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OPTIMAL UTILIZATION OF SERVICE FACILITY FOR A k-OUT-OF-n SYSTEM WITH REPAIR BY EXTENDING SERVICE TO EXTERNAL CUSTOMERS IN A RETRIAL QUEUE

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ABSTRACT. In this paper, we study a k-out-of-n system with single server who provides service to external customers also. The system consists of two parts:(i) a main queue consisting of customers (failed components of the k-out-of-n system) and (ii) a pool (of finite capacity M) of external customers together with an orbit for external customers who find the pool full. An external customer who finds the pool full on arrival, joins the orbit with probability γ and with probability $1-\gamma$ leaves the system forever. An orbital customer, who finds the pool full, at an epoch of repeated attempt, returns to orbit with probability δ (< 1) and with probability $1-\delta$ leaves the system forever. We compute the steady state system size probability. Several performance measures are computed, numerical illustrations are provided.

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1. Introduction

We study a k-out-of-n system with single server who provides service to external customers also as described in the following paragraphs.

The system consists of two parts:(i) a main queue consisting of customers (failed components of the k-out-of-n system) and (ii) a pool (of finite capacity M) of external customers together with an orbit for external customers who find the pool full. An external customer who finds the pool full on arrival, joins the orbit with probability γ and with probability $1-\gamma$ leaves the system forever. Customers in orbit, independently of each other, retry to access the

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server. Inter retrial times are assumed to have exponential distribution with parameter $n\theta$ when there are n customers in the orbit.

The arrival process. Arrival of main customers have inter occurrence time exponentially distributed with parameter λ_i, when the number of operational components of the k-out-of-n system is i. By taking $\lambda_i = \dfrac{\lambda}{i}$ we notice that the cumulative failure rate is a constant λ. We assume that the k-out-of-n system is COLD (components fail only when system is operational). The case of WARM and HOT system can be studied on the same lines (see Krishnamoorthy and Ushakumari [6]). External customers arrive according to a Markovian Arrival Process (MAP) with representation (D_0, D_1) where D_0 and D_1 are assumed to be matrices of order m. The fundamental arrival rate $\lambda_g = -\pi D_0 e$

Markovian arrival process (MAP) and its generalization Batch Markovian arrival process (BMAP) are special cases of semi-Markov processes with numerical tractability. A MAP is represented by a pair (D_0, D_1) of matrices which are square matrices of the same order, say, m. It can be described as follows : consider a Markov chain with m states. A particle moves from one state i to another state j with its sojourn time in state i having exponential distribution with parameter γ_{ij} $(i \neq j;\ i, j = 1, \ldots, m)$. This transition may trigger the occurrence of an event (say, an arrival) with probability p_1 and no event with probability p_0 $(p_0 + p_1 = 1)$. D_0 has its entries that are transition rates triggering no arrival, whereas D_1 has entries with transition rates combined with arrival. The diagonal entries of D_0 are all negative.

The service process. Service to the failed components of the main system is governed by the N-policy. That is at each epoch the system starts with all components operational (*ie.*, all n components are in operation), the server starts attending one by one the customers from the pool (if there is any). At the time when the number of failed components of the main system reaches N, no more customer from the pool is taken for service until there is no components of the main system waiting for repair. However service of the external customer, if there is any, will not be disrupted even when N components accumulate in the main queue (that is the external customer in service will not get pre-empted on realization of the event that N components of the main system failed and got accumulated; instead the moment the service of the present external customer is completed, the server is switched to the service of main customers).

Service time of main customers follow PH distribution of the order n_1 and representation (α, S_1) and that of external customers have PH distribution of order n_2 with representation (β, S_2); S_1^0 and S_2^0 are such that $S_i\mathbf{e} + s_i^0 = 0$, $i = 1, 2$ where $\mathbf{e}$ is a column vector of ones of appropriate order. The two service times are independent of each other and also independent of the failure of components of the main system as well as the arrival of external customers.

Objective. To utilize server idle time without affecting the system reliability.

Krishnamoorthy and Ushakumari [6] deals with the study of the reliability of a k-out-of-n system with repairs by server in a retrial queue. They do not give any priority to the failed components of the main system nor do they investigate any control policy. Krishnamoorthy, Ushakumari and Lakshmi [7] introduced the repair of failed components of a k-out-of-n system under the N-policy. For further details one may refer the paper Ushakumari and Krishnamoorthy [9]. Bocharov *et al* [2] examine an $M/G/1/r$ retrial queue with priority of primary customers. They obtain the stationary distribution of the primary queue size, an algorithm for the factorial moments of the number of retrial customers and an expression for the expected number of customers in the system. Nevertheless, we wish to emphasise that their paper does not distinguish between the priority and ordinary customers. This is distinctly done in this paper (our priority customers are the failed components of the k-out-of-n system):

We also consider an intermediate pool of finite capacity to which external customers join on encountering a busy server on arrival or after a successful retrial from the orbit. We expect that this intermediate pool from which an external customer can be selected for service, whenever the server becomes idle, will help us to decrease the server idle time.

The steady state distribution is derived. Note that the non-persistence of orbital customers together with the fact that an external customer, finding the pool full, may not join the pool ensures that even under very heavy traffic the system can attain stability. Several performance measures are obtained.
One can refer Deepak, Joshua, and Krishnamoorthy [4] for a detailed analysis of queues with pooled customers (postponed work).

2. Modelling and analysis

The following notations are used in the sequel:

$N_1(t) = \#$ of orbital customers at time t

$N_2(t) = \#$ of customers in the pool (including the one getting service, if any,) at time t.

