0	1.2857	1.2857	1.2857	0.1825	0.0583	0.0452	0.2857	0.2857	0.2857
8.5	1.1250	1.1250	1.1250	0.1808	0.0578	0.0448	0.2500	0.2500	0.2500
9.0	1.0000	1.0000	1.0000	0.1794	0.0573	0.0444	0.2222	0.2222	0.2222

TABLE 3.4. Variations in service rate μ. $\lambda = 4.5$, $\beta = 1.5$, $s = 5$, $S = 25$

Variations in arrival rate λ

	EC		EI		ED		EW	
	Models							
λ	I	II	I	II	I	II	I	II
4.4	1.6923	2.1562	13.8099	13.7164	4.4000	4.3999	0.3846	0.5077
4.6	1.9166	2.5298	13.7060	13.6016	4.5999	4.5999	0.4166	0.5720
4.8	2.1818	2.9996	13.6038	13.4875	4.7999	4.7999	0.4545	0.6527
5.0	2.4999	3.6059	13.5031	13.3738	4.9999	4.9997	0.4999	0.7565
5.2	2.8888	4.4155	13.4041	13.2597	5.2000	5.1991	0.5555	0.8948
5.4	3.3750	5.5469	13.3065	13.1419	5.3999	5.3963	0.6250	1.0876
5.6	3.9999	7.2323	13.2103	13.0063	5.5999	5.5843	0.7142	1.3739
5.8	4.8333	9.9987	13.1150	12.7930	5.7995	5.7330	0.8333	1.8419

	EC			EL			EW		
	Models								
λ	III	IV	V	III	IV	V	III	IV	V
4.4	1.6923	1.6923	1.6923	0.1735	0.0559	0.0436	0.3846	0.3846	0.3846
4.6	1.9166	1.9166	1.9166	0.1997	0.0632	0.0486	0.4166	0.4166	0.4166
4.8	2.1818	2.1818	2.1818	0.2279	0.0709	0.0537	0.4545	0.4545	0.4545
5.0	2.4999	2.4999	2.4999	0.2583	0.0792	0.0591	0.4999	0.4999	0.4999
5.2	2.8888	2.8888	2.8888	0.2908	0.0878	0.0646	0.5555	0.5555	0.5555
5.4	3.3750	3.3750	3.3750	0.3254	0.0970	0.0703	0.6250	0.6250	0.6250
5.6	3.9999	3.99998	3.9999	0.3622	0.1065	0.0762	0.7142	0.7142	0.7142
5.8	4.8333	4.8333	4.8333	0.4010	0.1165	0.0822	0.8333	0.8333	0.8333

TABLE 3.5. $\mu = 7$, $\beta = 1.5$, $s = 5$, $S = 25$

3.12.1. Numerical interpretation of the tables.

1. *Effect of the maximum inventory level S on various performance measures.*

   Table 3.1 shows that as S increases inventory level increases in all models. Due to the presence of more inventory the number of customers and hence the waiting time, decreases in Model II. Number of customers and waiting time is same in Model I as customers do not join when the inventory level is zero. In Models III, IV and V, due to local purchase there is no change in the number of customers. In all models the time interval to reach the reorder point increases due to more inventory and thus reorder rate decreases.

2. *Effect of the reorder level s on various performance measures.*

   From table 3.2 we realise that the behaviour of the system performance measures as s increases is similar to that corresponding to S, except that the reorder rate increases, the time interval to reach the reorder point decreases and so more orders are placed. The rate of local purchase in models III, IV and V decrease when s increases due to the availability of more inventory with the system.

3. *Effect of the replenishment rate β on various performance measures.*

   When β increases as we expect the inventory level increases in all models. In model II the number of customers and the waiting time decrease as inventory is available more quickly. In model I no change for number of customers and waiting time, while the number of customers do not join when the inventory level is zero decreases due to the availability of more inventory with the system. The same reasoning can be given for the decrease in the rate of local purchase in models III, IV and V. In model III, IV and V, the waiting time is not affected by the replenishment rate, as we make local purchase if a customer arrives when the inventory is zero (see table 3.3).

4. *Effect of the service rate μ on various performance measures.*

   Table 3.4 shows that as the service rate increases the number of customers and their waiting time decreases in all models. Re-order rate increases and the inventory level does not change in models I and II. Local purchase decreases in models III, IV and V.

5. *Effect of the arrival rate λ on various performance measures.*

   As the arrival rate increases the number of customers and their waiting time increase in all models. Inventory level decreases and the expected number of departure increases as λ increases in models I and II. Expected number of local purchase increases in models III, IV and V due to increased arrival (see table 3.5)

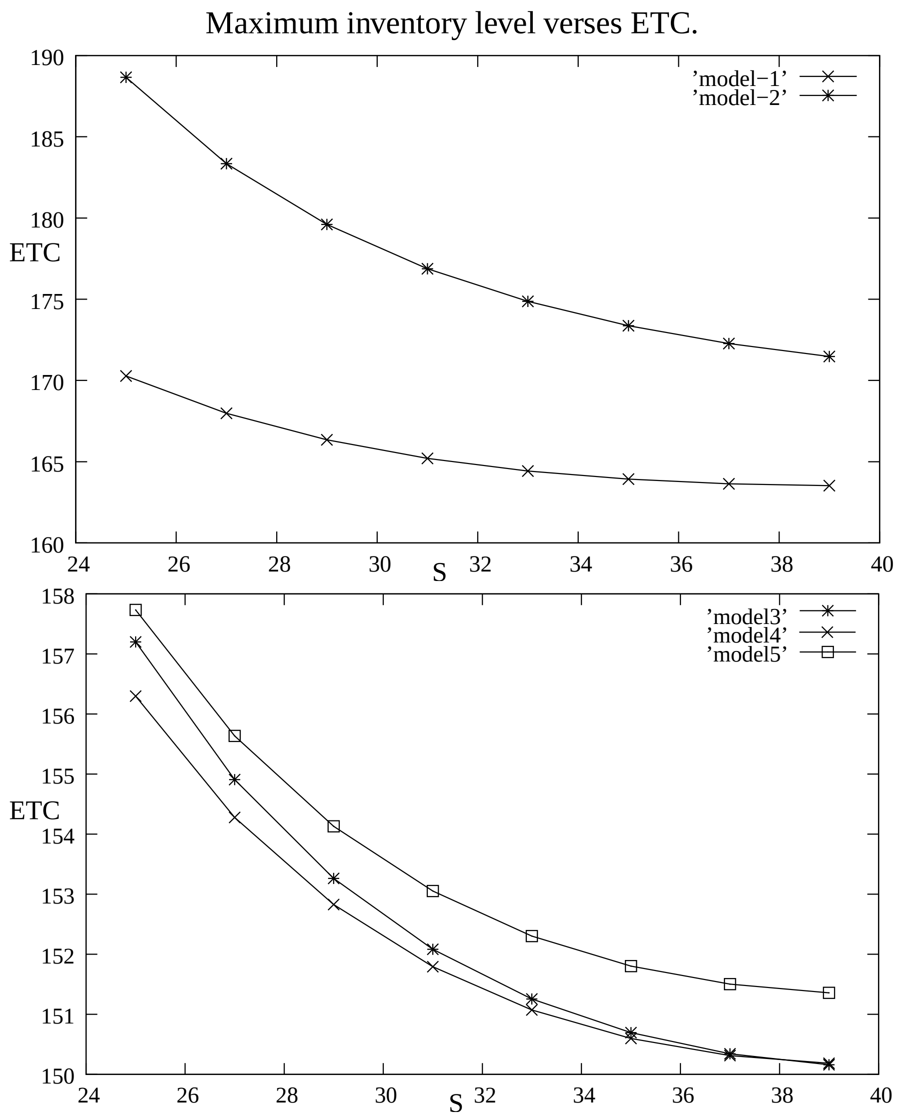


FIGURE 3.1. $\lambda = 4.5$, $\mu = 5$, $\beta = 1$, $s = 10$, $C = 100$, $C_1 = 20$, $C_2 = 1$, $C_3 = 3$, $C_4 = 7$

3.12.2. Interpretation of the graphs. The objective is to compare the five models and identify the one which is more profitable. For this, we compute the expected total cost per unit time by varying the parameters one at a time, keeping others fixed. From figure 3.1 we observe that as S increases the expected cost decreases, this can be attributed to the decrease in reorder rate. Figure 3.2 shows that the cost function is convex in s for model II, for all other models it increases as s increases. As β increases

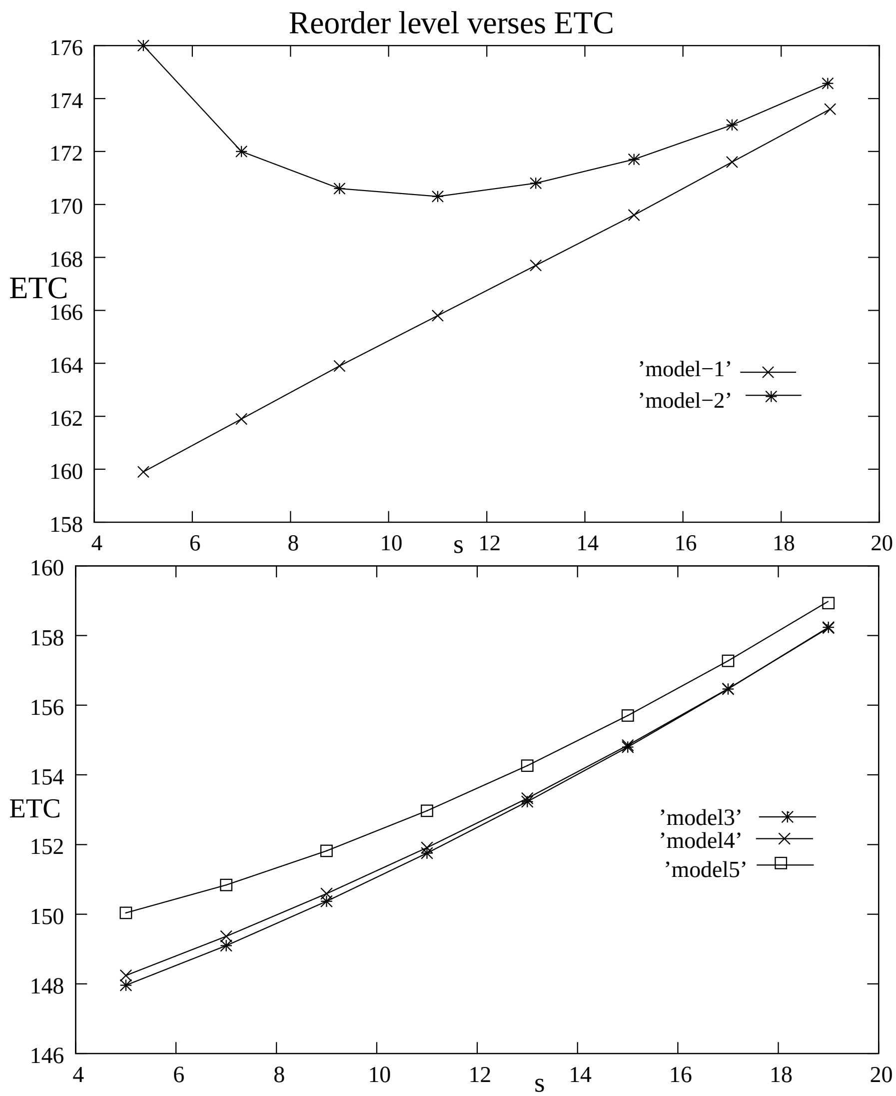


FIGURE 3.2. $\lambda = 4.5$, $\mu = 5$, $\beta = 1$, $S = 50$, $C = 100$, $C_1 = 20$, $C_2 = 1$, $C_3 = 3$, $C_4 = 7$

the expected cost for all models, except model II increases (see figure 3.3). Figure 3.4 shows that as the arrival rate increases the cost also increases in all models.

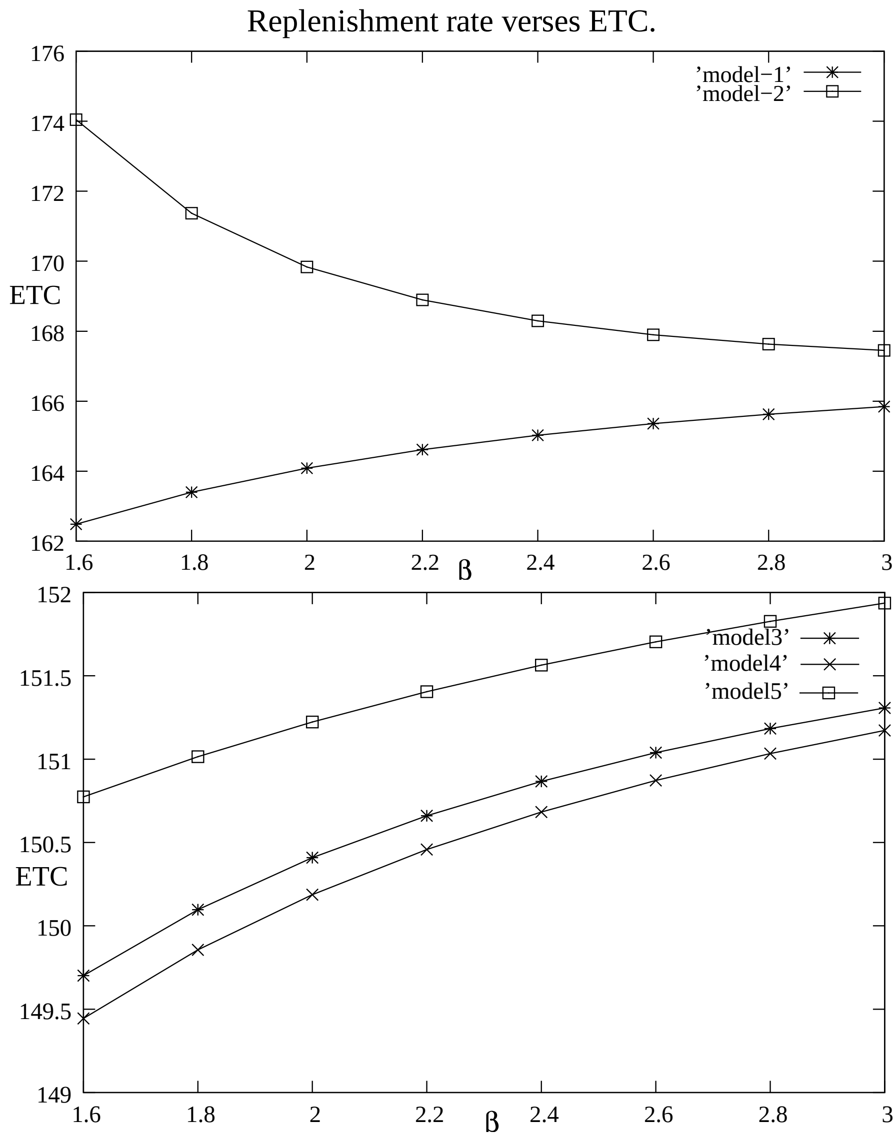


FIGURE 3.3. $\lambda = 4.5$, $\mu = 5$, $s = 5$, $S = 25$, $C = 100$, $C_1 = 20$, $C_2 = 1$, $C_3 = 3$, $C_4 = 7$

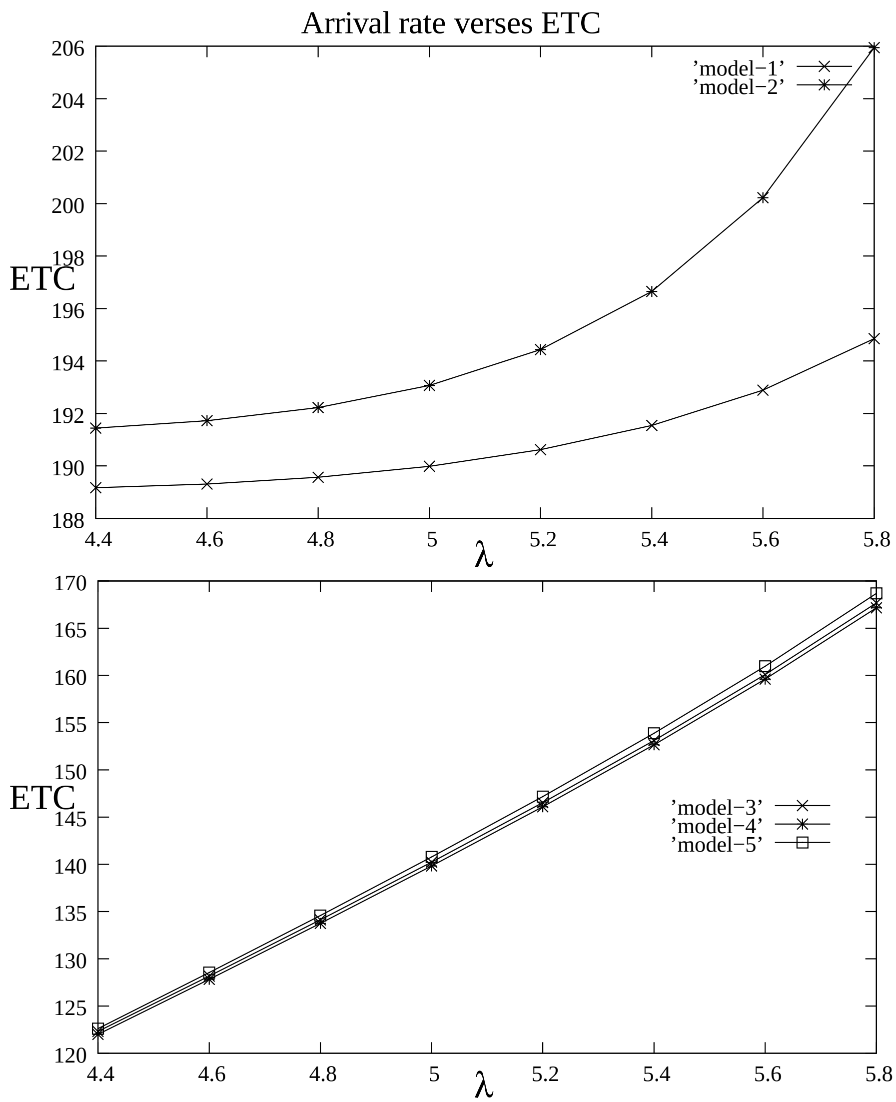


FIGURE 3.4. $\mu = 7$, $\beta = 1.5$, $s = 5$, $S = 25$, $C = 100$, $C_1 = 20$, $C_2 = 1$, $C_3 = 3$, $C_4 = 7$

3.13. Conclusion

We can compare the models by checking their total expected cost for different parameters. Between models I and II the cost of model I is less. That is, it is better not to allow the customers to join the system, when the inventory level is zero. Among models III, IV and V, the expected total cost of model IV is least, that is, it is best to make a local purchase of s units of inventory if a customer enters for service when the inventory level is zero. Again among all models model IV is more profitable. We compare all models with the given cost function and for given values of parameters.

