REFERENCE ONLY

STUDIES IN FIELD THEORIES AT FINITE TEMPERATURE AND FINITE BOSONIC DENSITY

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CERTIFICATE

Certified that the work presented in this thesis entitled *Studies in Field Theories at Finite Temperature and Finite Bosonic Density* is based on the bonafied research work done by Mr. V. J. Peter under my guidance, at the Department of Physics, Cochin University of Science and Technology, and has not been included in any other thesis submitted previously for the award of any other degree.

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PREFACE

The study of temperature and density dependent effects in field theories with spontaneous symmetry breaking (SSB) has attracted considerable attention during the last decades mainly because of its relevance to the understanding of several important issues in cosmology and condensed matter physics. We have undertaken studies of phase transitions in certain models of gauge and non-gauge theories at finite temperature and finite density (FTFD) using the perturbative as well as the non-perturbative methods of calculation. We find that in certain field theoretical models with SSB at finite temperatures, the increase of bosonic density or temperature induce multiple phase transitions. The thesis, which is spread over five chapters contain a detailed account of these studies.

Chapter 1 is an introductory chapter, which gives the necessary background for the work done in later chapters. In the earlier part of this chapter, we shall be concerned with the formal content of FTFD and the perturbative loop expansion method of evaluating effective potential using Feynman path integral technique. The basic ideas on phase transitions in field theories are also briefly mentioned. In the later part, we review the gaussian effective potential (GEP) method which is a non-perturbative approach used in field theory.

In chapter 2, we study the phase transitions in a self interacting, 2-component ϕ^4-theory with O(2) symmetry at finite bosonic densities, in the one-loop and improved one-loop approximations. The behavior of effective mass and effective coupling constant are analyzed. It is shown that at non-zero temperatures, as the

chemical potential μ corresponding to a conserved bosonic charge is increased from zero, symmetry restoration takes place at a finite value of μ. However as the density is increased further, the symmetry is again broken at a sufficiently large value of μ. A similar result is obtained as temperature is increased with a finite μ. The existence of two critical densities and critical temperatures are demonstrated in the high temperature limit. The validity of these calculations for all temperatures is established by numerical methods, making them applicable for cold dense matter as well. However, at zero temperatures chemical potential has no effect at least at the level of approximation considered. These one-loop results are shown to be valid in the improved one-loop calculations also, where the imaginary parts of effective potential are removed by taking into account the effect of multiple insertions of quadratically divergent bubbles.

In chapter 3, we study the U(1) invariant abelian Higgs model at FTFD at the one-loop level of approximation and show the existence of density induced double phase transitions. Following the real time formalism of finite temperature field theory, we define a FTFD dependent mass on the mass-shell of the particle, at temperature zero and chemical potential zero, by including momentum dependent self energy diagrams. The spontaneously broken symmetry is restored at a finite chemical potential is again broken at a further large value of μ. The critical temperatures and critical densities responsible for these phase transitions are calculated in the high temperature limit and the validity of these calculations for all ranges of temperatures is established by numerical methods. We also demonstrate the gauge invariance of these results. The effective potential is calculated in the unitary gauge. The finite density behaviour of effective scalar and effective gauge

coupling constants are studied and the results show an increase of gauge coupling constant with rise of μ , while the scalar coupling shows a reverse trend. However for an asymptotically large value of μ , both these coupling constants reach their tree level values.

Chapter 4 present a GEP study of FTFD phase transitions in $\lambda\phi^4$ model, making use of both the cutoff and autonomous versions of GEP. It is shown that in the presence of SSB at finite non-zero temperatures, increase of bosonic chemical potential induces a sequence of symmetry restorations and symmetry breakdowns. For an asymptotically large value of chemical potential, these multiple phase transitions end in a symmetry broken phase. Similarly in the case of SSB at a finite non-zero value of bosonic chemical potential, increase of temperature induces a series of symmetry restoring and symmetry breaking phase transitions. For an asymptotically large value of temperature, symmetry remains restored. With zero chemical potential, increase of temperature only restore the SSB phase, without any multiple phase transition. We also study the FTFD behaviour of effective Higgs mass. Possibility of multiple phase transitions as revealed by gaussian approximation contrasts with the double phase transition found in perturbative analysis and may be attributed to the non-perturbative effects.

Chapter 5 presents a brief summary of our work and some possible applications of the result obtained. After a brief discussion of possible cosmological applications of our results, their relevance in the context of high temperature superconductivity are discussed. The results of the specific heat calculations are presented and the experimental claims for the existence of reentrant superconduct-

ing transitions in high T_c materials which support our results regarding multiple phase transitions are discussed.

Most of the work presented in the thesis has been published or communicated for publication in the form of following papers:

1. Phase Transitions in ϕ^4 Theory at Finite Densities,
   V. J. Peter and M. Sabir, *Mod. Phys. Lett.* **A 4**, 783, (1989).
2. Phase Transitions in Abelian Higgs Model at Finite Densities,
   V. J. Peter and M. Sabir, *Int. J. Mod. Phys.* **A 6** , 4063 (1991).
3. Gaussian Effective Potential Study of Finite Density Phase Transitions in ϕ^4 Model,
   V. J. Peter and M. Sabir, *J. Phys.* **A**, (in press).
4. Finite Density Phase Transitions and High Temperature Superconductivity,
   V. J. Peter and M. Sabir, *Pramana - J. Phys.* , (communicated).

CONTENTS

Chapter 1

INTRODUCTION

Finite temperature and density field theories (FTDF) or statistical field theories had their origin in early works of relativistic many body theories [1-7]. The present interest in the amalgamation of field theory and statistical mechanics springs from the realization that many problems encountered experimentally and theoretically in particle physics have many body aspects [8]. In the context of spontaneously broken gauge theories (SBGT) the observation of symmetry restoration at finite temperature [4-10] has been one of the important reasons for the increased interest in FTDF since the early seventies. Since SBGT provide the basic frame work for unification of fundamental interactions the possibility of phase transitions in these theories at finite temperature / density has important implications especially regarding the early stages of the universe in the hot big bang model [9, 10]. Another important area of application of FTDF is in the study of phase transitions from hadronic to quark-gluon plasma.

In this introductory chapter, we outline the general formalism of

FTDF starting with the basic principles of statistical mechanics and field theory. This development is most elegantly accomplished by means of Feynmans functional integral formalism, which is used here to go from an expression for the time translation operator $\exp(-i\hat{H}t)$ to the partition function $Tr(\exp(-\beta(\hat{H}-\mu\hat{N})))$ by means of analytic continuation. Having a functional integral expression for the partition function then allows a straight forward derivation of diagrammatic rules for interacting theories. Following this we describe two important techniques useful for the study of finite temperature / density phase transitions. One is the effective potential method and its evaluation by the loop expansion procedure. The other one is the non-perturbative variational method of gaussian effective potential. Some general remarks on phase transitions in field theories are also given here.

1.1 Finite Temperature / Density Field Theory

1.1.1 Grand Canonical Partition Function

Consider a dynamical system characterized by a time independent Hamiltonian $\hat{H}$ and a set of conserved number operators $\hat{N}_i$. The equilibrium state of the system at rest in a large volume V is described by the grand-canonical density matrix [11-20]

$$\hat{\rho} = \exp\left(-\beta\left(\hat{H} - \mu_i\hat{N}_i\right)\right) \tag{1.1}$$

where μ_i is the chemical potential of the i^{th} species and

$$\beta = (k_B T)^{-1} = T^{-1} \tag{1.2}$$

in units with the Boltzmann constant k_B set equal to 1. The ensemble average of an operator $\hat{A}$ is

$$A = \frac{Tr\ \hat{\rho}\hat{A}}{Tr\ \hat{\rho}} \tag{1.3}$$

The grand canonical partition function is

$$Z = Tr\ \hat{\rho} \tag{1.4}$$

The function Z=Z(V, T, $\mu_1, \mu_2, \cdots$) is the single, most important function in thermodynamics. From it all other standard thermodynamic properties may be determined. For example, the pressure, particle number, entropy, and energy are in the infinite volume limit,

$$P = T\frac{\partial \ln Z}{\partial V}\ , \tag{1.5}$$

$$N_i = T\frac{\partial \ln Z}{\partial \mu_i}\ , \tag{1.6}$$

$$S = \frac{\partial (T\ \ln Z)}{\partial T}\ , \tag{1.7}$$

$$E = -P\,V + T\,S + \mu_i\ N_i\ . \tag{1.8}$$

The partition function (1.4) may be formulated as a path integral by the following series of steps [21]. First we take the independent states of the system to be the eigenstates of the Schrodinger picture field operators and introduce eigenstates $\mid \phi(x), t >$ of the Heisenberg picture field operators $\hat{\phi}(t,x)$ by

$$\hat{\phi}(t,x) \mid \phi(x), t > \ = \ \phi(x) \mid \phi(x), t >, \tag{1.9}$$

where $\phi(x)$ are complex functions. The Schrodinger picture field operators are $\hat{\phi}(t=0,x)$, and the corresponding eigenstates $\mid \phi(x), t=0 >$ are given by

$$\hat{\phi}(t=0,x) \mid \phi(x), t=0 > \ = \ \phi(x) \mid \phi(x), t=0 > . \tag{1.10}$$

Then the partition function of (1.4) may be written explicitly as a summation over the eigenstates:

$$Z = \sum < \phi(x), t = 0 \mid \exp\left(-\beta\left(\hat{H} - \mu\hat{N}\right)\right) \mid \phi(x), t = 0 > . \qquad (1.11)$$

Second, we make an analogy with the zero temperature field theory of a scalar field. The transition amplitude written as a path integral is

$$\begin{aligned} < \phi''(x), t'' \mid \phi'(x), t' > &= < \phi''(x), t = 0 \mid \exp(-i\hat{H}(t'' - t')) \mid \phi'(x), t = 0 > \\ &\propto \int \mathcal{D}\phi \int \mathcal{D}\pi \exp\left(i\int_{t'}^{t''} dt \int d^3x \left(\pi\frac{\partial\phi}{\partial t} - \mathcal{H}(\pi, \phi)\right)\right) \end{aligned} \qquad (1.12)$$

where the path integral is over all momentum functions $\pi(t, x)$, and over field functions $\phi(t, x)$ satisfying the boundary conditions

$$\phi(t'', x) = \phi''(x), \quad \phi(t', x) = \phi'(x). \qquad (1.13)$$

If, heuristically we introduce a imaginary time variable

$$\tau = it = ix^0 \qquad (1.14)$$

and take the limits of integration in (1.12) to be

$$t' = 0, \qquad t'' = -i\beta \qquad (1.15)$$

we obtain,

$$\begin{aligned} < \phi''(x), t = 0 \mid \quad & \exp\left(-\beta\ \left(\hat{H} - \mu\hat{N}\right)\right)\ \mid \phi'(x), t = 0 > \\ & \propto \int \mathcal{D}\phi^*\ \mathcal{D}\phi \int \mathcal{D}\pi^*\ \mathcal{D}\pi \exp\left(\int_0^\beta d\tau \int d^3x \right. \\ & \left. \left(i\pi^*\frac{\partial\phi^*}{\partial\tau} + i\pi\frac{\partial\phi}{\partial\tau} - \mathcal{H}(\pi^*,\ \pi,\ \phi^*,\ \phi\ ) + \mu\mathcal{N}\right)\right), \end{aligned} \qquad (1.16)$$

where ϕ and π are now regarded as functions of τ and x, and the path integral is over all functions $\pi(\tau, x)$ and over functions $\phi(\tau, x)$ satisfying the boundary conditions

$$\exp(\beta\mu)\ \phi(\beta, x) = \phi''(x), \qquad \phi(0, x) = \phi'(x). \tag{1.17}$$

The final step is to take

$$|\ \phi''(x), t = 0 > \ = \ |\ \phi'(x), t = 0 > \ = \ |\ \phi(x), t = 0 > \tag{1.18}$$

in (1.16) and sum over all eigenstates as in (1.11). Then we obtain

$$Z \propto \int_{\phi(\beta)=\phi(0)} \mathcal{D}\phi^*\ \mathcal{D}\phi\ \mathcal{D}\pi^*\ \mathcal{D}\pi \exp\left(\int_0^\beta d\tau \int d^3x \left(i\pi^* \frac{\partial\phi^*}{\partial\tau} + i\pi\frac{\partial\phi}{\partial\tau} - \mathcal{H} + \mu\mathcal{N}\right)\right). \tag{1.19}$$

The boundary conditions (1.17) together with (1.18) mean that the path integral is now restricted to functions $\phi(\tau, x)$ which are periodic in τ with

$$\phi(\tau = 0, x) \ = \ \exp(\beta\mu)\ \phi(\tau = \beta, x). \tag{1.20}$$

When the Lagrangian and Hamiltonian densities take the form

$$\mathcal{L}\,(\phi, \phi^*, \partial_\mu\phi, \partial_\mu\phi^*) = (\partial_\mu\phi)\,(\partial_\mu\phi)^* - f\,(\phi, \phi^*, \nabla\phi, \nabla\phi^*) \tag{1.21}$$

and

$$\mathcal{H} = \pi^*\pi - f\,(\phi, \phi^*, \nabla\phi, \nabla\phi^*) \tag{1.22}$$

the integration over π, may be carried out explicitly to obtain

$$\begin{aligned} Z \ &= \ N(\beta)\int_{periodic} \mathcal{D}\phi^*\ \mathcal{D}\phi \exp\left(-\int_0^\beta d\tau \int d^3x \left(\left(\frac{\partial\phi^*}{\partial\tau}\right)\left(\frac{\partial\phi}{\partial\tau}\right)\right.\right. \\ &\quad \left.\left. -f\,(\phi, \phi^*, \nabla\phi, \nabla\phi^*) + \mu\mathcal{N}\right)\right) \\ &= \ N(\beta)\int_{periodic} \exp\left(\int_0^\beta d\tau \int d^3x\,(\mathcal{L}\,(\phi, \phi^*, \partial_\mu\phi, \partial_\mu\phi^*) + \mu\mathcal{N})\right) \end{aligned} \tag{1.23}$$

where

$$\partial_\mu \phi = \left(i\frac{\partial \phi}{\partial \tau}, \nabla \phi \right)$$

and $N(\beta)$ is a temperature dependent normalization factor.

1.1.2 Finite Temperature and density (FTD) Greens Functions and Generating Functionals

FTD Greens functions are extremely useful objects which contain information about the equilibrium thermodynamic properties of the FTDF system [22-25]. For simplicity of presentation we shall restrict the discussion to a single complex scalar field. The 2-point FTD Greens function is defined by

$$G^{(2)}(\bar{x}_1, \bar{x}_2) \equiv < T_\tau \left(\hat{\phi}^\dagger(\bar{x}_1) \hat{\phi}(\bar{x}_2 \right) > \tag{1.24}$$

where

$$\bar{x} \equiv (-i\tau, x) \tag{1.25}$$

and T_τ is the τ- ordering operator which orders the fields from right to left in the order of increasing τ. The expectation value $< \quad >$ here means a thermal average rather than just a vacuum expectation value.

$$< T_\tau \left(\hat{\phi}^\dagger(\bar{x}_1) \hat{\phi}(\bar{x}_2) \right) > \equiv \frac{Tr \left[\exp\left(-\beta \left(\hat{H} - \mu \hat{N} \right) \right) \; T_\tau \left(\hat{\phi}(\bar{x}_1) \hat{\phi}(\bar{x}_2) \right) \right]}{Tr \left[\exp\left(-\beta \left(\hat{H} - \mu \hat{N} \right) \right) \right]} \tag{1.26}$$

where, the trace means to sum the matrix elements of the operators in the square bracket between all independent states of the system. The method described earlier for partition function can be easily extended to obtain a path integral representation

for the Greens functions. One finds:

$$G^{(2)}(\bar{x}_1,\bar{x}_2) \;=\; \frac{1}{Z}\int \mathcal{D}\phi^*\,\mathcal{D}\phi\;\phi^*\phi \exp\left(\int_0^\beta d\tau \int d^3x\,(\mathcal{L}(\phi,\phi^*,\partial_\mu\phi,\partial_\mu\phi^*)+\mu\mathcal{N})\right). \tag{1.27}$$

By analogy with zero temperature case, we now introduce a generating functional for FTD Green functions [26, 27]

$$\bar{W}[J] \;=\; \frac{1}{Z}\int \mathcal{D}\phi^*\,\mathcal{D}\phi \left(\exp\left(\int_0^\beta d\tau\int d^3x\,(\mathcal{L}(\phi,\phi^*,\partial_\mu\phi,\partial_\mu\phi^*)+\mu\mathcal{N})\right)\right. \tag{1.28}$$
$$\left.\exp\left(-\int_0^\beta d\tau (J^*(\bar{x})\phi+\phi^* J(\bar{x}))\right)\right),$$

where the sources J are complex functions of $\bar{x}$. The FTD Greens functions are obtained from $\bar{W}[J]$ by functional differentiation

$$G^{(2)}(\bar{x}_1,\bar{x}_2) = \frac{\delta^2\bar{W}[J]}{\delta J^*(\bar{x}_2)\delta J(\bar{x}_1)}\;|_{(J^*=J=0)}\;. \tag{1.29}$$

Usually, it is more convenient to work with Greens functions in momentum space. Fourier transformed FTD Green functions $\tilde{G}^{(2)}$ may be introduced through

$$\tilde{G}^{(2)}(\bar{k}_1,\bar{k}_2)\,\delta\left(\bar{k}_1+\bar{k}_2\right)(2\pi)^3\beta \;=\; \int d\bar{x}_1\int d\bar{x}_2 \exp\left(i\left(\bar{k}_1.\bar{x}_1+\bar{k}_2.\bar{x}_2\right)\right)G^{(2)}(\bar{x}_1,\bar{x}_2)\,, \tag{1.30}$$

where

$$d\bar{x} \equiv \int_0^\beta d\tau\int d^3x, \qquad \bar{k}\equiv(i\omega_n,K) \tag{1.31}$$

with

$$\omega_n \equiv \frac{2\pi n}{\beta} - i\mu, \qquad \bar{k}.\bar{x}\equiv\omega_n\tau - K.X. \tag{1.32}$$

and

$$\delta\left(\bar{k}_1 + \bar{k}_2\right) \equiv \delta_{\omega_1+\omega_2}\delta(K_1 + K_2)\,. \tag{1.33}$$

The discreteness of the frequencies ω_n is a consequence of the periodicity condition (1.20).

A generating functional $\bar{X}[J]$ for connected FTD Green functions $\mathcal{G}^{(2)}$ may be defined through

$$\bar{W}[J] = \exp(\bar{X}[J]) \tag{1.34}$$

with the relations

$$\mathcal{G}^{(2)}(\bar{x}_1, \bar{x}_2) = \frac{\delta^{(2)}\bar{X}[J]}{\partial J^*(\bar{x}_2)\ \partial J(\bar{x}_1)}\,|_{(J=0,\ J^*=0)} \tag{1.35}$$

and

$$\bar{X}[J] = \sum_{N=1}^{\infty} \frac{1}{N!} \int d\bar{x}_1 \cdots \int d\bar{x}_N\, \mathcal{G}^{(N)}(\bar{x}_1, \cdots \bar{x}_N). \tag{1.36}$$

1.1.3 Perturbation Theory and Feynman Rules

In the case of a free complex scalar field with Lagrangian density

$$\mathcal{L}_0\left(\phi, \bar{\partial}_\mu\phi\right) = -\left(\frac{\partial\phi}{\partial\tau}\right)\left(\frac{\partial\phi}{\partial\tau}\right) - \nabla\phi^*\nabla\phi - m^2\phi^*\phi \tag{1.37}$$

the generating functional can be evaluated exactly to yield the Greens function

$$G\left(\bar{x}' - \bar{x}\right) = \Delta\left(\bar{x}' - \bar{x}\right) \tag{1.38}$$

with

$$\Delta\left(\bar{x}' - \bar{x}\right) = \frac{1}{\beta}\sum_n \int \frac{d^3K}{(2\pi)^3} \exp\left(-ik(x' - x)\right) \bar{\Delta}(\bar{k}) \tag{1.39}$$

and

$$\bar{\Delta}(\bar{k}) = -\left(\omega_n^2 + K^2 + m^2\right)^{-1} \tag{1.40}$$

where ω_n is defined by (1.32).

If we have a Lagrangian of the form

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I(\phi)$$

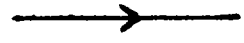
where $\mathcal{L}_I$ represents an interaction term which is higher than quadratic in ϕ the generating functional can not be evaluated exactly. But using the result

$$W[J] = \exp\left(i\int dx \mathcal{L}_I\left(-i\frac{\partial}{\partial J(\bar{x})}\right) W_0[J]\right) \tag{1.41}$$

a perturbation series can be developed by expanding the exponential operator as a power series. As is well known in the case of zero temperature field theory this leads to the perturbation evaluation of Greens function by means of Feynman diagrams.

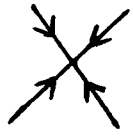
The usual approach used for ordinary Greens functions can be easily adopted to FTD Greens functions. The only difference arise because k_0 has been replaced by $i\omega_n$, and because various factors of i no longer occur in Greens functions and generating functionals, compared with the zero temperature case. The resulting Feynman rules for perturbative computation of the 2-point connected Greens functions in ϕ^4-theory are as follows [23, 25]:

1. Draw all possible connected, topologically distinct, graphs with 2 external lines ;
2. With each line carrying 'momentum' $\bar{k} = (i\omega_n, K)$ we associate a factor



$$: -(\omega_n^2 + K^2 + m^2)^{-1} ;$$

2. With each vertex of four lines carrying 'momenta' $\bar{k}_1, \bar{k}_2, \bar{k}_3, \bar{k}_4$ we associate a factor $(-\lambda)$, constraining the 'momenta' so that there is overall conservation



$$: \quad -\lambda \qquad \left(\bar{k}_1 + \bar{k}_2 + \bar{k}_3 + \bar{k}_4 = 0\right);$$

3. Integrate and sum over each independent internal loop 'momentum' $\bar{k} = (i\omega_n, K)$ with weight

$$\frac{1}{\beta}\sum_n \int \frac{d^3K}{(2\pi)^3}, \qquad \omega_n = \frac{2\pi n}{\beta} - i\mu .$$

The corresponding modifications are made to the Feynman rules for gauge fields. No factors of i for vertices or propagators,
$k_0 \to i\omega_n$ and $\int (d^4k/(2\pi)^4) \to (1/\beta) \sum_n (d^3K/(2\pi)^3)$.

1.1.4 Real Time Formalism

The discussion of FTDF has so far been in the so called imaginary time formalism or Matsubara formalism which includes a discrete set of energies. The study of dynamical problems requires the use of real time Greens functions which requires a continuation from the discrete energies to the real axis. In principle, this can be done by a process of analytic continuation. However, this process is mathematically difficult and sometimes result in ambiguities. The time path method [25, 27-31] has been developed as an alternative scheme for real time FTDF. Since our interest is not in formal developments and most of our calculation can be done satisfactorily in the imaginary time formalism, we do not describe these developments in detail. We

make use of real time Greens functions only in Chapter 3. The connection between the real and imaginary time Greens functions is summarized in an appendix.

1.1.5 The Effective Potential Method

The concept of effective potential is a very useful one in understanding many important questions in quantum field theory, particularly those related to spontaneous symmetry breaking (SSB) [32-36]. The effective potential includes all quantum corrections to the classical field theory potential. By minimizing the effective potential the true ground state of the theory can be identified.

Effective Potential at Zero Temperature

Let us consider the simple case of a scalar field with the generating functional

$$W[J] = \int \mathcal{D}\phi \exp\left(i \int d^4x \left(\mathcal{L}(\phi(x)) + J(x)\phi(x)\right)\right). \tag{1.42}$$

When $\ln W[J]$ is expanded in a functional Taylor series in J(x), the coefficients will be the connected Greens functions

$$\ln W[J] = \sum_n \frac{1}{n!} \int d^4x_1 \cdots d^4x_n \, G^{(n)}(x_1 \cdots x_n) J(x_1) \cdots J(x_n). \tag{1.43}$$

We define the classical field ϕ_c as the vacuum expectation value of the operator ϕ in the presence of the source J(x)

$$\phi_c(x) = \frac{\delta \ln W}{\delta J(x)} = \left(\frac{< 0 \mid \phi(x) \mid 0 >_J}{< 0 \mid 0 >_J}\right). \tag{1.44}$$

The effective action of the classical field $\Gamma(\phi_c)$ is defined by the functional Legendre transform

$$\Gamma(\phi_c) = \ln W[J] - \int d^4x\ J(x)\phi_c(x). \tag{1.45}$$

From this definition, it follows that

$$\frac{\delta\Gamma(\phi_c)}{\delta\phi_c} = -J(x). \tag{1.46}$$

We can also expand $\Gamma(\phi_c)$ in powers of ϕ_c

$$\Gamma(\phi_c) = \sum_n \frac{1}{n!}\int d^4x_1 \cdots d^4x_n\ \Gamma^{(n)}(x_1 \cdots x_n)\phi_c(x_1)\cdots\phi(x_n). \tag{1.47}$$

It is possible to show that $\Gamma^{(n)}(x_1 \cdots x_n)$ is the sum of all one-particle irreducible (1PI) Feynman diagrams with n external lines. Alternatively we can expand the effective action $\Gamma(\phi_c)$ in powers of momentum about zero momentum. In position space this expansion takes on the form

$$\Gamma(\phi_c) = \int d^4x \left[-V_{eff}(\phi_c) + \frac{1}{2}\left(\partial_\mu\phi_c\right)^2 Z(\phi_c) + \cdots\right] \tag{1.48}$$

The term without derivatives, $V_{eff}(\phi_c)$ is called the effective potential. If ϕ_c is constant in space and time, from (1.46) it follows that

$$\frac{dV_{eff}}{d\phi_c}\Big|_{(J=0)} = 0. \tag{1.49}$$

The vacuum expectation value of the full operator can be found by solving this equation. In this case V_{eff} can be expanded in terms of 1PI Greens functions, first writing $\Gamma^{(n)}$ in momentum space

$$\begin{aligned}\Gamma^{(n)}(x_1 \cdots x_n) \;=\; & \int \frac{d^4k_1}{(2\pi)^4} \cdots \frac{d^4k_n}{(2\pi)^4}\ (2\pi)^4\delta^4(k_1 \cdots k_n)\exp\left(i(k_1x_1 + \cdots k_nx_n)\right) \\ & \Gamma^{(n)}(k_1 \cdots k_n).\end{aligned} \tag{1.50}$$

Substitution of this into (1.47) gives

$$\begin{aligned}\Gamma(\phi_c) &= \sum_n \frac{1}{n!}\int d^4x_1\cdots d^4x_n \int \frac{d^4k_1}{(2\pi)^4}\cdots\frac{d^4k_n}{(2\pi)^4}\int d^4x \exp\left(i(k_1+\cdots k_n).x\right) \\ &\quad \exp\left(i(k_1.x_1+\cdots k_n.x_n)\right)\left(\Gamma^{(n)}(0,\cdots 0)\phi_c(x_n)\phi_c(x_n)+\cdots\right) \\ &= \int d^4x \sum_n \frac{1}{n!}\left(\Gamma^{(n)}(0,\cdots 0)\,(\phi_c)^n+\cdots\right). \qquad (1.51)\end{aligned}$$

Comparing (1.48) and (1.51) we find that $V_{eff}(\phi_c)$ which means sum of all 1PI diagrams with n external lines carrying zero momenta

$$V_{eff}(\phi_c) = -\sum_n \frac{1}{n!}\,\Gamma^{(n))}(0\cdots 0)\,(\phi_c(x))^n\,. \qquad (1.52)$$

An elegant method particularly suited for study of non-perturbative phenomena like SSB is the loop expansion technique. This is an expansion according to the increasing number of independent loops of connected Feynman diagrams. Thus the lowest order graphs will be the Born diagrams or tree graphs. The next order consists of the one-loop diagrams which have one integration over the internal momenta, etc. For the effective potential (1.52) each loop level still involves an infinite summation corresponding to all possible lines. The usual classical potential is simply the first term (the tree graphs) of $V_{eff}(\phi_c)$ in this loop expansion. In fact it can be shown that, the loop expansion can be identified as an expansion in powers of the plancks constant $\hbar$.