$N_3(t) = \#$ of failed components (including the one under repair, if any) at time t

$$N_4(t) = \begin{cases} 0 & \text{if the server is idle} \\ 1 & \text{if the server is busy with repair} \\ & \text{of a failed component of the main system} \\ 2 & \text{if the server is attending an external customer at time } t. \end{cases}$$

$N_5(t) =$ Phase of the arrival process,

$$N_6(t) = \begin{cases} & \text{Phase of service of the customer, if any, in service at } t \\ 0, & \text{if no service is going on at time } t. \end{cases}$$

It follows that $\{X(t) : t \geq 0\}$ where

$$X(t) = (N_1(t), N_2(t), N_3(t), N_4(t), N_5(t), N_6(t))$$

is a continuous time Markov chain on the state space

$$
\begin{aligned}
S = & \Big\{(j_1, 0, j_3, 0, j_5, 0)|j_1 \geq 0;\ 0 \leq j_3 \leq N-1;\ 1 \leq j_5 \leq m\Big\} \\
& \cup \Big\{(j_1, j_2, j_3, 1, j_5, j_6)|j_1 \geq 0,\ 0 \leq j_2 \leq M;\ 1 \leq j_3 \leq n-k+1; \\
& \quad 1 \leq j_5 \leq m; 1 \leq j_6 \leq n_1\Big\} \\
& \cup \Big\{(j_1, j_2, j_3, 2, j_5, j_6)|j_1 \geq 0;\ 1 \leq j_2 \leq M; \\
& \quad 0 \leq j_3 \leq n-k+1; 1 \leq j_5 \leq m;\ 1 \leq j_6 \leq n_2\Big\}
\end{aligned}
$$

Arranging the states lexicographically, and then partitioning the state space into levels $\underline{i}$, where each level $\underline{i}$ corresponds to the collection of states with $\underline{i}$ customers in the orbit, we get the infinitesimal generator of the above chain as

$$
Q = \begin{bmatrix} A_{10} & A_0 & 0 & 0\dots \\ A_{21} & A_{11} & A_0 & 0\dots \\ 0 & A_{22} & A_{12} & A_0\dots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}
$$

where

$$
A_{10} = \begin{bmatrix} W_0 & W_5 & & & & & \\ W_3 & W_1 & W_6 & & & & \\ & W_4 & W_1 & W_6 & & & \\ & & & & \ddots & & \\ & & & & W_4 & W_1 & W_6 \\ & & & & & W_4 & W_2 \end{bmatrix}.
$$

In the above

$$
W_0 = \begin{bmatrix} B_0 & B_8 & & & & & & & & & \\ B_4 & B_1 & B_9 & & & & & & & & \\ & B_5 & B_1 & B_9 & & & & & & & \\ & & \ddots & \ddots & \ddots & & & & & & \\ & & & B_5 & B_1 & B_9 & & & & & \\ & & & & B_5 & B_1 & B_{10} & & & & \\ & & & & & B_6 & B_2 & B_{11} & & & \\ & & & & & & B_7 & B_2 & B_{11} & & \\ & & & & & & & & \ddots & & \\ & & & & & & & & & B_2 & B_{11} \\ & & & & & & & & & B_7 & B_3 \end{bmatrix},
$$

$$
\begin{aligned}
& B_0 = D_0 - \lambda I_m, \quad B_1 = \begin{bmatrix} D_0 - \lambda I_m & 0 \\ 0 & D_0 \oplus S_1 - \lambda I_{mn_1} \end{bmatrix}, \\
& B_2 = D_0 \oplus S_1 - \lambda I_{mn_1}, \quad B_3 = D_0 \oplus S_1, \\
& B_4 = \begin{bmatrix} 0 \\ I_m \otimes S_1^0 \end{bmatrix}, \quad B_5 = \begin{bmatrix} 0 & 0 \\ 0 & I_m \otimes (S_1^0 \alpha) \end{bmatrix}, \quad B_6 = \begin{bmatrix} 0 & I_m \otimes (S_1^0 \alpha) \end{bmatrix} \\
& B_7 = I_m \otimes (S_1^0 \alpha), \quad B_8 = \begin{bmatrix} \lambda I_m & 0 \end{bmatrix}, \quad B_9 = \lambda I_{m+mn_1},
\end{aligned}
$$

$$B_{10} = \begin{bmatrix} I_m \otimes (\lambda\alpha) \\ \lambda I_{mn_1} \end{bmatrix}, \quad B_{11} = \lambda I_{mn_1};$$

$$W_1 = \begin{bmatrix} C_0 & C5 & & & & \\ C_3 & C_1 & C_6 & & & \\ & C_4 & C_1 & & & \\ & & & \ddots & & \\ & & & & C_1 & C_6 \\ & & & & C_4 & C_2 \end{bmatrix}$$

where $C_0 = D_0 \oplus S_2 - \lambda I_{mn_2}, \quad C_1 = C_2 - \lambda I_{m(n_1+n_2)}$

$$C_2 = \begin{bmatrix} D_0 \oplus S_1 & 0 \\ 0 & D_0 \oplus S_2 \end{bmatrix}, \quad C_3 = \begin{bmatrix} I_m \otimes (S_1^0\beta) \\ 0 \end{bmatrix}$$

$$C_4 = \begin{bmatrix} I_m \otimes (S_1^0\alpha) & 0 \\ 0 & 0 \end{bmatrix}, \quad C_5 = \begin{bmatrix} 0 & \lambda I_{mn_2} \end{bmatrix}, \quad C_6 = \lambda I_{m(n_1+n_2)}.$$

