STUDIES ON PSEUDOSCALAR MESON BOUND STATES AND SEMILEPTONIC DECAYS IN A RELATIVISTIC POTENTIAL MODEL

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DOCTOR OF PHILOSOPHY

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CERTIFICATE

Certified that the work presented in this thesis entitled **"Studies on Pseudoscalar Meson Bound States and Semileptonic Decays in a Relativistic Potential Model"** is the bonafide work done by Mr. A.P Jayadevan, under my guidance in the Department of Physics, Cochin University of Science and Technology, Cochin and that this work has not been included in any other thesis submitted previously for the award of any degree.

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February 22, 2005

Dr. Ramesh Babu .T
Supervising Teacher

DECLARATION

I hereby declare that the work presented in this thesis entitled **"Studies on Pseudoscalar Meson Bound States and Semileptonic Decays in a Relativistic Potential Model"** is based on the original work done by me under the guidance of Dr. Ramesh Babu.T, Reader, Department of Physics, Cochin University of Science and Technology and has not been included in any other thesis submitted previously for the award of any degree.

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Contents

Preface

In this thesis quark-antiquark bound states are considered using a relativistic two-body equation for Dirac particles. The mass spectrum of mesons includes bound states involving two heavy quarks or one heavy and one light quark. In order to analyse these states within a unified formalism, it is desirable to have a two-fermion equation that limits to one body Dirac equation with a static interaction for the light quark when the other particle's mass tends to infinity. A suitable two-body equation has been developed by Mandelzweig and Wallace. This equation is solved in momentum space and is used to describe the complete spectrum of mesons. The potential used in this work contains a short range one-gluon exchange interaction and a long range linear confining and constant potential terms. This model is used to investigate the decay processes of heavy mesons. Semileptonic decays are more tractable since there is no final state interactions between the leptons and hadrons that would otherwise complicate the situation. Studies on B and D meson decays are helpful to understand the nonperturbative strong interactions of heavy mesons, which in turn is useful to extract the details of weak interaction process. Calculation of form factors of these semileptonic decays of pseudoscalar mesons are also presented.

Chapter 1
Introduction and Overview

1.1 Motivation

A consistent and quantitative study of hadrons as strong interacting particles still poses one of the most interesting and challenging problems of modern particle physics. Although the elementary constituents are known to be quarks and gluons, a quantitative analysis of the underlying theory viz. Quantum chromodynamics (QCD) is possible only with a perturbative treatment in the high energy region. For example, the so called jet events in electron-positron annihilation at energies larger than 10GeV at LEP results from the creation of a quark-antiquark pair and can be successfully described in terms of the fundamental quark-gluon interactions.

The experimentally observable hadrons, however, have excitations with typical energies less than 1GeV. In this low energy regime the QCD coupling constant is large so that the theory becomes essentially nonperturbative. The fundamental equations can only be solved in lattice calculations with large numerical effort and crude approximations such as treating quarks as static. Satisfactory phenomenological models especially in the light quark sector therefore still attract considerable interest.

Gellman and Ne'eman [1] introduced the u,d,s quarks as a fundamental representation of flavour - SU(3) symmetry in order to classify the known baryon ground state as three quark objects and the mesons as quark-antiquark ($q\bar{q}$) states. The nonrelativistic quark model (NRQM) introduced by Isgur and Karl [2] showed that all hadron masses known at that time could be explained efficiently with nonrelativistic dynamics by means of constituent quarks being confined by a (linearly rising) potential. Thus quarks are the relevant degrees of freedom for the low energy hadron spectroscopy. Note however, that recently there have been evidences for glue ball candidates also from the proton antiproton annihilation events [3].

Despite its success in calculating mass spectra, the applicability of NRQM is questionable especially for light quarks. The large ratio of the level spacing as compared to the ground state energy shows that light quarks move with relativistic velocity because of their large 'binding energy'. Indeed a naive application of nonrelativistic decay formulae leads to completely wrong results especially for the ground state meson octet. Beyond these low energy phenomena there is at present a considerable experimental interest in the domain of 'pre-perturbative' medium energy of hadrons. The facilities at CEBAF for instance provide a large duty factor combined with high luminosity and allow for the precise measurement of meson electroproduction and therefore for the extraction of meson form factors. In the region of up to 6GeV electron energy the meson will recoil with relativistic velocity. Therefore a theoretical understanding of transitions at high momentum transfer must certainly be based on a treatment of mesons as relativistic quark-antiquark bound states.

In QED, the Bethe-Salpeter equation [4] provides a relativistic description of two-body bound state systems. This equation is an exact formulation based on quantum field theory and is a relativistic covariant generalization of the Lippman-Schwinger equation.

But in the ladder approximation, the Bethe-Salpeter equation does not yield the one-body Dirac equation, as one of the particles mass goes to infinity.

The various features of QED and QCD can be compared to emphasize the basic connections between these two theories. Both are relativistic gauge field theories with a gauge field and associated charge. Furthermore, both describe fundamental spin half particles (the various leptons and quarks) which can form a two-body bound state system, e.g. positronium in QED and mesons in QCD. Thus by using a theory based on QED, we may be able to cast some light on QCD aspects of mesons. There are also fundamental differences between QCD and QED. QCD is believed to be confining and generally much more complicated than QED due to the nonabelian nature of the local SU(3) color symmetry. It is the intractability of the nonperturbative aspects of QCD which necessitate model studies.

Mandelzwieg and Wallace [9] has developed a two-body relativistic equation for Dirac particles which was also motivated from QED. This two-body Dirac equation does indeed have the one-body Dirac equation limit as one of the particles mass goes to infinity. This is achieved by including the interactions with the negative energy states by incorporating Z graphs.

In view of these we present this work a relativistic potential model for the mesons based on the two-body Dirac equation due to Mandelzwieg and Wallace. In order to avoid the complications of the full four dimensional equation, we use a covariant formulation based on the Equal Time formalism, by integrating out the time components of relative momenta. This leads to a well defined eigen value problem for the meson masses and amplitudes. The main task of this work is to combine the calculation of mass spectra with a study on form factors of the semileptonic decays of pseudoscalar mesons.

1.2 Overview of thesis

In Chapter 2, the basic two-body equation for two spin half particles interacting in a potential is discussed. The equation is comprised of two basic components: a kinetic and a potential, both of which will be discussed. The kinetic term was derived by Mandelzweig and Wallace, who obtained a propagator that has a number of desirable physical properties not present in the usual Bethe-Salpeter equation. The potential is obtained from a simple phenomenological Cornell type potential which has the following r-space and Lorentz structure

$$V(r) = \kappa r + C + \frac{\alpha}{r}\gamma_1 \cdot \gamma_2$$

Since the analysis is in momentum-space, the confining potential is highly singular at zero relative momentum. Indeed the Fourier transform of the κr goes as p^{-4}. Particular care must be taken when treating such terms and we use the method developed by Spence and Vary [11, 12] to treat the singularity correctly.

In chapter 3, a brief description of the structure of the wave function is given. While the solutions of the two-body equation have been determined numerically, there are some basic analytical properties that can be obtained by using the symmetries of the equation, such as parity invariance, charge conjugation and rotational invariance. These analytical properties yield various relations between the wave functions and these relations offer a check of wave functions obtained numerically.

To test the numerical procedure for obtaining the wave functions and energy eigenvalues; first the Schrödinger equation describing the linear potential is solved as a nonrelativistic limit of the two-body equation. This demonstrates that an accurate spectrum can be obtained using the linear potential for the non-relativistic case. Next the Coulomb potential is examined for the case of heavy quarkonium. Finally the solution of the heavy quark meson spectrum in just the ++ channels of the two-body equation is discussed and compared with the solutions for the same system that have been obtained by Spence and Vary[11].

In Chapter 4, a covariant formulation of the two-body equation is discussed and is numerically solved in the centre of mass frame to obtain a description of the spectrum and wave functions of various mesons. In this thesis we employ a momentum space analysis. The potential used in this work contains a short range gluon exchange interaction that is vector and a long range potential with linear and constant terms that may be scalar or time-like vector. The numerical methods used in the analysis of these singular interactions are described. Meson mass spectra obtained in this work are compared with the experimental values. Also included in our comparisons are the results of Isgur-Godfrey [13] and Tiemeijer-Tjon [18].

The study of semileptonic decay of hadrons has been of great interest to particle physics since it helps not only in probing the quark structure of hadrons but also providing means to measure the CKM parameters necessary to realize the CP violating effects within the minimal standard model picture. In particular, the semileptonic decays of heavy flavoured mesons such as D and B have received considerable attention in recent years due to the emergence of new theoretical ideas such as heavy quark symmetries leading to many interesting model independent predictions in this sector. Significant progress has also been made through the ongoing efforts to acquire relatively more precise experimental data for these semileptonic processes. The theoretical analysis of such decays usually requires a detailed knowledge of transition form factors with their explicit q^2 (four momentum transfer squared) dependence. The form factors are in fact manifestations of QCD bound state characters of the quarks involved in the process are yet to be solved from the first principle. Although the Heavy Quark Effective Theory [37, 38, 39], which corresponds QCD in the limit of $\lambda_{QCD}/m_Q \rightarrow 0$, can relate different form factors to a single one called Isgur-Wise function, it is not possible to predict theoretically the q^2 dependence of this function except through an appeal to the nonperturbative techniques such as lattice QCD [40]. Therefore the weak decay form factors required to describe the semileptonic decays are usually obtained by various phenomenological bound-state models.

We therefore consider it is worthwhile to investigate the semileptonic decay of heavy pseudoscalar mesons into pseudoscalar mesons in this relativistic potential model. In Chapter 5, the first part provide a brief outline of the general formalism [56, 57] adopted here for the analysis of the decay. We describe the model conventions and realize the invariant transition matrix element as well as the relevant form factors with their appropriate q^2 dependence directly from the model. In the second part the dynamical boost of the bound state wave function that is needed to evaluate the transition matrix elements is discussed. A boost rule for the bound states of two scalar particles of equal mass satisfying the requirements of Poincare invariance was developed by Wallace [44, 45] has been described in detail. This has also been extended to scalar particles of unequal masses[43]. The last part of this Chapter embodies the results and discussions. The form factors obtained are in good agreement with the experimental results [36, 46, 51, 53, 57, 58].

The study of these decays represents a significant part of the experimental programmes at the Proton-Proton Accelerators and at the B-factories at SLAC and KEK.

Chapter 2
Two-Body Dirac Equation

2.1 Introduction

The study of mesons in terms of quarks and antiquarks requires a formulation of the bound state problem for two Dirac particles. The bound state formulation should have a clear relation to quantum field theory in order to incorporate the relevant features of QCD.

As mentioned in the introduction there are a number of significant features shared by QED and QCD, so it is useful to start with the Bethe-Salpeter equation which has some of these common features built in. The Bethe-Salpeter formalism provides a consistent frame work based on relativistic quantum field theory. However some desirable properties do not emerge in a simple way. One example is the single particle Dirac limit. It has been shown that the one-body Dirac equation is the correct limit of the two-body problem in the limit that one of the particles mass approaches infinity. This limits depends on cancellations between crossed and uncrossed Feynman graphs to all orders. Such a cancellation to all orders is an impractical demand so usually the BS kernel is truncated with the sacrifice of the one-body Dirac equation limit. A related inconvenience occurs in the high energy limit, where the sum of all crossed and uncrossed graphs limits to an exponential form identical to the non-relativistic eikonal approximation. Again the limit depends on cancellations between crossed and uncrossed graphs to all orders in perturbation theory. Both of these

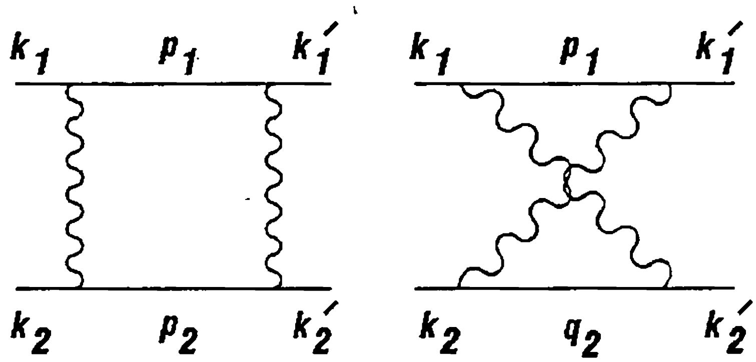

Fig 2.1 Box and crossed box graphs

limits can be incorporated in the quasipotential approach discussed by Mandelzwieg and Wallace [10].

The Bethe-Salpeter equation for quark-antiquark system has the following form,

$$G^{-1}(P,p)\Psi(P,p) = i\int \frac{d^4p'}{(2\pi)^4} V(P,p-p')\Psi(P,p') \tag{2.1}$$

where P and p are the total and relative momenta respectively of the two particle system. Since it is a four dimensional formulation, $V(P,p-p')$ and $\Psi(P,p)$ depend on the fourth component of relative momenta. In many circumstances, it is convenient to reduce the four dimensional equation to a three dimensional quasipotential equation. Original work along this line has been done by Blankenbecler and Sugar[8] and by Gross[6, 7]. In this chapter a reduction of this equation to three dimensions is discussed and then the potential is analysed in momentum space.

2.2 Two-Body Dirac Equation

The Bethe-Salpeter equation can be used to analyse the bound state of a two fermion system in terms of the interaction kernel specific to the field theory in question. The BS equation is the relativistic covariant generalization of the Lippmann-Schwinger (LS) equation for a two particle system and can be written as follows:

$$\Psi(P,p) = iG(P,p) \int \frac{d^4p'}{(2\pi)^4} V(P,p-p')\Psi(P,p') \tag{2.2}$$

where $\Psi(P,p)$ is the momentum space wave function for $q\bar{q}$ system and

$$G(P,p) = \frac{1}{(\gamma_1.p_1 - m_1 + i\eta)(\gamma_2.p_2 - m_2 + i\eta)} \tag{2.3}$$

with

$$p_1 = \frac{1}{2}P + p,\ p_2 = \frac{1}{2}P - p \tag{2.4}$$

For each quark there are now negative-energy states as well as the usual positive-energy states. Thus, the number of degrees of freedom has been doubled and this increases the numerical complexity of the problem. Also the wave function $\Psi(P,p)$ and potential $V(P,p-p')$ depend on the fourth component of the relative momentum p.

The quasipotential reduction amounts to assuming the $q\bar{q}$ potential, $V(p,p')$, does not depend on the relative energy in the centre of mass frame. This is equivalent to using a coordinate-space description in which the potential is calculated with the equal-time restriction, $x_0 = x_0'$. Defining,

$$\Psi\left(\overrightarrow{p}\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp^0 \Psi\left(P,p\right) \tag{2.5}$$

and neglecting retardations, the interaction becomes $V(\vec{p}, \vec{p}')$ and from Eq.(2.2) we obtain

$$\Psi(\vec{p}) = \left[\frac{i}{2\pi}\int_{-\infty}^{\infty} dp^0 G(P,p)\right] \int \frac{d^3p'}{(2\pi)^3} V(\vec{p}, \vec{p}')\Psi(P, \vec{p}') \tag{2.6}$$

which can be written as

$$\Psi(\vec{p}) = G(\vec{p}_1, \vec{p}_2) \int \frac{d^3p'}{(2\pi)^3} V(\vec{p}, \vec{p}')\Psi(\vec{p}') \tag{2.7}$$

where

$$G(\vec{p}_1, \vec{p}_2) = i\int_{-\infty}^{\infty} \frac{dp_1^0}{2\pi} G(p_1, p_2) \tag{2.8}$$

and Eq.2.7 is the Salpeter equation. However the Salpeter equation does not yield one-body Dirac equation limit. Mandelzweig and Wallace analysed this problem and showed that an improved equation is obtained by including an eikonal approximation to cross graphs. The resulting equation is,

$$\Psi(\vec{p}) = G_{MW}(\vec{p}_1, \vec{p}_2) \int \frac{d^3p'}{(2\pi)^3} V(\vec{p}, \vec{p}')\Psi(\vec{p}') \tag{2.9}$$

where

$$G_{MW}(\vec{p}_1, \vec{p}_2) = i\int_{-\infty}^{\infty} dp_1^0 \left[G(p_1, p_2) + G_{cross}(p_1, q_2)\right] \tag{2.10}$$

with

$$q_2 = \left(P^0 + p^0 - k^0 - k'^0, \vec{p}\right)$$

and

$$G_{cross}(p_1, q_2) = \frac{1}{(\gamma_1.p_1 - m_1 + i\eta)(\gamma_2.q_2 - m_2 + i\eta)} \tag{2.11}$$

In the Mandelzweig and Wallace equation[10], V must be redefined to omit parts of cross graphs which are automatically obtained from iterating the equation with G_{cross}. In the limit where V is the lowest elementary exchange, say one photon exchange, one obtains, in the eikonal approximation, the cross graphs and the Dirac equation limit. Rewriting the Dirac propagators using the identity

$$\frac{1}{(\gamma_i \cdot p_i - m + i\eta)} = \frac{\Lambda^+(\overrightarrow{p}_i)}{p_i^0 - \epsilon_i + i\eta} - \frac{\Lambda^-\left(\overrightarrow{-p_i}\right)}{p_i^0 + \epsilon_i - i\eta} \tag{2.12}$$

where

$$\epsilon_i = \sqrt{p_i^2 + m_i^2},\ i = 1,2 \tag{2.13}$$

and

$$\begin{aligned} \Lambda^+(\overrightarrow{p}_i) &= \sum_\lambda u^\lambda(\overrightarrow{p_i}) \otimes \overline{u}^\lambda(\overrightarrow{p}_i) = \frac{\not{p}_i + m_i}{2\epsilon_i} \\ \Lambda^-(-\overrightarrow{p_i}) &= -\sum_\lambda v^\lambda(-\overrightarrow{p_i}) \otimes \overline{v}^\lambda(-\overrightarrow{p_i}) = \frac{-\not{p}_i + m_i}{2\epsilon_i} \end{aligned} \tag{2.14}$$

are the usual projection operators. On performing the integration of Eq.2.10, the following quasipotential propagator is found

$$\begin{aligned} G(\overrightarrow{p_1}, \overrightarrow{p_2}, E_1, E_2) &= \frac{\Lambda_1^+(\overrightarrow{p_1})\Lambda_2^+(\overrightarrow{p_2})}{E_1 + E_2 - \epsilon_1 - \epsilon_2} - \frac{\Lambda_1^-(-\overrightarrow{p_1})\Lambda_2^+(\overrightarrow{p_2})}{E_1 - E_2 + \epsilon_1 + \epsilon_2} \\ &\quad - \frac{\Lambda_1^+(\overrightarrow{p_1})\Lambda_2^-(-\overrightarrow{p_2})}{-E_1 + E_2 - \epsilon_1 - \epsilon_2} - \frac{\Lambda_1^-(-\overrightarrow{p_1})\Lambda_2^-(-\overrightarrow{p_2})}{E_1 + E_2 + \epsilon_1 + \epsilon_2} \end{aligned} \tag{2.15}$$

Here $E_1$and E_2 are energy parameters constrained by the condition $E_1^2 - E_2^2 = m_1^2 - m_2^2$ and their sum is the bound state energy of the two particle system. The $++$ and $--$ terms in Eq.2.15 come from the uncrossed graph. These terms are familiar from the Salpeter equation which is the limit of BS equation for instantaneous photon exchange. The $+-$ and $-+$ terms, which are essential to obtaining the one-body Dirac limits, come from G_{cross} term in Eq.2.10. The eikonal approximation used to include the crossed Feynman graph only affects the $+-$ and $-+$ parts of G and these are small relative to the dominant

++ part. A calculation of G^{-1} yields

$$G^{-1} = \gamma_1^0\gamma_2^0 \left[(E_1 - h_1) \frac{h_2}{\epsilon_2} + (E_2 - h_2) \frac{h_1}{\epsilon_1} \right] \tag{2.16}$$

where

$$h_i = \overrightarrow{\alpha_i} . \overrightarrow{p_i} + \beta_i m_i, \ i = 1, 2. \tag{2.17}$$

The result may be checked by verifying that $G^{-1}G = GG^{-1} = 1$. This leads to a wave equation which can be used for bound state calculations. Thus the equation for the two Dirac particles can be written as

$$\left[(E_1 - h_1) \frac{h_2}{\epsilon_2} + (E_2 - h_2) \frac{h_1}{\epsilon_1} \right] \Psi (\overrightarrow{p}) = \int \frac{d^3p'}{(2\pi)^3} \hat{V}(\overrightarrow{p} - \overrightarrow{p}')\Psi(\overrightarrow{p}') \tag{2.18}$$

where $\hat{V} = \gamma_1^0\gamma_2^0 V$.