We now illustrate the calculation of effective potential in the simple case of $\lambda\phi^4$ theory in the one-loop approximation. The Lagrangian density is given by

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I \qquad (1.53)$$

with

$$\begin{aligned} \mathcal{L}_0 &= \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 , \\ \mathcal{L}_I &= -\frac{\lambda}{4!}\phi^4. \end{aligned} \tag{1.54}$$

To calculate the effective potential in eqn (1.52), we must sum all one-loop diagrams with an even number of external lines having zero momenta (see Fig. 1.1). The 1PI Greens function is given by

$$\Gamma^{2n}(0\cdots 0) = iS_n \int \frac{d^4k}{(2\pi)^4}\left((-i\lambda)\frac{i}{k^2 - m^2 + i\epsilon}\right)^n \tag{1.55}$$

where S_n is the symmetry factor

$$S_n = \frac{(2n)!}{2^n 2n} \tag{1.56}$$

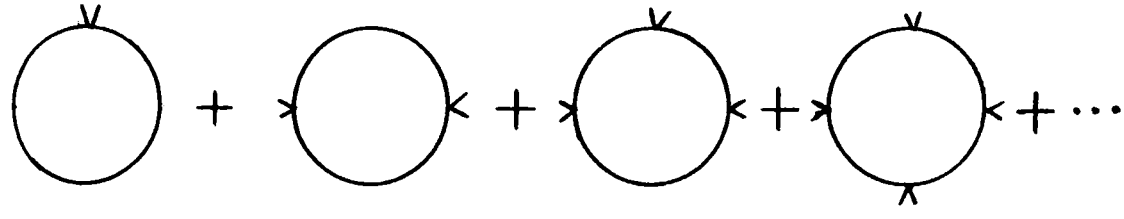


Fig. 1.1 One-loop diagrams contributing to the effective potential

corresponding to the fact that there are $(2n)!$ ways to distribute 2n particles to the external lines of the diagram and that interchanges of any two external lines at a given vertex or reflections and rotations of n vertices on the ring do not lead to new contributions. The no-loop and one-loop effective potential is then given by

$$\begin{aligned} V_{eff}(\phi_c) &= \frac{1}{2}m^2\phi_c^2 + \frac{\lambda}{4!}\phi_c^4 + i\int\frac{d^4k}{(2\pi)^4}\sum_{n=1}^{\infty}\frac{1}{2n}\left(\frac{(\lambda/2)\phi_c^2}{k^2-m^2+i\epsilon}\right)^n \\ &= \frac{1}{2}m^2\phi_c^2 + \frac{\lambda}{4!}\phi_c^4 + \frac{i}{2}\int\frac{d^4k}{(2\pi)^4}\ln\left(1+\frac{(\lambda/2)\phi_c^2}{k^2-m^2+i\epsilon}\right). \end{aligned} \tag{1.57}$$

The integral is divergent. If it is cut off at some large momentum Λ, we obtain

$$\begin{aligned} V_{eff}(\phi_c) &= \frac{1}{2}m^2\phi_c^2 + \frac{\lambda}{4!}\phi_c^4 + \frac{\Lambda^2}{32\pi^2}\left(m^2+\frac{\lambda}{2}\phi_c^2\right) \\ &\quad + \frac{1}{64\pi^2}\left(m^2+\frac{\lambda}{2}\phi_c^2\right)^2\left(\ln\left(\frac{m^2+(\lambda/2)\phi^2+i\epsilon}{\Lambda^2}\right)-\frac{1}{2}\right). \end{aligned} \tag{1.58}$$

To remove the cut off dependence we introduce counter terms which have the same structure as the original potential

$$V_{eff}^{ct}(\phi_c) = \frac{A}{2}\phi_c^2 + \frac{B}{4!}\phi_c^4\,, \tag{1.59}$$

so that the renormalized effective potential, given by

$$V_{eff}^{ren}(\phi_c) = V_{eff}(\phi_c) + V_{eff}^{ct}(\phi_c) \tag{1.60}$$

is finite and cut-off independent. The coefficients A and B in (1.59) can be determined by the renormalization conditions

$$m^2 = \frac{d^2V_{eff}}{d\phi_c^2}\,|_{(\phi_c=0)} \tag{1.61}$$

and

$$\lambda = \frac{d^4V_{eff}}{d\phi_c^4}\,|_{(\phi_c=0)}\,. \tag{1.62}$$

In this way we have

$$V_{eff}^{ren} = \frac{1}{2}m^2\phi_c^2 + \frac{\lambda}{4!}\phi_c^4 + \frac{1}{64\pi^2}\left(\left(m^2+\frac{\lambda}{2}\phi_c^2\right)^2 \ln\left(\frac{m^2+(\lambda/2)\phi_c^2}{m^2}\right) - \frac{\lambda}{2}m^2\phi_c^2 - \frac{3}{8}\lambda^2\phi_c^4\right). \tag{1.63}$$

The computation of one-loop effective potential can be easily extended to gauge models. However, the combinatorial method is limited to one-loop and to calculate the higher loop effects systematically functional techniques have been employed [20].

1.1.6 Finite Temperature Effective Potential (FTEP)

The technique of effective potential can be easily generalized to the case of finite temperature field theory by replacing the generating functional and Greens functions at zero temperature by those at finite temperatures [21]. Let us define a classical field $\phi_c(\bar{x})$ by

$$\phi_c = \frac{\partial \bar{X}[J]}{\partial J(\bar{x})}\,. \tag{1.64}$$

From (1.29)

$$\frac{\delta \bar{W}[J]}{\partial J(\bar{x})} = <\hat{\phi}(\bar{x})>_J \tag{1.65}$$

where $<\hat{\phi}(\bar{x})>_J$ is the expectation value (thermal average) of $\hat{\phi}(\bar{x})$ in the presence of the source J. Using (1.34)

$$\phi_c(\bar{x}) = \frac{<\hat{\phi}(\bar{x})>_J}{\bar{W}[J]}\,. \tag{1.66}$$

For zero source

$$\phi_c(\bar{x}) = <\hat{\phi}(\bar{x})>\,, \qquad J=0 \tag{1.67}$$

since

$$\bar{W}[0] = 1. \tag{1.68}$$

Moreover,

$$< \hat{\phi}(\bar{x}) > = \frac{Tr\left[\exp(-\beta\hat{H})\hat{\phi}(\tau,x)\right]}{Tr\left[\exp(-\beta\hat{H})\right]} \sim \frac{Tr\left[\exp(-\beta\hat{H})\hat{\phi}(0,x)\right]}{Tr\left[\exp(-\beta\hat{H})\right]}, \tag{1.69}$$

where we have used the connection between the field operator at time t and time zero, with $t \rightarrow -i\tau$. Combining (1.67) and (1.69),

$$\phi_c(\bar{x}) =< \hat{\phi}(0,x) > , \qquad J = 0. \tag{1.70}$$

Thus, for zero source, $\phi(\bar{x})$ is the expectation value (thermal average) of $\hat{\phi}(0,\bar{x})$, the Schrodinger picture field operator. An effective action is defined by

$$\bar{\Gamma}[\phi_c] = \bar{X}[J] - \int d\bar{x}\ J(\bar{x})\phi_c(\bar{x}) \tag{1.71}$$

and the source is given by

$$J(\bar{x}) = -\frac{\delta\bar{\Gamma}[\phi_c]}{\partial\phi_c(\bar{x})}\ . \tag{1.72}$$

1PI temperature Greens functions $\Gamma^{(n)}$, may be defined by the expansion

$$\bar{\Gamma}[\phi_c] = \sum_{N=1}^{\infty}\frac{1}{n!}\int d\bar{x_1},\cdots\int d\bar{x_N}\ \Gamma^{(N)}(\bar{x_1},\cdots\bar{x_N})\ \phi_c(\bar{x_1})\cdots\phi_c(\bar{x_N}) \tag{1.73}$$

and momentum space 1PI temperature Greens functions, $\tilde{\Gamma}^{(N)}$, by

$$\tilde{\Gamma}^{(N)}(\bar{k}_1,\cdots\bar{k}_N)(2\pi)^3\beta = \int d\bar{x}_1\cdots\int d\bar{x}_N\ \exp[i(\bar{k}_1.\bar{x}_1 + \cdots\bar{k}.\bar{x}_N)]\ \Gamma^{(N)}(\bar{x}_1,\cdots\bar{x}_N) \tag{1.74}$$

with $\delta(\bar{k}_1+\cdots+\bar{k}_N)$ as in (1.33). The FTEP $V^{\beta}_{eff}(\phi_c)$ may be defined by an expansion

$$\bar{\Gamma}[\phi_c] = \int d\bar{x}\ \left(-V^{\beta}_{eff}(\phi_c) + \frac{\bar{A}(\phi_c)}{2}\bar{\partial}_\mu\phi_c\bar{\partial}^\mu\phi_c + \cdots\right). \tag{1.75}$$

If the classical field has no spatial (or τ) dependence then only the $V^{\beta}_{eff}(\phi_c)$ term in the expansion (1.75) need to be retained, and (1.72) becomes

$$\frac{dV^{\beta}_{eff}}{d\phi_c} = J \,. \tag{1.76}$$

If we set the source term to zero, then from (1.70) ϕ_c has the significance of the expectation value (thermal average) of the field operator, and

$$\frac{dV^{\beta}_{eff}}{d\phi_c} = 0 \,. \tag{1.77}$$

Thus when it has no spatial variation, the expectation value of the field operator at finite temperature may be obtained by minimizing the FTEP.

Using the inverse of (1.74) in (1.73), the effective potential may be expanded in terms of Fourier transformed temperature Greens functions at zero 'momenta'

$$V^{\beta}_{eff}(\phi_c) = -\sum_{N=1}^{\infty}\left(\frac{1}{N!}\tilde{\Gamma}^{(N)}(0,\cdots 0)\,\phi_c^N\right) \,. \tag{1.78}$$

To actually compute the FTEP, we can use the loop expansion method discussed earlier for the zero temperature case. The diagrams which contribute to the FTEP are same as that of zero temperature effective potential. Making use of the Feynman rules discussed in Sec. 1.1.3 we get the effective potential. This is shown below by computing V^{β}_{eff} upto the one-loop terms for spinless particles.

Consider the simplest model of one self interacting bose field described by the Lagrangian (1.53). The tree level potential is

$$V_0(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 \,. \tag{1.79}$$

First we write the one-loop effective potential at zero temperature [37-39]. Diagrammatically we have

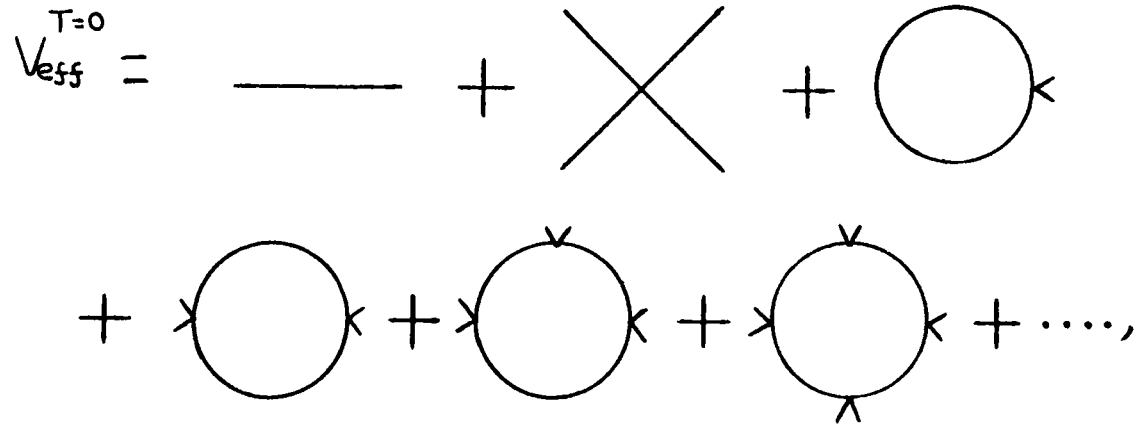


Fig. 1.2

so that $V_{eff}(\bar{\phi})$ where, $\bar{\phi}$ is a constant field, can be written as

$$\begin{aligned} V_{eff}^{T=0}(\bar{\phi}) &= V_0(\bar{\phi}) + i\sum_{n=1}^{\infty}\int\frac{d^4k}{(2\pi)^4}\frac{1}{2n}\left(\frac{(\lambda/2)\bar{\phi}^2}{k^2-m^2+i\epsilon}\right)^n \\ &= V_0(\bar{\phi}) - \frac{i}{2}\int\frac{d^4k}{(2\pi)^4}\ln\left(1-\frac{(\lambda/2)\bar{\phi}^2}{k^2-m^2+i\epsilon}\right). \end{aligned} \tag{1.80}$$

After a Wick rotation $k^0 = ik_E^0$, $k_E = (-ik_0, K)$, $k^2 = (k^0)^2 - K^2 = -k_E^{0\,2} - K^2 = -k_E^2$. Eqn. (1.80) can be cast as

$$V_{eff}^{T=0}(\bar{\phi}) = V_0(\bar{\phi}) + \frac{1}{2}\int\frac{d^4k_E}{(2\pi)^4}\ln\left(1+\frac{(\lambda/2)\bar{\phi}^2}{k_E^2+m^2}\right). \tag{1.81}$$

At finite temperature, the Fig. 1.2 can be computed with the help of the rules of Sec. 1.1.3:

$$V_{eff}^{\beta}(\bar{\phi}) = V_0(\bar{\phi}) + -\frac{1}{2\beta}\sum_{-\infty}^{\infty}\int\frac{d^3K}{(2\pi)^3}\ln\left(1-\frac{(\lambda/2)\bar{\phi}^2}{(-4\pi^2n^2/\beta^2)-K^2-m^2}\right)$$

$$= V_0(\bar{\phi}) + \frac{1}{2\beta}\sum_{-\infty}^{\infty}\int\frac{d^3K}{(2\pi)^3}\ln\left(\frac{4\pi^2n^2}{\beta^2}+E_M^2\right), \quad (1.82)$$

where $E_M^2 = K^2 + M^2$, $M^2 = m^2 + (\lambda/2)\bar{\phi}^2$.

The sum over n in (1.82) diverges, but the infinite part does not depend on $\bar{\phi}$. The finite part which contains the ϕ-dependence, can be computed to give

$$V_{eff}^{\beta}(\bar{\phi}) = V_0(\bar{\phi}) + \int\frac{d^3K}{(2\pi)^3}\left(\frac{E_M}{2}+\frac{1}{\beta}\ln(1-\exp(-\beta E_M))\right). \quad (1.83)$$

The first integral in (1.83) is just the one-loop effective potential at zero-temperature. It is

$$\int\frac{d^3K}{(2\pi)^3}\frac{E_M}{2} = \frac{-i}{2}\int\frac{d^4k}{(2\pi)^4}\ln\left(-k_0^2+K^2+m^2+\frac{\lambda}{2}\bar{\phi}^2-i\epsilon\right). \quad (1.84)$$

From (1.84) and (1.81) we see that

$$V_{eff}^{\beta}(\bar{\phi}) = V_{eff}^{T=0}(\bar{\phi}) + V_{eff}^{T\neq 0}(\bar{\phi}) \quad (1.85)$$

with $V_{eff}^{T=0}(\bar{\phi})$ given by (1.81) zero temperature one-loop effective potential and

$$V_{eff}^{T\neq 0}(\bar{\phi}) = \frac{1}{\beta}\int\frac{d^3K}{(2\pi)^3}\ln(1-\exp(-\beta E_M)). \quad (1.86)$$

1.2 Phase Transitions in Field Theories

A symmetry is said to be spontaneously broken if $\phi_c = <0 \mid \phi \mid 0> \neq 0$ which corresponds to a minimum of the effective potential. At finite temperatures the

ground state value of the field ϕ , does not correspond to the minimum of the effective potential $V_{eff}^{T=0}(\bar{\phi})$, but to the minimum of the FTEP $V_{eff}^{\beta}(\bar{\phi})$. Thus even if the minimum of the $V_{eff}^{T=0}(\bar{\phi})$ occurs at $\bar{\phi} = \sigma \neq 0$, very often, for sufficiently large temperatures, the minimum of $V_{eff}^{\beta}(\bar{\phi})$ is at $\bar{\phi} = 0$: this phenomenon is known as symmetry restoration at high temperature [2-6, 21, 40]. This gives rise to the phase transition from $\bar{\phi} = 0$ to $\bar{\phi} = \sigma$.

To illustrate the analysis of such phase transitions consider the simple model corresponding to (1.53) with $m^2>0$. At the tree level, the possible ground states of the theory determined by $\left(\partial V_0/\partial\bar{\phi}\right) = 0$ are $\bar{\phi} = 0$ and $\bar{\phi} = \pm\left(m/\sqrt{\lambda}\right)$. The state $\bar{\phi} = 0$ is a unstable local maximum and the energetically favored states corresponding to the minimum of $V(\bar{\phi})$, at $\bar{\phi} = \pm\left(m/\sqrt{\lambda}\right)$, are shown in Fig. 1.3. The symmetry $\bar{\phi} \rightarrow -\bar{\phi}$ of the Lagrangian (1.53) is spontaneously broken.

To investigate what happens to spontaneously broken symmetry at finite temperatures, one must compute the FTEP. At high temperatures this can be approximated as [5, 21]:

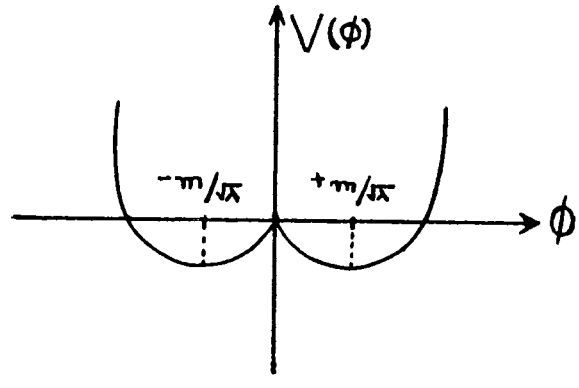


Fig. 1.3

$$V_{eff}^{\beta} = -\frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4 - \frac{\pi^2}{90\beta^4} + \frac{\left(m^2 + (\lambda/2)\bar{\phi}^2\right)}{24\beta^2} + O(\beta). \qquad (1.87)$$

The behaviour of V_{eff}^{β} is shown in Fig. 1.4 for a number of different temperatures. It is clear from Fig. 1.4 that as T rises, the equilibrium value of ϕ at the minimum of V_{eff}^{β} decreases and above some critical temperature T_c the only remaining minimum is the one at $\phi = 0$, i.e., symmetry is restored. Eqn.(1.87) then implies that the field ϕ decreases continuously to zero with rising temperature. The critical temperature for this phase transition is determined by the condition

$$m_{\beta}^2 = \frac{\partial V_{eff}^{\beta}}{\partial \phi^2}\Big|_{(\phi=0)} = 0\,, \qquad (1.88)$$

where m_{β} is called the temperature dependent mass. In this model we have

$$T_c^2 = \frac{1}{\beta_c^2} = \frac{-24m^2}{\lambda}\,. \qquad (1.89)$$

The method employed in studying the phase transitions in scalar models can be extended to theories involving gauge fields.

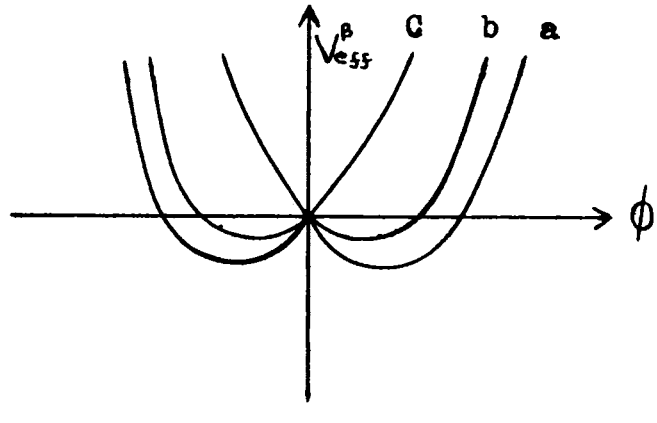


Fig. 1.4 (a)T=0 (b)0<T<T_c (c)T>T_c

1.3 Non Perturbative Study of FTDF using Gaussian Approximation

Gaussian approximation is a non-perturbative variational principle based on rather elementary and conceptually simple ideas well known from quantum mechanics of many body systems. Making use of this principle and a Gaussian trial wave functional for the ground state, a modified effective potential, namely the Gaussian effective potential (GEP) may be computed. In addition to its intuitive appeal [41, 42], this GEP is known to contain the one-loop and leading order 1/N results in the appropriate limiting cases [43-46]. The technique has been extensively developed, particularly in the last few years along the lines initiated by Stevenson, Consoli and others [41-91]. These various works differ in certain technicalities in the computation. It has been established that, in four dimensions a simple and viable non-trivial theory called precarious theory arising from a bare coupling constant of a particular negative infinitesimal form exist. Without regularization, this is stable, but does not possess SSB. However using a finite ultraviolet cutoff for the momentum integrals SSB can be induced. This cutoff version GEP method has been applied to various scalar and fermion models [92-96]. Ni et al [95, 97] have evaluated the GEP in cutoff version of certain gauge models including SU(2)xU(1) and obtained limits on Higgs boson mass.

Another version of GEP that has been shown to exist is the autonomous form, which possess SSB, and allows a positive value for the bare coupling constant [97-101]. But its stability depends strongly on the wave function renor-

malization and it is not definitely established whether autonomous theory is only an artifact of the Gaussian approximation or not. Extension of the GEP method to finite temperature field theory has been done by many authors [99-106]. In the evaluation of finite temperature GEP, puzzling nature of some of the earlier results were carefully analyzed by Stevenson and Hajj [101]. They had shown that, increase of temperature restores the spontaneously broken symmetry in scalar model. This section is devoted to review the finite temperature GEP (FTGEP) formalism.

