STUDIES ON INTEGRABILITY AND CHAOTIC BEHAVIOUR OF CERTAIN NONLINEAR SYSTEMS

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CERTIFICATE

Certified that the work reported in the present thesis is based on the bonafide work done by Mr. M. P. Joy, under my guidance in the Department of Physics, Cochin University of Science and Technology, and has not been included in any other thesis submitted previously for the award of any degree.

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DECLARATION

Certified that the work presented in this thesis is based on the original work done by me under the guidance of Dr. M. Sabir, Professor, Department of Physics, Cochin University of Science and Technology, and has not been included in any other thesis submitted previously for the award of any degree.

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PREFACE

Studies on nonlinear dynamical systems have been revived in the recent past and has led to the discovery of deterministic chaos. Availability of high speed computers and applications to wide ranging areas of study have given much importance to the subject. Deterministic nonlinear systems can exhibit a broad spectrum of behaviour varying from ordered, predictable, periodic motion on one end to completely disordered, unpredictable, random-like motion known as chaos at the other end. In contrast to chaotic systems integrable systems show regular motion or order. In this thesis we present some studies on different aspects of integrability and chaotic behaviour of nonlinear dynamical systems. Our main concern is with conservative Hamiltonian systems - finite dimensional and infinite dimensional.

In the first chapter we give a brief review of the basic facts concerning chaos and integrability in deterministic dynamical systems as a background for the remaining Chapters 2-6. After some opening remarks we define and explain integrability of dynamical systems in § 1.1. This is followed by a brief introduction to singular point analysis in the next section. Section 1.3 introduces chaos. Here the hierarchy of disorder properties shown by dynamical systems is described and a brief introduction to different characterisations of chaos is given. Chaos in quantum systems is reviewed in § 1.4 . Section 1.5 contains an introduction to Yang-Mills theories and monopoles. A brief review is given of the studies on integrability and chaotic behaviour of such field theoretic models and their importance highlighted.

There are no general tests for determining whether a system is integrable or not. However, there are some techniques of considerable practical utility for identifying integrable cases. Among these are Painlevé analysis due to Ablowitz, Ramani, and Segur (ARS) and its generalisations. Analysis of Kowalevskaya exponents (KE) and stability of straight line periodic solutions are also very useful in this regard. In Chapter 2 we describe these techniques. An account is then given of our work where we have combined Painlevé analysis, Yoshida's methods of calculating KE and stability analysis to show that, for two dimensional homogeneous potentials of degree 2m, integrability restricts KE and integrability coefficients to discrete sets of values. This has been made use of in the analysis of integrability of symmetric potentials with m=2, 3 and 4. Second integrals for the integrable cases identified are also constructed directly. We have also generalised the integrable potentials of arbitrary degree 2m by constructing the corresponding second integral.

Painlevé analysis have been generalised by Weiss, Tabor and Carnevale (WTC) to study partial differential equations (PDE). In Chapter 3 we apply this method to study the integrability of some field theoretic models. Question of integrability of non-Abelian gauge fields or Yang-Mills fields has attracted wide attention and has much importance in Particle Physics and Field theory. We have studied the integrability of spherically symmetric time dependent non-selfdual sector of SU(2) Yang-Mills (SSYM) and Yang-Mills-Higgs (SSYMH) theory using WTC method. They are shown to be nonintegrable. Various reductions of these systems to ODEs are also investigated and shown to be nonintegrable.

Chaos is characterised by the exponential divergence of nearby trajectories. Calculating Lyapunov exponents is a convenient way to characterise chaos quantitatively. If the maximal LE is positive the system is said to be chaotic. What causes chaos is a very complicated question. Our understanding of the origin of chaos is still rudimentary. If the Riemannian curvature of the manifold on which Hamiltonian flow can be considered as a geodesic flow is negative everywhere the system can be proved to be chaotic. Converse is, however, not true. For Riemannian curvature calculation we do not consider the potential boundary. There are systems which have positive curvature everywhere but are chaotic. In Chapter 4 we have studied such a two dimensional quartic oscillator system which goes from an integrable case to a highly chaotic one as a parameter changes from 0 to 1. We calculated the maximal LE and negative curvature of the potential boundary at different parameter values. We establish a direct correlation between the curvature and chaos of the system.

Yang-Mills classical mechanics is highly chaotic. Space time dependent Yang-Mills system also appears to be chaotic. Chaotic behaviour of Yang-Mills theories is relevant Why? to the problems of quark confinement, monopole stability, etc. in field theory. In Chapter 5 we have investigated the chaotic behaviour of spherically symmetric time dependent SU(2) Yang-Mills-Higgs (SSYMH) system in detail. We have studied the dynamics near the 't Hooft-Polyakov monopole solution. For our study we calculated the maximal LE of the system obtained by discretising the original PDE. We found that there is a transition from order to chaos as a parameter which depends on the self interaction constant of the scalar Coupling constant field increases. Presence of Higgs field reduces chaos of the original YM fields. We have shown the existence of space-time chaos in YMH system and the exponential instability of 't ? Hooft-Polyakov monopoles.

How classical chaos manifests in Quantum Mechanics is a controversial and difficult question. Different characterisations of quantum chaos have been proposed. There is a widespread belief that Quantum Mechanics suppresses chaos. Most of the studies concerning these questions are semiclassical in nature. In Chapter 6 we investigate the quantum chaos of the quartic Hamiltonian system studied in

Chapter 4 by the recently introduced method of Gaussian effective potential (GEP). GEP is an approximate potential describing the effect of quantum fluctuations on a classical system. It is not a semiclassical quantity. We have calculated GEP for different parameter values and for various values of Planck's constant $\hbar$. GEP becomes a regular potential as we increase the value of $\hbar$. But as the classical chaos of the system increases the value of $\hbar$ at which GEP becomes completely regular also increases showing the existence of signatures of chaos even with quantum fluctuations.

The results presented in the thesis have appeared in the form of following papers.

i. Joy M.P. and M.Sabir, Integrability of two dimensional homogeneous potentials, J. Phys. A : Math. Gen., 21 (1988) 2291-2299.

ii. Joy M.P. and M.Sabir, Non-integrability of SU(2) Yang-Mills and Yang-Mills-Higgs systems, J. Phys. A : Math. Gen., 22 (1989) 5153-5159.

iii. Joy M.P. and M.Sabir, Transition from order to chaos in Yang-Mills-Higgs system, Pramana - J. Phys., 39 (1992) L91-L94.

iv. Joy M.P. and M.Sabir, Chaotic behaviour and order to chaos transition of 't Hooft-Polyakov monopoles, J. Phys. A : Math. Gen., 25 (1992) (In Press).

v. Joy M.P. and M.Sabir, Chaos and curvature in a quartic Hamiltonian system, (Communicated to Pramana - J. Phys., 1992).

vi. Joy M.P. and M. Sabir, Chaos and quantum fluctuations in a quartic Hamiltonian system, (Communicated to Phys. Rev. A, 1992).

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CONTENTS

CHAPTER 1

INTRODUCTION

The last decade has witnessed a resurgence of interest in nonlinear dynamics. This is mainly attributable to the realization that the nonlinear dynamical systems can exhibit a variety of behaviour- from ordered, predictable, regular motion to completely disordered, unpredictable, irregular and stochastic motion commonly known as chaos (Lichtenberg and Lieberman 1983, Schuster 1984, Hao Bai Lin 1984, Berge et al 1986, Steeb and Louw 1986b, Gleick 1987, Tabor 1989). Generally these different types of behaviour occur in a system as some parameter of the system is varied. The basic characteristic of chaotic motion is a sensitive dependence on initial conditions. The resulting unpredictability of future behaviour in completely deterministic nonlinear systems has attracted much attention. Even an apparently simple nonlinear system of three differential equations can show such unexpected ways of behaviour. Seminal studies by Lorenz in 1963 of a truncation of hydrodynamical partial differential equations (Lorenz 1963) and by Henon and Heiles in 1964 of a model for the motion of a star under an axially symmetric galactic potential (Henon and Heiles 1964) were the harbingers of a new era. These models have now become the paradigms in the study of nonlinear dynamics. But the beginning of the subject can be traced to the end of the 19^{th} century. Poincaré was certainly aware of the possibility of such intricate motion in dynamical systems (Poincaré 1892). However significant contributions in this field have been made only in comparatively recent times. Among the various reasons for this has been the emergence of high speed computers. The dramatic developments in our understanding of nonlinear dynamics has touched almost all areas of science - Physics, Chemistry, Meteorology, Biology, Physiology, Ecology, Economics and Sociology.

Not all nonlinear systems exhibit chaotic behaviour. Integrable systems have regular phase space motion. How to identify integrable systems and how and when chaos arise in a nonintegrable system are questions yet to be answered in general. This thesis is concerned with some aspects of integrability and chaotic behaviour of deterministic nonlinear dynamical systems. We are mostly concerned with conservative, Hamiltonian systems. We study the integrability and chaotic behaviour of some field theoretic models also. These are related to the Yang-Mills theory and are of relevance in particle physics. The implications of chaos in Quantum Mechanics is another fundamental question with far from unambiguous answers. We touch upon some aspects of this question by analysing a simple model.

As a background for the discussion of our work in Chapters 2-6 we present an overview of various aspects of integrability, singular point analysis, chaotic behaviour and quantum chaos in the remaining sections of this chapter. An introduction to Yang-Mills theory is also given in the last section.

1.1. Dynamical systems and Integrability

Deterministic dynamical systems may be described as systems of variables which evolve with an independent variable such as time t and possibly with space variables x,y,z, according to definite rules. When there is only one independent variable we have a finite dimensional system modeled by ordinary differential equations (ODE). In the case of more than one independent variable a system is said to be infinite dimensional and is described by partial differential equations (PDE). We can define a finite dimensional dynamical system (DS) as a set of n-first order differential equations in n-variables, $\mathbf{x} = (x_1, x_2, \ldots\ldots, x_n)$ (Ozorio de Almeida 1988):

$$\frac{dx_i}{dt} \equiv \dot{x}_i = F_i(\mathbf{x},t), \quad i = 1,2,\ldots.n, \tag{1.1}$$

$\mathbf{x}$ denotes a point in the n-dimensional phase space and system (1.1) is said to be a DS of n-dimensions. A solution of the DS is a vector function $\mathbf{x}(\mathbf{x}_0,t)$ satisfying (1.1) and the initial condition

$$\mathbf{x}(\mathbf{x}_0,0) = \mathbf{x}_0 \tag{1.2}$$

If the vector field $\mathbf{F}$ is independent of time t we call the system autonomous. By redefining time t as a new variable we can consider an n-dimensional non-autonomous system as an (n+1)-dimensional autonomous system. In nonlinear systems $\mathbf{F}$ will be nonlinear functions of $\mathbf{x}$. DS can be described by higher order differential equations also. But in general we can transform an n^{th} order differential equation in to n-first order differential equations. If the independent variables are discrete and not continuous we have discrete dynamical systems described by difference equations. In the present work our main concern is with autonomous systems with continuous independent variables.

Dynamical systems may be classified mainly into two classes : conservative systems and dissipative systems. As the time evolves in dissipative systems, the phase space volume contracts whereas in conservative systems phase space volume remains constant (Lichtenberg and Lieberman 1983). Dynamical behaviour of the two types of systems are entirely different. We may define the divergence of a system as

$$\frac{dV}{dt} = \int_V d^n x \left(\sum_{i=1}^{n} \frac{\partial F_i}{\partial x} \right) . \tag{1.3}$$

If $dV/dt < 0$, the system is said to be dissipative. For conservative systems $dV/dt = 0$.

We shall concentrate our attention mainly on conservative (Hamiltonian) systems. Let us consider a system described by Hamilton's equations of motion (Goldstein 1980).

$$\dot{q}_i = \partial H/\partial p_i, \qquad \dot{p}_i = - \partial H/\partial q_i \ , \ i = 1,\ldots N \tag{1.4}$$

H is the Hamiltonian which is a function of generalized coordinates q_i and corresponding conjugate momenta p_i. The system is said to be a Hamiltonian system of N-degrees of freedom and the phase space is 2N-dimensional. Volume preservation is clear from Liouville's theorem :

$$\text{div } (\dot{q},\dot{p}) = \sum_i \left(\frac{\partial^2 H}{\partial q_i \partial p_i} - \frac{\partial^2 H}{\partial p_i \partial q_i}\right) = 0 \tag{1.5}$$

A Hamiltonian system of N-degrees of freedom is said to be integrable if there exist N time independent, analytic, single valued integrals of motion I_i, which are in involution. That is their Poisson brackets vanish.

$$[\ I_i, I_j] = 0,\ [\ H, I_i] = 0,\ i,j = 1,2,\ldots,N \qquad (1.6)$$

Integrals of motion are also known as constants of motion, first integrals, etc. If the Hamiltonian H is independent of time, it is a constant of motion. By Liouville-Arnold theorem an integrable system executes linear motion on an N-dimensional torus defined by these first integrals and their solutions can be obtained by quadratures (Abraham and Marsden 1978, Arnold 1978, Kozlov 1983). Completely integrable systems are exceptional. That is generic systems are nonintegrable (Moser 1973). Phase space motion will be regular and predictable in the case of integrable systems. In general we may be able to find action-angle variables J, θ and hence solve the Hamilton-Jacobi equation, for such systems. Action-angle variables J, θ are defined in such a way that, by a canonical transformation from the original variables, the transformed Hamiltonian H_0 depends only on the momenta J, the action variable. Now the equations of motion become

$$\dot{J} = \frac{-\partial H_0}{\partial \theta} = 0\ , \qquad \dot{\theta} = \frac{\partial H_0}{\partial J} = \omega(J) \qquad (1.7)$$

and can be easily integrated to

$$J = \text{constant}, \qquad \theta = \omega\ t + \delta\ . \qquad (1.8)$$

All one degree of freedom autonomous Hamiltonian systems are integrable. For a two degrees of freedom autonomous system to be integrable there must exist one more integral of motion. If there exist $p < N$ constants of motion one can reduce the effective degrees of freedom from N to N-p. The p constants of motion enter as parameters in the reduced system.