Let us write the above equation in a convenient fashion

$$A_{12}\Psi (\overrightarrow{p}) = 0 \tag{2.19}$$

where A_{12} is the operator corresponds to the kinetic and potential components. In general the operator A_{12} has the form of a sum of products

$$A_{12} = \sum_n A_n \Gamma_n (1) \Gamma_n (2) \tag{2.20}$$

where $\Gamma_n (1)$ is a Dirac matrix in the space of particle 1 and $\Gamma_n (2)$ is a Dirac matrix in the space of particle 2. In the Eq.2.19 $\Psi (\overrightarrow{p})$ may be treated either as a 16 component spinor or a 4×4 matrix. Thus we can write

$$A_{12}\Psi (\overrightarrow{p}) = \sum_n A_n \Gamma_n (1)_{\alpha\alpha'} \Gamma_n (2)_{\beta\beta'} \Psi_{\alpha'\beta'} (\overrightarrow{p}) = 0 \tag{2.21}$$

where $\alpha, \alpha', \beta, \beta'$ are Dirac matrix indices. This is equivalent to

$$\sum_n A_n \Gamma_n (1)_{\alpha\alpha'} \Psi_{\alpha'\beta'} (\overrightarrow{p}) \left(\Gamma_n^T (2)\right)_{\beta'\beta} = 0 \tag{2.22}$$

or in matrix notation

$$\sum_n A_n \Gamma_n \Psi (\overrightarrow{p}) \Gamma_n^T = 0. \tag{2.23}$$

Here a matrix to the left refers to the particle 1 Dirac space and a matrix to the right refers to the particle 2 Dirac space and the particle 2 matrices are always transposed.

The Eq.2.18 has the usual symmetries of parity invariance, time-reversal invariance, and it has solutions of good J provided the potential V commutes with the total angular momentum $\overrightarrow{J}$. For the bound state problem in the CM frame $E = E_1 + E_2$ is the eigenvalue and the subsidiary condition $E_1 - E_2 = (m_1^2 - m_2^2)/E$ is used to define the energy difference.

This is a more general equation than the usual Salpeter equation which does not involve the Z-graph terms, $\Lambda_1^- (-\overrightarrow{p}_1)\, \Lambda_2^+ (\overrightarrow{p}_2)$ and $\Lambda_1^+ (\overrightarrow{p}_1)\, \Lambda_2^- (-\overrightarrow{p}_2)$. Indeed the Salpeter equation can be further approximated in the case of heavy quark systems by dropping the double Z-graph term, i.e. $\Lambda_1^- (-\overrightarrow{p}_1)\, \Lambda_2^- (-\overrightarrow{p}_2)$. This is justified since by looking at the $--$ term in Eq.2.15 above and noting $E_1 + E_2 = M$, where M is just the mass of the state we have,

$$M + \epsilon_1 + \epsilon_2 \approx 4m_q \gg M - \epsilon_1 - \epsilon_2 \approx V \tag{2.24}$$

so we can drop the $--$ state compared to the $++$. The double Z-graph terms are also strongly suppressed by retardations. Spence and Vary have solved the Salpeter equation for $q\bar{q}$ with just $++$ channels. In the two-body equation written here the equation for the $++$

channel is

$$[(E_1 - \epsilon_1) + (E_2 - \epsilon_2)]\,\Psi(p) = \int \frac{d^3p}{(2\pi)^3} V^{++,++}(\vec{p} - \vec{p}')\Psi(\vec{p}'). \tag{2.25}$$

2.3 Partial Wave Analysis

The two-body equation can be written in the form,

$$[(E_1 - \epsilon_1\hat{\rho}_1)\hat{\rho}_2 + (E_2 - \epsilon_2\hat{\rho}_2)\hat{\rho}_1]\,\Psi(\vec{p}) = \int \frac{d^3\vec{p}'}{(2\pi)^3} \hat{V}(\vec{p}, \vec{p}')\Psi(\vec{p}') \tag{2.26}$$

where, $\hat{\rho}_i = h_i/\epsilon_i$, i=1,2 has eigenvalues $\rho_i = \pm 1$. It is convenient to analyse the wave function by using two bases, one called the ρ basis and the other called the plane wave basis. In the ρ basis the potential term is easier to manage while in the plane wave basis the kinetic term is easier to analyse. We will use the ρ basis only to analyse and obtain the potential contribution to the two-body equation. Since the kinetic term takes a rather intractable form in the ρ basis we will solve for the wave function and energies using the plane wave basis.

In section 2.4 the partial wave analysis is performed in the plane wave basis. In section 2.5 the ρ basis analysis is introduced. In this basis the potential term in Eq.2.26 can be easily evaluated. In section 2.6 the transformation linking the potential in the ρ basis to that in the plane wave basis is given and the equation which is a mixture of both the bases is obtained. A spline expansion is used to convert the momentum space integral equation to a numerically solvable matrix problem.

2.4 Partial Wave Analysis in the Plane Wave Basis

Since the two-body equation, Eq.2.18, describes two Dirac particles, each being represented by Dirac spinors in the space of particle 1 and 2 and their wave function will be a sixteen component matrix as mentioned earlier. We write this 16 component wave function in the plane-wave basis as

$$\Psi_J^M(\vec{p}) = \sum_{\rho_1\rho_2}\sum_{LS} u_1^{\rho_1}(\rho_1\vec{p})\,\mathcal{Y}_{LSJ}^M(\hat{p})\,u_2^{\rho_2}(-\rho_2\vec{p})^T\,G_{LS}^{\rho_1\rho_2}(p)\,. \tag{2.27}$$

The $\mathcal{Y}_{L'S'J}$ include the 2×2 matrices that span the spin space of the two Dirac particles and are discussed in more detail in Appendix A. In the notation used here, the particle 1 operator in Dirac space, u_1 is always written to the left along with any other operators acting on it and the particle 2 operator in Dirac space, u_2^T is always written to the right and transposed as well as any operators acting on it.

The plane-wave basis is a 4 $\times$2 matrix given by

$$u_i^+(\vec{p}) = \sqrt{\frac{\epsilon_i+m_i}{2\epsilon_i}}\begin{pmatrix}1\\ \frac{\vec{\sigma}.\vec{p}}{\epsilon_i+m_i}\end{pmatrix},\quad u_i^-(-\vec{p}) = \sqrt{\frac{\epsilon_i+m_i}{2\epsilon_i}}\begin{pmatrix}\frac{-\vec{\sigma}.\vec{p}}{\epsilon_i+m_i}\\ 1\end{pmatrix} \tag{2.28}$$

with

$$h_1(\vec{p})\,u_1^{\rho_1}(\rho_1\vec{p}) = \rho_1\epsilon_1 u_1^{\rho_1}(\rho\vec{p})\,, \tag{2.29}$$

$$h_2(-\vec{p})\,u_2^{\rho_2}(-\rho\vec{p}) = \rho_2\epsilon_1 u_1^{\rho_2}(-\rho_2\vec{p})\,. \tag{2.30}$$

and they are orthonormal in the sense that

$$u_i^{\rho\dagger}(\rho\vec{p})\,u_i^{\sigma}(\sigma\vec{p}) = \delta^{\rho\sigma} \tag{2.31}$$

Here $h_i(\overrightarrow{p_i}) = \overrightarrow{\alpha_i}.\overrightarrow{p_i} + \beta_i m_i$, $\hat{\rho}_i = h_1/\epsilon_1$ and $\hat{\rho}_i$ has eigenvalue ρ_i which has values $= \pm 1$ in this basis. The four sets $\rho_1\rho_2 = \{++, +-, -+, --\}$ determine four linearly independent matrices and together with angular momentum states $\mathcal{Y}^M_{LSJ}(\hat{p})$ the wave function in states of good JM can be expanded as given in Eq.2.27. It can also be seen that we are multiplying a 2×2 matrix with a 4×2 matrix u_1 from the left side and with a 2×4 matrix which is the transpose of matrix u_2 from the right side so that Ψ^M_J is a 4×4 matrix. The kinetic term is simpler to obtain in the plane-wave basis. Thus $(E_1 - \epsilon_1\hat{\rho}_1)\hat{\rho}_2\Psi^M_J(\overrightarrow{p})$ becomes $(E_1 - \epsilon_1\hat{\rho}_1)\Psi^M_J(\overrightarrow{p})\hat{\rho}_2^T$ where in the matrix $\Psi^M_J(\overrightarrow{p})$ the row index is the particle 1 label and column index as the particle 2 label. For example

$$(E_1 - \epsilon_1\hat{\rho}_1)u_1^{\rho_1}(\rho_1\overrightarrow{p})\mathcal{Y}^M_{LSJ}(\hat{p})u_2^{\rho_2}(-\rho_2\overrightarrow{p})^T\hat{\rho}_2^T =$$

$$u_1^{\rho_1}(\rho_1\overrightarrow{p})\mathcal{Y}^M_{LSJ}(\hat{p})u_2^{\rho_2}(-\rho_2\overrightarrow{p})^T(E_1 - \epsilon_1\rho_1)\rho_2 \tag{2.32}$$

Now inserting this into the two-body equation and using the orthonormality property of the plane-wave basis,

$$\int d\Omega_p Tr \quad \left[\left(u_1^{\rho_1}(\rho_1\overrightarrow{p})\mathcal{Y}^M_{LSJ}(\hat{p})u_2^{\rho_2}(-\rho_2\overrightarrow{p})^T\right)^\dagger u_1^{\sigma_1}(\sigma_1\overrightarrow{p})\mathcal{Y}^M_{L'S'J}(\hat{p})u_2^{\sigma_2}(-\sigma_2\overrightarrow{p})^T\right] =$$

$$\delta_{\rho_1\sigma_1}\delta_{\rho_2\sigma_2}\delta_{LL'}\delta_{SS'} \tag{2.33}$$

we obtain an equation for radial wave function $G(p)$ as follows

$$[(E_1 - \epsilon_1\rho_1)\rho_2 + (E_2 - \epsilon_2\rho_2)\rho_1]G^{\rho_1\rho_2}_{LS}(p) =$$

$$(2\pi)^{-3}\sum_{\sigma_1\sigma_2, L'S'}\int dp'p'^2 V^{\rho_1\rho_2,\sigma_1\sigma_2}_{LS,L'S'}(p,p')G^{\sigma_1\sigma_2}_{L'S'}(p') \tag{2.34}$$

where

$$V^{\rho_1\rho_2,\sigma_1\sigma_2}_{LS,L'S'}\left(\overrightarrow{p},\overrightarrow{p}'\right) = \int d\Omega_p d\Omega_{p'} Tr\left[\left(u_1^{\rho_1}\left(\rho_1\overrightarrow{p}\right)\mathcal{Y}^M_{LSJ}\left(\hat{p}\right)u_2^{\rho_2}\left(-\rho_2\overrightarrow{p}\right)^T\right)^\dagger \hat{V}\left(\overrightarrow{p}-\overrightarrow{p}'\right)u_1^{\sigma_1}\left(\sigma_1\overrightarrow{p}'\right)\mathcal{Y}^M_{L'S'J}\left(\hat{p}\right)u_2^{\sigma_2}\left(-\sigma_2\overrightarrow{p}'\right)^T\right] \quad (2.35)$$

This equation will be solved numerically by introducing an expansion of the radial wave function $G^{\sigma_1\sigma_2}_{L'S'}(p)$ in terms of cubic splines[65]. However the potential is more conveniently treated using the ρ basis.

2.5 Partial Wave Analysis in the ρ Basis

The following four linearly independent ρ matrices are used as a basis

$$\rho_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (2.36)$$

$$\rho_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \rho_4 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (2.37)$$

It is easy to show they are linearly independent matrices and thus form a basis. Also they obey the following orthonormality condition

$$Tr\left(\rho_i^\dagger \rho_j\right) = \delta_{ij} \quad (2.38)$$

The 4×4 matrix Ψ^M_J can be expanded in terms of a direct product of two subspaces of 2×2 matrices. One of these will be just the 2×2 matrices that span the spin space of the two Dirac particles and the other will be four matrices that span the other 2×2 subspace, the ρ spin matrices in Equations2.36 and 2.37. Therefore the wave function Ψ^M_J for states of good $J^2 = J(J+1)$ and $J_z = M$ is expanded in the ρ basis as follows

$$\Psi_J^M(\vec{p}) = \sum_{iLS} \rho_i \otimes \mathcal{Y}_{LSJ}^M(\hat{p}) F_{LS}^i(p) \tag{2.39}$$

where $F_{LS}^i(p)$ are a set of sixteen radial wave functions. The 16 basis states $\rho_i \otimes \mathcal{Y}_{LSJ}^M$ satisfy the following orthonormality condition

$$\int d\Omega_p Tr\left[\left(\rho_i \otimes \mathcal{Y}_{LSJ}^M(\hat{p})\right)^\dagger \rho_j \otimes \mathcal{Y}_{L'S'J}^M(\hat{p})\right] = \delta_{LL'}\delta_{SS'}\delta_{ij} \tag{2.40}$$

The potential in momentum space is written in terms of the usual Dirac matrices as

$$\hat{V}(q) = \gamma_1^0\gamma_2^0\left[V_r(q) + \gamma_1 \cdot \gamma_2 V_{1/r}(q)\right]. \tag{2.41}$$

where V_r is the Lorentz scalar piece coming from the linearly rising potential in the r-space and $V_{1/r}(q)$ is the Lorentz vector term coming from $1/r$ term in r-space. In the ρ basis $\hat{V}\Psi$ is written as follows

$$\hat{V}(\vec{p}, \vec{p}')\Psi(p) = \sum_{iLS} V_s^i(\vec{p}, \vec{p}')\rho_j \otimes \mathcal{Y}_{LSJ}^M(\hat{p}) F_{LS}^j(p) \tag{2.42}$$

To obtain the partial wave expansion for the wave function we simply substitute the ρ basis expansion into the two-body Eq.2.18

$$\sum_{iLS} \left[(E_1 - h_1)\frac{h_2}{\epsilon_2} + (E_2 - h_2)\frac{h_1}{\epsilon_1}\right]\rho_i \otimes Y_{LSJ}^M(\hat{p}) F_{LS}^i(p) =$$

$$\sum_i \int \frac{p'^2 dp'^2}{(2\pi^3)}\rho_i \otimes \sum_{LS}\left(\int d\Omega_{p'} V_s^i(\vec{p} - \vec{p}')\mathcal{Y}_{LSJ}^i(\hat{p}')\right) F_{LS}^i(p'). \tag{2.43}$$

In this equation it is not very convenient to evaluate the left hand side, as it leads to a complicated set of coupled equations in the ρ basis. However the potential term is much simpler because the angular integral in the potential term on the right hand side is straightforward to evaluate and gives,

$$\int d\Omega_{p'} V^i\left(\vec{p}, \vec{p}'\right) \mathcal{Y}^M_{LSJ}\left(\hat{p}\right) = \mathcal{Y}^M_{LSJ}\left(\hat{p}\right) V^i_{LS}\left(p, p'\right) \tag{2.44}$$

where

$$\begin{aligned} V^i_{LS}\left(p, p'\right) &= 2\pi \int_{-1}^{1} dx V^i_S\left(q^2\right) P_L\left(x\right), \\ &= \frac{\pi}{pp'} \int_{(p-p')^2}^{(p+p')^2} dq^2 V^i_S\left(q^2\right) P_L\left(\frac{p^2 + p'^2 - q^2}{2pp'}\right) \end{aligned} \tag{2.45}$$

with $q^2 = p^2 + p'^2 - 2pp'x$ and $P_L\left(x\right)$ are the usual Legendre polynomials and the $V^i_S\left(q\right)$ for $i = 1, \ldots 4$ in the ρ basis are,

$$V^1_S = -V_r + V_{1/r}\left(1 - C_S\right), \quad V^2_S = -V_r + V_{1/r}\left(1 + C_S\right) \tag{2.46}$$

$$V^3_S = V_r + V_{1/r}\left(1 + C_S\right), \quad V^4_S = V_r + V_{1/r}\left(1 - C_S\right) \tag{2.47}$$

Here $C_S = 2S(S+1) - 3$ and is $+1$ for $S = 1$ states and -3 for $S = 0$ states.

The wave function can be split into two sets of 8 radial functions, each with different parity. Using the orthonormality of the basis states, for either of these sets with different parity, we can project out the following 8 coupled equation for the radial wave function

$$\sum_{jL'S'} K^{ij}_{LS,L'S'}\left(p\right) F^j_{L'S'}\left(p\right) = \frac{1}{(2\pi)^3} \int_0^\infty dp' p'^2 V^i_{LS}\left(p, p'\right) F^i_{LS}\left(p'\right) \tag{2.48}$$

where the kinetic term is given by the expression

$$K^{ij}_{LS,L'S'}\left(p\right) = Tr \int d\Omega$$

$$\left[\left(\rho_i \mathcal{Y}^M_{LSJ}\left(\hat{p}\right)\right)^\dagger \left[\left(E_1 - h_1\left(p\right)\right)\frac{h_2}{\epsilon_2} + \left(E_2 - h_2\left(p\right)\right)\frac{h_1}{\epsilon_1}\right] \rho_j \otimes \mathcal{Y}^M_{L'S'J}\left(\hat{p}\right)\right] \tag{2.49}$$

and $V^i_{LS}\left(p, p'\right)$ have been defined above.