1.3.1 Principle of GEP

We begin with the zero temperature definition of GEP. For definiteness, we shall consider $\lambda\phi^4$ theory in 3+1 dimensions. It is described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_B^2\phi^2 - \lambda_B\phi^4 \tag{1.90}$$

which corresponds to a Hamiltonian density

$$\mathcal{H} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m_B^2\phi^2 + \lambda_B\phi^4. \tag{1.91}$$

Let us write the field ϕ as $\phi_0 + \hat{\phi}$, where ϕ_0 is a constant classical field, and $\hat{\phi}$ is a quantum free field of mass Ω. The state $|\ 0>$ is then defined as the vacuum state of this free field of mass Ω. Following the usual field theoretical conventions, we write

$$\phi = \phi_0 + \int \frac{d^3K}{(2\pi)^{3/2}\,\omega_k(\Omega)}\left(a_\Omega(K)\exp(-ikx) + a_\Omega^\dagger(k)\exp(ikx)\right) \tag{1.92}$$

and hence

$$\partial_\mu\phi = \int \frac{d^3K}{(2\pi)^{3/2}\,\omega_k(\Omega)}\left(-ik_\mu\left(a_\Omega(k)\exp(-ikx) - a_\Omega^\dagger(k)\exp(ikx)\right)\right), \tag{1.93}$$

where the energy component of the four vector k_μ is

$$k^0 = \omega_k(\Omega) \equiv \sqrt{K^2 + \Omega^2}. \tag{1.94}$$

The creation and annihilation operators obey the usual commutation relation

$$[a_\Omega(k), a^\dagger_\Omega(k')] = \delta_{kk'} \equiv 2\omega_k(\Omega)(2\pi)^3\delta^{(3)}(K - K'). \tag{1.95}$$

The state $| 0 >$ has the defining property

$$a_\Omega(k) \mid 0 > = 0 \tag{1.96}$$

and

$$< 0 \mid \phi \mid 0 > = \phi_0. \tag{1.97}$$

The GEP is then defined as

$$\bar{V}_G(\phi_0) \;=\; \overset{min}{\Omega}\; V_G(\phi_0, \Omega) \;=\; \overset{min}{\Omega}\; < 0 \mid \mathcal{H} \mid 0 > . \tag{1.98}$$

The evaluation of $V_G(\phi_0, \Omega)$ for the Hamiltonian (1.91) is a straight forward exercise [107, 108]. Term by term we have

$$< 0 \mid \frac{1}{2}[\dot{\phi}^2 + (\nabla\phi)^2] \mid 0 > = \int \frac{d^3K}{(2\pi)^{3/2}\,\omega_k}\left(\omega_k^2(\Omega) - \frac{1}{2}\Omega^2\right), \tag{1.99}$$

$$< 0 \mid \frac{1}{2}m_B^2\phi^2 \mid 0 > = \frac{1}{2}m_B^2\left(\phi_0^2 + \int \frac{d^3K}{(2\pi)^{3/2}\,\omega_k}\right), \tag{1.100}$$

and

$$< 0 \mid \lambda_B\phi^4 \mid 0 > = \lambda_B\left(\phi_0^4 + 6\phi_0^2\int \frac{d^3K}{(2\pi)^{3/2}\,\omega_k} + 3\int \frac{d^3K}{(2\pi)^{3/2}\,\omega_k}\int \frac{d^3K}{(2\pi)^{3/2}\,\omega_k}\right). \tag{1.101}$$

Introducing the notation

$$I_N(\Omega) \equiv \int \frac{d^3K}{(2\pi)^{3/2}\,\omega_k(k)} \left(K^2+\Omega^2\right)^N, \tag{1.102}$$

we can write the result as

$$V_G(\phi_0,\Omega) = I_1 + \frac{1}{2}(m_B^2-\Omega^2)I_0 + \frac{1}{2}m_B^2\phi_0^2 + \lambda_B\phi_0^4 + 6\lambda_B I_0\Phi_0^2 + 3\lambda_B I_0^2. \tag{1.103}$$

The GEP itself, $\bar{V}_G(\phi_0)$, is obtained by minimizing this expression with respect to the variational parameter Ω , in the range $0 < \Omega < \infty$. We denote the optimum value of Ω by $\bar{\Omega}$. Normally, $\bar{\Omega}$ will be given by the equation

$$\frac{d\bar{V}_G}{d\Omega}\,|_{(\Omega=\bar{\Omega}_0)} = 0, \tag{1.104}$$

which, using the result that

$$\frac{dI_N}{d\Omega} = (2N-1)\Omega I_{N-1} \tag{1.105}$$

leads to the '$\bar{\Omega}$-equation'

$$\bar{\Omega}^2 = \frac{m_B^2}{12} + \lambda_B\left(I_0(\bar{\Omega}) + \phi_0^2\right). \tag{1.106}$$

Using (1.106) we write $\bar{V}_G(\phi_0)$ in the form

$$\bar{V}_G(\phi_0) = I_1 - 3\lambda_B I_0^2 + \frac{1}{2}m_B^2\phi_0^2 + \lambda_B\phi_0^4. \tag{1.107}$$

The $\bar{\Omega}$-equation in (1.106) may have more than one solution, and care must be taken to choose the solution which is a minimum of V_G ; also the solution must not occur at the end point of the range $0 < \Omega < \infty$. The argument of the I_N integrals which is $\bar{\Omega}$, implicitly depends on ϕ_0 through (1.106).

1.3.2 Renormalization

The above expression for the GEP are full of divergent integrals and non-finite bare parameters m_B and λ_B. It is possible to re-express $V_G(\phi_0)$ as a manifestly finite function of ϕ_0 , by including renormalized parameters m_R and λ_R , except for a divergent, but ϕ_0 independent constant D

$$D \equiv \bar{V}_G(\phi_0 = 0) = I_1(\bar{\Omega}_0) - 3\lambda_B \left(I_0(\bar{\Omega}_0)\right)^2 \tag{1.108}$$

where $\bar{\Omega}_0$ is the solution to the $\bar{\Omega}$ equation at $\phi_0 = 0$. The constant D represents the vacuum energy density of the $\phi_0 = 0$ vacuum. The presence of this divergent constant has no physical consequences, since only energy differences, not absolute energies are measurable. A convenient choice for the two new parameters is to define

$$m_R^2 \equiv \frac{d^2\bar{V}_G}{d\phi_0^2}\Big|_{(\phi_0=0)} = \bar{\Omega}_0^2 \tag{1.109}$$

and

$$\lambda_R \equiv \frac{1}{4!}\frac{d^4\bar{V}_G}{d\phi_0^4}\Big|_{(\phi_0=0)} = \lambda_B \frac{[1 - 12\lambda_B I_{-1}(m_R)]}{[1 + 6\lambda_B I_{-1}(m_R)]} . \tag{1.110}$$

It can be shown that the GEP is exactly renormalization group invariant [59, 60]. That is, the physical content of the results are the same no matter how one chooses to parameterize them. Eliminating m_B and λ_B in favour of m_R and λ_R gives

$$\begin{aligned} V_G(\phi_0, \Omega) = {} & D + \frac{1}{2}m_R^2\phi_0^2 + \lambda_B\phi_0^4 + L_3(x)\frac{m_R^2}{32\pi^2} + \frac{3}{4}\lambda_B m_R^4 (x-1)^2 \left(I_{-1}(m_R)\right)^2 \\ & + \frac{1}{8}I_{-1}(m_R)\, m_R^4 (x-1)\left((x-1) - \frac{3\lambda_B}{2\pi^2}\left(L_2(x) + \frac{16\pi^2\phi_0^2}{m_R^2}\right)\right) \\ & - L_2(x)\frac{m_R^4}{32\pi^2}\left((x-1) - \frac{3\lambda_B}{8\pi^2}\left(L_2(x) + \frac{32\pi^2\phi_0^2}{m_R^2}\right)\right) \end{aligned} \tag{1.111}$$

where

$$x = \frac{\Omega^2}{m_R^2},$$

$$L_2(x) = x \ln x - x + 1,$$

and

$$L_3(x) = \frac{1}{4}(2x^2 \ln x - 2(x-1) - 3(x-1)^2).$$

Similarly the $\bar{\Omega}$ equation becomes

$$(x-1)(1 + 6\lambda_B I_{-1}(m_R)) = \frac{3\lambda_B}{4\pi^2}\left(L_2(x) + \frac{16\pi^2\phi_0^2}{m_R^2}\right). \tag{1.112}$$

Both these equations contain the divergent integral I_{-1}. We shall therefore treat I_{-1} as arbitrarily large and positive, taking the limit $I_{-1} \to \infty$. The relation between λ_B and λ_R, eqn.(1.110), allows the possibility that λ_B is a finite parameter. However a detailed analysis show that finite values, positive or negative do not lead to viable models. A positive value of λ_B which vanishes like $1/I_{-1}$ lead only to a trivial theory. A natural theory is, however, obtained if

$$\lambda_B = \frac{-1}{6I_{-1}}. \tag{1.113}$$

With this value of λ_B GEP (1.107) becomes

$$V_G(\phi_0, \Omega) = D + \frac{1}{2}xm_R^2\phi_0^2 - \frac{m_R^4}{16\lambda_R}(x-1)^2 + \frac{L_3(x)m_R^4}{32\pi^2} \tag{1.114}$$

dropping terms of order $1/I_{-1}$ or smaller. V_G is now finite for any Ω. The $\bar{\Omega}$ equation is also manifestly finite, since (1.113) inserted into(1.112) gives

$$(x-1) = \frac{\lambda_R}{4\pi^2}\left(L_2(x) + \frac{16\pi^2\phi_0^2}{m_R^2}\right). \tag{1.115}$$

It can be shown that the GEP (1.114) is bounded from above and below and describes a phase with massive particle interactions through an attractive force.

Cutoff Version of GEP

The above form of GEP has nothing to do with the Higgs mechanism, since it has no SSB. However, SSB can be induced in the model, by including an ultraviolet cutoff Λ for the momentum integrals. The stability property of cutoff ϕ^4 and ϕ^4 are the opposite of one another. In cutoff ϕ^4 , positive λ_B gives a bounded potential with a pair of minima at very large ϕ_0, corresponding to a SSB phase, while negative λ_B leads to an unbounded potential [42]. However in the absence of a UV cutoff the situation is the reverse: positive λ_B gives an unbounded potential, while a negative λ_B of the form $-1/(6I_{-1})$ leads to a stable, non-trivial theory, with unbroken symmetry.

More detailed discussion about cutoff version GEP will be given in chapter 4.

Autonomous Version of GEP

Stevenson and Tarrach [98] were the first to note that with a re-scaling of the field ϕ_0 , there exists a different renormalization method leading to a stable, non-trivial GEP with positive λ_B. Consider a renormalization of the theory in which

$$\lambda_B = \frac{1}{12 I_{-1}(\rho)} , \tag{1.116}$$

$$m_B^2 + 12\lambda_B I_0(0) = \frac{3}{2}\frac{m_0^2}{I_{-1}(\rho)} \tag{1.117}$$

and the field is rescaled by

$$\phi_0^2 = I_{-1}(\rho)\Phi_0^2 . \tag{1.118}$$

This leads to a GEP which, as $\Lambda \to \infty$, remains manifestly finite in terms of Φ_0, m_0 and ρ. By a straight forward calculation one obtains $\bar{V}_G(\Phi_0)$;

$$\bar{V}_G(\Phi_0) = D + \frac{1}{2}m_0^2\Phi_0^2 + \frac{\Phi_0^4}{144\pi^2}\left(\ln\left(\frac{2\Phi_0^2}{3\rho^2}\right) - \frac{3}{2}\right). \tag{1.119}$$

This GEP is valid for all values Φ_0. Eqn.(1.119) allows SSB when m_0^2 is negative or not too large. It is then convenient to rewrite it in terms of the vacuum value of Φ_0 , denoted by α , which is given by

$$\ln\left(\frac{2\alpha^2}{3\rho^2}\right) = 1 - \frac{36\pi^2 m_0^2}{\alpha^2} . \tag{1.120}$$

Eliminating ρ in favor of α gives

$$\bar{V}_G(\Phi_0) - D = \frac{1}{2}m_0^2\Phi_0^2\left(1 - \frac{\Phi_0^2}{2\alpha^2}\right) + \frac{\Phi_0^4}{144\pi^2}\left(\ln\left(\frac{\Phi_0^2}{\alpha^2}\right) - \frac{1}{2}\right). \tag{1.121}$$

This GEP is renormalization group invariant up to the rescaling of Φ_0 .

1.3.3 FTGEP

In equilibrium thermodynamics the GEP at finite values of temperature T is evaluated by Stevenson [101] by minimizing the Helmholtz free energy F of a quantum field system in a finite volume V calculated from the partition function Z

$$Z = Tr(\exp(-\beta H)) \tag{1.122}$$

and

$$F = -\frac{1}{\beta}\ln Z . \tag{1.123}$$

This amounts to a replacement of $I_1(\bar{\Omega})$ and $I_0(\bar{\Omega})$ integrals in (1.121) by $I_1(\bar{\Omega}) + I_1^\beta(\bar{\Omega})$ and $I_0(\bar{\Omega}) + I_0^\beta(\bar{\Omega})$ respectively, where

$$I_1^\beta(\bar{\Omega}) = \frac{1}{\beta}\int\frac{d^3K}{(2\pi)^3}\ln(1 - \exp(-\beta\omega_k)) \tag{1.124}$$

and

$$I_0^\beta(\bar{\Omega}) = \int \frac{d^3K}{(2\pi)^3} \frac{1}{\omega_k(\exp(\beta\omega_k) - 1)} \tag{1.125}$$

with

$$\omega_k^2 = K^2 + \bar{\Omega}^2 .$$

Following this rule it is shown that [101], for the autonomous ϕ^4 model the FTGEP is obtained in the form

$$\bar{V}_G^\beta(\Phi_0) - D = \frac{1}{2}m_0^2\Phi_0^2 + \frac{1}{144\pi^2}\Phi_0^4\left(\ln\frac{2\Phi_0^2}{3\nu^2} - \frac{3}{2}\right) + I_1^\beta\left(\bar{\Omega}^2 = \frac{2}{3}\Phi_0^2\right), \tag{1.126}$$

where ν is a parameter with the dimension of mass. The constant of integration D in (1.126) is temperature independent and is the usual divergent vacuum energy constant.

Chapter 2

Phase Transitions in ϕ^4 Theory

The study of FTD effects in field theories with SSB has attracted considerable attention during the last two decades, mainly because of its relevance to the understanding of several important cosmological questions [9]. Though most cosmological scenarios are based on models in which symmetries are restored at sufficiently high temperatures, there also exist models where symmetry always remains broken or is restored at a finite temperature but is broken again at a higher temperature [5, 6, 109-112]. The effect of fermion density increase has also been investigated. It happens that while in non-gauge models symmetry is restored at high densities, in gauge ones symmetry breaking increases further with increasing density [9, 26]. The nature of the phase transitions thus depends on the detailed relations among temperature, density and the various parameters of a model.

In spontaneously broken gauge models, in addition to fermionic matter, bosonic matter is present in the form of Higgs particles. Though direct experimental searches have failed to reveal the existence of Higgs particles of mass

[113] less than 5 GeV in the absence of viable alternatives, we take the view that such particles must be considered seriously. It then becomes significant to study the finite density effects of these particles in the early universe. With this background, we investigate in this chapter the effect of the bosonic chemical potential on phase transitions. By means of a perturbative analysis of a 2-component ϕ^4 theory with O(2) symmetry in the presence of finite chemical potential corresponding to a conserved bosonic charge, it is seen through the effective potential method that at non-zero temperatures as the chemical potential μ is increased from zero symmetry restoration takes place at finite values of μ . However as the density is increased further, the symmetry is again broken at a sufficiently large value of μ . The existence of two critical densities is demonstrated by one-loop and improved one-loop approximations at finite temperature and densities. It is further shown that at zero temperatures chemical potential has no effect, at least at the level of approximation considered here.

2.1 FTD Effective Potential

Consider the model of a self interacting 2-component spinless field with an O(2) invariant interaction described by the Lagrangian,

$$\mathcal{L}\left(\hat{\phi}_a(x)\right) = \frac{1}{2}\partial_\nu\phi_a\partial^\nu\phi_a - \frac{1}{2}m^2\phi_a\phi_a - \frac{\lambda}{4!}(\phi_a\phi_a)^2, \qquad a = 1,2. \tag{2.1}$$

We shall study a weak coupling case ($0 < \lambda << 1$) with tachyonic mass for the field ($m^2 < 0$) at finite temperature and density. The counter terms which must be

added to (2.1) are

$$\frac{-1}{2}\delta m^2\phi^2 - \frac{\delta\lambda}{4!}\phi^4. \tag{2.2}$$

To compute the effective potential $V_{eff}^{\beta,\mu}(\hat{\phi})$, we shift $\hat{\phi}_a$ in $\mathcal{L}(\hat{\phi}_a(x))$ by constant fields ϕ_a and drop all terms independent of linear in ϕ_a^3 . In quantum statistics at non-zero chemical potential, the effective potential $V_{eff}^{\beta,\mu}$ is given by the same 1PI vacuum diagrams as in field theory with ω_n in the integrals for the diagrams replaced by $(i\mu - 2\pi n/\beta)$ and integration over ω_n replaced by summation over n. The lowest order effective potential is the tree approximation,

$$V_0(\hat{\phi}) = \frac{1}{2}(m^2 + \delta m^2)\phi^2 + \frac{1}{4!}(\lambda + \delta\lambda)\phi^4 \tag{2.3}$$

which is FTD independent. The one-loop term is computed from Fig. 2.1 as

$$V_1^{\beta,\mu}(\hat{\phi}) = \frac{1}{2\beta}\sum_n \int \frac{d^3K}{(2\pi)^3} \ln(k^2 - m_1^2)(k^2 - m_2^2) \tag{2.4}$$

where

$$m_1^2 = m^2 + \delta m^2 + \frac{1}{2}(\lambda + \delta\lambda)\phi^2 ,$$

$$m_2^2 = m^2 + \delta m^2 + \frac{1}{6}(\lambda + \delta\lambda)\phi^2 ,$$

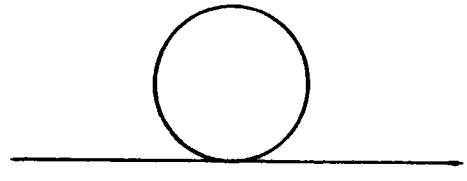


Fig. 2.1 The one-loop contribution to the effective potential

and

$$k = (i\omega_n, K) \quad with \quad \omega_n = i\mu - \frac{2\pi n}{\beta} .$$

Eqn.(2.4) can be expanded to give

$$V_1^{\beta,\mu} = \frac{1}{2\beta}\sum_n \int \frac{d^3K}{(2\pi)^3}\left[\ln(k^2 + M_1^2) + \ln(k^2 - M_2^2) - \frac{\delta M_1^2}{K^2 - M^2} - \frac{\delta M_2^2}{K^2 - M_2^2}\right], \tag{2.5}$$

where

$$M_1^2 = m^2 + \frac{\lambda}{2}\phi^2; \qquad \delta M_1^2 = \delta m^2 + \frac{\delta\lambda}{2}\phi^2$$
$$M_2^2 = m^2 + \frac{\lambda}{6}\phi^2; \qquad \delta M_2^2 = \delta m^2 + \frac{\delta\lambda}{6}\phi^2 .$$

In (2.5), we need consider only the first two terms, as the others get canceled in the next order of calculation. The first term is

$$\frac{1}{2\beta}\sum_n \int \frac{d^3K}{(2\pi)^3}\ln(k^2 - M_1^2) = \frac{1}{2\beta}\sum_n \int \frac{d^3K}{(2\pi)^3}\ln\left(\frac{-4\pi^2 n^2}{\beta^2} + \frac{4\pi n i\mu}{\beta} + \mu^2 - K^2 - M_1^2\right). \tag{2.6}$$

The summation may be done by defining

$$v = \sum_n \ln\left(\frac{-4\pi^2 n^2}{\beta^2} + \frac{4\pi n i\mu}{\beta} + \mu^2 - k^2 - M_1^2\right). \tag{2.7}$$

Hence

$$\frac{\partial v}{\partial K} = \sum_n \left(2K\beta^2 / \left(4\pi^2\left(n - \frac{\beta i}{2\pi}\left(\mu + \sqrt{K^2 + M_1^2}\right)\right)\left(n - \frac{\beta i}{2\pi}\left(\mu - \sqrt{K^2 + M_1^2}\right)\right)\right)\right). \tag{2.8}$$

Using the relation,

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n-x)(n-y)} = \frac{\pi}{y-x}\left(\cot(\pi x) - \cot(\pi y)\right) \tag{2.9}$$

we find

$$v = \ln\left(\sin^2\left(\frac{\beta i\mu}{2}\right)\cos^2\left(\frac{\beta i}{2}\sqrt{K^2+M_1^2}\right) - \cos^2\left(\frac{\beta i\mu}{2}\right)\sin^2\left(\frac{\beta i}{2}\sqrt{K^2+M_1^2}\right)\right). \tag{2.10}$$

A similar computation of the second term in (2.5) give the one-loop effective potential term in the form

$$\begin{aligned} V_1^{\beta,\mu}(\hat{\phi}^2) \;=\; & \frac{1}{4\pi^2\beta}\int_0^\infty dK\, K^2 \\ & \cdot\left[\ln\left[\sin^2\left(\frac{\beta i\mu}{2}\right)\cos^2\left(\frac{\beta i X_1}{2}\right) - \cos^2\left(\frac{\beta i\mu}{2}\right)\sin^2\left(\frac{\beta i X_1}{2}\right)\right]\right. \\ & \left.+\ln\left[\sin^2\left(\frac{\beta i\mu}{2}\right)\cos^2\left(\frac{\beta i X_2}{2}\right) - \cos^2\left(\frac{\beta i\mu}{2}\right)\sin^2\left(\frac{\beta i X_2}{2}\right)\right]\right] \end{aligned} \tag{2.11}$$

where $X_1^2 = K^2 + M^2$ and $X_2^2 = K^2 + M_2^2$. From (2.3) and (2.11), we find

$$V_{eff}^{\beta,\mu}(\hat{\phi}^2) = \frac{1}{2}m^2\hat{\phi}^2 + \frac{1}{4!}\lambda\hat{\phi}^4 + \frac{1}{2}B\hat{\phi}^2 + \frac{1}{2}C\hat{\phi}^4 + V_1^{\beta,\mu}(\hat{\phi}^2)\,, \tag{2.12}$$

where B and C are the renormalization constants given by

$$B = \frac{-\lambda}{2}\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2 - m^2 + i\epsilon} \tag{2.13}$$

and

$$C = \frac{3i\lambda^2}{2}\int\frac{d^4k}{(2\pi)^4}\left(\frac{1}{k^2 - m^2 + i\epsilon}\right)^2. \tag{2.14}$$

2.2 Effective Mass and Effective Coupling Constant

From the effective potential, it is possible to calculate many other physically interesting quantities. The temperature and chemical potential dependent mass $m^2_{\beta,\mu}$ and coupling constant $\lambda_{\beta,\mu}$ are given by

$$m^2_{\beta,\mu} = \frac{\partial^2 V^{\beta,\mu}_{eff}}{\partial\phi^2}\,|_{(\phi=0)} \tag{2.15}$$

and

$$\lambda_{\beta,\mu} = \frac{\partial^4 V^{\beta,\mu}_{eff}}{\partial\phi^4}\,|_{(\phi=0)} \ . \tag{2.16}$$

Eqns.(2.12), (2.15) and (2.16) yield the following

$$m^2_{\beta,\mu} = m^2 + \frac{\lambda}{12\pi^2}\int_0^\infty \frac{dK\ K^2 \sinh(\beta X)\ X^{-1}}{\cosh^2(\beta\mu/2)\sinh^2(\beta X/2) - \sinh^2(\beta\mu/2)\cosh^2(\beta X/2)} \tag{2.17}$$

and

$$\lambda_{\beta,\mu} = \lambda + \frac{5\lambda^2}{48\pi^2}\int_0^\infty \frac{dK\ K^2\,(\sinh(\beta X)\ X^{-3} - \beta\ \cosh(\beta X)\ X^{-2})}{\sinh^2(\beta\mu/2)\cosh^2(\beta X/2) - \cosh^2(\beta\mu/2)\sinh^2(\beta X/2)} \ . \tag{2.18}$$

where $X^2 = K^2 + m^2$. To analytically compute the effective mass and effective coupling constant from (2.17) and (2.18), we are forced to introduce some approximations, as the integrals cannot be evaluated in a closed form.

For small values of μ , we make a Taylor expansion of (2.17) and (2.18) as a function of μ . The first term is independent of chemical potential and is same as the corresponding finite temperature expression which is known in the high temperature limit [5, 116]. The second term of the expansion vanishes and

the third term can be computed in the high temperature limit, after a Binomial expansion in β. Neglecting terms of $O(\beta^2 m^2)$, we get

$$m^2_{\beta,\mu} = m^2 + \frac{\lambda}{18\beta^2} - \frac{m\lambda}{6\pi\beta} + \frac{\mu^2\lambda}{6\pi^2} + O(\beta^2 m^2) \tag{2.19}$$

and

$$\lambda_{\beta,\mu} = \lambda - \frac{5\lambda^2}{24\pi}\left[\frac{1}{m\beta} - \frac{1}{\pi}\ln(\frac{1}{m\beta}) + \frac{\mu^2}{12\pi\beta m^3}\right] + O(\beta^2 m^2)\ . \tag{2.20}$$

Eqns.(2.19) and (2.20) at fixed temperature shows an increase of effective mass and effective coupling constant with increase of chemical potential. Here, $m^2_{\beta,\mu}$ may increase with μ to such an extent that it becomes positive beyond a particular value of $\mu = \mu_{c1}$ restoring the spontaneously broken symmetry. The critical chemical potential μ_{c1} , at which this symmetry restoration is switched on can be calculated from (2.19)

$$\mu^2_{c1} = \frac{6\pi^2 m^2}{\lambda} - \frac{\pi^2}{3\beta^2} + \frac{\pi m}{\beta}\ . \tag{2.21}$$

The corresponding critical temperature is

$$\frac{1}{\beta^2_{c1}} = \frac{18m^2}{\lambda} - \frac{3\mu^2}{\pi^2} + \frac{3m}{\pi\beta}\ . \tag{2.22}$$

To study the behavior of (2.17) and (2.18) for large values of chemical potential, we introduce the approximations

$$\cosh^2\left(\frac{\beta\mu}{2}\right) \approx \exp\left(\frac{\beta\,|\,\mu\,|}{2}\right) \quad \textit{and} \quad \sinh^2\left(\frac{\beta\mu}{2}\right) \approx \exp\left(\frac{\beta\,|\,\mu\,|}{2}\right)\ . \tag{2.23}$$

Eqns.(2.17) and (2.18) can now be expanded in binomial series and on neglecting terms of $O(\beta^4 m^4)$, we get the effective mass and effective coupling constant at high

temperature as

$$m^2_{\beta,\,\mu} = m^2 + \frac{\lambda}{6\pi^2\beta^2\exp(\beta\mid\mu\mid)} \left[1 + \frac{\gamma}{2}\beta^2m^2 + \frac{\beta^2m^2}{2}\ln(\beta m) + \frac{\beta^3m^3}{2} + O(\beta^4m^4)\right] \qquad (2.24)$$

and

$$\lambda_{\beta,\,\mu} = \lambda + \frac{5\lambda^2}{12\pi^2\exp(\beta\mid\mu\mid)}\left[\frac{4\gamma}{3} + \frac{\ln(\beta m)}{2} - \frac{\beta m}{2} + \frac{\beta^2m^2}{8} + O(\beta^4m^4)\right] \qquad (2.25)$$

with $\gamma \approx 0.5772$. For a fixed temperature, (2.24) and (2.25) gives a decrease of $m^2_{\beta,\,\mu}$ and $\lambda_{\beta,\,\mu}$ with increase in chemical potential. Starting from the symmetry restored phase for a very large value of μ , $m^2_{\beta,\,\mu}$ can become negative to give back the original spontaneously broken phase. It is possible to define a critical chemical potential μ_{c2} corresponding to this symmetry breaking phase transition. Using only the leading terms of (2.24), we get

$$\mu_{c2} = \frac{1}{\beta}\ln\left[\frac{\lambda}{6\pi^2\beta^2m^2}\right]. \qquad (2.26)$$

Correspondingly, the μ- dependent critical temperature is

$$\frac{1}{\beta_{c2}} = \left[\left[\frac{2}{3\mid\mu\mid}\left[\cosh\left[\frac{1}{3}\cosh^{-1}\left[\frac{27\mu^2\lambda}{12\pi^2\mid m^2\mid} - 1\right]\right]\right]\right] - \frac{1}{3\mid\mu\mid}\right]^{-1}. \qquad (2.27)$$

For an asymptotically large value of $\mid\mu\mid$, the effective mass and effective coupling constant become the same as the tree level values.