In the case of non-Hamiltonian systems which are not derivable from a Hamiltonian the notion of integrability is not as well defined as that in Hamiltonian systems. A working definition can be given like this (Yoshida 1983, Puri 1990). A general dynamical system of n-dimensions may be said to be completely integrable if it can be reduced to final quadrature by the existence of n-1 time independent integrals of motion or it can be transformed by a change of variables into a set of linear ordinary differential equations (with variable coefficients). Here also completely integrable systems show regular behaviour. Integrals of motion can also be time dependent (Kus 1983, Steeb and Louw 1986b). In such a case n-dimensional system is integrable if there exist n time dependent constants of motion. In general a dissipative system can be put into a Hamiltonian system by doubling the number of coordinates (Steeb et al 1985a).

In the case of infinite dimensional systems, the concept of integrability is not as clear as that of finite dimensional systems. Integrability in such systems are related to the existence of soliton solutions or solvability by inverse scattering transform (IST) or to the existence of infinite number of conservation laws, or possibility of transforming (by a change of variables) into a system of linear partial differential equations (Puri 1990).

There are no general methods to identify integrable systems. One can not tell a priori for a given system of N degrees of freedom whether N integrals of motion would exist or not. Moreover no general technique is available for finding all the existing integrals of motion, or even for finding their total number unless some obvious symmetries are present. Sometimes numerical experiments suggest their presence and can in turn help us to construct these.

One can in principle directly search for integrals of motion by making use of the fact that they must be in involution. It is a very difficult task in the case of systems with more than two degrees of freedom (DOF). Even in the 2 DOF case the method is not exhaustive. Hietarinta (1987) has identified various classes of integrable 2 DOF systems. In this approach a particular functional form for the constants of motion are assumed and from the condition of vanishing Poisson brackets certain restrictions on the coefficients are obtained. The technique was first applied by Whittaker (1927) to a Hamiltonian of the form

$$H = (p_1^2 + p_2^2)/2 + V(q_1,q_2). \qquad (1.9)$$

He analysed a class of potentials V for which there exist an integral of motion up to quadratic order in the momenta p.

$$I(p,q) = a\, p_1^2 + b\, p_2^2 + c\, p_1 p_2 + e\, p_1 + f\, p_2 + g \qquad (1.10)$$

Here the coefficients are functions of q_i. From the requirement $[I,H] = 0$, one obtains a set of partial differential equations for the coefficients in terms of the potential V and its derivatives. Solution of these yields integrable potentials V and the associated first integrals I. The method can be extended to integrals which are higher order in p_i (Hall 1983, Holt 1982, Sen 1985,1987). In recent times computer programs for doing this have been developed (Schwarz 1985,1988).

Integrability can also be related to the existence of nontrivial symmetries or Lie symmetries (Bluman and Cole 1974, Lutzky 1979, Sahadevan and Lakshmanan 1986). Here one finds the infinitesimal symmetries of one parameter continuous transformations leaving the equations of motion invariant. By applying Noether's theorem one can construct constants of motion from the infinitesimals. Suppose the equations of motion,

$$\ddot{x} = \alpha_1(x,y) , \quad \ddot{y} = \alpha_2(x,y) \qquad (1.11)$$

are invariant under one parameter (ε) continuous transformations.

$$x \longrightarrow X = x + \varepsilon\, \eta_1(t,x,y,\dot{x},\dot{y}) + O(\varepsilon^2)$$
$$y \longrightarrow Y = y + \varepsilon\, \eta_2(t,x,y,\dot{x},\dot{y}) + O(\varepsilon^2)$$
$$t \longrightarrow T = t + \varepsilon\, \xi\, (t,x,y,\dot{x},\dot{y}) + O(\varepsilon^2)$$
$$\varepsilon << 1 \qquad (1.12)$$

where η_1, η_2 and ξ are infinitesimals. Then the invariant equations are

$$\ddot{\eta}_1 - \dot{x}\,\ddot{\xi} - 2\,\alpha_1\dot{\xi} = E(\alpha_1)$$
$$\ddot{\eta}_2 - \dot{y}\,\ddot{\xi} - 2\,\alpha_2\dot{\xi} = E(\alpha_2) \qquad (1.13)$$

Here E is the infinitesimal operator given by,

$$E = \xi\ \partial/\partial t + \eta_1\partial/\partial x + \eta_2\partial/\partial y + [\ \dot{\eta}_1 - \dot{x}\dot{\xi}\,]\partial/\partial \dot{x} + [\dot{\eta}_2 - \ \dot{y}\dot{\xi}\,]\partial/\partial \dot{y} \qquad (1.14)$$

From these equations one can find η_1, η_2 and ξ explicitly. Using these infinitesimals one can find the associated integrals of motion, if they exist.

Another technique is the Lax-pair approach where one searches for two matrices L and M so that $dL/dt = [L,M]$ is equivalent to the original Hamilton's equations of motion. If that is possible then the coefficients of g^n in the expansion of det $(L-g)$ are invariants and in involution. This method has also been successfully applied to identify some new integrable cases (Olshanetsky and Perelomov 1981).

A widely used and generally satisfactory method for establishing the integrability of finite as well as infinite dimensional systems is singular point analysis. More details on this technique are given in the next section and in Chapters 2 and 3.

1.2. Singular Point Analysis

Study of the analytic structure of dynamical systems reveal several details concerning its behaviour such as integrability and chaotic behaviour. Singular point analysis is the most widely used method for identifying integrable cases. The method relies on the conjecture that systems having the Painlevé property (PP) are integrable. A system is said to have the Painlevé property when the only movable singularities of its solution in the complex time plane are simple poles (Hille 1976, Davis 1962).

The recent revival of interest in the singularity structure aspects and integrability is mainly attributable to the works of Ablowitz, Ramani and Segur (1978,1980). But the idea has a long history and can be traced back to Sonya Kowalevskaya who formulated the idea and applied it to identify integrable cases of rigid body motion in 1889 (Cooke

1984, Tabor 1984).

Differential equations can have two types of singularities : fixed and movable (Ince 1956). Fixed ones are determined by the equation itself while the location of the movable singularities depend upon the initial conditions. Linear equations can have only fixed singularities. Nonlinear equations can have both movable and fixed singularities.

Painlevé (Ince 1956) investigated all first order ODEs, $dw/dz = f(z,w)$, with f rational in w and analytic in z whose only movable singularities are poles. Fuchs (1884) examined the question further. It was proved that the Riccati equation $dw/dz= f_0(z)+f_1(z)\, w + f_2(z)\, w^2$, is the only first order ODE which is free from movable critical points.

Sonya Kowalevskaya (1889,1890,1978) was the first to apply singularity analysis to a physical problem. Fuchs' works and the works of Jacobi on elliptical functions which are meromorphic functions motivated Kowalevskaya to study the integrability of a heavy rigid body rotating under the influence of gravity in connection with the singularity structure properties shown by the solutions in the complex time plane. She considered the equations of motion,

$$\begin{aligned} A\frac{dp}{dt} &= (B - C)\, q\, r - \beta\, z_0 + \gamma\, y_0 \\ B\frac{dq}{dt} &= (C - A)\, p\, r - \gamma\, x_0 + \alpha\, z_0 \\ C\frac{dr}{dt} &= (A - B)\, p\, q - \alpha\, y_0 + \beta\, x_0 \end{aligned} \qquad (1.15a)$$

$$\begin{aligned} \frac{d\alpha}{dt} &= \beta\, r - \gamma\, q \\ \frac{d\beta}{dt} &= \gamma\, p - \alpha\, r \\ \frac{d\gamma}{dt} &= \alpha\, q - \beta\, p \quad , \end{aligned} \qquad (1.15b)$$

where (p,q,r) and (α,β,γ) are the components of angular velocity and direction cosines respectively of the spinning top. (A,B,C) and (x_0,y_0,z_0) are constant parameters related to the components of moments of inertia and the position coordinates of the centre of gravity respectively. For the complete integrability of this system 4 integrals of motion are needed. The system has 3 first integrals

$$\begin{aligned} h_1 &= \frac{1}{2} (A p^2 + B q^2 + C r^2) + \alpha x_0 \\ h_2 &= A \alpha p + B \beta q + C \gamma r \qquad (1.16) \\ h_3 &= \alpha^2 + \beta^2 + \gamma^2 \end{aligned}$$

3 integrable special cases were known at the time of Kowalevskaya.

(i) $A = B = C$: the trivial case of complete kinetic symmetry.

$$h_4 = x_0 p + y_0 q + z_0 r \qquad (1.17)$$

(ii) $x_0 = y_0 = z_0$: Euler case

$$h_4 = A^2 p^2 + B^2 q^2 + C^2 r^2 \qquad (1.18)$$

and (iii) $A = B,\ x_0 = y_0 = 0$: Lagrange case

$$h_4 = r \qquad (1.19)$$

In the cases of Euler and Lagrange the solutions are expressed in terms of elliptic functions which are meromorphic. They do not have singular points other than poles in the finite complex time plane. Motivated with this she searched for parametric choices having this property and found that there are only four special cases satisfying this. 3 are the already known cases (i), (ii) and (iii) and for the fourth case

$$A = B = 2C, \quad z_0 = 0,$$

she found an additional integral also and hence proved the integrability. For the Kowalevskaya case

$$h_4 = (p^2 - q^2 - \alpha x_0)^2 + (2pq - \beta x_0)^2 \quad (1.20)$$

She explicitly integrated this special choice and obtained the general solution in terms of hyper-elliptic functions. No other integrable cases are known for the system till now (Golubev 1953).

Investigations by Painlevé and coworkers is a remarkable work in the study of singularity structure analysis in which they classified all second order ODEs of the form (Painlevé 1900,1902, Fuchs 1906, Gambier 1909),

$$\frac{d^2w}{dz^2} = F(z,w,w')$$

with F rational in w', algebraic in w and analytic in z, whose critical points are fixed. They identified 50 types with this property. Out of these 44 are integrable in terms of known functions including elliptic functions, by quadratures or by linearisation. The remaining six equations are now known as Painlevé transcendents and have transcendental meromorphic solutions. Classification of higher order systems with Painlevé property have been attempted but is not yet complete (Garnier 1912, Bureau 1964,1972).

Ablowitz, Ramani and Segur (ARS) observed that all similarity reductions of integrable PDEs are of Painlevé type. This observation prompted them to formulate a conjecture : every ODE obtained by an exact reduction of a PDE solvable by IST possesses the PP (Ablowitz and Segur 1977,1981). They also put forward an algorithm for testing whether a system of ODEs satisfy the necessary criteria for possessing the PP or not (Ablowitz et al 1980). Using this one can check whether the solution of the system in complex time plane can be expanded in terms of a Laurent series around a movable pole, with sufficient number of arbitrary coefficients. That is one looks for solutions of the form

$$w_i = (z-z_0)^{p_i} \sum_{m=0}^{\infty} a_i^{(m)} (z-z_0)^m \tag{1.21}$$

for the system

$$\frac{dw_i}{dz} = F_i(w_1,....,w_n,z), \; i = 1,....,n \tag{1.22}$$

The algorithm consists of three steps,

(1) the study of dominant or leading order behaviour,

(2) the determination of resonance values at which arbitrary constants enter in the Laurent expansion

and (3) checking whether sufficient number of arbitrary constants enter in the expansion.

The algorithm can also be applied to test the PP of systems not written in the above form of first order equations but formulated as systems with higher derivatives. The algorithm gives a necessary condition for the absence of movable branch points either algebraic or logarithmic, but occurrence of movable essential singularities can not be detected.

