STUDIES ON TRIANGLE NUMBER IN A GRAPH AND RELATED TOPICS

Thesis submitted to the
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By

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CERTIFICATE

This is to certify that the thesis entitled STUDIES ON TRIANGLE NUMBER IN A GRAPH AND RELATED TOPICS *submitted to the* Cochin University of Science and Technology *by* Sri B Radhakrishnan Nair *for the award of the degree of* Doctor of Philosophy *in the* Faculty of Science *is a bonafide record of studies done by him under my supervision. This report has not been submitted previously for considering the award of any degree, fellowship or similar titles elsewhere.*

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INTRODUCTION

Graph theory is a subject of study since 1736 when Euler solved the famous Königsberg bridge problem. It has a wide variety of applications in different branches of science, engineering and social science as described in [1], [3], [5], [6], [9], etc.

The concept of triangle, the smallest non-trivial complete graph and the smallest cycle, has been used in the graphical formulation of well-known theorems such as *Ramsey theorem*, *Friendship theorem* and *Kirkman's schoolgirl problem*. The concept of complementation is also an equally interesting and beautiful concept in *Graph* theory. These are the two main characters interlinking most of the results in this thesis.

1.1 DEFINITIONS AND PRELIMINARIES

In this section we give some preliminary ideas and definitions, some of which are new. We follow [4] and [8] for notations and terminology not given here.

We consider finite undirected graphs without loops and multiple edges. By a *graph* $G = G(p,q) = G(V,E)$, we generally mean a graph of *order* $p = p(G)$ and *size* $q = q(G)$ with *vertex set* $V = V(G)$ and *edge set* $E = E(G)$. $\langle S\rangle = \langle S\rangle_G$ denote the *subgraph* of G *induced* by $S \subseteq V(G)$. By writing uv, we mean an edge joining the vertices are u and v.

Definition 1.1 The *distance* $d(u,v) = d_G(u,v)$ between two vertices u and v in a graph G is the length of the shortest u-v path, the *eccentricity* $ecc(u) = ecc_G(u)$ of a vertex u is the distance to a vertex farthest from it. The *diameter* $diam(G)$ and the *radius* $rad(G)$ are respectively the maximum and minimum of the eccentricities of vertices in G. Vertices u and v in a graph G with $ecc(u) = diam(G)$ and $ecc(v) = rad(G)$ are respectively called *diametral vertex* and *central vertex*. Two vertices u and v with $d(u,v) = diam(G)$ are called *antipodal vertices*. Set of all central vertices in a graph G is called the *centre* of G. A graph G is said to be *self-centered* if each of its vertices is a central vertex, that is, if diameter and radius are equal.

Definition 1.2 A vertex u in a graph G is said to be a *neighbour* of another vertex v if they are adjacent. The set $N(u)$ of neighbours of u is called the *neighbourhood* of u, the set $N[u] = \{ u \} \cup N(u)$ is the *closed neighbourhood* and the set $E(u)$

is the set of all edges incident at u is the *edge neighbourhood*. A subset D of $V(G)$ is said to be a *dominating set* if every vertex of G is either in D or is adjacent to some vertex in D. The minimum of the cardinalities of the dominating sets in G is called the *domination number* of G and is denoted by $\gamma(G)$.

Definition 1.3 Two graphs are said to be *homeomorphic* if both can be obtained from the same graph by a sequence of subdivisions of edges. An *isomorphism* between two graphs is a bijection between the vertex sets which preserves adjacency. An *automorphism* of a graph G is an isomorphism of G onto itself.

Definition 1.4 The *complement* $\overline{G}$ of a graph G has $V(G)$ as its vertex set, and two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. A graph is *self-complementary* if it is isomorphic to its complement. If G is self-complementary, an isomorphism between G and $\overline{G}$ is called a *complementing permutation* and the set of all complementing permutations of G is denoted by $\mathcal{C}(G)$. A vertex in a self-complementary graph is said to be a *fixed-vertex* if there is a complementing permutation σ of G that maps the vertex onto itself. The set of all fixed vertices of G is denoted by $F(G)$. The set of all edges, in self-complementary graph G, such that there exists a complementing permutation σ mapping one of its end-vertices onto the other is denoted by $Z(G)$.

Definition 1.5 Two vertices (edges) in a graph are said to be *similar* if there is an automorphism that maps one of the vertices (edges) onto the other. A graph G is *vertex-symmetric* (*edge-symmetric*) if every pair of vertices (edges) are similar.

Definition 1.6 A graph G of order p is *strongly regular* with parameters (p, r, λ, μ) if it is regular of degree r, any two adjacent vertices have precisely λ common neighbors and any two non-adjacent vertices have precisely μ common neighbors.

Definition 1.7 The *join* $G + H$ of two graphs G and H has vertex set $V(G) \cup V(H)$ and edge set $E(G+H) = E(G) \cup E(H) \cup \{uv \mid u \in V(G),\ v \in V(H)\}$. The *cartesian product* $G \times H$ has vertex set $V(G) \times V(H)$ and two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent whenever $[\ u_1 = v_1$ and $u_2v_2 \in E(H)\ ]$ or $[\ u_2 = v_2$ and $u_1v_1 \in E(G)\ ]$. The *composition or lexicographic product* $G(H)$ also has vertex set $V(G) \times V(H)$ and two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent whenever $u_1v_1 \in E(G)$ or $[\ u_1 = v_1$ and $u_2v_2 \in E(H)\ ]$.

Definition 1.8 Let G be a graph and $\mathcal{F} = \{\ H_u \mid u \in V(G)\ \}$ be a family of graphs. The *G-join* $G(\mathcal{F})$ of $\mathcal{F}$ is the graph with vertex set $\{\ (u,v) \mid u \in V(G),\ v \in V(H_u)\ \}$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent if either $u_1u_2 \in E(G)$ or $u_1 = u_2$ and $v_1v_2 \in E(H_{u_1})$. The *S-join (star join)* is a special case of G-join when G is a star $K_{1,p}$ and $\mathcal{F}$ is a family of p graphs each of which corresponds to a pendent vertex of the star.

Definition 1.9 The *neighbourhood graph* $N(G)$ *of a graph* G is the intersection graph of the collection of neighbourhoods in G. That is, the graph with vertex set same as that of G and two vertices are adjacent whenever they have a common neighbour in G. A graph is a *neighbourhood graph* if it is the neighbourhood graph of some graph H. The *antipodal graph* $A(G)$ *of a graph* G

also has vertex set $V(G)$ and two vertices are adjacent if they are antipodal. A graph G is an *antipodal graph* if it is the antipodal graph of some graph H and is *self-antipodal* if it is the antipodal graph of itself.

Definition 1.10 The *S-antipodal graph* $A^*(G)$ *of a graph* G has its vertex set the diametral vertices of G and two vertices are adjacent whenever they are antipodal.

Definition 1.11 The number $t(u) = t_G(u)$ of triangles in a graph G containing a vertex u is called the *triangle number of the vertex* u. *Triangle number* $t(e)$ *of an edge* is the number of triangles containing e. The number $t(G)$ of triangles in a graph G is the *triangle number of the graph.* A vertex (an edge) is said to be *triangle positive* if its triangle number is non-zero. A *graph is triangle positive* if each of its edges is triangle positive. The set of all vertices with triangle number $k(k-1)$ in a self-complementary graph G of order $4k+1$ is denoted by $\hat{F}(G)$.

Definition 1.12 A connected graph without cut-vertices is *2-connected.* A graph G is *dense* if it is triangle positive, 2-connected and of diameter two. A graph which is not 2-connected is *separable.*

Definition 1.13 A graph G is *vertex triangle regular* (*VTR*) (*edge triangular regular* (*ETR*)) if all of its vertices (edges) have the same triangle number. In this case the common triangle number $t(u)$ ($t(e)$) is called the *vertex* (*edge*) *triangle number* of the *VTR* (*ETR*) graph G. A graph is *strongly vertex triangle regular* (*SVTR*) (*strongly edge triangle regular* (*SETR*)) if it is regular and *VTR* (*ETR*).

Definition 1.14 For a given positive integer p, let a_1, a_2, ..., a_k be a sequence of integers where $0 < a_1 < a_2 < \cdots \cdots < a_k < \frac{p+1}{2}$. Then the *circulant graph* $C(p;\ a_1, a_2, \cdots \cdots \cdots, a_k)$ is the graph on p vertices $u_0, u_1, \cdots \cdots \cdots, u_{p-1}$ with vertex u_i adjacent to each vertex $u_{i \pm a_j (\mathrm{mod}\, p)}$. The values a_i are called *jump sizes.*

Definition 1.15 An *isomorphic factorization* of a graph *G* is a partition of *G* into edge-disjoint isomorphic spanning subgraphs. A *graph G is divisible* by m if it can be factored into m isomorphic graphs and is denoted by m/G. The set of graphs which occur as factors in isomorphic factorizations of a graph *G* into exactly m factors is denoted by G/m. If *H* is a member of G/m, we write H/G. An isomorphism that maps between the factors in an isomorphic factorization is called *factorizing permutation.*

1.2 BACKGROUND OF THE WORK

Neighbourhood graphs were introduced and characterized by Acharya and Varthak [11]. The *neighbourhood graph* of a graph *G* is the graph having the same vertex set as *G* with an edge joining two vertices if and only if they have a common neighbour in *G*. A graph *H* is said to be a *neighbourhood graph* if there is a graph *G* such that *H* is isomorphic to the neighbourhood graph *N*(*G*) of *G*. These graphs have also been studied under the name of *2-step graphs* by Exoo and Harary [29] and Greenberg et al. [32]. Brigham and Dutton [15] analyzed these further and studied the class of graphs G for which $N(G) \cong K_p$, $N(G) \cong G$ and $N(G) \cong \overline{G}$. The following theorem is of interest to us.

Theorem 1.1 ([15]) The following are equivalent for a graph G of order $p \geq 3$.

(a) $N(G) \cong K_p$

(b) $diam(G) \leq 2$ and every edge of G is in a triangle

and (c) $\gamma(\overline{G}) \geq 3$. □

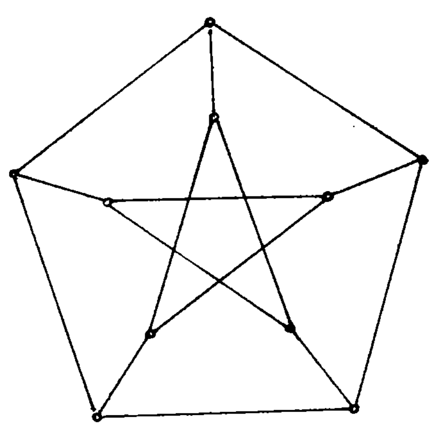
In [40], Koh and Sauer have defined a *dense graph* as a 2-connected graph with diameter less than or equal to two in which every edge is in a triangle. From theorem 1.1 and the definition of dense graphs, it follows that $N(G) \cong K_p$ for every dense graph G. But this condition is not sufficient for G to be dense, since there exist *non-dense graphs* G with $N(G) \cong K_p$. We have explored this class of graphs in [44]. Here we have renamed such graphs as *S-graphs* to avoid any possible confusion with an already existing concept of *F-graphs* [19].

Aravamudhan and Rajendran [12] have introduced the concept of antipodal graphs and characterized them. *Antipodal graph* of a graph G is the graph $A(G)$ having the same vertex set as G with an edge joining two vertices if and only if the distance between them in G is the diameter of G. A graph is *antipodal* if it is the antipodal graph of some graph. They obtained the conditions for $A(G) = G$, $A(G) = \overline{G}$, etc. and proved that $A(G) \subseteq \overline{G}$ for any graph $G \not\cong K_p$. These are also referred in the survey [16]. Acharya and Acharya [10] have studied self-antipodal graphs.

A graph G is said to be *self-complementary* if it is isomorphic to its complement. While proving some results on mean distances in self-complementary graphs of diameter three, Hendry [38] considered the graph G^*, whose vertices are those of G with eccentricity three and an edge joins two vertices of G^* if and only if the distance between them in G is three. He proved that G^* is bipartite.

All these developments have motivated us to generalize the definition of G^* to any graph G of diameter d and call it the *S-antipodal graph* $A^*(G)$ of G.

A graph G of order p is said to be *strongly regular* with parameters (p,r,λ,μ) if it is regular of degree r, any two adjacent vertices have precisely λ common neighbors and any two non-adjacent vertices have precisely μ common neighbors.



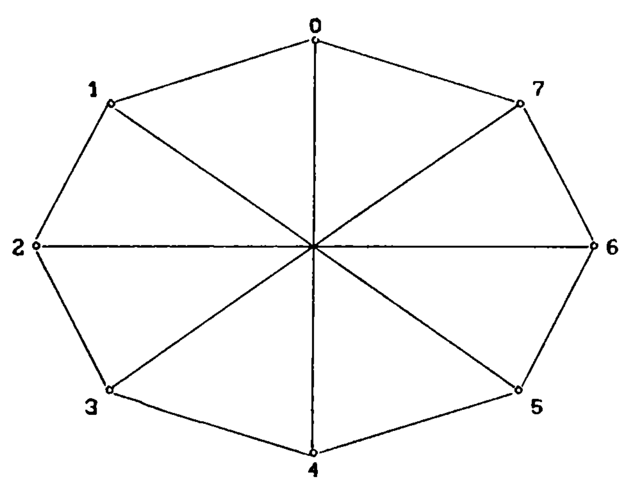
A strongly regular graph

Figure 1.1

This class of graphs is closely related to partial geometries and symmetric designs, and was studied extensively by many authors like Bose [14], Cameron [18], and Hubalt [39].

Lemma 1.2 ([18]) If G is strongly regular with parameters (p,r,λ,μ) then $\overline{G}$ is also strongly regular with parameters $(p, p-r-1, p-2r+\mu-2, p-2r+\lambda)$. □

A graph is said to be *vertex-symmetric* if every pair of its vertices are similar. This class of graphs had been studied under the name of 'vertex-transitive graphs' also. *Edge-symmetric* graphs are also defined in similar terms. *Circulant* graphs are a special type of vertex-symmetric graphs. A circulant graph is given fig. 1.2.



The circulant graph C(8; 1, 4)

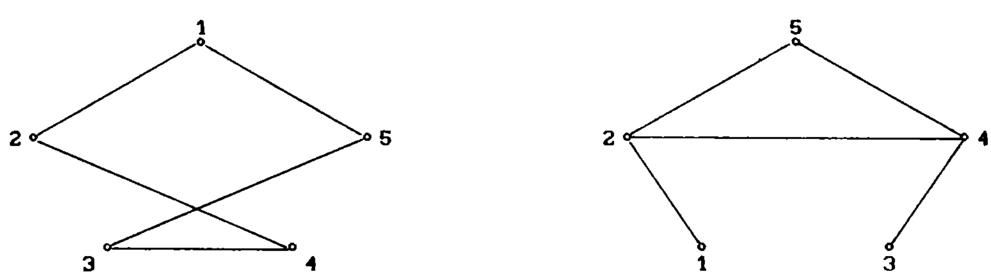
figure 1.2

Self-complementary graphs were introduced and its basic properties were studied independently and simultaneously by Ringel [56] and Sachs [62]. An immediate consequence of the definition is that, self-complementary graphs are of order p = 4k or p = 4k+1 for some natural number k. Also, self-complementary graphs exists for all such integral orders. The order of a regular self-complementary graph is 4k+1 and it exists for every such integral orders. Problems concerning the degree sequences, hamiltonicity, factorization, length of cycles and chains of self-complementary graphs were studied by Camion [20], Clapham [22, 23, 25], Rao [48, 50, 51, 52, 53], etc.

Clapham [24] introduced the concept of graphs *self-complementary in* K_n-e. They exist for orders p = 4k+2 and 4k+3, that is for which self-complementary graphs do not exist. These were independently studied by Das [27] in the name of *almost self-complementary graphs*. Enumeration of self-complementary graphs has been carried out by Read [55] and an asymptotic formula for the number of self-complementary graphs was given by Palmer [47] using Polya's enumeration theorem.

Self-centered self-complementary graphs [17], regular self-complementary graphs [37, 54], vertex-symmetric self-complementary graphs [54, 64] and strongly regular self-complementary graphs [26, 54, 57, 59] are also interesting. It is to be noted that the strongly regular self-complementary (*SRSC*) graphs coincide with a class of graphs investigated by us, namely strongly edge triangle regular self-complementary (SETRSC) graphs.

The diameter of a self-complementary graph is two or three and that of a regular self-complementary graph is two. A regular self-complementary graph will be self-centered also. If G is self-complementary, isomorphisms between G and $\bar{G}$ are nothing but permutations of $V(G)$ and are called *complementing permutations* of G. $\mathcal{C}(G)$ denotes the set of all such permutations. Ringel [56] and Sachs [62] proved that the length of a cycle of a complementing permutation is a multiple of four except exactly one of unit length when p is odd. A self-complementary graph may have more than one complementing permutation and non-isomorphic self-complementary graphs may have same complementing permutation (see fig. 1.3)



(1234)(5) (2453)(1)   (1234)(5) (2453)(1)

Self-complementary graphs and complementing permutations

figure 1.3

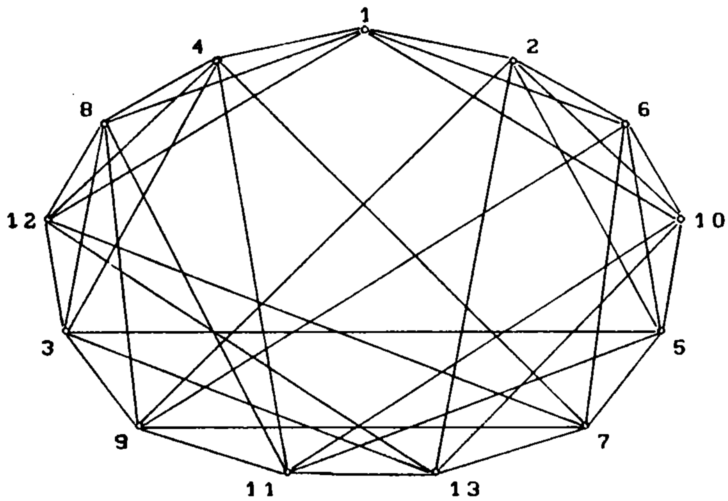
If there are more than one complementing permutation for a self-complementary graph, then the cycle structure of them need not be the same. In this connection, Kotzig asked (*problem 2,* [41]) " Is it true that, *for every regular self-complementary* *graph G, there is at least one complementing permutation σ such that, except for the cycle of length one, every cycle of σ is of length exactly four ?* ". Hartsfield [37] answered it negatively by giving a regular self-complementary graph (fig. 1.4) each of whose complementing permutation include a cycle of length eight.