From the above analysis, we find that in high temperature surroundings, a symmetry restoration takes place at a relatively small value of chemical potential and further at a large value, there is symmetry breaking. Now to extend these results for all temperatures, we adopt a numerical approach. Effective mass

and effective coupling constant can be calculated by numerical evaluation of the integrals in (2.17) and (2.18). The results are plotted in Figs. 2.2 and 2.3 for various temperatures. The one-loop expressions (2.17) and (2.18) are actually complex and we are taking only the real part here. The imaginary part will disappear when we include the higher loop effects in the diagram [114]. From Figs. 2.2 and 2.3, it is seen that effective mass increases with increase of chemical potential to produce symmetry restoration and then decreases with further increase of chemical potential to give back the symmetry broken phase. These effects are found to be present even at very low temperatures, which makes them applicable for cold dense matter like neutron stars and quark stars. However, a separate calculation shows that at zero temperature, chemical potential does not produce any effect.

2.3 Improved One-Loop Approximation

The results obtained above have all been deduced at the one-loop level. From finite temperature studies, it is known that one-loop approximation is not valid at the critical temperature [115]. The occurrence of imaginary part in the effective potential is also an indication of the inadequacy of the simple one-loop calculations. These remarks apply equally well to the one-loop study of finite density effects made here. It is easily seen that close to the critical densities μ_{c1} and μ_{c2} , the one-loop approximation breaks down. An imaginary part is also present in the effective potential (2.12).

To improve the present calculations, we adopt the method suggested

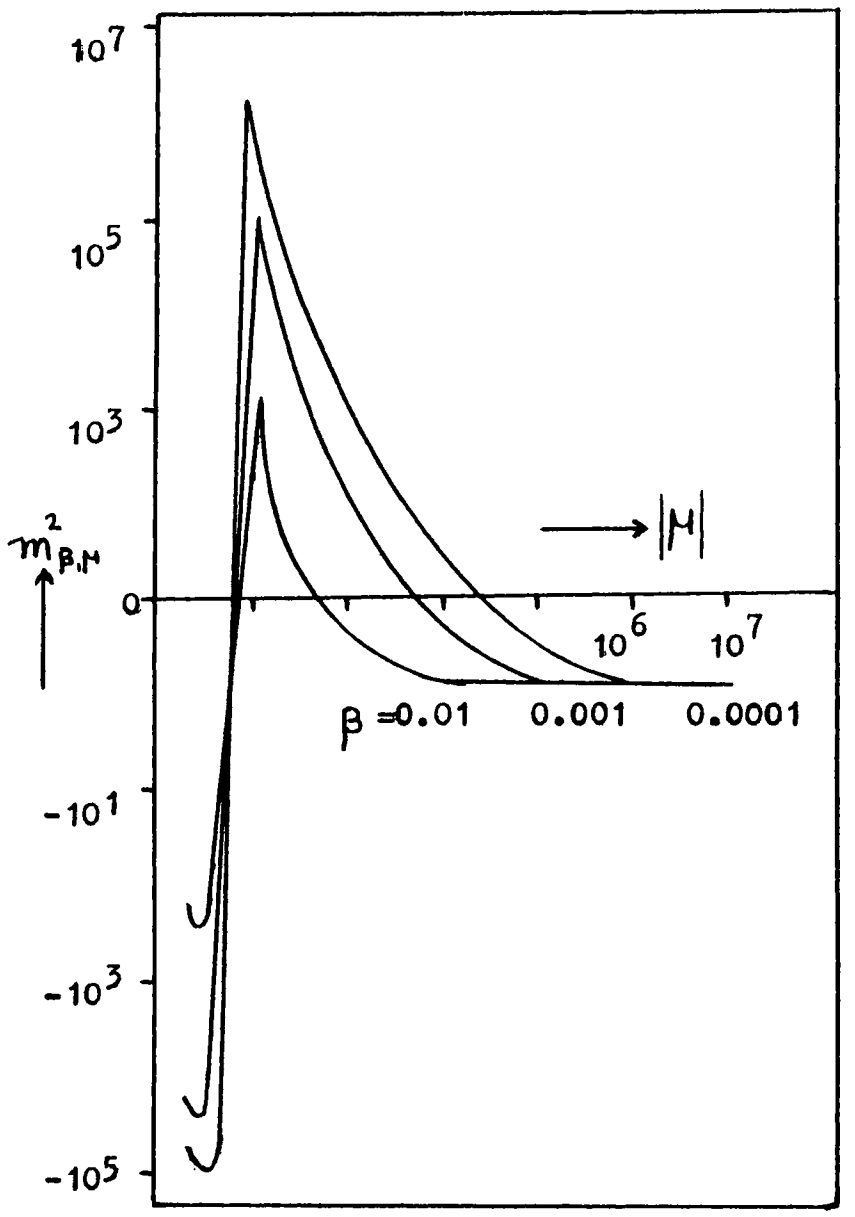


Fig. 2.2 Variation of effective mass with chemical potential

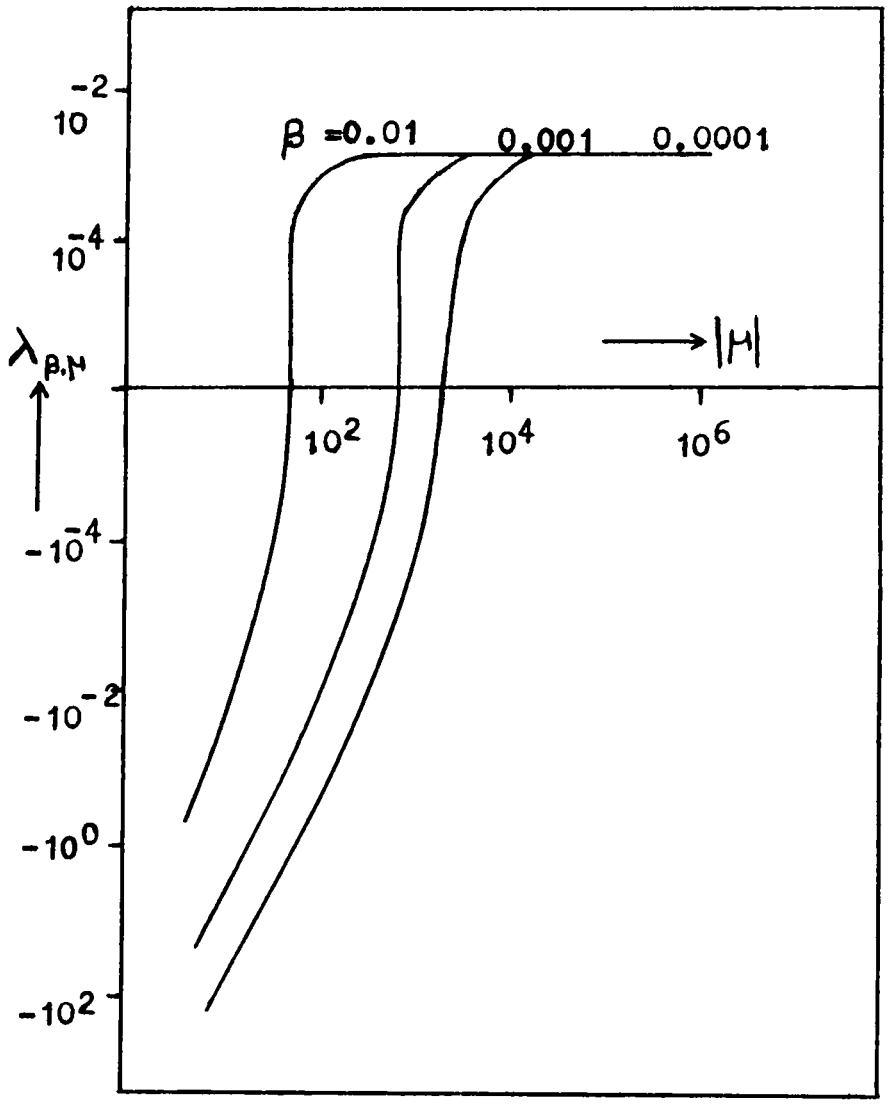


Fig. 2.3 Variation of effective coupling constant with chemical potential

by Fendley [114] to include multi-loop effects in the one-loop calculations. The effect of multiple insertions of quadratically divergent bubbles may be studied by considering the four simple contributions to the two point function in ϕ^4 theory shown in Fig. 2.4. It is obvious that at high temperatures, the largest graph to any order in the loop expansion are the ones with the minimum number of quadratically divergent loops, namely those like Fig. 2.5a. Another contribution to the effective potential is displayed in Fig. 2.5b. When we add a bubble on to any propagator that is not part of a quadratically divergent loop, the diagram is multiplied by $\lambda T^2/U^2$, where U is the mass scale of the theory. We can do this to any diagram as in Fig. 2.6. This means that for a one-loop approximation to be valid, we must have $\lambda T^2/U^2 << 1$, along with the usual $\lambda << 1$. Thus multiple insertions of quadratically divergent bubbles cannot be neglected.

To take into account all such diagrams, we make use of the expansion of the full propagator, shown diagrammatically in Fig. 2.7 and define an effective mass M such that

$$\frac{1}{k^2+M^2} = \frac{1}{k^2+m^2}\sum_{n=0}^{\infty}\frac{(-\sigma)^n}{(k^2+m^2)^n} = \frac{1}{k^2+m^2+\sigma}\,, \tag{2.28}$$

giving

$$M^2 = m^2 + \sigma\,. \tag{2.29}$$

σ is the 1PI self energy, which may be approximated as in Fig. 2.8:

$$\sigma \approx \frac{2\lambda}{\beta}\sum_n \int \frac{d^3K}{(2\pi)^3}\,\frac{1}{(i\mu-(2\pi n/\beta))^2-K^2} + counterterms. \tag{2.30}$$

The logarithmically divergent part of the diagram may be ignored as a sub-leading temperature dependence of the diagram and the T=0 counter term cancels the

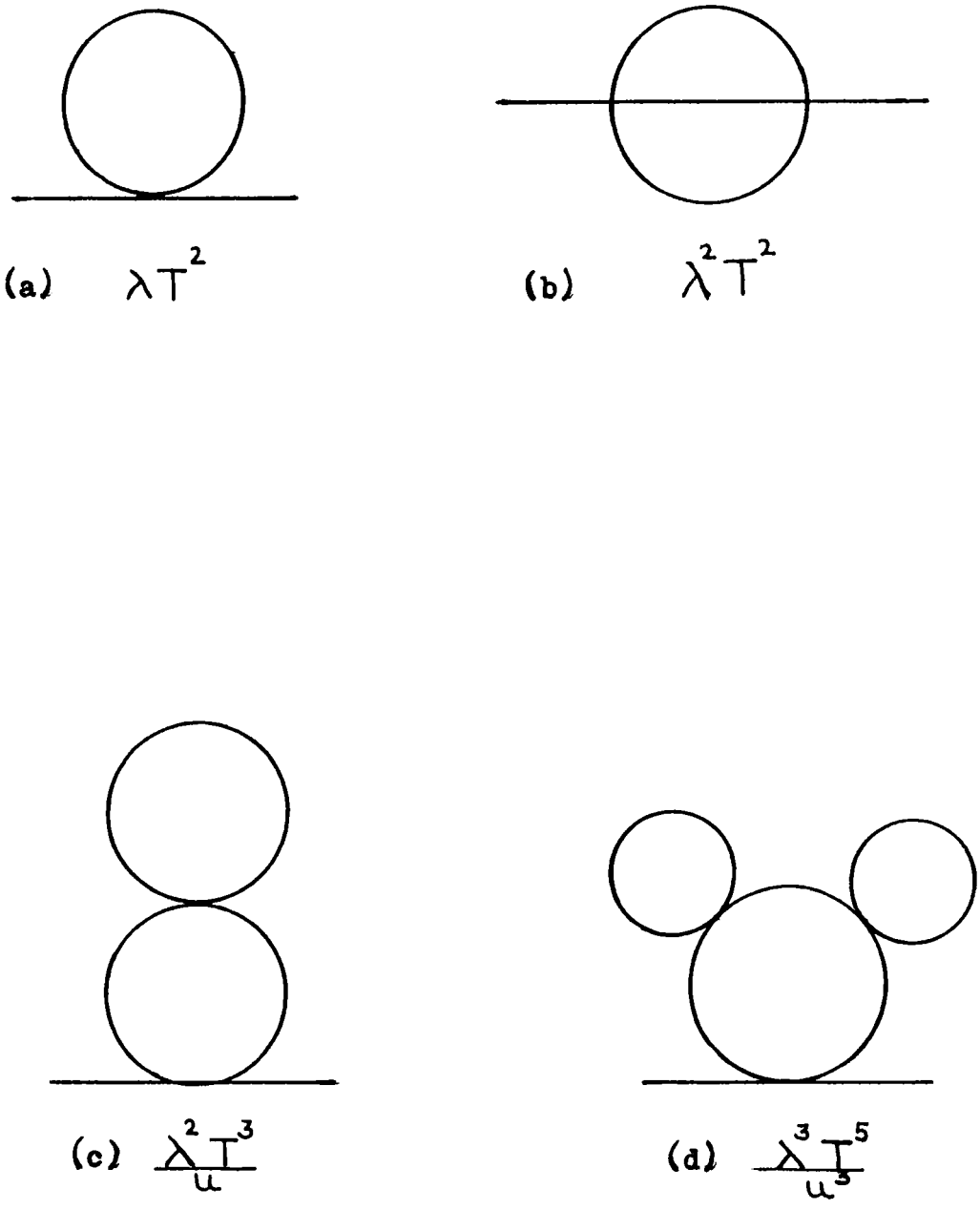
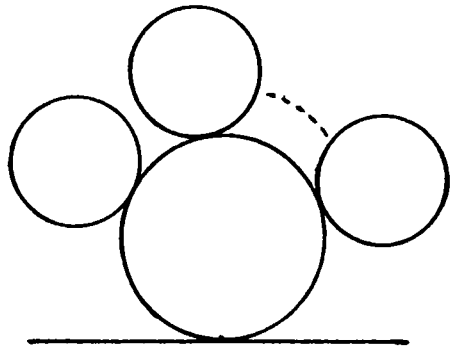
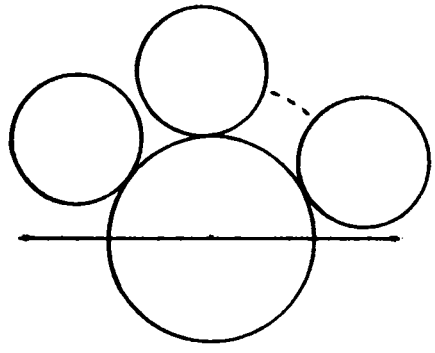


Fig. 2.4 Lowest order contributions to the self energy, with their orders of magnitude at high temperature



(a)

For n loops $\frac{\lambda^{n} T^{2n-1}}{u^{2n-1}}$



(b)

For (n+1) loops $\frac{\lambda^{n+1} T^{2n}}{u^{2n-2}}$

Fig. 2.5 Multi-loop self energy contributions and their orders of magnitude at high temperature

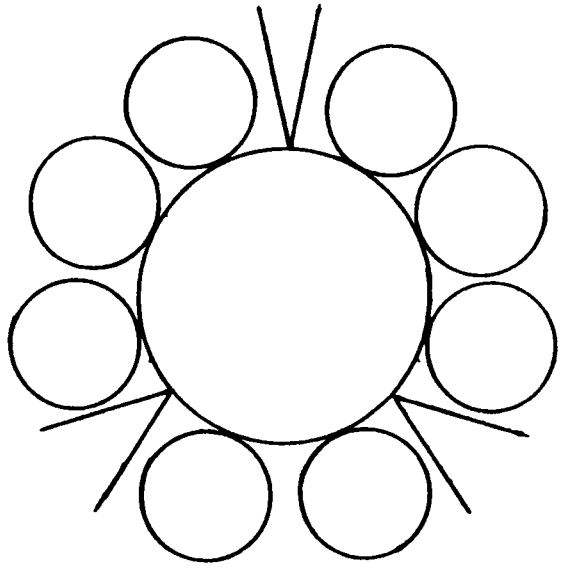


Fig. 2.6 A typical contribution to the effective potential, that is relevant at high temperature

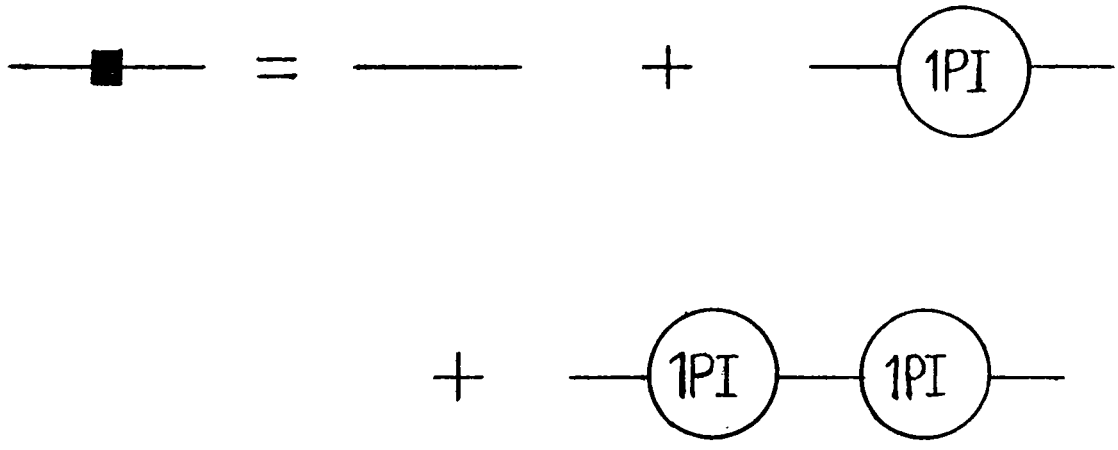


Fig. 2.7 Diagrammatic expansion of the full propagator

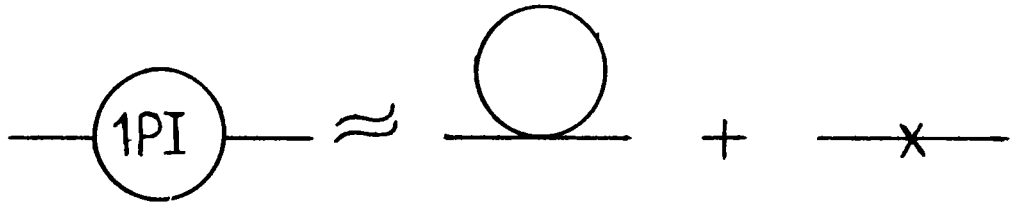


Fig. 2.8 Lowest-order corrections to the self energy

divergent piece. Now we replace m^2 with M^2 in the expression for the effective potential and perform one-loop calculations in this improved approximation. This includes the largest diagrams at high T, namely those given by the one-loop diagram with bubbles attached, as in Fig. 2.5a and 2.6. Diagrams we are neglecting are suppressed by some power of λ or $\lambda T/M$. Thus the improved expansion is valid if $\lambda T/M << $ and $\lambda << 1$. The validity of the one-loop calculation at and above critical temperature is restored with the additional condition $T << M/\lambda$. Notice that the expansion breaks down as $T \rightarrow T_c$, because $M \rightarrow 0$. Calculations of $m^2_{\beta,\mu}$ and $\lambda_{\beta,\mu}$ give the same results as in (2.19), (2.20), (2.24) and (2.25) with m^2 being replaced by M^2. The behavior of $m^2_{\beta,\mu}$ and $\lambda_{\beta,\mu}$ with respect to the variation of chemical potential can be studied, as done previously by numerical evaluation of the integrals appearing in their expressions. We have found the same pattern of behavior as in the simple one-loop study. The symmetry is restored at a finite density which is again broken at a higher density.

Chapter 3

Phase Transitions in the Abelian Higgs Model

It is well known that, the effective potential method when applied to gauge theories leads to a gauge dependent result for the finite temperature effective mass, which is an observable in the theory. The ensuing puzzles were solved, at least for the abelian Higgs model, by Ueda [117], who showed that by slightly modifying the definition of the effective mass, one can maintain gauge invariance even at a finite temperature. This is achieved by including certain momentum-dependent diagrams in the self energy which are absent in the effective potential treatment in some gauges.

In this chapter, following the approach of Ueda, we study the effective mass of the theory at a finite density using the real-time formalism. It is shown that, at non-zero temperatures, as the chemical potential μ corresponding to a conserved bosonic charge is increased from zero, symmetry restoration takes place at a finite

value of μ . However, as the density is increased further symmetry is again broken at a sufficiently large value of μ . The critical densities and critical temperatures responsible for these phase transitions are calculated. We also study the finite density behaviour of coupling constants present in the theory. The effective coupling constants are calculated at the one-loop level, using the vertex renormalization procedure, from their lowest order vertex diagrams. The scalar coupling constant λ first decreases and then increases with the increase of the chemical potential, while the gauge coupling constant e shows a reverse trend.

3.1 Effective Potential in the Unitary Gauge

We consider the locally gauge invariant abelian Higgs model of a two-component scalar field Φ and a vector field A_μ . The Lagrangian describing their O(2)-invariant interaction is

$$\mathcal{L} = \frac{-1}{4}(\partial_\nu A_\rho - \partial_\rho A_\nu)^2 - \mid (\partial_\nu - ieA_\nu)\,\Phi \mid^2 - m_1^2 \Phi^* \Phi - \frac{\lambda}{6}(\Phi^* \Phi)^2 . \qquad (3.1)$$

We shall consider a case of weak scalar coupling ($0 < \lambda << 1$) and weak gauge coupling ($e \sim \lambda^2$) with tachyonic mass ($m^2 < 0$) for the field at a finite temperature and density. Finite density effects are taken into account by introducing a chemical potential μ corresponding to the conserved bosonic charge of (3.1). The fields in Lagrangian (3.1) admits gauge transformations and it is necessary to fix a gauge. We choose the physically interesting unitary gauge for our computation. Expressing the Lagrangian (3.1) in terms of polar co-ordinates ϕ and θ

$$\Phi(x) = \frac{1}{\sqrt{2}}(\phi(x) + C) \exp\left(\frac{i\theta(x)}{C}\right) \quad with \quad < \phi(x)_0 >_0 = 0 \qquad (3.2)$$

and a new variable B_ν ,

$$B_\nu(x) = A_\nu(x) - \frac{1}{eC}\partial_\nu\theta(x) \ , \tag{3.3}$$

we obtain the Lagrangian in the unitary gauge:

$$\begin{aligned}\mathcal{L} \ = \ & \frac{-1}{4}(\partial_\nu B_\rho - \partial_\rho B_\nu)^2 - \frac{1}{2}(\partial_\nu\phi)^2 - \frac{1}{2}(eC)^2 B_\nu^2 - \frac{1}{2}\left(m_1^2 + \frac{\lambda}{2}C^2\right)\phi^2 \\ & -\frac{1}{2}e^2B^2\left(\phi^2 + 2C\phi\right) - \frac{\lambda}{4!}\phi^4 - \frac{\lambda}{6}C\phi^3 - C\phi\left(m_1^2 + \frac{\lambda}{6}C^2\right) \ . \end{aligned} \tag{3.4}$$

Using the corresponding effective Hamiltonian in the interaction representation and choosing $C^2 = (-6m_1^2/\lambda)$ leads to $< \phi >_0 = 0$ at the tree-level approximation. The B_μ and ϕ meson masses are respectively,

$$M^2 = (eC)^2 \quad and \quad m^2 = \frac{\lambda}{3}C^2 \ . \tag{3.5}$$

The Lagrangian (3.4) leads to the one-loop effective potential,

$$V_{eff}^{\beta,\mu} = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 - \frac{1}{2\beta}\sum_n \int \frac{d^3K}{(2\pi)^3}\left(3\ln\left(k^2 - M_1^2\right) + \ln\left(k^2 - M_2^2\right)\right) \ , \tag{3.6}$$

where $M_1^2 = e^2\phi^2$ and $M_2^2 = m^2 + \lambda\phi^2/2$. The sums appearing in (3.6) can be evaluated by following the steps described in Chapter 2 to yield,

$$\begin{aligned} V_{eff}^{\beta,\mu} \ = \ & \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 + \frac{1}{4\pi^2\beta}\int_0^\infty dK\ K^2 \\ & .\left(\ln\left(\sin^2\left(\frac{\beta i\mu}{2}\right)\cos^2\left(\frac{\beta i}{2}\sqrt{K^2 + M_1^2}\right)\right.\right. \\ & \left. - \cos^2\left(\frac{\beta i\mu}{2}\right)\sin^2\left(\frac{\beta i}{2}\sqrt{K^2 + M_2^2}\right)\right) \\ & +3\ln\left(\sin^2\left(\frac{\beta i\mu}{2}\right)\cos^2\left(\frac{\beta i}{2}\sqrt{K^2 + M_1^2}\right)\right. \\ & \left.\left. - \cos^2\left(\frac{\beta i\mu}{2}\right)\sin^2\left(\frac{\beta i}{2}\sqrt{K^2 + M_1^2}\right)\right)\right) . \end{aligned} \tag{3.7}$$