Segur (1980) revived the Kowalevskaya's approach of exploring the integrability of finite dimensional systems by investigating the integrable cases of the Lorenz equation through an application of the PP. Since then it has been used to identify integrable cases of several systems (Bountis et al 1982,1983,1984, Grammaticos et al 1982, Tabor and Weiss

1981). In the original algorithm due to Ablowitz, Ramani and Segur only integer leading orders can arise so that the movable branch points are excluded. This is now known as the strong PP. Ramani, Dorizzi and Grammaticos (1982) suggested the so called weak PP so that PP is generalized to include finite branching. Here the leading order can be a rational number. Many integrable systems possess only weak PP. In the generalised form PP has been widely used as a criterion for integrability (Dorizzi et al 1983,1984,1986, Grammaticos et al 1983,1984,1985, Hietarinta 1983, Lakshmanan and Sahadevan 1984,1985, Menyuk et al 1983, Ramani et al 1984,1985, etc. For more references see Sahadevan 1986, Steeb and Euler 1988, Ramani et al 1989). But there is no general proof connecting the PP to integrability. Adler and van Moerbeke (1982a,b,1988) proved for a class of Hamiltonian systems that PP is a necessary condition for algebraic integrability in terms of Abelian functions. No Hamiltonian systems having PP are found to be algebraically nonintegrable. Only a few rigorous results are available (See Ercolani and Siggia 1986,1989, Flaschka 1988). Though PP helps us to identify integrable cases integrability can be proved only by finding sufficient number of integrals of motion. Analytic structure of the solutions and the chaotic behaviour of the system also appear to be related (Chang et al 1981,1982,1983, Bountis et al 1987,1991, Bessis and Chafee 1986, Frish 1984, Thual and Frish 1985, Dombre et al 1986, Fournier et al 1988, Levine and Tabor 1988, Tabor 1989).

Yoshida (1983) introduced a method of finding

Kowalevskaya exponents (KE) and proved the following theorem.

A necessary condition for a similarity invariant system to be algebraically integrable is that all Kowalevskaya exponents be rational numbers.

In other words irrational or imaginary KE imply nonintegrability. KEs and Painlevé resonances are related to each other . More details on this will be discussed in Chapter 2.

Ziglin's work on nonintegrability by studying the variational equations contain rigorous results in this field (Ziglin 1983). Yoshida (1986,1987a,1989) and Ito (1985,1987) developed the theory further producing rigorous results and applying them to many dynamical systems (Yoshida 1987b,c,1988, Yoshida et al 1987a,b,1988, Grammaticos et al 1987). The approaches of Painlevé , Yoshida and Ziglin have some mutual connections and the details of whioh are described in Chapter 2.

Though ARS put forward the algorithm in connection with the integrability of PDEs, it cannot be applied directly for testing integrability of PDEs. One should reduce the PDE before applying the algorithm. Definite conclusions are also generally not possible. Weiss, Tabor and Carnevale (1983) generalized the P-test by introducing a notion of PP for PDEs. According to them a PDE possesses the PP if its solutions are single valued about a movable singularity manifold. Ward (1984) has given more precise definition and pointed out that the singularity manifold must not be a characteristic. We seek solutions of a PDE in the form

(1.23)

about the singularity manifold

$$\phi\ (z_1,\ldots .z_n) = 0\ . \tag{1.24}$$

The algorithm essentially goes like ARS algorithm, but the calculations are more complicated. Kruskal proposed a simplification where one take the singularity manifold in the form $\phi(z_1,z_2) = z_1 + \psi(z_2)$ with ψ an arbitrary analytic function and the u_j is a function of z_2 only (when there are only two independent variables z_1 and z_2). In many situations for PDE also we have to introduce the WPP concept. Several systems have been studied using this method (Weiss 1983,1984,1985,1986,1987, Chudnovsky et al 1983, Sahadevan et al 1986, Hlavaty 1985, Steeb and Euler 1987a,b,1990, Steeb and Louw 1985,1986a,1987, Steeb et al 1984,1986c, Webb 1990) and computer programs have also been developed for doing the P-test (Hlavaty 1986, Hereman and Van den Bulck 1988, Rand and Winternitz 1986).

The main advantage of WTC method is that the singular expansion obtained can be used for deriving Bäcklund transformations, Lax Pairs, etc. Further extensions lead to special solutions of nonintegrable equations also (Newell et al 1987, Gibbon et al 1985, Tabor and Gibbon 1986, Conte 1988, Conte and Mussette 1989, Carriello and Tabor 1989, 1991). Different aspects of this and an application of WTC method of P-test to test the integrability of time dependent spherically symmetric SU(2) Yang-Mills and Yang-Mills-Higgs systems are given in Chapter 3.

1.3. Chaotic behaviour

As mentioned in the previous sections integrable systems show regular behaviour. But they are rather exceptional. Most dynamical systems are nonintegrable and many of them exhibit chaotic behaviour. How and under what conditions chaos arises is the important problem yet to be understood fully. Hamiltonian systems, in general, have divided phase space ; regular and irregular phase space regions coexist. Before going into the details of such systems we take a look at the hierarchy of disorder properties a classical system can exhibit.

In ergodic theory dynamical systems are classified according to the degree of disorder shown by them (Arnold and Avez 1968, Lebowitz and Penrose 1970, Ford 1973). In increasing order of stochastic properties they are classified as recurrent systems, ergodic systems, systems with mixing, systems with n-fold mixing, Kolmogorov systems, C-systems and B-systems.

In a recurrent system trajectory returns to a given neighbourhood of a point an infinite number of times as time evolves. According to Poincaré's recurrence theorem any Hamiltonian system which maps a finite region of phase space onto itself is of this type. This result, however does not have much practical significance in view of the fact that the return time is generally of enormous magnitude. A system is generally characterised as ergodic if any trajectory fills its energy surface. In ergodic systems time averages can be replaced by phase space averages. Ergodic hypothesis is the foundation of classical statistical mechanics where the assumption is that ensemble and time averages are equal. Once it was thought that some nonintegrable perturbations would give rise to ergodicity. But it turns out that this is not true in general.

Let $z(t) = (p(t),q(t))$ represent a trajectory of a particle in the phase space. The time evolution of $z(t)$ can be written using the time shift operator $\hat{T}$ as

$$z(t+T) = \hat{T}z(t) \tag{1.25}$$

Then the time evolution of an arbitrary function of z is given by

$$f(z(t+T)) = \hat{S}_T f(z(t)) = f[\hat{T}z(t)], \tag{1.26}$$

where $\hat{S}_T$ is a time shift operator along the orbit.

A dynamical system is said to be ergodic if for every integrable function f,

$$\lim_{T\to\infty} \frac{1}{T}\int_t^{t+T} dt'\ f[z(t')] = \langle f\rangle = \frac{1}{\Omega}\int_\Omega d\Gamma(z)f(z) \tag{1.27}$$

where $\Gamma(z)$ is the invariant measure in the accessible phase space of volume Ω. Equation (1.27) means that time average is equal to the phase space average. In an ergodic system time average of any function is independent of the initial point.

A dynamical system is mixing if and only if, for every integrable functions f and g,

$$\lim_{T\to\infty} \langle\ \hat{S}_T f.g\ \rangle = \langle f\rangle.\langle g\rangle \tag{1.28}$$

This implies that $\hat{S}_T$ has a continuous spectrum. For any mixing system the motion of a particle in the phase space has infinite cycles and comes to be independent of the initial point as time passes. An example is given by Arnold and Avez (1968). A mixture of 20% rum and 80% cola is stirred a large number of times n ($n \longrightarrow \infty$). Then every part of it will contain 20% rum and 80% cola. Implication is that it is mixed.

One can easily show that mixing implies ergodicity. But the converse is not true; ergodicity does not imply mixing. Baker's transformation is an example for mixing.

A dynamical system is said to be n-fold mixing if and only if for every integrable $f_1,\ldots\ldots,f_n$,

$$\lim_{T_1,\ldots T_n\to\infty} \langle \hat{S}_{\tau_1} f_1 \ldots\ldots \hat{S}_{\tau_n} f_n \rangle = \langle f_1 \rangle \ldots\ldots \langle f_n \rangle \qquad (1.29)$$

Next in the hierarchy is the so called Kolmogorov system or a K-system which has a positive Kolmogorov-Sinai (KS) entropy. The KS entropy is defined as follows. Let us divide the phase space into a set $\{A_j(0)\}$ of small cells of finite measure at t=0. The backward evolution of the system by a unit time step transforms this set to $\{A_j(-1)\}$. The intersection $B(-1) \equiv \{ A_i(0) \cap A_j(-1)\}$ of this new set will typically have smaller measure than $\{A_j(0)\}$. Continuing the backward evolution, we can generate the elements of the set,

$$B(-2) \equiv \{A_i(0) \cap A_j(-1) \cap A_k(-2)\}, \text{ etc.}$$

We say that the system has positive KS-entropy if the average measure of each element of B decreases exponentially as $t \longrightarrow \infty$. Then the average exponential rate is defined as

$$h\{A_j(0)\} = - \lim_{t\to\infty} \frac{1}{t} \sum_i \mu\, [B_i(-t)] \ \log \mu\, [B_i(-t)] \ >0 \qquad (1.30)$$

where μ denotes measure. KS entropy is defined as the maximum of h over all initial measurable partitions of phase space.

In regions of connected chaos KS-entropy is the sum of all positive Lyapunov exponents (Pesin 1977). Consequently K-systems have very sensitive dependence on initial conditions. Kolmogorov and Sinai proved that K-systems are mixing (Lichtenberg and Lieberman 1983).

C-systems or Anosov systems are more stochastic than K-systems in the sense that C-systems are K-systems but not vice versa. In a C-system part of the tangent space is associated with exponential divergence and a part disjoint from that is associated with exponential convergence of trajectories. Arnold's cat map is an example of a C-system.

Bernoulli systems or B-systems are at the top of the hierarchy. They show behaviour indistinguishable from randomness as in the Bernoulli shift map.

Natural systems do not strictly belong to any of these classes. Usually the phase space is divided into intermingled regions of chaotic and orderly behaviour. As a first step towards the study of such systems let us consider a near integrable system ie. an integrable system H_0 perturbed with a nonintegrable part H_1.

$$H = H_0(J) + \varepsilon H_1(\theta, J) \tag{1.31}$$

where ε is a parameter characterising the strength of perturbation. J, θ are the action-angle variables of the integrable part. When $\varepsilon = 0$ system is integrable and the motion occurs on an N-torus. When the perturbation is small what happens to the invariant tori of the integrable Hamiltonian H_0, is addressed by the famous Kolmogorov-Arnold-Moser (KAM) theorem (Lichtenberg and Lieberman 1983, Berry 1978).

Suppose the perturbation is sufficiently smooth and small. Suppose also that the frequencies $\nu_i(J) = \frac{\partial H_0}{\partial J_i}$, associated with the unperturbed Hamiltonian are linearly independent or incommensurate. That is, for any set $(n_1, n_2, \ldots . n_N)$ of integers that are not all zero,

$$\sum_{j=1}^{N} n_j \nu_j(J) \neq 0 \tag{1.32}$$

KAM theorem says that, under these conditions, most of the N-tori (KAM-tori) of the unperturbed system are not destroyed, only distorted slightly. The theorem does not apply for tori with commensurate frequencies. A large enough perturbation ϵH_1 destroys all tori.

When the frequencies are commensurate the original tori decompose into smaller and smaller tori. Some of these new tori will again become stable according to KAM theorem. But between the stable tori the motion is completely irregular. According to Poincaré-Birkhoff theorem (Birkhoff 1927) the original torus with rational frequency ratio is not completely destroyed under a perturbation, but there remains an even number of fixed points; an alternating sequence of elliptic and hyperbolic fixed points. One can thus see that in conservative systems regular and irregular motion are densely interweaved. The destruction of KAM tori and the onset of stochasticity can be explained by the resonance overlap criterion proposed by Chirikov (1979).

In the case of two degrees of freedom regular regions and chaotic regions are separate, because a two dimensional KAM surface can divide the 3-dimensional energy surface into distinct regions. But when $N > 2$, this is not possible and hence irregular trajectories wander about between the preserved tori with incommensurate frequencies. They are not confined to separate regions of stochasticity as in the case of $N = 2$. This results in what is known as Arnold diffusion. Irregular trajectories form an interconnected `Arnold web'. The Arnold web can be arbitrarily close to any point of the energy surface and it can exist even as the perturbation strength approaches zero. But in many cases the diffusion rate may be extremely small (Nekhroshev 1977). In such cases we can consider the system as one with different stochastic regions and neglect the Arnold diffusion.

An important aspect of the study of dynamical systems is characterisation of chaos. The techniques of Poincaré surface of section and calculation of Lyapunov characteristic exponents are two useful tools in this regard (Tabor 1981). Making a Poincaré surface of section (Henon and Heiles 1964, Henon 1983) is a simple technique to study the chaotic behaviour especially in the case of systems with two degrees of freedom. In this method a two dimensional surface in the phase space is considered and the successive intersections of a trajectory with the surface along any particular direction are noted. When N=2 the trajectory lies on the 3-dimensional energy surface defined by $E = H(p_0,q_0)$ in the four dimensional phase space. This means that any of the four variables say p_2 can be obtained in terms of the other three. A convenient choice is a plane (p_1,q_1) at a point q_2 = constant. This plane (p_1,q_1) is now known as the

Poincaré surface of section. The successive points define a map called Poincaré map. Poincaré map produces iterates on a bounded area of the surface and it is area preserving, because of the volume preserving property of Hamiltonian flows. If the points after a sufficient number of iterations of the map lie on a closed curve, the trajectory corresponding to them lies on an invariant torus or KAM surface. Instead of this, if these points fill a two dimensional area in the surface, then the trajectory corresponding to them is chaotic. Choice of different initial conditions, depending on whether they belong to regular regions or irregular regions, result in different behaviour for the map. For systems with divided phase space there appears islands of closed curves and a chaotic sea. The concept of surface of section can be generalised to higher dimensional systems with $N > 2$ also. But the method is not of much utility in such cases.