We write $W_2 = W_1 + \bar{W}_1$. In the above

$$\bar{W}_1 = \begin{bmatrix} (1-\gamma)(D_1 \otimes I_{n_2}) & 0 \\ 0 & I_{n-k+1} \otimes \bar{\bar{W}}_1 \end{bmatrix}$$

$$\text{with} \quad \bar{\bar{W}}_1 = \begin{bmatrix} (1-\gamma)(D_1 \otimes I_{n_1}) & 0 \\ 0 & (1-\gamma)(D_1 \otimes I_{n_2}) \end{bmatrix}.$$

Next we have

$$W_3 = \begin{bmatrix} W_{30} & 0 & 0 \\ 0 & I_{N-1} \otimes W_{31} & 0 \\ 0 & 0 & I_{n-k-N+2} \otimes W_{32} \end{bmatrix}$$

with

$$W_{30} = I_m \otimes S_2^0, \quad W_{31} = \begin{bmatrix} 0 & 0 \\ I_m \otimes S_2^0 & 0 \end{bmatrix}_{m(n_1+n_2)\times m(n_1+n_2)},$$

$$W_{32} = \begin{bmatrix} 0 \\ I_m \otimes (S_2^o\alpha) \end{bmatrix}_{m(n_1+n_2)\times mn_1},$$

$$W_4 = \begin{bmatrix} E_0 & 0 & 0 \\ 0 & I_{N-1} \otimes E_1 & 0 \\ 0 & 0 & I_{n-k-N+2} \otimes E_2 \end{bmatrix},$$

$$E_0 = I_m \otimes (S_2^0\beta), \quad E_1 = \begin{bmatrix} 0 & 0 \\ 0 & I_m \otimes (S_2^0\beta) \end{bmatrix}_{m(n_1+n_2)\times m(n_1+n_2)},$$

$$E_2 = \begin{bmatrix} 0 & 0 \\ I_m \otimes (S_2^o\alpha) & 0 \end{bmatrix}.$$

Next we have

$$W_5 = \begin{bmatrix} F_0 & 0 & 0 \\ 0 & F_1 & 0 \\ 0 & 0 & F_2 \end{bmatrix}$$

where

$$F_0 = D_1 \otimes \beta, \quad F_1 = I_{N-1} \otimes F_1', \quad F_1' = \begin{bmatrix} 0 & D_1 \otimes \beta \\ D_1 \otimes I_{n_1} & 0 \end{bmatrix},$$
$$F_2 = I_{n-k+2-N} \otimes F_2' \quad \text{ where } F_2' = \begin{bmatrix} D_1 \otimes I_{n_1} & 0 \end{bmatrix}.$$

Further we have

$$W_6 = \begin{bmatrix} H_0 & 0 \\ 0 & I_{n-k+1} \otimes H_1 \end{bmatrix}$$

where

$$H_0 = D_1 \otimes I_{n_2}, \quad H_1 = \begin{bmatrix} D_1 \otimes I_{n_1} & 0 \\ 0 & D_1 \otimes I_{n_2} \end{bmatrix}$$

and

$$A_{1i} = A_{10} - \tilde{A}_{1i} \quad \text{ for } i \geq 1.$$

In the above

$$\tilde{A}_{1i} = \begin{bmatrix} i\theta I_{L_2} & 0 \\ 0 & i\theta(1-\delta) I_{L_1} \end{bmatrix},$$

where
$L_1 = (n-k+2)mn_2 + (n-k+1)mn_1$, $L_2 = Nm + (n-k+1)mn_1 + (M-1)L_1$;

$$A_{2i} = \begin{bmatrix} 0 & Z_i & 0 \\ 0 & 0 & i\theta I_{(M-1)L_1} \\ 0 & 0 & i\theta(1-\delta) I_{L_1} \end{bmatrix}, \quad i \geq 1;$$

$$Z_i = \begin{bmatrix} Z_{1i} & 0 & 0 \\ 0 & I_{N-1} \otimes Z_{2i} & 0 \\ 0 & 0 & I_{(n-k-N+2)} \otimes Z_{3i} \end{bmatrix}, \quad Z_{1i} = I_m \otimes (i\theta\beta),$$

$$Z_{2i} = \begin{bmatrix} 0 & I_m \otimes (i\theta\beta) \\ i\theta I_{mn_1} & 0 \end{bmatrix}, \quad Z_{3i} = \begin{bmatrix} i\theta I_{mn_1} & 0 \end{bmatrix}.$$

Finally

$$A_0 = \begin{bmatrix} 0 & 0 \\ 0 & \bar{A}_0 \end{bmatrix}$$

where

$$\bar{A}_0 = \begin{bmatrix} (\gamma D_1) \otimes I_{n_2} & 0 \\ 0 & I_{n-k+1} \otimes \bar{A}_0^{(1)} \end{bmatrix}, \quad \bar{A}_0^{(1)} = \begin{bmatrix} (\gamma D_1) \otimes I_{n_1} & 0 \\ 0 & (\gamma D_1) \otimes I_{n_2} \end{bmatrix}.$$

3. System stability

Theorem 1. *The assumption that after each retrial a customer may leave the system with probability* $1-\delta$ *makes the system stable irrespective of the parameter values.*