2.6 Partial Wave Analysis of the Two-Body Equation

Since both the basis form a complete set we can expand the plane-wave basis states in terms of the ρ basis states as

$$u_1^{\rho_1}\left(\rho_1 \overrightarrow{p}\right) \mathcal{Y}_{LSJ}^{M}\left(\hat{p}\right) u_1^{\rho_2}\left(-\rho_2 \overrightarrow{p}\right)^T = \sum_{iL'S'} M_{LS,L'S'}^{\rho_1\rho_2,i}\left(\overrightarrow{p}\right) \rho_i \otimes \mathcal{Y}_{L'S'J}^{M}\left(\hat{p}\right) \qquad (2.50)$$

And this can be used to determine the potential $V_{LS,L'S'}^{\rho_1\rho_2',\sigma_1\sigma_2}$ in the plane-wave basis, from the potential V_{LS}^{i} in the ρ basis. The basis transformation matrix $M\left(\overrightarrow{p}\right)$ is discussed in Appendix B. With this transformation the radial wave functions are related by the following relation,

$$F_{LS}^{i}\left(\overrightarrow{p}\right) = \sum_{\rho_1\rho_2 L'S'} G_{L'S'}^{\rho_1\rho_2}\left(p\right) M_{LS,L'S'}^{\rho_1\rho_2,i}\left(\overrightarrow{p}\right) \qquad (2.51)$$

From Eq.2.35, and using the first term in Eq.2.50, we can obtain the following expression,

$$\sum_{\sigma_1\sigma_2,L'S'} V_{LS,L'S'}^{\rho_1\rho_2,\sigma\sigma'}\left(p,p'\right) G_{L'S'}^{\sigma_1\sigma_2}\left(p\right) =$$

$$\int d\Omega_p d\Omega_{p'} Tr \sum_{iL''S''} \left(M_{LS,L''S''}^{\rho_1\rho_2,i}\left(p\right) \mathcal{Y}_{L''S''J}^{M}\left(\hat{p}\right) \rho_i\right)^{\dagger} \hat{V}\left(p,p'\right) \Psi\left(p'\right) \qquad (2.52)$$

using equation Eq.2.42 and Eq.2.51 this becomes,

$$\sum_{\sigma\sigma',L'S'} V_{LS,L'S'}^{\rho_1\rho_2,\sigma\sigma'}\left(p,p'\right) G_{L'S'}^{\sigma\sigma'}\left(p\right) = \int d\Omega_p d\Omega_{p'} Tr \left[\sum_{iL''S'';jL'''S'''} \left(M_{LS,L''S''}^{\rho_1\rho_2,i}\left(p\right) \mathcal{Y}_{L''S''J}^{M}\left(\hat{p}\right) \rho_i\right)^{\dagger}\right)$$

$$V_{L''S''}^{j}\left(p,p'\right) \rho_j \mathcal{Y}_{L'''S'''J}^{M}(\hat{p}' \sum_{\sigma\sigma',L'S'} M_{L'S',L'''S'''}^{\sigma\sigma',i}\left(p'\right) G_{L'S'}^{\sigma\sigma'}\left(p\right)\Bigg] \qquad (2.53)$$

Using Eq.2.44 and Eq.2.40, we obtain the potential in the plane-wave basis in terms of the potential in the ρ basis as

$$V_{LS,L'S'}^{\rho_1\rho_2,\sigma_1\sigma_2}\left(p,p'\right) = \sum_{iL''S''} M_{LS,L''S''}^{\rho_1\rho_2,i}\left(p\right) V_{L''S''}^{i}\left(p,p'\right) M_{L'S',L''S''}^{\sigma_1\sigma_2 i}\left(p'\right) \qquad (2.54)$$

Finally we have to solve the following equation for the wave function and eigenvalues,

$$[(E_1 - \epsilon_1\rho_1)\rho_2 + (E_2 - \epsilon_2\rho_1)\rho_1] G_{LS}^{\rho_1\rho_2}(p) =$$

$$\sum_{\sigma_1\sigma_2,L'S'} \int \frac{dp'p'^2}{(2\pi)^3} V_{LS,L'S'}^{\rho_1\rho_2,\sigma_1\sigma_2}(p,p') G_{L'S'}^{\sigma_1\sigma_2}(p') \tag{2.55}$$

which can be rewritten in the form

$$\frac{1}{2}(E_1 + E_2)[\rho_1 + \rho_2 - \Delta(\rho_1 - \rho_2)] G_{LS}^{\rho_1\rho_2}(p) =$$

$$(\epsilon_1 + \epsilon_2)\rho_1\rho_2 G_{LS}^{\rho_1\rho_2}(p) - \sum_{\sigma_1\sigma_2,L'S'} \int \frac{dp'p'^2}{(2\pi)^3} V_{LS,L'S'}^{\rho_1\rho_2,\sigma_1\sigma_2}(p,p') G_{L'S'}^{\sigma_1\sigma_2}(p') \tag{2.56}$$

where

$$\Delta = \frac{E_1^2 - E_2^2}{E_1 + E_2} = \frac{m_1^2 - m_2^2}{E_1 + E_2} \tag{2.57}$$

Since analytic solutions are not possible we have to solve the Eq.2.56 numerically. For this the radial momentum wave function $G_{LS}^{\rho_1\rho_2}$ is expanded in a linear combination of N interpolating spline functions as follows,

$$G_{LS}^{\rho_1\rho_2}(p) = \sum_{n=1}^{N} C_{LSn}^{\rho_1\rho_2} \frac{B_n(p)}{p} \tag{2.58}$$

where $B_n(p)$ are cubic B-splines with continuous derivatives upto second order and $C_{LSn}^{\rho_1\rho_2}$ are the spline coefficients to be determined. Multiplying Eq.2.58 by $pB_n(p)$ and integrating over p yields a matrix equation for the spline coefficients as follows

$$\sum_{n,m} \left[\left(\frac{1}{2}E(\rho_1 + \rho_2) - \Delta(\rho_1 - \rho_2)\right) I_{mn} - CS_{mn}\right] C_{LSn}^{\rho_1\rho_2} =$$

$$\sum_{n,m} \left[K_{mn} C_{LSn}^{\rho_1\rho_2} + \sum_{\sigma_1\sigma_2,L'S'} V_{LS,m,L'S',n}^{\rho_1\rho_2,\sigma_1\sigma_2} C_{L'S'n}^{\rho_1\rho_2}\right] \tag{2.59}$$

where

$$I_{mn} = \int_{p_l}^{p_u} dpB_m(p)\,B_n(p) \tag{2.60}$$

$$S_{mn} = \int_{p_l}^{p_u} dpB_m(p)\,B_n(p)\,\frac{m_1}{\epsilon_1}\frac{m_2}{\epsilon_2} \tag{2.61}$$

$$K_{mn} = \int_{p_l}^{p_u} dpB_m(p)\,B_n(p)\,(\epsilon_1+\epsilon_2) \tag{2.62}$$

and

$$V^{\rho_1\rho_2,\sigma_1\sigma_2}_{LS,m,L'S',n} = \int_{p_l}^{p_u} pdp\int_{p'_l}^{p'_u} p'dp'B_m(p)\,V^{\rho_1\rho_2,\sigma_1\sigma_2}_{LS,L'S'}B_n(p') \tag{2.63}$$

This is a homogeneous matrix equation for finding $C^{\rho_1\rho_2}_{LSm}$ and the condition for a nontrivial solution determine the mass eigen value $E_1 + E_2$. The potential $V^i_{LS}(p,p')$ is defined by Eq.2.45, Eq.2.46 and Eq.2.47. This equation will be solved to obtain the wave functions and the mass spectrum of various particles.

Chapter 3
Two-Body Dirac Equation Properties and Solution Techniques

3.1 Introduction

Two different sets of basis states are employed to solve the two-body equation. In the plane-wave basis the kinetic part takes on a very simple form. This basis is used to solve the two-body equation for the bound state energies and associated wave functions. In this section the plane wave-basis is used to expand the wave function and symmetry properties are established.

3.2 Parity Transformation

Along with the total angular momentum, $\overrightarrow{J}$, the other operators that distinguish the various states are the parity operator, $\mathcal{P}_P$ and the charge conjugation operator, $\mathcal{P}_C$. These will be examined analytically so that the numerical solutions can be checked for consistency. The radial wave function $G_{LS}^{\rho\rho'}(p)$ describes the magnitude dependence on relative momentum, p and $\mathcal{Y}_{LSJ}^{M}(\hat{p})$ describes the dependence on $\hat{p}$, which denotes the spherical angles in momentum space.

The action of the parity operator $\mathcal{P}_P$ on the state $\Psi_J^M(\overrightarrow{p})$ is defined as

$$\mathcal{P}_P \Psi_J^M(\overrightarrow{p}) = \gamma_1^0 \Psi_J^M(-\overrightarrow{p})\, \gamma_2^0 \pi_{q\bar{q}} \tag{3.1}$$

If the potential is invariant under parity transformation, various states of the two-body equation are split into those with parity $(-1)^J$ and $(-1)^{J+1}$. Here $\pi_{q\bar{q}}$the intrinsic parity of the bound state. For the quark-antiquark state we have $\pi_{q\bar{q}} = -1$, due to the presence of particle as well as antiparticle in the meson bound state.

For $q\bar{q}$ states of total parity $(-1)^{J+1}$ we obtain

$$\pi_{q\bar{q}}\gamma_1^0\Psi_J^M\left(-\overrightarrow{p}\right)\gamma_2^0 = (-1)^J\pi_{q\bar{q}}\Psi_J^M\left(\overrightarrow{p}\right) = (-1)^{J+1}\Psi_J^M(\overrightarrow{p}) \tag{3.2}$$

Using the plane wave expansion and the identity

$$\gamma_1^0 u_1^\rho\left(-\rho\overrightarrow{p}\right) = \rho u_1^\rho\left(\rho\overrightarrow{p}\right) \tag{3.3}$$

we can evaluate the left hand side of Eq.3.2 as

$$\gamma_1^0 u_1^\rho\left(-\rho\overrightarrow{p}\right)\mathcal{Y}_{LSJ}^M\left(-\hat{p}\right)u_2^{\rho'}\left(\rho'\overrightarrow{p}\right)\gamma_2^0 = \rho\rho'(-1)^L u_1^\rho\left(\rho\overrightarrow{p}\right)\mathcal{Y}_{LSJ}^M\left(\hat{p}\right)u_2^{\rho'}\left(-\rho'\overrightarrow{p}\right) \tag{3.4}$$

This gives the following identity that must be satisfied by non vanishing radial wave functions for states $q\bar{q}$ which have parity $(-1)^{J+1}$. We get

$$\rho\rho'(-1)^L = (-1)^J \tag{3.5}$$

3.3 Charge Conjugation

To examine how charge conjugation affects the solutions, we expand the wave function in the plane-wave basis as follows,

$$\Psi_J^M\left(\overrightarrow{p}\right) = \sum_{LS} u_1^\rho\left(\rho\overrightarrow{p}\right)\mathcal{Y}_{LSJ}^M\left(\hat{p}\right)\left[u_2^{\rho'}\left(-\rho'\overrightarrow{p}\right)\right]^T G_{LS}^{\rho\rho'}(p) \tag{3.6}$$

Here $u_1^{\rho}(\rho\overrightarrow{p})$ and $u_2^{\rho'}(\rho'\overrightarrow{p})$ are the plane-wave Dirac spinors introduced above and $G_{LS}^{\rho\rho'}(p)$ the radial wave functions. $\mathcal{Y}_{LSJ}^{M}(\hat{p})$ are angular functions which are discussed in appendix B.

The charge conjugation operator, $\mathcal{C}$, is defined as,

$$\mathcal{C} = \left(i\gamma_1^2\right)\left(i\gamma_2^2\right)K \tag{3.7}$$

where K is the complex conjugation operator. We can define the charge-conjugate wave function by the relation

$$\Psi_J^{M^C}(p) = \mathcal{C}\Psi_J^M(p) \tag{3.8}$$

We can then write the two-body equation as

$$\mathcal{C}\left[(E_1 - h_1)\frac{h_2}{\epsilon_2} + (E_2 - h_2)\frac{h_1}{\epsilon_1} - V\right]\mathcal{C}^{-1}\mathcal{C}\Psi = 0 \tag{3.9}$$

Since the potentials we are dealing with are charge-conjugation invariant we have

$$\mathcal{C}V\mathcal{C}^{-1} = V \tag{3.10}$$

Then it follows that since $\mathcal{C}h_i\mathcal{C}^{-1} = -h_i$ (note $\overrightarrow{p^*} = \left(i\overrightarrow{\partial}\right)^* = -\overrightarrow{p}$) the two-body equation under charge-conjugation operation becomes,

$$\left[(-E_1 - h_1)\frac{h_2}{\epsilon_2} + (-E_2 - h_2)\frac{h_1}{\epsilon_1} - V\right]\Psi^C = 0 \tag{3.11}$$

Thus Ψ^C is the negative-energy solution corresponding to an anti-particle bound state. To each positive-energy solution Ψ, with energy $E = E_1 + E_2$ there is an associated solution Ψ^c with energy $-E = -E_1 - E_2$, thus the spectrum is symmetric about zero energy.

3.4 Solution of Two-Body Equation

The two-body equation will be solved for check cases where results are known from other work. First the numerical procedure is discussed then we will compare the non-relativistic limit of the two-body equation with only a linear potential and compare solutions with the known solution of Schrödinger equation. This serves as a check on the most singular part of the potential. Next in the same non relativistic limit the Coulomb part of the potential is checked with positronium results. Finally the two-body will be compared with some results obtained by Spence and Vary[12] and for the Salpeter equation

3.5 Numerical Methods

As mentioned in Chapter 2, each wave function component is expanded in Eq.2.58 as follows,

$$G_{LS}^{\rho_1\rho_2}(p) = \sum_{n=1}^{N} C_{LSn}^{\rho_1\rho_2} \frac{B_n(p)}{p}$$

where $B_n(p)$ are cubic B-splines with continuous second derivatives and $C_{LSn}^{\rho_1\rho_2}$ are the spline coefficients to be determined. Each spline function vanishes outside of a finite range of the argument, p, which is controlled by selecting a sequence of knot points. The choice of knot points and spline functions used in this work follows closely that used by Spence and Vary in a similar momentum space analysis. Since cubic splines has continuous derivative upto second order neighbouring splines overlap such that the superposition can describe a smoothly varying wave function of the type expected for bound states.

Multiplying by $pB_m(p)$ and integrating over p we obtained a matrix equation Eq.2.59 for the spline coefficients as follows

$$\sum_{n,m} \left[\left(\frac{1}{2}E(\rho_1+\rho_2)-\Delta(\rho_1-\rho_2)\right) I_{mn} - CS_{mn}\right] C^{\rho_1\rho_2}_{LSn} =$$

$$\sum_{n,m}\left[K_{mn}C^{\rho_1\rho_2}_{LSn} + \sum_{\sigma_1\sigma_2,L'S'} V^{\rho_1\rho_2,\sigma_1\sigma_2}_{LS,m,L'S',n} C^{\rho_1\rho_2}_{L'Sn}\right]$$

where

$$I_{mn} = \int_{p_l}^{p_u} dpB_m(p)\,B_n(p)$$

$$S_{mn} = \int_{p_l}^{p_u} dpB_m(p)\,B_n(p)\,\frac{m_1}{\epsilon_1}\frac{m_2}{\epsilon_2}$$

$$K_{mn} = \int_{p_l}^{p_u} dpB_m(p)\,B_n(p)\,(\epsilon_1+\epsilon_2)$$

and

$$V^{\rho_1\rho_2,\sigma_1\sigma_2}_{LS,m,L'S',n} = \int_{p_l}^{p_u} pdp \int_{p'_l}^{p'_u} p'dp' B_m(p)\, V^{\rho_1\rho_2,\sigma_1\sigma_2}_{LS,L'S'} B_n(p')$$

For I_{mn}, S_{mn} and K_{mn} the integration range $[p_l, p_u]$ is the overlap region where both splines are nonzero. Thus they are banded matrices with regard to m and n indices because only splines in the neighbourhood regions overlap. For the potential terms the integration ranges cover the range where both spline functions are nonzero. Given the smooth properties of the splines, the potential singularities at $p = p'$ can be handled. The resulting matrix equation is solved using standard methods for the generalized eigenvalue problem to obtain the rest mass energy E which is the mass of the bound system. When the quark masses are unequal, the parameter $\Delta = (m_1^2 - m_2^2)/2E$ is nonzero and proportional to $1/E$. It is necessary to solve the problem with an approximate value of $\triangle$ and then to iterate using E from the previous solution. The iteration converges rapidly for all cases studied.

The potential is transformed to the plane wave basis from the ρ basis using the transformation matrix $M^{\rho_1\rho_2,i}_{L'S,LS}(p)$. To make the numerical computation simpler we combine the spline functions and transformation coefficients as follows,

$$f^{\rho_1\rho_2,j}_{LS,ls,m}(p) = B_m(p)\, M^{\rho_1\rho_2,j}_{LS,ls,m}(p) \tag{3.12}$$

Inserting specific forms for the scalar and vector terms, the potential in the ρ basis takes the form

$$V^{\rho_1\rho_2,\sigma_1\sigma_2}_{LS,m,L'S',n} = \int pdp \int p'dp' \sum f^{\rho_1\rho_2,j}_{LS,ls;m}(p) \left[a^j V^{sc}_l(p,p') + b^j_s V^{vec}_l(p,p')\right] f^{\sigma_1\sigma_2,j}_{ls,L'S';n}(p) \tag{3.13}$$

At this point, the labels for ρ−spin, spin and angular momentum are suppressed for clarity. The contribution of the scalar potential to Eq.3.13, apart from the coefficient a^j is written in the simpler form

$$V^{sc}_{mn} = \int pdp \int p'dp' f_m(p)\, V^{sc}_l(p,p')\, f_n(p) \tag{3.14}$$

and the contribution of the vector term, apart from the coefficient b^k_s is

$$V^{vec}_{mn} = \int pdp \int p'dp' f_m(p)\, V^{vec}_l(p,p')\, f_n(p) \tag{3.15}$$

A linear confining potential in coordinate space

$$V^{sc}(r) = \kappa \lim_{\mu \to 0} \left(-\frac{\partial}{\partial \mu}\right)^2 \frac{e^{-\mu r}}{r} \tag{3.16}$$

and this corresponds in momentum space to

$$V^{sc}(q) = 4\pi\kappa \lim_{\mu\to 0}\left(-\frac{\partial}{\partial\mu}\right)^2 \frac{1}{(\vec{q}^{\,2}+\mu^2)} \tag{3.17}$$

where $\vec{q} = \vec{p} - \vec{p}\,'$.Partial wave projection involves Legendre functions of the second kind as

$$V_l^{sc}(p,p') = 4\pi\kappa \lim_{\mu\to 0}\left(-\frac{\partial}{\partial\mu}\right)^2 \frac{Q_l(Z)}{2pp'} \tag{3.18}$$

where

$$Z = \left[\frac{p'^2 + p^2 + \mu^2}{2pp'}\right] \tag{3.19}$$

and

$$Q_0(Z) = \frac{1}{2}\ln\left|\frac{Z+1}{Z-1}\right| \tag{3.20}$$

along with the recursion relation

$$Q_{L+1}(Z) = \left(\frac{2L+1}{L+1}\right) Z Q_L(Z) - \left(\frac{L}{L+1}\right) Q_{L-1}(Z).$$

logarithmic factors in $Q_l(Z)$ of the form $\ln|(p-p')^2+\mu^2|$ become singular when $p = p'$ and $\mu = 0$. The resulting singularity of $\frac{\partial^2 Q_l}{\partial\mu^2}$ is $\frac{1}{(p-p')^2}$ in the limit $\mu \to 0$. When the splines in V_{mn} do not overlap, the singularity at $p = p'$ is not within the integration range and the integrations in V_{mn} may be performed directly.