A more convenient technique to analyse chaotic behaviour is the study of Lyapunov characteristic exponents (LCE). LCE give us the average rate of exponential divergence of nearby trajectories and is a quantitative measure of chaos, the sensitive dependence on initial conditions. Characterisation of chaos of a phase space trajectory in terms of exponential divergence of nearby trajectories was introduced by Henon and Heiles (1964). It has been further developed by various authors (Zaslavsky and Chirikov 1972, Froeschle and Scheidecker 1973, Ford 1983).

The theory of Lyapunov exponents was applied to characterise irregular trajectories by Oseledec (1968). The connections among Lyapunov exponents, chaos and Kolmogorov entropy have been established by Bennettin et al (1976), Pesin (1977) and others. Bennettin et al (1980, 1976) and Shimada and Nagashima (1979) gave an algorithm for calculating LCE. The method can be used to study chaotic behaviour of dissipative systems as well. Calculation of LCE from time series is given in Wolf et al (1985) along with FORTRAN program code. In the case of Hamiltonian systems the sum of LCE is zero and for dissipative systems it should be less than zero. For calculating LCE we have to solve the equations of motion along with the corresponding variational system. Chaotic behaviour in many systems have been studied by this method (Udry and Pfenniger 1988, Contopoulos et al 1987,1989, Cleary 1989). More details on Lyapunov exponents will be given in Chapter 4.

1.4. Quantum Chaos

How does classical chaos manifest in quantum mechanics? What are the differences between properties of a quantum system whose classical limit is regular and of a system whose classical limit is chaotic? These are the questions the emerging discipline called quantum chaos tries to answer. Whether there is actually something called quantum chaos is still a controversial, unsettled and interesting question (Berry 1983,1985, Hogg and Huberman 1982,1983, Ford et al 1991, Partovi 1992). Different characterisations and properties for quantum chaos have been suggested and pursued by different authors.

As mentioned in the last section classical chaos is well characterised by the sensitive dependence on initial conditions which can be quantitatively determined by calculating Lyapunov exponents. But in quantum mechanics the scenario is completely different. For studying the time evolution of a system we have Schrödinger equation, which is a linear differential equation. Nonlinearity is a necessary condition for classical chaos. Hence chaos in a quantum system can not be similar to that in a classical system. In quantum mechanics we deal with statistical quantities. The uncertainty principle rules out definite trajectories for particles in quantum mechanics. Hence the criterion of exponential divergence of nearby trajectories can not be used directly as a definition for quantum chaos. The uncertainty principle also implies the coarse graining of quantum phase space; we can not distinguish points in a 2N-dimensional phase space within a volume $\hbar^N$. Hence the finite value of $\hbar$ tends to suppress chaos.

Because of Bohr's correspondence principle by which the quantum and classical behaviour should coincide for macroscopic systems, we expect that some remnants of classical chaos must persist in quantum mechanics. Actually in the area of quantum chaos main interest is with study of the properties of a quantum system whose classical limit shows chaotic behaviour. Berry's terminology for this area of study is quantum chaology (Berry 1987).

The semiclassical quantisation of a classical system is still an unsolved problem. The problem of quantising nonintegrable systems was first realised by Einstein (1917). Integrable systems can be quantised using the Einstein-Brillouin-Keller-Maslov quantisation rule (Zaslavsky 1981, Eckhardt 1988). For a system with a Hamiltonian H with N degrees of freedom depending on the generalised coordinates and momenta $(q_1, q_2, \dots q_N)$ and $(p_1, p_2, \dots p_N)$ which performs a finite motion quantisation rules can be given under the following conditions :

(i) If the variables are separable the quantisation rule is given by

$$\oint p_i dq_i = n_i \hbar \ , \ i = 1,2\dots,N \qquad (1.33)$$

where $n_i > 0$ are arbitrary integers which are the quantum numbers. This is the Bohr-Sommerfeld quantisation rule.

(ii) If not separable but integrable we have EBKM quantisation rule

$$S_k \equiv \int_{C_k} \sum_{i=1}^{N} p_i dq_i = \left(n_k + \frac{\alpha_k}{4} \right) \hbar \ , \ k = 1,2,\dots,N \qquad (1.34)$$

where C_k are N closed contours defined on the N-dimensional invariant torus, which can not be reduced to each other by a continuous deformation. α_k are called the Maslov indices.

Using numerical methods and a semiclassical approach Gutzwiller (1971,1980,1990) developed a semi classical quantisation method making use of the periodic orbits of classical system. This is based on the Feynman's path integral formulation of quantum mechanics.

In 1973 Percival invoked the correspondence principle to conjecture that in the semiclassical limit, the energy level spectrum consists of regular and irregular part corresponding to regular and irregular regions of classical phase space (Percival 1973). Many authors have investigated these questions since then. Various characterisations have been proposed and techniques to identify quantum chaos have been suggested (Berry 1983, McDonald and Kaufmann 1979, Pechukas 1982,1983,1984, Yukawa 1985, Eckhardt 1988, Bohigas et al 1984,1985). Some of these are :

(1) the method of avoiding crossings of energy levels,

(2) the sensitivity of energy eigenvalues to perturbations,

(3) the statistical analysis of fluctuations in the spectral sequences,

(4) the distribution of nearest neighbour level spacings,

(5) the structure of eigenvectors,

(6) the study of nodal curves,

(7) the loss of memory of initial states,

(8) Hose-Taylor criterion for quantum chaos,

(9) quantum Poincare sections,

(10) quantum entropies and Lyapunov exponents,

(11) algorithmic complexity theory.

It seems from these studies that quantum chaos is not as strong as classical chaos.

Before discussing some of these methods in detail let us see what is meant by quantum integrability. As discussed above integrability in classical mechanics is a well defined concept related to the existence of N integrals of motion in involution.

$$[\, h_n,\ h_m]_{PB} = 0.$$

Correspondingly one may define quantum integrability on the basis of the existence of N operators $\hat{J}_1,\ \hat{J}_2,\ldots,\hat{J}_N$ whose commutator brackets vanish.

$$[\, \hat{J}_n,\ \hat{J}_m] = 0 \tag{1.35}$$

Here use may be made of the fact that one can replace an operator $\hat{A}(\hat{p},\hat{q})$ by a c-number function $A(p\ ,q\ )$.

$$\hat{A}(\hat{p}\ ,\hat{q}\ ) = \int d^N\theta\ d^N\tau\ d^Np\ d^Nq\ (2\pi\hbar)^{-2N}\ A(p,q)$$
$$\exp\{(i/\hbar)[\tau.(\hat{p}-p)+\theta.(\hat{q}-q)]\} \tag{1.36}$$

The correspondence between the c-number functions and the operators is one to one. The commutator bracket can be replaced by Moyal bracket.

$$\{A,B\}_M = 2/\hbar\ \sin\ [\hbar/2(\partial q_A.\partial p_B - \partial p_A.\partial q_B)]\ A(p_A,q_A)B(p_B,\ q_B) \tag{1.37}$$

Korsh (1982) has suggested that any classically integrable system is also quantum integrable in the above sense. However, Hietarinta (1982,1984) has shown that this is not true in general. Classical integrability does not imply quantum integrability and vice versa. We may have to add higher order terms in $\hbar$ for integrability.

One of the most widely used methods of studying quantum chaos is the analysis of eigenvalue statistics. Nonintegrability is reflected in the statistical properties of sequences of energy levels (Bohigas et al 1984,1985a,b). Let us consider a sequence of energy levels.

$$E_1\le\ E_2\le\ E_3\le\ldots\ldots\le\ E_k\le\ \ldots.$$

Assume that the eigenvalues are nondegenerate except for accidental degeneracies. If the Hamiltonian admits symmetries then the underlying Hilbert space has to be decomposed into invariant subspaces so that the eigenvalues are nondegenerate in these subspaces.

Before studying the eigenvalue statistics we have to "unfold" the spectrum (Bohigas et al 1985a). By unfolding we can eliminate the variation in average level spacing as a function of energy. This is done by mapping the spectrum $\{E_i\}$ onto the spectrum $\{\varepsilon_i\}$ through $\varepsilon_i = \overline{N}(E_i)$ where $\overline{N}$ is the smoothed cumulative density.

The simplest statistical measure is the distribution of nearest neighbour spacings

$$S_1, S_2, S_3, \ldots\ldots, S_k, \ldots. \tag{1.38}$$

where $S_k = \varepsilon_{k+1} - \varepsilon_k$. It is assumed that the sequence is sufficiently long so that statistical techniques can be applied . For the exact semiclassical limit we have to consider an infinite sequence of energy levels. From the above sequence one can calculate the normalised probability of finding an energy level spacing s. Using this we find the probability distribution P(s) of nearest neighbour level spacing s. P(s) is different for integrable and nonintegrable systems and can be used for distinguishing between them. In 1977 Berry and Tabor (1977) showed that the nearest neighbour level spacing distribution for an integrable system is Poissonian,

$$P(s) = \exp(-s). \tag{1.39}$$

There is level clustering; individual levels are uncorrelated and randomly distributed. Coupled harmonic oscillators are an exception to this rule.

In the case of chaotic systems we have level repulsion. Nearby levels are correlated and the distribution peaks at a nonzero value. Random matrix theory of nuclear physics has been used for correlating level distribution with chaotic properties (Mehta 1967, Brody et al 1981). If we consider a Gaussian orthogonal ensemble (GOE) of random matrix elements we obtain a distribution with linear level repulsion,

$$P(s) = \pi/2 \; s \exp(- \pi \; s^2/4) \qquad (1.40)$$

This is known as Wigner surmise (Wigner 1967). If we consider a Gaussian Unitary Ensemble (GUE) of matrix elements we obtain a distribution with quadratic level repulsion.

$$P(s) = (32/\pi) \; s^2 \exp(- 4 \; s^2/\pi) \qquad (1.41)$$

The exact nature of repulsion thus depend on the symmetry properties of the Hamiltonian. The important point here is that these distributions are generic properties of nonintegrable Hamiltonians. It is independent of the degree of nonintegrability.

In the generic Hamiltonian systems, where there is a mixed behaviour of order and chaos, the situation is more complicated. In such systems with divided phase space we have to consider superposition of statistically independent sequences of levels from each of the corresponding classical phase space regions; sequences from regular regions having Poisson distribution and those from irregular regions having Wigner distributions. P(s) depends on the Liouville measure of regular regions and chaotic regions (Berry and Robnik

1984).

First numerical study of eigenvalue statistics was carried out by McDonald and Kaufman (1979) for the stadium billiard. They found Poisson distribution corresponding to the integrable limit and Wigner distribution for nonintegrable cases. This has been confirmed by further studies of Casati, Valz-Griz and Guarneri (1980). Calculations of Berry (1981) on Sinai's billiard also gave similar results.

Higher order correlations such as Δ_3 and Q-statistics are also used to characterise quantum chaos. Δ_3-statistics or rigidity parameter measures the mean square deviation of the integrated density $n(\varepsilon)$ of states in an interval $[x, x+l]$ from a straight line (Mehta 1967).

$$\Delta_3(l, x) = \frac{1}{l} \min_{A,B} \int_x^{x+l} [n(\varepsilon) - A\varepsilon - B]^2 d\varepsilon \tag{1.42}$$

(i) For Poisson distributed levels

$$\Delta_3(l, x) = l/15 \tag{1.43}$$

(ii) For GOE $\Delta_3(l, x) = 1/\pi^2 \log l - 0.00695$ (1.44)

(iii) For GUE $\Delta_3(l, x) = 1/2\pi^2 \log l + 0.059$ (1.45)

in the limit of large l.

In contrast to the level spacing distribution where no semiclassical proof is known, Δ_3 can be characterised completely in terms of periodic orbits.

Sensitivity of energy level motion on external parameters have been studied by Pechukas (1983), Yukawa (1985) and Steeb and van Tonder (1988).

Recently Luna Acosta (1991) used a definition for sensitive dependence applicable both to classical and quantum systems and showed that sensitive dependence on initial conditions is absent in bounded quantum systems. Absence of sensitive dependence in quantum systems has also been shown by Partovi (1992). In a recent work Ford et al (1991) showed the possible failure of quantum correspondence principle in a specific example. Heller (1984) observed that quantum eigen states of stadium billiard have a structure with high amplitude along short unstable periodic orbits. This is known as quantum scars of classical periodic orbits and is explained as due to constructive quantum interference. Scarred wave functions have been numerically observed in many physically relevant systems as well as experimentally (Sridhar 1991).

In Chapter 6 we approach the problem of quantum chaos by calculating the Gaussian effective potential of a classically chaotic system.

1.5. Yang-Mills theories, Monopoles and Chaos

Non-Abelian gauge theories or Yang-Mills theories play an important role in our current understanding of fundamental interactions in nature (Abers and Lee 1973, Itzykson and Zuber 1980). Yang and Mills in 1954 introduced the notion of non-Abelian gauge fields by extending the concept of local gauge transformations to the non-Abelian gauge group SU(2) (Yang and Mills 1954). It was soon generalised to arbitrary non-Abelian groups. The unified theory of weak and electromagnetic interactions proposed by Weinberg, Salam and Glashow is a gauge theory with SU(2)xU(1) as the gauge group.