Proof. To prove the theorem we use a result due to Tweedie [8]. For the model under consideration we consider the following Lyapunov function:

$$\phi(s) = i \text{ if } s \text{ is a state belonging to level } i$$

The mean drift y_s for an s belonging to level $i \geq 1$ is given by

$$\begin{aligned} y_s &= \sum_{p \neq s} q_{sp}(\phi(p) - \phi(s)) \\ &= \sum_{s'} q_{ss'}(\phi(s') - \phi(s)) + \sum_{s''} q_{ss''}\Big(\phi(s'') - \phi(s)\Big) \\ &\quad + \sum_{s'''} q_{ss'''}\Big(\phi(s''') - \phi(s)\Big) \end{aligned}$$

where s', s'', s''' vary over the states belonging to levels $i-1, i, i+1$ respectively. Then by definition of ϕ, $\phi(s) = i$, $\phi(s') = i-1$, $\phi(s'') = i$, $\phi(s''') = i+1$
so that

$$y_s = -\sum_{s'} q_{ss'} + \sum_{s'''} q_{ss'''}$$

$$y_s = \begin{cases} -i\theta + \sum_{s'''} q_{ss'''}, & \text{if } s \in I_i \\ -i\theta(1-\delta) + \sum_{s'''} q_{ss'''}, & \text{if } s \in \bar{I}_i \end{cases}$$

where I_i denotes the collection of states in level i which corresponds to $N_2(t) < M$, and $\bar{I}_i$ denotes the collection of states in level i which correspond to $N_2(t) = M$.

We note that $\sum_{s'''} q_{ss'''}$ is bounded by some fixed constant for any s in any level $i \geq 1$. So, let $\sum_{s'''} q_{ss'''} < \kappa$, for some real number $\kappa > 0$, for all states s belonging to level $i \geq 1$. Also since $1-\delta > 0$ for any $\epsilon > 0$, we can find N' large enough that $y_s < -\epsilon$ for any s belonging to level $i \geq N'$.

Hence by Tweedie's result, the theorem follows. □

4. Steady state distribution

Since the process under consideration is an *LDQBD*, to calculate the steady state distribution, we use the methods described in Bright and Taylor [3].

By partitioning the steady state vector $\mathbf{x}$ as $\mathbf{x} = (x_0, x_1, x_2, \ldots)$ we can write

$$x_k = x_0 \prod_{l=0}^{k-1} R_l \quad \text{for } k \geq 1$$

where the family of matrices $\{R_k, k \geq 0\}$ is minimal non-negative solutions to the system of equations:

$$A_0 + R_k A_{1\ k+1} + R_k [R_{k+1} A_{2\ k+2}] = 0,\ k \geq 0 \tag{1}$$

x_0 is calculated by solving

$$x_0 [A_{10} + R_0 A_{21}] = 0 \tag{2}$$

such that

$$x_0 \mathbf{e} + x_0 \sum_{k=1}^{\infty} \left[\prod_{l=0}^{k-1} R_l \right] \mathbf{e} < \infty. \tag{3}$$

The calculation of the above infinite sums does not seem to be practical. So we approximate x_ks by $x_k(K^*)$s where $\left(x_k(K^*)\right)_j$, $0 \leq k \leq K^*$, is defined as the stationary probability that $X(t)$ is in the j^{th} state of level k, conditional on $X(t)$ being in level i, $0 \leq i \leq K^*$.

Then $x_k(K^*)$, $0 \leq k \leq K^*$ is given by

$$x_k(K^*) = x_0(K^*) \prod_{l=0}^{k-1} R_l \tag{4}$$

where $x_0(K^*)$ satisfies (2) and

$$x_0(K^*) \mathbf{e} + x_0(K^*) \left[\sum_{k=1}^{K^*} \left[\prod_{l=0}^{k-1} R_l \right] \right] \mathbf{e} = 1 \tag{5}$$

Here we have for all $i \geq 1$, and for all k, there exists j such that $[A_{2i}]_{k,j} > 0$. So we can construct a dominating process $\bar{X}(t)$ of $X(t)$ and can use it to find the truncation level K^* in the same way as in [3], as follows. By dominating process we mean a process that has tail probability at least as large as the one under consideration. For a strictly dominating process the tail probability will be higher. The dominating process $\bar{X}(t)$ has generator

$$\bar{Q} = \begin{bmatrix} A_{10} & A_0 & 0 & 0 & 0 & \dots \\ 0 & \bar{A}_{11} & \bar{A}_0 & 0 & 0 & \dots \\ 0 & \bar{A}_{22} & \bar{A}_{12} & \bar{A}_0 & 0 & \dots \\ 0 & 0 & \bar{A}_{23} & \bar{A}_{13} & \bar{A}_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}$$

where
$(\bar{A}_0)_{i,j} = \frac{1}{C}[(A_0 e)_{\max}]$, $(\bar{A}_{2k})_{i,j} = \frac{1}{C}\Big((A_{2,k-1})\mathbf{e}\Big)_{\min}$ for $k \geq 2$; $(\bar{A}_{1k})_{ij} = (A_{1k})_{ij}$, $j \neq i$, $k \geq 1$; and $C = Nm + (M+1)(n-k+1)mn_1 + M(n-k+2)mn_2$ is the dimension of level $i \geq 1$. We choose the truncation level K^* in such a

way that the probability of the process being in level higher than K^* is sufficiently small (any prescribed positive number, however small). This guarantees that by our truncation procedure no useful information is lost. Note that by this truncation procedure we are not chopping the system beyond level K^*. Instead it is assumed that the retrial rate, while in levels K^* or higher, remain the same. This means that we have the infinitesimal generator looking like a quasi-Toeplitz matrix since from level K^* onward we have "repetitive pattern" in the infinitesimal generator. The readers may also refer to Anisimov and Artalejo [1] and Falin and Templeton [5] on truncation procedures. The study of level-dependent quasi-birth and death process (LDQBD) is carried out using matrix analytic method by first choosing a truncation level. Since for large truncation levels it can be assumed that the system behaviour (for example retrial rate in the present set up) at any level above the truncation level can be assumed the same as when the system is at the truncated level, the entries in the infinitesimal generator starting from that level, will have a repetitive pattern.