But when the singularity is encountered, the general form of the integral to be evaluated is

$$A_{mn} = \int_{p_l}^{p_u} dp \int_{p'_l}^{p'_u} dp' f_m(p)\, A(p,p')\, f_n(p') \tag{3.21}$$

where $A(p,p')$ is symmetric with respect to interchange of p and p' and has a singularity at $p = p'$ of the form $1/(p-p')^2$. The limits of integration are the range $[p_l, p_u]$ where the spline function $f_m(p)$ is nonzero and $[p'_l, p'_u]$ where $f_n(p')$ is nonzero. Due to the symmetry of $A(p,p\prime)$ integral may be rewritten as

$$A_{mn} = \frac{1}{2}\int_a^b dp \int_a^b dp' \left[f_m(p) A(p,p\prime) f_n(p') + f_m(p) A(p,p\prime) f_n(p') \right] \tag{3.22}$$

Since the splines vanishes outside a finite range the new limits of integration $a = \min(p_l, p_l')$ and $b = \max(p_u, p_u')$ define somewhat larger integration region and at the integration limits, both spline functions vanish at least as fast as $(p-a)^3$ or $(p-b)^3$. Thus the integral can be rewritten as

$$\begin{aligned} A_{mn} &= \frac{1}{2}\int_a^b dp \int_a^b dp' \left[f_m(p) - f_m(p') \right] A(p,p\prime) \left[f_n(p) - f_n(p') \right] \\ &+ \int_a^b dp f_m(p) f_n(p) \int_a^b dp' A(p,p\prime) \end{aligned} \tag{3.23}$$

Because of the symmetry the second integral cancels the two terms from the first integral and the remaining terms are same as in Eq.3.22. Two powers of $p - p'$ arising from the differences of spline functions are sufficient to regulate the singularity in the first integral. For the potentials of interest, the second integral involving $\int dp' A(p,p\prime)$ may be done analytically.

First consider the linear potential for which $A(p,p\prime) = 4\pi\kappa \frac{\partial^2 Q_L(Z)}{\partial \mu^2}$. The singular parts are isolated by the use of the identity $Q_L(Z) = P_L(Z) Q_0(Z) - W_{L-1}(Z)$ where $Q_0(Z)$ has logarithmic singularity at $p = p\prime$ when $\mu = 0$ and

$$Q_0(Z) = \frac{1}{2} \ln \left| \frac{(p+p')^2 + \mu^2}{(p-p')^2 + \mu^2} \right| \tag{3.24}$$

$P_L(Z)$ is the Legendre Polynomial, $W_{L-1}(Z) = \sum_{m=1}^{L} (1/L)\, P_{m-1}(Z)\, P_{L-m}(Z)$ is also a polynomial. It follows that

$$\frac{\partial^2 Q_L(Z)}{\partial\mu^2} = \frac{\partial^2 Q_0(Z)}{\partial\mu^2} + \frac{1}{pp'} P_L'(1)\, Q_0(Z) + \frac{R_L(p,p')}{pp'} \tag{3.25}$$

where $R_L(p,p')$ is the nonsingular part,

$$R_L(p,p')\mid_{\mu=0} = [P_L(Z_0) - 1]\, Q_0'(Z_0) + [P_L'(Z_0) - P_L'(1)]\, Q_0(Z_0) - W_{L-1}''(Z_0) \tag{3.26}$$

and primes denote derivatives with respect to Z. Here Z_0 is Z evaluated at $\mu = 0$. This leads to

$$V_{mn}^{sc} = 4\pi\kappa \left[V_{mn}^{a} + P_L'(1)\, V_{mn}^{b} + \int_a^b dp \int_a^b dp' f_m(p)\, R_L(p,p')\, f_n(p') \right] \tag{3.27}$$

where the singular terms involving $\frac{\partial^2 Q_0(Z)}{\partial\mu^2}$ are evaluated in the limit $\mu \to 0$ using Eq.3.23 and we set

$$V_{mn}^{a} = -\frac{1}{2}\int_a^b dp \int_a^b dp' \left[f_m(p) - f_m(p')\right] \left[\frac{1}{(p+p')^2} - \frac{1}{(p-p')^2}\right] \left[f_n(p) - f_n(p')\right]$$

$$+ \int_a^b dp f_m(p)\, f_n(p) \left[\frac{2p}{p^2 - a^2} - \frac{2p}{p^2 - b^2}\right] \tag{3.28}$$

and the singular term involving just $Q_0(Z)$ yields

$$V_{mn}^{a} = -\frac{1}{2}\int_a^b dp \int_a^b dp' \left[\frac{f_m(p)}{p} - \frac{f_m(p')}{p'}\right] \frac{1}{2}\ln\left|\frac{(p+p')^2}{(p-p')^2}\right| \left[\frac{f_n(p)}{p} - \frac{f_n(p')}{p'}\right]$$

$$+ \int_a^b dp \frac{f_m(p)\, f_n(p)}{p^2} F(p,a,b) \tag{3.29}$$

where

$$F(p,a,b) = (p+b)\ln(p+b) - (p+a)\ln(p+a) -$$
$$(b-p)\ln(|b-p|) + (a-p)\ln(|a-p|) \quad (3.30)$$

Explicit results for the $\int dp' A(p,p\prime)$ contributions have been inserted based upon evaluating them with finite μ and then taking the limit as $\mu \rightarrow 0$. Owing to the vanishing of the spline functions at the limits of integration, the various integrals over p in Equations 3.28 and 3.29 remains finite.

The singularity in the case of $1/x$ potential is logarithmic and the vector interaction involves $A(p,p\prime) = 4\pi\alpha Q_0(Z)$ and the vector potential terms involves,

$$V_l^{vec}(p,p') = 4\pi\alpha\frac{Q_0(Z)}{2pp'} \quad (3.31)$$

The expressions discussed above allow all integrals to be performed numerically and we can determine the matrices required. Gaussian integration is used to evaluate integrals. This momentum space analysis reproduces the eigen values and wave functions for the case where the exact solutions are known. Typically our calculation use 40 spline terms for each component of wave function.

The spline functions $B_n(p)$ are defined by a recursion relation in terms of $N+4$ (distinct) knots $\{\tau_i\}$. These were chosen to be the zeros of a Chebyshev polynomial $\{x_j\}$

$$x_j = -\cos\left(\frac{(2j-1)}{2N}\pi\right) \quad (3.32)$$

Now τ_i are defined by the following mapping

$$\tau_{j+4} = g\sqrt{\frac{1+x_j}{1-x_j}} + \delta \quad (3.33)$$

and for $j \leq 4$, $\tau_j = 0$. The remaining knots were chosen symmetrically so that

$$\tau_{4-j} = -\tau_{j+4}, \quad j = 1, 2, 3 \tag{3.34}$$

The choices of g and δ found to give numerically stable results in all partial waves, were $g = 0.5GeV$. and $\delta = 0.025GeV$.

3.6 Comparison with Exact Results for Linear Potential

In momentum space linear potential has the form $1/q^4$, which is very singular at $q = 0$. In our momentum space calculation, we use the method of Eyre, Spence and Vary[11, 15] for this highly singular kernel. Since the Schrödinger equation with the linear potential has an analytic solution, the exact eigenvalues can be compared with the eigenvalues obtained numerically.

The two-body Dirac equation can be written in the form

$$[(E_1 - \epsilon_1 \hat{\rho}_1)\,\hat{\rho}_2 + (E_2 - \epsilon_2 \hat{\rho}_2)\,\hat{\rho}_1 - V]\,\Psi = 0 \tag{3.35}$$

To obtain the non-relativistic limit of this equation, the negative energy components are excluded, i.e., we keep only the ++, channel where $\rho_1 = \rho_2 = +1$. The bound-state energy for the non-relativistic case is

$$E_i^{NR} = E_i - m_i \tag{3.36}$$

When the following approximation is made,

$$\epsilon_i = \sqrt{p_i^2 + m_i^2} \simeq m_i + \frac{p_i^2}{2m_i} \tag{3.37}$$

the non-relativistic limit of the two-body equation takes the following form

$$\left\{\left[E_1 - \frac{p_1^2}{2m_2} - m_1\right] + \left[E_2 - \frac{p_2^2}{2m_2} - m_2\right] - V^{NR}\right\}\Psi^{NR} = 0 \tag{3.38}$$

This can be written as

$$\left\{\left[\frac{p_1^2}{2m_2} + m_1\right] + \left[\frac{p_2^2}{2m_2} + m_2\right] + V^{NR}\right\}\Psi^{NR} = (E_1 + E_2)\,\Psi^{NR}. \tag{3.39}$$

which is just the Schrödinger equation. Thus the coordinate-space equation is,

$$\left[\left(\frac{-\nabla_1^2}{2m_2}\right) + \left(\frac{-\nabla_2^2}{2m_2}\right) + V^{NR}\right]\Psi^{NR} = E^{NR}\Psi^{NR} \tag{3.40}$$

where $E^{NR} = E_1 + E_2 - m_1 - m_2$. Making the standard transformation to CM system and relative coordinates, we get the Schrödinger equation,

$$\left[-\frac{\hbar^2\nabla^2}{2\mu} + V^{NR}\right]\Psi^{NR}(\vec{r}) = E^{NR}\Psi^{NR}(\vec{r}). \tag{3.41}$$

Letting $V(r) = \kappa r$ and $E^{NR} = E$, for brevity, we get the usual 3-D Schrödinger equation,

$$\left[-\frac{\hbar^2\nabla^2}{2\mu} + \kappa r\right]\Psi^{NR}(\vec{r}) = E\Psi^{NR}(\vec{r}), \tag{3.42}$$

For $l = 0$ it can be written in spherical coordinates as

$$\left[-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} - \kappa r\right]U(r) = -EU(r) \tag{3.43}$$

Here the reduced wave function $U(r) = rR(r)$. The standard linear potential boundary conditions for the reduced wave function are given by

$$\begin{aligned} U(0) &= 0, \\ U(\infty) &= 0 \end{aligned} \tag{3.44}$$

When the reduced equations are solved we obtain the usual Airy function solutions of the linear potential[16]. To obtain the expression for the energy eigenvalues, let

$$\frac{2\mu\kappa}{\hbar^2} = \frac{1}{l^3} \tag{3.45}$$

$$\frac{2\mu E}{\hbar^2} = \frac{\lambda}{l^2} \tag{3.46}$$

to obtain

$$\left[-\frac{d^2}{dr^2} + \frac{1}{l^3}r\right] U(r) = \frac{\lambda}{l^2} U(r) \tag{3.47}$$

with the change of variable,

$$\xi = \frac{r}{l} - \lambda \tag{3.48}$$

we arrive at the non-dimensional form $U(r) = \widetilde{U}(\xi)$

$$\left[-\frac{d^2}{d\xi^2} + \xi\right] \widetilde{U}(\xi) = 0 \tag{3.49}$$

with the boundary conditions

$$\widetilde{U}(-\lambda) = 0 \tag{3.50}$$

$$\widetilde{U}(\infty) = 0 \tag{3.51}$$

The exact solutions are $\widetilde{U}(\xi) = A_i(\xi)$, where $A_i(\xi)$ is the Airy function. The eigenvalues are given by,

$$E_n = \lambda_n \left(\frac{\kappa}{\sqrt{2\mu}}\right)^{\frac{2}{3}} \tag{3.52}$$

where λ_n are the roots of the Airy function. In computation (see Table 3.1) we use typically $\kappa = 0.29\,(GeV)^2$. Let $\mu = \frac{1}{2}m_q$ we get

$$E_n = \lambda_n \left(\frac{\kappa}{\sqrt{m_q}}\right)^{\frac{2}{3}} \tag{3.53}$$

Table 3.1 Comparison of exact and calculated energies for the Schrodinger equation with linear potential, using $m_q = 25\ MeV$ and $m_q = 1250\ MeV$

$m_q = 25\ MeV$			
n	λ_n (MeV)	$Exact\ E_n$	E_n (40 splines)
1	2.338	3500	3505
2	4.088	6119	6128
3	5.521	8263	8276
4	6.787	10157	10174
5	7.944	11889	11909
6	9.023	13504	13526
7	10.040	15026	15052
8	11.009	16476	16504
9	11.936	17864	17896
10	12.829	19200	19236

n	λ_n(MeV)	Exact E_n	E_n (40 splines)
1	2.338	951	951
2	4.088	1662	1663
3	5.521	2245	2246
4	6.787	2759	2761
5	7.944	3231	3232
6	9.023	3670	3672
7	10.040	4083	4087
8	11.009	4477	4484
9	11.936	4854	4888
10	12.829	5222	5285

3.7 Comparison with Exact Results for Coulomb Potential

A test for precision of the eigenvalues for the $1/r$ part of the potential can also be performed in a similar manner. The situation now corresponds to positronium bound state, but with the mass now referring to the quark masses. The exact energy is given by the formula[17]

$$E_n^{theory} = \mu \frac{\alpha^2}{2n^2} \tag{3.54}$$

Here μ is just reduced mass and α in the coupling constant, which in QED is just the fine structure constant $\frac{1}{137}$. In this case, quark masses of 250 MeV and 25 MeV are chosen.

Table 3.2 Heavy mesonium with Coulomb Potential

n	$m_q(MeV)$	E_n^{theory}	E_n 40 splines
1	250	-6.320	-6.51
2	250	-1.581	-1.627
3	250	-0.702	-0.713
4	250	-0.395	-0.400
5	250	-0.253	-0.255
6	250	-0.175	-0.160
7	250	-0.129	-0.084

n	$m_q(MeV)$	E_n^{theory}	E_n 40 splines
1	25	-0.6320	-0.651
2	25	-0.1581	-0.161
3	25	-0.0702	-0.0713
4	25	-0.0395	-0.040
5	25	-0.0253	-0.0255
6	25	-0.0175	-0.0154
7	25	-0.0129	-0.0084

3.8 Heavy Meson Spectra using the Salpeter Equation

The inclusion of positive and negative energy states is very important for the lighter meson states, for which the energy gap between positive and negative energy states is relatively small. Now we are interested in heavy quark states for which we may ignore the negative energy states. This physical situation has been studied by Spence and Vary[12].

The basic equation Spence and Vary solve is the three-dimensional reduction of the Bethe-Salpeter equation in the instantaneous approximation for ++ states. They obtain the spectra for this equation and compare it to the known meson spectra. The two-body equation we are using reduces to the same equation when only the ++ channels are included.

An important test of the present analysis is to reproduce the results of Spence and Vary, which are known to approximate the spectra of the heavy quarks. The two body equation for ++ states takes the simple form,

$$[E_1 - \epsilon_1 + E_2 - \epsilon_2]\,\Psi(\vec{p}) = \int \frac{d^3q}{(2\pi)^3} V^{++,++}(\vec{p} - \vec{p}')\,\Psi(\vec{p}') \tag{3.55}$$

Spence and Vary solve this equation using the Coulomb plus a linear confinement potential

$$V(q) = \frac{-4\pi\alpha\gamma_0\gamma_\mu^{(1)} \otimes \gamma_0\gamma_\mu^{(2)}}{(\vec{q}^{\,2})} + 4\pi\kappa \lim_{\mu\to 0}\left(\frac{-\partial}{\partial\mu}\right)^2 \frac{\gamma_0 \otimes \gamma_0 1^{(2)} \otimes 1^{(2)}}{(\vec{q}^{\,2} + \mu^2)} \tag{3.56}$$

The eigenvalues from the present work are compared with those of Spence and Vary in Table 3.3 and Table 3.4. The parameters chosen are given below.

$V = \kappa r + \alpha/r$			
	quark mass	κ	α
$c\bar{c}$	1250 MeV	$0.2 Gev^2$	0.25
$b\bar{b}$	4580 MeV	$0.2 Gev^2$	0.25

Table 3.3 Charmonium mass spectra based on ++ equation. M_{SV} represents Spence and Vary's results.

Charmonium Masses in MeV					
particle	J^P	$N^{2S+1}L_J$	Expt	M_{SV}	This work
η_C	0^-	1^1S_0	2980	3049	3049
η_C'	0^-	2^1S_0	3594	3651	3651
J/ψ	1^-	1^3S_1	3097	3105	3103
ψ'	1^-	2^3S_1	3685	3691	3687
ψ''	1^-	1^3D_1	3770	3741	3736
ψ'''	1^-	3^3S_1	4030	4094	4102
ψ''''	1^-	2^3D_1	4160	4127	4128
ψ^V	1^-	4^3S_1	4415	4414	4404
χ_0	0^+	1^3p_0	3415	3437	
χ_1	1^+	1^3p_1	3510	3462	
χ_2	2^+	1^3p_2	3556	3528	

Table 3.4 Bottomonium mass spectra based on ++ equation.M_{SV} Spence and Vary's results.

Bottomonium Masses in MeV					
particle	J^P	$N^{2S+1}L_J$	*Exp*	M_{SV}	This work
Υ	1^-	1^3S_1	9490	9480	9478
Υ	1^-	2^3S_1	10023	10004	10003
	1^-			10099	10095
Υ	1^-	1^3D_1	10356	10384	10413
	1^-			10448	10413
Υ	1^-	3^3S_1	10573	10701	10800
χ_{b0}	0^+	1^3p_0	9860	9825	
χ_{b1}	1^+	1^3p_1	9892	9842	9841
χ_{b2}	2^+	1^3p_2	9813	9907	9965
χ_{b0}'	0^+	1^3p_0	10232	10229	
χ_{b2}'	1^+	1^3p_1	10255	10244	10247
χ_{b3}'	2^+	1^3p_2	10268	10299	

Chapter 4
Meson Spectra

4.1 Introduction

The relativistic properties of the quark-antiquark interaction potential play an important role in the analysis of different static and dynamic characteristics of heavy mesons. The Lorentz structure of the confining quark-antiquark interaction is also of particular interest. For a long time the scalar confining kernel has been considered to be the most appropriate one[24]. The main argument in favour of this choice is based on the nature of the heavy quark spin-orbit potential. The scalar potential gives a vanishing long range magnetic interaction, which is in agreement with the flux tube picture of quark confinement [26] and allows to get the fine structure of heavy quarkonia in accordance with experimental data. However the calculation of electro-weak decay rates of heavy mesons with a scalar confining potential alone yield results which are in worse agreement with data than with a vector potential[27, 28]. In this contest it is worth noting that the recent study of the $q\overline{q}$ interaction in the Wilson loop approach [29] indicate that it cannot be considered as purely scalar. Moreover, the found structure of spin independent relativistic corrections is not compatible with a scalar potential. A similar conclusion has been obtained by Szczepaniak et al. [30]on the basis of a Foldy-Wouthuysen reduction of the full Coulomb Gauge Hamiltonian of QCD. There the Lorentz structure of the confinement has been found to be of vector nature.

In previous chapters we have discussed the two-body equation for the case of equal mass quarks. Here the more general case of unequal mass quarks will be considered. We discuss the covariant form of the two-body equation first.

4.2 Two-Body Equation in Covariant Form

A covariant form [10] is readily obtained by rewriting the two-body Dirac equation in terms of suitable four vectors. Using the notation $p_i = (E_i, \overrightarrow{p_i})$ the total momentum $P = p_1 + p_2$ is a constant of the motion. Relative momentum $p = \frac{1}{2}(p_1 - p_2)$ is a dynamical variable. The kinematical constraint, $E_1^2 - E_2^2 = m_1^2 - m_2^2$, may be rewritten in an arbitrary frame as

$$p_1^2 - p_2^2 = m_1^2 - m_2^2 \tag{4.57}$$

or as

$$P.p = \frac{1}{2}\left(m_1^2 - m_2^2\right) \tag{4.58}$$

Thus the component of relative momentum parallel to P is fixed and the dynamics is reduced to three dimensions. Defining a unit four-vector parallel to the total momentum as

$$\hat{P} = \frac{P}{\sqrt{P \cdot P}} \tag{4.59}$$

the relative momentum may be split into four-vectors which are parallel and perpendicular to the total momentum as follows,

$$p_{\|} = \left(p.\hat{P}\right)\hat{P} \tag{4.60}$$

$$p_{\perp} = p - \left(p.\hat{P}\right)\hat{P} \tag{4.61}$$

Note that because of the constraint, $p_{\|}$ is a constant of motion. Moreover $p_\perp$ is space like ($p_\perp^2 < 0$) in all frames. The two-body equation rewritten in terms of these variables is

$$[(\gamma_1 \cdot p_1 - m_1)\,\lambda_2 + (\gamma_2 \cdot p_2 - m_2)\,\lambda_1 - V_{12}]\,\psi = 0$$

The relativistic bound state of two fermions is analysed using the above two-body Dirac equation, which has the covariant form

$$\left\{ \left(\not{p}_1 - m_1\right)\lambda_2 + \left(\not{p}_2 - m_2\right)\lambda_1 - \hat{V} \right\} \Psi = 0 \tag{4.62}$$

The operators λ_i are defined by

$$\lambda_i = \frac{m_i - \not{p}_{i\perp}}{\epsilon_i} \tag{4.63}$$

In the rest frame of the bound system, a simpler form of the equation is obtained, after multiplication by $\gamma_1^0\gamma_2^0$

$$\left\{ [E_1 - h_1(\vec{p})]\,\frac{h_2(-\vec{p})}{\epsilon_2} + [E_2 - h_2(-\vec{p})]\,\frac{h_1(\vec{p})}{\epsilon_1} - \gamma_1^0\gamma_2^0 V \right\} \psi = 0 \tag{4.64}$$

where $h_i(\vec{p}) = \alpha_i \cdot \vec{p} + \beta_i m_i$ is Dirac Hamiltonian for particle i and $\epsilon_i = \sqrt{m_i^2 + \vec{p}_i^{\,2}}$. The total energy is $E = E_1 + E_2$. Owing to the constraint the relative energy is fixed to $E_1 - E_2 \equiv 2\Delta$ where $\Delta = (m_1^2 - m_2^2)/2E$.