Grand unified theories which unify strong, weak and electro magnetic interactions into a single theory are also based on non-Abelian gauge theories. Strong interactions can be described using SU(3) as the gauge group. Non-Abelian gauge theories are intrinsically nonlinear and their integrability and chaotic behaviour aspects are of much importance in field theory and particle physics. Chaos in YM theories has got relevance in the explanation of problems like quark confinement, monopole stability, etc.

Gauge theories are characterised by the group of symmetry transformations (the gauge group) under which they remain invariant. Based on the type of gauge group gauge theories can be Abelian or non-Abelian. The simplest gauge group is U(1) and the corresponding Abelian gauge theory is used in the description of quantum electrodynamics.

An SU(2) Yang-Mills theory may be constructed in the following way (Actor 1979). Let $\phi(x)$ be a set of n scalar fields. Consider the globally symmetric Lagrangian

$$\mathscr{L} = (\partial_\mu \phi)^\dagger (\partial_\mu \phi) - m^2 \phi^\dagger \phi \qquad (1.46)$$

(In this section and Chapters 3 and 5 we use the Einstein summation convention). A global gauge transformation is defined by

$$\phi(x) \longrightarrow \phi'(x) = e^{-i\theta^a T^a} \phi(x) \qquad (1.47)$$

where T^a are the 3 generators of the SU(2) group in the n-dimensional representation satisfying the Lie algebra,

$$[T^a, T^b] = i\, \varepsilon_{abc} T^c \qquad (1.48)$$

θ^a are 3 arbitrary real parameters. If we make θ^a space-time dependent

$$\phi(x) \longrightarrow \phi'(x) = e^{-i\theta^a(x)T^a}\, \phi(x) = U(\theta(x))\phi(x) \qquad (1.49)$$

we obtain a local gauge transformation belonging to the group SU(2). The Lagrangian (1.46) is not invariant under the local gauge transformation (1.49). It can be made gauge invariant by replacing ordinary derivatives defined by

$$D_\mu \phi_n = (\partial_\mu \delta_{nm} - ig\, A^a_\mu\, T^a_{nm})\, \phi_m \qquad (1.50)$$

where A^a_μ are 3 vector fields called SU(2) gauge fields and g is the coupling constant. The transformation of the gauge fields can be obtained by requiring covariant derivatives to transform like the fields.

$$D_\mu\phi(x) \longrightarrow (D_\mu\phi(x))' = U\,(\,\theta(x))\; D_\mu\phi(x) \qquad (1.51)$$

Solving (1.51) we get

$$A^a_\mu T^a \longrightarrow A^{a\prime}_\mu\, T^a = U(\theta(x))A^a_\mu T^a U^{-1}(\theta(x)) - \frac{1}{g}(\partial_\mu U(\theta(x))U^{-1}(\theta(x))) \qquad (1.52)$$

The SU(2) gauge invariant Lagrangian is

$$\mathcal{L}' = (D_\mu\phi)^\dagger(D_\mu\phi) - m^2\phi^\dagger\phi \qquad (1.53)$$

To complete the Lagrangian one should add the kinetic energy term for the gauge fields. The simplest gauge invariant form of kinetic energy is

$$\mathcal{L} = -\frac{1}{4}\, F^a_{\mu\nu}\, F^{\mu\nu a} \qquad (1.54)$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\, \epsilon_{abc} A^b_\mu A^c_\nu \qquad (1.55)$$

It may be noted that a gauge invariant mass term is not present in the Lagrangian. This implies that the non-Abelian gauge fields are massless fields.

Systems with spontaneous symmetry breaking (SSB) are very important in field theory. Spontaneous breaking of symmetry occurs when there exist degenerate vacuum states.

The vacuum is not invariant under the symmetry group of transformations even though the Lagrangian is invariant. When a local gauge symmetry is broken some components of the gauge fields become massive and all Goldstone bosons disappear. This is known as Higgs mechanism (Higgs 1964). Goldstone boson is a massless spin zero particle created when a continuous global symmetry is spontaneously broken.

Classical solutions of field theories play an important role in the non-perturbative dynamics of the corresponding quantum field theories. Especially important are solutions with energy density confined to a small region in space, which can be interpreted as particles. These are coherent excitations of the basic fields and a consistent quantum theory exists for many of them. For obtaining physically relevant classical solutions, some conditions like finiteness of energy or action is imposed. This condition often defines a map between non-trivial topological spaces. Such maps fall into different equivalent classes known as homotopic classes, which are labelled by a number called winding number. Therefore we can classify all finite energy solutions with respect to their winding numbers (Goddard and Olive 1978). For a fixed winding number n, a solution having the lowest energy will be the stable one. n is always an integer.

Let us consider a non-Abelian example with SSB and Higgs mechanism. Consider an SU(2) gauge theory with Higgs triplet defined by the Lagrangian,

$$\mathcal{L} = -\frac{1}{4} F^{a}_{\mu\nu}F^{\mu\nu a} + \frac{1}{2} D_{\mu}\phi^{a}D^{\mu}\phi^{a} - V(\phi) \qquad (1.56)$$

where,

$$F^a_{\mu\nu} = \partial_\mu A^a - \partial_\nu A^a + g\,\varepsilon_{abc}\,A^b_\mu A^c_\nu$$

$$D_\mu \phi^a = \partial_\mu \phi^a + g\,\varepsilon_{abc} A^b_\mu \phi^c$$

and $V(\phi) = \frac{\lambda}{4}\left(\phi^a\phi^a - \frac{m^2}{\lambda} \right)^2.$

This is the Georgi-Glashow (1972) model. Here the SU(2) symmetry is broken down to U(1) by the Higgs triplet. The equations of motion are,

$$D_\nu F^{\mu\nu a} = -\,g\,\varepsilon_{abc}(\,D^\mu\phi_b)\,\phi_c \tag{1.57}$$

$$D_\mu D^\mu \phi_a = (\,m^2 - \lambda\,\phi^2)\,\phi_a \tag{1.58}$$

For $m^2 > 0$, the minimum of $V(\phi)$ corresponds to the value of ϕ given by the relation

$$\phi^2 = m^2/\lambda \tag{1.59}$$

All finite energy solutions assume the ground state configuration as $r \longrightarrow \infty$.

$$\text{ie.,}\quad \phi^2 \longrightarrow m^2/\lambda \quad \text{as } r \longrightarrow \infty \tag{1.60}$$

Here $\phi^2 = \phi^a\phi^a = \phi_1^2 + \phi_2^2 + \phi_3^2$ and $r^2 = x_1^2 + x_2^2 + x_3^2$.

't Hooft (1974) and Polyakov (1974) discovered monopoles as classical finite energy solutions of non-Abelian gauge theories. The winding number of monopole solutions originates from the finite energy condition (1.60) on the Higgs field ϕ_a. For magnetic monopoles the magnetic charge is related to the winding number. Sometimes winding number is referred to as the topological charge. Since the winding number is a conserved quantity magnetic charge is also conserved.

The 't Hooft-Polyakov monopole solution with winding number 1 is

$$A_0^a = 0, \qquad A_i^a = \frac{1}{g}\,\varepsilon_{ain} r_n \frac{1-K(r)}{r^2}$$

$$\phi = \frac{1}{g}\, r_a \frac{H(r)}{r^2} \tag{1.61}$$

where $r_n = x_n$ and r is the radial variable. This static spherically symmetric ansatz converts equations of motion to two coupled differential equations.

$$r^2 \frac{\partial^2 K}{\partial r^2} = K\,(\,K^2 - 1 + H^2\,)$$

$$r^2 \frac{\partial^2 H}{\partial r^2} = H\,(\,2K^2 - m^2 r^2 + \frac{\lambda}{g^2} H^2\,) \tag{1.62}$$

The energy integral in terms of the ansatz function is

$$E = \frac{4\pi}{g^2}\int_0^\infty dr\,\{\,K'^2 + \frac{(rH'-H)^2}{2r^2} + \frac{(K^2-1)^2}{2r^2} + \frac{K^2H^2}{r^2} + \frac{\lambda r^2}{4g^2}\,(\,\frac{H^2}{r^2} - \frac{g^2 m^2}{\lambda}\,)^2 \tag{1.63}$$

where $H' = \partial H/\partial r$ and $K' = \partial K/\partial r$. For finiteness of the above integral (1.63) the ansatz function should satisfy the condition

$$H \longrightarrow 0,\ K \longrightarrow 1 \text{ as } r \longrightarrow 0 \tag{1.64}$$

$$H \longrightarrow gmr/\sqrt{\lambda},\quad K \longrightarrow 0 \text{ as } r \longrightarrow \infty \tag{1.65}$$

There is SSB in this theory because the minimum of $V(\phi)$ corresponds to values of ϕ_a on

$$S^2 = \{\,\phi_1, \phi_2, \phi_3 : \phi_1^2 + \phi_2^2 + \phi_3^2 = m^2/\lambda\,\} \tag{1.66}$$

Choosing any one out of these degenerate minima breaks the symmetry spontaneously. However any arbitrarily chosen vacuum is still invariant under SO(2) symmetry. Since SO(2) $\approx$ U(1) we can say that the U(1) symmetry survives and the gauge field corresponding to this symmetry is long ranged.

Exact solutions of the system (1.62) are not available except in the limit of the vanishing Higgs potential. In this limit, known as Prasad-Sommerfield (PS) limit $m \rightarrow 0$ and $\lambda \rightarrow 0$ with m^2/λ finite. The spontaneous symmetry breaking survives because a classical solution can have its Higgs field assuming non zero value $m/\sqrt{\lambda}$ as $r \rightarrow \infty$. The solution in the PS limit is (Prasad and Sommerfield 1975)

$$K(r) = \beta r/ \sinh \beta r, \quad H(r) = \beta r \coth \beta r - 1 \qquad (1.67)$$

where $\beta = M_w = gm/\sqrt{\lambda}$.

K and H approach the boundary condition (1.65) in the following way

$$\text{As } r \longrightarrow \infty, \quad K(r) \longrightarrow O(e^{-M_w r})$$

$$H(r) \longrightarrow gmr/\sqrt{\lambda} + O(e^{-\mu r})$$

where $\mu = \sqrt{2}\, m$ is the mass of the massive Higgs particle. The 't Hooft-Polyakov monopole has a definite size determined by the Compton wavelength of massive fields. The massive fields exist inside the core and outside they vanish exponentially leaving a field configuration exactly similar to that of the Dirac monopole.

Yang-Mills theories are intrinsically nonlinear and hence they can exhibit chaotic behaviour. Integrability and chaotic behaviour aspects of such nonlinear field theories has attracted much attention during the last decade. Matinyan et al (1981a) studied a simplified Yang-Mills model and established the chaotic nature of YM fields. They considered the case of spatially homogeneous fields. Following this various authors have investigated such models in detail (Savvidy 1984, Chang 1984, Karkowski 1991). Spatially homogeneous models of YM theory with spontaneous symmetry breaking have also been investigated (Matinyan et al 1981b, Nagarajakumar and Khare 1989). Here there is an order to chaos transition as a system parameter is changed. More details on these are given in Chapter 3. However only a few studies are available on the more important and realistic space-time dependent systems. First such study was made by Matinyan et al (1986,1988). They investigated numerically the chaotic behaviour of spherically symmetric time dependent SU(2) Yang-Mills (SSYM) system. We have studied the integrability of SSYM using Painlevé analysis and the results are described in Chapter 3. The nonintegrability aspects of spherically symmetric time dependent SU(2) Yang-Mills-Higgs (SSYMH) system obtained using the time dependent version of the ansatz (1.61) in (1.56) is also discussed there. Studies on chaos in SSYMH is described in Chapter 5.

Chaos in non-Abelian gauge theories are of much importance in Quantum Chromodynamics. This is because of the result that presence of random fields in the vacuum is a necessary and sufficient condition for quark confinement (Nielsen and Olsen 1979, Olsen 1982). Though quantum regime of classically chaotic systems is not well understood it is expected that chaos may somehow show up in the quantum case. Multiparticle hadron production processes may also be related to chaos in such systems (Carruthers et al 1989).

CHAPTER 2

INTEGRABILITY OF TWO DIMENSIONAL HOMOGENEOUS POTENTIALS

2.1. Introduction

Recently considerable attention has been paid to the question of integrability of dynamical systems. As mentioned in Chapter 1 it is not possible to say whether a given Hamiltonian system is integrable or not except when one can construct integrals of motion directly. Singular point analysis due to Ablowitz, Ramani and Segur (1980) and Yoshida (1983) is an extremely useful technique for the study of integrability and for the identification of integrable cases. Ziglin's theory connecting nonintegrability and properties of monodromy matrices of certain periodic solutions give us some rigorous results in this area (Ziglin 1983). Yoshida has also proved some important theorems relating integrability of dynamical systems and the Kowalevskaya exponents and stability of straight line periodic solutions (Yoshida 1986,1987a,1989). These three methods have some connections among themselves. A number of candidates for integrable systems have been identified by these methods and their combinations.