A vertex u in a self-complementary graph is said to be *fixed-vertex* if $\sigma(u) = u$ for some complementing permutation σ. Ringel and Sachs proved that for each $\sigma \in \mathcal{C}(G)$, there exists a unique fixed vertex if G is of order $p = 4k+1$ and none if G is of order 4k. Three sets $F(G)$, $\hat{F}(G)$ and $Z(G)$ defined in connection with a regular self-complementary graph of order $p = 4k+1$ are:

$$F(G) = \{ u \in V(G) \ / \ \exists \ \sigma \in \mathcal{C}(G) \text{ such that } \sigma(u) = u \},$$

$$\hat{F}(G) = \{ u \in V(G) \ / \ t(u) = k(k-1) \}$$

and $$Z(G) = \{ uv \in E(G) \ / \ \exists \ \sigma \in \mathcal{C}(G) \text{ such that } \sigma(u) = v \}$$



A self-complementary graph whose complementing permutations always include a cycle of length eight
σ = (1) (2 3 4 5) (6 7 8 9 10 11 12 13)

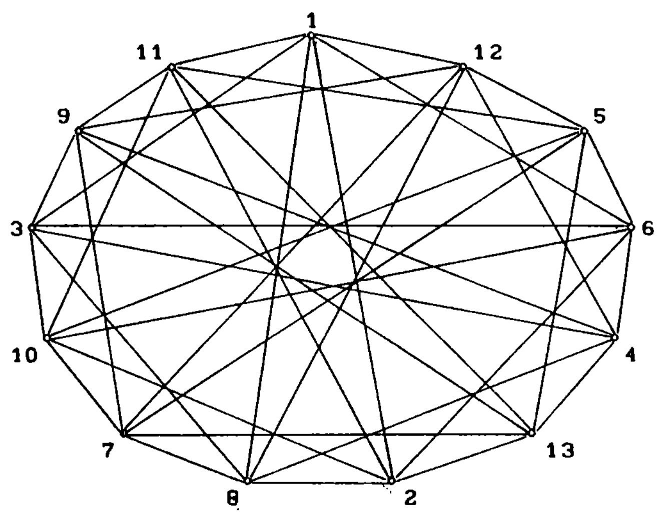
figure 1.4

Kotzig [41] observed that $F(G) \leq \hat{F}(G)$ for every regular self-complementary graph and conjectured that:

(1) *A self-complementary graph is strongly regular if and only if* $F(G) = V(G)$ *and* $Z(G) = E(G)$.

(2) $F(G) = \hat{F}(G)$ *for every regular self-complementary graph.*

Ruiz [59] has disproved the first conjecture by giving a regular self-complementary graph G (fig. 1.5) with $F(G) = V(G)$ and $Z(G) = E(G)$ but is not strongly regular. Rao [54] also has independently disproved the same. Along with the first, he has disproved the second by constructing counterexamples. We observed some mistakes in these. Rao has characterized the set $F(G)$ also.



A self-complementary graph with $F(G) = V(G)$ and $Z(G) = E(G)$
but not strongly regular
$\sigma = (1\ 2\ 3\ 4)\ (5\ 6\ 7\ 8)\ (9\ 10\ 11\ 12)\ (13)$

figure 1.5

Hence, to study more on this conjecture, we have first considered the idea of *triangle number*, the number of triangles in a graph containing a vertex or an edge.

The first result, in our belief, on the triangle number is the following.

Theorem 1.3 (Goodman [31]) For any graph G of order p,

$$t(G) + t(\overline{G}) \geq \begin{cases} \frac{k(k-1)(k-2)}{3} & \text{when } p = 2k \\ \frac{2k(k-1)(4k+1)}{3} & \text{when } p = 4k+1 \\ \frac{2k(k+1)(4k+1)}{3} & \text{when } p = 4k+3 \end{cases}$$

for some natural number k, where $t(G)$ and $t(\overline{G})$ denote the number triangles in G and $\overline{G}$ respectively □

Some other significant results in this direction are:

Theorem 1.4 (Lorden [42]) For any graph G on p vertices

$t(G) + t(\overline{G}) = \binom{p}{3} - \frac{1}{2} \sum_{u \in V(G)} d(u)(p-d(u)-1)$, where $d(u)$ is the degree of u. □

Theorem 1.5 (Clapham [21]) If G is a self-complementary graph of order p, then

$$t(G) \geq \begin{cases} \frac{2k(k-1)(2k-1)}{3} & \text{when } p = 4k \\ \frac{k(k-1)(4k+1)}{3} & \text{when } p = 4k+1 \end{cases}$$ □

Theorem 1.6 (Rao [49]) If G is a self-complementary graph of order p, then

$$t(G) \leq \begin{cases} \dfrac{k(k-1)(8k-1)}{3} & \text{when} \quad p = 4k \\ \binom{2k}{2} + \dfrac{k(k-1)(8k-1)}{3} & \text{when} \quad p = 4k+1. \end{cases}$$
□

Two other interesting results concerning the range of number of triangles in self-complementary graphs given in [49] are:

Theorem 1.7 Let t be an integer. There is a self-complementary graph G of order 4k with $t(G) = t$ if and only if t is even and $\frac{2}{3}\, k(k-1)(2k-1) \leq t \leq \frac{1}{3}\, k(k-1)(8k-1)$. □

Theorem 1.8 Let t be an integer. There is a self-complementary graph G of order 4k+1 with $t(G) = t$ if and only if $\frac{1}{3}\, k(k-1)(4k+1) \leq t \leq \binom{2k}{2} + \frac{1}{3}\, k(k-1)(8k-1)$ unless k = 2 and $t \in \{ 9, 12, 13 \}$ or k = 3 and $t \in \{ 33, 41, 49, 54, 57 \}$. □

Even though the work is not in our lines, it seems worth mention the concept of *triangle graphs* introduced by Egawa and Ramos [28]. They defined the *triangle graph* $R(G)$ of a graph G as the graph whose vertices are the triangles in G and two vertices are adjacent if the corresponding triangles have a common edge in G.

The concept of G-join was introduced by Sabidussi [61] and was also studied by Ruiz [58]. The following theorem of Ruiz is interesting.

Theorem 1.9 Let G be a self-complementary graph with complementing permutation σ and let $\mathcal{F} = \{ H_u \,/\, u \in V(G) \}$ be a family of graphs such that $H_{\sigma(u)} = \bar{H}_u$ for all $u \in V(G)$. Then the G-join $G(\mathcal{F})$ is also self-complementary. □

Remark 1.10 It is to be noted that the G-join is not only a generalization of composition, introduced by Harary [34], but also of the join [4, 8] and sequential join [8] of graphs.

If G is a self-complementary graph of order p, then G and $\bar{G}$ form a factorization of K_p into two isomorphic factors. Harary, Robinson and Wormald [35] investigated the existences of isomorphic factorization of K_p into $m \geq 2$ factors. To them, a *graph G is divisible by m* if it can be factored into m isomorphic factors. They have proved the following theorems.

Theorem 1.11 If m divides $\frac{p(p-1)}{2}$ and $(m,p) = 1$ or $(m,p-1) = 1$, then K_p is divisible by m, where (m,p) denotes the g.c.d. of the integers m and p. □

Theorem 1.12 (Divisibility theorem) The complete graph K_p is divisible by m if and only if m divides $\frac{p(p-1)}{2}$. □

Study of isomorphic factorization in which the factors have certain prescribed properties have also been attempted by many authors. These include, isomorphic factorization into factors with given diameter, isomorphic factorization of K_p where each factor is regular of degree two etc. The details of such work are in [35] and [36].

We have thus given a survey of results relevant to the work reported in this thesis. The definitions and results given in this thesis are either generalizations, byproducts or have been motivated by earlier results, especially on dense graphs, antipodal graphs, self-complementary graphs, circulants, strongly regular graphs and vertex-symmetric graphs.

1.3 GIST OF THE THESIS

The thesis consists of five chapters including this introductory chapter.

In the second chapter, we first discuss S-graphs followed by S-antipodal graphs and S-antipodal graphs of S-graphs and trees. Some of the results in this chapter are:

1) Let G be a connected graph of order $p \geq 5$ then the following are equivalent.

(a) G is an S-graph

(b) G has exactly one cut vertex and is adjacent to all other vertices and no block of G is isomorphic to K_2 .

(c) G is an S-join of a family of graphs.

2) If G is an S-graph, then the S-antipodal graph $A^*(G)$ is self-centered with diameter two and hence $A^*(A^*(G)) = \overline{A^*(G)}$.

3) Let G be an S-graph of order p with k blocks. Then the following are equivalent.

(a) $A^*(G)$ is dense

(b) $N(G) \cong K_{p-1}$

(c) either $k \geq 3$ or there is a block B in G such that each vertex in $B\backslash v$ has degree at most $|V(B)| - 2$, where v is the cut vertex of G.

4) Every graph without isolated vertices is the S-antipodal graph of a hamiltonian graph and every eulerian graph of even order is the S-antipodal graph of an eulerian graph.

5) Characterizations of S-antipodal graphs of S-graphs and trees.

In the third chapter, we derive expressions for the triangle number of a vertex in a graph, for vertices and edges under some graph operations and introduce the concepts of strongly vertex triangle regular graph and strongly edge triangle regular graph. We also deduce some results of Clapham [21], Kotzig [41], Lorden [42], Rao [54], Rosenberg [57] and the well-known relationship between the parameters of a strongly regular graph. Some of the results proved in this chapter are :

(6) $$t(u) + \bar{t}(u) = \binom{p-d(u)-1}{2} - q + \sum_{v \in N(u)} d(u)$$

for any vertex u in a (p,q) graph.

(7) The triangle number of (u,v) in the composition $G(H)$ of the graphs $G(p_1,q_1)$ and $H(p_2,q_2)$ is given by

$$t(u,v) = q_2 d(u) + p_2 d(u) d(v) + p_2^2 \, t(u)$$

(8) For any edge e joining (u_1,v_1) and (u_2,v_2) in $G(H)$

$$t(e) = \begin{cases} t(e_1) + d(v_1) + d(v_2) & \text{when } u_1u_2 \in E(G),\ e_1 = u_1u_2 \\ t(e_2) + p_2 d(u_1) & \text{when } u_1 = u_2,\ v_1v_2 \in E(H),\ e_2 = v_1v_2 \end{cases}$$

(9) G is strongly regular if and only if both G and $\bar{G}$ are strongly edge triangle regular.

(10) A self-complementary graph is strongly edge triangle regular if and only if it is strongly regular.

In the fourth chapter, we restrict our analysis to self-complementary graphs to initiate the discussion of a conjecture of Kotzig, namely $F(G) = \hat{F}(G)$ for a regular self-complementary graph of order p, which is trivially true for p = 5. Rao has given counterexample to this in [54]. But, we have observed in [45] that the argument works only for p = 9 and hence the conjecture was made open for p = 4k+1, k ≥ 3. Attempts in pursuance of this conjecture are mentioned in this chapter. Some of the observations included in the chapter are:

(11) $\hat{F}(G) = \{ u \in V(G) / t(u) = \bar{t}(u) \}$ and hence $\hat{F}(G) = \hat{F}(\bar{G})$ for every regular self-complementary graph G.

(12) Composition of vertex symmetric self-complementary graphs with strongly vertex triangle regular self-complementary graphs which are not vertex-symmetric results in latter type of graphs.

(13) Strongly vertex triangle regular self-complementary graphs which are not vertex-symmetric are counterexamples to the conjecture of Kotzig.

(14) Graphs of the type stated in (12), of order p exists for p = 17 and p = 33 also and hence for $p = 9^{\alpha}17^{\beta}33^{\gamma}p_1^{\delta}$ where α, β, γ and δ are integers with at least one of the first three is non-zero and p_1 is such that vertex-symmetric self-complementary graphs of order p_1 exist.

The main aim in the last chapter is to extend a construction of self-complementary graphs given by Gibbs [30] to obtain a construction of the factors in an isomorphic factorization of complete graphs into more than two factors and thereby obtain a simpler proof of a theorem by Harary et al. [35]. The chapter ends with a concluding remark and suggestions for further study.

As remarked earlier, ' triangles ' and ' complementation ' which are the main characters of the thesis and old heroes of many branches of graph theory have been brought again to a common stage. We sincerely believe that reasonable success has been achieved in this attempt.

*

CHAPTER

II

2

S-GRAPHS AND S-ANTIPODAL GRAPHS

In this chapter we study a class of non-dense graphs called *S*-graphs, the concept of *S*-antipodal graphs and the *S*-antipodal graphs of *S*-graphs and trees. Some results of this chapter are in [44]

2.1 S-GRAPHS

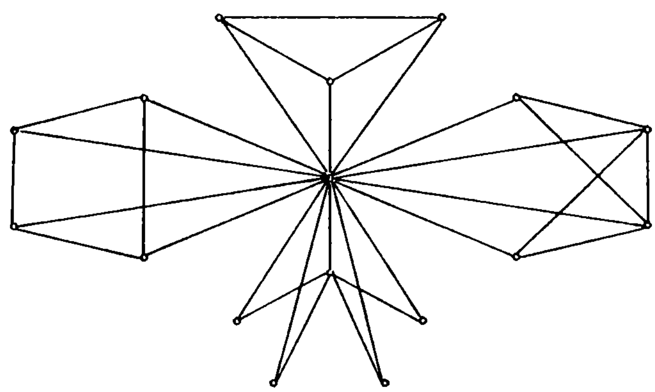
Let *G* be a graph. A vertex *u* of *G* is *triangle positive* if its triangle number is non-zero. *Triangle positive edge* is similarly defined. If *G* is without isolated vertices and every edge is triangle positive, then every vertex will also be triangle positive.

A graph in which every edge is triangle positive is called a *triangle positive graph.*

Consider a graph *G* and its neighborhood graph *N*(*G*). Brigham and Dutton [15] have proved theorem 1.1, part of which in our terminology can be stated as "The neighborhood graph *N*(*G*) of a graph *G* of order $p \geq 3$ is K_p if and only if diam(*G*) ≤ 2 and

G is triangle positive". In [40], Koh and Sauer have defined dense graphs which, using our terms, reads as "a graph G of order $p \geq 3$ is *dense* if G is 2-connected, triangle positive and $diam(G) \leq 2$". It follows that the neighborhood graph of any dense graph of order p is K_p. But, there are non-dense graphs G of order p such that $N(G) \cong K_p$. Obviously, these are the connected separable triangle positive graphs whose diameter is at most two. In fact, the diameter of such graphs will always be two, since $p(G) \geq 3$.

A graph G is an *S-graph* if it is separable, triangle positive and $diam(G) = 2$.



An S-graph

figure 2.1

Lemma 2.1 An S-graph G has exactly one cut-vertex and it is adjacent to all other vertices.

Proof: Let G be an S-graph. Existence of a cut-vertex follows from the definition. If G has more than one cut-vertex, we could find a block B of G containing two or more cut-vertices of G. Let u and v be two distinct cut-vertices in the block B and let A and C be two distinct blocks of G such that $u \in V(A)$ and $v \in V(C)$. Consider any $x \in V(A \backslash u)$ and $y \in V(C \backslash v)$. Then each path joining x and y contains u and v. So $d(x,y) \geq 3$ which is impossible. Hence G has exactly one cut-vertex.

Now, let v be the cut-vertex of G and u be a vertex not adjacent to v. Then $d(u,v) \geq 2$. Let B be the block of G containing u and let $w \in V(G \backslash B)$. Then every path joining u and w contains the vertex v and hence $d(u,w) \geq 3$. But $diam(G) = 2$. ■

Remark 2.2: By definition, an S-graph doesn't have a block isomorphic to K_2 and hence these graphs have at least five vertices.

Recall that, the *S-join* (*star join*) $G = S(\mathcal{F})$ of a family $\mathcal{F}$ of p-1 non-trivial graphs obtained by replacing each pendent vertex u_i of the star $K_{1,p}$ by $G_i \in \mathcal{F}$ where $\mathcal{F} = \{ G_1, G_2, G_3, \cdots\cdots\cdots, G_p \}$ is a family of p non-trivial connected graphs and joining each vertex of G_i to the central vertex u_0 of the star.

Theorem 2.3 Let G be a connected graph of order $p \geq 5$. Then the following are equivalent.

(a) G is an S-graph.

(b) G has exactly one cut-vertex which is adjacent to all other vertices.

and (c) G is the S-join of a family of two or more non-trivial connected graphs.

Proof: Consider a connected graph G of order $p \geq 5$.

(a) ⇒ (b) By lemma 2.1.

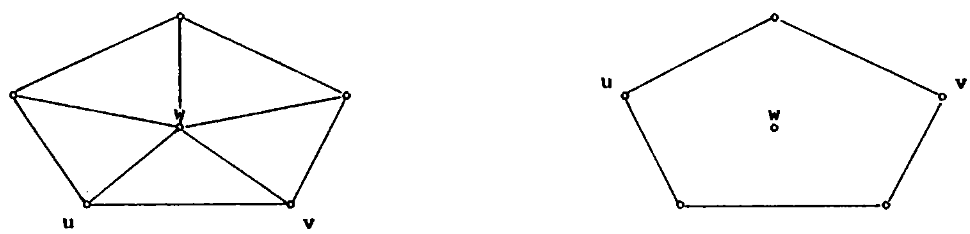
(b) ⇒ (c) Let v be the cut-vertex and consider the family $\mathcal{F}$ of the components of $G\backslash v$. Since G has no member isomorphic to K_2, no member of $\mathcal{F}$ is trivial and obviously G is the S-join of $\mathcal{F}$.

(c) ⇒ (a) Let G be the S-join of a family $\mathcal{F}$ of two or more non-trivial connected graphs and v be the central vertex of the star. By definition of S-join, $ecc(v) = 1$, v is a common neighbor to any pair of vertices u and u' other than v and there exist at least one pair of non-adjacent vertices in G. Hence $diam(G) = 2$. Obviously, v is a cut vertex in G. Hence G is separable. Since none of the members in $\mathcal{F}$ is trivial, for every edge e containing v there exists a vertex u which forms a triangle with e. For each edge not containing v, v is a common neighbor to its end vertices. So $t(e)$ is non-zero for every edge e in G. Hence G is an S-graph. ■

Lemma 2.4 If G is an S-graph, then the domination number of $\overline{G}$ is three.

Proof: Let G be an S-graph and v be its cut-vertex. Then $\gamma(\overline{G}) \geq 3$ by theorem 1.1. Since the cut vertex of G is adjacent to all other vertices, it will be an isolated vertex in $\overline{G}$. So it is in any dominating set of $\overline{G}$. Now, consider two distinct blocks A and B of G and let $u \in V(A\backslash v)$ and $w \in V(B\backslash v)$. Then, in $\overline{G}$, each vertex in $V(G\backslash A)$ will be adjacent to u and each vertex in $V(G\backslash B)$ will be adjacent to w . So $\{ u, v, w \}$ form a dominating set of $\overline{G}$. Thus $\gamma(\overline{G}) = 3$. ■

It is clear that S-graphs are extremal for the inequality in theorem 1.1. Then the following question arises: ' Are S-graphs the only graphs satisfying both $N(G) \cong K_p$ and $\gamma(\overline{G}) = 3$? ' The answer is negative, as there are other graphs H with $\gamma(\overline{H}) = 3$. One such graph is the wheel $H = W_6 = K_1 + C_5$ given in fig. 2.2. For it $\overline{H} = K_1 \cup C_5$ and $\gamma(\overline{H}) = 3$.



H $\overline{H}$

the wheel on six vertices and its complement

figure 2.2

Remark 2.5: It is to be noted that the friendship graphs (fig. [illegible]) are the simplest S-graphs.

2.2 S-ANTIPODAL GRAPHS

S-antipodal graph of a graph G is the graph $A^*(G)$ whose vertices are those of G with maximum eccentricity and two vertices are adjacent if their distance in G is maximum. A graph G is S-antipodal if it is the S-antipodal graph of some graph H.

Remark 2.6 $A^*(G)$ may be disconnected.

Lemma 2.7 Let G be any graph, $A(G)$ be its antipodal graph and $A^*(G)$ its S-antipodal graph. Then

(a) $E(A^*(G)) = E(A(G))$.

(b) $V(A^*(G)) = V(G)$ if and only if G is self-centered.

(c) $A^*(G) = A(G)$ if and only if G is self-centered.