CHAPTER 4

Analysis and Comparison of Some Retrial Inventory Models

4.1. Introduction

Retrial queues (queues with repeated calls, returning customers *etc.*) are a type of network with re-servicing after blocking. Inventory systems in which arriving customers who find all items are out of stock, may retry for the items after a period of time, are called retrial inventory. Artalejo, Krishnamoorthy and Lopez-Herrero [**9**] were the first to attempt to study inventory policies with positive lead time and retrial of customer who could not get the items during their earlier attempts. In 2007, Parthasarathy and Sudheesh [**56**] obtained transient solution using continued fraction approach to a single server retrial queue in which arrival and retrial rates are state dependent.

This chapter is an extension of the last chapter. Here we introduce retrial of unsatisfied customers into the models discussed in chapter 3, with the assumption that there is no waiting space for the customers at the service station except the one under going service. Customers arrive to a single server system according to a Poisson process with rate λ and service times are exponentially distributed with parameter μ. One unit of item is demanded by each customer. An order for replenishment of $Q = S - s$ quantity of goods is placed when the inventory level depletes to s. The lead time follows an exponential distribution with parameter β. An arriving customer who finds the server busy, proceeds to an orbit of infinite capacity and tries its luck to access the server from there. The inter-retrial times follow an exponential distribution with constant rate θ. In Model I, customers do not join the orbit when the inventory level is zero. In Model II, customers join the orbit even when the inventory level is zero. In Model III and IV it is assumed that a local purchase of one unit and s units of the item, respectively, at a higher cost if a customer (orbital customer or primary customer) enters for service when the inventory level is zero. In Model V, under the same situation a local purchase of S units of the item is made cancelling the existing order. The time required to make a local purchase is assumed to be negligible. Local purchase is made to decrease the waiting time of the customers thereby earning the goodwill of the customers.

4.2. Mathematical Formulation of Model I

Problem I is described as follows: Arrival of customer to a single server system forms a Poisson process with rate λ. Service times are identically and independently distributed exponential random variables with parameter μ. When the inventory level depletes to s due to demands, an order for replenishment for $Q = S - s$ quantity is placed where S is the maximum capacity of the system. The lead time is exponentially distributed with parameter β. An arriving customer, who finds the server busy, proceeds to an orbit of infinite capacity and tries its luck from there. Customers do not join the orbit when the inventory level is zero. The inter retrial times follow an exponential distribution with parameter θ.It is assumed that retrial rate is the same,independent of the number of customers in the orbit. This is possible, for example, by assuming that a queue of customers is formed in the orbit (see Gomez-Corral [**20**])

Let $N(t)$ be the number of customers in the orbit, $I(t)$ be the inventory level and $C(t)$, the server state at time t.

$$\text{Here } C(t) = \begin{cases} 1 & \text{if the server is busy} \\ 0 & \text{if the server is idle} \end{cases}$$

Then $\{(N(t), C(t), I(t)), t \geq 0\}$ is a Continuous Time Markov Chain (CTMC) on the state space $\{(i, 0, j), 0 \leq j \leq S\} \cup \{(i, 1, j), 1 \leq j \leq S\}$, $i \geq 0$. The above model can be studied as Linearly Independent Quasi-Birth and Death (LIQBD) process. The infinitesimal generator $\overline{Q}$ of the process has the following form

$$\overline{Q} = \begin{bmatrix} A_{10} & A_0 & 0 & 0 & 0 & \cdots \\ A_2 & A_{11} & A_0 & 0 & 0 & \cdots \\ 0 & A_2 & A_{11} & A_0 & 0 & \cdots \\ 0 & 0 & A_2 & A_{11} & A_0 & \cdots \\ \vdots & & \vdots & & & \end{bmatrix} \tag{4.2.1}$$

where A_{10}, A_{11}, A_0, A_2 are square matrices of order $(2S + 1)$ and they are given by $A_0 = \begin{bmatrix} 0 & 0 \\ 0 & \lambda I_S \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & E_1 \\ 0 & 0 \end{bmatrix}$ where

$$E_1 = \begin{bmatrix} 0 \\ \theta I_S \end{bmatrix}_{(S+1)\times S}$$

$$A_{1i} = \begin{bmatrix} M_1 & 0 & M_2 & M_3 & 0 & 0 \\ 0 & M_4 & 0 & 0 & M_5 & 0 \\ 0 & 0 & M_6 & 0 & 0 & M_7 \\ M_8 & 0 & 0 & M_9 & 0 & M_{10} \\ M_{11} & M_{12} & 0 & 0 & M_{13} & 0 \\ 0 & M_{14} & M_{15} & 0 & 0 & M_{16} \end{bmatrix} \quad i = 0, 1 \tag{4.2.2}$$

where M_1 is a square matrix of order $(s + 1)$ whose non-zero entries are given by $M_1(1, 1) = -\beta$ and $M_1(j, j) = -(\lambda + \beta + i\theta)$, $j = 2$ to $s + 1$,

M_2 is a square matrix of order $(s+1)$ whose non-zero entries are given by $M_2(j, j) = \beta$, $j = 1$ to $s + 1$,

M_3 is of order $(s + 1) \times s$ whose non-zero entries are given by $M_3(j + 1, j) = \lambda$, for $j = 1$ to s,

M_4 is a square matrix of order $(S - 2s - 1)$ where the non-zero entries are given by $M_4(j, j) = -(\lambda + i\theta)$, $j = 1$ to $S - 2s - 1$,

M_5 is a square matrix of order $(S - 2s - 1)$ whose non-zero entries are given by $M_5(j, j) = \lambda$, $j = 1$ to $S - 2s - 1$,

M_6 is a square matrix of order $(s + 1)$ where the non-zero elements are given by

$M_6(j,j) = -(\lambda + i\theta)$,

M_7 is a square matrix of order $(s+1)$ whose non-zero entries are given by $M_7(j,j) = \lambda$,

M_8 is of order $s \times (s+1)$ whose non-zero elements are given by

$M_8(j,j) = \mu$, $j = 1$ to s,

M_9 is a square matrix of order s whose non-zero entries are given by

$M_9(j,j) = -(\lambda + \mu + \beta)$, $j = 1$ to s,

M_{10} is of order $s \times (s+1)$ where non-zero entries are given by

$M_{10}(j, j+1) = \beta$, $j = 1$ to s,

M_{11} is of order $(S - 2s - 1) \times (s+1)$ whose non-zero entries are given by

$M_{11}(1, s+1) = \mu$,

M_{12} is a square matrix of order $(S - 2s - 1)$ where non-zero elements are given by

$M_{12}(j+1, j) = \mu$, $j = 1$ to $S - 2s - 2$,

M_{13} is a square matrix of order $(S - 2s - 1)$ whose non-zero entries are given by

$M_{13}(j,j) = -(\lambda + \mu)$, $j = 1$ to $S - 2s - 1$,

M_{14} is of order $(s+1) \times (S - 2s - 1)$ where non-zero elements are given by

$M_{14}(1, S - 2s - 1) = \mu$,

M_{15} is a square matrix of order $(s+1)$ whose non-zero entries are given by

$M_{15}(j+1, j) = \mu$, $j = 1$ to s,

M_{16} is a square matrix of order $(s+1)$ where non-zero entries are given by

$M_{16}(j,j) = -(\lambda + \mu)$, $j = 1$ to $s+1$,.

4.3. Mathematical Formulation of Model II

The only difference of this model from the first one is that customers join the orbit even when the inventory level is zero. Here also $\{(N(t), C(t), I(t)), t \geq 0\}$ is a CTMC on the state space $\{(i,0,j),\ 0 \leq j \leq S\} \cup \{(i,1,j), 1 \leq j \leq S\}$,$i \geq 0$. Then the generator has the form (4.2.1) where A_{10}, A_{11}, A_0, A_2 are square matrices of order $(2S+1)$ and they are given by $A_0 = \begin{bmatrix} E_2 & 0 \\ 0 & \lambda I_S \end{bmatrix}$ where $E_2 = \begin{bmatrix} \lambda e_1 \\ 0 \end{bmatrix}_{(S+1)\times(S+1)}$, where e_j is a row vector with 1 in the jth place and zeros elsewhere. A_2 is the same as in the problem I, and $A_{1i},\ i = 0,1$ have the form of (4.2.2) where all the sub-matrices, except M_1 is same as in the first model and here $M_1(1,1) = -(\lambda+\beta)$ and $M_1(j,j) = -(\lambda+\beta+i\theta)$, $j = 2$ to $s+1$ and M_1 is of order $(s+1)\times(s+1)$

4.4. Analysis of Models I and II

4.4.1. System stability. Define the generator matrix A(for each model)as $A = A_0 + A_{11} + A_2$ and $\pi = (\pi(0,0), \pi(0,1), \cdots, \pi(0,S), \pi(1,1), \pi(1,2), \cdots, \pi(1,S))$ where π is a the steady state probability vector of A. From the relation $\pi A = 0$ we get the following solution:

$$\pi(1,k) = \begin{cases} \left(\frac{\lambda+\beta+\theta}{\mu}\right)^{k-1} \left(\frac{\mu+\beta}{\lambda+\theta}\right)^{k-1} \frac{\beta}{\mu}\pi(0,0), \\ \qquad \text{for } k = 1,2,\ldots,s \\ \left(\frac{\lambda+\beta+\theta}{\mu}\right)^{s} \left(\frac{\mu+\beta}{\lambda+\theta}\right)^{s} \frac{\beta}{\mu}\pi(0,0), \\ \qquad \text{for } k = s+1,\ldots,Q \end{cases}$$

$$\pi(1,Q+k) = \pi(1,Q) - \pi(1,k),\ k = 1,2,\ldots,s$$

$$\pi(0,k) = \begin{cases} \left(\frac{\lambda+\beta+\theta}{\mu}\right)^{k-1} \left(\frac{\mu+\beta}{\lambda+\theta}\right)^{k} \frac{\beta}{\mu}\pi(0,0), \\ \qquad \text{for } k = 1,2,\ldots,s \\ \left(\frac{\lambda+\beta+\theta}{\mu}\right)^{s} \left(\frac{\mu+\beta}{\lambda+\theta}\right)^{s} \frac{\beta}{\lambda+\theta}\pi(0,0), \\ \qquad \text{for } k = s+1,\ldots,Q \end{cases}$$

$$\pi(0,Q+k) = \frac{\mu}{\lambda+\theta}\pi(1,Q) - \pi(0,k),\ k = 1,2,\ldots,s$$

$\pi(0,0)$ can be obtained from $\pi e = 1$

THEOREM 4.4.1. *The system in model I is stable if and only if $\lambda^2 < \theta(\mu-\lambda)$. The system in model II is stable if and only if*

$$\lambda\left(\frac{\mu}{\lambda+\beta+\theta}\right)^s < Q\left(\frac{\beta+\mu}{\lambda+\theta}\right)^s \beta\left(\frac{\theta}{\lambda+\theta} - \frac{\lambda}{\mu}\right).$$

PROOF. For the positive recurrence of $\overline{Q}$ we must have $\pi A_0 e < \pi A_2 e$ (see Neuts [**53**]). Simplifying this we get the indicated results. $\square$

4.4.2. Steady-state analysis. Let $X = (x(0), x(1), \ldots)$ be the steady state probability vector of $\overline{Q}$. Then $X\bar{Q} = 0$ together with $Xe = 1$ result in $x(i)$ having the matrix geometric solution:

$$x(i) = x(1)R^{i-1} \text{ for } i \geq 2 \tag{4.4.1}$$

where R is the minimal non negative solution of the matrix equation $A_0 + RA_{11} + R^2 A_2 = 0$. $x(0)$ and $x(1)$ are calculated from the equations

$$x(0)A_{10} + x(1)A_2 = 0 \tag{4.4.2}$$

$$x(0)A_0 + x(1)(A_{11} + RA_2) = 0 \tag{4.4.3}$$

subject to the normalizing condition $Xe = 1$,
that is, $x(0)e + x(1)(1 - R)^{-1} = 1$.
Having found, $x(1)$ we can find $x(i)$, $i \geq 2$ from (4.4.1).

4.5. System Performance Measures

Let $X = (x(0), x(1), \ldots)$ be the steady-state probability vector of $\overline{Q}$(for each model) and $x(i)$, $i \geq 0$, be partitioned as

$$x(i) = (y_{i00}, y_{i01}, \ldots, y_{i0S}, y_{i11}, y_{i12}, \ldots y_{i1S}).$$

Then we have the following performance measures.

(1) Expected number of customers in the orbit EC is given by

$$\text{EC} = \sum_{i=1}^{\infty} i x(i) e$$

(2) Expected inventory level EI is given by

$$\text{EI} = \sum_{i=1}^{\infty} \sum_{j=1}^{S} j(y_{i0j} + y_{i1j})$$

(3) Expected re-order rate ER is given by

$$\text{ER} = \mu \sum_{i=1}^{\infty} y_{i,1,s+1}$$

(4) Overall retrial rate OR is given by

$$\text{OR} = \theta \sum_{i=1}^{\infty} x(i)e$$

(5) *S*uccessful retrial rate SR is given by

$$\text{SR} = \theta \sum_{i=1}^{\infty} \sum_{j=1}^{S} y_{i,0,j}$$

(6) Probability that the server is busy is given by

$$P(B) = \sum_{i=1}^{\infty} \sum_{j=1}^{S} y_{i1j}$$

Model I

(7) Expected waiting time EW is given by EW = $\frac{\text{EC}}{\lambda}$.

Model II

(8) Expected waiting time EW is given by EW

$$= \frac{\text{EC}}{\lambda[1 - \sum_{i=0}^{\infty} y_{i00}]}.$$

Model I

(9) Expected number of customers EJ not joining the orbit when the inventory level is zero, is given by

$$\text{EJ} = \lambda \sum_{i=0}^{\infty} y_{i00}$$

4.6. Cost Function

To construct cost function we define the costs as follows:

C = fixed ordering cost

C_1 = procurement cost/unit

C_2 = holding cost of inventory/unit/unit time

C_3 = shortage cost of inventory/unit/unit time

The total expected cost function ETC is given as follows:

Model I

$\text{ETC} = [C + QC_1]\text{ER} + C_2\text{EI} + C_3\text{EJ}$

Model II

$\text{ETC} = [C + QC_1]\text{ER} + C_2\text{EI}$

4.7. Mathematical Formulation of Model III

In addition to assumptions in problem II, here a local purchase of one unit of the commodity is made if a customer enters for service when the inventory level is zero. Let $N(t)$ be the number of customers in the orbit, $I(t)$ be the inventory level and $C(t)$ be the server state at time t.

$$C(t) = \begin{cases} 1 & \text{if the server is busy} \\ 0 & \text{if the server is idle} \end{cases}$$

Then $\{(N(t), C(t), I(t)), t \geq 0\}$ is a CTMC on the state space.

$$\{(i, 0, j), 0 \leq j \leq S\} \cup \{(i, 1, j), 1 \leq j \leq S\},\ i \geq 0.$$

Then the generator has the form (4.2.1) where A_{10}, A_{11}, A_0, A_2 are square matrices of order $(2S+1)$ and they are given by

$$A_0 = \begin{bmatrix} 0 & 0 \\ 0 & \lambda I_S \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & E_3 \\ 0 & 0 \end{bmatrix}, \text{ where } E_3 = \begin{bmatrix} \theta & 0 & \cdots & 0 \\ \theta & 0 & \cdots & 0 \\ 0 & \theta & \cdots & 0 \\ & \cdots & \cdots & \\ 0 & 0 & \cdots & \theta \end{bmatrix}_{(S+1)\times S} ;$$

A_{1i}, $i = 0, 1$ is given by (4.2.2) in which all the sub-matrices except M_1 and M_3 are same and they are given as follows: M_1 a square matrix of order $(s+1)$ whose non-zero entries are given by $M_1(j, j) = -(\lambda + \beta + i\theta)$, $j = 1$ to $s+1$; M_3 is of order $(s+1) \times s$ where non-zero entries are given by $M_3(1, 1) = \lambda$ and $M_3(j+1, j) = \lambda$, $j = 1$ to s.