In this chapter we carry out singular point analyses and related studies on systems with Hamiltonian of the form,

$$H = \frac{1}{2}(P_x^2 + P_y^2) + V(x,y) \tag{2.1}$$

with V(x,y) a homogeneous polynomial potential of even degree 2m. Hamiltonians of these types are used in lattice dynamics, condensed matter physics, field theory, astrophysics, etc., and special cases of these have been studied in the existing literature (Bountis et al 1982, Dorizzi et al 1983, Grammaticos et al 1983, Lakshmanan and Sahadevan 1985, Steeb et al 1985b). To make the discussion self-contained we describe briefly the ARS algorithm for Painlevé analysis, Kowalevskaya exponents (KE) analysis and related theorems of Yoshida and the stability analysis of Yoshida in the next section. The mutual connections among these approaches are pointed out. Combining these methods we deduce a stronger condition for integrability as a restriction on the possible Kowalevskaya exponents (KE) and integrability coefficients (IC). Singularity and stability analyses of symmetric homogeneous potentials with $m = 2,3$ and 4 is carried out and possible integrable cases are identified. A second integral is also constructed directly in those cases suggested by these analyses. We have generalised the integrable cases to a potential of arbitrary degree 2m by constructing the corresponding second integral. These results are presented in section 2.3. Section 2.4 summarises our conclusions.

2.2. Singularity, Stability and Integrability

2.2.1. Painlevé analysis

According to the extended Painlevé conjecture (Ramani et al

1982) a sufficient condition for integrability is the weak Painlevé property (WPP). A system of equations is said to have the strong Painlevé property (PP) when the only movable singularities of the solutions in the complex time plane are poles. In weak Painlevé case certain algebraic branch points are also allowed. A strong necessary condition for the PP or WPP is provided by the Painlevé analysis (P-analysis) (Ablowitz et al 1980, Graham et al 1985, Steeb and Euler 1988). This consists in checking whether a solution can be expanded in terms of a Laurent series with sufficient number of arbitrary coefficients in the following form.

$$w_i = \sum_{m=0}^{\infty} a_i^{(m)} \tau^{m+p_i} , \quad \tau = z-z_0 \tag{2.2}$$

We describe the algorithm with sufficient modifications for the inclusion of WPP. Let us consider a system of n first order ordinary differential equations,

$$\frac{dw_i}{dz} = F_i(z,\mathbf{w}), \qquad i=1,2,\ldots\ldots n. \tag{2.3}$$

where $\mathbf{F}$ is real, analytic in z and algebraic in $\mathbf{w}$. ARS test does not consider the presence of movable essential singularities hence it gives only a necessary condition for PP. The algorithm consists of 3 steps.

Step 1. Leading order behaviour or Dominant behaviour

The main assumption of ARS test is that the dominant behaviour of the solutions in the neighbourhood of a movable singularity is of the form,

$$w_i \approx \alpha_i \tau^{p_i}, \qquad \tau \longrightarrow 0 \tag{2.4}$$

where $\tau = z-z_0$, z_0 is arbitrary with some $Re(p_i) < 0$. Substituting (2.4) in (2.3) we can find the possible values of p_i for which two or more terms in each equation balance each other, while the rest can be ignored as $\tau \longrightarrow 0$. The balancing terms are known as leading order terms or dominant terms. For each choice of p_i this requirement also determines the corresponding values of the α_i. There can be different leading behaviours with different p_i. One must find and examine separately all possible dominant behaviours.

From this step the following observations can be made.

(a) If all p_i are negative integers, then (2.4) may represent the first term of the Laurent series for each p_i and this may be an indication of the strong PP ($\alpha_i = a_i^{(0)}$).

(b) If p_i are not integers, but rational fractions it may be associated with WPP. The solution will have a movable algebraic branch point. In some cases it may be possible to transform the system to one without algebraic branch points by a simple change of variables.

If any of the p_i are irrational or imaginary it indicates that the system is non P-type.

For an n^{th} order system there are $(n-1)$ arbitrary constants to be sought among the $a_i^{(m)}$ in (2.2) for the expansion to be generic. z_0 is the first free integration constant of the system. The powers at which they arise are known as resonances and in the next step we turn to find these.

Step 2. Resonances

In this step we substitute,

$$w_i = \alpha_i \tau^{p_i} (1 + \gamma_i \tau^r), \quad i=1,2,\ldots,n, \; r>0. \qquad (2.5)$$

in the system containing only the leading order terms. We retain only the terms linear in γ_i which can be written as

$$Q(r).\gamma = 0 \qquad \gamma \equiv (\gamma_1, \ldots, \gamma_n) \tag{2.6}$$

where Q(r) is an n x n matrix, with r entering only in its diagonal elements at most linearly. The r roots of the equation,

$$\det Q(r) = (r+1)(r^{n-1}+A_2 r^{n-2}+\ldots.. + A_{n-1}) = 0 \tag{2.7}$$

determine the resonance values. Some general remarks :

(a) One root is always -1, representing the arbitrariness of z_0.

(b) The resonance r=0 corresponds to the arbitrariness of one of the leading order coefficients $\alpha_i = a_i^{(0)}$.

(c) Any resonance with Re(r)<0 (except r=-1) must be ignored because it violates the leading order hypothesis.

(d) Any resonance with Re(r)>0, but r not an integer, indicates that $z=z_0$ is a movable branch point and in general the system is non P-type. We must check whether it can be removed by coordinate transformations.

(e) In the case where p_i is itself rational, the appearance of a rational r with same denominator as p_i indicates a finite branching and is related to the WPP.

(f) If for every possible (p_i, α_i) from step 1, all of the resonances r except -1 and 0 are positive real integers then there are no algebraic branch points.

For generic solutions we must have (n-1) nonnegative real resonances. If any of the resonance is irrational or imaginary the algorithm is terminated at this point.

Step 3. The constants of integration

In this step we check whether arbitrary constants enter in the expansion at the resonance values without introducing logarithmic terms. For this we substitute into the full system (2.3), for every dominant behaviour (2.4), the truncated expansion,

$$w_i = \alpha_i \tau^{p_i} + \sum_{m=1}^{r_s} a_i^{(m)} \tau^{p_i+m}, \tag{2.8}$$

where r_s is the largest positive resonance value. We then equate the terms order by order in powers of τ to obtain,

$$Q(m)\, a^{(m)} = R^{(m)}(\, z_0; a^{(j)}), \; j=1,\ldots,m-1 \tag{2.9}$$

with $m=1,\ldots.r_s$, $R = (R_1,\ldots,R_n)^T$; $a = (a_1,\ldots\ldots,a_n)^T$.

(i) For $m < r_1$, r_1 being the smallest positive resonance (2.9) determines $a^{(m)}$.

(ii) At $m = r_1$, for (2.8) to have a solution (ie., for $a^{(r_1)}$ to have one arbitrary component, assuming r_1 is a simple root of equation (2.7)) the following compatibility condition must be satisfied.

$$\det Q^{(k)}(r_1) = 0 \quad k=1,2,\ldots,n \tag{2.10}$$

where $Q^{(k)}(r_1)$ is the matrix $Q(r_1)$ with its k^{th} column replaced by $R^{(r_1)}$.

(iii) If (2.10) is satisfied, then for $r_1 < m < r_2$, the next smallest positive resonance, equation (2.9) again determines $a^{(m)}$.

(iv) The same procedure must be repeated at each higher resonance upto the largest one. In the case of multiple resonance it must be ensured that the number of arbitrary components of $a^{(r)}$ is equal to the multiplicity of the resonance r.

(v) When (2.10) is not satisfied at some resonance r, one or more expansions (2.2) will have to be altered by introducing logarithmic terms as follows,

$$w_i = \sum_{m=0}^{r-1} a_i^{(m)} \tau^{p_i+m} + (a_i^{(r)} + b_i^{(r)} \ln \tau) \tau^{p_i+r} + \ldots . \quad (2.11)$$

If even with this $a_i^{(r)}$ is not arbitrary we introduce more singular terms like $(\ln \tau)^2$, etc., until the coefficient $a_i^{(r)}$ becomes arbitrary. (2.11) signals the presence of movable logarithmic branch points and then the system is non-Painlevé type. This algorithm does not exclude the possibility of movable essential singularities, hence the sufficiency of single valuedness has to be checked by some other methods.

2.2.2. Kowalevskaya exponents

Consider a similarity invariant system of ODEs

$$\frac{dx_i}{dt} = F_i(x_1, \ldots, x_n) \quad i=1,2,\ldots,n \quad (2.12)$$

where F_i are rational functions of x. A system is said to be similarity invariant under the similarity transformation

$$t \longrightarrow \alpha^{-1} t, \; x_i \longrightarrow \alpha^{g_i} x_i \quad (2.13)$$

where g_i are rational numbers and α is a constant if

$$F_i(\alpha^{g_1} x_1, \ldots, \alpha^{g_n} x_n) = \alpha^{g_i+1} F_i(x_1, \ldots, x_n) \quad (2.14)$$

for arbitrary x and α. A function $\phi(t,x_1,\ldots,x_n)$ is said to be weighted homogeneous of weighted degree M if it satisfies

$$\phi(\alpha^{-1} t, \alpha^{g_1} x_1, \ldots, \alpha^{g_n} x_n) = \alpha^M \phi(t, x_1, \ldots, x_n). \quad (2.15)$$

Differentiating (2.14) with respect to α and putting $\alpha=1$, we have

$$\sum_{j=1}^{n} g_j x_j \frac{\partial F_i}{\partial x_j}(x_1, \ldots, x_n) = (g_i + 1) F_i(x_1, \ldots, x_n) . \quad (2.16)$$

These linear algebraic equations determine the unknowns $g_1,\ldots,g_n$ from $F_i(x)$.

A similarity invariant system (2.12) in general admits a special type of particular solutions

$$x_1 = k_1 t^{-g_1}, \ \ldots, x_n = k_n\, t^{-g_n} \tag{2.17}$$

with $k_i g_i \neq 0$ for at least one i. We can see that the solution (2.17) satisfy (2.12) when the constants are a set of solutions of

$$F_i(k_1\ ,\ldots,k_n) = -\ g_i k_i \quad i=1,\ldots,n \tag{2.18}$$

Consider now the variational equations about the reference solution (2.17),

$$\frac{dy_i}{dt} = \sum_{j=1}^{n} \frac{\partial F_i}{\partial x_j}\,(k_1 t^{-g_1},\ldots,k_n t^{-g_n})y_j,\ i=1,2,\ldots,n \tag{2.19}$$

Here $\partial F_i/\partial x_j(k_1,\ldots,k_n)$ denotes $\partial F_i(x)/\partial x_j$ at $x_1=k_1,\ldots,x_n=k_n$. This notation is followed throughout this chapter. Differentiating (2.14) with respect to x and putting $\alpha=t^{-1}$, $x_1= k_1,\ldots,x_n= k_n$, we have

$$t^{-g_j}\,\frac{\partial F_i}{\partial x_j}\,(k_1 t^{-g_1},\ldots,k_n t^{-g_n}) = t^{-g_i-1}\,\frac{\partial F_i}{\partial x_j}\,(k_1,\ldots,k_n). \tag{2.20}$$

Hence the variational equations can be written as

$$\frac{dy_i}{dt} = \sum_{j=1}^{n} \frac{\partial F_i}{\partial x_j}\,(k_1,\ldots,k_n)t^{g_j-g_i-1}\ y_j,\ i=1,2,\ldots,n. \tag{2.21}$$

It can be shown that

$$y_1 = y_{1,0}t^{\rho-g_1},\ldots,y_n = y_{n,0}t^{\rho-g_n}, \tag{2.22}$$

satisfy equation (2.21) when the constant ρ is an eigenvalue and the constant column vector $\boldsymbol{y} = (y_{1,0},\ldots,y_{n,0})^T$ is an eigenvector of a n x n constant matrix $K = (K_{ij})$ with matrix elements

$$K_{ij} = \frac{\partial F_i}{\partial x_j}(k_1,\ldots,k_n) + \delta_{ij} g_i \quad . \tag{2.23}$$

The characteristic polynomial

$$K(\rho) = \det_{1 \le i,j \le n} \left(\rho\delta_{ij} - K_{ij} \right), \tag{2.24}$$

is called the Kowalevskaya determinant and the eigenvalues of the matrix K which are roots of the equation $K(\rho)=0$ are called the Kowalevskaya exponents (KE).

Theorems relating KE and integrals of motion have been proved by Yoshida. Weighted degree of a homogeneous first integral appear as a Kowalevskaya exponent. Its gradient should not vanish at $x_i=k_i$. If there are two independent weighted homogeneous first integrals of the same weighted degree M then ρ=M becomes a KE with multiplicity two. Their gradients at $k_1,\ldots,k_n$ must be nonvanishing and linearly independent. In Hamiltonian systems KE come in pairs (ρ, $g_H-1-\rho$) and the pair ($-1,g_H$) is always present. Here g_H is the weighted degree of the weighted homogeneous Hamiltonian. Yoshida proved also a very important result relating KE to integrability. Yoshida's theorem : *In order that a given similarity invariant system (2.12) with rational functions* ***F(x)*** *be algebraically integrable, it is necessary that every possible KE is a rational number.* This means that existence of an imaginary or irrational KE is a sufficient condition for nonintegrability. These results have been widely used in the study of integrability. Since we get an idea about the weighted degree of the integral of motion, it can be used for searching integrals of motion also. KE can also be defined for nonsimilarity invariant sytems.