Proof: (a) $uv \in E(A^*(G)) \Rightarrow d_G(u,v) = \mathrm{diam}(G)$

$\Rightarrow uv \in E(A(G))$

Conversely,

$$uv \in E(A(G)) \Rightarrow d_G(u,v) = \mathrm{diam}(G)$$

$$\Rightarrow \begin{cases} \mathrm{ecc}_G(u) = \mathrm{ecc}_G(v) = \mathrm{diam}(G) \\ \text{and } d_G(u\ v) = \mathrm{diam}\ G) \end{cases}$$

$$\Rightarrow u,\ v \in V(A^*(G)) \text{ and } uv \in E(A^*(G)$$

(b) $V(A^*(G)) = V(G) \Rightarrow \mathrm{ecc}_G(u) = \mathrm{diam}(G)$ for every $u \in V(G)$

$\Rightarrow G$ is self-centered.

Conversely,

G is self-centered $\Rightarrow \mathrm{ecc}_G(u) = \mathrm{diam}(G)$ for every $u \in V(G)$

$\Rightarrow u \in V(A^*(G))$ for every $u \in V(G)$

$\Rightarrow V(A^*(G)) = V(G)$

(c) Follows from (a), (b) and the fact that $V(A^*(G)) = V(G)$.

■

Lemma 2.8 If G is K_p or $\overline{K}_p$, then $A^*(G) \cong K_p$.

Proof: In both cases, G is self-centered and hence $V(A^*(G)) = V(G)$. When $G \cong K_p$, $\mathrm{diam}(G) = 1$ and each pair of vertices are at a distance of one. So $A^*(G) \cong K_p$. When $G \cong \overline{K}_p$, $\text{diam}(G) = \infty$ and hence $d(u,v) = \infty$ for every pair of vertices. So $A^*(G) \cong K_p$. ■

Lemma 2.9 Let G be a connected graph of diameter d. Then $E(A^*(G)) = E(\overline{G^{d-1}})$ if and only if $d \geq 2$.

Proof: Let G be a connected graph of diameter d.

When $d = 1$, $A^*(G) = G$ by lemma 2.8. Thus the condition is necessary.

Conversely, let $\text{diam}(G) \geq 2$ and $uv \in E(A^*(G))$. Then $ecc_G(u) = ecc_G(v) = d_G(u,v) = d$. Hence u and v are not adjacent in G^{d-1} by its definition. So $uv \in E(\overline{G^{d-1}})$. On the other hand if $uv \in E(\overline{G^{d-1}})$, then $d_G(u,v) \geq d$. But $d_G(u,v) = d$. Hence $ecc_G(u) = ecc_G(v) = d$. Thus $u,v \in V(A^*(G))$ and $uv \in E(A^*(G))$. ■

Theorem 2.10 $E(A^*(G)) \subseteq E(\overline{G})$ if and only if $\text{diam}(G) \geq 2$ and equality holds if and only if either $\text{diam}(G) = 2$ or G is disconnected and every component of it is complete.

Proof: $E(\overline{G^{d-1}}) \subseteq E(\overline{G})$ since $E(G) \subseteq E(G^{d-1})$. Hence by lemma 2.9, $E(A^*(G)) \subseteq E(\overline{G})$ if $\text{diam}(G) \geq 2$.

Conversely, if $\text{diam}(G) = 1$, then $E(A^*(G)) = E(G)$. Hence $\text{diam}(G) \geq 2$ is necessary.

For the equality, the necessity of $\text{diam}(G) \geq 2$ follows from lemma 2.8.

Now, let G be a connected graph of diameter $d \geq 3$. Then there exist at least one pair $\{ u, v \}$ of vertices in G with $d_G(u,v) = 2$. These vertices will not be adjacent in $A^*(G)$

even if they are in $V(A^{*}(G))$. But they will be adjacent in $\overline{G}$. Thus $E(A^{*}(G))$ does not contain $E(\overline{G})$ when G is connected and $\text{diam}(G) \geq 3$.

Let G be disconnected and $\{u, v\}$ be a pair of non-adjacent vertices in one of the components. Then $d_G(u,v) < \infty$ while $\text{diam}(G) = \infty$. So $uv \notin E(A^{*}(G))$. But $uv \in E(\overline{G})$. Hence $E(A^{*}(G))$ does not contains $E(\overline{G})$ if there is a component of G which is not complete. Hence the conditions for equality. ■

Since $A^{*}(G)$ and $A(G)$ are same if G is self-centered, we can deduce the following properties of $A^{*}(G)$ from that of $A(G)$ given by Aravamudhan and Rajendran [12].

Property 2.11 $A^{*}(G)$ is complete k-partite if G is disconnected with k components. □

Property 2.12 $A^{*}(G) = G$ if and only if $G \cong K_p$. □

Property 2.13 $A^{*}(G) = \overline{G}$ if and only if G is self-centered of diameter 2 or G is disconnected and each component of G is complete. □

Theorem 2.14 $A^{*}(G) \cong G$ if either $G \cong K_p$, an odd cycle or a self-complementary graph of diameter two.

Proof: We have seen that $A^{*}(G) = G$ if $G \cong K_p$.

Now, let G be an odd cycle of length 2n+1. Then $\text{diam}(G) = n$. For every vertex u, there are exactly two vertices at a distance of n from it. Hence the degree of u in $A^{*}(G)$ will be 2 and $A^{*}(G)$ will be connected also. Hence $A^{*}(G)$ will also be a cycle of length 2n+1.

When G is a self-complementary graph of diameter 2, the statement follows from property 2.13, since every such graph is self-centered. ∎

Corollary 2.15 If G is a regular self-complementary graph, then $A^{\star}(G) = \overline{G} \cong G$. □

Theorem 2.16 $A^{\star}(G) = A^{\star}(\overline{G})$ if and only if G is either complete or totally disconnected.

Proof: Let $A^{\star}(G) = A^{\star}(\overline{G})$. Then exactly one of G and $\overline{G}$ is complete. Because otherwise, by theorem 2.10, we have

$$\operatorname{diam}(G) \geq 2 \Rightarrow E(A^{\star}(G)) \subseteq E(\overline{G})$$

$$\operatorname{diam}(\overline{G}) \geq 2 \Rightarrow E(A^{\star}(\overline{G})) \subseteq E(G).$$

But, $E(A^{\star}(G)) = E(A^{\star}(\overline{G}))$ by hypothesis and this set simultaneously belongs to both $E(G)$ and $E(\overline{G})$, which is a contradiction . Thus, either G or $\overline{G}$ is K_p . ∎

Converse is obvious by lemma 2.8.

Corollary 2.17 $A^{\star}(G) \cong A^{\star}(\overline{G})$ if G is either K_p or $\overline{K}_p$ □

Theorem 2.18 Every graph without isolated vertices is the S-antipodal graph of a hamiltonian graph of diameter two.

Proof: Let G be a graph of order p without isolated vertices. Consider the graph $H = \overline{G} + K_p$. Then

(1) diameter of H is two and $A^{\star}(H) = G$

and (2) H is Hamiltonian.

Proof of (1): Since G is without isolated vertices, for every vertex u in G, there is a vertex v in G adjacent to it. So u and v are not adjacent in $\overline{G}$ and hence in H. Thus, for every $u \in V(G)$

there exists a $v \in V(G)$ such that $d_H(u,v) \geq 2$. Hence $ecc_H(u) \geq 2$ for every $u \in V(G)$. But each vertex u' in K_p is a common neighbour to every pair of vertices in H. So $d_H(u,v) \leq 2$ for every pair $u, v \in V(G)$. Every $u' \in V(K_p)$ is adjacent to all other vertices in H. So $ecc_H(u') = 1$ for every $u' \in V(K_p)$. Hence $diam(H) = 2$ and $V(A^*(H)) = V(G)$ and hence $E(A^*(H)) = E(\overline{H})$ by theorem 2.10 and $E(\overline{H}) = E(G)$ by definitions of complement and H. So $A^*(H) = G$.

Proof of (2) Label the vertices of G by 1, 2, ··· ··· ··· ···, p and those of K_p by 1′, 2′, ··· ··· ···, p′; then 11′22′ ··· ··· ··· pp′1 is a spanning cycle of H. ■

Theorem 2.19 A graph is S-antipodal if and only if it has no isolated vertices.

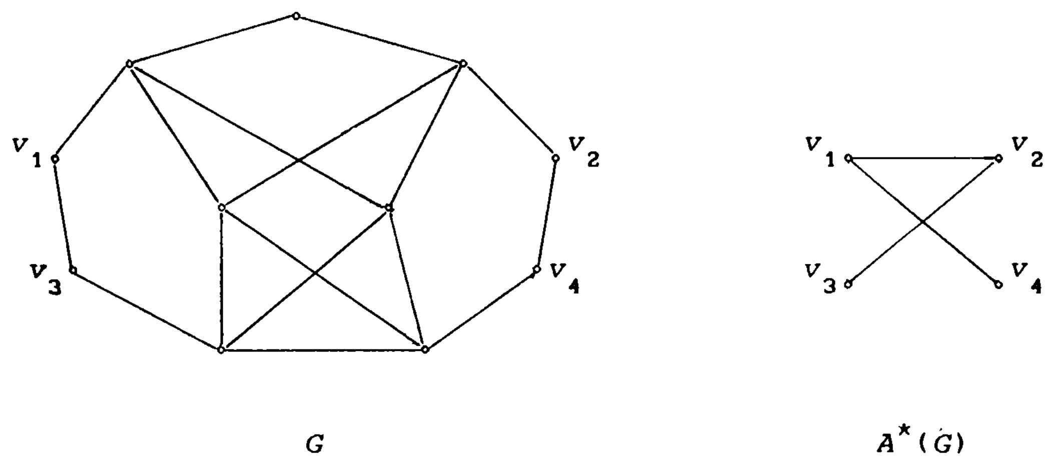
Proof: Let G be an S-antipodal graph and $u \in V(G)$. Then there exists a graph H such that $A^*(H) = G$ and $u \in V(H)$ with $ecc(u) = diam(H)$. So, there should be a vertex $v$in H with $d(u,v) = diam(H)$ and hence. Thus u is not an isolated vertex in G.

Converse follows from theorem 2.18. ■

Theorem 2.20 Every eulerian graph of even order is the S-antipodal graph of an eulerian graph.

Proof: Let G be an eulerian graph of even order p. Being Eulerian, the degree $d_G(u)$ is even for every vertex u. But $d_G(u) + d_{\overline{G}}(u) = p-1$, odd. So $d_{\overline{G}}(u)$ is odd. Consider the graph $H = \overline{G} + K_1$. Clearly, degree of every vertex in H is even and hence H is eulerian. Now, consider a vertex u in G. Since G is eulerian and hence connected, there will be a vertex v in G. Such that u and v are adjacent in G and hence $d_H(u,v) \geq 2$. For every $w \in V(G)$ the vertex of K_1, say θ, is a common neighbour to u and w in H. So $d_H(u,w) \leq 2$ and $d(\theta,w) = 1$ for every $w \in V(G)$. Hence Hence $ecc_H(u) = 2$ for every $u \in V(G)$ and $ecc_H(\theta) = 1$. Thus $V(A^*(H)) = V(G)$ and $E(A^*(H)) = E(\overline{H}) = E(G)$ by theorem 2.10. ■

Remark 2.21 The S-antipodal graph of an eulerian graph need not be eulerian. The following example illustrates this.



Eulerian graph along with its non-eulerian S-antipodal graph

figure 2.3

2.3 S-ANTIPODAL GRAPH OF S-GRAPHS

Let G be an S-graph and v be its cut-vertex. Then $diam(G) = 2$ and v is adjacent to all other vertices. Hence $ecc(v) = 1$. Because of the separability of G, all other vertices are of eccentricity two. So $V(A^*(G)) = V(G \setminus v)$ and by theorem 2.10, $E(A^*(G)) = E(\overline{G})$. So $A^*(G) = \overline{G}\backslash v$. Here we discuss some properties of $A^*(G)$ and obtain its characterization.

Theorem 2.22 Let G be an S-graph with k blocks and every block of G is complete, then $A^*(G)$ is a complete k-partite graph.

Proof: Let G be an S-graph with each of its blocks are complete. Let v be the cut-vertex, $B_1, B_2, B_3, \cdots\cdots\cdots, B_k$ be the blocks of G and $V_i = V(B_i\backslash v)$; $i = 1, 2, \cdots\cdots\cdots, k$. Then $V_i \cap V_j = \emptyset$ for every i and j, $i \neq j$ and $\bigcup_{i=1}^{k} V_i = V(A^*(G))$. Since each B_i is complete in G, each $\langle V_i \rangle$ is totally disconnected in $\overline{G}\backslash v = A^*(G)$. Obviously, $\overline{G}\backslash v = A^*(G)$, each vertex in V_i is adjacent to all other vertices in V_j for every i and j, $i \neq j$. So $A^*(G)$ is complete k-partite. ∎

Corollary 2.23 If G is an S-graph, then

(a) $A^*(G)$ has a complete k-partite spanning subgraph,

(b) $A^*(G)$ is 2-connected.

Proof: (a) Since the cut vertex v is adjacent to every other vertex in G, $A^*(G) = \overline{G}\backslash v$ has a complete k-partite subgraph since G is a subgraph of an S-graph whose every block is complete.

(b) Follows from (a). ∎

Theorem 2.24 If G is an S-graph, then $A^*(G)$ is self-centered with diameter two.

Proof: Let G be an S-graph and v be its cut-vertex. Then we have $A^*(G) = \overline{G}\backslash v$. To prove that the eccentricity of any vertex in $A^*(G)$ is two. Let $e^*(u)$ and $d^*(u,w)$ respectively denote the eccentricity and distance in $A^*(G)$. Let u be any vertex in $A^*(G)$ and B be the block of G in which u belongs. Then $d^*(u,w) = 1$ for every $w \in V(G\backslash B)$ since $d_G(u,w) = 2$ and u and v being in distinct blocks of G. Also $d^*(u,w) \le 2$ for every $w \in V(B\backslash v)$ since each vertex in $V(G\backslash B)$ is common neighbour to u and w in $A^*(G)$. Hence $e^*(u) \le 2$. Since G has no block isomorphic to K_2, there exist at least one vertex $w \in V(B\backslash v)$ adjacent to u in G. So this w is not adjacent to u in $A^*(G)$ and hence $d^*(u,w) \ge 2$. Thus $e^*(u) \ge 2$. ∎

Theorem 2.25 Let G be an S-graph with cut-vertex v. Every edge of $A^*(G)$ is triangle positive if and only if either G has at least three blocks or there is a block B in G such that every vertex in $B\backslash v$ has degree at most $|V(B)| - 2$ in G.

Proof: Let G be an S-graph and v be its cut-vertex. Then we have $A^*(G) = \overline{G}\backslash v$. Let $t^*(e)$ denote the triangle number of an edge e in $A^*(G)$.

Let G has at least three blocks. If the end vertices of an edge e in $A^*(G)$ are in distinct blocks of G, then every vertex in other blocks is common neighbour to them. So $t^*(e) > 0$. If both ends are in the same block of G, then also vertices in other blocks are common neighbours to them in $A^*(G)$. Thus $t^*(e) > 0$ in this case also.

If there are only two blocks and each vertex in $B\backslash v$ has degree at most $|V(B)| - 2$ in G, then for each $u \in V(B\backslash v)$ there is a vertex w in $V(B\backslash v)$ not adjacent to u in G. Consider an edge e in $A^*(G)$ whose end vertices are in distinct blocks of G. There should be edges in $A^*(G)$ with one end coinciding with that of e, say u, and other end, say w, lying in the block containing u. Then the other end of e is common neighbour to u and w. Thus there is a triangle in $A^*(G)$ containing e, since each vertex in a block of G is adjacent, in $A^*(G)$, to all other vertices in each of the remaining blocks.

Conversely, suppose G has only two blocks B_1 and B_2 and every edge of $A^*(G)$ is triangle positive. Let $u_i \in V(B_i \backslash v)$ be such that degree of u_i in G is $|V(B_i)| - 1$; i = 1, 2. Then the edge u_1u_2 of $A^*(G)$ fails to be in a triangle since none of the vertices in $V(B_i \backslash v)$ is adjacent to u_i in $A^*(G)$, i = 1, 2. Because, for a triangle containing the edge u_1u_2, either u_1 has a neighbour in B_1 or u_2 has a neighbour in B_2. ■

Theorem 2.26 Let G be an S-graph of order p with k blocks. Then the following are equivalent

(a) $A^*(G)$ is dense.

(b) $N(A^*(G)) \cong K_{p-1}$.

and (c) Either $k \geq 3$ or there is a block B in G such that each vertex in $B \backslash v$ has degree at most $|V(B)| - 2$, where v is the cut-vertex of G.

Proof: Let G be an S-graph of order p with k blocks and v be its cut-vertex. We have $A^*(G) = \overline{G} \backslash v$ and $\mathrm{diam}(A^*(G)) = 2$ by theorem 2.24.

(a) ⇒ (b) Follows from the definition of dense graph and theorem 1.1.

(b) ⇒ (c) Follows from theorems 1.1 and 2.25.

(c) ⇒ (a) Follows from corollary 2.23, theorems 2.24 and 2.25

2.4 S-ANTIPODAL GRAPHS OF TREES

Here we discuss the properties of S-antipodal graphs of trees and characterize them. As usual, we call a tree *unicentral* or *bicentral* according as its centre is K_1 or K_2. In the latter case, the edge joining the central vertices is called the *central edge*.

Lemma 2.27 Let T be a unicentral tree homeomorphic to a star having the centre same as that of the star. Then the S-antipodal graph $A^{*}(T)$ of T is a complete graph.

Proof: Let T be a tree satisfying the hypothesis. Then T has at least two longest tails, (by a 'tail' we mean a path whose one end vertex is the centre and the other is a pendent vertex of T), because otherwise, T will be bicentral or have a different centre. Now, the vertices of $A^{*}(T)$ are precisely the pendent vertices of T corresponding to its longest tails and each pair of such vertices are at a distance of diam(T) in T. Thus every pair of vertices are adjacent in $A^{*}(T)$. ∎

Lemma 2.28 The S-antipodal graph of a unicentral tree is either complete or complete multipartite.

Proof: Let T be a tree. If T satisfies the hypothesis of lemma 2.27, then $A^{*}(T)$ is complete. Otherwise, define an equivalence relation on $V(A^{*}(T))$ as: two vertices are equivalent if the path in T joining them does not contain the central vertex of T. Let S_1, S_2, ··· ··· ···, S_k be the equivalence classes. Then $d_T(u,v)$ < diam(T) for u, $v \in S_i$ and $d_T(u,v)$ = diam(T) for

$u \in S_i$ and $v \in S_j$, i $\neq$ j. So the graph $A^*(T)$ is complete k-partite with partite sets S_1, S_2, ……, S_k. ∎

Lemma 2.29 S-antipodal graph of a bicentral tree is complete bipartite.

Proof: Let T be a bicentral tree. Define an equivalence relation on $V(A^*(T))$ as: two vertices are equivalent if the path in T joining them does not contain the central edge of T. Then there are exactly two equivalence classes. For, if possible, consider any three distinct equivalence classes S_1, S_2 and S_3. Let u_i be the vertex in S_i nearest to an end-vertex of the central edge. Clearly u_i's are distinct for i = 1, 2 and 3, being the members of distinct equivalence classes. Then each of the u_i-u_j paths, i, j = 1, 2, 3; i $\neq$ j, in T contains the central edge. So, we can traverse through these paths so that any two of these paths have a common vertex before the central edge is reached. Without loss of generality let us assume that the u_1-u_3 path and the u_2-u_3 path (if these are not the paths, we can achieve this by re-labelling the vertices u_i) have such a common vertex. Then the u_1-u_2 path does not contain the central edge. Thus the number of equivalence classes is at most two. The number of equivalence classes is one only if T has no longest path containing the central edge, but it is not so. Thus, there are exactly two equivalence classes.