4.8. Mathematical Formulation of Model IV

In this model we make a local purchase of s units of inventory if a customer enters for service when the inventory level is zero. Here also $\{(N(t), C(t), I(t)), t \geq 0\}$ is a CTMC with the state space

$$\{(i, 0, j), 0 \leq j \leq S\} \cup \{(i, 1, j), 1 \leq j \leq S\},\ i \geq 0.$$

The infinitesimal generator $\overline{Q}$ has the form of (4.2.1) where A_{10}, A_{11}, A_0, A_2 are square matrices of order $(2S+1)$ and they are given by

$$A_0 = \begin{bmatrix} 0 & 0 \\ 0 & \lambda I_S \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & E_4 \\ 0 & 0 \end{bmatrix} \text{ where}$$

$$E_4 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ \vdots \\ \vdots \\ S \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & \cdots & s & \cdots & S \end{array} \\ \begin{pmatrix} 0 & 0 & \cdots & \theta & \cdots & 0 \\ \theta & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \theta & \cdots & 0 & \cdots & 0 \\ & & & & & \\ & \cdots & & & \cdots & \\ & \cdots & & & \cdots & \\ 0 & 0 & \cdots & 0 & \cdots & \theta \end{pmatrix}_{(S+1)\times S} \end{array},$$

A_{1i}, $i = 0, 1$ is given by (4.2.2), where all sub-matrices except M_1 and M_3 are same in the first model and they are given as follows.

M_1 is a square matrix of order $(s+1)$ whose non zero entries are given by

$M_1(j, j) = -(\lambda + \beta + i\theta)$, $j = 1$ to $s+1$.

M_3 is of order $(s+1)\times s$ whose non zero entries are given by
$M_3(1,s)=\lambda$, $M_3(j+1,j)=\lambda$,$j=1$ to s.

4.9. Mathematical Formulations of Model V

The main difference of this model from third and fourth model is that here we make a local purchase of maximum capacity of inventory S units, if a customer enters for service while the inventory is zero, which results in the cancellation of the existing order,as the maximum capacity of the inventory is S. The infinitesimal generator $\overline{Q}$ has the form of (4.2.1) where A_{10}, A_{11}, A_2, A_0 are square matrices of order $(2S+1)$ they are given by $A_0=\begin{bmatrix}0 & 0\\ 0 & \lambda I_S\end{bmatrix}$, $A_2=\begin{bmatrix}0 & E_5\\ 0 & 0\end{bmatrix}$ where

$$E_5=\begin{bmatrix}0 & 0 & \cdots & \theta\\ \theta & 0 & \cdots & 0\\ 0 & \theta & \cdots & 0\\ \cdots & & \cdots & \\ \cdots & & \cdots & \\ 0 & 0 & \cdots & \theta\end{bmatrix}_{(S+1)\times S}$$

$$A_{0i}=\begin{bmatrix}M_1 & 0 & M_2 & M_3 & 0 & M_{17}\\ 0 & M_4 & 0 & 0 & M_5 & 0\\ 0 & 0 & M_6 & 0 & 0 & M_7\\ M_8 & 0 & 0 & M_9 & 0 & M_{10}\\ M_{11} & M_{12} & 0 & 0 & M_{13} & 0\\ 0 & M_{14} & M_{15} & 0 & 0 & M_{16}\end{bmatrix}\qquad i=0,1.$$

where all the sub matrices M_2 to M_{16} is same as those in Model I. M_1 and M_{17} are given as follows:

M_1 is a square matrix of order $(s+1)$ where the non zero entries are given by
$M_1(j,j)=-(\lambda+\beta+i\theta)$, $j=1$ to $s+1$.

M_{17} is also a square matrix of order $(s+1)$ whose only non zero entry is given by $M_{17}(1,s+1)=\lambda$.

4.10. Analysis of Models III,IV and V

4.10.1. System stability. Define the generator matrix A (for each model) as $A=A_0+A_{11}+A_2$ and $\pi=(\pi(0,0),\pi(0,1),\cdots,\pi(0,S),\pi(1,1),\pi(1,2),\cdots,\pi(1,S))$ where π is the steady state probability vector of A. From the relation $\pi A=0$ we get the following values for each model.

Model III.

$$\pi(1,1)=\frac{\lambda+\beta+\theta}{\mu}\pi(0,0),$$

$$\pi(1,k)=\begin{cases}\left(\frac{\lambda+\beta+\theta}{\mu}\right)^{k-1}\left(\frac{\mu+\beta}{\lambda+\theta}\right)^{k-2}\left[\left(\frac{\mu+\beta}{\lambda+\theta}\right)\left(\frac{\lambda+\beta+\theta}{\mu}\right)-1\right]\pi(0,0), \\ \qquad k=2,\ldots,s \\ \left(\frac{\lambda+\beta+\theta}{\mu}\right)^{s}\left(\frac{\mu+\beta}{\lambda+\theta}\right)^{s-1}\left[\left(\frac{\mu+\beta}{\lambda+\theta}\right)\left(\frac{\lambda+\beta+\theta}{\mu}\right)-1\right]\pi(0,0), \\ \qquad k=s+1,\ldots,Q\end{cases}$$

$$\pi(1,Q+k)=\pi(1,Q)+\frac{\lambda+\theta}{\mu}\pi(0,0)-\pi(1,k)$$

$$k=1,2,\ldots,s,$$

$$\pi(0,k)=\begin{cases}\left(\frac{\lambda+\beta+\theta}{\mu}\right)^{k-1}\left(\frac{\mu+\beta}{\lambda+\theta}\right)^{k-1}\left[\left(\frac{\mu+\beta}{\lambda+\theta}\right)\left(\frac{\lambda+\beta+\theta}{\mu}\right)-1\right]\pi(0,0), \\ \qquad k=1,2,\ldots,s \\ \left(\frac{\lambda+\beta+\theta}{\lambda+\theta}\right)^{s}\left(\frac{\mu+\beta}{\mu}\right)^{s-1}\left[\left(\frac{\mu+\beta}{\lambda+\theta}\right)\left(\frac{\lambda+\beta+\theta}{\mu}\right)-1\right]\pi(0,0), \\ \qquad k=s+1,\ldots,Q\end{cases}$$

$$\pi(0,Q+k)=\frac{\mu}{\lambda+\theta}\pi(1,Q)+\pi(0,0)-\pi(0,k),\ k=1,2,\ldots,s-1$$

$$\pi(0,S)=\frac{\mu}{\lambda+\theta}\pi(1,Q)-\pi(0,s)$$

Model IV.

$$\pi(1,k)=\begin{cases}\left(\frac{\lambda+\beta+\theta}{\mu}\right)^{k}\left(\frac{\mu+\beta}{\lambda+\theta}\right)^{k-1}\pi(0,0), \\ \qquad\qquad\qquad\qquad \text{for } k=1,2,\ldots,s \\ \left(\frac{\lambda+\beta+\theta}{\mu}\right)\left[\left(\frac{\mu+\beta}{\lambda+\theta}\right)^{s}\left(\frac{\lambda+\beta+\theta}{\mu}\right)^{s}-1\right]\pi(0,0), \\ \qquad\qquad\qquad\qquad \text{for } k=s+1,\ldots,Q\end{cases}$$

$$\pi(1,Q+k)=\pi(1,Q)+\frac{\lambda+\theta}{\mu}\pi(0,0)-\pi(1,k),\ \text{for } k=1,\ldots,s$$

$$\pi(0,k)=\left(\frac{\lambda+\beta+\theta}{\mu}\right)^{k}\left(\frac{\mu+\beta}{\lambda+\theta}\right)^{k}\pi(0,0),\ \text{for } k=1,\ldots,s-1$$

$$\pi(0,s)=\left[\left((\frac{\mu+\beta}{\lambda+\theta})(\frac{\lambda+\beta+\theta}{\mu})\right)^{s}-1\right]\pi(0,0),$$

$$\pi(0,k)=\frac{\lambda+\beta+\theta}{\lambda+\theta}\pi(0,s),\ k=s+1,\ldots,Q$$

$$\pi(0,Q+k)=\frac{\mu}{\lambda+\theta}\pi(1,Q)+\pi(0,0)-\pi(0,k),\ k=1,\ldots,s-1$$

$$\pi(0,S)=\frac{\mu}{\lambda+\theta}\pi(1,Q)-\pi(0,s)$$

Model V.

$$\pi(1,k)=\begin{cases}\left(\frac{\lambda+\beta+\theta}{\mu}\right)^{k}\left(\frac{\mu+\beta}{\lambda+\theta}\right)^{k-1}\pi(0,0), \\ \qquad\qquad\qquad\qquad \text{for } k=1,\ldots,s \\ \left(\frac{\lambda+\beta+\theta}{\mu}\right)^{s+1}\left(\frac{\mu+\beta}{\lambda+\theta}\right)^{s}\pi(0,0), \\ \qquad\qquad\qquad\qquad \text{for } k=s+1,\ldots,Q\end{cases}$$

$$\pi(1,Q+k)=\pi(1,Q)+\frac{\lambda+\theta}{\mu}\pi(0,0)-\pi(1,k),\ k=1,\ldots,s$$

$$\pi(0,k)=\begin{cases}\left(\frac{\lambda+\beta+\theta}{\mu}\right)^{k}\left(\frac{\mu+\beta}{\lambda+\theta}\right)^{k}\pi(0,0),\ k=1,\ldots,s \\ \frac{\mu}{\lambda+\theta}\left(\frac{\lambda+\beta+\theta}{\mu}\right)^{s+1}\left(\frac{\mu+\beta}{\lambda+\theta}\right)^{s}\pi(0,0), \\ \qquad\qquad\qquad\qquad \text{for } k=s+1,\ldots,Q\end{cases}$$

$$\pi(0, Q+k) = \frac{\mu}{\lambda+\theta}\pi(1, Q) + \pi(0,0) - \pi(0,k),\ k = 1, \dots, s-1$$

$$\pi(0, S) = \frac{\mu}{\lambda+\theta}\pi(1, Q) - \pi(0,s)$$

We can find $\pi(0,0)$ from the equation $\pi e = 1$. Here $Q = S - s$.

THEOREM 4.10.1. *The systems in models 3 to 5 are stable if and only if* $\lambda^2 < \theta(\mu - \lambda)$.

PROOF. For the positive recurrence of $\overline{Q}$ we must have $\pi A_0 e < \pi A_2 e$ (see Neuts [**53**]). Simplifying this leads to the above condition. □

4.10.2. Steady-state analysis. Let $X = (x(0), x(1), \dots)$ be the steady state probability vector of $\overline{Q}$(for each model). Then $X\bar{Q} = 0$, $Xe = 1$ and $x(i)$ are given by

$$x(i) = x(1)R^{i-1} \text{ for } i \geq 2 \tag{4.10.1}$$

where R is the minimal non negative solution of the matrix equation $A_0 + RA_{11} + R^2A_2 = 0$. $x(0)$ and $x(1)$ are calculated from the equation

$$x(0)A_{10} + x(1)A_2 = 0 \tag{4.10.2}$$

$$x(0)A_0 + x(1)(A_{11} + RA_2) = 0 \tag{4.10.3}$$

subject to the normalizing condition $Xe = 1$,
That is, $x(0)e + x(1)(1-R)^{-1}e = 1$. Then we can find $x(i)$, $i \geq 2$ from (4.10.1)

4.11. System Performance Measures

Let $X = (x(0), x(1), \dots)$ be the steady-state probability vector of $\overline{Q}$(for each model)and $x(i)$, $i \geq 0$ partitioned as

$$x(i) = (y_{i00}, y_{i01}, \dots, y_{i0S}, y_{i11}, y_{i12}, \dots y_{i1S}).$$

Then we have the following performance measures:

(1) Expected number of customers EC in the orbit is given by

$$\mathrm{EC} = \sum_{i=1}^{\infty} i x(i) e$$

(2) Expected inventory level EI is given by

$$\mathrm{EI} = \sum_{i=1}^{\infty} \sum_{j=1}^{S} j(y_{i0j} + y_{i1j})$$

(3) Expected re-order rate ER is given by

$$\mathrm{ER} = \mu \sum_{i=1}^{\infty} y_{i,1,s+1}$$

(4) Overall retrial rate OR is given by

$$\mathrm{OR} = \theta \sum_{i=1}^{\infty} x(i) e$$

(5) Successful retrial rate SR is given by

$$\mathrm{SR} = \theta \sum_{i=1}^{\infty} \sum_{j=0}^{S} y_{i0j}$$

(6) Probability that the server is busy is given by

$$P(B) = \sum_{i=1}^{\infty} \sum_{j=1}^{S} y_{i1j}$$

(7) Expected waiting time EW is given by EW = $\frac{\mathrm{EC}}{\lambda}$.

(8) Expected rate of local purchase EL is given by

$$\mathrm{EL} = \lambda \sum_{i=0}^{\infty} y_{i00} + \theta \sum_{i=1}^{\infty} y_{i00}$$

4.12. Cost Function and Numerical Examples

To construct the cost function we define the cost as follows:

C = fixed ordering cost

C_1 = procurement cost/unit

C_2 = holding cost of inventory/unit/unit time

$(1+k)C_1 zEL$=total cost of local purchase of z units of inventory with a hike of k times C_1/unit.

In model V as we make a local purchase of S units and thus cancelling the existing order,the system losses the ordering cost already paid and $(ER - EL)$=the remaining rate of ordering inventory.

The total expected cost function ETC is given as follows.

Model III

$\text{ETC} = [C + QC_1]\text{ER} + C_2\text{EI} + (1 + l)C_1\ \text{EL}$

Model IV

$\text{ETC} = [C + QC_1]\text{ER} + C_2\text{EI} + (1 + m)C_1 \times s \times\ \text{EL}$

Model V

$\text{ETC} = C\ \text{ER} + QC_1[\text{ER} - \text{EL}] + C_2\text{EI} + (1 + n)C_1 \times S \times\ \text{EL}$, where l, m, n are proper fractions and $l > m > n > 0$, when the local purchase is made in higher quantity the hike in price decreases.

S	EC		EI		ER		EW	
	Models							
	I	II	I	II	I	II	I	II
25	3.0796	8.0407	8.3948	12.7346	0.0508	0.0660	3.0796	8.5173
27	3.0774	7.0780	9.0775	13.8438	0.0454	0.0585	3.0774	7.4426
29	3.0750	6.4241	9.7462	14.9201	0.0410	0.0524	3.0750	6.7163
31	3.0729	5.9516	10.4061	15.9766	0.0373	0.0475	3.0728	6.1938
33	3.0710	5.5946	11.0643	17.0200	0.0342	0.0434	3.0709	5.8005
35	3.0694	5.3155	11.7220	18.0544	0.0315	0.0399	3.0693	5.4940
37	3.0680	5.0914	12.3794	19.0821	0.0292	0.0370	3.0679	5.2487

S	EC			EI			EL		
	Models								
	III	IV	V	III	IV	V	III	IV	V
25	3.0481	3.0481	3.0481	8.5524	13.3693	14.0130	0.0110	0.0440	0.0093
27	3.0481	3.0481	3.0481	9.2298	14.3772	15.0167	0.0097	0.0391	0.0082
29	3.0481	3.0481	3.0481	9.9033	15.3688	16.0101	0.0087	0.0352	0.0074
31	3.0481	3.0481	3.0481	10.5737	16.3502	16.9963	0.0079	0.0320	0.0067
33	3.0481	3.0481	3.0481	11.2420	17.3346	17.9847	0.0072	0.0293	0.0061
35	3.0481	3.0481	3.0481	11.9085	18.3228	18.9761	0.0067	0.0270	0.0056
37	3.0481	3.0481	3.0481	12.5738	19.3135	19.9693	0.0062	0.0251	0.0052

TABLE 4.1. Variations in Maximum inventory level S. $\lambda = 1$, $\mu = 1.7$, $\beta = .2$, $\theta = 3$, $s = 10$

s	EC		EI		EW		ER	
	Models							
	I	II	I	II	I	II	I	II
4	3.0888	8.9757	12.1534	18.1750	3.0888	9.6208	0.0508	0.0660
6	3.0794	6.8542	12.4594	18.9329	3.0794	7.2119	0.0454	0.0585
8	3.0720	5.6123	12.8708	19.7436	3.0720	5.8246	0.0411	0.0524
10	3.0662	4.8275	13.3651	20.6148	3.0661	4.9606	0.0337	0.0475
12	3.0618	4.3092	13.9257	21.5322	3.0618	4.3962	0.0342	0.0434
14	3.0584	3.9566	14.5391	22.4816	3.0584	4.0154	0.0315	0.0399
16	3.0557	3.7118	15.1721	23.4521	3.0557	3.7528	0.0292	0.0370

s	EI			EL			ER		
	Models								
	III	IV	V	III	IV	V	III	IV	V
4	12.2697	19.0075	20.4261	0.0549	0.0215	0.0117	0.0202	0.0203	0.0267
6	12.6075	19.5254	20.6553	0.0409	0.0125	0.0086	0.0217	0.0220	0.0283
8	13.0518	20.1237	20.9970	0.0304	0.0081	0.0063	0.0235	0.0236	0.0304
10	13.5697	20.8027	21.4616	0.0201	0.0056	0.0047	0.0256	0.0254	0.0329
12	14.1375	21.5643	22.0511	0.0169	0.0040	0.0035	0.0279	0.0274	0.0359
14	14.7386	22.4107	22.7636	0.0126	0.0029	0.0027	0.0304	0.0297	0.0393
16	15.3615	23.2966	23.5539	0.0095	0.0022	0.0021	0.0330	0.0323	0.0428

TABLE 4.2. Variations in re-order level s. $\lambda = 1$, $\mu = 1.7$, $\beta = .2$, $\theta = 3$, $S = 40$

β	EI		EW		EJ	EC	
	Models						
	I	II	I	II	I	I	II
0.3	9.4650	14.6231	3.0580	3.8882	0.0124	3.0580	3.8286
0.4	10.0335	15.4863	3.0516	3.2780	0.0042	3.0517	3.2615
0.5	10.3793	15.9950	3.0494	3.1256	0.0016	3.0494	3.1198
0.6	10.6103	16.3313	3.0486	3.0774	0.0006	3.0487	3.0751
0.7	10.7755	16.5706	3.0483	3.0601	0.0003	3.0484	3.0591
0.8	10.8995	16.7496	3.0482	3.0533	0.0001	3.0482	3.0529
0.9	10.9960	16.8887	3.0481	3.0504	0.0001	3.0482	3.0503
1.0	11.0733	16.9999	3.0481	3.0492	0.0000	3.0482	3.0491