2.2.3. Ziglin's theory and Integrability coefficient

Ziglin (1983) obtained conditions for the integrability for the variational equations, about a particular solution. Using Ziglin's theory Yoshida developed conditions for nonintegrability for some Hamiltonian systems.

Consider a Hamiltonian system with N degrees of freedom

$$H = 1/2\ p^2 + V(q) \tag{2.25}$$

where potential V(q) is homogeneous polynomial of integer degree k. In general the system has straight line periodic solutions of the form

$$q = c\ \phi(t), \quad p = c\ \dot{\phi}(t) \tag{2.26}$$

where $\phi(t)$ is a solution of the differential equation

$$\frac{d^2\phi}{dt^2} + \phi^{k-1}=0 \tag{2.27}$$

and the constant vector $c = (c_1,\ldots\ldots,c_N)^T$ is a solution of the algebraic equation

$$c = \frac{\partial V}{\partial q}(c). \tag{2.28}$$

The linear variational equation around the solution (2.26) are with $\delta q = \xi$

$$\frac{d^2\xi}{dt^2} = -\phi(t)^{k-2}V_{qq}(c)\xi \tag{2.29}$$

where $V_{qq}(c)$ is the Hessian matrix of V(q) evaluated at $q=c$. Since $V_{qq}(c)$ is symmetric by a change of variables $\xi = U\xi'$ with an orthogonal matrix U (2.29) is diagonalised or separated to

$$\frac{d^2\xi'}{dt^2} = -\phi(t)^{k-2}\mathrm{diag}(\lambda_1,\lambda_2,\ldots,\lambda_N)\xi' \tag{2.30}$$

where $\lambda_1,\lambda_2,\ldots\ldots,\lambda_N$ are the eigenvalues of $V_{qq}(c)$.

In the case of two degrees of freedom Hamiltonian systems there is only one nontrivial eigenvalue given by

$$\lambda = \mathrm{Tr}\, V_{qq}(c) - (k-1). \tag{2.31}$$

The normal variational equation (NVE) can be written as

$$\frac{d^2\xi}{dt^2} + \lambda\phi(t)^{k-2}\xi = 0 \tag{2.32}$$

The quantity λ is called the integrability coefficient (IC). Using the monodromy groups of the NVE and Ziglin's theorem regarding nonintegrability Yoshida proved the following theorem.

Theorem : *If the integrability coefficient λ lies in the region S_k defined below , the system is nonintegrable. The regions S_k are defined as follows :*

(i) $k \geq 3$

$$S_k = \{ \lambda < 0, \quad 1<\lambda<k-1, \quad k+2<\lambda<3k-2, \ldots\ldots, \quad j(j-1)k/2+j < \lambda < j(j+1)k/2-j, \ldots \} \tag{2.33}$$

(ii) $S_1 = \mathbb{R} - \{0,1,3,6,10,\ldots\ldots, j(j+1)/2,\ldots\}$ (2.34)

(iii) $S_{-1} = \mathbb{R} - \{1,0,-2,-5,-9,\ldots,-j(j+1)/2+1,\ldots\}$ (2.35)

(iv) $k \leq -3$

$$S_k = \{ \lambda>1,\ 0>\lambda>-|k|+2,\ -|k|-1>\lambda>-3|k|+3,\ -3|k|-2>\lambda>-6|k|+4,\ldots,-j(j-1)|k|/2-(j-1)>\lambda>-j(j+1)|k|/2+(j+1),\ \ldots \} \tag{2.36}$$

Nonintegrability of various Hamiltonian systems have been proved using these results.

2.2.4. Restrictions on KE and IC

If all the k_i are nonzero the KEs and resonances are the same. When some of k_i are zero there will be a difference of an additive term between them. But when resonances are not KEs they can even be irrational or imaginary. More details on the connection between P-analysis, KE and Ziglin's theory are given by Yoshida et al (1987) and Ramani et al (1989).

In the case of homogeneous polynomial potential of degree 2m with two degrees of freedom the three methods discussed above can be combined to yield more specific results on integrability. In P-analysis we try to find solutions around a movable singularity at t_0 in the complex time plane in the form,

$$x(t) = \sum_{j=0}^{\infty} a_j \tau^{-p+j/s} \tag{2.37}$$

$$y(t) = \sum_{j=0}^{\infty} b_j \tau^{-q+j/s} \tag{2.38}$$

Here p and q are positive rational numbers with a common integer denominator $s>0$. $s \neq 1$ corresponds to WPP. For the system to have PP, $j=rs$ must be integers, where r is the resonance.

Hamiltonians with homogeneous potentials of degree 2m are invariant under the similarity transformation

$$t \longrightarrow \alpha^{-1}t, \quad x \longrightarrow \alpha^{g}x, \qquad y \longrightarrow \alpha^{g}y \tag{2.39}$$
$$P_x \longrightarrow \alpha^{g'}P_x, \quad P_y \longrightarrow \alpha^{g'}P_y$$

where $g = 1/(m-1)$ and $g' = m/(m-1)$. For such a system Kowalevskaya determinant is given by

$$K(\rho)= (\rho+1)(\rho-g_H)\ [\ \rho^2 - \rho(2g+1) + 2(g+1)^2 + D_m\ ], \tag{2.40}$$

where

$$D_m = \nabla^2 V(k_1,k_2), \tag{2.41}$$

the Laplacian of V at $x=k_1$ and $y=k_2$ and $g_H = 2m/(m-1)$ is the weighted degree of the Hamiltonian. k_1 and k_2 are determined from the equations,

$$gg'k_1 = -\frac{\partial V}{\partial x}(k_1,k_2) \text{ and } gg'k_2 = -\frac{\partial V}{\partial y}(k_1,k_2) \tag{2.42}$$

Roots of the equation $K(\rho) = 0$ gives the KE. It can be shown that KEs are the same as resonances of the P-analysis when $p = q = g$.

Integrability coefficient for the system is given by

$$\lambda_m = \nabla^2 V(c_1, c_2) - (2m-1) , \qquad (2.43)$$

where $\nabla^2 V$ is Laplacian of V and c_1 and c_2 are solutions of

$$\frac{\partial V}{\partial x}(c_1, c_2) = c_1 , \qquad \frac{\partial V}{\partial y}(c_1, c_2) = c_2 \qquad (2.44)$$

Exponential instability occurs when

$$\lambda_m < 0; \quad 1 < \lambda_m < 2m-1; \quad 2m+2 < \lambda_m < 6m-2 ;\ldots\ldots\ldots\ldots$$
$$\ldots; \ j(j-1)m+j < \lambda_m < j(j+1)m-j \quad ;\ldots\ldots\ldots \qquad (2.45)$$

So the Hamiltonian system is non-integrable in the corresponding regions.

The relationship between the resonance of a Painlevé singularity and the KEs have been clarified by Roekaerts and Schwarz (1987). More detailed discussions are given by Yoshida et al (1987) and Ramani et al (1989). In general Painlevé leading singularities having no counter part in Yoshida's theorem might exist. Here we consider the case where they can be compared. The theorems of KE can be translated into theorems on resonances.

(i) If an integral of motion of weighted degree g_I exists such that its gradient is not zero at k_1, k_2, both nonzero, then there is a resonance $r=g_I$ associated with the corresponding Painlevé leading singularity with $p=q=g$.

(ii) If an integral of motion of weighted degree g_I exists such that its gradient is not zero on a solution (2.42) with $k_1=0$, $k_2 \neq 0$ or $k_1 \neq 0$, $k_2=0$, then there is resonance $r = 2(g_I - 1/(m-1) + 1)$ associated with the corresponding leading singularity with $p < q = 1/(m-1)$ or $q < p = 1/(m-1)$ respectively.

(iii) A necessary condition for the existence of an algebraic second invariant is that all resonances with Painlevé leading singularities with $p = 1/(m-1)$ or $q = 1/(m-1)$ are rational numbers.

The restrictions imposed on the resonances by the extended P-conjecture imply that (i) all KEs associated with solutions k_1, k_2, both nonzero ($p = q = g$) must be integral multiples of $1/(m-1)$, and (ii) all KEs associated with solutions k_1 and k_2 with $k_1 = 0$, $k_2 \neq 0$ ($k_1 \neq 0$, $k_2 = 0$) for $p < q = g$ ($q < p = g$) must be integral multiples of $1/2s$ where $s = 1/n(m-1)$ and n is a fixed integer specific to a particular Hamiltonian. In case (i) $\rho = r$ and in case (ii) $\rho = 1/(m-1) - (r-1)/2$.

2.2.5. Further restrictions

We now combine singularity analysis with stability analysis to obtain further restrictions on KEs. For homogeneous potentials it follows from (2.40) and the results of Roekaerts and Schwarz (1987) that the solutions of the equation

$$\rho^2 - \left(\frac{m+1}{m-1}\right)\rho + 2\left(\frac{m}{m-1}\right)^2 + D_m = 0 \tag{2.46}$$

must be integral multiples of $1/(m-1)$ in case (i) discussed above. Hence, for integrability, D_m must be given by

$$-D_m = [k(k-m-1) + 2m^2]/(m-1)^2 \tag{2.47}$$

where k is an integer. Comparing equations (2.42) and (2.44) and making use of equations (2.41) and (2.43) we find that λ_m

is directly related to D_m by

$$\lambda_m = -D_m/gg' - (2m-1) . \tag{2.48}$$

Consequently λ_m is also restricted to a set of discrete values

$$\lambda_m = k(k-m-1)/m + 1. \tag{2.49}$$

Expressing k modulo m by

$$k = n\,m + i \tag{2.50}$$

where n is an integer and $i = 0,1,2,\ldots\ldots,m-1$, we have

$$\lambda_m = j(j-1)m + j \tag{2.51}$$

for $n=j$ and $i=1$ $(k=jm+1)$

and $$\lambda_m = j(j+1)m - j \tag{2.52}$$

for $n=j+1$ $,i=0$ $(\ k=(j-1)m\ )$.

For $jm+1 < k < (j+1)m$,

$$j(j-1)m + j < \lambda_m < j(j+1)m - j \tag{2.53}$$

and hence is in the unstable region. It follows that for integrability k can assume only the values jm or jm+1 for arbitrary j. In other words apart from -1 and g_H the only values KEs in case (i), can assume are 0, 1 (mod m) in units of 1/(m-1). The integrability coefficient λ_m, then assumes only the values corresponding to boundaries separating stable and unstable regions.

In case (ii) the solution of equation (2.46) must be a multiple of 1/2s. Hence for integrability

$$-D_m = [k/2n\ (k/2n - m - 1\ ) + 2m^2]/(m-1)^2 \tag{2.54}$$

where k is an integer. Correspondingly the integrability coefficient is

$$\lambda_m = (k/2n)\ (\ k/2n - m-1\ ) + 1\ . \tag{2.55}$$

If $k = 2nj$ eq.(2.55) is formally the same as eq.(2.49) with $k \longrightarrow j$. By the repetition of the previous reasoning it will then follow that integrable cases correspond to $j=0, 1 \pmod{m}$. However, there can also exist other integrable cases with $k \neq 2nj$ depending on the values of n and m .

2.3. Integrable Potentials

We have performed Painlevé analysis and calculated KEs for symmetric quartic, sextic and octic potentials with a view to identifying possible integrable cases in the light of the above results. Direct construction of second integral of motion is also given in some cases. A generalisation of the integrable cases to potential of arbitrary degree 2m is also obtained.

2.3.1. Quartic potentials

Consider a system with Hamiltonian

$$H = \frac{1}{2}(P_x^2 + P_y^2) + A(x^4 + y^4) + B(x^3y + x\,y^3) + C\,x^2y^2,$$
$$A, B, C \neq 0 \qquad (2.55)$$

and equations of motion

$$\dot{x} = P_x \quad , \quad \dot{y} = P_y$$
$$\dot{P}_x = -[4Ax^3 + B(3x^2y + y^3) + 2Cxy^2] \qquad (2.56)$$
$$\dot{P}_y = -[4Ay^3 + B(x^3 + 3xy^2) + 2Cx^2y]$$

To perform Painlevé analysis we look for dominant behaviour near a singularity of the form (2.37,2.38). Substituting in (2.56) give $p = q = 1$ with $b_0 = \alpha\, a_0$, where α can assume one of the four possible values

$$\alpha_{1,2} = \pm 1 \quad , \qquad (2.57)$$
$$\alpha_{3,4} = \{(4A-2C) \pm [(4A-2C) - 4B]^{1/2}\}/2B.$$

Correspondingly

$$a_0^2 = -2/(4A + 3B\alpha + 2C\alpha^2 + B\alpha^3). \qquad (2.58)$$

Solutions of (2.56) can be expanded in the form

$$x(t) = \sum_{j=0}^{\infty} a_j \tau^{-1+j} \qquad (2.59)$$

$$y(t) = \sum_{j=0}^{\infty} b_j \tau^{-1+j} \qquad (2.60)$$

This is a strong P-case. Resonances are found to be $-1,1,2,4$ ($\alpha = \pm 1$) and sufficient arbitrary constants enter with the above type of solutions, when $C = 6A$, A and B arbitrary.