4.3 Quark-antiquark Interaction

The best understood part of the quark antiquark interaction is due to one gluon exchange at short distance. In this work, the confining interaction is modeled by a simple phenomenological potential that may be a scalar or time-like vector. For the case of scalar confinement, the interaction is as follows,

$$\hat{V}\left(\overrightarrow{p},\overrightarrow{p}'\right) = V^{conf}\left(\overrightarrow{p}-\overrightarrow{p}'\right) + \gamma_1\cdot\gamma_2 V^{vec}\left(\overrightarrow{p}-\overrightarrow{p}'\right) \tag{4.65}$$

This yields

$$V^{j,k}_{ls,l's'}\left(p,p'\right) = \delta_{ll'}\delta_{ss'}\left\{a^j V^{conf}_l\left(p,p'\right) + b^j_s V^{conf}_l\left(p,p'\right)\right\} \tag{4.66}$$

The coefficients a^j and b^j_s used above are given in the following Table. The subscript in b^j_s indicates that it depends on the spin quantum number s

j	a^j	b^j_s	c^j
1	-1	$4-2s\left(s+1\right)$	+1
2	-1	$-2+2s\left(s+1\right)$	+1
3	+1	$-2+2s\left(s+1\right)$	+1
4	+1	$4-2s\left(s+1\right)$	+1

For the time-like vector confinement the first term of Eq.4.65 becomes

$$\gamma^0_1\gamma^0_2 V^{conf}\left(\overrightarrow{p}-\overrightarrow{p}'\right) \tag{4.67}$$

and Eq.4.66 is altered by the replacement of the coefficients a_j by c_j as given in the table. An admixture of scalar and time-like vector confinement may be obtained by using a linear combination of a_j and c_j.

When parity is conserved by the interactions, half of the 16 partial wave components vanish in any given state. The parity of the basis function is $\rho_1\rho_2\left(-1\right)^L$, where ρ_1 and ρ_2 factors account for the intrinsic parity of Dirac spinors. For a fermion-antifermion pair, we treat antifermion as a positive energy antiparticle rather than as a negative energy state propagating backward in time. This convention assigns positive energies and an extra intrinsic parity factor of $P_{q\bar{q}} = -1$ to each $q\bar{q}$ state, i.e. the total parity is $\rho_1\rho_2\left(-1\right)^L P_{q\bar{q}}$. With this convention, the $\bar{q}q$ states of parity $\left(-1\right)^L P_{q\bar{q}}$ involve nonzero values for the following eight components :

$$G_{L,S}^{\rho_1\rho_2} = G_{J,0}^{++}, G_{J,1}^{++}, G_{J+1,1}^{+-}, G_{J-1,1}^{+-}, G_{J+1,1}^{-+}, G_{J-1,1}^{-+}, G_{J,0}^{--}, G_{J,1}^{--}$$

Similarly states of parity $(-1)^{J+1} P_{q\bar{q}}$ involve nonzero values for the remaining eight components:

$$G_{L,S}^{\rho_1\rho_2} = G_{J+1,1}^{++}, G_{J-1,1}^{++}, G_{J,0}^{+-}, G_{J,1}^{+-}, G_{J,0}^{-+}, G_{J,1}^{-+}, G_{J+1,1}^{--}, G_{J-1,1}^{--}$$

An exception is $J = 0$ states which have only four nonvanishing components that are obtained by omitting the $L = J - 1$ and $L = J, S = 1$ components.

When an instant, scalar confining interaction is used in this two body equation, imaginary values of the energy[14] are possible. This is due to the couplings due to the Ψ^{--}components of the relativistic wave functions. In our momentum space analysis, we use only $++$ components of the wave function and we always get real and positive energy eigen values.

Gaussian integration is used to evaluate the integrals. The momentum space analysis reproduces the analytical eigenvalues and wave functions. Typically our calculation use 40 spline terms for each component of wave function. The results are compared with that of Gofrey and Isgur[13, 48] and Tiemeijer and Tjon[18].

Table 4.1 GI represents parameters used by Godfrey and Isgur and TT means corresponding parameters by Tiemeijer and Tjon

$V = \kappa r + \alpha/r + C$			
	This Work	GI	TT
$m_u\,(GeV)$	0.230	0.220	0.250
$m_s\,(GeV)$	0.350	0.419	0.390
$m_c\,(GeV)$	1.457	1.628	1.719
$m_b\,(GeV)$	4.730	4.977	5.096
$\kappa\, GeV^2$	0.219024	0.18	0.33
α	-0.333	-0.8	-0.8
$CGeV$	-0.529	-0.253	-1

Table 4.2 Masses for Charm Mesons

Charm(masses in MeV)						
Particle	J^P	$N^{2S+1}L_J$	Expt	this work	M_{TT}	M_{GI}
η_C	0^-	1^1S_0	2980	2954	2969	2970
η_C'	0^-	2^1S_0	3590	3693	3742	3620
J/ψ	1^-	1^3S_1	3097	3108	3096	3100
ψ'	1^-	2^3S_1	3685	3720	3810	3680
ψ''	1^-	1^3D_1	3770	3777	3873	3820
ψ'''	1^-	3^3S_1	4030	4098	4370	4000
ψ''''	1^-	2^3D_1	4160	4190	4409	4190
ψ^{V}	1^-	4^3S_1	4420	4467	4370	4450
ψ^{Vi}	1^-	2^3D_1		4499		4520
h_{c1}	1^+	1^1p_1	3526	3540	3517	3520
χ_{c0}	0^+	1^3p_0	3415	3514	3461	3440
χ_{c1}	1^+	1^3p_1	3511	3573	3526	3510
χ_{c2}	2^+	1^3p_1	3556	3508	3572	3550

Table 4.3 Masses for Bottom Mesons

Bottom (Masses in MeV)						
Particle	J^P	$N^{2S+1}L_J$	Expt	This work	M_{TT}	M_{GI}
η_b	0^-	1^1S_0		9470	9401	9400
η_b'	0^-	2^1S_0			10067	9980
Υ	1^-	1^3S_1	9460	9489	9460	9460
Υ	1^-	2^3S_1		10001		10000
Υ'	1^-	1^3D_1	10023	10134	10099	10140
Υ'	1^-	3^3S_1	10355	10654		10350
Υ''	1^-	2^3D_1	10355	10688	10206	10440
Υ'''	1^-	4^3S_1		10654		10630
	1^-	3^3D_1		10809	10556	10700
Υ''''	1^-	5^3S_1		10885		10880
	1^-	3^3D_1	10870	10923	10629	11100
Υ^V	1^-	6^3S_1		10999		
Υ^V	1^-	3^3D_1	11020	11354	10943	
h_{b1}	1^+	1^1p_1		9888	9881	9880
χ_{b0}	0^+	1^3p_0	9860	9901	9862	9850
χ_{b1}	1^+	1^3p_1	9892	9954	9890	9880
χ_{b0}'	0^+	1^3p_0	10232	10222	10363	10230

Table 4.4 Masses for Mesons in $\pi - \rho$ and Φ families

$\pi - \rho$ (masses in MeV)						
Particle	J^P	$N^{2S+1}L_J$	Expt	This work	M_{TT}	M_{GI}
π	0^-	1^1S_0	135	137	439	150
π'	0^-	2^1S_0	1300	1349	1441	1300
π''	0^-	3^3S_0		1802	2246	1880
ρ	1^-	1^3S_1	768	809	798	770
ρ'	1^-	1^3D_1		1498		
ρ'	1^-	2^3S_1	1450	1509	1454	1450
ρ''	1^-	2^3D_1		1912		
ρ''	1^-	3^3S_1	1700	1883	1653	1660
ρ'''	1^-	3^3D_1		2261	2185	2000
ρ'''	1^-	4^3S_1		2365	2367	2150
b_1	1^+	1^1P_1	1230	1193	1091	1220
a_0	0^+	1^3P_0	983	1034	993	1090
a_1	1^+	1^3P_1	1260	1278	1126	1240
a_2	2^+	1^3P_2	1318	1231	1297	1310
π_2	2^-	1^1D_2	1670	1536	1524	1680

	J^P	$N^{2S+1}L_J$	Exp	this work	M_{TT}	M_{GI}
ϕ	1^-	1^3S_1	1019	1049		
ϕ'	1^-	1^3D_1		1728		
ϕ'	1^-	2^3S_1		1768		
ϕ''	1^-	2^3D_1		2201		
ϕ''	1^-	3^3S_1		2788		

Table 4.5 Masses for K mesons

Kaon family (Masses in MeV)						
Particle	J^P	$N^{2S+1}L_J$	Expt	This work	M_{TT}	M_{GI}
k	0^-	1^1S_0	493	541	593	470
k'	0^-	2^1S_0		1478	1457	1450
k''	0^-	3^3S_0		1895	2069	2020
k^*	1^-	1^3S_1	896	977	896	900
$k^{*\prime}$	1^-	2^3S_1		1633	1584	1583
$k^{*\prime\prime}$	1^-	2^3S_1	1714	1954	1680	1780
k_1	1^+	1^1P_1	1270		1282	1340
k_0^*	0^+	1^3P_0	1430	1305	1262	1240
k_1	1^+	1^3P_1	1400	1389	1333	1380
k_2^*	2^+	1^3P_2	1425	1400	1376	1430
k_1'	1^+	2^1P_1	1650	1809		1900
$k_0^{*\prime}$	0^+	2^3P_0	1945	1925		1890
k_1'	1^+	2^3P_1		1929		1930
$k_2^{*\prime}$	2^+	2^3P_2	1975	1980		1940
k_2	2^-	1^1D_2		1765		1780
	2^-	1^3D_2		1792		1810
k_2'	2^-	2^1D_2	1770	2120	1672	2230

Table 4.6 Masses for D and B mesons

D mesons(masses in MeV)						
particle	J^P	$N^{2S+1}L_J$	$Expt$	this work	M_{TT}	M_{GI}
D	0^-	1^1S_0	1864	1879	1868	1880
D^*	1^-	1^3S_1	2007	2013	2015	2040
D_1	1^+	1^1P_1	2420	2322	2388	2444
D_0^*	0^+	1^3P_0		2384	2321	2400
D_1'	1^+	1^3P_1		2411	2415	2490
D_2^*	2^+	1^3P_2	2459	2456	2458	2500
D_s	0^-	1^1S_0	1969	1914	1952	1980
D_s^*	1^-	1^3S_1		2156	2104	2130
D_{s1}	1^+	1^1P_1	2537	2301	2500	2530
D_{s0}^*	0^+	1^3P_0		2480	2427	2480
D_{s1}'	1^+	1^3P_1		2532	2516	2570
D_{s2}^*	2^+	1^3P_2		2493	2569	2590

B *mesons* (masses in MeV)						
Particle	J^P	$N^{2S+1}L_J$	Expt	This work	M_{TT}	M_{GI}
B	0^-	1^1S_0	5279	5286	5302	5310
B^*	1^-	1^3S_1	5324	5408	5391	5370
B_1	1^+	1^1P_1		5580		
B_0^*	0^+	1^3P_0		5669		
B_1'	1^+	1^3P_1		5709		
B_2^*	2^+	1^3P_2		5623		5800
B_s	0^-	1^1S_0		5486	5371	5390
B_s^*	1^-	1^3S_1		5596	5434	5450
B_{s1}	1^+	1^1P_1		5762		
B_{s0}^*	0^+	1^3P_0				
B_{s1}'	1^+	1^3P_1		5872		
B_{s2}^*	2^+	1^3P_2				5880

The calculated spectra are in good agreement with the experimental data. The parameter values we have are reasonable and comparable to other models of similar type.

Chapter 5
Semileptonic Decays

5.1 Introduction

Today improvement in both theory and experiment have made semileptonic decays a main focus of attention. They provide an important tool to investigate quark dynamics and to determine CKM matrix elements. Hadron dynamics is contained in form factors which are not yet calculated from the first principles of QCD. Thus various potential models, sum rules and lattice calculations have been proposed. Recently considerable progress have been achieved in describing heavy meson decays by the use of heavy quark effective theory (HQET)[37, 38, 39]. So far there are many models[46, 57, 58, 51, 53, 55, 61, 64] giving wide ranging predictions on the exclusive semileptonic decays of heavy flavoured mesons. In the non-relativistic constituent quark model of Isgur, Scora, Grinstein and Wise (ISGW)[46] all the weak decay form factors computed with the overlap integral of the nonrelativistic meson wave functions[13], have the same exponential q^2 dependence which is not entirely compatible with the predictions of the heavy quark symmetry. Altamari and Wolfenstein(AW)[56] in a similar non-relativistic approach,determine the form factors at $q^2 = q^2_{\mathrm{max}}$ and extrapolate them down to $q^2 = 0$ postulating the q^2 dependence through monopole forms. However in their calculation one of the form factors is found to be less trustworthy because of the exclusion of the significant effects due to the quadratic and higher order terms involving the daughter meson momentum. Gilman and

Singleton(GS)[58] use a modified quark model based on an approach similar to AW and suggest rescaling the form factors in order to fit the available data. In a relativistic calculation of Bauer, Stech and Wirbel (BSW)[50] the form factors having the q^2 dependence in the monopole ansatz with the normalization at $q^2 = 0$ are computed from the overlap integrals of light-cone wave functions[51, 52]. As an extension of this work Korner and Schuler (KS)[52] adopt a monopole or dipole ansatz for the q^2 dependence of the form factors. But such relativistic treatments are not totally free from objections. Unlike the quark potential models, the phenomenology in this case is yet to be tuned. Second, the computation of these form factors normalized at $q^2 \rightarrow 0$, requires the knowledge of the infinite momentum frame wave functions near the end points where they are usually small or least understood. Recently Barik and Dash[55] have studied the form factors of exclusive semileptonic decays based on relativistic independent quark model. There exists some discrepancy between their prediction and currently available experimental data. Therefore it appears that a completely consistent calculation of the weak decay form factors in the framework of quark model has not been accomplished yet. This may be mainly due to the fact that in the calculation of the hadronic matrix element, the truly bound state relativistic character of the relevant hadrons has not been adequately represented.

We therefore consider it worthwhile to investigate the semileptonic decay of heavy mesons D and B in our relativistic potential model. It is important to note that since B and D mesons contain light quarks, relativistic effects are quite significant. The first part provides a brief outline of the general formalism[52] adopted here for the analysis these decays. In the next section we describe the model conventions and realize the invariant

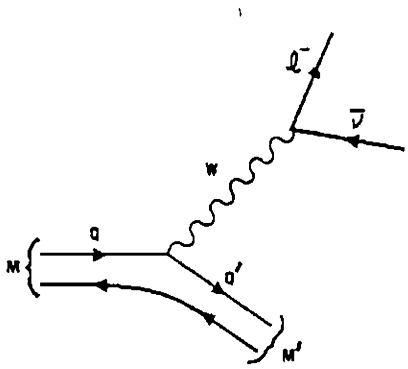

Fig 5.1

transition matrix element as well as the relevant form factors with their appropriate q^2 dependence directly from the model.

5.2 General Formalism

We are interested in the exclusive semileptonic decay of heavy flavoured pseudoscalar mesons $D^0$and B^0 into pseudoscalar mesons K and D respectively. Such a process is described in Fig. 5.1 through the decay of the heavy quark Q in the parent meson M into a less heavy or light quark Q' in the daughter meson M' along with the virtual W boson which ultimately decays into a charged lepton and its neutrino, where the antiquark $\bar{q}$ remains as a spectator.

A brief outline of the general formalism is described here. For the decay process the invariant transition matrix element is generally written as

$$\mathcal{M} = \frac{G_F}{\sqrt{2}} V_{Qq} L^{\mu} H_{\mu} \tag{5.1}$$

where G_F is the effective Fermi Coupling constant and V_{Qq} is the CKM parameter. The leptonic and Hadronic parts of the amplitude here are

$$L^{\mu} = \overline{u_{\nu}}\gamma^{\mu}(1-\gamma^{5})v_{e}$$

$$H_{\mu} = \langle M'(P', S_{M'})|J_{\mu}^{h}(0)|M(P, S_{M})\rangle \tag{5.2}$$

where $J_{\mu}^{h} = V_{\mu} - A_{\mu}$. Here (P, P') the four momentum and $(S_M, S_{M'})$ the spin projection of the parent M and the daughter M' meson respectively. The hadronic matrix element is conventionally expressed in terms of the form factors. For the semileptonic transition of the type $(0^- \to 0^-)$ where the pseudoscalar meson is in the final state, only the hadronic vector current contributes, which is expressed as

$$\langle M'(P')|V_{\mu}|M(P)\rangle = f_{+}(q^2)(P+P')_{\mu} + f_{-}(q^2)(P-P')_{\mu} \tag{5.3}$$

where $q^{\mu} = (P-P')^{\mu}$ and $f_{+}(q^2)$ and $f_{-}(q^2)$ are the form factors. For these quantum numbers, the hadronic current J_{μ}^{h} has no axial vector contribution and can also be written

$$\langle M'(P')|V_{\mu}|M(P)\rangle = F_1(q^2)\left[(P+P')^{\mu} - \frac{M^2 - M'^2}{q^2}q^{\mu}\right] + F_0(q^2)\frac{M^2 - M'^2}{q^2}q^{\mu} \tag{5.4}$$

Here $F_0(0) = F_1(0)$. So that there is no singular behaviour at $q^2 = 0$.

On the other hand, for transitions of the type $(0^- \to 1^-)$ where a vector meson is in final state the corresponding matrix elements are given by

$$\langle M'(P', \epsilon^*)|V_{\mu}|M(P)\rangle = ig(q^2)\,\epsilon_{\mu\nu\rho\sigma}\epsilon^{*\nu}(P+P')^{\rho}(P-P')^{\sigma} \tag{5.5}$$

$$\langle M'(P',\epsilon^*)|A_\mu|M(P)\rangle = f\left(q^2\right)\epsilon^*_\mu \quad +a_+\left(q^2\right)(\epsilon^*\cdot P)(P+P')_\mu$$

$$+a_-\left(q^2\right)(\epsilon^*\cdot P)(P-P')_\mu \qquad (5.6)$$

where $\epsilon^* \equiv (\epsilon_0^*, \hat{\epsilon}^*)$ represents the vector meson polarization and $\epsilon^*\cdot P' = 0$.