3.2 Effective Mass

For the purpose of studying the phase transitions at a finite temperature and a finite density, we calculate the effective mass according to Ueda's prescription [117]: We define a temperature-and chemical potential- dependent mass $m^2_{\beta,\mu}$, on the mass shell of the particle at temperature zero and chemical potential zero, as

$$m^2_{\beta,\mu} = m^2 + Re\ \Pi_{\beta,\mu}(P^2 = -m^2)\ , \tag{3.8}$$

where $\Pi_{\beta,\mu}$ is the temperature- and chemical potential- dependent scalar meson self energy and Re refers to the real part. This definition of the effective mass by Ueda differs from the standard one by the inclusion of momentum dependent self energy diagrams. These diagrams which contribute to the scalar-meson self energy at the one-loop level are shown in Fig. 3.1. Following the standard Feynman rules for the Lagrangian (3.4), we compute the self energy of the Higgs boson at the one-loop level as,

$$\begin{aligned} m^2_{\beta,\mu} &= m^2 + \lambda\int\frac{d^4k}{(2\pi)^4}\,\Delta(k) + 2e^2\int\frac{d^4k}{(2\pi)^4}\,\Delta_{\omega}(k) \\ &\quad + \frac{3im^2\lambda}{2}\int\frac{d^4k}{(2\pi)^4}\,\Delta(k)\Delta(p-k) + 2ie^2M^2\int\frac{d^4k}{(2\pi)^4}\,\Delta_{\nu\rho}(p-k)\ , \end{aligned} \tag{3.9}$$

where $\Delta(k)$ and $\Delta_{\nu\rho}(k)$ are the temperature- and chemical potential-dependent propagators for the scalar meson and vector meson respectively and m and M are the scalar meson- and vector meson- masses respectively. In the real time formalism of finite temperature field theory, these propagators in momentum space are given by

$$\Delta(k) = \left[\frac{i}{k^2 - m^2 - i\epsilon} + \frac{2\pi\,\delta(k^2+m^2)}{\exp(\beta(k_0+\mu)) - 1}\right], \tag{3.10}$$

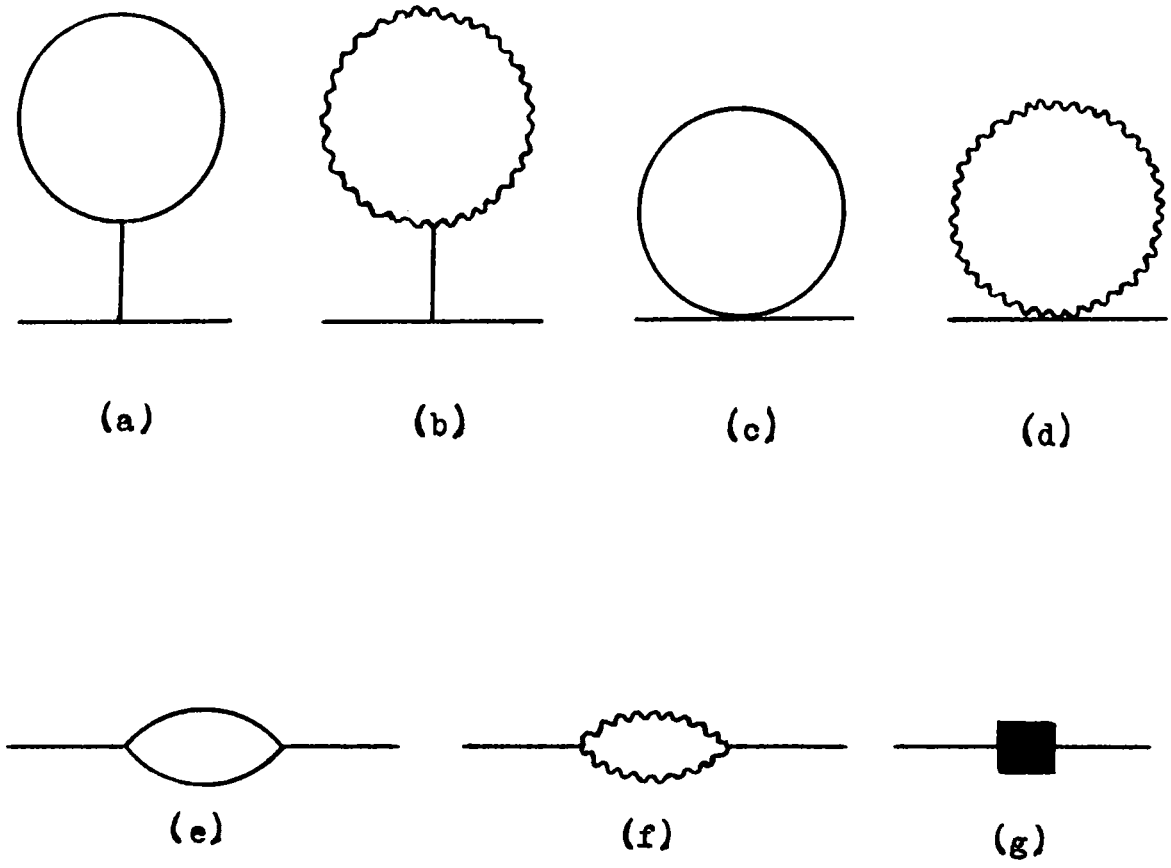


Fig. 3.1 Diagrams contributing to the scalar-meson propagator in the unitary gauge in the one-loop approximation

$$\Delta_{\nu\rho}(k) = \left[\frac{i}{k^2 + M^2 - ie} + \frac{2\pi\,\delta(K^2 + M^2)}{\exp(\beta(k_0 + \mu)) - 1}\right]\left(\delta_{\nu\rho} - \frac{k_\nu k_\rho}{M^2}\right)\,, \qquad (3.11)$$

where μ is the chemical potential corresponding to the conserved bosonic charge in this system and we follow the metric $k^2 = K^2 - k_0^2$. The real-time formalism has the advantage of keeping separate the temperature and chemical potential dependent terms from zero temperature- and zero chemical potential terms. Performing renormalization at T=0 and $\mu = 0$ and considering only the temperature and chemical potential dependent real terms in self energy, we write (3.9) as

$$\begin{aligned} m^2_{\beta,\,\mu} \;=\; & m^2 + \lambda\int\frac{d^4k}{(2\pi)^4}\,\frac{2\pi\,\delta(k^2+m^2)}{\exp(\beta(k_0+\mu))-1} + 6e^2\int\frac{d^4k}{(2\pi)^4}\,\frac{2\pi\,\delta(k^2+M^2)}{\exp(\beta(k_0+\mu))-1} \\ & +3\pi\lambda m^2\int\frac{d^4k}{(2\pi)^4}\left(\frac{\delta(k^2+m^2)}{((P-K)^2+m^2)\,(\exp(\beta(k_0+\mu))-1)}\right. \\ & \left.+\frac{\delta((p-k)^2+m^2)}{(k^2+m^2)(\exp(\beta(k_0+\mu))-1)}\right) \\ & +4\pi e^2M^2\int\frac{d^4k}{(2\pi)^4}\left(\frac{\delta(k^2+M^2)}{((p-k)^2+M^2)(\exp(\beta(k_0+\mu))-1)}\right. \\ & \left.+\frac{\delta((p-k)^2+M^2)}{(k^2+M^2)(\exp(\beta(k_0+\mu))-1)}\right)\left(\delta_{\nu\rho}-\frac{k_\nu k_\rho}{M^2}\right) \\ & .\left(\delta_{\nu\rho}-\frac{(p-k)_\nu(p-k)_\rho}{M^2}\right). \end{aligned} \qquad (3.12)$$

Since the effective mass is expected to be a measurable quantity, we have considered only the real terms in obtaining that equation. Performing the k_0 integration in the rest frame of the scalar meson using

$$((p-k)^2 + m^2) = (k^2 + m^2),$$

$$p^2 - 2(p.k) = -m^2 + 2mk_0\,, \quad etc.,$$

we get

$$m^2_{\beta,\,\mu} \;=\; m^2 + \frac{\lambda}{4\pi^2}\int_m^\infty \frac{dy_1\,\sqrt{y_1^2-m^2}}{\exp(\beta(y_1+\mu))-1} + \frac{3e^2}{2\pi^2}\int_m^\infty \frac{dy_2\,\sqrt{y_2^2-M^2}}{\exp(\beta(y_2+\mu))-1}$$

$$+\frac{3\lambda m^2}{4\pi^2}\int_m^\infty \frac{dy_1\,\sqrt{y_1^2-m^2}}{(\exp(\beta(y_1+\mu))-1)(4y_1^2-m^2)}$$
$$+\frac{e^2}{M^2}\int_M^\infty \frac{dy_2\,\sqrt{y_2^2-M^2}}{\exp(\beta(y_2+\mu))-1}\left(\frac{3M^4+m^2(y_2^2-M^2)}{4y_2^2-m^2}\right), \qquad (3.13)$$

where $y_1^2 = K^2 + m^2$ and $y_2^2 = K^2 + M^2$. Since the integrals in (3.13) are difficult to evaluate in a closed form without further approximation, we perform a numerical computation. Fig. 3.2 plots the results of the numerical evaluation of $m^2_{\beta,\,\mu}$ as a function of μ , for various fixed values of temperatures. We see that, at finite temperatures , starting from a spontaneously broken symmetry $m^2_{\beta,\,\mu}$ increase with μ to such an extent that it becomes positive beyond a critical value of $\mu = \mu_{c1}$, restoring the symmetry, and then decreases with further increase of μ to break the symmetry at another critical value of $\mu = \mu_{c2}$.

To calculate the critical chemical potentials and critical temperatures responsible for these phase transitions, we calculate the effective mass from (3.13) in the high temperature approximation in the region of small value of μ where there is symmetry restoration and then for large values of μ for further symmetry breaking. For small values of μ it is possible to expand (3.13) as a Taylor series in μ . This gives

$$\begin{aligned} m^2_{\beta,\,\mu} \;=\; & m^2 + \frac{\lambda}{4\pi^2}\int_m^\infty \frac{dy_1\,\sqrt{y_1^2-m^2}}{a_1} \\ & +\frac{3e^2}{2\pi^2}\int_M^\infty \frac{dy_2\,\sqrt{y_2^2-M^2}}{a_2} + \frac{3\lambda m^2}{16\pi^2}\int_m^\infty \frac{dy_1\,\sqrt{y_1^2-m^2}}{y_1^2\,a_1} \\ & +\frac{e^2}{\pi^2 M^2}\int_M^\infty \frac{dy_2\,y_2(y_2^2-M^2)}{a_2(4y_2^2-m^2)}\left[3M^4+m^2(y_2^2-M^2)\right] \\ & -\frac{\mu\beta\lambda}{4\pi^2}\int_m^\infty \frac{dy_1\,y_1\,\sqrt{y_1^2-m^2}\exp(\beta y_1)}{a_1^2} \end{aligned}$$

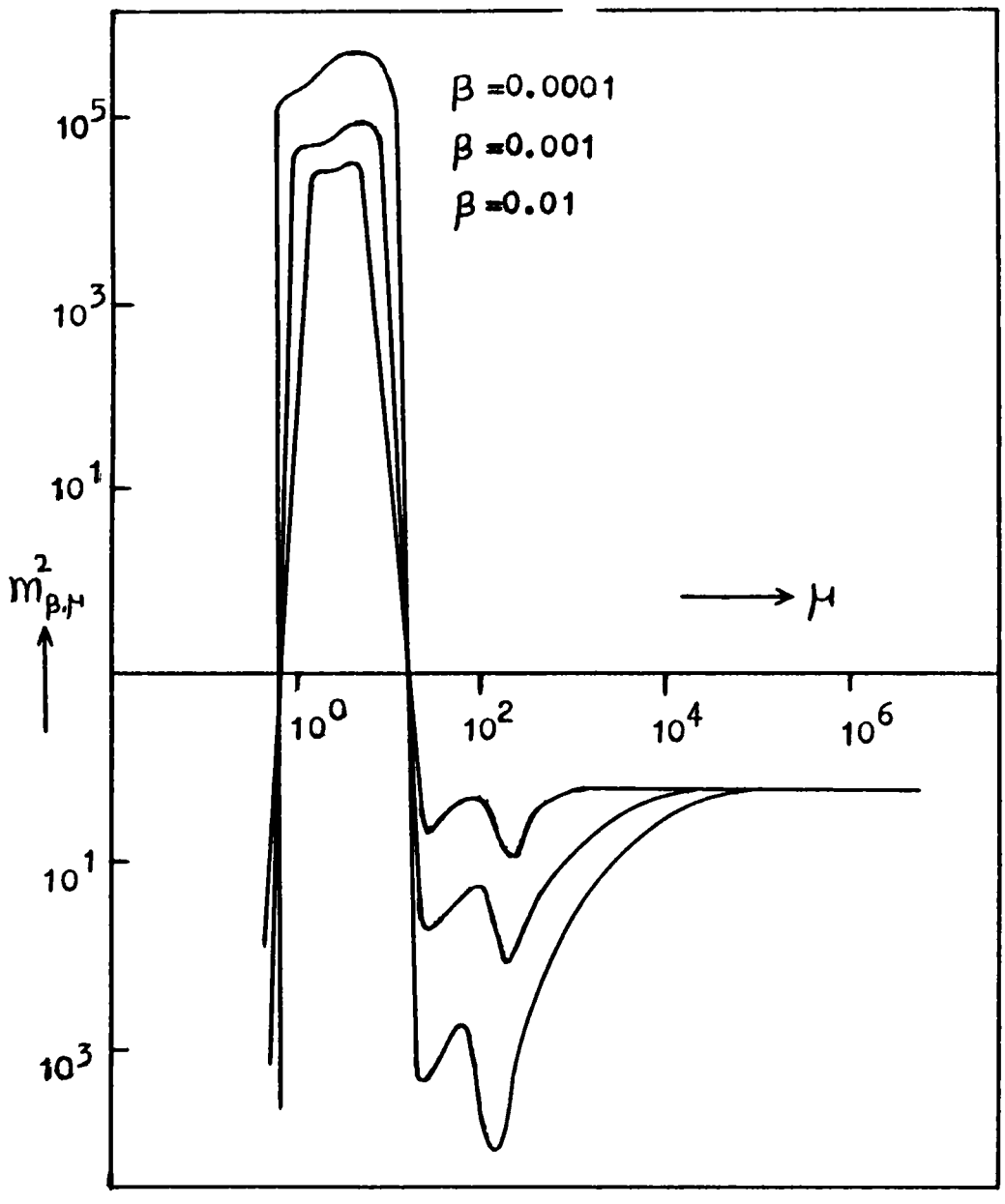


Fig. 3.2 Variation of the effective mass with the chemical potential

$$
\begin{aligned}
&-\frac{3\mu\beta e^2}{2\pi^2}\int_M^\infty \frac{dy_2\, y_2\sqrt{y_2^2-M^2}\exp(\beta y_2)}{a_2^2} \\
&-\frac{3\lambda m^2\beta\mu}{16\pi^2}\int_m^\infty \frac{dy_1\sqrt{y_1^2-m^2}\exp(\beta y_1)}{y_1^2\, a_1^2} \\
&-\frac{e^2\beta}{\pi^2 M^2}\int_M^\infty \frac{dy_2\, y_2\sqrt{y_2^2-M^2}[3M^4+m^2(y_2^2-M^2)]\exp(\beta y_2)}{(4y_2^2-m^2)a_2^2} \\
&-\frac{\lambda\beta^2\mu^2}{8\pi^2}\int_m^\infty dy_1\sqrt{y_1^2-m^2}\left(\frac{\exp(\beta y_1)}{a_1^2}-\frac{2\exp(2\beta y_1)}{a_1^3}\right) \\
&-\frac{3e^2\beta^2\mu^2}{4\pi^2}\int_M^\infty dy_2\sqrt{y_2^2-M^2}\left(\frac{\exp(\beta y_2)}{a_2^2}-\frac{2\exp(2\beta y_2)}{a_2^3}\right) \\
&-\frac{3\lambda\mu^2\beta^2 m^2}{32\pi^2}\int_m^\infty \frac{dy_1\sqrt{y_1^2-m^2}}{y_1^2}\left(\frac{\exp(\beta y_1}{a_1^2}-\frac{2\exp(2y_1)}{a_1^3}\right) \\
&-\frac{\mu^2 e^2\beta^2}{2\pi^2 M^2}\int_M^\infty \frac{dy_2\sqrt{y_2^2-M^2}[3M^4+m^2(y_2^2-M^2)]}{(4y_2^2-m^2)} \\
&\cdot\left(\frac{\exp(\beta y_2)}{a_2^2}-\frac{2\exp(2\beta y_2)}{a_2^3}\right), \qquad (3.14)
\end{aligned}
$$

where $a_1 = (\exp(\beta y_1)-1)$ and $a_2 = (\exp(\beta y_2)-1)$. At high temperatures, expanding the numerators of the integrands in (3.14) as a binomial series to $O(\beta m)$

$$
m^2_{\beta,\,\mu} = m^2 + \frac{\lambda}{18\beta^2} + \frac{\mu^2\lambda}{6\pi^2} + \frac{e^2}{4\beta^2} - \frac{3e^2\mu}{2\pi^2\beta} - \frac{\mu\lambda}{3\pi^2\beta} + \frac{\lambda m}{6\pi\beta} + \frac{me^2}{\pi\beta} + O(\beta m)\,. \quad (3.15)
$$

This shows an increase in $m^2_{\beta,\,\mu}$ with increase of μ at a finite temperature, indicating a restoration of spontaneously broken symmetry. Using (3.15) we calculate the critical chemical potential μ_{c1} , at which this symmetry restoration occurs, as

$$
\mu^2_{c1} = \frac{6\pi^2}{\lambda}\left(-m^2 - \frac{\lambda}{18\beta^2} - \frac{e^2}{4\beta^2} - \frac{\lambda m}{6\pi\beta} - \frac{me^2}{\pi\beta}\right). \qquad (3.16)
$$

The corresponding critical temperature is

$$
\frac{1}{\beta^2_{c1}} = \frac{36}{2\lambda+9e^2}\left(-m^2 - \frac{\mu^2\lambda}{6\pi^2}\right). \qquad (3.17)
$$

For large values of μ , it is possible to introduce the approximation

$$\exp(\beta(y+\mu)) - 1 \approx \exp(\beta(y+\mu)) \; . \tag{3.18}$$

Using this, we find that eqn.(3.13) becomes

$$\begin{aligned} m^2_{\beta,\,\mu} \;=\; & m^2 + \exp(-\beta\mu)\left[\frac{\lambda}{4\pi^2}\int_m^\infty \frac{dy_1\,\sqrt{y_1^2-m^2}}{\exp(\beta y_1)} + \frac{3e^2}{2\pi}\int_M^\infty \frac{dy_2\,\sqrt{y_2^2-M^2}}{\exp(\beta y_2)}\right. \\ & + \frac{3\lambda m^2}{16\pi^2}\int_m^\infty \frac{dy_1\,\sqrt{y_1^2-m^2}}{y_1^2\,\exp(\beta y_1)} + \frac{e^2}{4M^2}\int_M^\infty \frac{dy_2\,\sqrt{y_2^2-M^2}}{y_2^2\,\exp(\beta y_2)} \\ & \left. .\left[3M^4 + m^2(y_2^2-M^2)\right]\right] . \end{aligned} \tag{3.19}$$

Binomial expansion of the numerators of (3.19) at high temperatures gives to $O(\beta m)$,

$$m^2_{\beta,\,\mu} = m^2 + \frac{1}{\beta\exp(\beta\mu)}\left[\frac{\lambda}{3\pi^2\beta} + \frac{\lambda m}{3\pi^2} + \frac{3e^2}{2\pi^2\beta} + \frac{3e^2M}{2\pi^2}\right] + O(\beta m) \; . \tag{3.20}$$

This shows a decrease of the effective mass with increase of μ . Thus, starting from a symmetry restored phase it is broken at a very large value of μ . The critical chemical potential μ_{c2} , at which this symmetry breaking occurs, is obtained from (3.20) as

$$\mu_{c2} = \frac{1}{\beta}\ln\left(\frac{-1}{m^2\beta^2\,\pi^2}\left(\frac{\lambda}{3}(1+m\beta) + \frac{3e^2}{2}(1+M\beta)\right)\right) . \tag{3.21}$$

The corresponding critical temperature is

$$\frac{1}{\beta_{c2}} = \left[\sqrt{\frac{4b_1}{3}}\sinh\left(\frac{1}{3\sinh^{-1}b_2}\right) - \frac{1}{3\mu}\right]^{-1} , \tag{3.22}$$

where

$$b_1 = \frac{\lambda}{3\pi^2 m\mu} + \frac{3e^2M}{2\pi^2m^2\mu} - \frac{1}{3\mu^2} \; , \tag{3.23}$$

$$b_2 = \frac{\sqrt{27}}{2b_1}\left[\frac{\lambda}{9\pi^2 m\mu^2} - \frac{2}{27\mu^3} + \frac{e^2M}{2\pi^2m^2\mu^2} - \frac{\lambda}{3\pi^2m^2\mu} - \frac{3e^2}{2\pi^2m^2\mu}\right] . \tag{3.24}$$

To study the gauge dependence of the above FTD phase transitions, we now compute the same scalar meson mass in a ghost-free gauge. The Lagrangian

$$\mathcal{L} = \frac{-1}{2}e^2\phi_i^2 A_\mu^2 - \frac{\lambda}{4!}(\phi_i^2)^2 + eA_\mu(\phi_2\,\partial_\mu\phi_1 - \phi_1\,\partial_\mu\phi_2) - \frac{1}{2}m^2\phi_i^2 - \frac{1}{2}(\partial_\mu\phi_i)^2, \quad i = 1,2, \tag{3.25}$$

in terms of shifted fields χ and ϕ is expressed as

$$\begin{aligned}\mathcal{L} &= \frac{-1}{2}e^2\chi^2 A_\mu^2 - \frac{1}{2}e^2\phi^2 A_\mu^2 - \frac{1}{24}(\phi^4 + \chi^4 + 2\phi^2\chi^2) - e^2C\chi A_\mu^2 \\ &\quad + eA_\mu(\partial_\mu\chi\,\phi - \chi\,\partial_\mu\phi) - \frac{\lambda C}{6}\chi^3 - \frac{\lambda C}{6}\phi^2\chi - \frac{1}{2}M^2A_\mu^2 \\ &\quad - \frac{1}{2}m_\chi^2\chi^2 - \frac{1}{2}m_2^2\phi^2 - MA_\mu\,\partial_\mu\phi - \frac{1}{2}(\partial_\mu\chi)^2 - \frac{1}{2}(\partial_\mu\phi)^2 \\ &\quad - \frac{1}{4}(F_{\mu\nu})^2 - \frac{1}{2\alpha}(\partial_\mu A_\mu)^2,\end{aligned} \tag{3.26}$$

where $\phi_1 = \chi + C$, $\phi_2 = \phi$, $m_\chi^2 = m^2 + (\lambda/2)C^2$, $m_2^2 = m^2 + (\lambda/6)C^2 = 0$ and $M = eC$. The last term in (3.26) is added for fixing a gauge. The FTD propagators taking into account the $A_\mu - \phi$ mixing effects to all orders of λ are

$$\Delta^\chi(k) = \left[\frac{-i}{k^2 + m_\chi^2 - i\epsilon} + \frac{2\pi\,\delta(k^2 + m_\chi^2)}{\exp[\beta(k_0 + \mu)] - 1}\right], \tag{3.27}$$

$$\Delta^\phi(k) = \left[\frac{-i}{k^2 - i\epsilon} + \frac{2\pi\,\delta(k^2)}{\exp[\beta(k_0 + \mu)] - 1} + \frac{\alpha m^2}{(k^2)^2}\right], \tag{3.28}$$

$$\begin{aligned}\Delta_{\mu\nu}(k) &= \left[\left[\frac{-i}{k^2 + M^2 - i\epsilon} + \frac{2\pi\delta(k^2 + M^2)}{\exp[\beta(k_0 + \mu)] - 1}\right]\left[\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right]\right. \\ &\quad \left. - \left[\frac{2\pi\delta(k^2)}{\exp[\beta(k_0 + \mu)] - 1}\right]\frac{k_\mu k_\nu}{M^2} + \frac{\alpha k_\mu k_\nu}{(k^2)^2}\right].\end{aligned} \tag{3.29}$$

The diagrams contributing to the scalar meson self energy are in Fig. 3.3 and the Feynman rules in Fig. 3.4. Contributions from each of these diagrams can be split up into three parts, namely, α^2, α and $\alpha-$independent terms. On summation of these different diagrams, α^2 and α terms cancel separately and we need to consider