In $A^*(G)$, two vertices are adjacent if and only if they are in different equivalence classes, since every longest path of a tree contains the centre. Thus $A^*(G)$ is a complete bipartite graph. ∎

Theorem 2.30 A graph G is the S-antipodal graph of a tree T if and only if G is complete or complete multipartite.

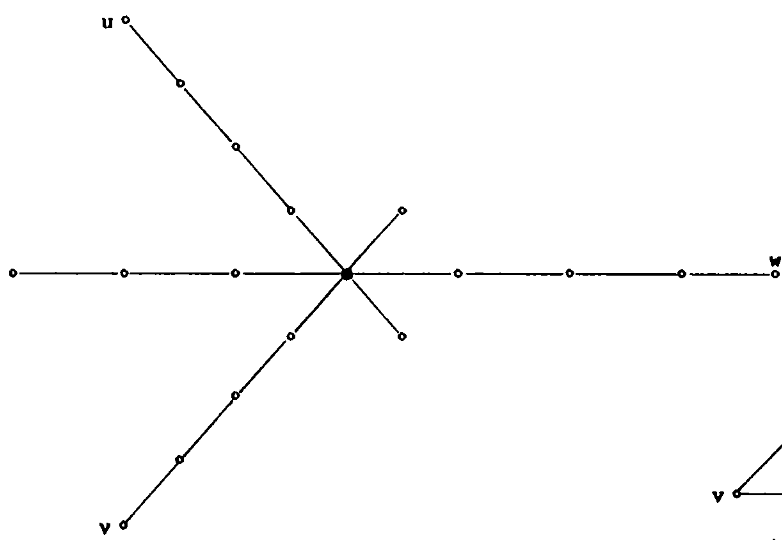
Proof: Necessary part follows from lemmas 2.27, 2.28 and 2.29.

Conversely, Let G be a complete graph of order p. Then $G = A^*(T)$ for a star T of order p+1. Now, let G be a complete k-partite graph and $S_1, S_2, \cdots\cdots, S_k$ be its partite sets. Construct a tree T with $V(T) = \{ u_0, u_1, \cdots\cdots u_k \} \cup (\bigcup_{i=1}^{k} S_i)$ and join each vertex u_i, i = 1, 2, $\cdots\cdots$, k to u_0 and to every vertex in S_i. The center of T is the vertex u_0 and diametral vertices are those in $\bigcup_{i=1}^{k} S_i$. The diameter of T is four and $d(u,v) = 4$ if and only if they belong to different sets S_i. Hence $A^*(T) = G$. ■

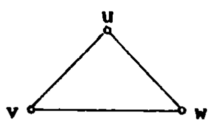
T_1 $A^*(T_1)$

A star and its S-antipodal graph

figure 2.4



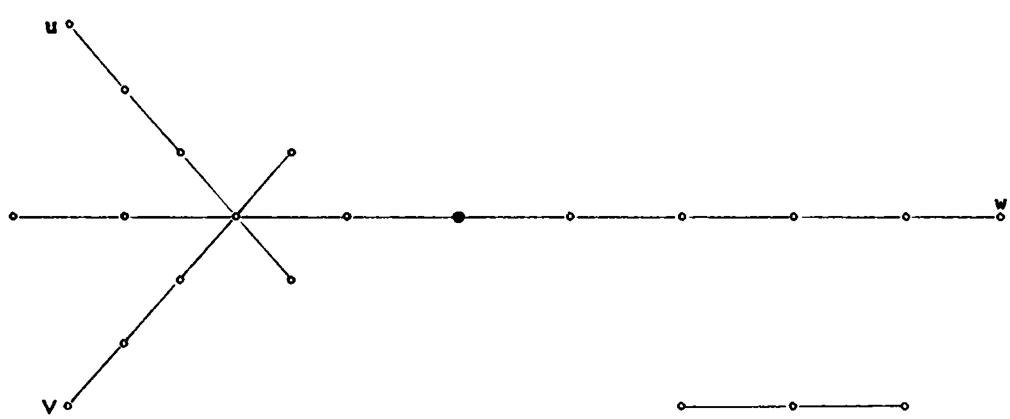
T_4

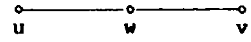


$A^*(T_4)$

A unicentral tree homeomorphic to the star $T_2 = K_{1,6}$ whose centre coincides with that of the star

figure 2.5



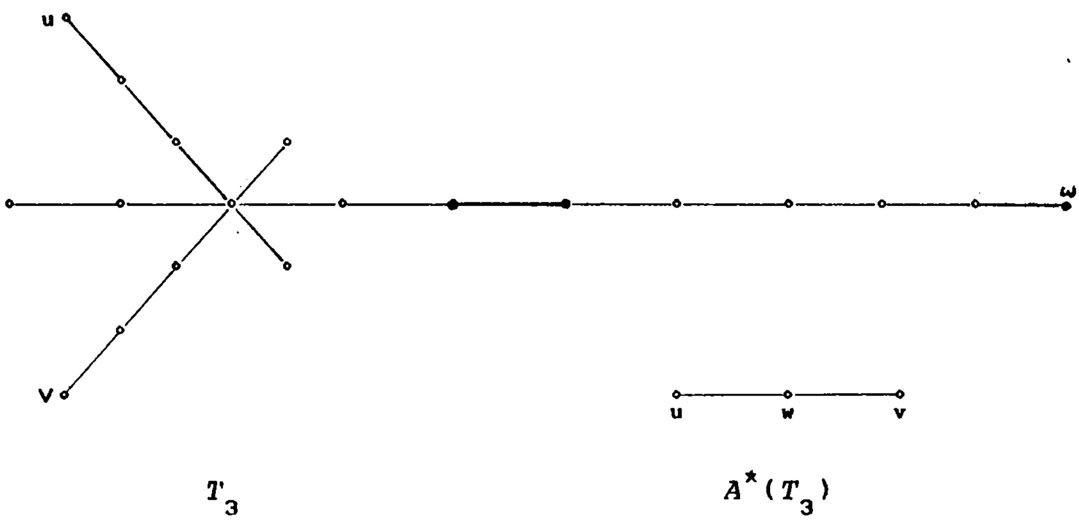


T_2

$A^*(T_2)$

A unicentral tree homeomorphic to the star $T_2 = K_{1,6}$ whose centre does not coincide with that of the star

figure 2.6



A bicentral tree homeomorphic to $K_{1,6}$

figure 2.7

* *

3

TRIANGLE NUMBER AND TRIANGLE REGULARITY

This chapter focuses on the triangle number. An expression for $t(u) + \bar{t}(u)$ is derived and several known results including the well-known relationship between the parameters of a strongly regular graph and some other results are deduced. It is an important observation that the expression for $t(G) + t(\bar{G})$ given by Lorden [42] follows from ours. Expressions for the triangle number of vertices and edges in the join, cartesian product and composition of graphs are also derived. Properties of strongly vertex triangle regular and strongly edge triangle regular graphs are also discussed. Some of the results are in [45] and [46].

3.1 TRIANGLE NUMBER.

The *triangle number* $t(u)$ of a vertex u in a graph G is the number of triangles in G containing u. Triangle number $t(e)$ of an edge e is defined similarly. The number of triangles in G is the *triangle number of G*, denoted by $t(G)$. The triangle number of u in $\bar{G}$ will be denoted by $\bar{t}(u)$. Clearly, for a vertex

u, $t(u)$ is the size of the subgraph of G induced by the neighbourhood $N_G(u)$ of u, $\bar{t}(u)$ is the size of the subgraph of $\bar{G}$ induced by $V(G)\setminus N_G[u]$, that is the number of non edges of G in the subgraph $\langle V(G)\setminus N_G[u]\rangle$ of G and $t(e)$ is the number of common neighbors of the end vertices of e. The following lemma is an immediate consequence of these definitions.

Lemma 3.1 Let G be a graph, then

$$\text{(a)} \quad t(u) = \frac{1}{2} \sum_{e\in E(u)} t(e) \quad , \text{ for any } u \in V(G) \qquad \cdots \cdots \cdots (3.1)$$

and

$$\text{(b)} \quad t(G) = \frac{1}{3} \sum_{u\in V(G)} t(u). \qquad \cdots \cdots \cdots (3.2)$$

Proof: (a) For each triangle containing the vertex u, two of its edges are incident at u. So each triangle contributes two to the sum $\sum_{e\in E(u)} t(e)$. Hence $\sum_{e\in E(u)} t(e) = 2\, t(u)$.

(b) Each triangle in G is counted once at each of its vertices. So $\sum_{u\in V(G)} t(u) = 3\, t(G)$. ■

Along the lines of results by Goodman [31] and Lorden [42], we have:

Theorem 3.2 Let G be a (p,q)-graph and $u \in V(G)$. Then

$$t(u) + \bar{t}(u) = \binom{p-d(u)-1}{2} - q + \sum_{v\in N(u)} d(v) \qquad \cdots \cdots \cdots (3.3)$$

Proof: Let G be a (p,q)-graph and u be a vertex in it with degree $d = d(u)$ and neighbourhood $N = N(u)$. Also, let $\bar{N} = V(G)\setminus N[u]$. Then $|N| = d$, $|\bar{N}| = p-d-1$, $t(u)$ is the number of edges in $\langle N\rangle$ and $\bar{t}(u)$ is the number of non-edges in $\langle\bar{N}\rangle$.

Now let $D = \sum_{v\in N} d(v)$ and $\bar{D} = \sum_{v\in \bar{N}} d(v)$. Then,

$$D + \overline{D} + d = \sum_{v \in N} d(v) + \sum_{v \in \overline{N}} d(v) + d(u) = \sum_{v \in V(G)} d(v) = 2q \quad \cdots \cdots (3.4)$$

The contribution to D by the d edges in G incident at u is d and by the t(u) edges in $\langle N \rangle$ is $2t(u)$. So the number of edges in G with one end in N and the other end in $\overline{N}$ is

$$D - d - 2\, t(u) \quad \cdots \cdots \cdots (3.5)$$

The number of edges in $\langle \overline{N} \rangle$ is $\binom{p-d-1}{2} - \overline{t}(u)$ and the contribution of these edges to $\overline{D}$ is $2\binom{p-d-1}{2} - 2\,\overline{t}(u)$. So the number of edges in G with one end in $\overline{N}$ and other end in N is

$$\overline{D} - 2\binom{p-d-1}{2} + 2\,\overline{t}(u) \quad (3.6)$$

Obviously, the quantities given by (3.5) and (3.6) are equal and consequently we get

$$t(u) + \overline{t}(u) = \frac{1}{2}\left[2\binom{p-d-1}{2} + D - \overline{D} - d \right]$$

$$= \binom{p-d-1}{2} + \frac{1}{2}[\, D - (2q-D)\,] \qquad \text{by } (3.4)$$

$$= \binom{p-d-1}{2} - q + D$$

Thus,

$$t(u) + \overline{t}(u) = \binom{p-d(u)-1}{2} - q + \sum_{v \in N(u)} d(v).$$

■

Corollary 3.3 If G is an r-regular graph of order p, then $t(u) + \overline{t}(u) = \binom{p-1}{2} - \frac{3}{2}\, r(p-r-1)$ for every $u \in V(G)$. ··· (3.7)

Proof: Let G be an r-regular graph of order p. Then $d(u) = r$ for every $u \in V(G)$. So $q = \frac{1}{2}pr$ and $\sum_{v \in N(u)} d(v) = r^2$ and the result follows. ■

Corollary 3.4 For every vertex u in a regular self-complementary graph of order 4k+1,

$$t(u) + \bar{t}(u) = 2k(k-1) \qquad \cdots\cdots\cdots (3.8)$$

Proof: If G is a regular self-complementary graph of order p = 4k+1, then its regularity is 2k and size is k(4k+1). Substituting these in (3.3) we get (3.8). ∎

We can also deduce the following known results.

Corollary 3.5 (Lorden [42]) If G is a (p,q)-graph, then

$$t(G) + t(\bar{G}) = \binom{p}{3} - (p-1)q + \frac{1}{2} \sum_{u \in V(G)} [d(u)]^2 \qquad \cdots\cdots\cdots (3.9)$$

Proof: $t(G) + t(\bar{G}) = \frac{1}{3} \sum_{u \in V(G)} [t(u)+\bar{t}(u)]$ by (3.2)

$$= \frac{1}{3} \sum_{u \in V(G)} \left[\frac{(p-d(u)-1)(p-d(u)-2)}{2} - q + \sum_{v \in N(u)} d(v) \right]$$

$$= \frac{1}{3} \sum_{u \in V(G)} \left[\frac{(p-1)(p-2)}{2} - \frac{d(u)}{2}(2p-3) + \frac{[d(u)]^2}{2} - q + \sum_{v \in N(u)} d(v) \right]$$

$$= \frac{p(p-1)(p-2)}{3 \times 2} - \frac{2p-3}{3} \frac{1}{2} \sum_{u \in V(G)} d(u) + \frac{1}{3 \times 2} \sum_{u \in V(G)} [d(u)]^2 - \frac{pq}{3} + \frac{1}{3} \sum_{u \in V(G)} \left[\sum_{v \in N(u)} d(v) \right]$$

$$= \binom{p}{3} - \frac{1}{3}(2p-3)\, q - \frac{pq}{3} + \frac{1}{2 \times 3} \sum_{u \in V(G)} [d(u)]^2 + \frac{1}{3} \sum_{u \in V(G)} [d(u)]^2$$

Thus, $t(G) + t(\bar{G}) = \binom{p}{3} - (p-1)\, q + \frac{1}{2} \sum_{u \in V(G)} [d(u)]^2.$ ∎

Corollary 3.6 (Clapham [21]) The number of triangles in a regular self-complementary graph of order 4k+1 is

$$\frac{1}{3}\, k(k-1)(4k+1).$$

3.2 TRIANGLE NUMBER AND SOME BINARY GRAPH OPERATIONS.

Here we consider the composition, join and cartesian product of two graphs and derive expressions for the triangle number of vertices and edges in them.

a) COMPOSITION OF GRAPHS.

The composition $F = G(H)$ of two graphs G and H has vertex set $V(F) = \{ (u,v) \,/\, u \in V(G),\ v \in V(H) \}$ and edge set $E(F) = \{(u,v)(u',v') \,/\text{either } uu' \in E(G) \text{ or } u = u' \text{ and } vv' \in E(H)\}$. This operation is discussed in [34] and [60].

Remark 3.10 $G(H)$ can be obtained by replacing each vertex u_i of G by a copy of H and each edge u_iu_j of G by all the possible edges between the copies of H corresponding to the vertices u_i and u_j of G.

Theorem 3.11 Let $G(p_1,q_1)$ and $H(p_2,q_2)$ be two graphs. Then the triangle number of a vertex (u,v) in $G(H)$ is

$$t(u,v) = t(v) + q_2 d(u) + p_2 d(u) d(v) + p_2^2 t(u) \quad \cdots (3.12)$$

Proof: Consider two graphs $G(p_1,q_1)$, and $H(p_2,q_2)$. Let (u,v) be a vertex in $G(H)$. The triangles at (u,v) in $G(H)$ are formed precisely in the following ways.

1) A triangle at v in H is also a triangle at (u,v) in $G(H)$. The number of such triangles at (u,v) is $t(v)$.

2) An edge in a copy of H corresponding to a neighbour of u in G forms a triangle at (u,v) in $G(H)$. The number of such triangles is $q_2 d(u)$.

3) Each edge of H at v forms a triangle in $G(H)$ with each of the vertices in the copy of H that corresponds to a neighbour of u in G. This contributes $p_2 d(u)d(v)$ to $t(u,v)$.

and 4) Each triangle in G at u contributes p_2^2 triangles in $G(H)$ at (u,v). The number of triangles so formed is $p_2^2 t(u)$.

So, $t(u,v) = t(v) + q_2 d(u) + p_2 d(u)d(v) + p_2^2 t(u)$. ■

Corollary 3.12 If there are t_1 triangles in $G(p_1,q_1)$ and t_2 triangles in $H(p_2,q_2)$. Then the number of triangles in $F = G(H)$ is given by $t(F) = p_1 t_2 + p_2^3 t_1 + 2p_2 q_1 q_2$.

Proof: $t(F) = \frac{1}{3} \sum_{u \in V(G)} \sum_{v \in V(H)} t(u,v)$

$$= \frac{1}{3} \sum_{u \in V(G)} \sum_{v \in V(H)} [\ t(v) + q_2 d(u) + p_2 d(u)d(v) + p_2^2 t(u)\ ]$$

$$= \frac{1}{3} \sum_{u \in V(G)} [\ 3t(H) + q_2 d(u) p_2 + p_2 d(u) 2q_2 + p_2^3 t(u)\ ]$$

$$= \frac{1}{3} [\ p_1 t(H) + p_2 q_2 2q_1 + 2p_2 q_2 2q_1 + p_2^3 3t(G)\ ]$$

Thus $t(F) = p_1 t_2 + p_2^3 t_1 + 2\ p_2 q_1 q_2$. ■

Theorem 3.13 Let G and H be graphs of order p_1 and p_2 respectively. Then the triangle number of an edge e in $G(H)$ joining the vertices (u_1,v_1) and (u_2,v_2) is given by

$$t(e) = \begin{cases} p_2 t(e_1) + d(v_1) + d(v_2) & \text{when } u_1 \neq u_2,\ e_1 = u_1 u_2 \in E(G) \\ t(e_2) + p_2 d(u_1) & \text{when } u_1 = u_2,\ e_2 = v_1 v_2 \in E(H) \end{cases}$$

Proof: Let G and H be graphs of order p_1 and p_2 respectively and e be an edge in $G(H)$ joining (u_1,v_1) and (u_2,v_2).

Case (i) $u_1 \neq u_2$

Then u_1 and u_2 are adjacent in G and let $e_1 = u_1u_2$. The triangles in $G(H)$ containing the edge e are precisely of the following types:

1) Since each vertex u in G is replaced by a copy of H in $G(H)$, each triangle u_1u_2u in G containing $u_1u_2 = e_1$ give rise to $p_2t(e_1)$ triangles in $G(H)$.

2) Each edge v_1v incident at v_1 in the copy of H corresponding to $u_1 \in V(G)$ form a triangle with (u_2,v_2) in $G(H)$ containing e. Number of such triangles is $d(v_1)$.

and 3) Similarly, the $d(v_2)$ edges at v_2 in the copy of H corresponding to u_2 form $d(v_2)$ triangles in $G(H)$ containing e.

Thus, the total number of triangles in $G(H)$ containing the edge e is $p_2t(e_1) + d(v_1) + d(v_2)$.

Case (ii) $u_1 = u_2$

Then v_1 and v_2 are adjacent vertices in the same copy of H. The triangles in $G(H)$ containing e are precisely of the following types:

1) The $t(e_1)$ triangles in the copy of H, where $e_1 = v_1v_2$

and 2) Each vertex in the copy of H corresponding to u_1 is adjacent to all vertices in the copies of H corresponding to the neighbors of u_1 in G. The number of such triangles formed in $G(H)$ is $p_2d(u_1) = p_2d(u_2)$.

Thus $t(e) = t(e_1) + p_2d(u_1)$ in this case. ∎

b) JOIN OF GRAPHS

The join $G + H$ of two graphs G and H is the graph with vertex set $V(G+H) = V(G) \cup V(H)$ and edge set

$$E(G+H) = E(G) \cup E(H) \cup \{ uv \,/\, u \in V(G),\ v \in V(H) \}.$$

Theorem 3.14 The triangle numbers of a vertex u and an edge e in the join $G + H$ of the graphs G and H are given by

$$t(u) = \begin{cases} t_G(u) + d_G(u)p(H) + q(H) & \text{when } u \in V(G) \\ t_H(u) + d_H(u)p(G) + q(G) & \text{when } u \in V(H) \end{cases}$$

and $$t(e) = \begin{cases} t_G(e)+p(H) & \text{when } e \in E(G) \\ t_H(e)+p(G) & \text{when } e \in E(H) \\ d_G(u)+d_H(v) & \text{when } e = uv \text{ with } u \in V(G),\ v \in V(H). \end{cases}$$

Proof: Let G and H be any two graphs and J be their join. Consider any vertex u in J. Then either $u \in V(G)$ or $u \in V(H)$.