β	EL			EW			EI		
	Models								
	III	IV	V	III	IV	V	III	IV	V
0.3	0.0138	0.0043	0.0040	3.0481	3.0481	3.0481	9.5748	14.5876	14.7974
0.4	0.0049	0.0018	0.0018	3.0481	3.0481	3.0481	10.1216	15.3125	15.3862
0.5	0.0019	0.0008	0.0008	3.0481	3.0481	3.0481	10.4565	15.7839	15.8117
0.6	0.0008	0.0003	0.0003	3.0481	3.0481	3.0481	10.6814	16.1110	16.1222
0.7	0.0003	0.0001	0.0001	3.0481	3.0481	3.0481	10.8427	16.3496	16.3543
0.8	0.0001	0.0000	0.0000	3.0481	3.0481	3.0481	10.9639	16.5307	16.5328
0.9	0.0000	0.0000	0.0000	3.0481	3.0481	3.0481	11.0584	16.6727	16.6737
1.0	0.0000	0.0000	0.0000	3.0481	3.0481	3.0481	11.1341	16.7870	16.7875

TABLE 4.3. Variations in replenishment rate β. $\lambda = 1$, $\mu = 1.7$, $\theta = 3$, $s = 10$ $S = 25$

λ	EC		EW		ER		EJ
	Models						
	I	II	I	II	I	II	I
0.4	0.1408	0.1494	0.3522	0.3744	0.0079	0.0266	0.0003
0.5	0.2535	0.2865	0.5083	0.5763	0.0126	0.0333	0.0012
0.6	0.4304	0.5293	0.7215	0.8923	0.0185	0.0400	0.0034
0.7	0.7124	0.9750	1.0291	1.4201	0.0255	0.0466	0.0078
0.8	1.1851	1.8618	1.5101	2.3977	0.0336	0.0533	0.0152
0.9	2.0653	3.9763	2.3646	4.6104	0.0428	0.0599	0.0265
1.0	4.1094	12.3876	4.2925	13.0882	0.0531	0.0645	0.0426

λ	EC			EW			EL		
	Models								
	III	IV	V	III	IV	V	III	IV	V
0.4	0.1401	0.1401	0.1401	0.3503	0.3503	0.3503	0.0004	0.0004	0.0004
0.5	0.2513	0.2513	0.2513	0.5027	0.5027	0.5027	0.0014	0.0011	0.0010
0.6	0.4253	0.4253	0.4253	0.7088	0.7088	0.7088	0.0037	0.0022	0.0020
0.7	0.7027	0.7027	0.7027	1.0038	1.0038	1.0038	0.0083	0.0038	0.0034
0.8	1.1692	1.1692	1.1692	1.4615	1.4615	1.4615	0.0161	0.0058	0.0051
0.9	2.0420	2.0420	2.0420	2.2689	2.2689	2.2689	0.0281	0.0081	0.0071
1.0	4.0784	4.0784	4.0784	4.0784	4.0784	4.0784	0.0453	0.0107	0.0091

TABLE 4.4. Variations in arrival rate λ. $\lambda = 1$, $\mu = 1.7$, $\beta = .2$, $s = 10$ $S = 25$.

4.12.1. Interpretations of the Numerical Results.

1. *Effect of the maximum inventory level S on various performance measures:*

   As S increases in all models considered, expected inventory level increases. The number of customers and hence the waiting time decrease in model I and II. As more inventory is with the system the time interval to reach the re-order level increases, so re-order rate decreases in model I and II. Due to the same reason local purchases in models III, IV and V decreases. The number of customers in III, IV and V is same due to local purchase. (see table 4.1)

2. *Effect of the re-order level s on various performance measures.*

   From table 4.2 one may conclude that the behaviour of system performance measures as s increases, is similar to that of S, except that the re-order rate increases, the time interval to reach the re-order point decreases and so more orders are placed.

3. *Effect of the replenishment rate β on various performance measures.*

   As we expect when β increases the inventory level increases in all models. In models III, IV and V, as replenishment takes place at a higher rate the rate of local purchase decreases. Due to local purchase the waiting time of customers do not increase. The number of customers who do not join when the inventory level is zero, also decreases in model I. The number of customers and their waiting time decreases in models I and II as β measures (see table 4.3)

4. *Effect of the arrival rate λ on various performance measures.*

   Table 4.4 shows that when the arrival rate increases the number of customers and their waiting time increases in all models. Reorder rate increases in models I and II. The number of customers who do not join when the inventory level is zero also increases in model I. Due to more arrivals, the rate of local purchase also increases in models III, IV and V.

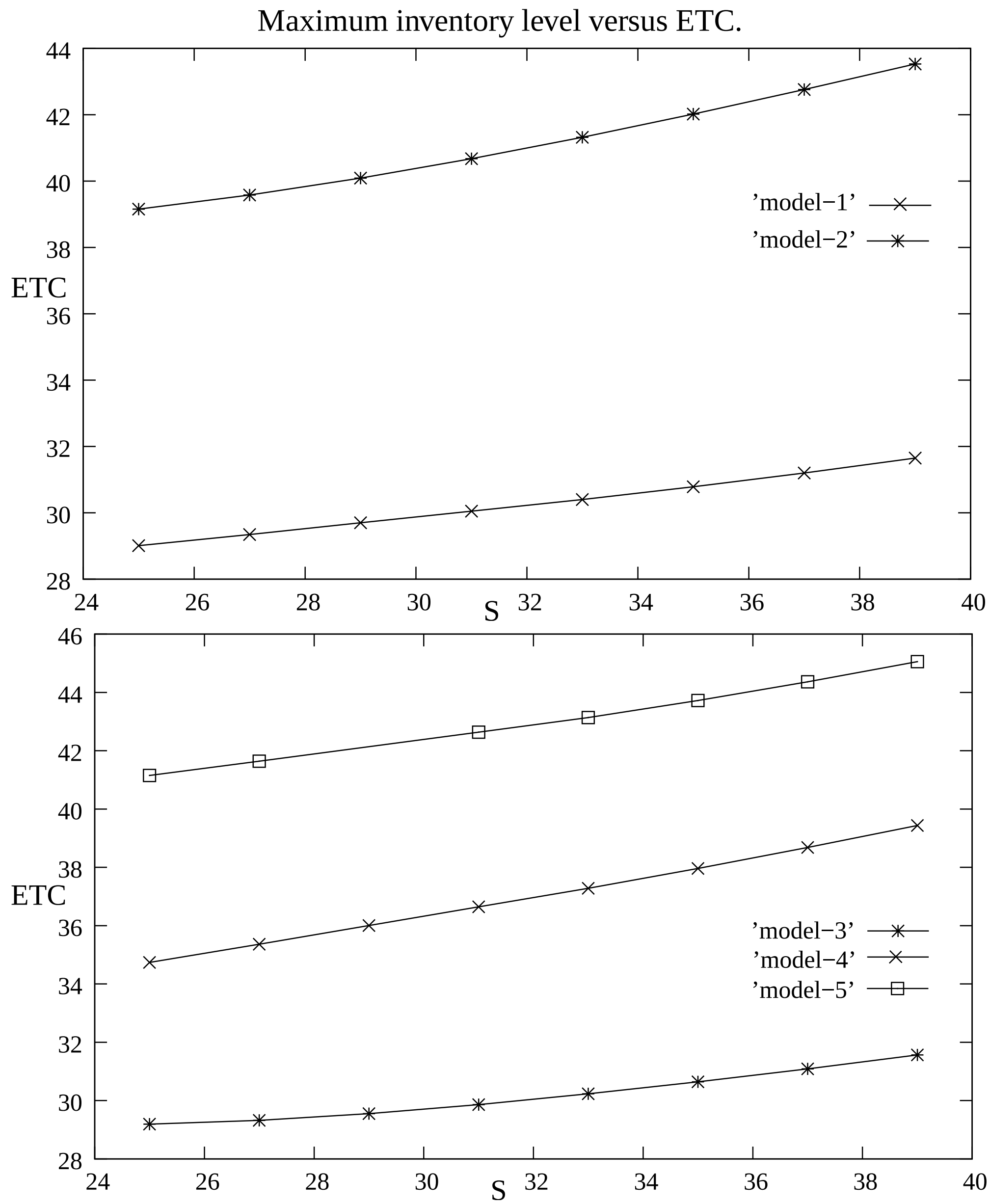


FIGURE 4.1. $\lambda = 1$, $\mu = 1.7$, $\beta = .2$, $\theta = 3$, $s = 10$, $C = 100$, $C_1 = 20$, $C_2 = 1$, $C_3 = 7$, $l = .75$, $m = .5$, $n = .25$

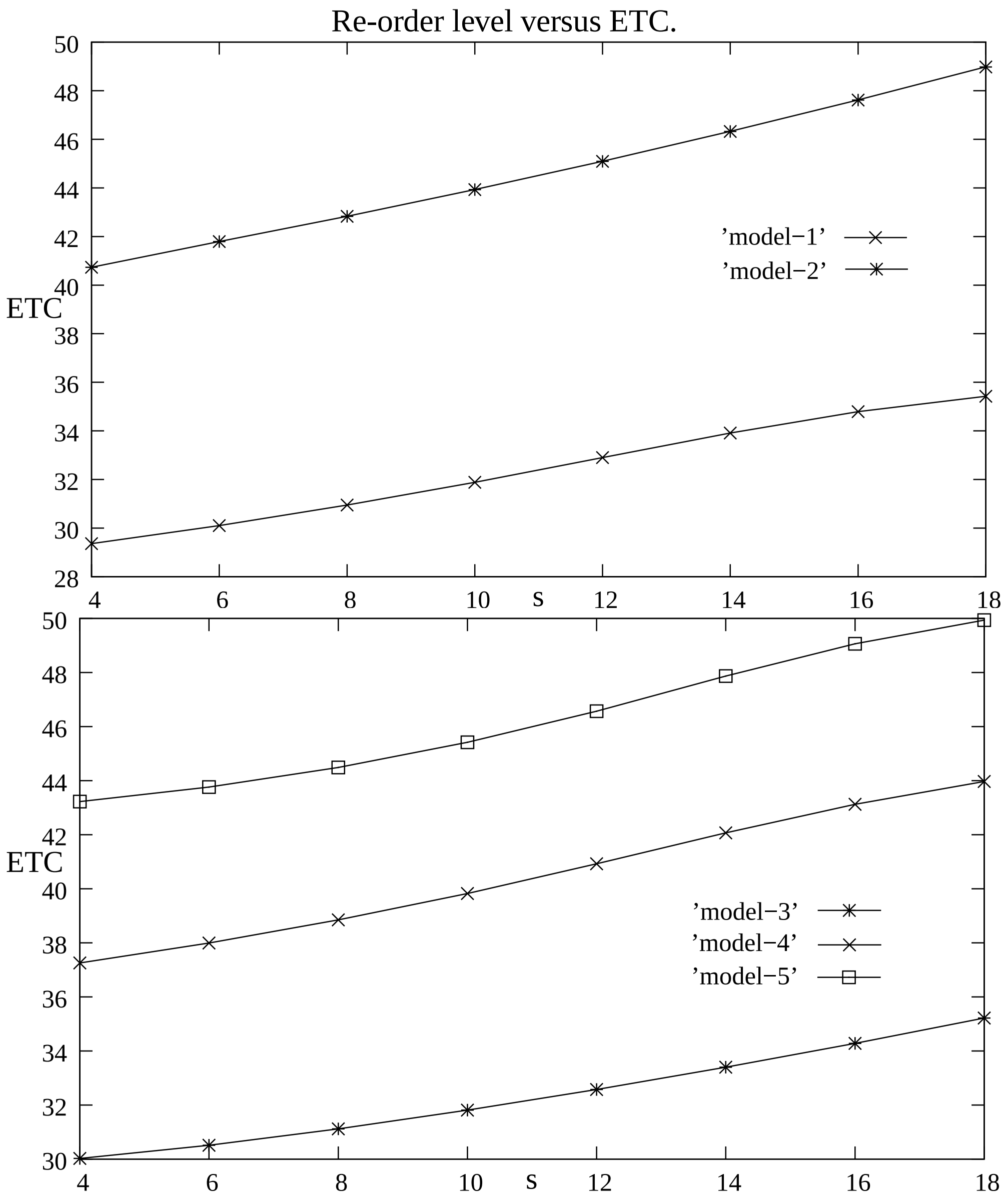
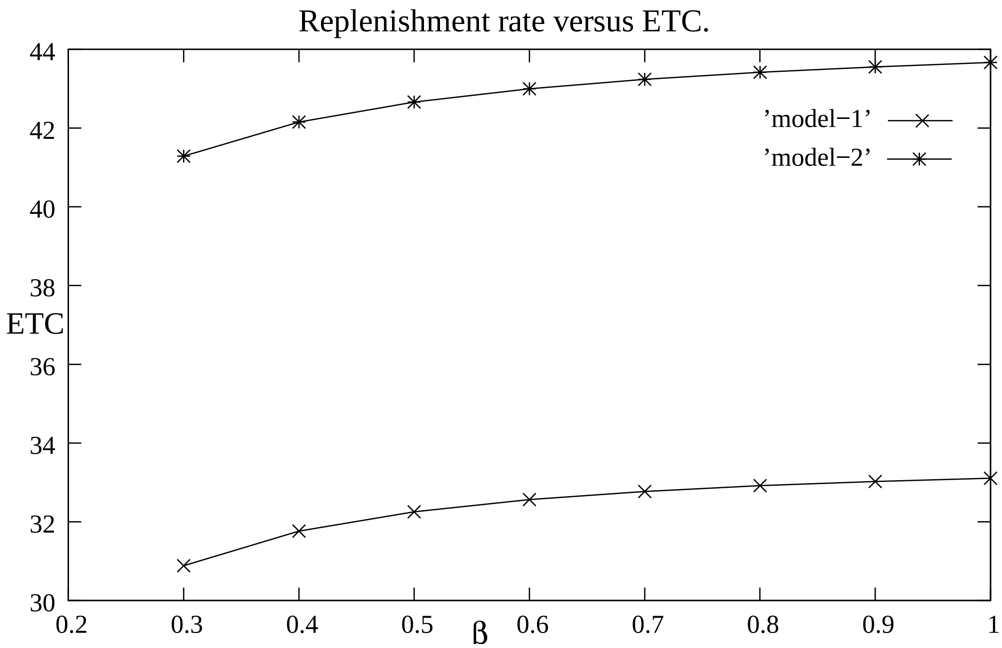


FIGURE 4.2. $\lambda = 1$, $\mu = 1.7$, $\beta = .2$, $\theta = 3$, $S = 40$, $C = 100$, $C_1 = 20$, $C_2 = 1$, $C_3 = 7$, $l = .75$, $m = .5$, $n = .25$



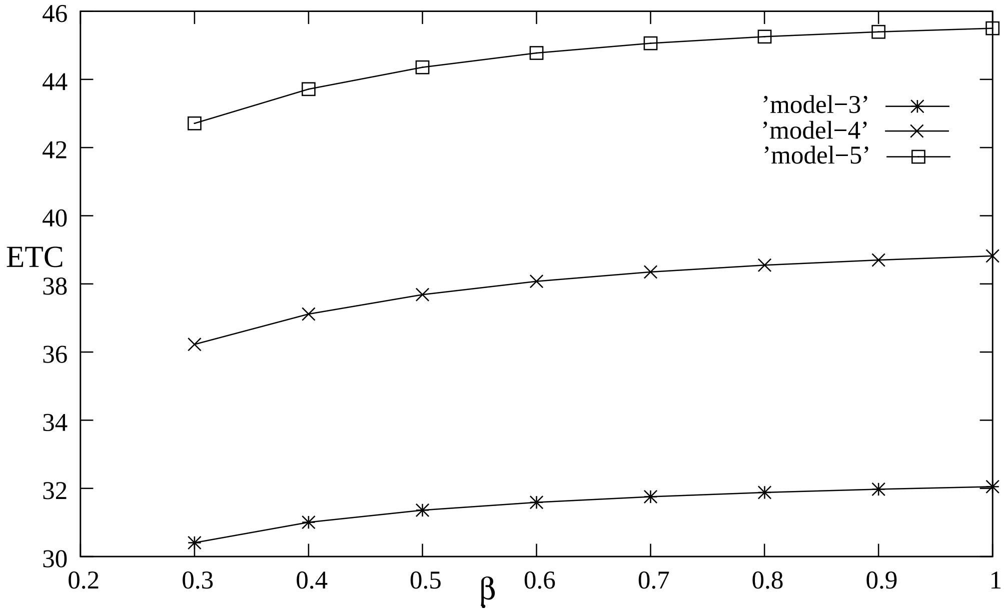


FIGURE 4.3. $\lambda = 1$, $\mu = 1.7$, $\theta = 3$, $s = 10$ $S = 25$, $C = 100$, $C_1 = 20$, $C_2 = 1$, $C_3 = 7$, $l = .75$, $m = .5$, $n = .25$