5. Performance measures

We partition the steady state vector $\mathbf{x}$ as $\mathbf{x} = (x_0, x_1, x_2, \dots)$ where the subvectors x_js are again partitioned as $x_j = x(j_1, j_2, j_3, j_4)$ which correspond to $N_i(t) = j_i$, $1 \le i \le 4$.

(1) Fraction of time the system is down is given by

$$\mathcal{P}_{\text{down}} = \sum_{j_1=0}^{K^*} \sum_{j_2=0}^{M} \sum_{j_4=1}^{2} x(j_1, j_2, n-k+1, j_4)\mathbf{e}.$$

(2) System reliability, defined as the probability that at least k components are operational, $\mathcal{P}_{\text{rel}}$ is given by

$$\mathcal{P}_{\text{rel}} = 1 - \mathcal{P}_{\text{down}}.$$

(3) Average number of external units waiting in the pool is given by

$$\mathcal{N}_{\text{pool}} = \sum_{j_2=1}^{M} j_2 (\sum_{j_1=0}^{K^*} \sum_{j_3=1}^{n-k+1} x(j_1, j_2, j_3, 1)\mathbf{e} + \sum_{j_2=2}^{M} (j_2 - 1) \sum_{j_1=0}^{K^*} \sum_{j_3=0}^{n-k+1} x(j_1, j_2, j_3, 2)\mathbf{e}.$$

(4) Average number of external units in the orbit is given by

$$\mathcal{N}_{\text{orbit}} = \sum_{j_1=1}^{K^*} j_1 [x(j_1)\mathbf{e}].$$

(5) Average number of failed components is given by

$$\mathcal{N}_{\text{faic}} = \sum_{j_3=1}^{n-k+1} j_3 \left(\sum_{j_1=0}^{K^*} \sum_{j_2=1}^{M} x(j_1, j_2, j_3, 2)\mathbf{e} + \sum_{j_1=0}^{K^*} \sum_{j_2=0}^{M} x(j_1, j_2, j_3, 1)\mathbf{e} \right) + \sum_{j_3=1}^{N-1} j_3 \sum_{j_1=0}^{K^*} x(j_1, 0, j_3, 0)\mathbf{e}.$$

(6) The probability that an external unit on its arrival joins the queue in the pool, is given by

$$\mathcal{P}_{\text{queue}} = \frac{1}{\lambda_g} \left\{ \sum_{j_1=0}^{K^*} \sum_{j_2=1}^{M-1} \sum_{j_3=1}^{n-k+1} \sum_{j_4=1}^{2} x(j_1, j_2, j_3, j_4)[D_1 \otimes I_{n_{j_4}}]\mathbf{e} + \sum_{j_1=0}^{K^*} \sum_{j_3=1}^{n-k+1} x(j_1, 0, j_3, 1)(D_1 \otimes I_{n_1})\mathbf{e} \right\}.$$

(7) The probability that an external unit on its arrival gets service directly, is given by

$$\mathcal{P}_{\text{ds}} = \frac{1}{\lambda_g} \left\{ \sum_{j_1=0}^{K^*} \sum_{j_3=0}^{N-1} x(j_1, 0, j_3, 0) D_1 \mathbf{e} \right\}.$$

(8) The probability that an external unit on its arrival enters orbit, is given by

$$\mathcal{P}_{\text{orbit}} = \frac{1}{\lambda_g} \left\{ \sum_{i=0}^{K^*} x(i) A_0 \mathbf{e} \right\}.$$

(9) Fraction of time the server is busy with external customers is given by

$$\mathcal{P}_{\text{exbusy}} = \sum_{j_1=0}^{K^*} \sum_{j_2=1}^{M} \sum_{j_3=0}^{n-k+1} x(j_1, j_2, j_3, 2)\mathbf{e}.$$

(10) Probability that the server is found idle is given by

$$\mathcal{P}_{\text{idle}} = \sum_{j_1=0}^{K^*} \sum_{j_2=0}^{N-1} x(j_1, 0, j_2, 0).$$

(11) Probability that the server is found busy is given by

$$\mathcal{P}_{\text{busy}} = 1 - \mathcal{P}_{\text{idle}}.$$

(12) Expected loss rate of external customers is given by

$$\boldsymbol{\lambda}_{\text{loss}} = \sum_{j_1=0}^{K^*} \sum_{j_2=1}^{n-k+1} x(j_1, M, j_2, 1)(1-\gamma)(D_1 \otimes I_{n_1})\mathbf{e} + \sum_{j_1=0}^{K^*} \sum_{j_2=0}^{n-k+1} x(j_1, M, j_2, 2)(1-\gamma)(D_1 \otimes I_{n_2})\mathbf{e} + \sum_{j_1=1}^{K^*} \sum_{j_2=1}^{n-k+1} (1-\delta) j_1 \theta x(j_1, M, j_2, 1)\mathbf{e} + \sum_{j_1=1}^{K^*} \sum_{j_2=0}^{n-k+1} (1-\delta) j_1 \theta x(j_1, M, j_2, 2)\mathbf{e}$$

(13) We construct a cost function with the following costs: C_1 is the holding cost per unit time per customer waiting in the pool, C_2 is the loss per unit time due to the system becoming down, C_3 is the loss per unit time due to a customer leaves the system without taking service, C_4 is the holding cost per unit time per failed component in the system, C_5 is the loss per unit time due to the server becoming idle and C_6 is the profit per unit time due to the server becoming busy with an external customer.