Let $u \in V(G)$. Then each triangle in G containing u is also a triangle in J containing u. In J, each vertex of G is adjacent to all vertices of H, and hence each edge of G forms $p(H)$ triangles in J with the vertices of H. So there are $p(H)d_G(u)$ such triangles in J containing u due to the $d_G(u)$ edges of G incident at u. In J, each edge of H forms a triangle with u. Such $q(H)$ triangles are there in J.

Thus $t_J(u) = t_G(u) + d_G(u)p(H) + q(H)$.

Similarly we can derive the expression for $t_J(u)$ when $u \in V(H)$.

Now, let e be an edge in J. Then either $e \in E(G) \cup E(H)$ or e is an edge joining a vertex of G and a vertex of H.

Let $e \in E(G)$. The $t_G(e)$ triangles in G are triangles in J, containing e, also. Each vertex of H is a common neighbour to the end vertices of e in J. Such $p(H)$ triangles are there in J containing e. These are the only triangles in J containing the edge e. Thus $t_J(e) = t_G(e) + p(H)$.

Similarly $t_J(e) = t_H(e) + p(G)$ when $e \in E(H)$

Let one of the end vertices, say u, of e be in G and the other, say v, be in H. In J, each edge incident at u in G forms a triangle with each of the vertices of H and each edge incident at v in H forms a triangle with each of the vertices of G. So, in J, every edge of G at u forms a triangle with v and every edge of H at v forms a triangle with u. Obviously both of these triangles contain the edge e. These are the only triangles containing e. Thus

$t_J(e) = d_G(u) + d_H(v)$ when $e = uv$, $u \in V(G)$ and $v \in V(H)$. ■

Corollary 3.15 The triangle number of the join of two graphs G and H is given by

$$t(G+H) = t(G) + t(H) + p(G)q(H) + p(H)q(G).$$

Proof: Let G and H be any two graphs. Then

$$t(G+H) = \frac{1}{3}\Big[\sum_{u \in V(G)} t_G(u) + d_G(u)p(H) + q(H) + \sum_{v \in V(H)} t_H(v) + d_H(v)p(G) + q(G)\Big]$$

$$= \frac{1}{3}\left[3t(G) + p(H)2q(G) + p(G)q(H) + 3t(H) + p(G)2q(H) + p(H)q(G)\right]$$

$$= t(G) + t(H) + p(G)q(H) + p(H)q(G).$$ ■

c) **CARTESIAN PRODUCT OF GRAPHS.**

The cartesian product $G \times H$ of two graphs G and H has vertex set $V(G{\times}H) = V(G) \times V(H)$ and edge set $E(G{\times}H) =$ $\{ (u_1,v_1)(u_2,v_2) \ / \ u_1 = u_2 \text{ and } v_1v_2 \in E(H) \text{ or } v_1 = v_2 \text{ and } u_1u_2 \in E(G) \}$

Remark 3.16 The cartesian product of G and H can be viewed as the graph obtained by replacing each vertex of G by a copy of H and joining the corresponding vertices in two copies of H at $u \in V(G)$ and $v \in V(G)$ if and only if $uv \in E(G)$.

Theorem 3.17 The triangle number of a vertex (u,v) and an edge e in the cartesian product of two graphs G and H are given by $t(u,v) = t_G(u) + t_H(v)$

and
$$t(e) = \begin{cases} t_H(e) & \text{when } e = (u,v_1)(u,v_2) \\ t_G(e) & \text{when } e = (u_1,v)(u_2,v) \end{cases}$$

Proof: Let G and H be two graphs and their cartesian product be F. While constructing the cartesian product, the additional edges introduced between two copies of H corresponding to two adjacent vertices of G form only a matching. Hence no new triangles are formed in F, except for the multiplicity due to the replacement of a vertex in G by the $p(H)$ vertices of H. Hence the expressions. ∎

Corollary 3.18 For any two graphs G and H, the triangle number of $G \times H$ is $p(G)t(H) + p(H)t(G)$.

Proof: Follows from $t(G{\times}H) = \frac{1}{3} \sum_{u \in V(G)} \sum_{v \in V(H)} [t_G(u) + t_H(v)]$.

∎

3.3 TRIANGLE NUMBER AND THE G-JOIN OF A FAMILY OF GRAPHS.

Sabidussi [61] has introduced the concept of the G-join of a family of graphs and Ruiz [58] has studied this in connection with self-complementary graphs. This is also discussed by Golumbic [7].

Let G be a graph and $\mathcal{F} = \{ H_u \ / \ u \in V(G) \}$ be a family of graphs, then the G-join of the family $\mathcal{F}$ is the graph $J = G(\mathcal{F})$ with the vertex set $\{ (u,v) \ / \ u \in V(G),\ v \in V(H_u) \}$ and edge set $\{ (u_1,v_1)(u_2,v_2) \ / \text{ either } u_1 = u_2 \text{ and } v_1v_2 \in E(H_u) \text{ or } u_1u_2 \in E(G) \}$.

We have observed the following properties of G-join.

Lemma 3.19 Let G be a graph and $\mathcal{F} = \{ H_u \ / \ u \in V(G) \}$ be a family of graphs. For $u_1 \neq u_2$, (u_1,v_1) and (u_2,v_2) belong to the same component of the G-join $J = G(\mathcal{F})$ if and only if u_1 and u_2 belong to the same component of G.

Proof: Let u_1 and u_2 belong to the same component of G. Then, we have a u_1-u_2 path, say $u_1u'_1u'_2u'_3 \cdots \cdots \cdots u'_mu_2$ in G. For any $v_1 \in V(H_{u_1})$, $v_2 \in V(H_{u_2})$ and $v'_j \in V(H_{u'_j})$, consider the sequence (u_1,v_1), (u'_1,v'_1), (u'_2,v'_2), $\cdots \cdots$, (u'_m,v'_m), (u_2,v_2). Each of the consecutive vertices in the sequences are adjacent in $G(\mathcal{F})$ since the vertices corresponding to the first co-ordinates of the consecutive ordered pairs are adjacent in G. Thus there is a $(u_1,v_1)-(u_2,v_2)$ path in J and hence (u_1,v_1) and (u_2,v_2) belong to the same component of $G(\mathcal{F})$

Conversely, let (u_1,v_1) and (u_2,v_2) belong to the same component of $G(\mathcal{F})$. Then there is a path $(u_1,v_1)(u'_1,v'_1)(u'_2,v'_2)$ ... ... $(u'_n,v'_n)(u_2,v_2)$ in $G(\mathcal{F})$. If two of the symbols u_1, u'_1, u'_2, ..., u'_n, u_2 ; say $\overline{u}_i$ and $\overline{u}_j$, are identical, the sequence obtained by deleting all the vertices between $(\overline{u}_i,\overline{v}_i)$ and $(\overline{u}_j,\overline{v}_j)$ including exactly one of them also form a (u_1,v_1)-(u_2,v_2) path. Because, if $\overline{u}_i = \overline{u}_j = u$, then neighbourhoods of $(u,\overline{v}_i)$ and $(u,\overline{v}_j)$ from outside H_u are identical. Repeat this process till we get a sequence (u_1,v_1), (u'_1,v'_1), (u'_2,v'_2), ..., (u'_k,v'_k), (u_2,v_2) in $G(\mathcal{F})$ with u_1, u'_1, u'_2, ..., u'_k, u_2 are all distinct. So the adjacencies between the consecutive vertices of the new path in $G(\mathcal{F})$ is due to the adjacencies of their first coordinates in G. Thus $u_1u'_1u'_2 \cdots\cdots u'_ku_2$ form a path in G and hence u_1 and u_2 belongs to the same component of G. ■

Theorem 3.20 Let G be a non-trivial graph of order p and $\mathcal{F}$ be a family of p graphs. Then $G(\mathcal{F})$ is connected if and only if G is connected.

Proof: Consider a graph G and a family $\mathcal{F} = \{ H_u \,/\, u \in V(G) \}$ of graphs. If G is a connected, by lemma 3.19, every pair of vertices of $G(\mathcal{F})$ belongs to the same component of $G(\mathcal{F})$. Hence $G(\mathcal{F})$ connected. Similar arguement for the converse also. ■

Lemma 3.21 The G-join $G(\mathcal{F})$ of a family $\mathcal{F}$ of graphs is complete if and only if G and each member of $\mathcal{F}$ is complete.

Proof: Let G and each member of $\mathcal{F}$ be complete. Consider the vertices (u_1,v_1) and (u_2,v_2) in $G(\mathcal{F})$. If $u_1 \neq u_2$, then u_1 and u_2 are adjacent in G and hence (u_1,v_1) and (u_2,v_2) are adjacent in $G(\mathcal{F})$. If $u = u_1$ ($= u$ say), then v_1, $v_2 \in V(H_u)$ are adjacent in H_u. Then also (u_1,v_1) and (u_2,v_2) are adjacent in $G(\mathcal{F})$.

Conversely, if u_1 and u_2 are not adjacent in G, then none of the vertices in H_{u_1} is adjacent to the vertices in H_{u_2}, in $G(\mathcal{F})$. If v_1, $v_2 \in V(H_u)$, for some $u \in V(G)$, are not adjacent in H_u. Then (u_1,v_1) and (u_2,v_2) should not be adjacent in $G(\mathcal{F})$. Thus $G(\mathcal{F})$ is complete only if G and each member of $\mathcal{F}$ is complete. ∎

Lemma 3.22 Let G be a non-trivial graph and (u_1,v_1), (u_2,v_2) be two vertices in the G-join of a family $\mathcal{F}$. Then

$$d_J((u_1,v_1),(u_2,v_2)) = \begin{cases} d_G(u_1,u_2) & \text{if } u_1 \neq u_2 \\ 1 & \text{if } u_1 = u_2 \text{ and } v_1v_2 \in E(H_{u_1}) \\ 2 & \text{if } u_1 = u_2 \text{ and } v_1v_2 \notin E(H_{u_1}) \\ d_H(v_1,v_2) & \text{if } u_1 = u_2 \text{ is an isolated vertex in } G \end{cases}$$

Proof: Let $u_1 \neq u_2$.

If (u_1,v_1) and (u_2,v_2) are in the distinct components of G, the result follows by lemma 3.19.

Now, let (u_1,v_1) and (u_2,v_2) be in the same component of $G(\mathcal{F})$ and $u_1u_1'u_2' \cdots\cdots\cdots u_m'u_2$ be a shortest u_1–u_2 path in G. Then $(u_1,v_1)(u_1',v_1')(u_2',v_2') \cdots\cdots\cdots\cdots (u_m',v_m')(u_2,v_2)$, for some

$v_1 \in V(H_{u_1})$, $v_i' \in V(H_{u_i})$ and $v_2 \in V(H_{u_2})$; i = 1,2, ⋯ ⋯ ⋯, m, will be a shortest path in $G(\mathcal{F})$ joining (u_1,v_1) and (u_2,v_2). Because, if it is not so, it will contradict the choice of the shortest u_1-u_2 path in G.

So, $d_J\big((u_1,v_1),(u_2,v_2)\big) = d_G(u_1,u_2)$.

Let $u_1 = u_2 = u$ and v_1 and v_2 are adjacent in H_u. Then, (u_1,v_1) and (u_2,v_2) are adjacent in J and so $d_J\big((u_1,v_1),(u_2,v_2)\big) = 1$. If v_1 and v_2 are not adjacent in H_u, (u_1,v_1) and (u_2,v_2) must have a common neighbour in J if u is not an isolated vertex in G and hence $d_J\big((u_1,v_1),(u_2,v_2)\big) = 2$ in this case. If u is an isolated vertex, H_u will be a component of J and hence $d_J\big((u_1,v_1),(u_2,v_2)\big) = d_H(v_1,v_2)$. ■

Theorem 3.23 Let G be a non-trivial graph and $J = G(\mathcal{F})$ be the G-join of the family $\mathcal{F} = \{ H_u \,/\, u \in V(G) \}$ of graphs. Then,

$$\operatorname{diam}(J) = \begin{cases} 1 & \text{if } \operatorname{diam}(G) = 1 \text{ and } \operatorname{diam}(H_u) = 1 \ \forall\ u \in V(G) \\ \operatorname{diam}(G) & \text{if } \operatorname{diam}(G) \geq 2 \\ 2 & \text{otherwise} \end{cases}$$

Proof: Case(i) $\operatorname{diam}(G) = 1$ and $\operatorname{diam}(H_u) = 1$ for every $H_u \in \mathcal{F}$.

Then G and every H_u is complete, by lemma 3.21, and hence $\operatorname{diam}(J) = 1$.

Case (ii) $\operatorname{diam}(G) \geq 2$.

If G is disconnected, then $\operatorname{diam}(J) = \operatorname{diam}(G)$ by theorem 3.20.

Let G be connected and $\text{diam}(G) = d$. Then there exist two vertices u_1 and u_2 in G with $d_G(u_1,u_2) = d \geq 2$. Hence $d_J\big((u_1,v_1),(u_2,v_2)\big) = d$, by lemma 3.22 and so $\text{diam}(J) \geq d$. Also, since G is non-trivial, $d_J\big((u,v_1),(u,v_2)\big) = 2 \leq d$ for every $u \in V(G)$ and $v_i \in V(H_u)$. Hence $\text{diam}(G) = d$.

Case(iii) $\text{diam}(G) = 1$ and $\text{diam}(H_u) \geq 2$ for at lest one u.

Then G is complete and at least one H_u in the family $\mathcal{F}$ is not complete. By the completeness of G, pairs of vertices of the form $\{ (u_1,v_1), (u_2,v_2) \mid u_1 \neq u_2 \}$ are adjacent in J. Now consider an H_u which is not complete. Let $v_1, v_2 \in V(H_u)$ be not adjacent in H_u. Clearly, each vertex in each of the remaining member of the family $\mathcal{F}$ is common neighbour to both (u,v_1) and (u,v_2) for every $v_1, v_2 \in V(H_u)$. Hence $d_J\big((u,v_1),(u,v_2)\big) = 2$. for every non-adjacent vertices v_1 and v_2 in H_u.

Hence $\text{diam}(J) = 2$ in this case. ■

Theorem 3.24 Let G be any graph and $\mathcal{F} = \{ H_u \mid u \in V(G) \}$, a family of graphs. Then the G-join $J = G(\mathcal{F})$ is self-centered if any of the following conditions is satisfied.

(1) G is self-centered and $\text{diam}(G) \geq 2$,

(2) G and each member of $\mathcal{F}$ is complete,

(3) G is complete and each member of $\mathcal{F}$ is self-centered of diameter 2.

Proof: (1) Let G be self-centered and $\text{diam}(G) = d \geq 2$.

When G is disconnected, the theorem follows from theorem 3.20. Let G be connected. Then, for every $u_1 \in V(G)$, there exists $u_2 \in V(G)$ such that $d_G(u_1,u_2) = d$. Now consider any vertex (u_1,v_1) in J and $v_2 \in V(H_{u_2})$. Then $d_J((u_1,v_1),(u_2,v_2)) = d$ by lemma 3.22 since $u_1 \neq u_2$. So $ecc_J(u_1,v_1) = d$ for every $(u_1,v_1) \in V(J)$. Thus J is self-centered.

(2) If G and each member of $\mathcal{F}$ is complete, then $G(\mathcal{F})$ is also complete and hence self-centered.

(3) Let G be complete and each member of $\mathcal{F}$ is self-centered of diameter 2. Then $d_J((u_1,v_1),(u_2,v_2)) = 1$ for every $u_1, u_2 \in V(G)$, $u_1 \neq u_2$. For every $v_1 \in V(H_u)$, there exists $v_2 \in V(H_u)$ such that $d_H(v_1,v_2) = 2$. So, $d_J((u_1,v_1),(u_2,v_2)) = 2$ for that u_1 and u_2 by lemma 3.22. Hence $ecc_J(u,v_1) = 2$ for every $u \in V(G)$ and every $v_1 \in V(H_u)$. Thus J is self-centered in this case also.

Theorem 3.25 The degree of a vertex (u,v) in the G-join of a family of graphs $\mathcal{F} = \{ H_u \,/\, u \in V(G) \}$ is $d_{H_u}(v) + \sum_{u_i \in N_G(u)} p_{u_i}$ where p_u is the order of H_u.

Proof: Let G be graph, $\mathcal{F} = \{ H_u \,/\, u \in V(G) \}$ be a family of graphs, $u \in V(G)$, $v \in V(H_u)$ and $J = G(\mathcal{F})$. The neighbours of (u,v) in J of the form (u,v'), $v' \in V(H_u)$ are $d_{H_u}(v)$ in number. For each neighbour u_i of u in G, all the p_{u_i} vertices (u_i,v_i'), $v_i' \in V(H_{u_i})$ are neighbours of (u,v) in J. These are the only neighbours of (u,v) in J. So its degree is $d_{H_u}(u) + \sum_{u_i \in N_G(u)} p_{u_i}$.

■

Corollary 3.26 $G(\mathcal{F})$ is regular if G is regular and members of $\mathcal{F}$ are of same order and regular of same degree.

Proof: Let G be regular of degree r and each graph in $\mathcal{F}$ is regular of degree r′ and order p′. Then, degree of (u,v) in J is r′+ rp′ for every $u \in V(G)$ and $v \in V(H_u)$. This expression is independent of u and v. Thus J is regular. ∎

Theorem 3.27 Let G be any graph and $\mathcal{F} = \{ H_u \,/\, u \in V(G) \}$ be a family of graphs each of whose members is of order p. Then the triangle number of a vertex (u,v) in $G(\mathcal{F})$ is given by

$$t(u,v) = p^2 t_G(u) + t_{H_u}(v) + p\, d_G(u)\, d_{H_u}(v) + \sum_{u_i \in N_G(u)} q(H_{u_i})$$

Proof: Let G be a graph and $\mathcal{F} = \{ H_u \,/\, u \in V(G) \}$ be a family of graphs such that the order of each H_u is p. Consider $u \in V(G)$ and $v \in V(H_u)$. Then the triangles in the G-join $G(\mathcal{F})$ containing the vertex (u,v) are formed precisely in the following ways:

(1) A triangle in H_u containing v is a triangle in J containing (u,v). There are such $t_{H_u}(v)$ triangles in J.

(2) Each triangle in G containing u transforms to p^2 triangles in J containing (u,v), since each edge of G is replaced by p^2 edges in J. Such triangles are $p^2 t_G(u)$ in number.

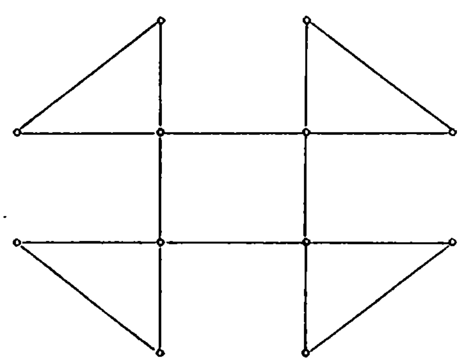
(3) Each edge in H_u incident at v form a triangle in J with each of the vertices in H_{u_i} corresponding to each neighbour u_i of u in G. Triangles so obtained are $d_G(u) \times d_{H_u}(v) \times p$ in number.

and (4) For every neighbour u_i of u in G, each edge in H_{u_i} form a triangle in J containing the vertex (u,v). The number of triangles so formed is $\sum_{u_i \in N_G(u)} q(H_{u_i})$.