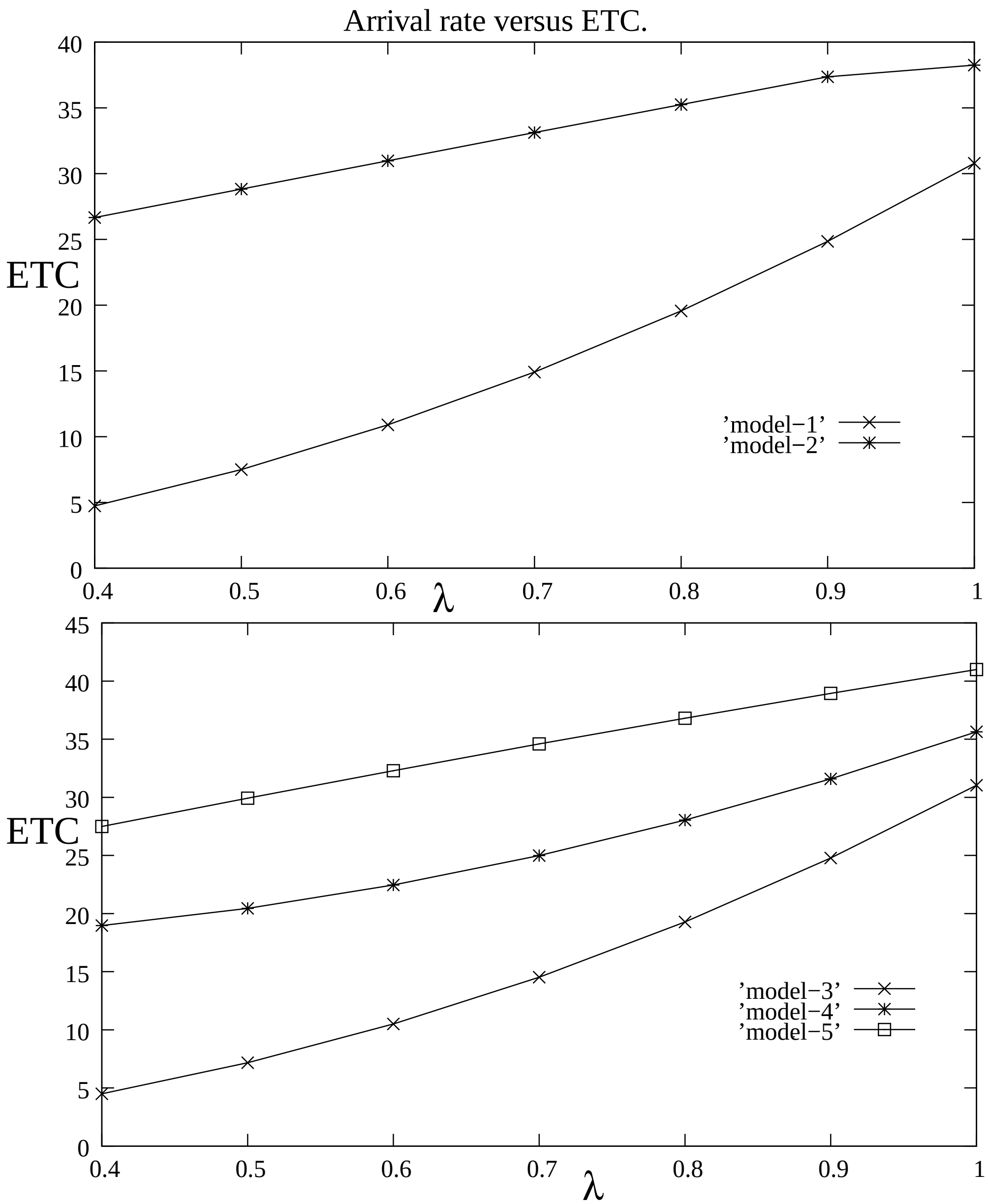


FIGURE 4.4. $\lambda = 1$, $\mu = 1.7$, $\beta = .2$, $s = 10$ $S = 25$, $C = 100$, $C_1 = 20$, $C_2 = 1$, $C_3 = 7$, $l = .75$, $m = .5$, $n = .25$

4.12.2. Interpretation of the graphs. In order to find the most profitable model, we compute the expected total cost per unit time for each model by varying the parameters one at a time keeping others fixed.

Figure 4.1 shows that as the maximum inventory level increases the total expected cost increases, this is primarily due to the increase in the holding cost of inventory. When the re-order level increases then also the cost increases as the inventory level increases (see 4.2). As the replenishment rate increases, here also cost increases due to the same reason (see figure 4.3). Figure 4.4 shows that the cost function is directly propotional to the arrival rate.

4.13. Conclusion

From all the graphs we understand that comparing models I and II, the cost is less in model I. Comparing models III, IV and V the cost involved in model III is least. That is local purchase by one unit is profitable. Among all the models the cost is least for model III. So model III is the best with the given cost function and given values of parameter. However, the input parameters do influence the total expected cost. Hence the models are sensitive to input parameters.

CHAPTER 5

Inventory Systems with Disasters

5.1. Introduction

In all inventory models discussed earlier in this thesis we have not brought in the role of perishability and disasters.In several practical situations,these factors play important roles in decision making. For example in a firm where there is a possibility of occurrence of disaster, it is to be decided about the maximum quantity that can be kept so that the inventory lost due to disasters is minimum and at the same time efficient running of the system is ensured.

Krishnamoorthy and Varghese [**43**] analyzed an inventory model where the items are damaged due to decay and disasters.They assumed that the lead time is zero and the service time is negligible. Arivarignan et.al [**1**] discussed a continuous review (s, S) inventory system with perishable items, where lead time and life time of items are exponentially distributed. They obtained both steady state and transient solutions. An extensive survey on perishable inventory can be seen in Nahmias [**52**]. Subsequently there followed several further investigations. Nevertheless these were all on with negligible service times. Krishnamoorthy and Anbazhagan [**30**] discussed a system with finite capacity for waiting space where the inventory is served according to an exponentially distributed time. Further they assume perishability of items on stock.

In this chapter we consider two models of (s, S) inventory systems where the commodities are destroyed by disasters. Customers arrive to a single server counter according to a Poisson process with rate λ. Service times of customers are independent and identically distributed exponential random variables with parameter μ. Lead time follows an exponential distribution with parameter β. The interval between disasters have exponential distribution with parameter δ. Each customer requires one unit of item. As a result of service, when the inventory level reaches s we place an order for $Q = S - s$

quantity of the item. If disaster occurs when the inventory level is between 0 and s there is no need to place the order again, and we place an order if the inventory level is between $s+1$ and S. That is only one existing order is allowed. We assume that customers register their names for the product. Since there is a chance of disasters physical presence of customers at the service station is thus avoided. In Model I we assume that customers do not join the system when the inventory level is zero: whereas in model II customers are assumed to join the system even when the inventory level is zero.

5.2. Mathematical Description of Model I

Let $N(t)$ be the number of customers in the system and $I(t)$ be the inventory level at time t. We assume that disaster destroys all the inventoried items present at that epoch; however it is assumed that the customers are not affected by disaster. Customers do not join the system when the inventory level is zero. Those who are already present stays there. It follows that the $\{((N(t), I(t)), t \geq 0\}$ is a LIQBD process on the state space $\{(i,j); i \geq 0, 0 \leq j \leq S\}$. The infinitesimal generator $\overline{Q}$ of the process is a block tridiagonal matrix having the following form:

$$\overline{Q} = \begin{bmatrix} A_{00} & A_0 & 0 & 0 & 0 & 0 & \cdots \\ A_2 & A_1 & A_0 & 0 & 0 & 0 & \cdots \\ 0 & A_2 & A_1 & A_0 & 0 & 0 & \cdots \\ 0 & 0 & A_2 & A_1 & A_0 & 0 & \cdots \\ & & \vdots & & \vdots & & \end{bmatrix} \tag{5.2.1}$$

where the blocks A_{00}, A_0, A_1, A_2 are square matrices of order $(S+1)$; they are given by

$$A_0 = \begin{bmatrix} 0 & 0 \\ 0 & \lambda I_S \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 0 \\ \mu I_S & 0 \end{bmatrix}$$

$$
A_{00} = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ s \\ s+1 \\ \vdots \\ \vdots \\ S \end{array}
\begin{array}{c}
\begin{array}{cccccccccc} 0 & 1 & \cdots & s & s+1 & \cdots & S-s & \cdots & & S \end{array} \\
\begin{pmatrix}
-\beta & & & & & & \beta & & & \\
\delta & -\Omega & & & & & & \beta & & \\
\cdots & & \ddots & & & & & & \ddots & \\
\delta & & & -\Omega & & \cdots & & & & \beta \\
\delta & & & & -\omega & & & & & \\
\vdots & & & & & & \ddots & & & \\
 & & & & & & & & & \\
\delta & & & & & & & & & -\omega
\end{pmatrix}
\end{array}
$$

where $\Omega = \lambda + \beta + \delta$, $\omega = \lambda + \delta$.

$$
A_1 = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ s \\ s+1 \\ \vdots \\ S \end{array}
\begin{array}{c}
\begin{array}{cccccccccc} 0 & 1 & \cdots & s & s+1 & \cdots & S-s & \cdots & & S \end{array} \\
\begin{pmatrix}
-\beta & & & & & & \beta & & & \\
\delta & -\Omega' & & & & & & \beta & & \\
\cdots & & \ddots & & & & & & \ddots & \\
\delta & & & -\Omega' & & \cdots & & & & \beta \\
\delta & & & & -\omega' & & & & & \\
\vdots & & & & & \ddots & & & & \\
\delta & & & & & & & & & -\omega'
\end{pmatrix}
\end{array}
$$

with $\Omega' = \lambda + \beta + \delta + \mu$, $\omega' = \lambda + \delta + \mu$

5.3. Analysis of Model I

5.3.1. System stability. Define the generator matrix A as $A = A_0 + A_1 + A_2$. Let $\pi = (\pi_0, \pi_1, \cdots, \pi_S)$ be the steady state probability vector of A. Then we have $\pi A = 0$ and $\pi e = 1$. Solving $\pi A = 0$ we get

$$
\pi_0 = \frac{\mu}{\beta + \delta}\pi_1 + \frac{\delta}{\beta + \delta}
$$

$$
\pi_k = (\frac{\beta + \delta + \mu}{\mu})^{k-1}\pi_1 \text{ for } k = 2, \ldots, s+1
$$

$$\pi_{s+k} = (\frac{\delta+\mu}{\mu})^{k-1}(\frac{\beta+\delta+\mu}{\mu})^s\pi_1 \text{ for } k = 2,\ldots,S-2s$$

$$\pi_{Q+k} = \frac{\delta+\mu}{\mu}\pi_{Q+k-1} - \frac{\beta}{\mu}\pi_{k-1}, \text{ where } Q = S-s, \quad k = 1,2,\ldots,s,$$

Here $\pi_0, \pi_2, \ldots, \pi_S$ are all expressed in terms of π_1. From $\pi e = 1$ we can find π_1 and hence $\pi_0, \pi_2, \ldots, \pi_S$.

THEOREM 5.3.1. *The Markov chain described by the model is stable if and only if $\lambda < \mu$.*

PROOF. From the well known results (see Neuts [**53**]) on positive recurrence of $\overline{Q}$ which states that $\pi A_0 e < \pi A_2$. Simplifying this we get $\lambda < \mu$ □

5.3.2. Steady state analysis. Let $X = (x(0), x(1), \ldots)$ be the steady state probability vector of the Markov chain. Since the model considered here is a LIQBD process, its steady state distribution has a matrix-geometric solution to the equations $X\overline{Q} = 0$ and $Xe = 1$. Then $x(i)$ has the matrix geometric form

$$x(i) = x(1)R^{i-1} \text{ for } i \geq 2, \tag{5.3.1}$$

where R is the minimal non-negative solution of the matrix equation $A_0 + RA_1 + R^2A_2 = 0$. $X\overline{Q} = 0$ gives

$$x(0)A_{00} + x(1)A_2 = 0 \tag{5.3.2}$$

$$x(0)A_0 + x(1)(A_1 + RA_2) = 0. \tag{5.3.3}$$

Solving the above equations we can find vectors $x(0)$ and $x(1)$ subject to the normalizing condition $Xe = 1$.
That is $x(0)e + x(1)(1-R)^{-1} = 1$. Having found $x(1)$, $x(i)$, $i \geq 2$ can be found from (5.3.1).

5.4. Performance Measures

Having computed the system state probabilities, we proceed to find out how the system performs. Let $X = (x(0), x(1), \ldots)$ be the steady-state probability vector of $\overline{Q}$ where $x(i) = (y_{i0}, y_{i1}, \ldots, y_{iS})$.

(1) Expected number of customers, EC in the system is given by

$$\text{EC} \ = \sum_{i=1}^{\infty} i x(i) e$$

(2) Expected inventory level EI is given by

$$\text{EI} \ = \sum_{i=0}^{\infty} \sum_{j=1}^{S} j y_{ij}$$

(3) Expected waiting time in the system EW is given by

$$\text{EW} = \frac{\text{EC}}{\lambda}.$$

(4) Expected re-order rate ER is given by

$$\text{ER} \ = \mu \sum_{i=1}^{\infty} y_{i,s+1} + \delta \sum_{i=0}^{\infty} \sum_{j=s+1}^{S} y_{ij}$$

(5) Expected number of inventory ET, lost due to disaster is given by $ET = \delta EI$.

(6) Expected number of customers EJ not joining the system when the inventory level is zero is given by

$$\text{EJ} = \lambda \sum_{i=0}^{\infty} y_{i0}.$$

(7) Expected rate of departure ED after completing service is given by

$$\text{ED} \ = \mu \sum_{i=0}^{\infty} \sum_{j=1}^{S} y_{ij}$$

5.5. Cost Function

To construct the cost function we define the following costs as

$C =$ fixed ordering cost

$C_1 =$ procurement cost/unit

$C_2 =$ holding cost of inventory/unit/unit time

$C_3 =$ revenue from service/unit/unit time

$C_4 =$ disaster cost/unit

$C_5 =$ shortage cost of inventory/unit/unit time

Then the total expected cost is defined as

$\mathrm{ETC} = [C + QC_1]\mathrm{ER} + C_2\mathrm{EI} - C_3\mathrm{ED} + C_4\mathrm{ET} + C_5\mathrm{EJ}$

5.6. Mathematical Description of Model II

The only difference of this model from the first model is that customers join the system even when the inventory level is zero. The infinitesimal generator $\overline{Q}$ of the process has the form of (5.2.1) where the blocks A_{00}, A_0, A_1, A_2 are square matrices of order $(S+1)$ and they are given by

$$A_0 = \lambda I_{S+1}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ \mu I_S & 0 \end{bmatrix},$$

$$A_{00} = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ s \\ s+1 \\ \vdots \\ S \end{array} \begin{array}{c} \begin{array}{ccccccccc} 0 & 1 & \cdots & s & s+1 & \cdots & S-s & \cdots & S \end{array} \\ \left(\begin{array}{ccccccccc} -\Delta & & & & & & \beta & & \\ \delta & -\Omega & & & & & & \beta & \\ \vdots & \ddots & & & & & & & \ddots \\ \delta & & & -\Omega & & \cdots & & & \beta \\ \delta & & & & -\omega & & & & \\ \vdots & & & & & & \ddots & & \\ \delta & & & & & & & & -\omega \end{array}\right) \end{array}$$

with $\Delta = \lambda + \beta$, $\Omega = \lambda + \beta + \delta$, $\omega = \lambda + \delta$.

$$
A_1 = \begin{array}{c} \\ \begin{array}{c} 0 \\ 1 \\ \vdots \\ s \\ s+1 \\ \vdots \\ \vdots \\ S \end{array} \end{array}
\begin{array}{c}
\begin{array}{cccccccccc} 0 & 1 & \cdots & s & s+1 & \cdots & S-s & \cdots & & S \end{array} \\
\left(\begin{array}{cccccccccc}
-\Delta & & & & & & \beta & & & \\
\delta & -\Omega' & & & & & & \beta & & \\
\vdots & \ddots & & & & & & & \ddots & \\
\delta & & & -\Omega' & & \cdots & & & & \beta \\
\delta & & & & -\omega' & & & & & \\
\vdots & & & & & \ddots & & & & \\
 & & & & & & & & & \\
\delta & & & & & & & & & -\omega'
\end{array}\right)
\end{array}
$$

where $\Omega' = \lambda + \beta + \delta + \mu$, $\omega' = \lambda + \delta + \mu$, $\Delta = \lambda + \beta$.

5.7. Analysis of Model II

5.7.1. System stability. Define the generator matrix A as $A = A_0 + A_1 + A_2$. Let $\pi = (\pi_0, \pi_1, \cdots, \pi_S)$ be the steady state probability vector of A. Then we have $\pi A = 0$ and $\pi e = 1$. Solving $\pi A = 0$ we get

$$
\begin{aligned}
\pi_0 &= \frac{\mu}{\beta+\delta}\pi_1 + \frac{\delta}{\beta+\delta} \\
\pi_k &= (\frac{\beta+\delta+\mu}{\mu})^{k-1}\pi_1 \text{ for } k = 2, \dots, s+1 \\
\pi_{s+k} &= (\frac{\delta+\mu}{\mu})^{k-1}(\frac{\beta+\delta+\mu}{\mu})^{s}\pi_1 \text{ for } k = 2, \dots, S-2s \\
\pi_{Q+k} &= \frac{\delta+\mu}{\mu}\pi_{Q+k-1} - \frac{\beta}{\mu}\pi_{k-1}, \quad \text{where } Q = S-s, \quad k = 1, 2, \dots, s,
\end{aligned}
$$

Here $\pi_0, \pi_2, \dots, \pi_S$ are all expressed in terms of π_1. From $\pi e = 1$ we can find π_1 and hence $\pi_0, \pi_2, \dots, \pi_S$.