6. Numerical illustration

Set $\theta = 15.0$, $\lambda = 1.0$, $\gamma = 0.7$, $\delta = 0.7$, $n = 11$, $k = 4$, $M = 5$, $N = 4$, $S_1 = \begin{bmatrix} -6.5 & 4.0 \\ 1.5 & -4.5 \end{bmatrix}$, $S_2 = \begin{bmatrix} -5.06 & 2.06 \\ 4.0 & -6.5 \end{bmatrix}$, $S_1^0 = \begin{bmatrix} 2.5 \\ 3.0 \end{bmatrix}$, $S_2^0 = \begin{bmatrix} 3.0 \\ 2.5 \end{bmatrix}$, $\alpha = (0.5, 0.5)$, $\beta = (0.5, 0.5)$,
$C_1 = 10.0$, $C_2 = 1500.0$, $C_3 = 100.0$, $C_4 = 20.0$, $C_5 = 50.0$, $C_6 = 200.0$.

Effect of correlation : The additional parameters for table 1 are the following

$$D_0 = \begin{bmatrix} -5.5 & 3.5 \\ 1.0 & -3.5 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1.0 & 1.0 \\ 1.0 & 1.5 \end{bmatrix}. \tag{A1}$$

For this pair average arrival rate = 2.34615, correlation =-0.00029

$$D_0 = \begin{bmatrix} -4.05 & 1.55 \\ 3.5 & -5.5 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 2.05 & 0.45 \\ 1.0 & 1.0 \end{bmatrix}. \tag{A2}$$

For this pair average arrival rate = 2.34615, correlation =0.00029

$$D_0 = \begin{bmatrix} -6.5 & 4.0 \\ 1.5 & -4.5 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1.5 & 1.0 \\ 1.0 & 2.0 \end{bmatrix}. \tag{B1}$$

For this pair average arrival rate = 2.83333, correlation =-0.00042

$$D_0 = \begin{bmatrix} -5.06 & 2.06 \\ 4.0 & -6.5 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 2.56 & 0.44 \\ 1.0 & 1.5 \end{bmatrix}. \tag{B2}$$

For this pair average arrival rate = 2.83333, correlation =0.00042

$$D_0 = \begin{bmatrix} -6.6 & 4.05 \\ 1.55 & -4.6 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1.55 & 1.0 \\ 1.0 & 2.05 \end{bmatrix}. \tag{C1}$$

For this pair average arrival rate = 2.88224, correlation =-0.00041

$$D_0 = \begin{bmatrix} -5.15 & 2.1 \\ 4.05 & -6.6 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 2.6 & 0.45 \\ 1.0 & 1.55 \end{bmatrix}. \tag{C2}$$

For this pair average arrival rate = 2.88224, correlation =0.00041

In the above correlation is between two inter-arrival times.

TABLE 1

	$\mathcal{P}_{\text{down}}$	$\mathcal{N}_{\text{pool}}$	$\mathcal{N}_{\text{orbit}}$	$\mathcal{N}_{\text{faic}}$	$\mathcal{P}_{\text{exbusy}}$	$\mathcal{P}_{\text{idle}}$	Cost
A1	$.2805 \times 10^{-2}$	3.262	0.1204	2.2281	0.5620	0.0842	37.8228
A2	$.2803 \times 10^{-2}$	3.2572	0.1207	2.2278	0.5612	0.0850	38.1696
B1	$.2923 \times 10^{-2}$	3.6689	0.1822	2.2431	0.5940	0.0522	68.2556
B2	$.2922 \times 10^{-2}$	3.6647	0.1824	2.2429	0.5935	0.0526	68.4537
C1	$.2932 \times 10^{-2}$	3.7031	0.1888	2.2442	0.5964	0.0497	71.6377
C2	$.2931 \times 10^{-2}$	3.6992	0.1890	2.2440	0.5960	0.0502	71.8214

The table 1 shows that as the external arrival rate increases the system down probability increases; but this increase is narrow as compared to the decrease in server idle probability. Also as expected, the expected number in the pool, in the orbit and the expected number of failed components and the fraction of time the server is found busy with an external customer, increases as the external arrival rate increases. The table also shows that as the correlation changes from negative to positive, there is a slight increase in cost and in the server idle probability. Also when correlation changes from negative to positive, the expected number of pooled customers and failed components decrease while the expected number in the orbit increases. The increase in probability $\mathcal{P}_{\text{exbusy}}$ being small compared to the increase in other parameters can be thought of as the reason behind increase in cost. But all these changes are narrow as the difference between negative and positive correlation is small.

Effect of component failure rate: Take $\theta = 20.0$, $\gamma = 0.7$, $\delta = 0.7$, $n = 11$, $k = 4$, $M = 5$, $N = 4$.

Arrival process is as in (A1).