Thus the total number of triangles in J containing (u,v) is $p^2 t_G(u) + t_{H_u}(v) + p\, d_G(u)\, d_{H_u}(v) + \sum_{u_i \in N_G(u)} q(H_{u_i})$ ∎

3.4 STRONGLY VERTEX TRIANGLE REGULAR AND STRONGLY EDGE TRIANGLE REGULAR GRAPHS.

A graph G is *vertex triangle regular* (*VTR*) if all its vertices have same triangle number and in this situation, the triangle number of a vertex in G is called the *vertex triangle number of the graph G*. G is *strongly vertex triangle regular* (*SVTR*) if it is regular also. If G is *SVTR* of order p, regularity r and has vertex triangle number t, then we say that G is an *SVTR* graph with parameters (p, r, t).



A VTR graph which is not SVTR

figure 3.1

Theorem 3.28 If *G* and *H* are *SVTR* graphs, then their composition is also *SVTR*.

Proof: Let *G* and *H* be *SVTR* graphs with parameters (p_1, r_1, t_1) and (p_2, r_2, t_2). Then $p(G) = p_1$, $p(H) = p_2$, $d_G(u) = r_1$, $d_H(v) = r_2$, $t_G(u) = t_1$ and $t_H(v) = t_2$ for every $u \in V(G)$ and $v \in V(H)$. So, $G(H)$ is regular of degree $r_2 + r_1p_2$.

Now by (3.12),

$$t(u,v) = t_2 + \frac{1}{2} p_2r_2r_1 + p_2r_1r_2 + p_2^2t_1 \text{ for every } (u,v) \in V(G(H)).$$

$$= t_2 + p_2^2t_1 + \frac{3}{2} p_2r_1r_2 ,$$ which is independent of the choice of *u* and *v*. Hence *G(H)* is an *SVTR* graph. ∎

Remark 3.29 Parameters of *G(H)* are

$$\left(p_1p_2 ,\ r_2+r_1p_2 ,\ t_2+ p_2^2t_1+\frac{3}{2} p_2r_1r_2 \right)$$

Theorem 3.30 If *G* is an *SVTR* graph with parameters (p, r, t), then $\overline{G}$ is also *SVTR* with parameters $\left(p ,\ p-r-1 ,\ \binom{p-1}{2}-\frac{3}{2}r(p-r-1)-t \right)$.

Proof: Let *G* be an *SVTR* graph with parameters (p, r, t). Then $d(u) = r$, and $t(u) = t$ for every $u \in V(G)$. Hence $d_{\overline{G}}(u) = p-r-1$ and $t_{\overline{G}}(u) = \binom{p-1}{2} - \frac{3}{2}r(p-r-1) - t$, by (3.7) for every vertex *u* in *G*. Hence $\overline{G}$ is an *SVTR* graph with these parameters ∎

Lemma 3.31 Parameters of a strongly vertex triangle regular self-complementary graph are (4k+1, 2k, k(k-1)) for some natural number k.

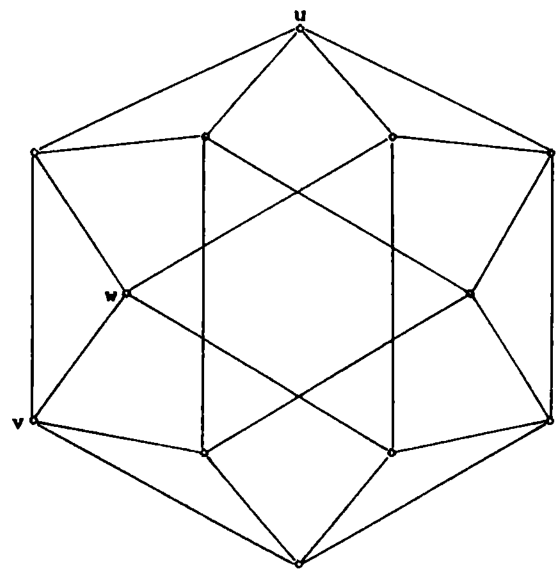
Proof: Let G be an *SVTRSC* graph with parameters (p, r, t). Then, due to regularity, p = 4k+1 and r = 2k. Clearly G has at least one fixed vertex and, by (*3.10*), its triangle number is k(k-1). Hence t = k(k-1), due to triangle regularity. ∎

A graph G is *edge triangle regular* (*ETR*) if all edges have the same triangle number. G is *strongly edge triangle regular* (*SETR*) if it is regular also. In this situation, the common triangle number of edges in G is called the *edge triangle number* of G. If G is *SETR* of order p, degree of regularity r and edge triangle number t, then we say that G is *SETR* with parameters (p, r, t).

Lemma 3.32 Every *SETR* graph with parameters (p, r, t) is *SVTR* with parameters (p, r, $\frac{1}{2}$rt).

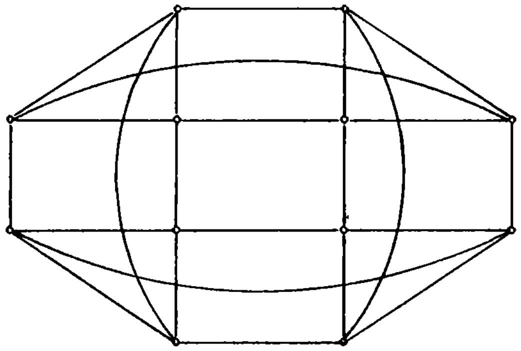
Proof: Let G be *SETR* with parameters (p, r, t). Then d(u)= r and t(e) = t for every $u \in V(G)$ and $e \in E(G)$. So, by (*3.1*), t(v) = $\frac{1}{2}$ rt for every $u \in V(G)$, since G is regular of degree r. Hence G is *SVTR* with parameters (p, r, $\frac{1}{2}$rt). ∎

Remark 3.33 Result analogous to theorem 3.30 does not hold for *SETR* graphs. Fig. 3.2 illustrates this - the graph G is *SETR* with parameters (12, 4, 1). But $\overline{G}$ is not, because t(uv) = 3 and t(uw) = 4. The converse of lemma 3.32 is also not true. Fig 3.3 illustrates this.



A SETR graph G with paramers(12, 4, 1)
whose complement is not SETR

fgure 3.2



A SVTR graph with parameters (12, 4, 1) which is not SETR

figure 3.3

Theorem 3.34 A graph G is strongly regular if and only if both G and $\overline{G}$ are *SETR*.

Proof: Let G be a strongly regular graph with parameters (p, r, λ, μ). Then, by lemma 1.2, $\overline{G}$ is also strongly regular with parameters $(p, p-r-1, p-2r+\mu-2, p-2r+\lambda)$. Hence $d_G(u) = r$, $t_G(e) = \lambda$, $d_{\overline{G}}(u) = p-r-1$ and $t_{\overline{G}}(e) = p-2r+\mu-2$ for every vertex u and edge e in the respective graphs. Thus both G and $\overline{G}$ are *SETR* with parameters (p, r, λ) and $(p, p-r-1, p-2r+\mu-2)$ respectively.

Conversely, let G and $\overline{G}$ be *SETR* with parameters (p, r, t) and $(p, p-r-1, t')$ respectively. Then $d_G(u) = r$ for every vertex u and any two adjacent vertices in G has t common neighbours and any two adjacent vertices in $\overline{G}$ has t′ common neighbours. So any two non-adjacent vertices in G has $2r+t'+2-p$ common neighbours. Hence G is strongly regular with parameters $(p, r, t, 2r+t'+2-p)$. ∎

Theorem 3.35 A self-complementary graph is *SETR* if and only if it is strongly regular with parameters $(4k+1, 2k, k-1, k)$ for some natural number k.

Proof: Let G be a self-complementary graph. If G is *SETR* then $\overline{G}$ is also *SETR* and hence G is strongly regular by theorem 3.34. Conversely, if G is strongly regular, then G is *SETR*.

Now, let (p, r, λ, μ) be the parameters of G. Then $p = 4k+1$ and $r = 2k$ for every $u \in V(G)$, for some natural number k, since G is regular. Further, by lemma 3.32, G is *SVTR* with vertex triangle number $\frac{1}{2}r\lambda$. But, by lemma 3.31, the vertex triangle number of a *SVTRSC* graph is $k(k-1)$. So $\lambda = k-1$. Then,

$$(4k+1-2k-1)\mu = 2\,k(2k-k+1-1) \qquad \text{by } (3.13)$$

$$\mu = k.$$ ∎

Corollary 3.36 (Rao [54]) If *G* is an edge-symmetric self-complementary graph, then *G* is strongly regular with parameters (4k+1, 2k, k-1, k) for some natural number k.

Proof: Let *G* be an edge symmetric self-complementary graph. Then *G* is regular and edge triangle regular. Hence *G* is *SETRSC* and so *G* is strongly regular with parameters

(4k+1, 2k, k-1, k), by the theorem. ■

* * *

4

A CONJECTURE OF KOTZIG ON SELF-COMPLEMENTARY GRAPHS

This chapter deals with one of the main aim of the thesis, to discuss a conjecture of Kotzig on self-complementary graphs. Some of the results are reported in [45] and [46].

4.1 KOTZIG'S CONJECTURE

Recall that, a vertex in a self-complementary graph is a *fixed vertex* if it is mapped onto itself by a complementing permutation. The set of all fixed vertices in a self-complementary graph is denoted by $F(G)$ and the set of all vertices with triangle number $k(k-1)$ in a regular self-complementary graph of order $4k+1$ is denoted by $\hat{F}(G)$. Two vertices u and v are said to be *similar*, written as $u \sim v$, if there exists an automorphism of G that maps u onto v. Clearly $\sim$ is an equivalence relation on $V(G)$. The equivalence classes under $\sim$ are called *G-orbits*. A vertex- symmetric graph has only one G-orbit

Kotzig [41] observed that $F(G) \subseteq \hat{F}(G)$ and asked about the possible characterization of $F(G)$ and gave the following:

KOTZIG'S CONJECTURE

$F(G) = \hat{F}(G)$ for any regular self complementary graph G.

In the subsequent sections, we recall the significant contribution made by Rao [54], characterize $\hat{F}(G)$ which motivates its definition being extended to any graph G and construct more counterexamples to the conjecture.

4.2 EARLIER ATTEMPT.

Rao has characterized $F(G)$ and constructed counterexamples to the conjecture in [54]. For convenience, we reproduce some of his results and a figure.

Theorem 4.1 (part of the lemma 2.1 in [54]) If G is a self-complementary graph of order **4k**+1, then exactly one of the G-orbits of $V(G)$ is of odd cardinality. □

Theorem 4.2 (part of the theorem 2.2 in [54]) If G is a regular self-complementary graph of order 4k+1, then $F(G)$ is the unique G-orbit of odd cardinality. □

Theorem 4.3 (theorem 4.1 in [54]) The following are equivalent for a self-complementary graph G of order ≥ 5.

(1) G is vertex-symmetric;

(2) $F(G) = V(G)$;

(3) $Z(G) = E(G)$. □

Theorem 4.4 (part of the theorem 4.2 of [54]) Let G_1, G_2 be two graphs and $G = G_1(G_2)$. Then the following hold.

(1) If G_1, G_2 are regular, then so is G;

(2) If G_1, G_2 are self-complementary, then so is G;

(3) If G_1, G_2 are vertex-symmetric, then so is G. □

Theorem 4.5 (theorem 2.3 in [54]) For every integer $k \geq 2$, there is a regular self-complementary graph G of order $4k+1$, such that $|F(G)| = 1$ but $|\hat{F}(G)| \geq 2k+1$.

Proof: Define a graph $G = G(4k+1)$ with $V(G) = \{ 0, 1, 2, \cdots\cdots\cdots, 4k+1 \}$ and $E(G) = \bigcup_{i=1}^{4} A_i$, where A_i, $1 \leq i \leq 4$ is given below:

$A_1 = \{ \{0, 2i+1\}, \{2i+1, 2i+2\}$, for every i, $0 \leq i \leq 2k-1$; $\{4j+2, 4j+4\}$ for every j, $0 \leq j \leq k-1 \}$,

$A_2 = \{ \{4i+1, 4j+2\}, \{4i+3, 4j+4\}$, for every i, j, $0 \leq i, j \leq k-1, i \neq j \}$,

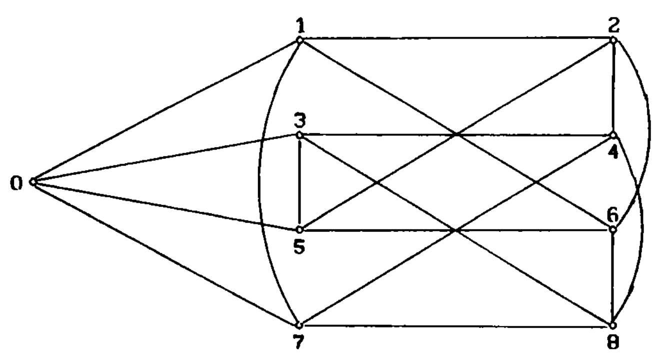
$A_3 = \{ \{4i+1, 4j+3\}$; for every i, j, $0 \leq i, j \leq k-1, i \neq j \}$

and $A_4 = \{ \{4i+2, 4j+2\}, \{4i+4, 4j+4\}$ for every i, j, $0 \leq i, j \leq k-1, i \neq j \}$.

It can be checked that G is a self-complementary graph of order $4k+1$ under $\sigma = (0)\prod_{i=0}^{k-1}(4i+1, 4i+2, 4i+3, 4i+4)$. Further the neighbourhood of 0 induces a regular graph of order $2k$ and degree $k-1$ and $0 \in F(G)$. It can be also checked that the neighbourhood of 2 induces a complete bipartite graph with bipartition $\{ 1, 5, 9, 13, \cdots\cdots\cdots, 4k-3 ; 6, 10, 14, \cdots\cdots\cdots, 4k-2 \}$ together with the isolated vertex 4, which clearly has $k(k-1)$

edges and is not regular. Further, for any i, $1 \le i \le 2k$, the induced subgraph on the neighbourhood of 2i is isomorphic to that on the neighbourhood of 2. Therefore $\hat{F}(G)$ contains the set $\{ 0, 2, 4, \cdots\cdots, 4k \}$. By Theorem 2.2 (theorem 4.2 here) and the fact that $0 \in F(G)$, it follows that for no i, $1 \le i \le 2k$, the vertex $2i \in F(G)$. The set $F(G)$ being a G-orbit (namely the unique G-orbit of odd length) it is the union of some cycles of the above σ. This implies that $F(G) = \{0\}$. ∎

Note that in case k = 2 for the graph $G(9)$, $F(G) = \{0\}$ and $\hat{F}(G) = V(G)$. However, for $k \ge 3$ and $G = G(4k+1)$, $F(G) = \{0\}$ and $\hat{F}(G) = \{ 0, 2, 4, \cdots\cdots 4k \}$.



The graph $G(9)$

figure 4.1

4.3 THE SET $\hat{F}(G)$

Recall that $\hat{F}(G)$ is the set of vertices in a regular self-complementary graph G of order 4k+1 with triangle number k(k-1).

Theorem 4.6 A vertex u in a regular self-complementary graph G is in $\hat{F}(G)$ if and only if $t(u) = \bar{t}(u)$.

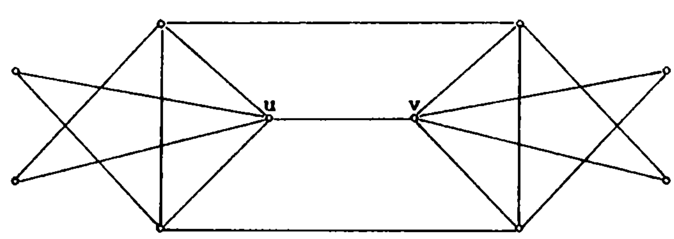
Proof: Let G be a regular self-complementary graph of order $p = 4k+1$, $k \in \mathbb{N}$ and let $u \in \hat{F}(G)$. Then $t(u) = k(k-1)$ and hence , by (3.8), $\bar{t}(u) = k(k-1)$.

Conversely, let $t(u) = \bar{t}(u)$ for some $u \in V(G)$. Then $t(u) = \bar{t}(u) = k(k-1)$ by (3.8). So, $u \in \hat{F}(G)$. ∎

An important and natural consequence of theorem 4.6 is that, $\hat{F}(G)$, which was defined only for regular self-complementary graphs can be extended to any graph.

Definition: Let G be a simple graph. Then the set $\hat{F}(G)$ is defined as $\hat{F}(G) = \{ u \in V(G) \mid t(u) = \bar{t}(u) \}$.

The graph G in fig. 4.2 is not self-complementary. The triangle number of each of the vertices labelled u and v is 3 in both G and $\bar{G}$ and that of other vertices are different in G and $\bar{G}$. Hence $\hat{F}(G) = \{ u, v \}$.



G

figure 4.2

Theorem 4.7 $\hat{F}(G) = \hat{F}(\overline{G})$ for any graph G. ... ... (4.1)

Proof: $u \in F(G) \Leftrightarrow \bar{t}(u) = t(u)$

$$\Leftrightarrow u \in \hat{F}(\overline{G})$$ ■

Theorem 4.8 A vertex u in a (p,q)-graph G is in $\hat{F}(G)$ if and only if the size of $\langle N(u) \rangle$ in G is

$$\frac{1}{2}\left[\binom{p-d(u)-1}{2} - q + \sum_{v \in N(u)} d(u) \right].$$

Proof: Let G be a graph and $u \in \hat{F}(G)$. Then $t(u) = \bar{t}(u)$. But, we have, $t(u) + \bar{t}(u) = \binom{p-d(u)-1}{2} - q + \sum_{v \in N(u)} d(u)$. So, $2\,t(u) = \binom{p-d(u)-1}{2} - q + \sum_{v \in N(u)} d(u)$. Hence the necessary part.

Conversely, let $u \in V(G)$ be such that

$t(u) = \frac{1}{2}\left[\binom{p-d(u)-1}{2} - q + \sum_{v \in N(u)} d(u) \right]$. Then $\bar{t}(u)$ is also $\frac{1}{2}\left[\binom{p-d(u)-1}{2} - q + \sum_{v \in N(u)} d(u) \right]$ by (3.1). Hence $t(u) = \bar{t}(u)$.

■

Corollary 4.9 Let G be a regular graph of order p and degree of regularity r, then a vertex u is in $\hat{F}(G)$ if and only if $t(u) = \frac{1}{2}\binom{p-1}{2} - \frac{3}{4}\,r(p-r-1)$. □

The proof being a routine one is omitted.

Remark 4.10 It follows from lemma 3.31 that, if G is a regular self-complementary graph then, $\hat{F}(G) = V(G)$ if and only if G is *SVTR*.

4.4 PRESENT ATTEMPT.

Here, we mention a fallacy in the proof of theorem 4.5 and identify a class of counterexamples. A construction of such graphs of order p, for an infinite number of values of p, is also carried out.

While analyzing the counterexamples of Rao, we came to know that they are wrong except for k = 2. Because, the claim in the proof of theorem 4.5 "the neighbourhood of 2 induces the complete bipartite graph with bipartition $\{ 1, 5, 9, \cdots\cdots\cdots, 4k-3 ; 6, 10, \cdots\cdots\cdots, 4k-2 \}$ together with the isolated vertex 4" is wrong for $k \geq 3$. In fact $\{ 6, 10, \cdots\cdots\cdots, 4k-2 \}$ induces a complete subgraph due to the edges $\{4i+1, 4j+2\}$, $0 \leq i, j \leq k-1$, $i \neq j$. So $t(2) = k(k-1) + \frac{1}{2} - (k-1)(k-2)$ and

$$t(1) = (k-1) + \frac{1}{2}(k-1) = k(k-1) - \frac{1}{2}(k-1)(k-2) \quad \text{for}$$

every $k \geq 2$ and consequently $\hat{F}(G) = \{ 0 \}$ for $k \geq 3$.