THEOREM 5.7.1. *The Markov chain is stable if and only if* $\lambda < \mu(1-\pi_0)$ *where*

$$
\pi_0 = \frac{\delta+\mu+\delta M}{(\delta+\mu+\beta)+(\delta+\beta)M},
$$

and

$$M = \Big(\frac{\beta+\delta+\mu}{\beta+\delta}\Big)(x^{s-1}-1) + \frac{\mu}{\delta}x^s(y^{S-2s}-1) + \frac{\mu}{\delta}\Big[\frac{\beta}{\beta+\delta}(x^s-1) - (\frac{x}{y})^s + 1\Big]$$

with $x = \frac{\beta+\delta+\mu}{\mu}$, $y = \frac{\delta+\mu}{\mu}$

PROOF. From the well known results (see Neuts [**53**]) on positive recurrence of $\overline{Q}$ which states that $\pi A_0 e < \pi A_2 e$. Simplifying this we get $\lambda < \mu(1-\pi_0)$

□

5.7.2. Steady state analysis. Let $X = (x(0), x(1), \dots)$ be the steady state probability vector of the Markov chain. Here again the model is a LIQBD process, its steady state probability distribution has a matrix-geometric solution to the equations $X\overline{Q} = 0$ and $Xe = 1$. Then $x(i)$ has the matrix geometric form

$$x(i) = x(1)R^{i-1} \text{ for } i \geq 2 \tag{5.7.1}$$

where R is the minimal non-negative solution of the matrix equation
$A_0 + RA_1 + R^2A_2 = 0$. $X\overline{Q} = 0$ gives

$$x(0)A_{00} + x(1)A_2 = 0 \tag{5.7.2}$$

$$x(0)A_0 + x(1)(A_1 + RA_2) = 0 \tag{5.7.3}$$

Solving the above equations we can find vectors $x(0)$ and $x(1)$ subject to the normalizing condition $Xe = 1$, that is $x(0)e + x(1)\ (1-R)^{-1} = 1$, then $x(i)$, for $i \geq 2$, can be obtained from (5.7.1).

5.8. Performance Measures

Having computed the system state probabilities, we proceed to find out how the system performs. Let $X = (x(0), x(1), \dots)$ be the steady-state probability vector of $\overline{Q}$ where $x(i) = (y_{i0}, y_{i1}, \dots, y_{iS})$.

(1) Expected number of customers, EC in the system is given by

$$\text{EC} = \sum_{i=1}^{\infty} i x(i) e$$

(2) Expected inventory level EI is given by

$$\text{EI} = \sum_{i=0}^{\infty} \sum_{j=1}^{S} j y_{ij}$$

(3) Expected waiting time in the system EW is given by

$$\text{EW} = \frac{\text{EC}}{\lambda[1 - \sum_{i=0}^{\infty} y_{i0}]}.$$

(4) Expected re-order rate ER is given by

$$\text{ER} = \mu \sum_{i=1}^{\infty} y_{i,s+1} + \delta \sum_{i=0}^{\infty} \sum_{j=s+1}^{S} y_{ij}$$

(5) Expected number of inventory, ET lost due to disaster is given by $\text{ET} = \delta$ EI.

(6) Expected number of departures, ED after completing service is given by

$$\text{ED} = \mu \sum_{i=0}^{\infty} \sum_{j=1}^{S} y_{ij}$$

5.9. Cost Function and Numerical Examples

To construct the cost function we define the following costs as

$C =$ fixed ordering cost

$C_1 =$ procurement cost/unit

$C_2 =$ holding cost of inventory/unit/unit time

$C_3 =$ revenue from service/unit/unit time

$C_4 =$ disaster cost/unit

Then the total expected cost is defined as

$\text{ETC} = [C + QC_1]\text{ER} + C_2\text{EI} - C_3\text{ED} + C_4\text{ET}$

S	EC I	EC II	EI I	EI II	ER I	ER II	ET I	ET II
	Models							
25	2.4999	4.0927	14.6654	14.6998	1.1973	1.3005	1.4665	1.4699
26	2.4999	4.0925	14.9948	15.0375	1.1343	1.2384	1.4994	1.5037
27	2.4999	4.0923	15.3514	15.3956	1.0742	1.1785	1.5351	1.5395
28	2.4999	4.0921	15.7344	15.7743	1.0169	1.1208	1.5734	1.5774
29	2.4999	4.0920	16.1429	16.1739	0.9624	1.0653	1.6142	1.6173
30	2.4999	4.0919	16.5761	16.5941	0.9105	1.0120	1.6576	1.6594
31	2.4999	4.0917	17.0333	17.0349	0.8612	0.9609	1.7033	1.7034

TABLE 5.1. Variations in maximum inventory level S. $\lambda = 1$, $\mu = 1.4$, $\beta = 1$, $\delta = 0.1$, $s = 10$

s	EC I	EC II	EI I	EI II	ER I	ER II	ET I	ET II
	Models							
8	2.4999	4.0934	16.36773	16.26279	0.6888	0.7769	1.6367	1.62628
9	2.4999	4.0925	16.39407	16.35455	0.7911	0.8864	1.6394	1.63545
10	2.4999	4.0919	16.57617	16.59414	0.9105	1.0120	1.6576	1.65941
11	2.4999	4.0915	16.93898	16.99960	1.0495	1.1557	1.6939	1.69996
12	2.4999	4.0913	17.51132	17.58924	1.2114	1.3194	1.7511	1.75892
13	2.4999	4.0911	18.32643	18.37972	1.3997	1.5050	1.8326	1.83797
14	2.4999	4.0910	20.33167	19.38010	1.6186	2.0842	2.0331	1.93801

TABLE 5.2. Variations in reorder level s. $\lambda = 1$, $\mu = 1.4$, $\beta = 1$, $\delta = 0.1$, $S = 30$

μ	EC I	EC II	EI I	EI II	ER I	ER II	EW I	EW II
	Models							
1.5	2.0000	3.0921	14.6654	14.7139	1.1973	1.3020	2.0000	3.4016
1.6	1.6666	2.4918	14.6654	14.7260	1.1973	1.3033	1.6666	2.7412
1.7	1.4287	2.0916	14.6654	14.7367	1.1973	1.3045	1.4285	2.3009
1.8	1.2500	1.8057	14.6654	14.7461	1.1973	1.3055	1.2500	1.9865
1.9	1.1111	1.5913	14.6654	14.7545	1.1973	1.3065	1.1111	1.7506
2.0	1.0000	1.4246	14.6654	14.7619	1.1973	1.3073	1.0000	1.5672
2.1	0.9090	1.2912	14.6654	14.7686	1.1973	1.3080	1.0000	1.4205

TABLE 5.3. Variations in service rate μ. $\lambda = 1$, $\beta = 1$, $\delta = 0.1$, $s = 10$, $S = 25$

	EC		ER		EW		EJ
	Models						
λ	I	II	I	II	I	II	I
1.7	1.3076	2.0780	1.8584	2.0267	0.7692	1.3463	0.1557
1.8	1.5000	2.4432	1.9505	2.1269	0.8333	1.4956	0.1653
1.9	1.7272	2.8995	2.0423	2.2268	0.9090	1.6823	0.1750
2.0	2.0000	3.4860	2.1336	2.3266	0.9999	1.9224	0.1849
2.1	2.3333	4.2672	2.2246	2.4262	1.1110	2.2425	0.1949
2.2	2.7500	5.3596	2.3152	2.5250	1.2500	2.6903	0.2051
2.3	3.2857	6.9943	2.4054	2.6206	1.4285	3.3602	0.2156

TABLE 5.4. Variations in arrival rate λ. $\mu = 3$, $\beta = 1$, $\delta = 0.1$, $s = 10$, $S = 25$

	EC		EI		ER		ET	
	Models							
δ	I	II	I	II	I	II	I	II
.05	2.0000	2.4929	15.7786	15.8006	1.1225	1.1735	0.7889	0.7900
.10	2.0000	3.0921	14.6654	14.7139	1.1973	1.3020	1.4665	1.4713
.15	1.9999	3.8463	13.6755	13.7484	1.2297	1.3910	2.0513	2.0622
.20	1.9999	4.8356	12.8101	12.9005	1.2291	1.4485	2.5620	2.5801
.25	2.0000	6.2035	12.0604	12.1556	1.2052	1.4824	3.0151	3.0389
.30	2.0000	8.2363	11.4124	11.4814	1.1670	1.4971	3.4237	3.4444
.35	2.0000	11.6023	10.8508	10.7863	1.1210	1.6471	3.7977	3.7752

TABLE 5.5. Variations in disaster rate δ. $\lambda = 1$, $\mu = 1.5$, $\beta = 1$, $s = 10$, $S = 25$

	EC		EI		EW		EJ
	Models						
β	I	II	I	II	I	II	I
1.1	2.4999	3.8735	14.8726	14.9154	2.4999	4.2258	0.0833
1.3	2.4999	3.5717	15.2012	15.2531	2.4999	3.8465	0.0714
1.5	2.4999	3.3751	15.4498	15.5054	2.4999	3.6001	0.0625
1.7	2.4999	3.2375	15.6446	15.7010	2.4999	3.4280	0.0555
1.9	2.4999	3.1363	15.8013	15.8571	2.4999	3.3014	0.0500
2.1	2.4999	3.0589	15.9300	15.9845	2.4999	3.2046	0.0454
2.3	2.4999	2.9979	16.0377	16.0905	2.4999	3.1283	0.0416

TABLE 5.6. Variations in replenishment rate β. $\mu = 1.4$, $\lambda = 1$, $\delta = 0.1$, $s = 10$, $S = 25$

5.9.1. Interpretation of the Numerical results.

1. *Effect of the maximum inventory level S on various performance measures*

From table 5.1 we conclude that as S increases, inventory level and thus the inventory lost due to disaster increase. Due to the availability of more inventory, reorder rate decreases as the time interval to reach the reorder level increases. The number of customers do not change in model I as customers do not join when the inventory level is zero, while in model II it decreases.

2. *Effect of the reorder level s on various performance measures*

   Table 5.2 shows that the changes on various performance measures as s changes is similar to that of S, except the reorder rate. Here when s increase reorder rate increases as the time interval to reach the reorder level decreases and more orders are placed.

3. *Effect of the service rate μ on various performance measures*

   When service rate increases as we expect the number of customers and hence the waiting time of customers decrease in both models. Reorder rate and hence inventory increase in model II as customers join even when inventory is zero, while in Model I both remain the same, as customers do not join when inventory level is zero (see table 5.3).

4. *Effect of the arrival rate λ on various performance measures*

   When arrival rate increases as we expect the number of customers, the waiting time and the reorder rate increase in both models. The number of customers who do not join when the inventory level is zero also increases (see table 5.4).

5. *Effect of the disaster rate δ on various performance measures*

   Table 5.5 shows that as δ increases the inventory lost due to disaster increases and so the inventory decrease in both models. Number of customers and reorder rate increase in model II as customers join even when the inventory is zero, while in model I, number of customers is same as customers do not join the system when the inventory level is zero. In Model I reorder rate increases first and then decreases as customers do not join when the inventory level zero. (see the formula for ER)

6. *Effect of the replenishment rate β on various performance measures*

   From table 5.6 we can understand that as β increases inventory level increases in both models. The number of customers and hence the waiting time of customers in model II decrease as the replenishment rate of inventory increase, while in model I no change as customers do not join the system when the inventory is zero. As more inventory is with the system, in model I, the number of customers who do not join when the inventory level is zero (EJ) decreases.

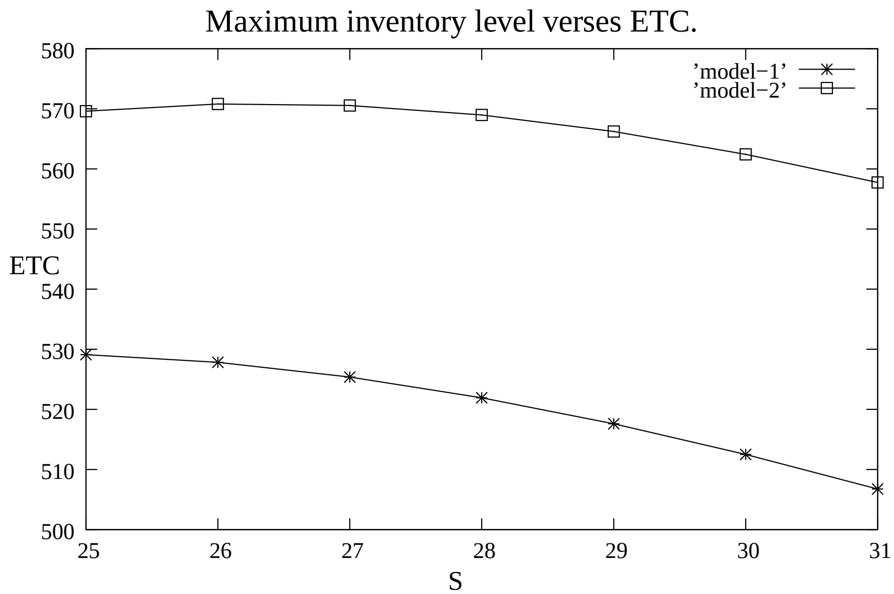


FIGURE 5.1. $\lambda = 1$, $\mu = 1.4$, $\beta = 1$, $\delta = 0.1$, $s = 10$, $C = 100$, $C_1 = 20$, $C_2 = 1$, $C_3 = 5$, $C_4 = 27$, $C_5 = 5$

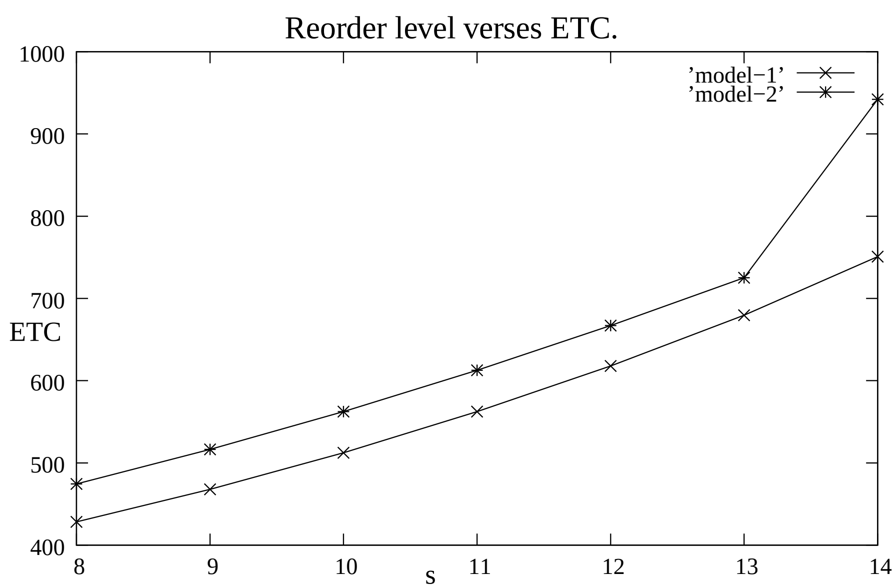


FIGURE 5.2. $\lambda = 1$, $\mu = 1.4$, $\beta = 1$, $\delta = 0.1$, $S = 30$, $C = 100$, $C_1 = 20$, $C_2 = 1$, $C_3 = 5$, $C_4 = 27$, $C_5 = 5$

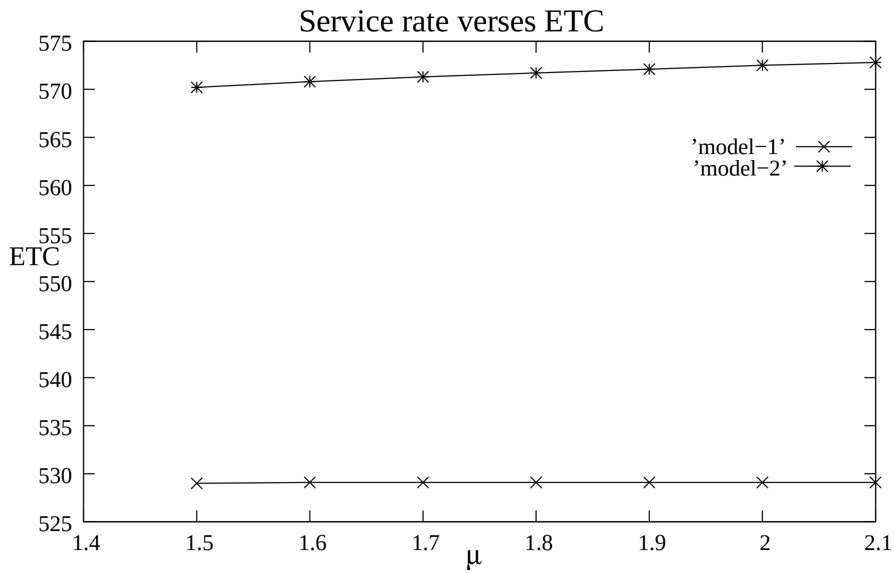


FIGURE 5.3. $\lambda = 1$, $\beta = 1$, $\delta = 0.1$, $s = 10$, $S = 25$ $C = 100$, $C_1 = 20$, $C_2 = 1$, $C_3 = 5$, $C_4 = 27$, $C_5 = 5$

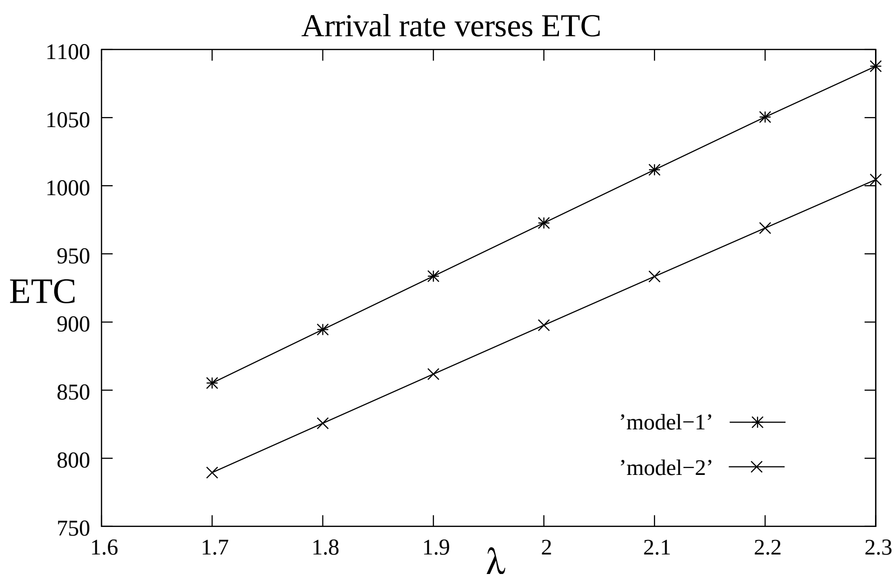


FIGURE 5.4. $\mu = 3$, $\beta = 1$, $\delta = 0.1$, $s = 10$, $S = 25$, $C = 100$, $C_1 = 20$, $C_2 = 1$, $C_3 = 5$, $C_4 = 27$, $C_5 = 5$.