Table 2 shows that when the component failure rate λ increases, the system down probability as well as expected number of failed components increase and the idle time probability of the server decreases as expected. But note that as λ increases, the fraction of time the server is found busy with an external customer

TABLE 2. Effect of component failure rate

λ	$\mathcal{P}_{\text{down}}$	$\mathcal{N}_{\text{pool}}$	$\mathcal{N}_{\text{orbit}}$	$\mathcal{N}_{\text{faic}}$	$\mathcal{P}_{\text{exbusy}}$	$\mathcal{P}_{\text{idle}}$	Cost
0.05	$.196 \times 10^{-8}$	2.1163	0.0285	1.5266	0.7513	0.2310	-67.3177
0.1	$.5933 \times 10^{-7}$	2.1765	0.0311	1.5538	0.7432	0.2213	-63.3658
1.0	$.2801 \times 10^{-2}$	3.2399	0.0907	2.2276	0.5607	0.0855	38.4979
2.0	0.04702	4.2095	0.1748	3.5505	0.3029	0.0208	261.502
3.0	0.17207	4.7390	0.2362	5.1091	0.1149	0.0038	580.397

decreases and as a result the expected pool size increases. Also note that the expected orbit size is small, which shows that the orbital customers are either transfered to the pool (when λ is small) or leaves the system forever (when λ is large). Since the probability $\mathcal{P}_{\text{down}}$ increases and the probability $\mathcal{P}_{\text{exbusy}}$ decreases, as λ increases, the cost also increases.

Effect of N policy level: $\theta = 20.0$, $\lambda = 2.0$, $n = 13$, $k = 4$, $M = 5$

The other parameters are same as for table 2.

Table 3 shows that the system performance measure which is most affected

TABLE 3. Effect of N-policy level

N	$\mathcal{P}_{\text{down}}$	$\mathcal{N}_{\text{pool}}$	$\mathcal{N}_{\text{orbit}}$	$\mathcal{N}_{\text{faic}}$	$\mathcal{P}_{\text{exbusy}}$	$\mathcal{P}_{\text{idle}}$	Cost
4	0.02245	4.2521	0.1802	3.8666	0.2866	0.01969	203.559
5	0.02795	4.2249	0.1801	4.2456	0.2869	0.02325	219.258
6	0.03528	4.1968	0.1796	4.6087	0.2882	0.02717	237.002
7	0.04509	4.1658	0.1787	4.9473	0.2910	0.03135	257.358
8	0.05830	4.1300	0.1771	5.2518	0.2959	0.03577	281.200

by the N-policy level is the expected number of failed components. This is expected because as N increases, time for the service of failed components to be started, once the system started with all components operational, increases so that during this time more components may fail. For the same reason a pooled customer has a better chance of getting service and as a result $\mathcal{P}_{\text{exbusy}}$ increases, $\mathcal{N}_{\text{pool}}$ and $\mathcal{N}_{\text{orbit}}$ decrease. Also note that the server idle probability is small. The increase in $\mathcal{N}_{\text{faic}}$ might be the reason behind the increase in cost.

Effect of retrial rate θ: Take $\lambda = 1.0$, $n = 11$, $k = 4$, $M = 5$, $N = 4$

The other parameters are the same as in table 2.

Table 4 shows that as θ increases, expected number of customers in the orbit decreases; but the expected pool size also decreases which tells that retrying customers may be leaving the system. Note that the idle probability of the server is very small and the expected pool size is also close to the maximum pool capacity so that retrying customers may choose to leave the system after a failed retrial. Also this can be thought of as the reason behind the decrease in the fraction of time the server is found busy with an external customer and the increase in cost as θ increases.

Effect of pool size M: $\theta = 10.0$, $\lambda = 1.0$

The other parameters are same as for table 2.

Table 5 shows that as M, the pool size, increases, expected number of pooled customers increases and as a result the expected number of failed components, the system down probability and the fraction of time the server is found busy with and external customer increase. But the expected number in the orbit decreases, which is expected because as M increases more customers can join the pool. As expected, the idle probability of the server decreases as M increases.

Comparison with the case where no external customers are allowed: Below we compare the k-out-of-n-system with a k-out-of-n system where no external customers are allowed.

Case 1. k-out-of-n system where no external customers are allowed (see tables 6 and 7)

Case 2. k-out-of-n system

$\theta = 10.0$, $\lambda = 1.0$, $\gamma = 0.7$, $\delta = 0.7$, $n = 11$, $k = 4$, $N = 4$

$$D_0 = \begin{bmatrix} -5.5 & 3.5 \\ 1.0 & -3.5 \end{bmatrix} \quad D_1 = \begin{bmatrix} 1.0 & 1.0 \\ 1.0 & 1.5 \end{bmatrix},$$
$$S_1 = \begin{bmatrix} -7.5 & 2.0 \\ 2.1 & -7.7 \end{bmatrix} \quad S_2 = \begin{bmatrix} -5.06 & 2.06 \\ 4.0 & -6.5 \end{bmatrix},$$
$$S_1^0 = \begin{bmatrix} 5.5 \\ 5.6 \end{bmatrix} \quad S_2^0 = \begin{bmatrix} 3.0 \\ 2.5 \end{bmatrix},$$
$$\alpha = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix} \quad \beta = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}.$$

Table 6(a) shows that compared to the increase in the fraction of time the server is found busy, the increase in the system down probability is not high, if we provide service to external customers in a k-out-of-n system To make these statements more clear we consider the cost function

$$ID_{\text{cost}} = C_{11} \cdot \mathcal{P}_{\text{down}} - C_{12} \cdot \mathcal{P}_{\text{busy}}$$

Table 4. Effect of retrial rate

θ	$\mathcal{P}_{\text{down}}$	$\mathcal{N}_{\text{pool}}$	$\mathcal{N}_{\text{orbit}}$	$\mathcal{N}_{\text{faic}}$	$\mathcal{P}_{\text{exbusy}}$	$\mathcal{P}_{\text{idle}}$	cost
5.0	$.2832 \times 10^{-2}$	3.3908	0.3501	2.2315	0.5704	0.07579	33.688
10.0	$.2813 \times 10^{-2}$	3.3008	0.1790	2.2290	0.5644	0.08176	36.612
15.0	$.2805 \times 10^{-2}$	3.2620	0.1204	2.2281	0.5620	0.08415	37.823
20.0	$.2801 \times 10^{-2}$	3.2399	0.0907	2.2276	0.5607	0.08546	38.498
25.0	$.2798 \times 10^{-2}$	3.2255	0.0728	2.2272	0.5598	0.08630	38.932