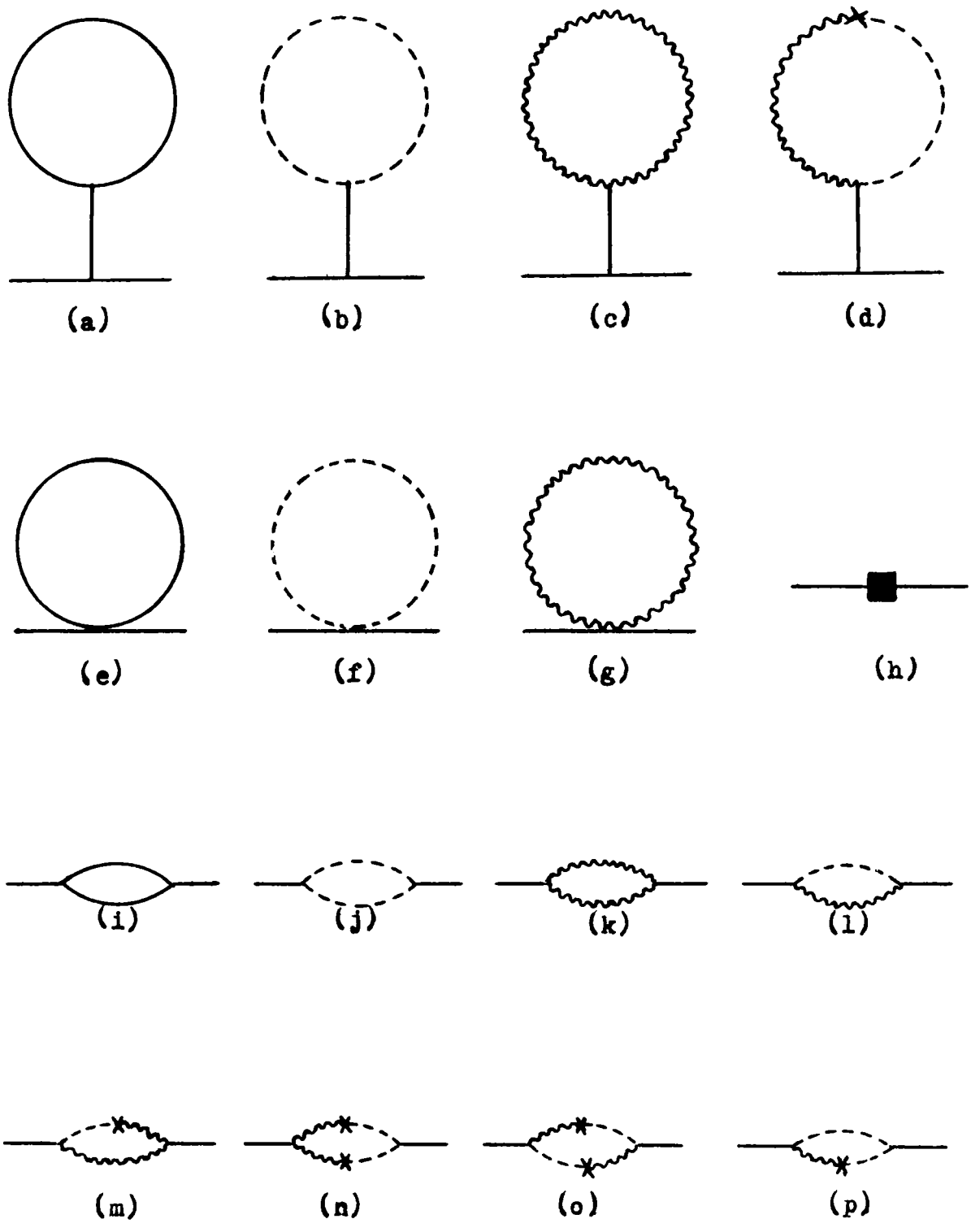


Fig. 3.3 Diagrams contributing to the scalar-meson propagator in the ghost-free gauge in the one-loop approximation

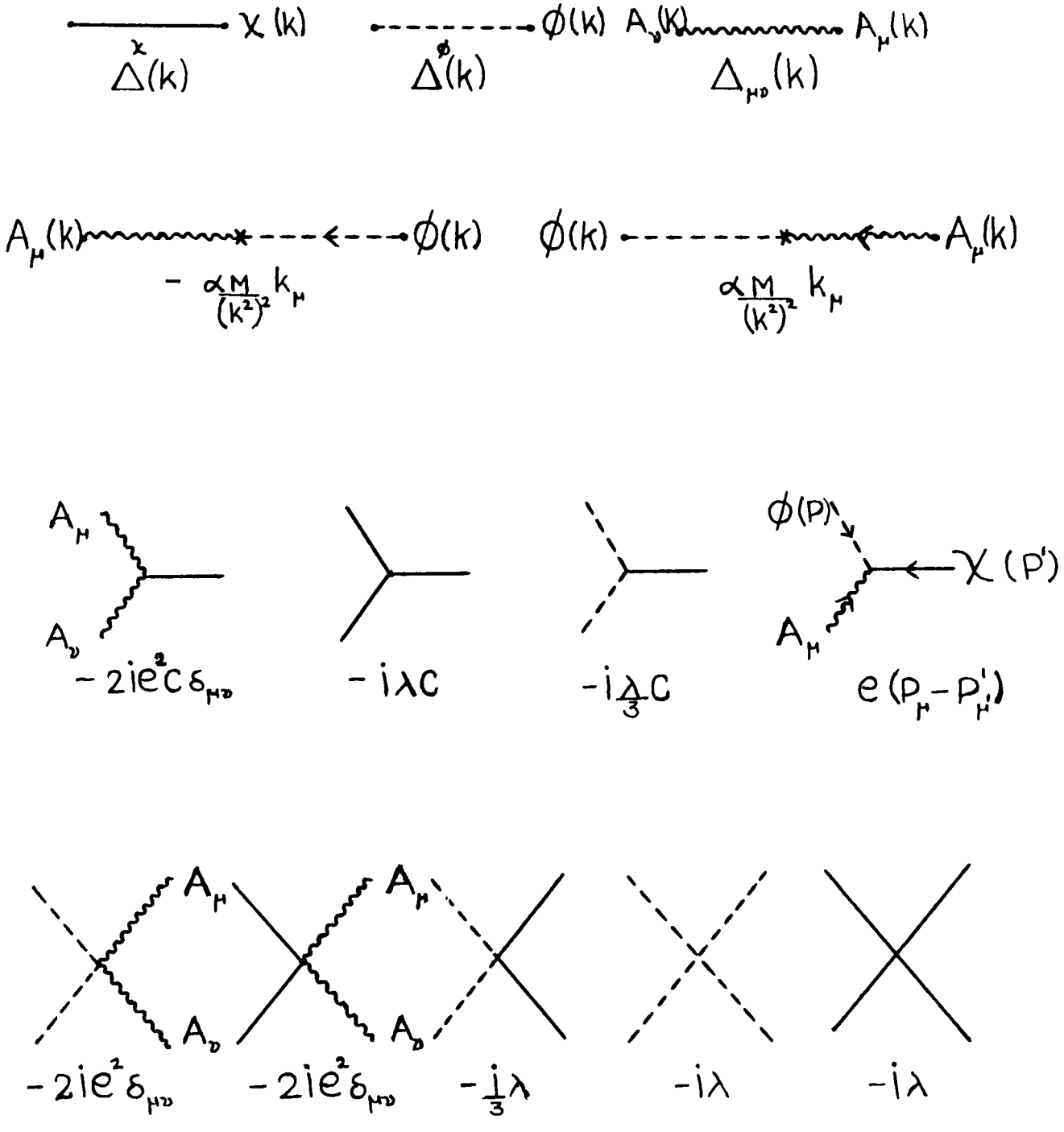


Fig. 3.4 Feynman rules in the ghost-free gauge

only α independent terms which do not have any $A_\mu - \phi$ mixing. The result obtained by summing all these terms is found to be the same as that from the unitary gauge, indicating gauge invariance.

3.3 Effective Coupling Constants

The abelian Higgs model (3.1) contains two unrenormalized coupling constants, the scalar coupling constant λ and the gauge coupling constant e. The renormalized coupling constants can be obtained either by the vertex-renormalization or from the effective potential. From finite temperature studies at the one-loop level it is known that, in the absence of SSB, the effective scalar coupling constant decreases and the gauge one increases with temperature [118].

Diagrams contributing to the vertex correction of the scalar coupling constant to the order λ^2 and the gauge one to the order e^4 are shown in Figs. 3.5 and 3.6 respectively, and they yield

$$\lambda_{\beta,\mu} = \lambda - \frac{3\lambda^2}{2i}\int d^4k\ \Delta(k)\Delta(2p-k)\ , \tag{3.30}$$

$$e^2_{\beta,\mu} = e^2 - \frac{e^4}{(2\pi)^4}\int d^4k\ K^2\ \Delta(k)\Delta(p-k)_{\mu,\nu}\ . \tag{3.31}$$

Using the propagators (3.10) and (3.11) and the mass-shell condition for the external momentum gives the real parts of the above equations as

$$\lambda_{\beta,\mu} = \lambda - \frac{3\lambda^2}{8\pi^2}\int_0^\infty \frac{dK\ K^2}{(K^2+m^2)^{3/2}\left(\exp(\beta\sqrt{K^2+m^2}+\beta\mu)-1\right)}\ , \tag{3.32}$$

$$e^2_{\beta,\mu} = e^2 + \frac{e^4}{2\pi^2}\int_0^\infty \frac{dK\ K^2}{(K^2+m^2)^{3/2}\left(\exp(\beta\sqrt{K^2+m^2}+\beta\mu)-1\right)}\ . \tag{3.33}$$

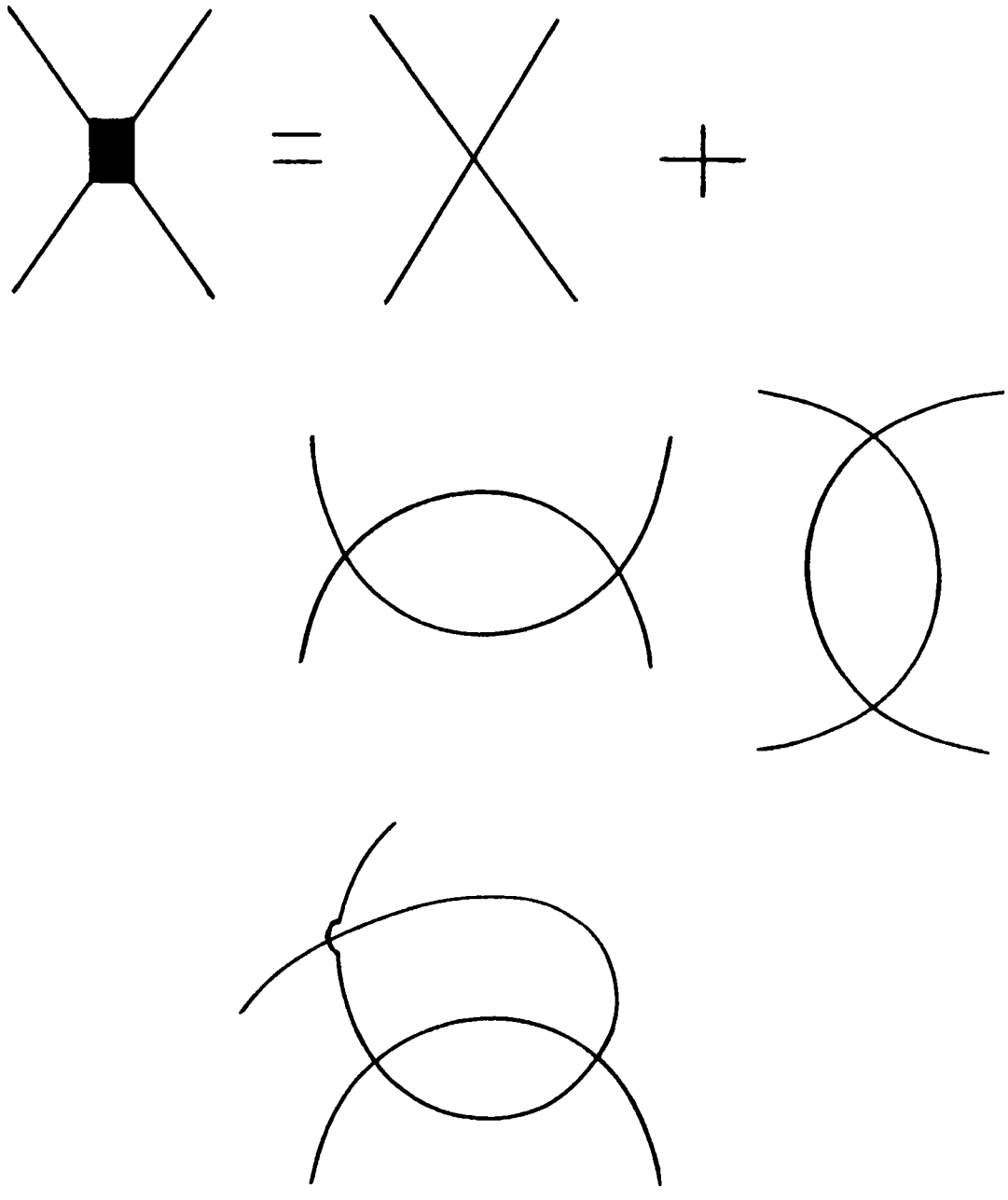


Fig. 3.5 Vertex diagrams contributing to the scalar coupling constant in the one-loop approximation

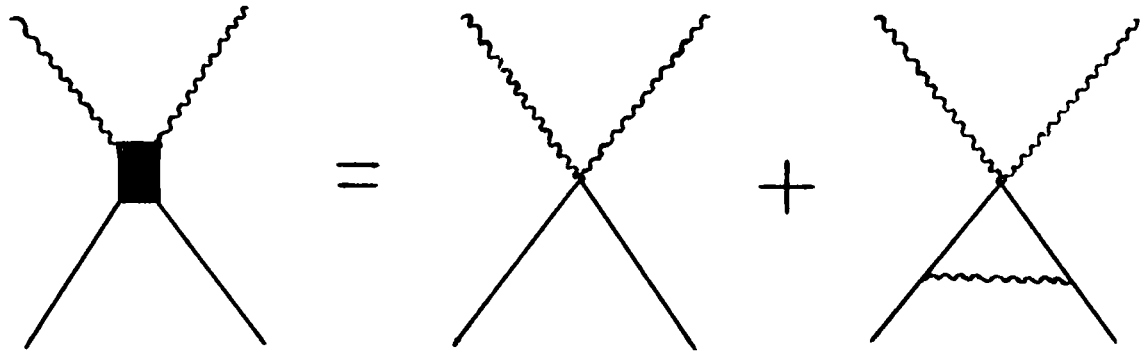


Fig. 3.6 Vertex diagrams contributing to the gauge coupling constant in the one-loop approximation

To study the finite density behaviour of $\lambda_{\beta,\mu}$ and $e^2_{\beta,\mu}$ from (3.32) and (3.33), we first use the numerical methods, and the results are plotted in Figs. 3.7 and 3.8 respectively for various values of temperatures. From these figures one finds that, at fixed temperatures, $\lambda_{\beta,\mu}$ first decreases and then increases with the increase of μ while $e^2_{\beta,\mu}$ shows the opposite behaviour. These characteristics can be clarified by evaluating $\lambda_{\beta,\mu}$ and $e^2_{\beta,\mu}$ in the small value and large value regions of μ separately in the high temperature limit.

For small values of μ , we make a Taylor expansion of (3.32) and (3.33) as a function of μ . Using the high temperature approximation after a binomial expansion of $(K^2+m^2)^{3/2}$, we get to $O(\beta^2 m^2)$

$$\lambda_{\beta,\mu} = \lambda - \frac{3\lambda^2 C_1}{8\pi^2} + O(\beta^2 m^2) \,, \tag{3.34}$$

$$e^2_{\beta,\mu} = e^2 + \frac{e^4 C_1}{2\pi^2} + O(\beta^2 m^2) \,, \tag{3.35}$$

where

$$C_1 = \frac{\pi}{2\beta m} + \frac{1}{2}\ln\left(\frac{\beta m}{4\pi}\right) - \frac{\beta\mu}{4}\left(\beta m - \ln(\beta m)\right) + \frac{\mu^2\beta^2}{2}\left(\beta m - \ln(\beta m)\right) \,. \tag{3.36}$$

According to (3.22) and (3.35), at a fixed temperature $\lambda_{\beta,\mu}$ decreases with μ while $e^2_{\beta,\mu}$ shows the reverse behaviour.

For large values of μ , by means of the approximation (3.18) at high temperatures we get to $O(\beta^2 m^2)$

$$\lambda_{\beta,\mu} = \lambda - \frac{3\lambda^2\, C_2}{8\pi^2 \exp(\beta\mu)} + O(\beta^2 m^2) \,, \tag{3.37}$$

$$e^2_{\beta,\mu} = e^2 + \frac{e^4 C_2}{2\pi^2 \exp(\beta\mu)} + O(\beta^2 m^2) \,, \tag{3.38}$$

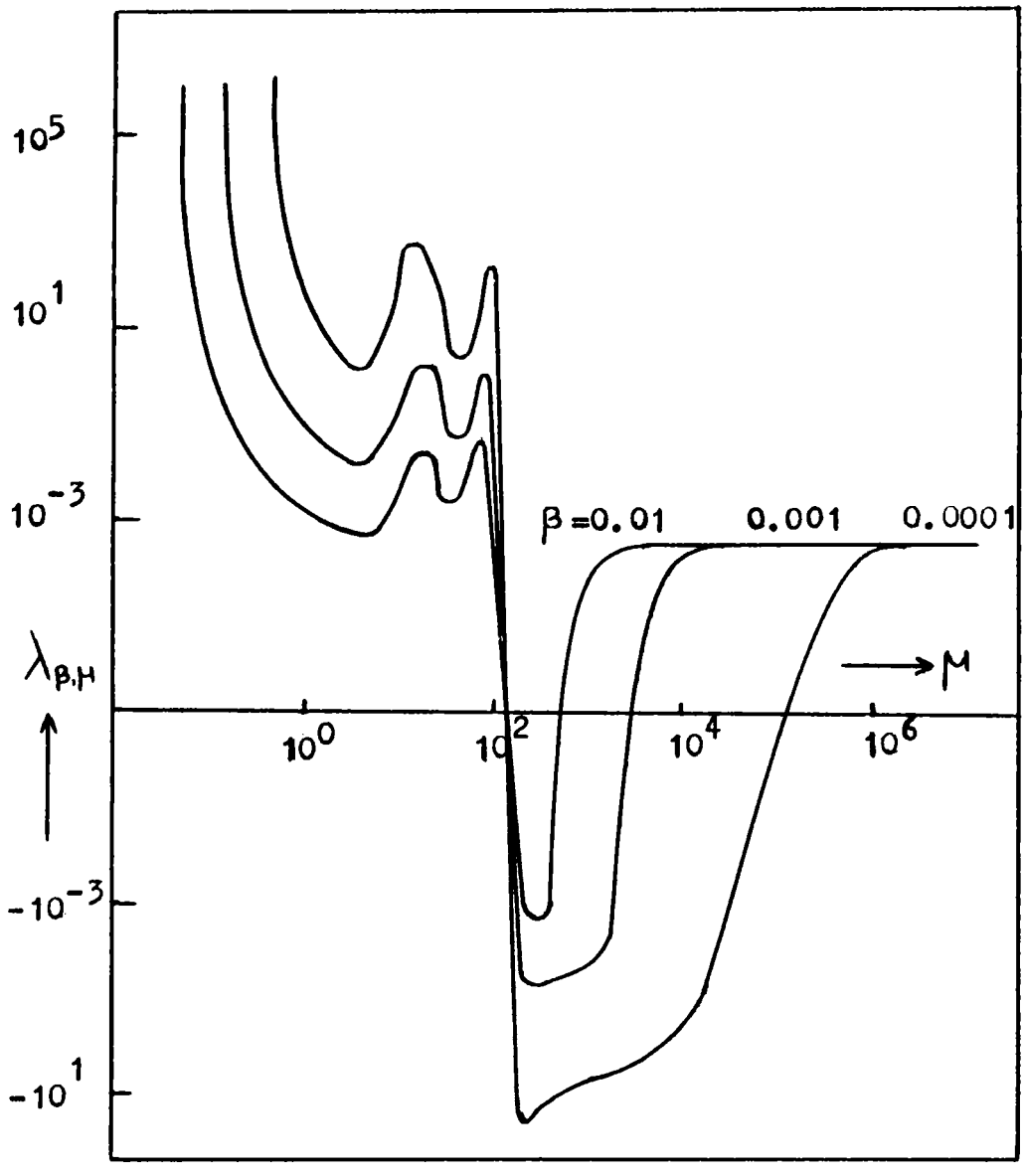


Fig. 3.7 Variation of the effective scalar coupling constant with the chemical potential

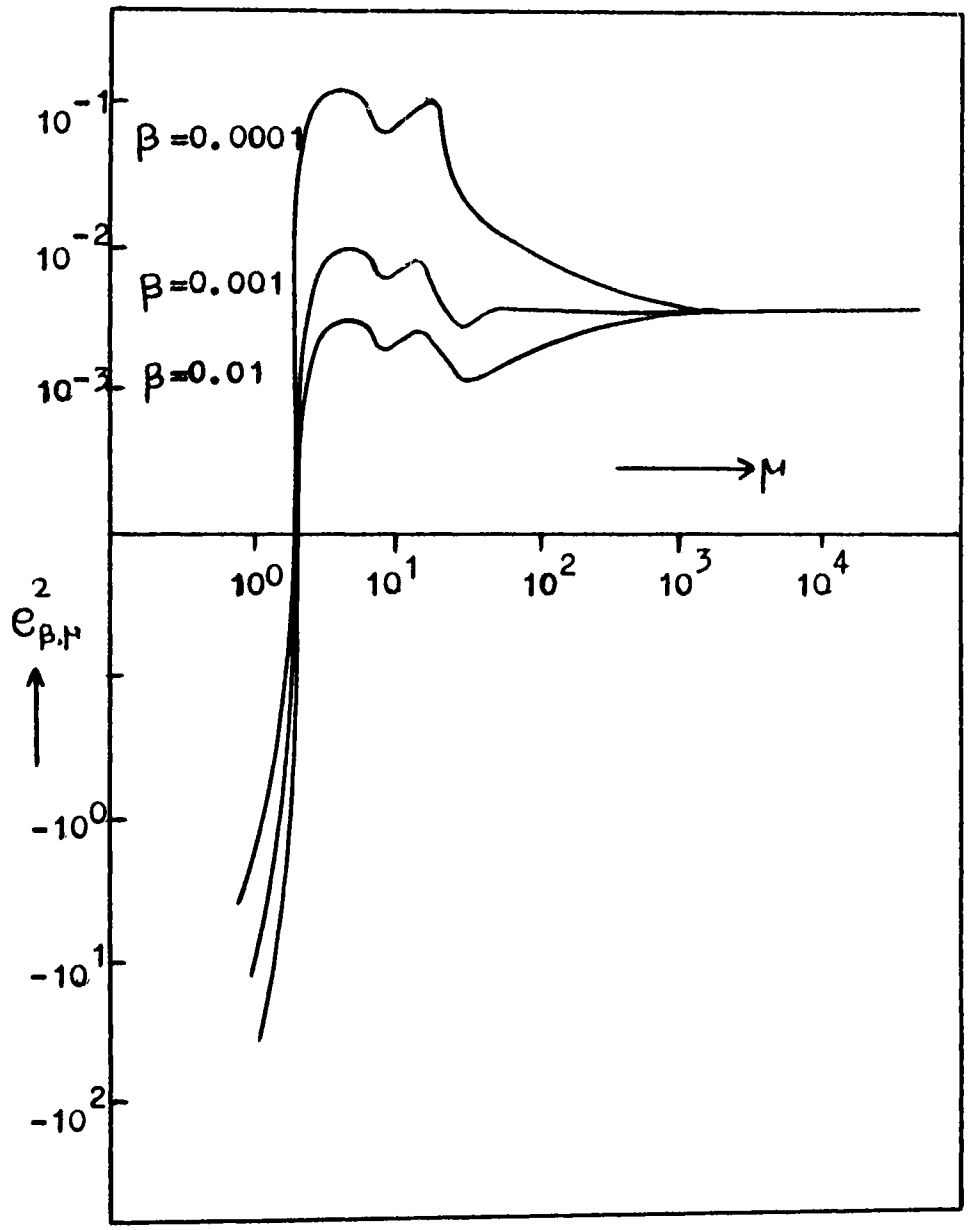


Fig. 3.8 Variation of the effective gauge coupling constant with the chemical potential

where

$$C_2 = 0.7886 - \frac{\beta m}{2} - \frac{\beta^2 m^2}{8} \,. \tag{3.39}$$

Equations (3.37) and (3.38) at a fixed temperature indicate an increase of $\lambda_{\beta,\,\mu}$ with the increase of μ while $e^2_{\beta,\,\mu}$ shows the reverse behaviour. For an asymptotically large value of μ both $\lambda_{\beta,\,\mu}$ and $e^2_{\beta,\,\mu}$ approach their tree level values. Note that we evaluated these coupling constants from the vertex diagrams using the mass-shell condition for the external momentum. Hence all reasons we indicated for the gauge independence of the effective mass hold also for the gauge independence of these effective coupling constants.

Chapter 4

GEP Study of FTD Phase Transitions in ϕ^4 Model

The Higgs mechanism plays an important role in standard unified model of fundamental interactions [119, 120]. However, there is some worry about its basis as some authors have shown that the pure quantum $\lambda\phi^4$ model may be trivial (i.e., $\lambda \to 0$, no interaction exists at all) in four space-time dimensions [85, 89]. Recently, aiming at the revival of $\lambda\phi^4$ theory some effort has been made using the non-perturbative GEP approach [41-43, 81, 121], the important features of which have been presented in chapter 1. One method is to introduce explicitly a large but finite momentum cutoff Λ and treat the $\lambda\phi^4$ model as an effective model at low energy [92-95]. On the other hand, in the so called autonomous theory [98, 101, 122], after performing a special type of wave function renormalization while keeping the bare coupling parameter λ_B positive but infinitesimal ($\lambda_R \to 0^+$), one can let $\Lambda \to \infty$ and regain a meaningful $\lambda\phi^4$ model. From the practical point of view, there is a benefit in the former kind of theory (with finite cutoff Λ) as some bounds on the mass of the elusive Higgs boson could be found when the gauge fields are included [95, 123-133], whereas no observable restriction exists in the latter kind of theory (with $\Lambda \to \infty$).

In this chapter, we make a GEP study of FTD phase transitions in $\lambda\phi^4$ making use of both the cutoff and autonomous versions of GEP. It is shown that

in the presence of SSB at finite non-zero temperatures, increase of bosonic chemical potential induces a sequence of symmetry restorations and symmetry breakdowns. For an asymptotically large value of chemical potential, these multiple phase transitions end in a symmetry broken phase. Similarly in the case of SSB at a finite non-zero value of bosonic chemical potential, increase of temperature induces a series of symmetry restoring and symmetry breaking phase transitions. For an asymptotically large value of temperature, symmetry remains restored. With zero chemical potential increase of temperature only restore the SSB phase, without any multiple phase transition. We also study the FTD behaviour of effective scalar boson mass. Possibility of these multiple phase transitions as revealed by gaussian approximation contrasts with the double phase transition found in perturbative analysis and may be attributed to the non-perturbative effects. In the remaining part of this chapter, we first discuss the FTD study of the cut off version of GEP and then the autonomous version of it.