In this work we concentrate on $(0^- \to 0^-)$ transitions. In general J_μ in terms of quark operators is

$$J_\mu = \overline{Q}'\Gamma^\mu Q \qquad (5.7)$$

Denoting our two-body wave function by $\psi_{\alpha\beta}$ and $\psi'_{\alpha\beta}$ with α index for quark Q the matrix element is

$$\begin{aligned}\langle M'(P')|J_\mu|M(P)\rangle &= \int d^3p\left(\gamma^0\psi'\right)^\dagger_{\beta\alpha'}\Gamma^\mu_{\alpha'\alpha}\psi_{\alpha\beta} \\ &= \int d^3p Tr\left\{\psi'^\dagger\gamma^0\Gamma^\mu\psi\right\} \qquad (5.8)\end{aligned}$$

The integration is over the relative momenta. If the initial meson is at rest its wave function is $\psi(\vec{p})$. We need the wave function of the daughter meson ψ' at momentum $\vec{P}' = -\vec{q}$. Now the current matrix element can be written as

$$\int d^3p Tr\left\{\psi'^\dagger\left(\vec{P}',\vec{p}\right)\gamma^0\Gamma^\mu\psi\left(\vec{p}\right)\right\} \qquad (5.9)$$

In order to calculate this matrix element we require boosted wave functions.

5.3 Boost Transformations

The two-body boost problem in the instant form of relativistic quantum mechanics is to constrain the interaction so that the Poincaré generators satisfy the commutation rules that are required for Poincaré invariance. The central issue in boosting an instant form of dynamics was identified by Dirac[41] : the generator of boosts, $\overrightarrow{K}$, and the Hamiltonian, H, both must contain the interaction, v. Such a boost is called dynamical and the commutation rule between the dynamical boost generator and Hamiltonian involves v^2 terms.

In quantum field theory, the boost of a mass eigen state such as a Bethe-Salpeter vertex function is kinematical. There is a linear relation between the interactions in different frames that depends upon the Lorentz transformation of momenta and spins. Also in this case the boost implicitly depends upon the interaction through the mass eigenvalue, M, which enters the boost velocity $\beta = \overrightarrow{P}/E_{\overrightarrow{P}}$ where $E_{\overrightarrow{P}} = \sqrt{M^2 + \overrightarrow{P}^2}$.

If the Bethe-Salpeter formalism for two particles is reduced to three-dimensions by integrating out the time-component of relative momentum the resulting formalism is quite similar to an instant form of relativistic quantum theory. Wallace[44] show that use of an approximate boost generator is sufficient to derive a simple boost rule for the interaction, v, such that the instant two-body problem has eigenvalue $E_{\overrightarrow{P}}$ corresponding to a fixed value of mass. The boost generator satisfies all but one of the required commutator relations. The analysis provides very simple and direct relationships of vertex functions, wave functions and t- matrices of the two-body problem in different frames. The results are applicable to calculations of matrix elements of an external current, for example form factor calculations based upon the ET formalism, where the initial state or final state, or both, must have nonzero total momentum.

The basic requirement of Poincaré invariance is that states must transform under a unitary representation of the Poincaré group. The ten generators for translations in time, translations in space, boosts and rotations are respectively the Hamiltonian operator H, the operator for total linear momentum $\overrightarrow{P}$, the boost operator $\overrightarrow{K}$ and the angular momentum operator $\overrightarrow{J}$. They obey the well known commutations relations [42]. When there is no interaction, all the required commutation relations may be satisfied by taking each generator to be a sum of single particle generators. But with interaction there are additional constraints. The interaction must be translationally and rotationally invariant and also must satisfy additional nontrivial constraints from the commutation relations

$$[\overrightarrow{K_i}, \overrightarrow{P}_j] = i\overrightarrow{H}\delta_{ij} \tag{5.10}$$

$$[\overrightarrow{K}, H] = i\overrightarrow{P} \tag{5.11}$$

$$[\overrightarrow{K}_i, \overrightarrow{K}_j] = -i\epsilon_{ijk}\overrightarrow{J}_k \tag{5.12}$$

where $[\overrightarrow{r}_j, \overrightarrow{p}_k] = i\delta_{jk}$.

For spinless particles in the instant form of dynamics, total momentum and angular momentum operators are, $\overrightarrow{P} = \overrightarrow{p}_1 + \overrightarrow{p}_2$ and $\overrightarrow{J} = \overrightarrow{r}_1 \times \overrightarrow{p}_1 + \overrightarrow{r}_2 \times \overrightarrow{p}_2$. If the particles are of equal masses, the Hamiltonian is

$$\overrightarrow{H} = \epsilon_1 + \epsilon_2 + v = \overrightarrow{H}_0 + v \tag{5.13}$$

where ϵ_i is the kinetic energy of the i^{th} particle,

$$\begin{aligned} \epsilon_1 &= \epsilon_1(\overrightarrow{p}; \overrightarrow{P}) \equiv \sqrt{m^2 + \left(\frac{1}{2}\overrightarrow{P} + \overrightarrow{p}\right)^2} \\ \epsilon_2 &= \epsilon_2(\overrightarrow{p}; \overrightarrow{P}) \equiv \sqrt{m^2 + \left(\frac{1}{2}\overrightarrow{P} - \overrightarrow{p}\right)^2} \end{aligned} \tag{5.14}$$

where

$$\vec{p} = \frac{1}{2}(\vec{p}_1 - \vec{p}_2)$$

Bakamjian and Thomas[66] have derived the free boost operator,

$$\vec{K}_0 = \frac{1}{2}(\vec{r}_1\epsilon_1 + \epsilon_1\vec{r}_1) + \frac{1}{2}(\vec{r}_2\epsilon_2 + \epsilon_2\vec{r}_2) \tag{5.15}$$

When the interaction v is present in the Hamiltonian, there is an interaction part of the boost operator that is given approximately by

$$\vec{K}_v = \frac{1}{2}\left(\vec{R}v + v\vec{R}\right) \tag{5.16}$$

where $\vec{R} = \frac{1}{2}(\vec{r}_1 + \vec{r}_2)$. These expressions exactly satisfy Eq.5.10 but not Eq.5.12. In that sense the boost generator $\vec{K}$ is approximate. However the interaction v take a form consistent with Eq.5.11. Noting that the free-boost operator, $\vec{K}_0$, and free Hamiltonian H_0, obey Eq. 5.11, the interaction-dependent terms in that commutation relation must sum to zero, i.e.,

$$[\vec{K}_0, v] + [\vec{K}_v, H_0] + [\vec{K}_v, v] = 0. \tag{5.17}$$

It can be shown that this constraint on the form of the interaction is equivalent to,

$$[\vec{R}, H_0]v + v[\vec{R}, H_0] + \frac{1}{2}H[\vec{R}, v] + \frac{1}{2}[\vec{R}, v]H + \frac{1}{2}(\epsilon_1 - \epsilon_2)\vec{r}v - \frac{1}{2}v\vec{r}(\epsilon_1 - \epsilon_2) = 0 \tag{5.18}$$

where $\vec{r} \equiv \vec{r_1} - \vec{r_2}$

In the paper [44] Wallace has approximately determined H such that its eigenvalue equation is

$$H|E_{\vec{p}}\rangle = E_{\vec{p}}|E_{\vec{p}}\rangle. \tag{5.19}$$

After some algebra this is equivalent to determining the interaction v satisfying the equation

$$\langle E_{\vec{P}} | H_{int} | E_{\vec{P}} \rangle = 0 \tag{5.20}$$

where

$$H_{int} = \left(1 + \frac{H_0}{E_{\vec{P}}}\right) \frac{[\vec{R}, H_0]}{2E_{\vec{P}}} v + v \frac{[\vec{R}, H_0]}{2E_{\vec{P}}} \left(1 + \frac{H_0}{E_{\vec{P}}}\right) + [\vec{R}, v] + \frac{\vec{p} \cdot \vec{P}}{E^2_{\vec{P}}} \vec{r} v - v \vec{r} \frac{\vec{p} \cdot \vec{P}}{E^2_{\vec{P}}}. \tag{5.21}$$

This equation can be solved in momentum space in order to determine the required form of the interaction. In momentum space the equation takes the form

$$\int \frac{d^3p'}{(2\pi)^3} \frac{d^3p}{(2\pi)^3} \langle E_{\vec{P}} | \vec{p}\,'; \vec{P} \rangle \mathcal{B}(\vec{p}\,', \vec{p}; \vec{P}) \langle \vec{p}; \vec{P} | E_{\vec{P}} \rangle = 0 \tag{5.22}$$

where

$$\mathcal{B}(\vec{p}\,', \vec{p}; \vec{P}) \equiv \left[\mathcal{C}(\vec{p}\,'; \vec{P}) + \mathcal{C}(\vec{p}; \vec{P}) + \mathcal{D}_{op}\right] v(\vec{p}\,', \vec{p}; \vec{P}) \tag{5.23}$$

$$\mathcal{C}(\vec{p}\,'; \vec{P}) = \left(1 + \frac{\epsilon_1(\vec{p}\,'; \vec{P}) + \epsilon_2(\vec{p}\,'; \vec{P})}{E_{\vec{P}}}\right) \frac{1}{2E_{\vec{P}}} \frac{\partial d}{\partial \vec{P}} \tag{5.24}$$

$$\mathcal{D}_{op} \equiv \frac{\partial}{\partial \vec{P}} + \frac{\vec{p}\,' \cdot \vec{P}}{E^2_{\vec{P}}} \frac{\partial}{\partial \vec{p}\,'} + \frac{\vec{p}\,' \cdot \vec{P}}{E^2_{\vec{P}}} \frac{\partial}{\partial \vec{p}} \tag{5.25}$$

The form of this partial differential equation for v suggests that the solution should have the form

$$v(\vec{p}\,', \vec{p}; \vec{P}) = f(\vec{p}\,'; \vec{P}) \tilde{v}(\vec{p}\,', \vec{p}; \vec{P}) f(\vec{p}; \vec{P}) \tag{5.26}$$

using 5.24 and 5.25 and making $\mathcal{B}(\vec{p}', \vec{p}; \vec{P}) = 0$ we can get

$$\mathcal{C}(\vec{p}; \vec{P}) + \frac{1}{f(\vec{p}; \vec{P})}\mathcal{D}_{op} f(\vec{p}; \vec{P}) + \mathcal{C}(\vec{p}'; \vec{P}) + \tag{5.27}$$

$$\frac{1}{f(\vec{p}'; \vec{P})}\mathcal{D}_{op} f(\vec{p}'; \vec{P}) + \frac{1}{\tilde{v}(\vec{p}', \vec{p}; \vec{P})}\mathcal{D}_{op}\tilde{v}(\vec{p}', \vec{p}; \vec{P}) = 0 \tag{5.28}$$

With the boundary condition $\tilde{v}(\vec{p}; \vec{P} = 0)$ as the CM frame interaction and $f(\vec{p}; \vec{P} = 0) = 1$. We can write $\tilde{v}$ as

$$\tilde{v}(\vec{p}', \vec{p}; \vec{P}) = v_c(\vec{p}'_c, \vec{p}_c) \tag{5.29}$$

where v_c is the interaction in the CM frame. Here

$$\vec{p}_c \equiv \vec{p} - \frac{(\vec{p} \cdot \vec{P})\vec{P}}{E_{\vec{P}}(E_{\vec{P}} + M)}, \tag{5.30}$$

and

$$\vec{p}'_c \equiv \vec{p}' - \frac{(\vec{p}' \cdot \vec{P})\vec{P}}{E_{\vec{P}}(E_{\vec{P}} + M)} \tag{5.31}$$

In the CM frame, $\vec{p}_c \to \vec{p}$ and $\vec{p}'_c \to \vec{p}'$ are the standard relative momenta. If the total momentum is in the z direction, the z-component of relative momentum is simply $p_{cz} = p_z/\gamma$, where $\gamma = E_{\vec{P}}/M$. Thus, the relative momentum is Lorentz contracted along the direction of $\vec{P}$. The components of relative momenta perpendicular to the total momentum are unaffected: $\vec{p}_{c\perp} = \vec{p}_\perp$. The same rule applies to $\vec{p}'_c$. The form of $f(\vec{p}; \vec{P})$ can be obtained by a consistency condition on the transformation of vertex function $\Gamma(\vec{p}; \vec{P})$ and it has the form

$$\Gamma(\vec{p}'; \vec{P}) = f(\vec{p}'; \vec{P})\Gamma_c(\vec{p}'_c). \tag{5.32}$$

where

$$f^2(\vec{p};\vec{P}) = \frac{M}{E_{\vec{P}}}\left(\frac{E_{\vec{P}}-\epsilon_1(\vec{p};\vec{P})-\epsilon_2(\vec{p};\vec{P})}{M-2\epsilon_c(\vec{p}_c)}\right)$$
$$= \frac{M}{E_{\vec{P}}}\left(\frac{M+2\epsilon_c(\vec{p}_c)}{E_{\vec{P}}+\epsilon_1(\vec{p};\vec{P})+\epsilon_2(\vec{p};\vec{P})}\right)\frac{1}{1-\Delta^2/E^2_{\vec{P}}}. \tag{5.33}$$

where

$$\epsilon_c(\vec{p}_c) \equiv \sqrt{m^2+\vec{p}_c^{\,2}}$$

and

$$\Delta = \epsilon_1 - \epsilon_2 \tag{5.34}$$

Thus for the two-body problem, if an arbitrary, rotationally and translationally invariant interaction in the CM frame of the form $v_c(\vec{p_c}', \vec{p}_c)$ defines the mass eigenvalue M by solution of the CM frame equation,

$$[M - 2\epsilon_c'(\vec{p}_c')]\Psi_c(\vec{p}_c') = \int \frac{d^3p_c}{(2\pi)^3} v_c(\vec{p}_c', \vec{p}_c)\Psi_c(\vec{p}_c) \tag{5.35}$$

then in another frame corresponding instant-form equation is

$$[E_{\vec{P}} - \epsilon_1(\vec{p}';\vec{P}) - \epsilon_2(\vec{p}';\vec{P})]\Psi(\vec{p}';\vec{P}) = \int \frac{d^3p}{(2\pi)^3} v(\vec{p}', \vec{p};\vec{P})\Psi(\vec{p};\vec{P}) \tag{5.36}$$

where

$$v(\vec{p}', \vec{p};\vec{P}) = f(\vec{p}';\vec{P})v_c(\vec{p}_c', \vec{p}_c)f(\vec{p};\vec{P}) \tag{5.37}$$

and the momenta $\vec{p}_c$ and $\vec{p}_c'$ in this equation are defined in terms of total momentum $\vec{P}$ and relative momentum $\vec{p}$ in the arbitrary frame as in Eqs.5.30 and 5.31, while $f(\vec{p};\vec{P})$ is defined as in Eq.5.33.

Now with the wave function in the CM frame as $\Psi_c(\vec{p}_c)$, the wave function $\Psi(\vec{p};\vec{P})$ in any arbitrary frame can be written as

$$\begin{aligned}\Psi(\vec{p};\vec{P}) &= f(\vec{p};\vec{P})\left(\frac{M-2\epsilon_c(\vec{p}_c)}{E_{\vec{P}}-\epsilon_1(\vec{p};\vec{P})-\epsilon_2(\vec{p};\vec{P})}\right)\Psi_c(\vec{p}_c) \\ &= \frac{M}{E_{\vec{P}}f(\vec{p};\vec{P})}\Psi_c(\vec{p}_c) \end{aligned} \tag{5.38}$$

This shows that the wave function in an arbitrary frame is a factor times the CM frame wave function evaluated at the appropriate arguments. This is a simple form of unitary transformation that guarantees the preservation of the normalization in the following sense, since $d^3p_c = \left(M/E_{\vec{P}}\right)d^3p$,

$$\int \frac{d^3p}{(2\pi)^3}\left|\sqrt{\frac{E_{\vec{P}}}{M}}f\left(\vec{p};\vec{P}\right)\Psi\left(\vec{p};\vec{P}\right)\right|^2 = \int \frac{d^3p_c}{(2\pi)^3}|\Psi_c(\vec{p}_c)|^2. \tag{5.39}$$

We have extended this formulation to scalar particles of unequal masses [43]. If m_1 and m_2 are the masses of the particles Eq.5.30 is written as,

$$\vec{p}_c = \vec{p} - \beta\vec{P} \tag{5.40}$$

where

$$\beta = \frac{(m_1^2-m_2^2)}{2ME_{\vec{P}}} + \frac{\vec{p}\cdot\vec{P}}{E_{\vec{P}}\left(E_{\vec{P}}+M\right)}. \tag{5.41}$$

Then

$$\Psi_c\left(\vec{p}_c\right) = \sum_{iLS}\rho_i\otimes\mathcal{Y}^M_{LSJ}\left(\hat{p}_c\right)F^i_{LS}\left(p_c\right) \tag{5.42}$$

where $p_c = |\vec{p}_c| = \sqrt{p^2+\beta^2P^2-2\beta pP\cos\theta}$ and $\hat{p}_c = (\theta_C,\phi_C)$. Since $\vec{P}$ is along Z axis, $\phi_c = \phi$ and $\cos\theta_c = \left(p\cos\theta - \beta\vec{P}\right)/p_c$.

Consider the decay process in which one quark remains as spectator. If we start in the CM frame with total momentum $\vec{P} = 0$ and absorb the momentum $\vec{q}$ by particle 1, then $\overrightarrow{P}' = q$ and $\overrightarrow{p}' = \overrightarrow{p} + \left(\frac{1}{2}\right)\overrightarrow{q}$. Then the expression for $\overrightarrow{p}_c$ in Eq.5.40 is modified as

$$\overrightarrow{p}_c = \overrightarrow{p}' - \beta\overrightarrow{P} \tag{5.43}$$

and the corresponding β is

$$\beta = \frac{(m_1^2 - m_2^2)}{2ME_{\overrightarrow{P}}} + \frac{\overrightarrow{p}' \cdot \overrightarrow{P}}{E_{\overrightarrow{P}}\left(E_{\overrightarrow{P}} + M\right)} \tag{5.44}$$

This can also be represented as

$$\overrightarrow{p}_c = \overrightarrow{p} - \beta'\overrightarrow{P} \tag{5.45}$$

where

$$\beta' = -\frac{1}{2} + \frac{(m_1^2 - m_2^2)}{2ME_{\overrightarrow{P}}} + \frac{\left(\overrightarrow{p} + \frac{\overrightarrow{P}}{2}\right) \cdot \overrightarrow{P}}{E_{\overrightarrow{P}}\left(E_{\overrightarrow{P}} + M\right)} \tag{5.46}$$

Then Eq.5.42 is

$$\Psi_c\left(\overrightarrow{p}_c\right) = \sum_{iLS} \rho_i \otimes \mathcal{Y}_{LSJ}^M\left(\hat{p}_c\right) F_{LS}^i\left(p_c\right) \tag{5.47}$$

where $p_c = |\overrightarrow{p}_c| = \sqrt{p^2 + \beta'^2 P^2 - 2\beta' pP\cos\theta}$ and $\hat{p}_c = (\theta_C, \phi_C)$. Since $\overrightarrow{P}$ is along Z axis, $\phi_c = \phi$ and $\cos\theta_c = \left(p\cos\theta - \beta'\overrightarrow{P}\right).(1/p_c)$.