Table 5. Effect of pool size

M	$\mathcal{P}_{\text{down}}$	$\mathcal{N}_{\text{pool}}$	$\mathcal{N}_{\text{orbit}}$	$\mathcal{N}_{\text{faic}}$	$\mathcal{P}_{\text{exbusy}}$	$\mathcal{P}_{\text{idle}}$	cost
3	$.2655 \times 10^{-2}$	1.9658	0.2155	2.2090	0.5084	0.1377	65.402
4	$.2743 \times 10^{-2}$	2.6238	0.1942	2.2201	0.5410	0.1051	55.047
5	$.2813 \times 10^{-2}$	3.3008	0.1790	2.2290	0.5644	0.0818	36.612

where C_{11} is the loss per unit time the system being down and C_{12} is the profit per unit time due to the server being busy.

Table 6(b) shows that when $M = 1$ and $\lambda \leq 1.5$, ID_{cost} is smaller in case 2 than in case 1 even when C_{11} is 1000 times bigger than C_{12}. But when $\lambda = 2.0$ and 2.5, ID_{cost} is larger in case 2 than case 1 when C_{11} is 100 times larger than C_{12}. When $M = 4$ and $\lambda \leq 1.0$, the table shows that ID_{cost} is smaller in case 2 than in case 1 even when C_{11} is 1000 times bigger than C_{12}. But when $\lambda = 2.0$ and 2.5, ID_{cost} is larger in case 2 than case 1, when C_{11} is 100 times larger than C_{12}.

Table 6(b) indicates that we are able to utilize server idle time without much effecting the system reliability.

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Table 6. **(a)** Comparison with the k-out-of-n system where no external customers are allowed.

			$\lambda = 0.1$	$\lambda = 0.5$	$\lambda = 1.0$	$\lambda = 1.5$	$\lambda = 2.0$	$\lambda = 2.5$
$M = 1$	$\mathcal{P}_{\text{down}}$	Case 1	$< 10^{-13}$	$.3956 \times 10^{8}$	$.9124 \times 10^{-6}$	$.2081 \times 10^{-4}$	$.1822 \times 10^{-3}$	$.9335 \times 10^{-3}$
		Case 2	$.129 \times 10^{-7}$	$.2379 \times 10^{-4}$	$.4329 \times 10^{-3}$	$.2039 \times 10^{-2}$	$.5728 \times 10^{-2}$	.01237
	$\mathcal{P}_{\text{busy}}$	Case 1	0.0180	0.0901	0.1802	0.2703	0.3603	0.4501
		Case 2	0.5347	0.5836	0.6415	0.6958	0.7458	0.7914
$M = 2$	$\mathcal{P}_{\text{down}}$	Case 1	$< 10^{-13}$	$.3956 \times 10^{8}$	$.9124 \times 10^{-6}$	$.2081 \times 10^{-4}$	$.1822 \times 10^{-3}$	$.9335 \times 10^{-3}$
		Case 2	$.1801 \times 10^{-7}$	$.3289 \times 10^{-4}$	$.5952 \times 10^{-3}$	$.2782 \times 10^{-3}$	$.7689 \times 10^{-2}$	$.1616 \times 10^{-1}$
	$\mathcal{P}_{\text{busy}}$	Case 1	0.0180	0.0901	0.1802	0.2703	0.3603	0.4501
		Case 2	0.7500	0.7941	0.8434	0.8848	0.9179	0.9433

(b) Variation in ID_{cost}

	ID_{cost}		$\lambda = 0.1$	$\lambda = 0.5$	$\lambda = 1.0$	$\lambda = 1.5$	$\lambda = 2.0$	$\lambda = 2.5$
$M = 1$	$C_{11} = 100$	Case 1	−0.1800	−0.9010	−1.8019	−2.7009	−3.5848	−4.4077
	$C_{12} = 10$	Case 2	−5.3470	−5.8336	−6.3717	−6.7541	−6.8852	−6.6770
	$C_{11} = 1000$	Case 1	−0.1800	−0.9010	−1.8011	−2.6822	−3.4208	−3.5675
	$C_{12} = 10$	Case 2	−5.3470	−5.8122	−5.9821	−4.9190	−1.7300	4.4560
	$C_{11} = 10000$	Case 1	−0.1800	−0.9010	−1.7929	−2.4949	−1.7810	4.8340
	$C_{12} = 10$	Case 2	−26.7349	−28.9421	−27.7460	−14.4000	19.9900	84.1300
$M = 4$	$C_{11} = 100$	Case 1	−0.1800	−0.9010	−1.8019	−2.7009	−3.5848	−4.4077
	$C_{12} = 10$	Case 2	−7.5000	−7.9377	−8.3745	−8.5698	−8.4101	−7.8170
	$C_{11} = 1000$	Case 1	−0.1800	−0.9010	−1.8011	−2.6822	−3.4208	−3.5675
	$C_{12} = 10$	Case 2	−7.5000	−7.9081	−7.8388	−6.0660	−1.4900	6.7270
	$C_{11} = 10000$	Case 1	−0.1800	−0.9010	−1.7929	−2.4949	−1.7810	4.8340
	$C_{12} = 10$	Case 2	−7.4998	−7.6121	−2.4820	18.9720	67.7110	152.167

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