4.1 Cutoff Version GEP

The precarious form of ϕ^4 has nothing to do with the Higgs mechanism, since it has no SSB. However, SSB can be induced in the model, by including an ultraviolet cutoff for the momentum in the integrals. This cutoff version GEP has been applied to various scalar and fermion models [53, 54]. The stability property of cutoff ϕ^4 and ϕ^4 are the opposite of one another. In cutoff ϕ^4, positive λ_B gives a bounded potential with a pair of minima at very large ϕ_0 , corresponding to a SSB phase, while negative λ_B leads to an unbounded potential [53]. In this section we evaluate the GEP at finite temperature and density (FTDGEP) by introducing explicitly a finite and very large cut off Λ for the momentum. This allows the bare coupling constants to assume a positive value and to have SSB for the ground state. Following the discussion of Sec. 1.3, the GEP at zero temperature and chemical potential is defined by

$$V_G(\phi_0) = \overset{min}{\Omega}\ V_G(\phi_0, \Omega) = \overset{min}{\Omega}\ {}_{\Omega,\phi_0}\langle 0 \mid \mathcal{H} \mid 0\rangle_{\Omega,\phi_0} \tag{4.1}$$

where $\mathcal{H}$ is the Hamiltonian density and $\mid 0 >_{\Omega,\, \phi_0}$ is a normalized gaussian wave functional centred on $\phi = \phi_0$. We first study the model of a self interacting 2-component spinless field in 3+1 dimensions with an O(2) invariant interaction described by the Lagrangian

$$\mathcal{L}(\phi_a(x)) = \frac{1}{2}\ \partial_\nu\phi_a\ \partial^\nu\phi_a - \frac{1}{2}m_B^2\ \phi_a\phi_a - \lambda_B(\phi_a\phi_a)^2\,, \quad a = 1,2. \tag{4.2}$$

A study of GEP for this model with a cut off Λ shows that the ground state of (4.2) possess a SSB phase under the conditions [96]

$$0 < \lambda_B\ < \frac{4}{J_2(\rho/\Lambda)} \quad and \quad -m_B > \eta_c\Lambda^2\,, \tag{4.3}$$

where

$$J_2(\rho/\Lambda) = \frac{1}{2\pi^2}\left(\ln\left(\frac{\Lambda}{\rho} + \sqrt{\left(\frac{\Lambda}{\rho}\right)^2 + 1}\right) - \frac{(\Lambda/\rho)}{\sqrt{(\Lambda/\rho)^2+1}}\right), \tag{4.4}$$

$$\eta_c = \left(\frac{\chi_c}{\sqrt{\chi_c^2+1}}\right)\frac{\lambda_B}{16\pi^2} \leq \frac{\lambda_B}{16\pi^2}\,, \tag{4.5}$$

with

$$\chi_c = \frac{1}{2}\exp\left(\frac{8\pi^2}{\lambda_B} + 1\right) \gg 1\,. \tag{4.6}$$

We shall follow Stevensons method [53] in calculating the GEP by writing the field ϕ as $\phi_0 + \hat{\phi}$ where ϕ_0 is a constant classical field and $\hat{\phi}$ is a quantum free field of mass Ω . This yields the GEP at zero temperature and zero density for the model described by (4.2) as

$$\begin{aligned} V_G(\phi_0) \ &= \ \frac{1}{2}m_B^2\ \phi_0^2 + \lambda_B\ \phi_0^4 + I_1(\Omega) + \frac{1}{2}\left(m_B - \Omega^2\right) I_0(\Omega) \\ &\quad + 6\lambda_B I_0(\Omega)\phi_0^2(\Omega) + 3\lambda_B I_0^2(\Omega) \end{aligned} \tag{4.7}$$

where the I_N integrals are given by (1.102). Minimizing the expression (4.7) with respect to the variable parameter Ω give the optimum value of Ω to be used and is a solution of the gap equation

$$\bar{\Omega}^2 = \frac{m_B^2}{12} + \lambda_B\left(I_0(\bar{\Omega}) + \phi_0^2\right). \tag{4.8}$$

This method can be alternatively described as a calculation of the vacuum energy density to first order in the perturbation theory generated by

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{int} \,, \tag{4.9}$$

$$\mathcal{H}_0 = \frac{1}{2}\left(\dot{\hat{\phi}}^2 + (\nabla\phi)^2 + \Omega^2\hat{\phi}^2\right), \tag{4.10}$$

$$\mathcal{H}_{int} = -\frac{1}{2}\Omega^2\hat{\phi}^2 + \frac{1}{2}m_B^2\left(\phi_0 + \hat{\phi}\right)^2 + \lambda_B\left(\phi_0 + \hat{\phi}\right)^4 \,. \tag{4.11}$$

Here the parameters ϕ_0 and Ω arise from the split up of the Hamiltonian and the I_N integrals arise as

$$I_1 = {}_\Omega\!<0 \mid \mathcal{H}_0 \mid 0>_\Omega \,, \quad I_0 = {}_\Omega\!<0 \mid \hat{\phi}^2 \mid 0>_\Omega \,. \tag{4.12}$$

Even though the method uses techniques of perturbation theory, this is not really a perturbative scheme in the sense that no expansion in λ is used. This method will be applied to obtain the finite temperature / density generalization of GEP.

Method of Computing FTDGEP

The question of generalizing GEP from zero temperature to finite temperature had led to some earlier confusions which were cleared by Hajj and Stevenson [101]. Following their approach we outline the calculation of FTGEP. We then extend the method to finite density as well. We consider a fixed spatial volume V surrounded by a heat bath at a fixed temperature T. A quantum field ϕ is defined in the region V and obeys some suitable (eg., periodic) boundary conditions. The set of modes of the ϕ field constitute the 'system' in the thermodynamic sense. Thermal equilibrium is achieved when the system has minimized its (Helmholtz) free energy

$$F = E - TS \,, \tag{4.13}$$

where E is the (internal) energy, and S is the entropy of the system. Thus the physically meaningful effective potential for the system corresponds to the function of ϕ_0 obtained by minimizing the free energy F , subject to the constraint that $<\phi>= \phi_0$. To use the energy E instead, as was done in some early works, would

correspond to the case where the entropy, rather than the temperature, is somehow held fixed [101].

The proper finite temperature generalization of the GEP is therefore obtained as follows. Calculate F to first order in the perturbation theory specified by the free part of the Hamiltonian; divide out by an overall volume factor V; and minimize with respect to Ω. Standard thermodynamics [134] gives both F and E in terms of the partition function Z:

$$Z \equiv Tr\left(\exp\left(-\beta H\right)\right) = \sum_{\alpha} < \alpha \mid \exp\left(-\beta\left(H\right)\right) \mid \alpha > \tag{4.14}$$

$$F = -\frac{1}{\beta} \ln Z \tag{4.15}$$

$$E = -\frac{d}{d\beta} \ln Z = \frac{Tr\left(\exp\left(-\beta H\right) H\right)}{Tr\left(\exp\left(-\beta H\right)\right)} \tag{4.16}$$

where $\mid \alpha >$ denotes an eigenstate of H. Calculating Z to first order in H_{int} , trace involves a summation over the unknown eigenstates of the full H.

$$\begin{aligned} Z &= Tr\left(\exp\left(-\beta H_0\right)\left(1 - \beta H_{int}\right)\right) \\ &= Z_0\left(1 - \beta < H_{int} >_T\right) , \end{aligned} \tag{4.17}$$

where

$$Z_0 \equiv Tr\left(\exp\left(-\beta H_0\right)\right) = \sum_{\alpha} < \alpha_0 \mid \exp\left(-\beta H_0\right) \mid \alpha_0 > \tag{4.18}$$

and the notation $< A >_T$ stands for the 'thermal average' of the operator A. Inserting (4.17) in (4.15), the free energy to first order in H_{int} is

$$F = -\frac{1}{\beta} \ln Z_0 + < H_{int} >_T \tag{4.19}$$

From the preceding discussion, the FTDGEP

$$\bar{V}_G^{\beta}(\phi_0) = \stackrel{min}{\Omega} \; V_G^{\beta}(\phi_0, \bar{\Omega}) \tag{4.20}$$

is obtained from the free energy of (4.19) as

$$V_G^{\beta}(\phi_0, \bar{\Omega}) \equiv \frac{F}{V} = -\frac{1}{\beta V} \ln Z_0 + < H_{int} >_T \; . \tag{4.21}$$

The first term is a free field theory quantity and represents the finite temperature generalization of the I_1 integral. A straight forward computation from first principles gives

$$\begin{aligned} Z_0 &= \exp(-\beta I_1 V)\left(\left(\sum_{n_1=0}^{\infty}\exp(-\beta n_1\omega_1)\right)\left(\sum_{n_2=0}^{\infty}\exp(-\beta n_2\omega_2)\right)\cdots\right) \\ &= \exp(-\beta I_1 V)\prod_i\left((1-\exp(-\beta\omega_i))^{-1}\right). \end{aligned} \tag{4.22}$$

Hence by defining

$$\frac{-1}{\beta V}\ln Z_0 \equiv I_1^{FTD}(\bar{\Omega}) =\equiv I_1(\bar{\Omega}) + I_1^{\beta}(\bar{\Omega})\ , \tag{4.23}$$

we have

$$I_1^{\beta}(\bar{\Omega}) = \frac{1}{\beta}\int\frac{d^3K}{(2\pi)^3}\ln(1-\exp(-\beta\omega_k))\ , \tag{4.24}$$

where the discrete modes ω_i have been replaced by $\omega_k = \sqrt{K^2+\bar{\Omega}^2}$ and the sum replaced by the integration $\sum_i \to V\int d^3K/(2\pi)^3$.

The evaluation of $< H_{int} >_T$, the other term in V_G^{β} in (4.21) involves the calculation of $< \hat{\phi}^2 >_T$ and $< \hat{\phi}^4 >_T$. At zero temperature these would give $I_0(\bar{\Omega})$ and $3I_0^2(\bar{\Omega})$ respectively. Hajj and Stevenson have shown by detailed calculation that for finite temperature generalization of these results $I_0(\bar{\Omega})$ is to be replaced by $I_0^{FT}(\bar{\Omega})$, where

$$I_0^{FT}(\bar{\Omega}) \equiv I_0(\bar{\Omega}) + I_0^{\beta}(\bar{\Omega}) =< \hat{\phi}^2 >_T \tag{4.25}$$

with

$$I_0^{\beta} = \int\frac{d^3K}{(2\pi)^{3/2}\,\omega_k}\frac{1}{(\exp(\beta\omega_k)-1)}\ . \tag{4.26}$$

The term $< \hat{\phi}^4 >_T$ equals $3\left(< \hat{\phi}^2 >_T\right)^2$ up to a volume suppressed term, as at zero temperature. A similar result holds for higher powers.

We now define FTDGEP by a direct extension of the technique described above. At finite densities $\omega_k \to \omega_k - \mu$ and so replace

$$I_1^{FT}(\Omega) \to I_1^{FTD}(\Omega) = I_1(\Omega) + I_1^{\beta,\,\mu} \tag{4.27}$$

and

$$I_0^{FT}(\Omega) \rightarrow I_0^{FTD}(\Omega) = I_0(\Omega) + I_0^{\beta,\,\mu}(\Omega) \tag{4.28}$$

where

$$I_1^{\beta,\,\mu}(\bar{\Omega}) = \frac{1}{\beta}\int \frac{d^3K}{(2\pi)^3} \ln\left(1 - \exp\left(-\beta\omega_k + \beta\mu\right)\right) , \tag{4.29}$$

$$I_0^{\beta,\,\mu} = \int \frac{d^3K}{(2\pi)^{3/2}\,\omega_k} \frac{1}{\left(\exp\left(\beta\omega_k - \beta\mu\right) - 1\right)} . \tag{4.30}$$

The final result can be summarized thus: the FTDGEP is obtained from the GEP by the replacements

$$I_1(\bar{\Omega}) \rightarrow I_1^{FTD}(\bar{\Omega}), \quad I_0(\bar{\Omega}) \rightarrow I_0^{FTD}(\bar{\Omega}) . \tag{4.31}$$

Furthermore, the relation

$$\frac{dI_1^{FTD}(\bar{\Omega})}{d\bar{\Omega}} = \bar{\Omega} I_0^{FTD} \tag{4.32}$$

holds, so that the rule $I_0(\bar{\Omega}) \rightarrow I_0^{FTD}(\bar{\Omega})$ also applies to the $\bar{\Omega}$ equation (4.8).

For the ϕ^4 -model described by eqn. (4.2) making use of the FTD conversion expressions (4.31) in (4.7) we evaluate FTDGEP and obtain

$$\begin{aligned}\overline{V_G^{\beta,\,\mu}}(\bar{\Omega}) \;=\; & I_1(\bar{\Omega}) + \frac{1}{2}\left(m_B^2 - \bar{\Omega}^2\right) I_0(\bar{\Omega}) + \frac{1}{2} m_B^2 \phi_0^2 + \lambda_B \phi_0^4 + I_1^{\beta,\,\mu}(\bar{\Omega}) \\ & + \frac{1}{2}\left(m_B^2 - \bar{\Omega}^2\right) I_0^{\beta,\,\mu}(\bar{\Omega}) + \lambda_B \Big((6 I_0(\bar{\Omega}) \phi^2 + 3 I_0^2(\bar{\Omega}) + 6\phi_0^2\, I_0^{\beta,\,\mu} \\ & + 6 I_0(\bar{\Omega}) I_0^{\beta,\,\mu}(\bar{\Omega}) + 3 (I_0(\bar{\Omega}))^2 \Big)\end{aligned} \tag{4.33}$$

Multiple Phase Transitions

The complete FTD dependence of the effective potential (4.33) is carried by the integrals $I_0^{\beta,\,\mu}(\bar{\Omega})$ and $I_1^{\beta,\,\mu}(\bar{\Omega})$. But it is difficult to analytically evaluate these integrals in a closed form and hence to understand the characteristics of (4.33), we resort to numerical methods. In Figs. 4.1-4.3 we show the results of numerical studies made on the expression for the effective potential in the cutoff method.

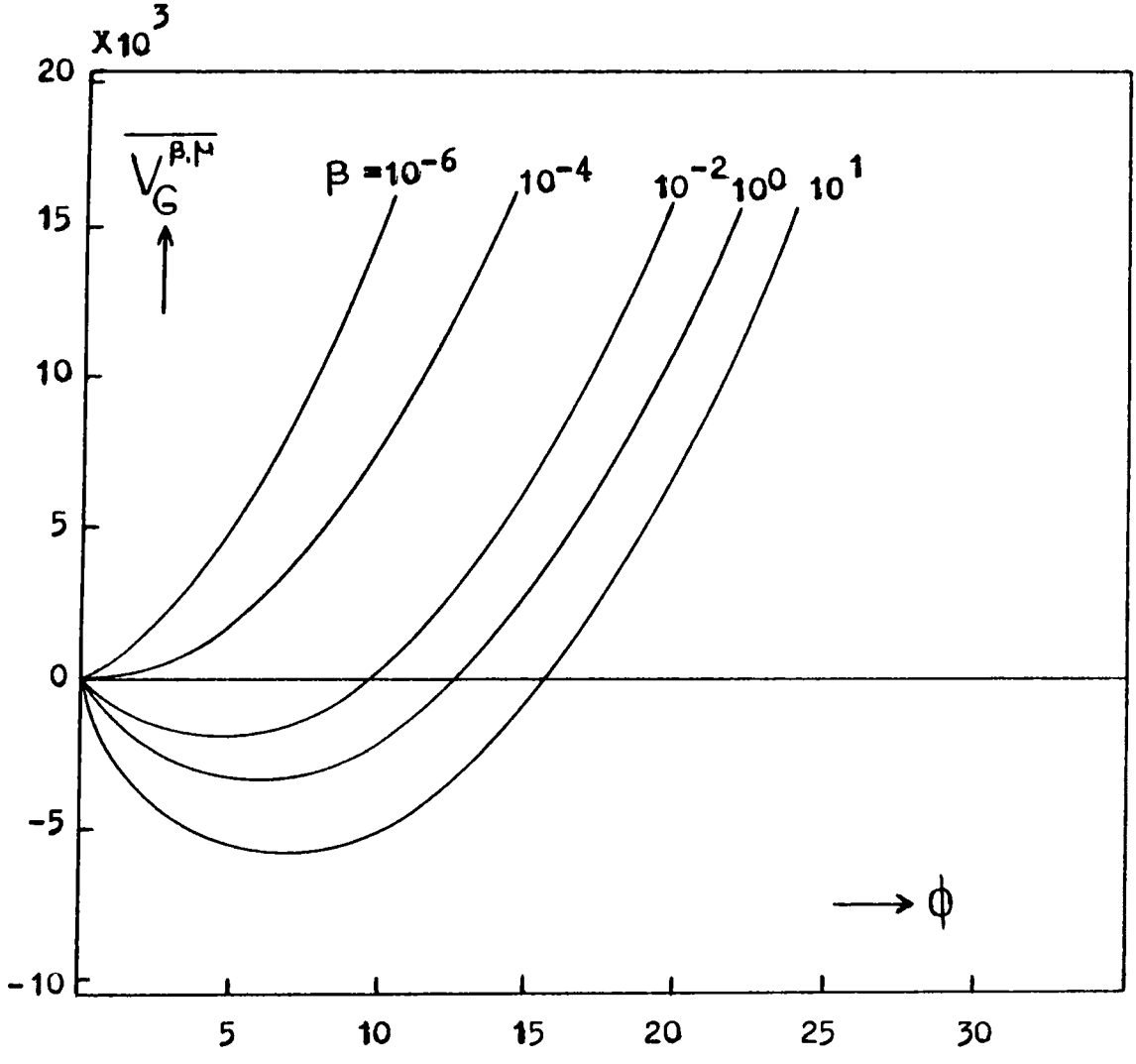


Fig. 4.1 Variation of FTDGEP with temperature in the absence of any chemical potential

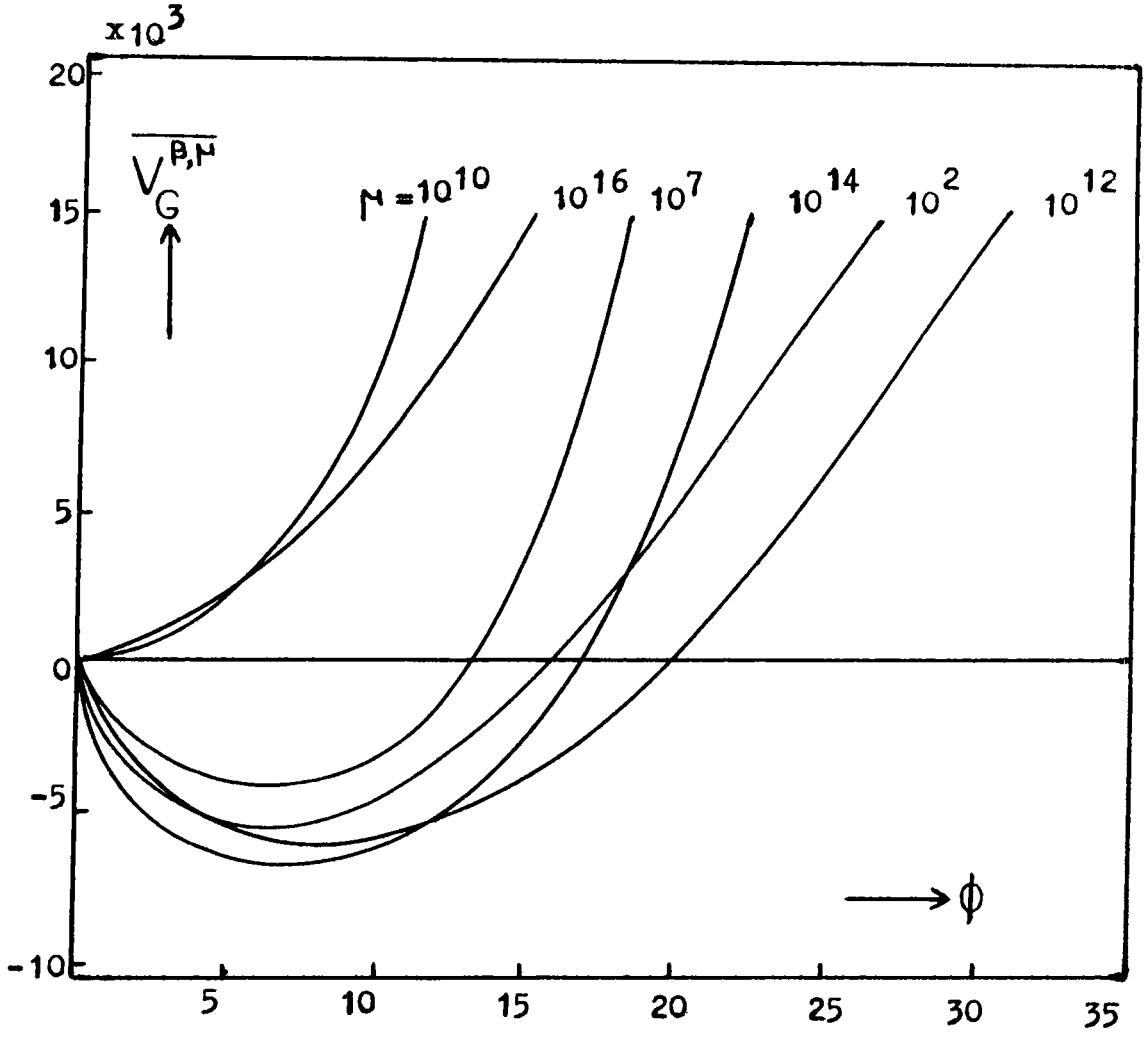


Fig. 4.2 FTDGEP for various chemical potentials at a non-zero temperature

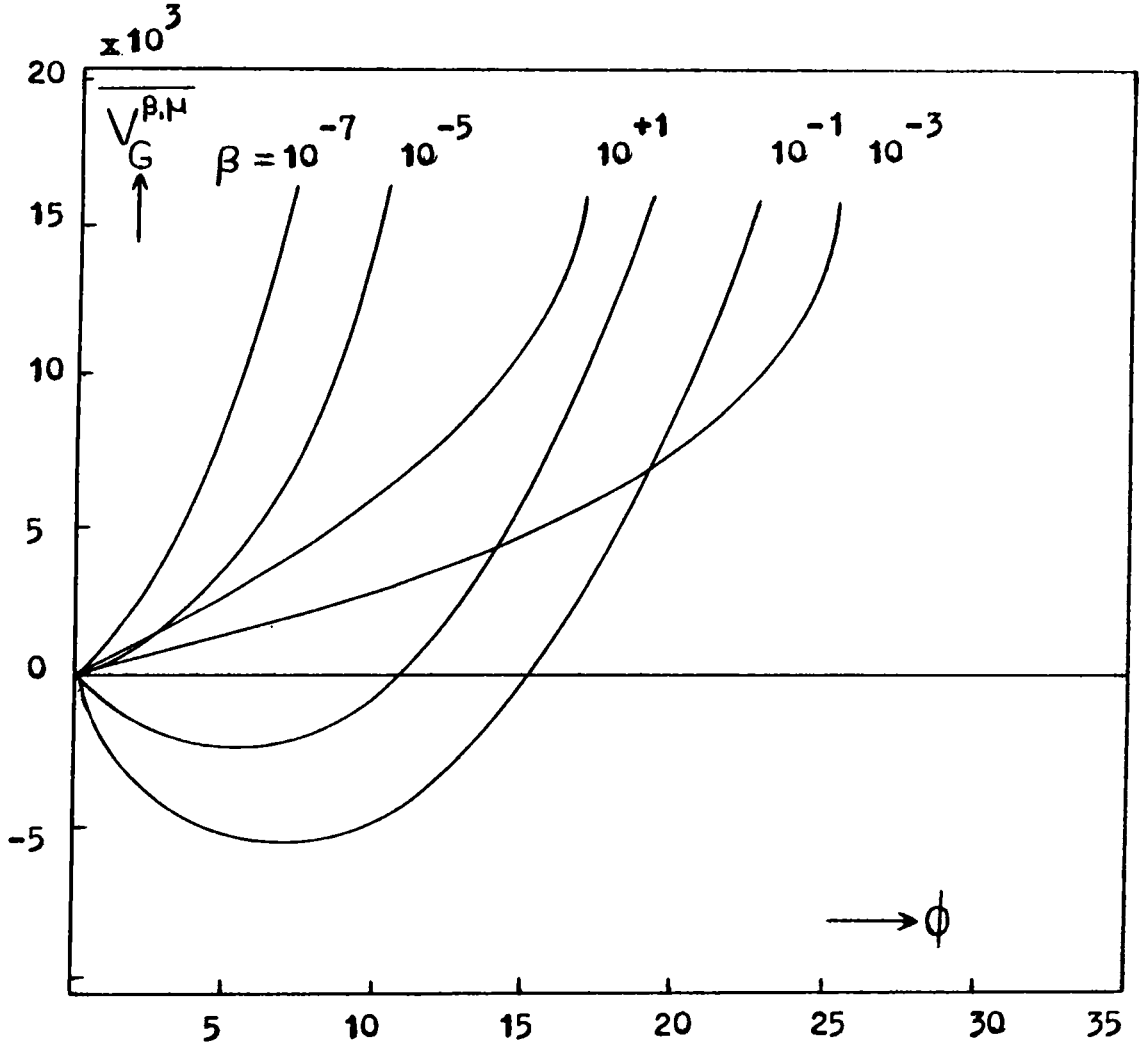


Fig. 43 FTDGEP for various temperatures at a non-zero chemical potential

Values of the various parameters in (4.33) are selected in accordance with (4.3), so as to have SSB. In Fig. 4.1, we plot the $\overline{V_G^{\beta,\,\mu}}$ for various temperatures in the absence of any chemical potential. It shows the usual finite temperature behaviour; starting from a SSB phase, the increase of temperature restore the symmetry at a particular critical temperature. In Fig. 4.2, $\overline{V_G^{\beta,\,\mu}}$ is plotted for various chemical potentials at a non-zero temperature. We see that, with the increase of chemical potential the symmetry is restored and is broken a number of times and for an asymptotically large value of μ , the symmetry remains broken. In Fig. 4.3, we show the $\overline{V_G^{\beta,\,\mu}}$ for various temperatures at non-zero chemical potentials. As in Fig. 4.2, this also show the existence of a temperature induced multiple phase transition. For a very large value of temperature the symmetry is eventually restored, whatever be the chemical potential. At zero temperature, variation of the chemical potential has no influence on the spontaneously broken symmetry.