Now we choose an approximate form for the boosted wave function as following

$$\psi\left(\overrightarrow{p}, \overrightarrow{P}\right) = \psi_c\left(\overrightarrow{p}_c\right) \tag{5.48}$$

and it satisfies the normalization condition

$$\int d^3pTr\left\{\psi^\dagger\left(\overrightarrow{P}, \overrightarrow{p}\right)\psi\left(\overrightarrow{P}, \overrightarrow{p}\right)\right\} = 2(2\pi)^3\sqrt{\overrightarrow{P}^2 + M^2} \tag{5.49}$$

if the normalization in the CM frame is

$$\int d^3pTr\left\{\psi_c^\dagger\left(\overrightarrow{p}_c\right)\psi_c\left(\overrightarrow{p}_c\right)\right\} = 2(2\pi)^3 M. \tag{5.50}$$

For the evaluation of the matrix element of the current given by Eq.5.9 ,now we can write it in the form

$$\int d^3pTr\left\{\psi'^{\dagger}\left(\vec{p}_c\right)\gamma^0\Gamma^{\mu}\psi\left(\vec{p}\right)\right\} \tag{5.51}$$

We consider the following general expression for an arbitrary k_μ,

$$\psi^{\dagger(2)}\left(\vec{p}_c\right)\gamma^0 k_\mu\gamma^\mu\psi^{(1)}\left(\vec{p}\right) =$$

$$\left[\rho_1^{\dagger}F_{00}^1\left(p_c\right)Y_{000}^{0\dagger}\left(\hat{p}_c\right)+\rho_2^{\dagger}F_{00}^2\left(p_c\right)Y_{000}^{0\dagger}\left(\hat{p}_c\right)+\rho_3^{\dagger}F_{00}^3\left(p_c\right)Y_{110}^{0\dagger}\left(\hat{p}_c\right)+\rho_4^{\dagger}F_{11}^4\left(p_c\right)Y_{110}^{0\dagger}\left(\hat{p}_c\right)\right]$$

$$\gamma^0 k_\mu\gamma^\mu$$

$$\left[\rho_1 Y_{000}^0\left(\hat{p}\right)F_{00}^1\left(p\right)+\rho_2 Y_{000}^0\left(\hat{p}\right)F_{00}^2\left(\hat{p}\right)+\rho_3 Y_{110}^0\left(\hat{p}\right)F_{11}^3\left(p\right)+\rho_4 Y_{110}^0\left(\hat{p}\right)F_{11}^4\left(p\right)\right] \tag{5.52}$$

1 and 2 labels on left hand side corresponds to parent and daughter mesons and considering ++channel only. $\rho_{1,}\rho_{2,}\rho_{3,}\rho_4$ are defined in Eq.2.36 and2.37. The Pauli spin matrices are

$$\begin{aligned} \sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \\ \sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{5.53}$$

$\rho's$ expressed in terms of $\sigma's$

$$\rho_1 = \frac{\sigma_1}{\sqrt{2}}, \rho_2 = \frac{i\sigma_2}{\sqrt{2}}, \rho_3 = \frac{\sigma_3}{\sqrt{2}}, \rho_4 = \frac{\sigma_4}{\sqrt{2}} \tag{5.54}$$

Also we get the relations

$$\rho_1\rho_2 = -\frac{1}{\sqrt{2}}\rho_3, \rho_2\rho_3 = -\frac{1}{\sqrt{2}}\rho_1, \rho_3\rho_1 = \frac{1}{\sqrt{2}}\rho_2 \tag{5.55}$$

$$\rho_1^2 = -\rho_2^2 = \rho_3^2 = \rho_4^2 = \frac{\sigma_4}{\sqrt{2}} \tag{5.56}$$

$$\rho_2^\dagger = -\rho_2, \gamma^5 = \sqrt{2}\rho_1, \overrightarrow{\alpha} = \rho_1 \otimes \overrightarrow{\sigma} \tag{5.57}$$

$$\alpha = \begin{pmatrix} 0 & \overrightarrow{\sigma} \\ \overrightarrow{\sigma} & 0 \end{pmatrix}, \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{5.58}$$

right hand side of Eq.5.52 is

$$\begin{aligned} = \quad & \left[\left(\rho_1^\dagger F_{00}^{1\dagger}(p_c) + \rho_2^\dagger F_{00}^{2\dagger}(p_c)\right) Y_{000}^{0\dagger}(\hat{p}_c) + \left(\rho_3^\dagger F_{00}^{3\dagger}(p_c) + \rho_4^\dagger F_{11}^{4\dagger}(p_c)\right) Y_{110}^{0\dagger}(\hat{p}_c)\right] \\ & \left[\left(\rho_1 F_{00}^1(p) + \rho_2 F_{00}^2(p)\right) Y_{000}^0(\hat{p}) + \left(\rho_3 F_{00}^3(p) + \rho_4 F_{11}^4(p)\right) Y_{110}^0(\hat{p})\right] k_0 - \\ & \left[\left(\rho_1^\dagger F_{00}^{1\dagger}(p_c) + \rho_2^\dagger F_{00}^{2\dagger}(p_c)\right) Y_{000}^{0\dagger}(\hat{p}_c) + \left(\rho_3^\dagger F_{00}^{3\dagger}(p_c) + \rho_4^\dagger F_{11}^{4\dagger}(p_c)\right) Y_{110}^{0\dagger}(\hat{p}_c)\right] \sqrt{2}\rho_1 \overrightarrow{\sigma} \cdot \overrightarrow{k} \\ & \left[\left(\rho_1 F_{00}^1(p) + \rho_2 F_{00}^2(p)\right) Y_{000}^0(\hat{p}) + \left(\rho_3 F_{00}^3(p) + \rho_4 F_{11}^4(p)\right) Y_{110}^0(\hat{p})\right] \end{aligned} \tag{5.59}$$

performing the trace operation and we get the following relations

$$Tr\left[\rho_i^\dagger \rho_j\right] = \delta_{ij} \tag{5.60}$$

$$Tr\left[\rho_i^\dagger \rho_1 \rho_j\right] = \frac{1}{\sqrt{2}}\left[\delta_{i4}\delta_{j1} - \delta_{i3}\delta_{j2} - \delta_{i2}\delta_{j3} + \delta_{i1}\delta_{j4}\right] \tag{5.61}$$

therefore right hand side of Eq.5.59 is

$$= k_0\left[F1 \times T1 + F2 \times T2\right] - \left[F3 \times T3 + F4 \times T4\right] \tag{5.62}$$

where

$$F1 = \left(F_{00}^{1}(p_c)\,F_{00}^{1}(p) + F_{00}^{2}(p_c)\,F_{00}^{2}(p)\right) \tag{5.63}$$

$$F2 = \left(F_{11}^{3}(p_c)\,F_{11}^{3}(p) + F_{11}^{4}(p_c)\,F_{11}^{4}(p)\right) \tag{5.64}$$

$$F3 = \left(F_{00}^{1}(p_c)\,F_{11}^{4}(p) - F_{00}^{2}(p_c)\,F_{11}^{3}(p)\right) \tag{5.65}$$

$$F4 = \left(F_{11}^{4}(p_c)\,F_{00}^{1}(p) - F_{11}^{3}(p_c)\,F_{00}^{2}(p)\right) \tag{5.66}$$

and

$$T1 = Tr\left(Y_{000}^{0\dagger}(\hat{p}_c)\,Y_{000}^{0}(\hat{p})\right) \tag{5.67}$$

$$T2 = Tr\left(Y_{110}^{0\dagger}(\hat{p}_c)\,Y_{110}^{0}(\hat{p})\right) \tag{5.68}$$

$$T3 = Tr\left(Y_{000}^{0\dagger}(\hat{p}_c)\,\vec{\sigma}\cdot\vec{k}\,Y_{110}^{0}(\hat{p})\right) \tag{5.69}$$

$$T4 = Tr\left(Y_{110}^{0\dagger}(\hat{p}_c)\,\vec{\sigma}\cdot\vec{k}\,Y_{000}^{0}(\hat{p})\right) \tag{5.70}$$

The trace $T1, T2, T3$ and $T4$ can be obtained out as follows

$$\begin{aligned} T1 &= Tr\left(Y_{000}^{0\dagger}(\hat{p}_c)\,Y_{000}^{0}(\hat{p})\right) \\ &= Tr\left(\frac{i\sigma_2}{\sqrt{2}}Y_{00}^{*}(\hat{p}_c)\,Y_{00}(\hat{p})\,\frac{i\sigma_2}{\sqrt{2}}\right) \\ &= Y_{00}^{*}(\hat{p}_c)\,Y_{00}(\hat{p}) \\ &= 1/4\pi \end{aligned} \tag{5.71}$$

$$\begin{aligned} T2 &= Tr\left(Y_{110}^{0\dagger}(\hat{p}_c)\,Y_{110}^{0}(\hat{p})\right) \\ &= Tr\left(-\frac{i\sigma_2}{\sqrt{2}}\sum_{\mu}\sigma_{\mu}^{\dagger}C_{\mu}^{*}Y_{1,-\mu}^{*}(\hat{p}_c)\sum_{\mu'}C_{\mu'}^{*}Y_{1,-\mu'}^{0}(\hat{p})\,\sigma_{\mu'}\frac{i\sigma_2}{\sqrt{2}}\right) \\ &= \frac{1}{3}\sum_{\mu}Y_{1,-\mu}^{*}(\hat{p}_c)\,Y_{1,-\mu}(\hat{p}) \\ &= \frac{1}{4\pi}\left(\sin\theta_c\sin\theta + \cos\theta_c\cos\theta\right) \end{aligned} \tag{5.72}$$

where $C\mu = \langle 1, 1, -\mu, \mu \mid 00 \rangle$ are the Clebsch-Gordon Coefficients. We use the following properties to calculate $T3$ and $T4$

$$Tr\left[\sigma_{\mu}^{\dagger}\sigma_{\mu'}\right] = 2\delta_{\mu\mu'} \tag{5.73}$$

$$\vec{\sigma} \cdot \vec{k} = \sum_{\mu} (-)^{\mu} \sigma_{\mu} k_{\mu} = \sum_{\mu} \sigma_{-\mu}^{\dagger} k_{\mu}, \quad \sigma_{\mu}^{\dagger} = (-)^{\mu} \sigma_{-\mu} \tag{5.74}$$

Also

$$\begin{aligned}
T3 &= Tr\left(Y_{000}^{0\dagger}(\hat{p}_c)\, \vec{\sigma} \cdot \vec{k}\, Y_{110}^{0}(\hat{p})\right) \\
&= Tr\left[\left(-\frac{i\sigma_2}{\sqrt{2}}\right) Y_{00}^{*}(\hat{p}_c)\, \vec{\sigma} \cdot \vec{k} \sum_{\mu} C_{\mu} Y_{1,-\mu}(\hat{p})\, \sigma_{\mu} \frac{i\sigma_2}{\sqrt{2}}\right] \\
&= \frac{1}{2} Y_{00}^{*}(\hat{p}) \sum k_{\mu'} C_{\mu} Y_{1,-\mu}(\hat{p})\, Tr\left[\sigma_{-\mu'}^{\dagger}\sigma_{\mu}\right] \\
&= \frac{1}{\sqrt{3}} Y_{00}^{*}(\hat{p}_c)\left[Y_{1,-1}(\hat{p})\, k_{-} - Y_{1,0}(\hat{p})\, k_Z + Y_{11}(\hat{p})\, k_{+}\right] \qquad (5.75) \\
&= \frac{k_Z}{2} \cos\theta \qquad (5.76)
\end{aligned}$$

doing similar simplification

$$\begin{aligned}
T4 &= Tr\left(Y_{110}^{0\dagger}(\hat{p}_c)\, \vec{\sigma} \cdot \vec{k}\, Y_{000}^{0}(\hat{p})\right) \\
&= -\frac{1}{\sqrt{3}}\left[Y_{1,-1}^{*}(\hat{p}_c)\, Y_{00}(\hat{p})\, k_{+} + Y_{1,0}^{*}(\hat{p}_c)\, Y_{00}(\hat{p})\, k_Z + Y_{1,1}^{*}(\hat{p}_c)\, Y_{00}(\hat{p})\, k_{-}\right] \\
&= \frac{k_Z}{2} \cos\theta_c \qquad (5.77)
\end{aligned}$$

From 5.62 the integral to be evaluated now (after ϕ integration) is

$$\begin{aligned}
I &= \int_0^{\infty} p^2 dp \int_0^{\pi} \sin\theta d\theta \left\{ F1\left(\frac{k_0}{2}\right) + F2\left(\frac{k_0}{2}\right) \cos(\theta_c - \theta) \right. \\
&\quad \left. + \left[F3\left(\frac{k_Z}{2}\right) \cos\theta\right] + \left[F4\left(\frac{k_Z}{2}\right) \cos\theta_c\right] \right\} \qquad (5.78)
\end{aligned}$$

Expressing on a convenient way the above expression is

$$I = k_0 I_1 + k_Z I_2 \tag{5.79}$$

where I_1 corresponds to the above integral with the first two terms without k_0 and I_2 is the integral involving only the last two terms without k_Z. The integral $I_2 \rightarrow 0$ as $\overrightarrow{P} \rightarrow 0$. Also $p_c \rightarrow p$ and $\theta_c \rightarrow \theta$ as $\overrightarrow{P} \rightarrow 0$

The integral can be evaluated numerically. Form Factors are obtained for semileptonic decays of pseudoscalar mesons ($0^- \rightarrow 0^-$). as explained below. The results are tabulated and are found to be in agreement with experimental values.

5.4 Calculation of Form Factors

Since the four momentum of the parent meson is $P = (M, 0)$ and that of the daughter meson is $P' = (E_P, 0, 0, P_Z)$ we can write,

$$A \equiv P^{\mu} J_{\mu} = f_+ \left(q^2\right) \left(M^2 + P \cdot P'\right) + f_- \left(q^2\right) \left(M^2 - P \cdot P'\right) \tag{5.80}$$

$$B \equiv P'^{\mu} J_{\mu} = f_+ \left(q^2\right) \left(P \cdot P' + M'^2\right) + f_- \left(q^2\right) \left(P \cdot P' - M'^2\right) \tag{5.81}$$

From these relations we get

$$f_+ \left(q^2\right) = \frac{(A + B)\, P \cdot P' - M'^2 A - M^2 B}{C} \tag{5.82}$$

$$f_- \left(q^2\right) = \frac{(B - A)\, P \cdot P' + M^2 B - M'^2 A}{C} \tag{5.83}$$

where

$$C = \left[\left(P \cdot P' - M'^2\right)\left(M^2 + P \cdot P'\right) - \left(M^2 - P \cdot P'\right)\left(P \cdot P' + M'^2\right)\right] \tag{5.84}$$

Also A and B can be represented in simple form

$$A = P^0 I_1 + P_Z I_2 = M I_1 \tag{5.85}$$

$$B = P'^0 I_1 + P_Z I_2 = E_P I_1 + P_Z I_2 \tag{5.86}$$

We numerically evaluate $f_+\left(q^2\right)$ and $f_-\left(q^2\right)$ using 5.82 and 5.83 and are expressed in terms of $F_1\left(q^2\right)$.

5.5 Results

The semileptonic decays investigated are $D^0 \to K^- e^+ \nu_e$ and $B^0 \to D^+ e^- \overline{\nu_e}$. Model parameters chosen are given in Chapter 4. On the basis of which we numerically evaluated the form factors with their q^2 dependence. The results obtained are compared with experimental values and also with the results predicted by other models. This is tabulated below in Table 5.1. $F_1\left(0\right)$ is the form factor at $q^2 = 0$ and $F_1\left(q^2_{\max}\right)$ is the form factor at $q^2 = q^2_{\max}$ (zero recoil case).

Table 5.1

$D^0 \to K^- e^+ \nu_e$	This work	ISGW[46]	GS[58, 57]	BSW[50]	WJ[53]	BD[55]	Expt.
$F_1(0)$	0.7911	0.80	0.70	0.75	0.70	0.80	0.75

$F_1\,(q^2_{max})$ obtained in this work is 0.81 which agrees with the predictions of HQET[39].

Table 5.2

$B^0 \to D^+ e^- \overline{\nu_e}$	This work	ISGW[46]	GS[58, 57]	BSW[50]	WJ[53]	BD[55]	Expt
$F_1(0)$	0.7093	–	–	0.69	0.67	0.97	–

$F_1\,(q^2_{max})$ obtained in this work is 1.22 which approximately agrees with the predictions of HQET[39] which is 1.13

5.6 Summary of the Work

In this thesis a quasipotential reduction of the Bethe-Salpeter equation is used to obtain a relativistic two body equation for a quark and anti-quark. This equation is solved in momentum space and is used to describe the spectrum of mesons..We use a scalar confining potential of the form $\kappa r + C$ and a vector potential of the form α/r. Wave functions for the various mesons are obtained and they obey charge conjugation, parity and time reversal symmetry. Varying the potential parameters κ, α and C as well as the masses of the quarks, the masses of the light mesons such as π, ρ, K etc.,as well as the D, B mesons and the heavy J/Ψ and Υ family of mesons are described quite well. The mass spectra obtained in this work is compared with other works. It can be seen that the model of Isgur and Godfrey has an additional six parameters for relativistic effects. We are able to include all relativistic effects *a priori*. This significantly enhances the predictive power of our model. For example, we predict new states in the present model. It seems that experiments may confirm these predictions in the near future.

This work opens up many avenues of investigation. Of primary importance is the application of the model to decay processes of heavy mesons. In particular the calculation of form factors that describe the semileptonic decays,not only for decays to pseudoscalars and vectors, but also to excited states, are of great interest. In this work we considered only the case of pseudoscalar mesons in the final state. The results are encouraging. Other approaches of calculating form factors are QCD sum rules and lattice QCD. Each of the above methods has only limited range of applicability. QCD sum rules are suitable for describing the low q^2 region of the form factors. The higher q^2 region is hard to get and higher order calculation are not likely to give real progress because of the appearance of many new parameters. The accuracy of the method cannot arbitrarily improved because of the necessity to isolate the contribution of the states of interest from others. Lattice QCD give good results for the higher q^2 region, but because of the many numerical extrapolations involved this method does not provide for a full picture of the form factors and for the relations between the various decay channels. Quark models do provide such relations and give the form factor in full q^2 region. But relativistic quark models work surprisingly well for the description meson spectra and form factors.

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Appendix A
Angular Momentum Spin Matrices

In this appendix the 2×2 matrix representation of the $\mathcal{Y}_{LSJ}^{M}$ will be presented. First the usual spinor basis will be written in terms of four 2×2 matrices. Then the total angular momentum 2×2 matrix representation will be defined and finally we will then show that it is an orthonormal basis.

A.1 Spin Angular Momentum in 2×2 Matrix Representation

The spin momentum functions for two spin half particles of total spin S functions are obtained.