Thus the conjecture was made open for $p = 4k+1$, $k \geq 2$

Theorem 4.11 (A class of counterexamples) If G is a self-complementary graph which is strongly vertex triangle regular and not vertex symmetric, then it is a counterexample to the conjecture.

Proof: Let G be a self–complementary graph. Then $F(G) = V(G)$ if and only if G is vertex-symmetric (theorem 4.3) and $\hat{F}(G) = V(G)$ if G is strongly vertex triangle regular (remark 4.10). So if G is *SVTR* and not vertex-symmetric, then $\hat{F}(G) = V(G) \neq F(G)$. Hence this class provides counterexamples to the conjecture. ∎

Remark 4.12 It is interesting to see that the counter-example G_9 , of Rao is also of the type specified in the theorem 4.11.

Theorem 4.13 Let G be an *SVTRSC* graph which is not vertex-symmetric and and H be a *VSSC* graph. If there are two vertices u and u' in G such that $\langle N(u)\rangle$ is regular and $\langle N(u')\rangle$ is not regular, then $G(H)$ is *SVTRSC* but not vertex-symmetric.

Proof: Let G be a *SVTRSC* graph which is not vertex-symmetric and H be a *VSSC* graph. Then clearly H is *SVTR* and hence $G(H)$ is *SVTRSC*.

Now, let $G_1 = \langle N(u)\rangle_G$ and $G_2 = \langle N(u')\rangle_G$ where u and u' are as in the hypothesis. Then G_1 is regular and G_2 is not regular. It is obvious that $\langle N(u,v)\rangle$ in $G(H)$ is $G_1(H)$ and $\langle N(u',v)\rangle$ is $G_2(H)$. Because of the regularity of G and H, $G(H)$ is also regular, but $G_2(H)$ is not regular since G_2 is not. So $\langle N(u,v)\rangle \not\cong \langle N(u',v)\rangle$ in $G(H)$. Hence $G(H)$ is not vertex-symmetric.

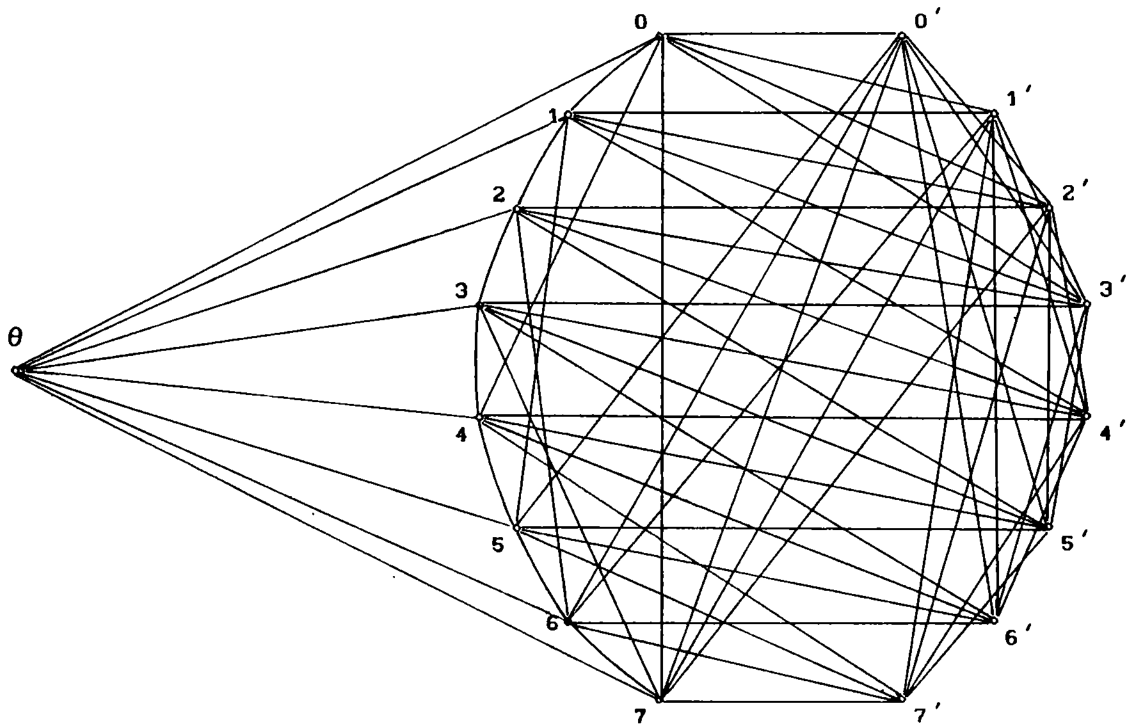
■

Remark 4.14 If G is a counterexample to the conjecture and H is a vertex-symmetric self-complementary graph. If there are vertices u and u' in G such that $\langle N(u)\rangle$ is regular and $\langle N(u')\rangle$ is not regular, then, by theorem 4.12, $G(H)$ and $H(G)$ are also counterexamples.

CONSTRUCTION OF COUNTEREXAMPLES

(1) Counterexample of order 17

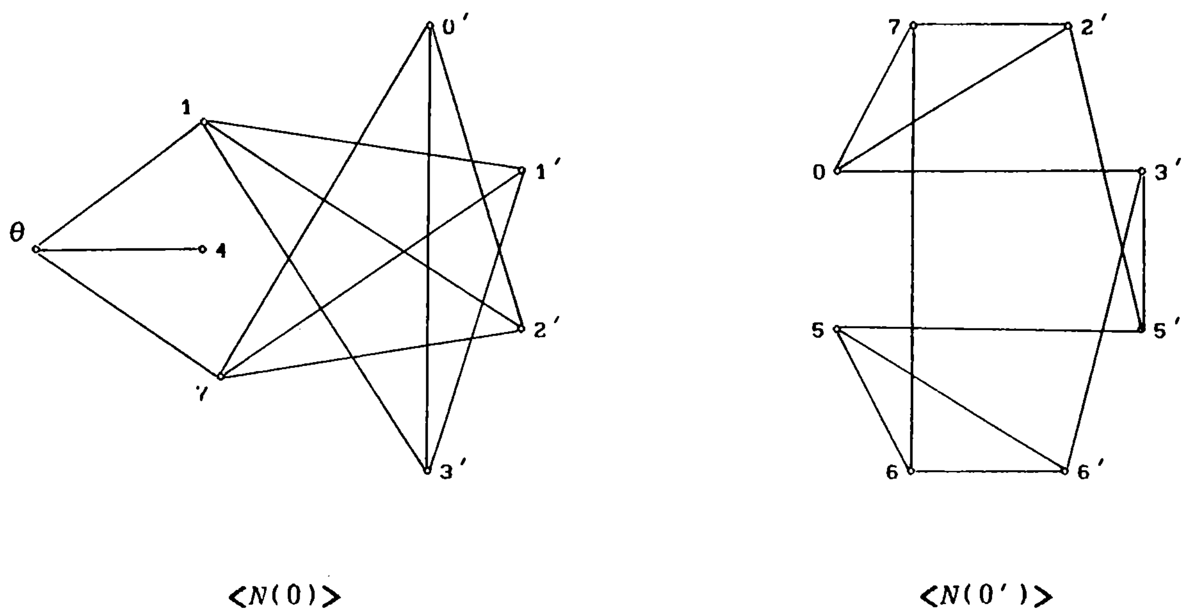
Take a single vertex *0*, a copy of the circulant graph *C*(8; 1, 4) with vertices labelled *0, 1, 2,* ……, *7* and a copy of its complement *C*(8; 2, 3) with vertices labelled *0′, 1′, 2′,* ……, *7′*. Join each vertex *i* to *θ, i′, i′+1, i′+2* and *i′+3,* addition being taken modulo 8 and $i'+j$ is to mean $(i+j)'$. The graph G_{17} so obtained is self-complementary, a complementing permutations is (θ)(*0 0′ 1 1′ 2 2′* … *7 7′*). From the figure of G_{17}, it's strong vertex triangle regularity is clear.



G_{17} : a counterexample of order 17

figure 4.3

It is not vertex-symmetric because the subgraph induced by the neighbourhood of *0* is the circulant graph $C(8;\ 1,\ 4)$ which is not isomorphic to the subgraph induced by the neighbourhood of any of the other vertices. Further $\langle N(i)\rangle$ and $\langle N(j')\rangle$ are also non-isomorphic for every i and j' (see fig. 4.4)

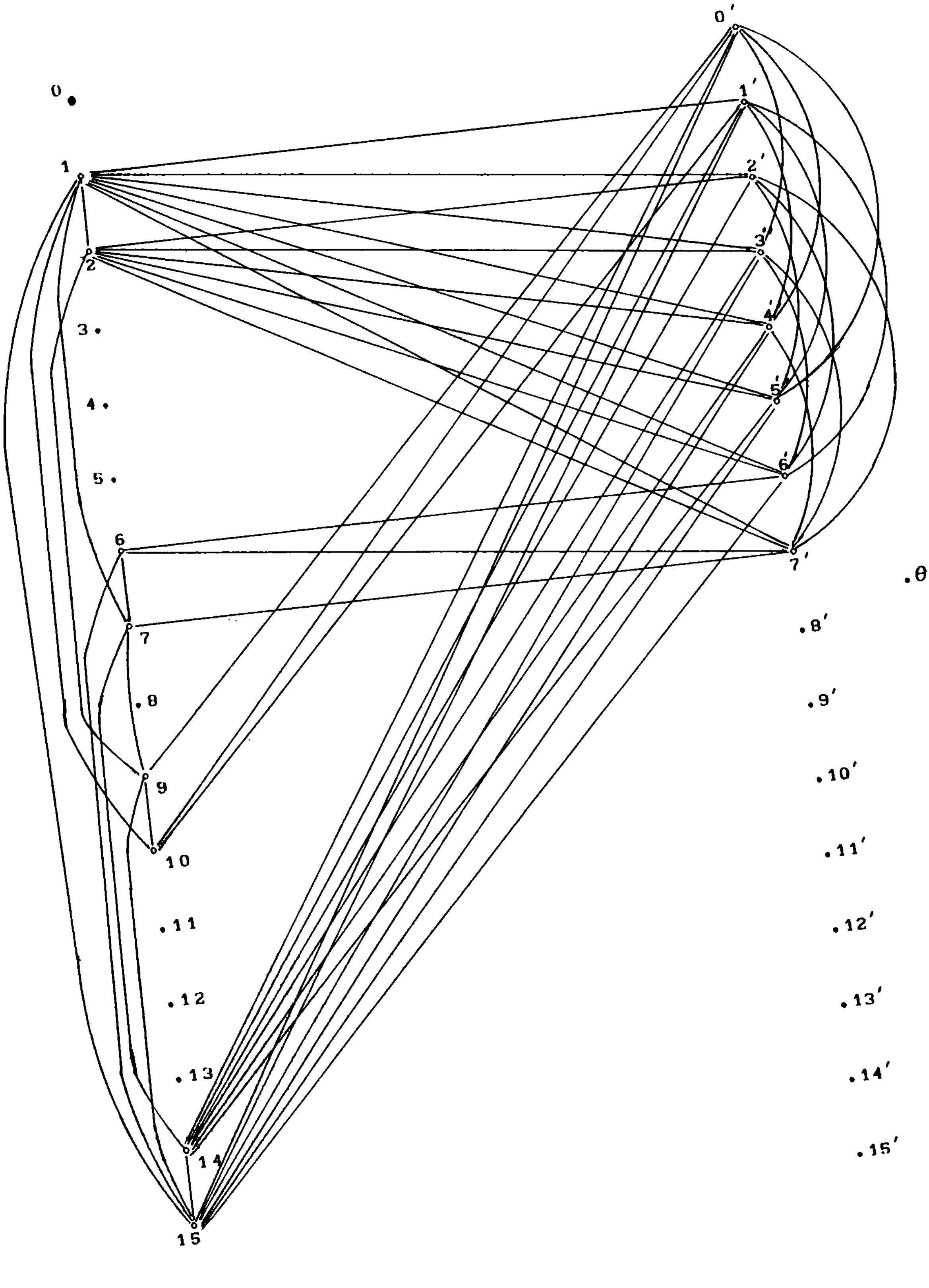


subgraphs of G_{17} induced by the neighbourhoods of *0* and *0'*

figure 4.4

Counterexample of order 33

Take a single vertex labelled θ, a copy of the circulant graph $C(16;\ 1,\ 2,\ 6,\ 7)$ with vertices labelled $0,\ 1,\ 2,\ \cdots\cdots\cdots,\ 15$ and a copy of its complement $C(16;\ 3,\ 4,\ 5,\ 8)$ with vertices labelled $0',\ 1',\ 2',\ \cdots\cdots\cdots,\ 15'$. Join each vertex i to $i',\ i'+1,\ i'+2,\ \cdots\cdots\cdots,\ i'+7$ and each i' to θ. Additions being taken modulo 16. The resulting graph G_{33} is self-complementary


0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
0′
1′
2′
3′
4′
5′
6′
7′
8′
9′
10′
11′
12′
13′
14′
15′
θ

under the complementing permutation $(\theta)(0\ 0'\ 1\ 1'\ 2\ 2'\ \cdots\ \cdots\ 15\ 15')$ and strongly vertex triangle regular. But it is not vertex-symmetric, since the subgraph induced by the neighbourhood of θ is the circulant graph $C(16;\ 3,\ 4,\ 5,\ 8)$ which is not isomorphic to the subgraph induced by the neighbourhood of any of the other vertices (see fig. 4.5) and $\langle N(j')\rangle$ are also non-isomorphic for every i and j'.

PRESENT STATUS OF THE CONJECTURE.

The conjecture is trivially true for p = 5. We have seen that strongly vertex triangle regular self-complementary graphs which are not vertex-symmetric form counterexamples. We have one such graph is $G(9)$ (fig. 4.1) and of order 17 (fig. 4.3) and 33 by the above construction. Hence by theorem 4.12, counterexamples of order $p = 9^{\alpha}17^{\beta}33^{\gamma}p_1^{\delta}$ where p_1 is an integer for which *VSSC* graph of order p_1 exists and α, β, γ and δ are integers such that at least one of α, β and γ is non-zero. Thus, the conjecture is false for $p = 9^{\alpha}17^{\beta}33^{\gamma}p_1^{\delta}$ where p_1, α, β, γ and δ are integers as above. We are examining the conjecture for other orders also and expect that our construction can be extended to graphs of order $p = 4k+1$ where $k = 2^n$, $n \in \mathbb{N}$. Then theorem 4.13 can be applied to get still more counterexamples.

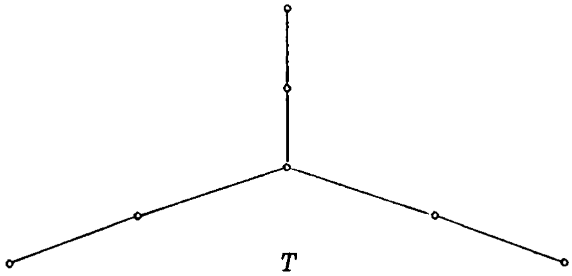
* * * *

5

ISOMORPHIC FACTORIZATION OF COMPLETE GRAPHS

5.1 ISOMORPHIC FACTORIZATION

A *factorization* of a graph is a partition of it into edge disjoint spanning subgraphs. A factorization in which any two factors are isomorphic is called an *isomorphic factorization*. A graph G is said to be divisible by an integer m if it can be factored into exactly m isomorphic factors and we write m/G. If G is divisible by m, then the set of all graphs H such that G can be factored into m isomorphic copies of H is denoted by G/m.



T

figure 5.1

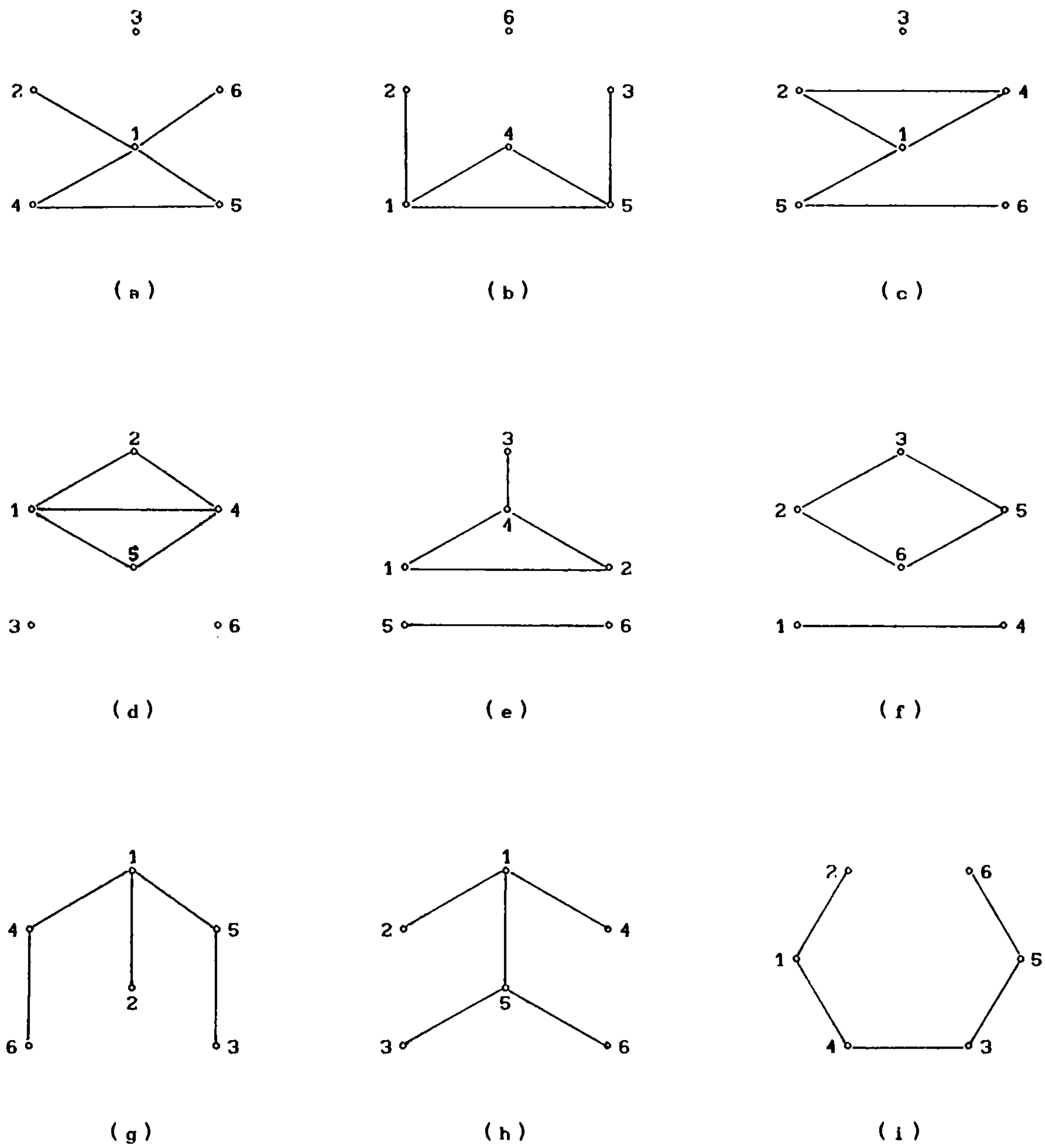
If G has q edges, G/m will be empty unless m/q. This necessary condition is not in general sufficient as in the case of the tree T in fig 5.1, which has six edges, yet $T/2$ is empty.