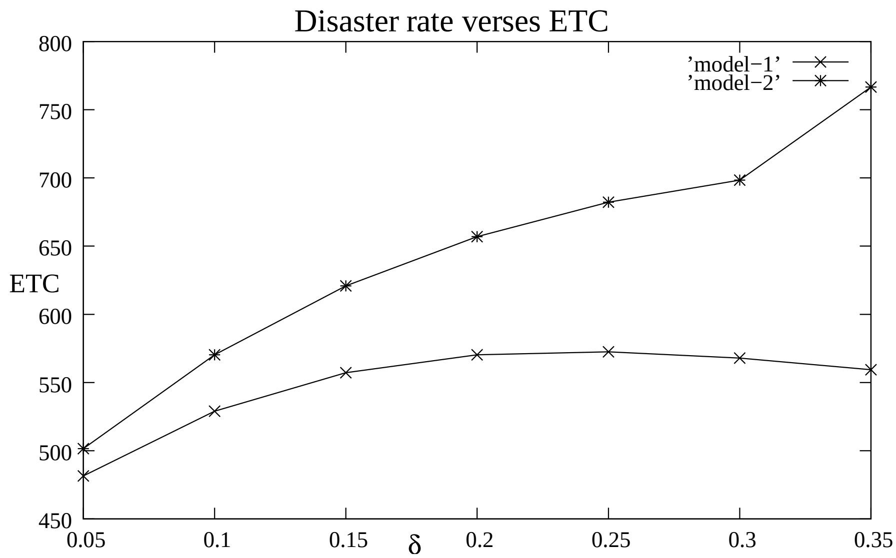


FIGURE 5.5. $\mu = 1.5$, $\beta = 1$, $s = 10$, $S = 25$, $C = 100$, $C_1 = 20$, $C_2 = 1$, $C_3 = 5$, $C_4 = 27$, $C_5 = 5$

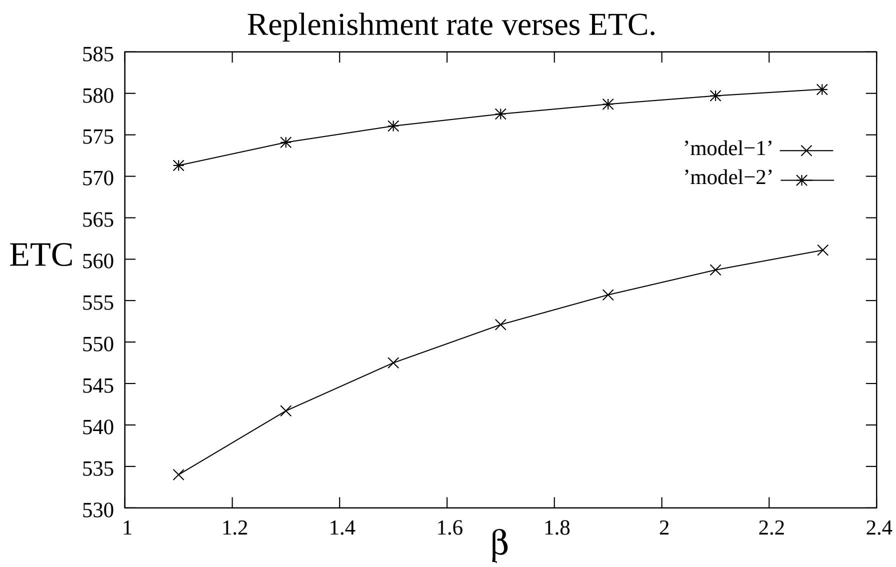


FIGURE 5.6. $\mu = 1.4$, $\lambda = 1$, $\delta = 0.1$, $s = 10$, $S = 25$, $C = 100$, $C_1 = 20$, $C_2 = 1$, $C_3 = 5$, $C_4 = 27$, $C_5 = 5$.

5.9.2. Interpretation of the graphs. Figure 5.1 shows the variation of the cost (ETC) with the maximum inventory level S. When S increases the cost decreases, it may be due to the decrease in the reorder rate. The cost increases when s increases as the reorder rate increases (see figure 5.2). From figure 5.3, we can understand that service rate does not affect the cost function in model I as the reorder rate and inventory is same, in model II the cost slightly increases. Total expected cost increases as arrival rate increases (see figure 5.4). Figure 5.5 shows that when the disaster rate increases in model II the cost increases as the reorder rate increases. In model I as the reorder rate first increases and then decreases, the cost function also behaves like that. Figure 5.6 shows that when replenishment rate increases the expected cost increases. This may be due to the increase in the reorder rate.

5.10. Conclusion

From all the graphs we may conclude that the expected cost of model I is less than model II. So model I is profitable with the given cost function and parameters. That is it is better for the system to not allow the customers to join when the inventory is zero.

CHAPTER 6

Inventory with Positive Service Time—Multi-Server Retrial Model

6.1. Introduction

Compared to single server retrial queue the study of multiserver retrial queue is more involved and needs much sophisticated tools. Artalejo et.al [**4, 5**] analyzed both single server and multiserver retrial queues. Artalejo et.al [**7**] studied the numerical solution of the multi-server retrial queues where the retrial rate is assumed to be constant. Krishnakumar et.al [**28**] discussed a multi-server retrial queue in which the server takes a Bernoulli vacation and obtained the solution using matrix-geometric technique. In 2002 Artalejo and Pozo [**10**] modelled a multiserver retrial queue in which inter retrial times follow an exponential distribution. They introduced a new approximation technique by assuming that the retrial rate depends on the system state (i, j) where i denotes the number of busy servers and j, the number of customers in the orbit.

In this chapter we consider a multiserver inventory model with retrial of customers. Customers arrive according to a Poisson process with rate λ. There are c identical servers and service times are exponentially distributed with parameter μ. We follow (s, S) inventory policy and lead times are exponentially distributed with parameter β. When the inventory level depletes to s (≥ 0) we place an order for $Q = S - s$ quantity of inventory. We assume that there is no waiting space in the system except for those undergoing service. If an arriving customer finds all servers busy it proceeds to an orbit and makes repeated attempts until it finds at least one of the c servers idle. We assume that customers do not join the orbit when the inventory level is zero. The inter retrial times follow exponential distribution with constant rate θ. Each demand is exactly for one item of the inventory. The number of servers c is assumed to satisfy the condition $c < s$. The purpose is to ensure that at the beginning part of the lead time itself a few servers should not be forced to be idle for want of item for service.

Let

$N(t) =$ number of customers in the orbit at time t

$C(t) =$ number of busy servers at time t.

$I(t) =$ Inventory level at time t.

Then $\{(N(t), C(t), I(t));\ t \geq 0\}$ is a CTMC on the state space $\{(i, k, j), i \geq 0, 0 \leq k \leq c, k \leq j \leq S\}$. The system can be studied as a LIQBD. The infinitesimal generator $\overline{Q}$ of this Markov chain is a block tri diagonal matrix and it has the following form

$$\overline{Q} = \begin{bmatrix} B_0 & A_0 & 0 & 0 & 0 & \cdots \\ A_2 & A_1 & A_0 & 0 & 0 & \cdots \\ 0 & A_2 & A_1 & A_0 & 0 & \cdots \\ 0 & 0 & A_2 & A_1 & A_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \end{bmatrix}$$

where the blocks B_0, A_0, A_1, A_2 are square matrices of order $\frac{c+1}{2}[\{2(S+1)-c\}]$; these are given as follows

$$B_0 = \begin{bmatrix} \overline{B}_{10} & B_{00} & & & & & \\ B_{21} & B_{11} & B_{01} & & & & \\ & B_{22} & B_{12} & B_{02} & & & \\ & & & \ddots & & & \\ & & & & B_{2,c-1} & B_{1,c-1} & B_{0,c-1} \\ & & & & & B_{2,c} & B_{1,c} \end{bmatrix},$$

$$A_0 = \begin{bmatrix} 0 & & & & \\ & M_1 & & & \\ & & M_2 & & \\ & & & \ddots & \\ & & & & M_c \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & K_0 & 0 & \cdots & 0 \\ 0 & 0 & K_1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & 0 & \cdots & K_{c-1} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} \overline{A}_{10} & A_{00} & & & & & \\ A_{21} & A_{11} & A_{01} & & & & \\ & A_{22} & A_{12} & A_{02} & & & \\ & & & & \ddots & & \\ & & & & A_{2,c-1} & A_{1,c-1} & A_{0,c-1} \\ & & & & & A_{2c} & A_{1c} \end{bmatrix}.$$

Next we describe the entries in the above matrices

$$\overline{B}_{10} = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ s \\ s+1 \\ \vdots \\ S \end{array} \begin{array}{c} \begin{array}{ccccccccc} 0 & 1 & \cdots & s & s+1 & \cdots & S-s & \cdots & S \end{array} \\ \begin{pmatrix} -\beta & & & & & & \beta & & \\ & -(\lambda+\beta) & & & & & & \ddots & \\ & & & & & & & & \\ & & \ddots & -(\lambda+\beta) & & & & & \beta \\ & & & & -\lambda & & & & \\ & & & & & & \ddots & & \\ & & & & & & & & -\lambda \end{pmatrix}_{(S+1)\times(S+1)} \end{array}$$

$$B_{1i} = \begin{array}{c} \\ i \\ \vdots \\ s \\ s+1 \\ \vdots \\ S \end{array} \begin{array}{c} \begin{array}{cccccccc} i & \cdots & s & s+1 & \cdots & S-s & \cdots & S \end{array} \\ \begin{pmatrix} -\Omega & & & & & \beta & & \\ & \ddots & & & & & \ddots & \\ & & -\Omega & & & & & \beta \\ & & & -\omega & & & & \\ & & & & & \ddots & & \\ & & & & & & & -\omega \end{pmatrix}_{(S-i+1)\times(S-i+1)} \end{array}$$

where $\Omega = \lambda + i\mu + \beta$, $\omega = \lambda + i\mu$, $1 \le i \le c$.

$$B_{0i} = \begin{bmatrix} 0 \\ \lambda I_{S-i} \end{bmatrix}_{(S-i+1)\times(S-i)} \quad \text{for } 0 \le i \le c-1.$$

$B_{2i} = [i\mu I_{S-i+1} \;\; 0]_{(S-i+1)\times(S-i+2)}$ for $1 \le i \le c$.

$M_i = \begin{bmatrix} \lambda e_1 \\ 0 \end{bmatrix}_{(S-i+1)\times(S-i+1)}$ where e_j is a row vector with 1 in the j^{th} place, and zeros elsewhere, for $1 \le i \le c$.

$M_c = \lambda I_{S-c+1}$.

Further $K_i = \begin{bmatrix} 0 \\ \theta I_{S-i} \end{bmatrix}_{(S-i+1)\times(S-i)}$ for $0 \le i \le c-1$.

$$\overline{A}_{10} = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ s \\ s+1 \\ \vdots \\ S \end{array} \begin{array}{c} \begin{array}{ccccccccc} 0 & 1 & \cdots & s & s+1 & \cdots & S-s & \cdots & S \end{array} \\ \begin{pmatrix} -\beta & & & & & & \beta & & \\ & -\Delta & & & & & & & \\ & & \ddots & & & & & \ddots & \\ & & & -\Delta & & \cdots & & & \beta \\ & & & & -\psi & & & & \\ & & & & & & \ddots & & \\ & & & & & & & & -\psi \end{pmatrix}_{(S+1)\times(S+1)} \end{array}$$

where $\Delta = \lambda + \beta + \theta$, $\psi = \lambda + \theta$.

$$A_{1i} = \begin{array}{c} \\ i \\ i+1 \\ \vdots \\ s \\ s+1 \\ \vdots \\ S \end{array} \begin{array}{c} \begin{array}{ccccccccc} i & i+1 & \cdots & s & s+1 & \cdots & S-s & \cdots & S \end{array} \\ \begin{pmatrix} -\Omega & & & & & & \beta & & \\ & -\delta & & & & & & & \\ & & \ddots & & & & & \ddots & \\ & & & -\delta & & \cdots & & & \beta \\ & & & & -\phi & & & & \\ & & & & & \ddots & & & \\ & & & & & & & & -\phi \end{pmatrix}_{(S-i+1)\times(S-i+1)} \end{array}$$

for $1 \le i \le c-1$ with $\Omega = \lambda + i\mu + \beta$, $\delta = \lambda + i\mu + \beta + \theta$, $\phi = \lambda + i\mu + \theta$.

$$
A_{1c} = \begin{array}{cc}
 & \begin{array}{ccccccccc} c & c+1 & \cdots & s & s+1 & \cdots & S-s & \cdots & S \end{array} \\
\begin{array}{c} c \\ c+1 \\ \vdots \\ s \\ s+1 \\ \vdots \\ S \end{array} &
\begin{pmatrix}
-\Omega & & & & & & \beta & & \\
 & -\Omega & & & & & & & \\
 & & \ddots & & & & & \ddots & \\
 & & & -\Omega & & \cdots & & & \beta \\
 & & & & -\omega & & & & \\
 & & & & & & \ddots & & \\
 & & & & & & & & -\omega
\end{pmatrix}_{(S-c+1)\times(S-c+1)}
\end{array}
$$

where $\Omega = \lambda + i\mu + \beta$, $\omega = \lambda + i\mu$.

$A_{2i} = \begin{bmatrix} i\mu I_{S-i+1} & 0 \end{bmatrix}_{(S-i+1)\times(S-i+2)}$ for $1 \leq i \leq c$.

$A_{0i} = \begin{bmatrix} 0 \\ \lambda I_{S-i} \end{bmatrix}_{(S-i+1)\times(S-i)}$ for $0 \leq i \leq c-1$

6.2.1. Stability condition. The matrix $A = A_0 + A_1 + A_2$ is the generator matrix of the Markov chain

Let $\pi = \left(\pi_{(0,0)}, \pi_{(0,1)}, \ldots, \pi_{(0,S)}, \pi_{(1,1)}, \ldots, \pi_{(1,S)}, \pi_{(2,2)} \ldots \pi_{(2,S)} \ldots \pi_{(c,c)} \ldots \pi_{(c,S)}\right)$

be the stationary probability vector of A. By solving the equation $\pi A = 0$ and $\pi e = 1$ we get π. The system is stable if and only if it satisfies the drift condition $\pi A_0 e < \pi A_2 e$ (see Neuts [**53**]). After some calculations this reduces to

$$\lambda \left[\sum_{i=1}^{c} \pi(i,i) + \sum_{j=c+1}^{S} \pi(c,j) \right] < \theta \sum_{i=0}^{c-1} \sum_{j=i+1}^{S} \pi(i,j).$$

Let $X = (x(0), x(1), x(2), \ldots)$ be the steady-state probability vector of $\overline{Q}$ such that

$$X\overline{Q} = 0 \text{ and } Xe = 1. \tag{6.2.1}$$

Using the matrix-geometric theorem [**53**] we have

$$x(i) = x(0)R^i, \quad i \geq 1, \tag{6.2.2}$$

where R is the rate matrix which is the minimal non-negative solution of $A_0 + RA_1 + R^2A_2 = 0$. Then equation (6.2.1) becomes

$$x(0)(B_0 + RA_2) = 0 \text{ and } x(0)(I - R)^{-1}e = 1. \tag{6.2.3}$$

R can be obtained using the successive iterative method:
$R(n+1) = -(A_0 + R^2(n)A_2)A_1^{-1}$, $n = 0, 1, 2, \ldots$; with $R(0) = 0$ and $R(n)$ are computed until

$$\max_{i,j}[R_{ij}(n+1) - R_{ij}(n)] < \epsilon$$

where $\epsilon > 0$ and sufficiently small. Then $x(i)$, $i \geq 0$ can be uniquely determined from (6.2.2) and (6.2.3).

6.3. System Performance Measures

We partition the stationary probability vector X of $\overline{Q}$ as $X = (x(0), x(1), \ldots)$ where each $x(i)$ is further partitioned as

$$x(i) = (y_{i00}, y_{i01}, \ldots, y_{i0S}, y_{i11}, \ldots, y_{i1S}, y_{i22}, \ldots, y_{i2S}, \ldots, y_{icc}, \ldots, y_{icS}).$$

Then we have the following performance measures.