For $S = 1,\ S_Z = 1$ we obtain

$$
\begin{aligned}
|0,0\rangle &= \frac{1}{\sqrt{2}}(\alpha_1 \otimes \beta_2 - \beta_1 \otimes \alpha_2) \\
|0,0\rangle &= \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}(0\ 1) - \begin{pmatrix}0\\1\end{pmatrix}(1\ 0)\right] = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}
\end{aligned}
\tag{A.1}
$$

Thus the singlet spin state can be written in the 2×2 triplet matrix representation,

$$
|0,0\rangle = \frac{1}{\sqrt{2}} i\sigma_2 \tag{A.2}
$$

For $S = 1,\ S_Z = 1, 0, -1$ We obtain

$$
\begin{aligned}
|1,0\rangle &= \frac{1}{\sqrt{2}}(\alpha_1 \otimes \beta_2 + \beta_1 \otimes \alpha_2) \\
&= \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}(0\ \ 1) + \begin{pmatrix}0\\1\end{pmatrix}(1\ \ 0)\right] = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \\
|1,0\rangle &= \frac{1}{\sqrt{2}}i\sigma_1 = \sigma_3\frac{1}{\sqrt{2}}i\sigma_2 \qquad \text{(A.3)} \\
|1,1\rangle &= \alpha_1 \otimes \alpha_2 \\
|1,1\rangle &= \begin{pmatrix}0\\1\end{pmatrix}(1\ \ 0) = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} = \frac{1+\sigma_3}{2} \\
&= \frac{\sigma_1+\sigma_2}{\sqrt{2}}\frac{i\sigma_2}{\sqrt{2}} = \sigma_+\frac{i\sigma_2}{\sqrt{2}} \\
|1,-1\rangle &= \beta_1 \otimes \beta_2 \\
&= \begin{pmatrix}0\\1\end{pmatrix}(0\ \ 1) = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} = \frac{1-\sigma_3}{2} \\
&= \frac{\sigma_1-\sigma_2}{\sqrt{2}}\frac{i\sigma_2}{\sqrt{2}} = \sigma_-\frac{i\sigma_2}{\sqrt{2}} \qquad \text{(A.4)}
\end{aligned}
$$

So the triplet spin state can be written in the matrix representation,

$$|1, S_z\rangle = \sigma_{S_Z}\frac{i\sigma_2}{\sqrt{2}} \qquad \text{(A.5)}$$

A.2 Total Angular momentum in 2×2 Matrix Representation

The usual orbital angular momentum spherical harmonics are as follows

$$\langle\hat{p}|LM\rangle = Y_{LM}(\hat{p}) \qquad \text{(A.6)}$$

$$\int d\Omega_p Tr\left(Y_{ls}^{M}(\hat{p})Y_{l's'}^{M}(\hat{p})\right) = \delta_{ll'}\delta_{ss'} \qquad \text{(A.7)}$$

Then we define the total angular momentum spherical harmonics are as follows

$$\mathcal{Y}_{LSJ}^{\mathcal{M}}(\hat{p}) = \sum_{S_z}\langle L,S;M-S_z,S_z|JM\rangle Y_{L,M-S_z}^{M}(\hat{p})\sigma_{S_Z}\frac{i\sigma_2}{\sqrt{2}}, \quad if\ S=1 \tag{A.8}$$

$$\mathcal{Y}_{LSJ}^{\mathcal{M}}(\hat{p}) = \delta_{LJ}Y_{JM}\frac{i\sigma_2}{\sqrt{2}} \quad if\ S=0 \tag{A.9}$$

Here the wave function $\hat{\Psi}$ is a matrix as indicated by the hat.

$$\Psi^\dagger\Psi = \sum_{\alpha\beta}\Psi^*_{\alpha\beta}\Psi_{\alpha\beta} = \sum_{\alpha\beta}\hat{\Psi}^\dagger_{\beta\alpha}\hat{\Psi}_{\alpha\beta} = Tr\hat{\Psi}^\dagger\hat{\Psi} \tag{A.10}$$

Thus the orthogonality condition for $\mathcal{Y}_{LSJ}^{\mathcal{M}}$ becomes

$$\int d\Omega_p Tr\left(Y_{LSJ}^{M*}(\hat{p})Y_{L'S'J}^{M}(\hat{p})\right) = \delta_{SS'}\delta_{LL'} \tag{A.11}$$

To show this we note that the integration over the $Y^*_{L,M-S_Z}Y_{L',M-S'_z} = \delta_{LL'}\delta_{S_ZS'_Z}$. Also if $S=S'$ we must evaluate

$$\begin{aligned} Tr\left[\left(\frac{\sigma_{S_Z}\sigma_2}{\sqrt{2}}\right)^\dagger\frac{\sigma_{S_Z}\sigma_2}{\sqrt{2}}\right] &= \frac{1}{2}Tr\left(\sigma_2\sigma^\dagger_{S_Z}\sigma_{S'_Z}\sigma_2\right) \\ &= \frac{1}{2}Tr\left(\sigma_2\sigma_2\sigma_{S_Z}\sigma_{S'_Z}\right) \\ &= \frac{1}{2}Tr\left(\sigma_{S_Z}\sigma_{S'_Z}\right) \\ &= \delta_{S_ZS'_Z} \end{aligned} \tag{A.12}$$

If $S=S'=0$ we must evaluate

$$Tr\left[\left(\frac{i\sigma_2}{\sqrt{2}}\right)^\dagger\frac{i\sigma_2}{\sqrt{2}}\right] = \frac{1}{2}Tr\left(\sigma_2^2\right) = 1 \tag{A.13}$$

and for $S \neq S'$

$$Tr\left[\left(\frac{i\sigma_2}{\sqrt{2}}\right)^\dagger\frac{\sigma_{S_Z}}{\sqrt{2}}\frac{i\sigma_2}{\sqrt{2}}\right] = Tr\left(\sigma_{S_Z}\right) = 0 \tag{A.14}$$

Therefore,

$$\int d\Omega_p Tr\left(Y^M_{LSJ}(\hat{p})Y^M_{L'S'J}(\hat{p})\right) = \delta_{SS'}\delta_{LL'}\sum_{S_z}|\langle L,S;M-S_z,S_z|JM\rangle|^2$$

Finally we must note $S=0$

$$|\langle L.0;M,0|JM\rangle|^2 = 1 \tag{A.15}$$

and for $S=1$

$$\sum_{S_z}|\langle L,1;M-S_z,S_z|JM\rangle|^2 = 1 \tag{A.16}$$

and with this we have shown our basis is orthonormal.

A.3 Potential in ρ basis

$$\hat{V} = \gamma_1^0\gamma_2^0\left[V_{Sc} + \gamma_1\cdot\gamma_2 V_{Vc}\right] \tag{A.17}$$

Writing $\gamma^0\overrightarrow{\gamma} = \gamma^5\overrightarrow{\sigma}$ and $\gamma_1\cdot\gamma_2 = \gamma_1^0\gamma_2^0 - \overrightarrow{\gamma_1}\cdot\overrightarrow{\gamma_2}$ this becomes

$$\begin{aligned}\hat{V} &= \gamma_1^0\gamma_2^0 V_{Sc} + V_{Vc} - V_{Vc}\overrightarrow{\gamma_1}\cdot\overrightarrow{\gamma_2} \\ &= \gamma_1^0\gamma_2^0 V_{Sc} + V_{Vc} - V_{Vc}\gamma_1^5\overrightarrow{\sigma}_1\gamma_2^5\overrightarrow{\sigma}_2\end{aligned}$$

and we obtain the following identities for the ρ matrices.

$$\rho_1 = \frac{1}{\sqrt{2}}\gamma^5, \rho_1 = \frac{1}{\sqrt{2}}\gamma^0\gamma^5, \rho_3 = \frac{1}{\sqrt{2}}\gamma^0, \rho_4 = \frac{1}{\sqrt{2}} \tag{A.18}$$

The wave function is expanded as follows,

$$\Psi_J^M(\hat{p}) = \sum_{iLS}\rho_i\otimes\mathcal{Y}^M_{LSJ}(\hat{p})F^i_{LS}(p) \tag{A.19}$$

Then using the rule that particle 1 operator acts on the matrix form of Ψ from the left and particle 2 operator transposed acts in Ψ from the right, we find

$$\hat{V}\Psi = \sum_{iLS}\left[\left(V_{Sc}2\rho_3\rho_i\rho_3^T + V_{Vc}\rho_i\right)\mathcal{Y}^{\mathcal{M}}_{\mathcal{LSJ}} - V_{Vc}2\rho_1\rho_i\rho_1^T\sigma_1\mathcal{Y}^{\mathcal{M}}_{\mathcal{LSJ}}\sigma_2\right]F^i_{LS}(p) \quad \text{(A.20)}$$

Simple calculation shows that

$$2\rho_3\rho_i\rho_3^T = (-\rho_1, -\rho_2, \rho_3, \rho_4) \text{ and } 2\rho_1\rho_i\rho_1^T = (\rho_1, -\rho_2, -\rho_3, \rho_4) \quad \text{(A.21)}$$

$$\sigma_1\mathcal{Y}^{\mathcal{M}}_{\mathcal{LSJ}}(\hat{p})\sigma_2^T = C_S\mathcal{Y}^{\mathcal{M}}_{\mathcal{LSJ}}(\hat{p})$$

Here $C_S = 2S(S+1) - 3$ and is $+1$ for $S = 1$ states and -3 for $S = 0$ states.

$$\hat{V}(p - p')\Psi(p) = \sum_{iLS} V^i_{LS}\left(p - p'\right)\mathcal{Y}^{\mathcal{M}}_{\mathcal{LSJ}}(\hat{p})F^i_{LS}(p) \quad \text{(A.22)}$$

the V^i for i=1,....4 in the ρ basis are,

$$V^1 = -V_{Sc} + V_{Vc}(1 - C_S), V^2 = -V_{Sc} + V_{Ve}(1 + C_S) \quad \text{(A.23)}$$

$$V^3 = V_{Sc} + V_{Ve}(1 + C_S), V^4 = V_{Sc} + V_{Ve}(1 - C_S) \quad \text{(A.24)}$$

A.4 Evaluation of $i\sigma^2\mathcal{Y}^{\mathcal{M}}_{\mathcal{LSJ}}(\hat{p})^*\left(i\sigma^2\right)^T$

From chapter 3 we wish to find $i\sigma^2\mathcal{Y}^{\mathcal{M}}_{\mathcal{LSJ}}(\hat{p})^*\left(i\sigma^2\right)^T$ for $LSJ = 110$ and $LSJ = 000$. We have

$$\begin{aligned} i\sigma^2\mathcal{Y}^0_{110}(\hat{p})^*\left(i\sigma^2\right)^T &= i\sigma^2\left[\sum_{\mu=\{-1,0,1\}}\langle 1,1,-\mu,\mu|00\rangle Y_{1,-\mu}(\hat{p})\left(\sigma_\mu\frac{i\sigma^2}{\sqrt{2}}\right)\right]^*\left(i\sigma^2\right)^T \\ &= \sum_{\mu=\{-1,0,1\}}\langle 1,1,-\mu,\mu|00\rangle Y^*_{1,-\mu}(\hat{p})\, i\sigma^2\left(\sigma_\mu\frac{i\sigma^2}{\sqrt{2}}\right)(-i\sigma^2) \quad \text{(A.25)} \end{aligned}$$

Now

$$Y^*_{1,-\mu}(\hat{p}) = (-1)^\mu Y_{1,-\mu}(\hat{p})$$

and noting

$$i\sigma^2\left(\sigma_\mu\frac{i\sigma^2}{\sqrt{2}}\right)\left(-i\sigma^2\right) = (-1)^\mu\sigma_{-\mu}\frac{i\sigma^2}{\sqrt{2}} \tag{A.26}$$

we find

$$\begin{aligned} i\sigma^2\mathcal{Y}^0_{110}(\hat{p})^*\left(i\sigma^2\right)^T &= -\sum_{\mu=\{-1,0,1\}}\langle 1,1,-\mu,\mu|00\rangle(-1)^\mu Y_{1,\mu}(\hat{p})(-1)^\mu\sigma_{-\mu}\frac{i\sigma^2}{\sqrt{2}} \\ &= -\sum_{\mu=\{-1,0,1\}}\langle 1,1,-\mu,\mu|00\rangle Y^*_{1,-\mu}(\hat{p})\left(\sigma_\mu\frac{i\sigma^2}{\sqrt{2}}\right) \\ &= -\mathcal{Y}^0_{110}(\hat{p}) \end{aligned}$$

For $\mathcal{Y}^0_{000}(\hat{p})$ we note that

$$\begin{aligned} i\sigma^2\mathcal{Y}^0_{000}(\hat{p})^*\left(i\sigma^2\right)^T &= i\sigma^2\left[Y_{00}(\hat{p})\left(\frac{i\sigma^2}{\sqrt{2}}\right)\right]^*\left(i\sigma^2\right)^T \\ &= Y_{00}(\hat{p})\frac{i\sigma^2}{\sqrt{2}} = \mathcal{Y}^0_{000}(\hat{p}) \end{aligned}$$

Appendix B
Basis Transformation Matrix

The plane wave can be written in the following form,

$$u_i^{\rho_i}\left(\rho_i\vec{p}\right) = N_i \begin{pmatrix} \frac{1+\rho_i}{2} - \frac{1-\rho_i}{2}\frac{\vec{\sigma}_i\cdot\vec{p}}{\epsilon_i+m_i} \\ \frac{1+\rho_i}{2}\frac{\vec{\sigma}_i\cdot\vec{p}}{\epsilon_i+m_i}\frac{1-\rho_i}{2} \end{pmatrix}, \qquad N_i = \sqrt{\frac{\epsilon_i+m_i}{2\epsilon_i}},\ i=1,2 \tag{B.1}$$

The transformation matrix $M_{L'S',LS}^{\rho\rho_i'i}\left(\vec{p}\right)$ between the plane wave and ρ basis is defined by the following expression which also allows it to be evaluated.

$$\begin{aligned} u^{\rho_1}\left(\rho_1\vec{p}\right) \qquad & \mathcal{Y}_{LSJ}^{M}(\hat{p})\, u^{\rho_2}\left(\rho_2\vec{p}\right) \\ = {} & N_1N_2 \begin{pmatrix} \frac{1+\rho_1}{2} - \frac{1-\rho_1}{2}\frac{\vec{\sigma}_1\cdot\vec{p}}{\epsilon_1+m_1} \\ \frac{1+\rho_1}{2}\frac{\vec{\sigma}_1\cdot\vec{p}}{\epsilon_1+m_1}\frac{1-\rho_1}{2} \end{pmatrix} \mathcal{Y}_{LSJ}^{M}\left(\vec{p}\right) \\ & \left(\frac{1+\rho_2}{2} - \frac{1-\rho_2}{2}\frac{\vec{\sigma}_2\cdot\vec{p}}{\epsilon_2+m_2}, \frac{1+\rho_2}{2}\frac{\vec{\sigma}_2\cdot\vec{p}}{\epsilon_2+m_2}\frac{1-\rho_2}{2}\right) \\ = {} & \sum_{iL'S'} M(p)\,\rho^i \mathcal{Y}_{LSJ}^{M}\left(\vec{p}\right) \end{aligned} \tag{B.2}$$

Expanding this equation and using the following identities,

$$\begin{aligned} \vec{\sigma}\cdot\vec{p}\,\mathcal{Y}_{LSJ}^{M}(\hat{p}) &= \sum_{L'S'} pL_{LL'}^{SS'}\mathcal{Y}_{L'S'J}^{M}(\hat{p}) \\ \mathcal{Y}_{LSJ}^{M}(\hat{p})\,\vec{\sigma}^{T}\cdot\vec{p} &= \sum_{L'S'} pR_{LL'}^{SS'}\mathcal{Y}_{L'S'J}^{M}(\hat{p}) \\ \vec{\sigma}\cdot\vec{p}\,\mathcal{Y}_{LSJ}^{M}(\hat{p})\,\vec{\sigma}^{T}\cdot\vec{p} &= p^2\sum_{L'S'} T_{LL'}^{SS'}\mathcal{Y}_{L'S'J}^{M}(\hat{p}) \end{aligned}$$

$$L_{J,J+1}^{01} = L_{J+1,J}^{10} = L_{J,J-1}^{11} = L_{J-1,J}^{11} = -\sqrt{\frac{J+1}{2J+1}} \tag{B.3}$$

$$L^{01}_{J,J-1} = L^{10}_{J-1,J} = -L^{11}_{J,J+1} = -L^{11}_{J+1,J} = \sqrt{\frac{J}{2J+1}} \tag{B.4}$$

$$L^{SS'}_{LL'} = (-1)^{S+S'} R^{SS'}_{LL'} \tag{B.5}$$

$$T^{00}_{J,J} = -T^{11}_{J,J} = -1, T^{11}_{J-1,J+1} = T^{11}_{J+1,J-1} = \frac{2\sqrt{J(J+1)}}{2J+1} \tag{B.6}$$

$$T^{11}_{J-1,J-1} = -T^{11}_{J+1,J+1} = \frac{1}{2J+1} \tag{B.7}$$

with all other $L^{ss'}_{LL'}, R^{ss'}_{LL'}, T^{ss'}_{LL'}$ equal to zero. Let

$$a = \sqrt{\frac{J+1}{2J+1}}, b = \sqrt{\frac{J}{2J+1}}, c = \frac{2\sqrt{J(J+1)}}{2J+1}, d = \frac{1}{2J+1}$$

and

$$p_i = \frac{p}{\epsilon_i + m_i}, p_\pm = p_1 \pm p_2$$

we obtain for $P = (-1)^J P_{q\bar{q}}$ case, we obtain for $M^{\rho_1\rho_2,j}_{LS,L'S'}(p)\sqrt{2}/(N_1N_2)$ the following matrix

$$\begin{pmatrix}
1-p_1p_2 & 1-p_1p_2 & -ap_+ & ap_+ & 0 & 0 & bp_+ & -bp_+ \\
1+p_1p_2 & -1-p_1p_2 & ap_- & ap_- & 0 & 0 & -bp_- & -bp_- \\
ap_+ & ap_+ & 1-dp_1p_2 & -1+dp_1p_2 & bp_- & bp_- & cp_1p_2 & -cp_1p_2 \\
-ap_- & ap_- & 1+dp_1p_2 & 1+dp_1p_2 & -bp_+ & bp_+ & -cp_1p_2 & -cp_1p_2 \\
0 & 0 & -bp_- & bp_- & 1+p_1p_2 & 1+p_1p_2 & -ap_- & ap_- \\
0 & 0 & bp_+ & bp_+ & 1-p_1p_2 & -1+p_1p_2 & ap_+ & ap_+ \\
-bp_+ & -bp_+ & cp_1p_2 & -cp_1p_2 & ap_- & ap_- & 1+dp_1p_2 & -1-dp_1p_2 \\
bp_- & -bp_- & -cp_1p_2 & -cp_1p_2 & -ap_+ & ap_+ & 1-dp_1p_2 & 1-dp_1p_2
\end{pmatrix}$$

For the case when $P = (-1)^{J+1} P_{q\bar{q}}$ case, we obtain for $M^{\rho_1 \rho_2, j}_{LS, L'S'}(p)\,\sqrt{2}/\left(N_1 N_2\right)$ the following matrix

$$
\begin{pmatrix}
1 - dp_1p_2 & 1 - dp_1p_2 & -ap_+ & ap_+ & cp_1p_2 & cp_1p_2 & -bp_- & bp_- \\
1 + dp_1p_2 & -1 - dp_1p_2 & ap_- & -ap_- & -cp_1p_2 & cp_1p_2 & bp_+ & bp_+ \\
ap_+ & ap_+ & 1 - p_1p_2 & -1 + p_1p_2 & -bp_+ & -bp_- & 0 & 0 \\
-ap_- & ap_- & 1 + p_1p_2 & 1 + p_1p_2 & bp_- & -a\left(p_1 + p_2\right) & 0 & 0 \\
cp_1p_2 & cp_1p_2 & bp_+ & -bp_+ & 1 + dp_1p_2 & 1 + dp_1p_2 & -ap_- & ap_- \\
-cp_1p_2 & cp_1p_2 & -bp_- & -bp_- & 1 - dp_1p_2 & 1 - dp_1p_2 & ap_+ & ap_+ \\
bp_- & bp_- & 0 & 0 & ap_- & ap_- & 1 + p_1p_2 & -1 - p_1p_2 \\
-bp_+ & bp_+ & 0 & 0 & -ap_+ & ap_+ & 1 - p_1p_2 & 1 - p_1p_2
\end{pmatrix}
$$