5.2 ISOMORPHIC FACTORIZATION OF COMPLETE GRAPHS.

Isomorphic factorization of complete graphs into m factors is a generalization of self-complementation. If K_p is divisible by two, then the members of $K_p/2$ are the self-complementary graphs of order p. Even though self-complementary graphs are connected, the elements of K_p/m need not be so for $m \geq 3$. For example see fig. 5.2. If the members of K_p/m are of size q, then $mq = \frac{p(p-1)}{2}$ and so $\frac{p(p-1)}{2m}$ is an integer. The result in the converse direction was independently proved by Guidotti [33] and Harary et al.[35].

Here, we give a simpler proof by generalizing a method of constructing self complementary graphs given by Gibbs [30]. The proof given in [35] essentially involves permutations of the p vertices and the $\frac{p(p-1)}{2}$ edges of K_p, while we use permutations of the vertices only.

When m/K_p, there are isomorphisms, that is permutations of $V(K_p)$, that maps between the factors. We call such a permutation σ as *factorizing permutation*. We label the m factors in an isomorphic factorization of K_p by G_0, G_1, G_2, ⋯ ⋯, G_{m-1} so that a factorizing permutation σ of $V(K_p)$ maps G_i onto $G_{i+1(\text{mod}\,m)}$, i = 1, 2, ⋯ ⋯ ⋯, m-1.



The nine members of $K_6/3$

Factorizing permutation for each factorization is (123)(456)

figure 5.2

Theorem 5.1 ([35]) If $m \,/\, \frac{p(p-1)}{2}$ and $(p,m) = 1$ or $(p-1,m) = 1$, then K_p is divisible by m.

Proof(by construction): Let m and p be such that $m \,/\, \frac{p(p-1)}{2}$ and either $(p,m) = 1$ or $(p-1,m) = 1$. We have to find m isomorphic factors of K_p.

CONSTRUCTION

Case (i) m is odd.

If there is an m-factorization, the edges in the subgraph of K_p spanned by each cycle of a factorizing permutation σ is to be distributed equally in the factors, every cycle of σ should be of length multiple of m except in the case of a 1-cycle when $p \equiv 1 (\mathrm{mod}\ m)$. But it is sufficient to consider the permutations with cycle length power of m. Because, if there is a cycle of σ not of this form, that is of length αm where α is not a multiple of m then the permutation σ^{α} will be of the required form and will be a factorizing permutation (not necessarily in the same order in which σ acts).

Consider a permutation σ of p symbols with each of its cycles is of length power of m, except one cycle of length one when $p \equiv 1$ (mod m). Assume without loss of generality that the symbols in σ are numbered consecutively from 1 to p and that the cycles are of non-decreasing length k_1, k_2, k_3, ⋯ ⋯ except the 1-cycle (p), if exists, at the end. It is to be noted that each k_i is a power of m. Now, the symbols 2, 3, ⋯ ⋯ ⋯, $\frac{k_1+1}{2}$ of the first cycle, the first k_1 symbols of each of the other cycle and the symbol p if (p) is a 1-cycle constitute the range of the symbol 1. We shall construct the graphs $G_0, G_1, G_2, \cdots\cdots\cdots, G_{m-1}$ with vertices labelled 1, 2, 3, ⋯ ⋯ ⋯, p and hence identify the symbols in σ with the vertices in G_j ; j = 0, 1, 2, ⋯ ⋯ ⋯, m-1. For each unordered pair { 1, j }, where j is in the range of 1, arbitrarily decide the graph G_i in which 1 and j are adjacent. Once these choices have been made, the symbols $\sigma^k(1)$ and $\sigma^k(j)$ are adjacent in $G_{i+k(\text{mod}\,m)}$, k = 1, 2, ⋯ ⋯ ⋯, k_j where j belongs to a cycle of length k_j . A table of the following form is helpful. In the first column of the table, the symbols 1, 2, ⋯ ⋯ ⋯, k_1 is to be repeated $\frac{k_j}{k_1}$ times where k_j is the maximum cycle length of σ.

vertex u	neighbours of u in the factor G_0	G_1	⋯	G_{m-1}
1	$v_{01}, v_{02}, \cdots$	$v_{11}, v_{12}, \cdots$	⋯	$v_{m-1\,1}, v_{m-1\,2}, \cdots$
2	$\sigma(v_{m-1\,1}), \cdots\cdots$	$\sigma(v_{01}), \cdots\cdots$	⋯	$\sigma(v_{m-2\,1}), \cdots\cdots\cdots$
3	$\sigma(v_{m-2\,1}), \cdots\cdots$	$\sigma(v_{m-1\,1}), \cdots$	⋯	$\sigma(v_{m-3\,1}), \cdots\cdots\cdots$
:	⋯ ⋯ ⋯	⋯ ⋯ ⋯	⋯	⋯ ⋯ ⋯ ⋯ ⋯
:				
k_1	⋯ ⋯	⋯ ⋯ ⋯ ⋯	⋯	⋯ ⋯ ⋯ ⋯ ⋯
1	:	:	:	:
2	:	:	:	:
:	:	:	:	:

Adjacency table for the isomorphic factoriztion of complete graphs

table 5.1

This completes the first stage of the algorithm. In the next stage, reduce the permutation σ to σ_1 on p-k_1 symbols by deleting the first cycle and do the process for the symbol k_1+1. Continue the process till all the cycles of non-unit length has been considered.

Case (ii) m is even.

In this case it is sufficient to consider permutations σ with cycle length powers of 2m only. Arrange σ so that the cycles are in the order of non-decreasing length except the one 1-cycle (p), if exists, at the end. Let k_1, k_2, ... be the cycle lengths and 1, 2, ..., p be the symbols in the permutation. The **range** of 1 consists of the symbols 2, 3, ..., $\frac{k_1}{2}+1$, the first k_1 symbols of the remaining cycles and the symbol p, if (p) is a 1-cycle. The rest of the algorithm is same as the first case.

Now we have to prove that the algorithm will produce a well defined isomorphic factorization.

Claim: As a result of performing the construction algorithm, (1) the adjacency relation between vertices is well-defined (2) every pair of vertices is assigned an adjacency relation and (3) the graphs G_0, G_1, ... G_{m-1} thus obtained are isomorphic.

Proof of (1) The pair { 1, j } cannot be sent to itself by σ^k when k is not a multiple of m, because $\sigma^k(j) \neq j$ for $k \neq M(m)$, except for the trivial case of the 1-cycle (p) and if $\sigma^k(1) = j$, then j is the symbol 1+k in the first cycle of σ and $\sigma^k(j) = 1+2k \neq 1$ since $k \neq M(m)$. Thus the pair { 1, j } can never be assigned simultaneous adjacency and non-adjacency.

The same argument applies to $\{ \sigma^{i}(1), \sigma^{i}(j) \}$ and carries over for all stages of the algorithm.

Proof of (2) Here we have to consider the two cases separately.

Case (i) m is odd

From the definition of range of the symbol 1, we have assigned adjacency to each pair $\{ 1, i \}$ when $2 \le i \le \frac{k_1+1}{2}$. For every j in the first cycle, symbols in its range from the first cycle are $j+1, j+2, \cdots\cdots\cdots, j+\frac{k_1-1}{2}$. Now $\frac{k_1+3}{2} = \sigma^{\frac{k_1+1}{2}}(1)$ and so the range of $\frac{k_1+3}{2}$ contains the symbols $\frac{k_1+5}{2}, \cdots\cdots\cdots, \frac{k_1+3}{2}+\frac{k_1-1}{2} = k_1+1 = 1$ of the first cycle. Hence 1 is in the range of $\frac{k_1+3}{2}, \cdots\cdots\cdots, k_1$. Thus the adjacency between 1 and every other symbol in the first cycle are defined if we fix the adjacency of 1 and those symbols in its range. This argument carries to all other symbols in the first cycle and for the adjacencies of the other symbols with those in the same cycle. Consider the cycle σ_j of length k_j, $j \neq 1$. We initially fix the adjacencies of the first k_1 symbols. But $\sigma^{k_1}(1) = 1$ and if $k_j > k_1$, then σ^{k_1} will give the adjacencies of next k_1 symbols in σ_j with 1. Our construction algorithm insists on continuing the process at least $\frac{k_j}{k_1}$ times. Thus the adjacency of 1 with each symbol in the cycle σ_j is defined. This is also applicable to all symbols in the first cycle and to all steps of the algorithm.

Case (ii) m is even.

Here the range of 1 is 2, 3, ⋯ ⋯ ⋯ ⋯, $\frac{k_1}{2}+1$ and that of any j in the first cycle is j+1, j+2, ⋯ ⋯ ⋯ ⋯, $j+\frac{k_1}{2}+1$. Now, $\frac{k_1}{2}+2 = \sigma^{\frac{k_1}{2}+1}(1)$ and hence the range of $\frac{k_1}{2}+2$ is $\frac{k_1}{2}+3$, $\frac{k_1}{2}+4$, ⋯ ⋯, $\frac{k_1}{2}+\frac{k_1}{2}+1 = k_1+1 = 1$ and the remaining arguments are similar to that in the first case.

Proof of (3) Now, we have shown that all the adjacencies are well defined and all possible adjacencies are determined. Clearly σ, σ^2, ⋯ ⋯ σ^{m-1} are isomorphism from G_0 to G_1, G_2, ⋯ , G_{m-1} respectively. ■

ILLUSTRATIONS

(i) ISOMORPHIC FACTORIZATION OF K_7 INTO THREE FACTORS

corresponding to the permutation $\sigma = (123)(456)(7)$

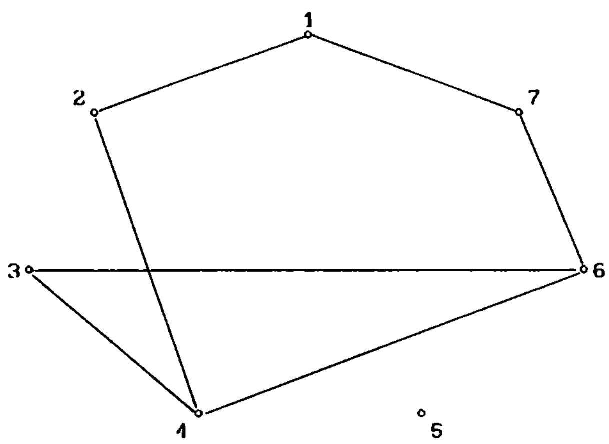
Stage 1: The the range of 1 is { 2, 4, 5, 6, 7 }. Let the vertex labelled 1 be adjacent to 2 and 7 in G_0, to 4 in G_1 and to 5 and 6 in G_2. The corresponding adjacency table is given in table 5.2.

Stage 2 The reduced permutation to be considered is $\sigma_1 = (456)(7)$. The range of 4 is { 5, 7 }. Let the vertex labelled 4 be adjacent to 5 and 7 in G_1. The adjacency table is given in table 5.3.

vertex u	neighbours of u in the factor G_0	G_1	G_2
1	2, 7	4, 5	6
2	4	3, 7	5, 6
3	6, 4	5	1, 7

Adjacency table at stage 1 for an isomorphic factorization of K_7 into three factors corresponding to the permutation (123)(456)(7)

table 5.2



The factor G_0 of K_7 resulting from the above construction

figure 5.3

vertex u	neighbours of u in the factor G_0	G_1	G_2
4		5, 7	
5	–		6, 7
6	4, 7		

Adjacency table at stage 2 for an isomorphic factorization of K_7 into three factors corresponding to the permutation (123)(456)(7)

table 5.3

(ii) ISOMORPHIC FACTORIZATION OF K_{20} INTO THREE FACTORS

corresponding to the permutation

σ = (1 2 3 4)(5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20)

Stage 1: The range of 1 is { 2, 3, 5, 6, 7, 8 }. Let the vertex labelled 1 be adjacent to 3 in G_0, to 6 and 7 in G_1 to 2, 5 and 8 in G_2 and none in G_3. The adjacency table is given in table 5.4.

Stage 2: The reduced permutation to be considered in this stage is σ_1=(5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20). The range of 5 is { 6, 7, 8, 9, 10, 11, 12, 13 }. Let the vertex labelled 5 be adjacent to 6 and 13 in G_0, 7 and 12 in G_1, 8 and 11 in G_2 and 9 and 10 in G_3. The adjacency table is given table 5.5

vertex u	neighbours of u in the factor G_0	G_1	G_2	G_3
1	3	6, 7	2, 5, 8	
2		4	7, 8	3, 6, 9
3	4, 7, 10		1	8, 9
4	9, 10	1, 8, 11		2
1	3	10, 11	2, 9, 12	
2		4	11, 12	3, 10, 13
3	4, 11, 14		1	12, 13
4	13,14	1, 12, 15		2
1	3	14, 15	2, 13, 16	
2		4	15, 16	3, 14, 17
3	4, 15, 18		1	16, 17
4	17, 18	1, 16, 19		2
1	3	18, 19	2, 17, 20	
2		4	19, 20	3, 18, 5
3	4, 19, 6		1	20, 5
4	5, 6	1, 20, 7		2

Adjacency table at stage 1 for an isomorphic factorization of K_{20} into four factors corresponding to the permutation (1 2 3 4) (5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20)

table 5.4

vertex u	neighbours of u in the factor G_0	G_1	G_2	G_3
5	6, 13	7, 12	8, 11	9, 10
6	10, 11	7, 14	8, 13	9, 12
7	10, 13	11, 12	8, 15	9, 14
8	10, 15	11, 14	12, 13	9, 16
9	10, 17	11, 16	12, 15	13, 14
10	14, 15	11, 18	12, 17	13, 16
11	14, 17	15, 16	12, 19	13, 18
12	14, 19	15, 18	16, 17	13, 20
13	14, 5	15, 20	16, 19	17, 18
14	18, 19	15, 6	16, 5	17, 20
15	18, 5	19, 20	16, 7	17, 6
16	18, 7	19, 6	20, 5	17, 8
17	18, 9	19, 8	20, 7	5, 6
18	6, 7	19, 10	20, 9	5, 8
19	6, 9	7, 8	20, 11	5, 10
20	6, 11	7, 10	8, 9	5, 12

Adjacency table at stage 2 for an isomorphic factorization of K_{20} into four factors corresponding to the permutation
(1 2 3 4) (5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20)

table 5.5

5.3 CONCLUDING REMARK AND SUGGESTIONS FOR FURTHER STUDY

This thesis is an attempt to shed more light on a conjecture of Anton Kotzig on self-complementary graphs. During this process, we have obtained several results relating the concepts of triangle and self-complementation, spread over the different chapters of this thesis. The survey of earlier results have been done to the extent possible and any serious omission due to oversight may kindly be pointed out.

Results of the thesis are far from complete. We list below some of the problems which we have either not attempted or found the answers to be difficult.

1. ANTIPODAL ITERATION NUMBER (ain.)

Consider a graph G and its antipodal graph $A(G)$. Let $G_0 = G$ and G_{i+1} be the graph obtained by superimposing $A(G_i)$ on G_i, for i = 0, 1, 2, ⋯ ⋯ ⋯ . If G is not complete, this process ultimately results in a complete graph since $E(A(G)) \subseteq E(\overline{G})$. The minimum value of i for which G_i is complete is called the *antipodal iteration number* (***ain.***) of G. It is obvious that $\text{ain}(K_p) = 0$ and $\text{ain}(G) = 1$ if $\text{diam}(G) = 2$. If G is disconnected, then its ain. is 1 if every component of G is complete and 2 otherwise. A formula for $\text{ain}(G)$ can be attempted.

2. S-ANTIPODAL GRAPH OF GRAPHS WITH A GIVEN PROPERTY

We have characterized $A^*(G)$ when G is a tree. Similar analysis can be done for a graph G with a given property P, where P could be maximal outer planar, hamiltonian, eulerian, chordal, etc. The question whether eulerian graph of odd order is the S-antipodal graph of some eulerian graph remains to be settled. We have answered (theorem 2.20) a similar question for even order.

3. TRIANGLE SEQUENCE

Similar to the results on degree sequences [63], the concept of *triangle sequence* could be investigated and characterization of an integer sequence being the triangle sequence of a graph may be attempted.

4. TRIANGLE NUMBER IN THE G-JOIN

Expression for the triangle number of a vertex / edge in the G-join of a family of graphs in the general setting is worth studying. See theorem 3.27 for our observation.

5. COUNTER EXAMPLES TO KOTZIG'S CONJECTURE

Our method of construction of counterexamples of order 17 and 33 could be extended to that of order $p = 4k+1$, where $k = 2^n$, $n \in \mathbb{N}$.

INDEX OF SYMBOLS AND ABBREVIATIONS

$A(G)$	antipodal graph of G
$A^{*}(G)$	S-antipodal graph of G
$D(G)$	dominating set in G
$E(u)$	set of edges incident at u
$E(G)$	edge set of G
$F(G)$	fixed vertices in a SC graph
$\hat{F}(G)$	set of vertices with same triangle number in G and its complement
G, H, ··· ···	graphs
$G(V,E)$	graph with vertex set V and edge set E
$G(p,q)$	graph of order p and size q
$\overline{G}$	complement of G
G/m	set of graphs each of which is a factor in some factorisation of G in-to m isomorphic factors
$G(H)$	composition of the graphs G and H
$G + H$	join of the graphs G and H
$G \times H$	cartesian product of G and H
$G(\mathcal{F})$	the G-join of a family $\mathcal{F}$ of graphs
H/G	H belongs to G/m for some integer m
K_p	complete graph on p vertices
$M(m)$	multiple of m
$N(G)$	neighbourhood graph of G
$N(u)$ or $N_G(u)$	neighbourhood u in G
$N[u]$ or $N_G[u]$	closed neighbourhood of u in G
$R(G)$	triangle graph of a graph G

$V(G)$	vertex set of G
$Z(G)$	set of edges in a *SC* graph one of whose end is mapped on to the other by a complementing permutation
$d(u)$ or $d_G(u)$	degree of a vertex u
$d(u,v)$ or $d_G(u,v)$	distance between two vertices
$\text{diam}(G)$	diameter of G
e	an edge in a graph
$\text{ecc}(u)$ or $\text{ecc}_G(u)$	eccentricity of a vertex u in G
$\text{ain}(G)$	antipodal iteration number of G
m/G	G can be factored in to m isomorphic factors (G is dividible by m)
p or $p(G)$	order of G
q or $q(G)$	size of G
r or $r(G)$	degree of a regular graph G
$t(u)$ or $t_G(u)$	triangle number of a vertex u
$\bar{t}(u)$	triangle number of a vertex u in $\bar{G}$
$t(e)$	triangle number of an edge e
$t(G)$	triangle number of G
$u, v, \cdots \cdots$	vertices in a graph
$u \sim v$	u and v are similar vertices
$\mathbb{N}$	the set of natural numbers
$\mathfrak{C}(G)$	set of all complementing permutations of G
$\mathcal{F}$	a family of graphs
σ	complementing permutation of a self-complementary graph
$(m,p) = 1$	m and p are relatively prime integers
$\langle S \rangle = \langle S \rangle_G$	subgraph of G induced by $S \subseteq V(G)$

ETR	edge triangle regular
RSC	regular self-complementary
SC	self-complementary
SETR	strongly edge triangle regular
SETRSC	strongly edge triangle regular self-complementary
SR	strongly regular
SRSC	strongly regular self-complementary
SVTR	strongly vertex triangle regular
SVTRSC	strongly vertex triangle regular self-complementary
VSSC	vertex-symmetric self-complementary
VTR	vertex triangle regular
ain.	antipodal iteration number

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