(1) Expected number of customers EC in the orbit, is given by

$$\text{EC } = \sum_{i=0}^{\infty} ix(i)e.$$

(2) Expected inventory level EI is given by

$$\text{EI } = \sum_{i=0}^{\infty}\sum_{j=0}^{c}\sum_{k=j}^{S} ky_{ijk}$$

(3) Expected waiting time of a customer in the orbit EW is given by

$$\text{EW} = \frac{\text{EC}}{\lambda}.$$

(4) Probability that an arriving primary customer goes to orbit

$$= \sum_{i=0}^{\infty} \sum_{k=c}^{S} y_{ick}$$

(5) The overall retrial rate OR is given by

$$\text{OR} = \theta \sum_{i=i}^{\infty} x(i)e.$$

(6) The successful retrial rate SR a given by

$$\text{SR} = \theta \sum_{i=1}^{\infty} \sum_{j=0}^{c-1} \sum_{k=j+1}^{S} y_{ijk}.$$

(7) Probability that all servers are idle

$$= \sum_{i=0}^{\infty} \sum_{k=0}^{S} y_{i0k}.$$

(8) Expected re-order rate ER is given by

$$\text{ER} \ = c\mu \sum_{i=0}^{\infty} \sum_{j=1}^{c} y_{i,j,s+1}$$

(9) Expected number of departures ED after completing service, is given by

$$\text{ED} \ = c\mu \sum_{i=0}^{\infty} \sum_{j=1}^{c} \sum_{k=j}^{S} y_{ijk}.$$

(10) Expected number of customers EJ not joining the systems when the inventory level is zero is given by

$$\text{EJ} = \lambda \sum_{i=0}^{\infty} y_{i00}.$$

(11) Mean number of busy servers ES is given by

$$\text{ES} = \sum_{i=0}^{\infty} \sum_{j=1}^{c} \sum_{k=j}^{S} j y_{ijk}.$$

6.4. Cost Function and Numerical Examples

To construct the cost function we define the following cost

$C =$ fixed ordering cost

$C_1 =$ procurement cost/unit

$C_2 =$ holding cost of inventory/unit/unit time

$C_3 =$ revenue from service/unit/unit time

$C_4 =$ shortage cost of inventory/unit/unit time

$C_5 =$ server cost/unit/unit time

$c =$ the number of servers

The total expected cost ETC is defined as

$$\text{ETC} = [C + QC_1]\text{ER} + C_2\text{EI} - C_3\text{ED} + C_4\text{EJ} + C_5\text{c}$$

	EC			EI			ER		
S	$c=3$	$c=4$	$c=5$	$c=3$	$c=4$	$c=5$	$c=3$	$c=4$	$c=5$
22	2.3064	0.7316	0.5067	5.2436	1.9675	1.0704	0.4134	0.2376	0.1861
25	2.0852	0.6142	0.4024	5.8656	2.0937	1.0583	0.3200	0.1547	0.0934
28	1.9480	0.5402	0.3365	6.4989	2.2472	1.0787	0.2625	0.1165	0.0586
31	1.8547	0.4898	0.2919	7.1373	2.4132	1.1142	0.2227	0.0934	0.0409
34	1.7872	0.4532	0.2597	7.7785	2.5861	1.1575	0.1934	0.0780	0.0308

TABLE 6.1. Variations in maximum inventory level S. $\lambda = 6$, $\mu = 4$, $\beta = 1$, $\theta = 4$, $s = 10$

	EC			ENB			EW		
μ	$c=3$	$c=4$	$c=5$	$c=3$	$c=4$	$c=5$	$c=3$	$c=4$	$c=5$
4.2	9.5283	1.3136	0.8319	1.0796	0.5328	0.3410	1.5880	0.2189	0.1386
4.4	5.7130	1.1230	0.7460	0.9532	0.4626	0.2990	0.9521	0.1871	0.1243
4.6	3.9978	0.9769	0.6755	0.8394	0.4044	0.2642	0.6663	0.1628	0.1125
4.8	3.0301	0.8618	0.6166	0.7425	0.3557	0.2350	0.5050	0.1436	0.1027
5.0	2.4126	0.7691	0.5667	0.6599	0.3147	0.2103	0.4021	0.1281	0.0944
5.2	1.9868	0.6930	0.5239	0.5891	0.2799	0.1892	0.3311	0.1155	0.0873

TABLE 6.2. Variations in service rate μ. $\lambda = 6$, $\beta = 1$, $\theta = 2$, $s = 5$, $S = 25$

λ	EC			EW			EJ		
	$c=3$	$c=4$	$c=5$	$c=3$	$c=4$	$c=5$	$c=3$	$c=4$	$c=5$
5.0	0.9451	0.3118	0.2093	0.1890	0.0623	0.0418	0.1810	0.1661	0.1623
5.2	1.1823	0.3756	0.2492	0.2273	0.0722	0.0479	0.2136	0.1952	0.1905
5.4	1.4898	0.4506	0.2951	0.2759	0.0834	0.0546	0.2499	0.2274	0.2215
5.6	1.8990	0.5387	0.3475	0.3391	0.0962	0.0620	0.2901	0.2629	0.2557
5.8	2.4631	0.6425	0.4075	0.4246	0.1107	0.0702	0.3343	0.3017	0.2929
6.0	3.2809	0.7654	0.4759	0.5468	0.1275	0.0793	0.3827	0.3440	0.3334

TABLE 6.3. Variations in arrival rate λ. $\mu = 4$, $\beta = 1$, $\theta = 3$, $s = 10$, $S = 25$

β	EC			EJ			EW		
	$c=3$	$c=4$	$c=5$	$c=3$	$c=4$	$c=5$	$c=3$	$c=4$	$c=5$
1.0	3.2809	0.7654	0.4759	0.3827	0.3440	0.3334	0.5468	0.1275	0.0793
1.2	2.8933	0.6139	0.3461	0.2378	0.2144	0.2078	0.4822	0.1023	0.0576
1.4	2.6420	0.5139	0.2607	0.1515	0.1371	0.1330	0.4403	0.0856	0.04346
1.6	2.4734	0.4463	0.2031	0.0987	0.0897	0.0871	0.4122	0.0743	0.0338
1.8	2.3572	0.3996	0.1635	0.0655	0.0599	0.0582	0.3928	0.0666	0.0272
2.0	2.2755	0.3668	0.1357	0.0443	0.0407	0.0396	0.3792	0.0611	0.0226

TABLE 6.4. Variations in replenishment rate β. $\lambda = 6$, $\mu = 4$, $\theta = 3$, $s = 10$, $S = 25$

θ	SR			EW			EC		
	$c=3$	$c=4$	$c=5$	$c=3$	$c=4$	$c=5$	$c=3$	$c=4$	$c=5$
2.0	1.1838	0.5042	0.2838	1.8608	0.1882	0.1061	11.1653	1.1295	0.6371
2.1	1.1990	0.5049	0.2840	1.4708	0.1785	0.1021	8.8248	1.0715	0.6128
2.2	1.2059	0.5056	0.2841	1.2207	0.1701	0.0985	7.3246	1.0207	0.5911
2.3	1.2099	0.5062	0.2842	1.0468	0.1626	0.0952	6.2809	0.9758	0.5716
2.4	1.2126	0.5069	0.2844	0.9188	0.1559	0.0923	5.5129	0.9358	0.5540
2.5	1.2168	0.5075	0.2845	0.7430	0.1500	0.0896	4.4580	0.9000	0.5380

TABLE 6.5. Variations in retrial rate θ. $\lambda = 6$, $\mu = 4$, $\beta = 1$, $s = 10$, $S = 25$

6.4.1. Interpretation of the Numerical Results.

1. *Effect of the maximum inventory level S on various performance measures.*

   Table 6.1 shows that as S increases the inventory level increases; it is least when $c = 5$ and most in $c = 3$. The number of customers decreases as the number of servers increases. When S increases the time interval to reach the reorder level increases and so the reorder rate decreases.

2. *Effect of the service rate μ on various performance measures*

   Expected number of customers and the waiting time decrease as the service rate increases. As the service rate increases the number of busy servers decreases (see table 6.2).

3. *Effect of the arrival rate λ on various performance measures*

   From table 6.3 we can understand that as λ increases the number of customers and the waiting time increase. When the number of servers increases (from $c = 3$ to $c = 5$) the number of customers and their waiting time decrease. As the arrival rate increases the number of customers not joining the orbit when the inventory level is zero also increases.

4. *Effect of the replenishment rate β on various performance measures*

   Table 6.4 shows that when β increases the number of customers and the waiting time decreases as the system get the inventory quickly. The number of customers who do not join when the inventory level is zero, also decreases as the system has more inventory.

5. *Effect of the retrial rate θ on various performance measures*

   As retrial rate increases successful retrial rate increases, so the number of customers and waiting time decreases. When the number of servers increases ($c = 3$ to $c = 5$) the number of customers and their waiting time decreases (see table 6.5)

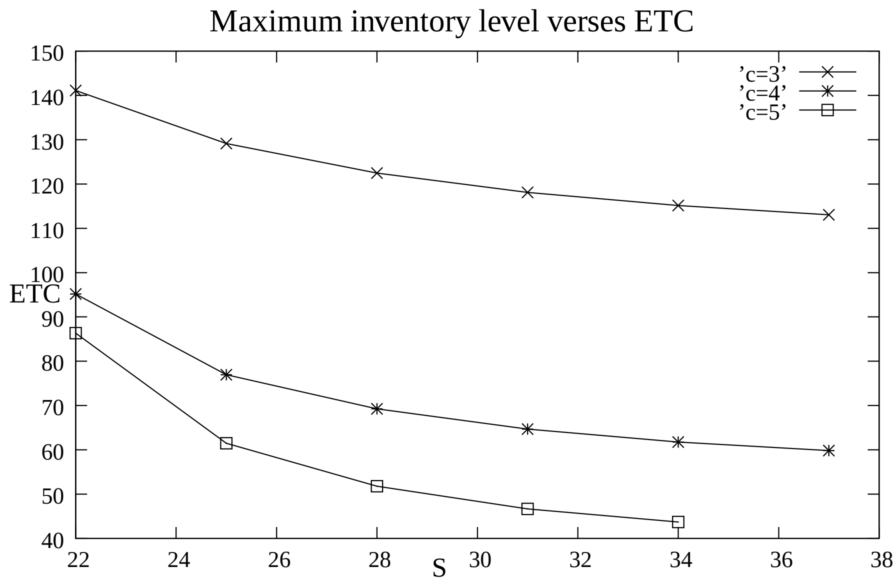


FIGURE 6.1. $\lambda = 6$, $\mu = 4$, $\beta = 1$, $\theta = 4$, $s = 10$, $C = 100$, $C_1 = 20$, $C_2 = 1$, $C_3 = 5$, $C_4 = 5$, $C_5 = 6$

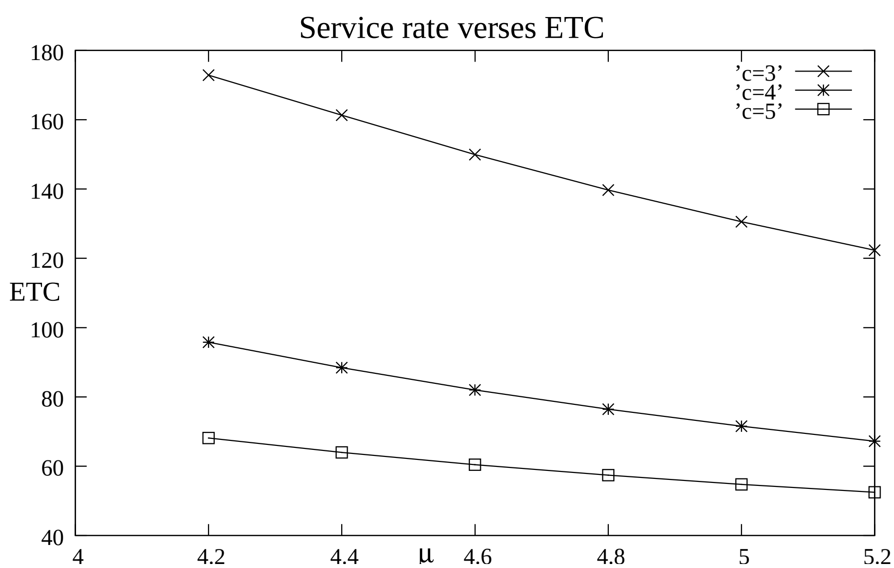


FIGURE 6.2. $\lambda = 6$, $\beta = 1$, $\theta = 2$, $s = 5$, $S = 25$, $C = 100$, $C_1 = 20$, $C_2 = 1$, $C_3 = 5$, $C_4 = 5$, $C_5 = 6$

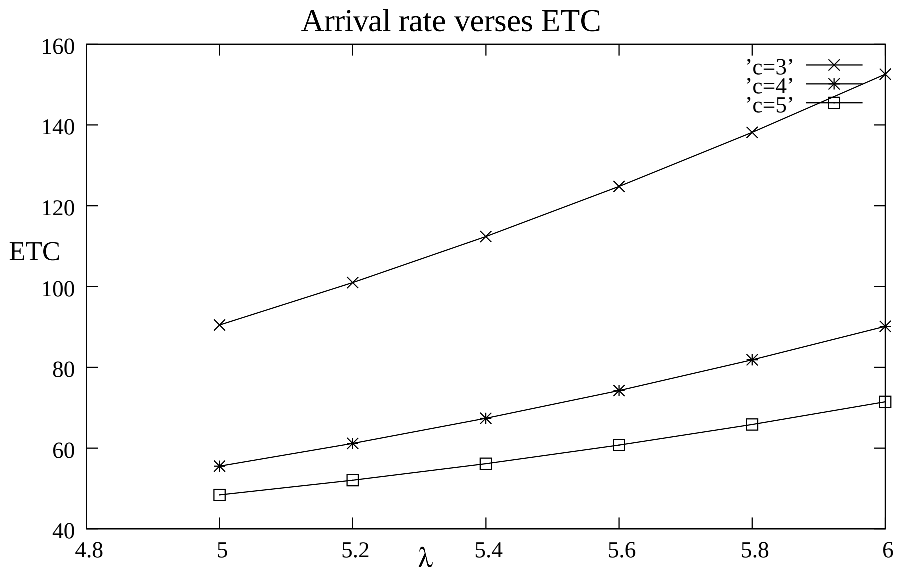


FIGURE 6.3. $\mu = 4$, $\beta = 1$, $\theta = 3$, $s = 10$, $S = 25.C = 100$, $C_1 = 20$, $C_2 = 1$, $C_3 = 5$, $C_4 = 5$, $C_5 = 6$

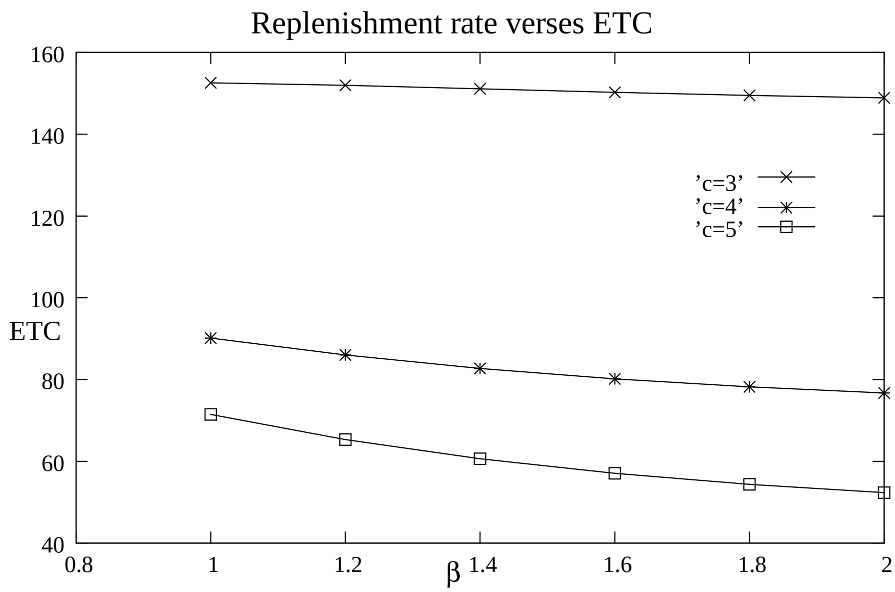


FIGURE 6.4. $\lambda = 6$, $\mu = 4$, $\theta = 3$, $s = 10$, $S = 25$, $C = 100$, $C_1 = 20$, $C_2 = 1$, $C_3 = 5$, $C_4 = 5$, $C_5 = 6$

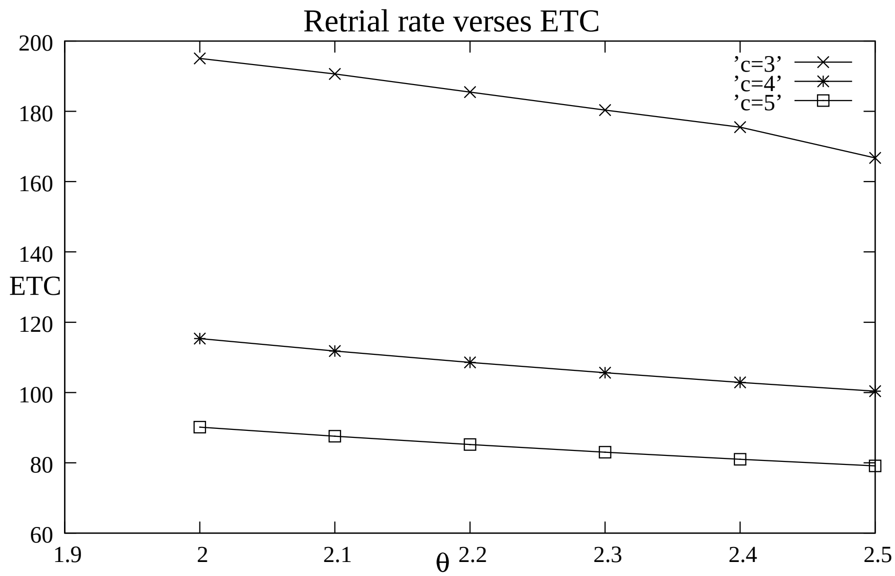


FIGURE 6.5. $\lambda = 6$, $\mu = 4$, $\beta = 1$, $s = 10$, $S = 25$ $C = 100$, $C_1 = 20$, $C_2 = 1$, $C_3 = 5$, $C_4 = 5$, $C_5 = 6$

6.4.2. Interpretation of the graphs. Figure 6.1 shows the changes of the total expected cost (ETC) with the maximum inventory level S and ETC decreases as S increases. From figure 6.2 we observe that as the service rate μ increases the total expected cost decreases. As the arrival rate λ increases the cost increases (see figure 6.3). Figure 6.4 shows that as the replenishment rate β increase the cost decreases. The cost decrease when the retrial rate increases (see figure 6.5). From all the figures we can understand that the total expected cost is minimum when the number of servers is 5 ($c = 5$), for the given cost function and parameters. So it is better for the system to have more servers.

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