Another method of demonstrating the existence of multiple phase transition is to compute the FTD dependent effective scalar boson mass $m_{\beta,\,\mu}$ from (4.33) by means of the relations

$$m^2_{\beta,\,\mu} = \frac{\partial^2\,\overline{V_G^{\beta,\,\mu}}}{\partial\phi^2}\;\Big|_{(\phi=0)}\;. \tag{4.34}$$

This gives

$$m^2_{\beta,\,\mu} = m_B^2 + \lambda_B\left(I_0(\bar{\Omega}) + I_0^{\beta,\,\mu}(\bar{\Omega})\right)\;. \tag{4.35}$$

Numerical evaluation of the expression (4.35) can be done and the results are shown in Figs. 4.4 and 4.5. The variation of $m^2_{\beta,\,\mu}$ with chemical potential at a non-zero temperature is shown in Fig. 4.4. Fig. 4.5 shows the variation of $m^2_{\beta,\,\mu}$ with temperature at non-zero chemical potential. These figures also indicate the same the FTD behaviour we found in Figs. 4.1-4.3.

To confirm the existence of temperature and density induced multiple phase transitions, we shall now carry out an analytical study of effective potential (4.33) employing a high temperature approximation. To do this, we Taylor expand

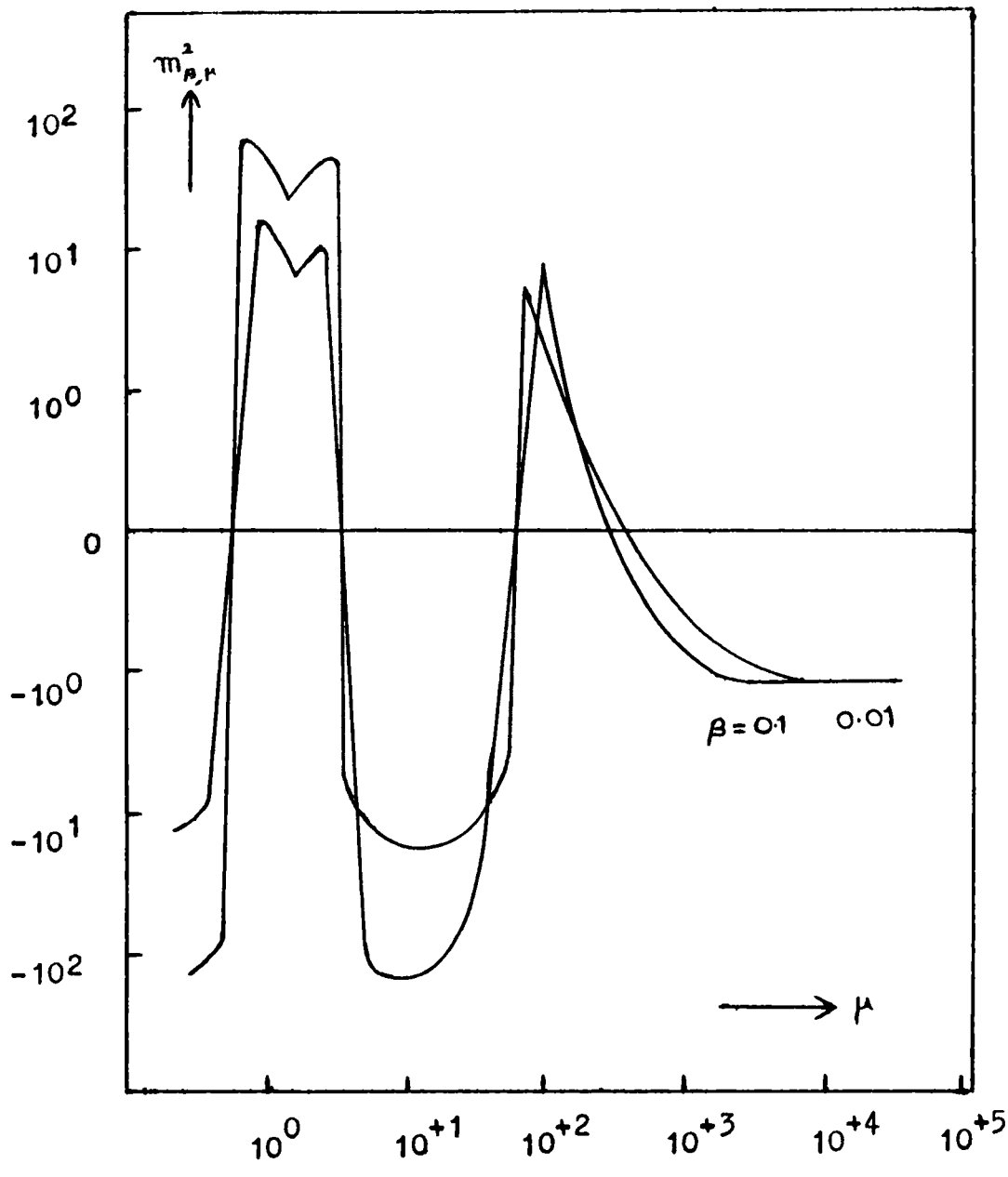


Fig. 4.4 **Variation of effective mass with chemical potential at a non-zero temperature**

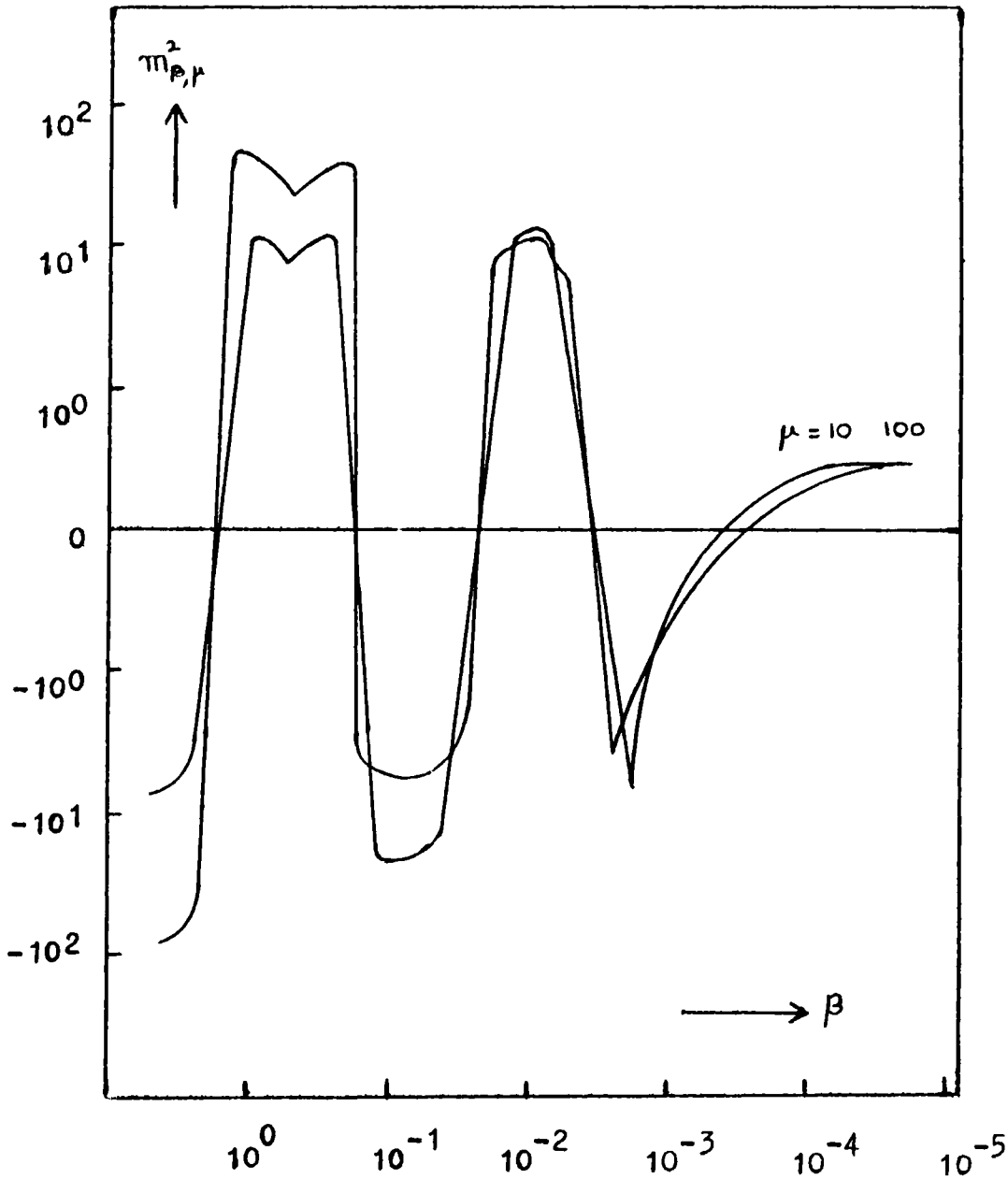


Fig. 4.5 **Variation of effective mass with temperature at a non-zero chemical potential**

the integrals $I_0^{\beta,\mu}(\bar{\Omega})$ and $I_1^{\beta,\mu}(\bar{\Omega})$ in β to $O(\beta^2)$ to obtain

$$I_0^{\beta,\mu}(\bar{\Omega}) = \frac{1}{2\pi^2\beta}\left(\frac{\pi^2}{6\beta} + \mu\beta(\bar{\Omega}-\mu) - \mu\ln\left(\exp\left(\beta(\bar{\Omega}-\mu)\right)-1\right) - \frac{\bar{\Omega}^2\pi}{4(\bar{\Omega}-\mu)}\right) + \frac{\bar{\Omega}^2}{8+\pi^2}\ln\left(\frac{\beta(\bar{\Omega}-\mu)}{4\pi}\right) + O(\beta^2) \tag{4.36}$$

and

$$I_1^{\beta,\mu}(\bar{\Omega}) = \frac{\bar{\Omega}^2}{24\beta^2} + \frac{\bar{\Omega}^2\mu(\bar{\Omega}-\mu)}{4\pi^2} - \frac{\bar{\Omega}^2}{8\pi\,\beta(\bar{\Omega}-\mu)} - \frac{\mu}{2\pi^2\beta}\ln\left(\exp\left(\beta(\bar{\Omega}-\mu)\right)-1\right) - \frac{\bar{\Omega}^2}{8\pi^2}\ln\left(\frac{\beta(\bar{\Omega}-\mu)}{4\pi}\right) + O(\beta^2)\,. \tag{4.37}$$

The integrals $I_0(\bar{\Omega})$ and $I_1(\bar{\Omega})$ are evaluated using a cutoff Λ for the momentum [96]. They yield

$$I_0(\bar{\Omega}) = \frac{1}{4\pi^2}\left(\frac{\Lambda}{\bar{\Omega}}\sqrt{\left(\frac{\Lambda}{\bar{\Omega}}\right)^2+1} - \ln\left(\frac{\Lambda}{\bar{\Omega}} + \sqrt{\left(\frac{\Lambda}{\bar{\Omega}}\right)^2+1}\right)\right) \tag{4.38}$$

and

$$I_1(\bar{\Omega}) = \frac{1}{8\pi^2}\left(\frac{\Lambda}{\bar{\Omega}}\left(\left(\frac{\Lambda}{\bar{\Omega}}\right)^2+1\right)^{3/2} - \frac{\Lambda}{2\bar{\Omega}}\sqrt{\left(\frac{\Lambda}{\bar{\Omega}}\right)^2+1} - \frac{1}{2}\ln\left(\frac{\Lambda}{\bar{\Omega}} + \sqrt{\left(\frac{\Lambda}{\bar{\Omega}}\right)^2+1}\right)\right)\,. \tag{4.39}$$

We calculate the critical values of temperature and density for these phase transitions from the condition for vanishing of the effective mass $m_{\beta,\mu}$ in (4.35). For μ=0, the high temperature approximation (4.36) yield $I_0^{\beta,\mu}(\bar{\Omega}) \approx 1/(12\beta^2)$. Hence the critical temperature is

$$\frac{1}{\beta_c^2} = -6m_B^2 - 6\lambda_B I_0(\bar{\Omega})\,. \tag{4.40}$$

There is only one critical temperature and this temperature correspond to the symmetry restoration transition. However for $\mu \neq 0$, the integral $I_0^{\beta,\mu}(\bar{\Omega})$ given in (4.36) contain terms corresponding to $1/\beta^2$, $1/\beta$, β, β^2,.... Inclusion of each higher order term in $I_0^{\beta,\mu}(\bar{\Omega})$ raise the power of $1/\beta$ term and μ term in (4.35) and hence leads to the existence of multiple solutions. This will result in the possibility of temperature and density dependent multiple phase transitions.

4.2 FTDGEP in the Autonomous Version

As has been discussed in Sec. 1.3.2 the GEP allows a set of renormalization conditions in (1.116)-(1.118) leading to what is called the autonomous version. Within this scheme the FTD substitution $I_0 \rightarrow I_0^{FTD}$ gives the following modified $\bar{\Omega}$ equation:

$$\bar{\Omega}^2 = \frac{2}{3}\,\Phi_0^2 + \frac{1}{I_{-1}(\rho)}\left(m_0^2 + \frac{\bar{\Omega}^2}{24\pi^2}\left(\ln\left(\frac{\bar{\Omega}}{\rho^2}\right) - 1\right) + \frac{2}{3} I_0^{\beta,\mu}(\bar{\Omega})\right). \qquad (4.41)$$

FTDGEP can now be computed by the first derivative method of Hajj and Stevenson [101] to yield

$$\overline{V_G^{\beta,\mu}}(\Phi_0) \;=\; D + \frac{1}{2}\, m_0^2\, \Phi_0^2 + \frac{1}{144\pi^2}\,\Phi_0^4\left(\ln\left(\frac{\Phi_0^2}{3\rho^2/2}\right) - \frac{3}{2}\right) + \; I_1^{\beta,\mu}(\bar{\Omega}) \qquad (4.42)$$

evaluated at $\bar{\Omega}^2 = (2/3)\,\Phi_0^2$, where D is the FTD independent vacuum energy constant. A numerical study of (4.42) made by us demonstrates the same FTD behaviour as we have come across in the cutoff version of GEP. This is also analytically verified using the high temperature approximation (4.37) for $I_1^{\beta,\mu}(\bar{\Omega})$ in eqn.(4.42).

Chapter 5

Conclusions and Applications

The central result of this thesis has been the demonstration that in theories with a non-zero bosonic chemical potential the variation of temperature can lead to multiple phase transitions. This has been shown using the one-loop effective potential for complex ϕ^4 and abelian Higgs model and also using the gaussian effective potential method for ϕ^4-model. We expect these results would hold even in models involving non-abelian gauge fields for the reason that symmetry breaking mechanism is common to all these models. In this concluding chapter, we examine the significance of the results obtained and discuss their probable applications. We envisage two broad classes of applications. One is to the cosmological models of the early Universe and the other is to solid state phenomena, particularly to superconductivity. While we are able to make only some general observations on cosmological applications, we make somewhat more detailed statements on applications to superconductivity and make contact with some recent experimental results on high T_c superconductors.

5.1 Cosmological Applications

In cosmological models based on grand unification the Universe begins in a highly symmetric state and as it expands and cools several phase transitions occur, each one

corresponding to one stage of symmetry breaking. If the Higgs mechanism is taken literally and assume that Higgs particles may have existed with significant density in the early Universe, it becomes necessary to investigate the consequences of the finite chemical potential of such particles. Our investigations assume significance in this context and the results we have obtained would imply that for each stage of symmetry breakdown there must have occurred several phase transitions. It will be the last one in the sequence which will be mainly responsible for whatever relics we observe today. If this scenario is actually realized it can result modifications to many cosmological estimates. For example, the GUT monopole and its production rate will be affected by multiple transitions. The inflationary models will also be affected by this possibility. Whether these modifications will be significant or whether they will produce any observable signature can be decided only after detailed studies on models involving non-abelian gauge models. This will also require a better understanding of the Higgs particles.

5.2 High Temperature Superconductivity

As is well known the complex ϕ^4-model discussed in the previous chapters is a covariant version of the Ginzburg-Landau (GL) theory of phase transitions in superconductivity [2]. With the inclusion of electromagnetic fields this becomes the abelian Higgs model. An accurate phenomenological description of the familiar superconducting states is given by these theories. ϕ may be considered as an order parameter or in a microscopic point of view related to the wave functions of Cooper pairs. The low temperature with SSB is the superconducting state. As temperature is raised symmetry is restored and superconductivity disappears above the critical temperature T_c.

One of the exciting developments in physics during the last decade has been the discovery of a new class of oxide materials which have high superconducting transition temperatures (high T_c)[137-139]. Many questions relating to nature and mechanism of superconductivity in these materials remain unanswered. Many

theoretical models, phenomenological as well as microscopic, have been proposed. Whether a simple extension of the BCS theory will be sufficient is a hotly debated question. Here we describe briefly some unusual features of the superconducting state of the high T_c materials.

A superconducting phase transition is characterized by a specific heat anomaly near the transition temperature. The sharp break is somewhat smoothened by thermal fluctuations near T_c and all ordinary superconductors show a peak in their specific heat versus temperature curve. Surprisingly, in high T_c superconductors several experimenters have reported the existence of two neighbouring peaks [140-142]. The interpretation of these two peaks has no unanimity in literature. One of the difficulties in coming to a definite conclusion is the lack of accurate published data in which an adequate variety of experiments, eg., IR absorption, tunneling, critical field, specific heats, resistivity were performed on the same sample. The experimental situation upto 1990 has been reviewed by Choy et al [143] who concluded that the double peaks represent two genuine superconducting transitions. A recent experimental measurement by Seyoum et al [144] of resistivity on high T_c thallium based superconducting sample gives direct evidence for superconducting - normal - superconducting transitions. This is referred to as reentrant superconductivity.

A phenomenological explanation of the double transition has been developed by Choy et al by an extended GL theory with a two component order parameter. There have also been attempts to explain this through a BCS approach on an anisotropic layered superconductor. However the quantitative success of these models in accounting for all the features of the transition have not so far been demonstrated.

We shall now attempt to relate the double transition or reentrant transition observed in high T_c superconductors to the results reported earlier in this thesis. With the inclusion of the bosonic chemical potential we have observed the existence of multiple phase transitions. To show this more explicitly in relation to experimental transitions let us compute the specific heat in the ϕ^4 model. In the finite temperature and density gaussian effective potential method we had obtained

the free energy of the ϕ^4 model (see Sec. 4.1) from which the specific heat may be obtained

$$C_{\beta,\mu} = \frac{-\partial}{\partial(1/\beta)}\left(\frac{1}{\beta}\frac{\partial \overline{V_G^{\beta,\mu}}}{\partial(1/\beta)}\right) \tag{5.1}$$

$$= D(\Omega) - \frac{\lambda^2}{\beta}(F(\Omega))^2 + E(\Omega)\left(\frac{-\Omega^2}{2} + \lambda_B^2\left(\phi^2 + I_0(\Omega) + I_0^{\beta,\mu}(\Omega)\right) - \frac{m_B}{2}\right), \tag{5.2}$$

where

$$E(\Omega) = \int_0^\infty \frac{dK\,K^2}{2\pi^2\,\omega_k}\frac{G}{((G-1)/\beta)^3}\left(\frac{(G-1)}{\beta} - (\omega_k - \mu)(G+1)\right), \tag{5.3}$$

$$F(\Omega) = \int_0^\infty \frac{dK\,K^2}{2\pi^2\,\omega_k}\frac{\beta^2 G(\omega_k-\mu)}{(G-1)^2}, \tag{5.4}$$

$$D(\Omega) = \int_0^\infty \frac{dK\,K^2}{2\pi^2}\left(\left(\frac{\beta(\omega_k-\mu)}{G-1}\right)^2 G + \frac{\beta(\omega_k-\mu)}{G-1} - \ln(1-\frac{1}{G})\right) \tag{5.5}$$

and

$$G = \exp(\beta(\omega_k-\mu)); \quad \omega_k = \sqrt{K^2+\Omega^2}\,. \tag{5.6}$$

We have evaluated this numerically and plotted the result in Fig. 5.1. The existence of multiple peaks is clear.

As already noted with a finite bosonic chemical potential the ϕ^4 or abelian Higgs model will exhibit multiple or reentrant transitions. In the context of superconductivity these corresponds to superconducting - normal - superconducting transition. The temperature separation of these transitions, the number of such transitions and other details relating to a high T_c superconductor can be obtained only if we are somehow able to fix the relevant parameters in this model. Such studies are not attempted here.

As far as we know reentrant transitions or multiple transitions are exhibited by two classes of materials, conventional granular superconductors and high T_c superconductors. Granular superconductors are arrays of Josephson junctions and their properties depend on the tunneling mechanism to a certain extent.

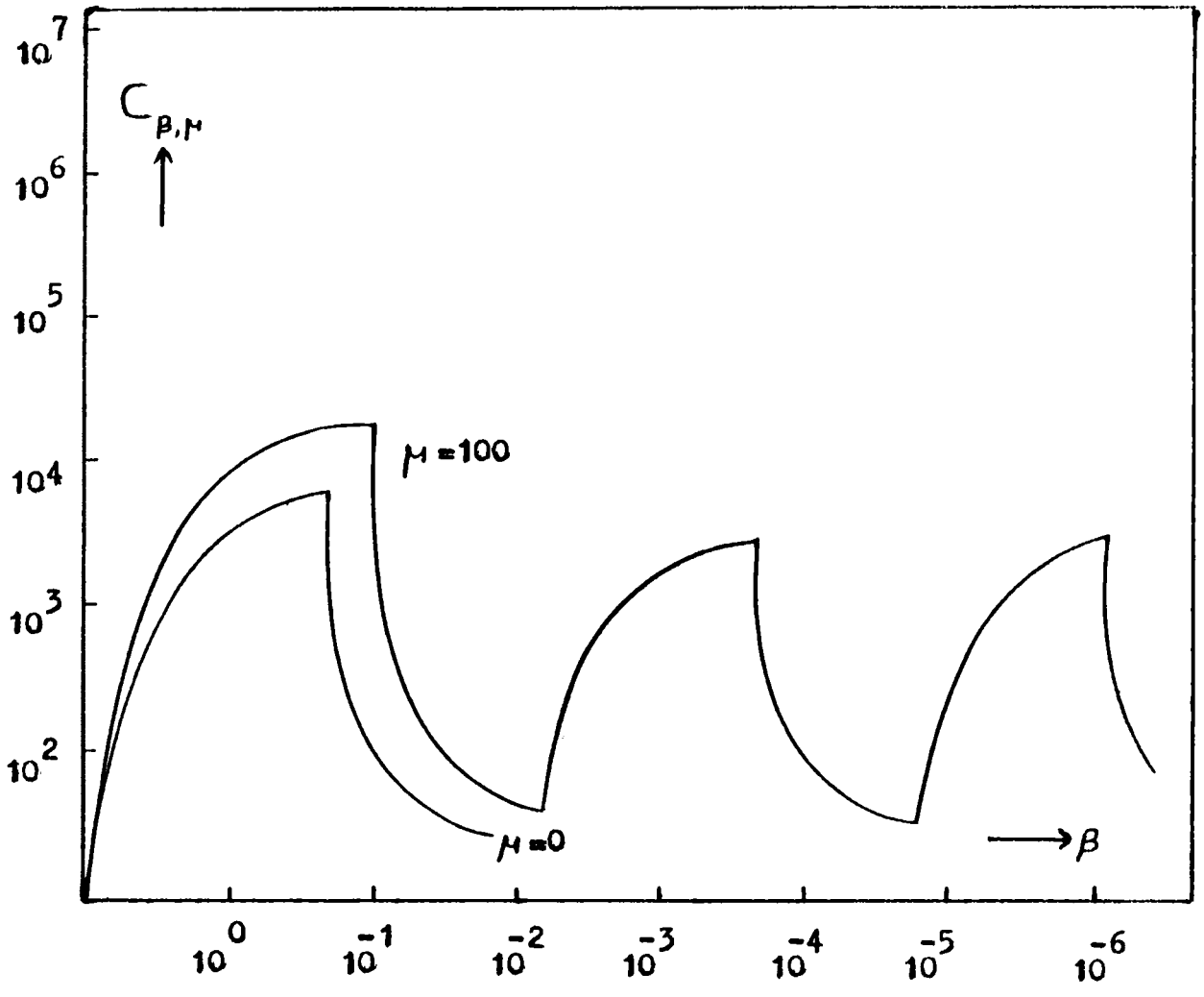


Fig. 5.1 Variation of specific heat with temperature

Our interest here is only on high T_c nongranular materials which exhibit reentrant transitions. The question arise as to why conventional superconductors do not exhibit this behaviour. In our approach, this becomes the question as to why bosonic chemical potential is relevant to high T_c superconductors but not to conventional ones. An explanation for this fact can be given in terms of the mechanism involved in the superconducting phenomenon.

The conventional superconductors are described by BCS theory. The ground state of a superconductor can be looked upon as a condensate of Cooper pairs (bosons). However, the Cooper pairs exist only below T_c and disappear at temperatures above T_c. Such bosons can not be assigned a chemical potential.

The mechanism of superconductivity in oxide superconductors is not yet clear. Various proposals have been made [145-147]. A major question is whether it is due to the usual BCS type pairing of fermions in k-space or due to real space pairs pre-existing at temperatures well above T_c. There are reasons to suppose that the latter may be the possibility. The bipolaron model of superconductivity is a realization of such a mechanism [148]. In this model the superconductivity is quite analogues to the superconductivity in He^4. In such a real space boson condensation, bosonic chemical potential plays an important role. Seen from this point, the inclusion of chemical potential in the GL approach (as we have done) can be justified if the mechanism is one of a real space condensation. We hope that further elaboration of the work reported here will be useful in the phenomenological understanding of high T_c superconductivity.

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