To calculate KEs and integrability coefficients we note that for the system (2.55), $g = 1$, $g' = 2$ and $m=2$. A solution of (2.42) is $k_2 = \alpha k_1$ (correspondingly $c_2 = \alpha c_1$ in eq.(2.44)) and

$$k_1^2 = - g\, g' c_1^2 = a_0^2 \qquad (2.61)$$

By the restrictions mentioned in section 2 KEs (in case(i) with k_1, k_2 both nonzero) can only be 1,2,3,...., that is, D_2 can have values $-6,-8,-12, -18,\ldots$ and corresponding values of λ_2 are 0, 1, 3, 6,..... for any choice of solutions. For the P-case, $C=6A$ (A and B arbitrary) KEs are $-1,1,2,4$ for $\alpha = \pm 1$ ($D_2 = -6$) and $-1,-1,4,4$, for ($D_2 = -12$) and the corresponding values of λ_2 are 0 and 3 respectively. Of the possible integrable cases corresponding to the allowed values of D_2, for the P-case, we have been able to construct the following second integral of motion directly from the Poisson bracket condition $[H,I] = 0$, assuming the weighted degree ≤ 4.

$$I = P_x P_y + B\,(x^4+y^4+ 6x^2y^2) + 4A\,(x^3y+xy^3) \qquad (2.62)$$

The special cases of the Hamiltonian (2.55) with $B=0$ has been discussed by Steeb et al (1985b).

2.3.2. Sextic potentials

Consider the Hamiltonian

$$H = \frac{1}{2}(P_x^2 + P_y^2)+A(x^6+y^6)+B(x^5y+xy^5)+C(x^4y^2+ x^2y^4)+Dx^3y^3,$$

$$A,B,C,D \neq 0 \qquad (2.63)$$

Equations of motion are

$$\dot{x} = P_x \ , \ \dot{y} = P_y$$

$$\dot{P}_x = -[6Ax^5+B(5x^4y + y^5)+C(4x^3y^2+ 2xy^4)+3Dx^2y^3] \quad (2.64)$$

$$\dot{P}_y = -[6Ay^5+B(x^5+ 5xy^4)+C(2x^4y +4x^2y^3)+3Dx^3y^2]$$

For this system, we have a singularity with dominant behaviour $p = q = 1/2$ and $b_0 = \alpha a_0$, where α is a root of the equation

$$B(\alpha^6-1)+(2C-6A)\ \alpha\ (\alpha^4 -1)+(3D-5B)\alpha^2(\alpha^2 -1) = 0 \quad (2.65)$$

$\alpha = \pm 1$ is a root of the equation. Correspondingly,

$$a_0^4 = -3/[4(6A+5B\alpha +B\alpha^5+2C\alpha^4+4C\alpha^2+3D\alpha^3)]$$

Solutions of (2.64) will be of the form

$$x(t) = \sum_{j=0}^{\infty} a_j\tau^{-1/2+j/2} \qquad (2.66)$$

$$y(t) = \sum_{j=0}^{\infty} b_j\tau^{-1/2+j/2} \qquad (2.67)$$

This is a weak P-case. The resonances are found to be -1, 1/2, 3/2, 3 (with $\alpha = \pm 1$) and sufficient number of arbitrary constants enter in the solution when C=15A and 10B = 3D, A and B arbitrary.

For the system (2.63) $g = 1/2$, $g' = 3/2$ and $m = 3$. $k_2 = \alpha k_1$ is a choice of solution of (2.42) (correspondingly $c_2 = \alpha c_1$ in eq.(2.44)) and $k_1^4 = -gg'c_1^4 = a_0^4$. In order that the system be integrable D_3 has to be -15/4, -18/4, -30/4, -39/4,.... and corresponding values of λ_3 are 0,1,5,8,.. for any choice. For C=15A and 10B=3D (A and B arbitrary) KEs are

$-1, 1/2, 3/2$, and 3 ($\alpha = \pm 1$) and $\lambda_3 = 0$.

Looking for an integral of motion with weighted degree ≤ 3 we find in the P-case

$$I = P_x P_y + B\,[x^6+y^6+15(x^4y^2+x^2y^4)] + A\,[6(x^5y + xy^5)+20x^3y^3] \tag{2.68}$$

Special cases of the Hamiltonian of the form (2.63) with B=D=0 have been discussed by Graham et al (1985).

2.3.3. Octic potentials

For the Hamiltonian

$$H = \frac{1}{2}(P_x^2+P_y^2) + A(x^8+y^8) + B(x^7y+xy^7) + C(x^6y^2+x^2y^6) + D(x^5y^3+x^3y^5) + Ex^4y^4 \tag{2.69}$$

It is found that C=28A, E=70A and D=7B, A and B arbitrary, is a P-case. For this system $g=1/3$, $g' = 4/3$ and $m=4$. D_4 can take values $-28/9, -32/9, -56/9, -76/9, \ldots$ and corresponding values of λ_4 are $0, 1, 7, 12, 22, \ldots$ for any choice of solutions. For the P-case we have a solution for which KEs are $-1, 1/3, 4/3$ and $8/3$ and $\lambda_4 = 0$ yielding an integrable case. We can also identify the following non-integrable cases.

(i) B=D=0 (except when (a) C=28A, E=70A (b) C=4A, E=6A and (c) C=E=0), (ii) A=B=C=D=0 , (iii) A=C=D=E=0 and (iv) A=B=C=E=0.

Searching for an integral of motion with weighted degree $\leq 8/3$ we have, when (a) C=28A, E=70A and D=7B

$$I = P_xP_y + A[8(x^7y+xy^7) + 56(x^5y^3+x^3y^5)] + B[x^8+y^8+28(x^6y^2+x^2y^6) + 70x^4y^4] \tag{2.70}$$

(b) C=4A, E=6A and B=D=0 $\quad I = P_xy - P_yx \quad$ (2.71)

and (c) B=C=D=E=0

2.3.4 Generalisation

We can generalise the integrable cases to arbitrary m ($m \geq 2$). The general form of an integrable symmetric Hamiltonian with homogeneous potential of degree 2m is

$$H = \frac{1}{2}(P_x^2 + P_y^2) + A\,V_m + B\,J_m \tag{2.73}$$

and its integral of motion with a weighted degree 2m/(m-1) is

$$I = P_x P_y + B\,V_m + A\,J_m \tag{2.74}$$

where

$$V_m = \sum_{j=0}^{m} \alpha_j\, x^{2m-2j}\, y^{2j} \tag{2.75}$$

$$\alpha_j = \begin{cases} \binom{2m}{2j} & \text{when } 2j \leq m \\ \binom{2m}{2m-2j} & \text{when } 2j > m \end{cases}$$

$$J_m = \sum_{j=0}^{m-1} \beta_j\, x^{2m-2j-1}\, y^{2j+1} \tag{2.76}$$

$$\alpha_j = \begin{cases} \binom{2m}{2j+1} & \text{when } 2j+1 \leq m \\ \binom{2m}{2m-(2j+1)} & \text{when } 2j+1 > m \end{cases}$$

Integrable cases (2.62),(2.68) and (2.70) are special cases of (2.74) for m= 2,3and 4 respectively.

2.4. Conclusion

In this chapter an attempt was made to combine singularity and stability analyses for a Hamiltonian system with a homogeneous potential. A new restriction on KEs, which may be used as an effective tool in the search for integrable systems, has been obtained. Applying this to symmetric quartic, sextic and octic potentials we have identified possible candidates for integrability. However it happens that the cases where we have been able to construct a second integral of motion directly are not genuinely new integrable systems. This is because potential of the integrable form (2.73) can be, by a rotation through an angle $\pi/4$ and scaling (Hietarinta 1987), reduced to known integrable potentials of the form

$$V = x^n + a\, y^n. \tag{2.77}$$

It is known that the general form of integrable symmetric potentials are $V = f(x^2 + y^2)$ and $V = f(x)+f(y)$ with integrals of motion $I = P_x y - P_y x$ and $I = P_x^2 + 2f(x)$ or $P_y^2 + 2f(y)$ respectively (Hietarinta 1987). Integrals of motion (2.71) and (2.72) are also special cases of these. The question whether these exhausts the integrable cases or there can exist an additional integral in the rest of the cases is yet to be answered completely.

CHAPTER 3

NON-INTEGRABILITY OF SU(2) YANG-MILLS AND YANG-MILLS-HIGGS SYSTEMS

3.1. Introduction

Recently the question of integrability of non-Abelian gauge fields has attracted wide attention (Chang 1984, Furusawa 1987, Ichtiaroglou 1989, Matinyan et al 1981a,b,1986,1988, Savvidy 1984, Villarroel 1988). It has been shown that chaos can appear in classical theory of non-Abelian gauge fields, at least under certain approximations. This is of particular significance in view of the result obtained by Olsen (1982) that the presence of random fields in the vacuum is a necessary and sufficient condition of quark confinement in Quantum Chromodynamics.

Most studies made so far have confined themselves to the finite dimensional subsystems depending only on time variable. Classical Yang-Mills theory depending only on time (YM Classical Mechanics) has been shown to be non-integrable and chaotic by various techniques (Matinyan et al 1981a, Nikolaevskii and Schur 1982,1983, Gorski 1984, Steeb et al 1986d, Karkowski 1990,1991). However, with regard to the general 3+1 field systems the situation is not fully understood. By the Painlevé criterion SU(2) self dual Yang-Mills equations have been shown to be integrable (Jimbo et al 1982, Ward 1984). But such analysis has not been carried out in the general case. On the other hand Matinyan et al (1986,1988) have shown that space time dependent spherically symmetric Yang-Mills system can exhibit dynamical chaos. They employed the Fermi-Pasta-Ulam (1955) method in which continuous equations are replaced by a set of discrete equations which are then numerically analysed. But it is well known that the discretisation itself can be a cause for chaotic behaviour. Also the continuum limit of a discrete model exhibiting chaos can be non chaotic.

In this chapter an attempt will be made to clarify the question of nonintegrability and space-time chaos in the spherically symmetric non self-dual-sector of SU(2) Yang-Mills and Yang-Mills-Higgs theory without introducing discretisation. We apply singular point analysis to test the integrability of the PDEs as well as of the ODEs obtained by symmetry reduction and by other means. Our results show that these systems are generally non-integrable.

The partial differential equations corresponding to the SU(2) theory and the ODEs obtained from them are described in section 3.2. A brief review of results regarding integrability and chaos of YM and YMH Classical Mechanics is also given. In section 3.3 we briefly describe the WTC algorithm for singular point analysis. The results are also presented in this section. Section 3.4 is a summary of results and conclusions. Expansions of recursion relations at various orders for spherically symmetric Yang-Mills (SSYM) and Yang-Mills-Higgs (SSYMH) systems are given in the Appendix.

3.2. Yang-Mills and Yang-Mills-Higgs systems

3.2.1. YMCM and SSYM

The SU(2) Yang-Mills system is described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F^{a}_{\mu\nu} F^{a}_{\mu\nu} \tag{3.1}$$

where
$$F^{a}_{\mu\nu} = \partial_{\mu} A^{a}_{\nu} - \partial_{\nu} A^{a}_{\mu} + g\varepsilon_{abc} A^{b}_{\mu} A^{c}_{\mu} \tag{3.2}$$

$$\mu\ ,\ \nu\ =\ 0,1,2,3 \quad ; \quad a,b,c = 1,2,3$$

The equations of motion have the form,

$$\partial_{\mu} F^{a}_{\mu\nu} + g\ \varepsilon_{abc} A^{b}_{\mu} F^{c}_{\mu\nu} = 0 \tag{3.3}$$

Let us look for a class of solutions of the system (3.3), for which the Poynting vector in some system vanishes (Baseyan et al 1979),

$$T_{0j} = F^{a}_{0i} F^{a}_{ji} = 0. \tag{3.4}$$

Here $T_{\mu\nu} = - F^{a}_{\mu\lambda} F^{a}_{\nu\lambda} + \frac{1}{4} g_{\mu\nu} F^{a^2}_{\lambda\rho}$ is the energy momentum tensor of the field. Choosing the gauge $A^{a}_{0} = 0$ the equation (3.3) and (3.4) reduce to the following set of equations,

$$\ddot{A}^{a}_{i} - F^{a}_{ji,j} + g\ \varepsilon_{abc} A^{b}_{j} F^{c}_{ji} = 0 \tag{3.5a}$$

$$N^{a} = \varepsilon_{abc} A^{b}_{i} \dot{A}^{c}_{i} = 0 \tag{3.5b}$$

and
$$\dot{A}^{a}_{i} F^{a}_{ij} = 0 \tag{3.6}$$

From (3.5b) and (3.6) we get,

$$\dot{A}^{a}_{i} \left(A^{a}_{j,i} - A^{a}_{i,j} \right) = 0 \quad . \tag{3.7}$$

Now let us consider the special case of spatially homogeneous Yang-Mills fields which satisfy (3.4) and depend only on the time coordinate. ie.,

$$A^{a}_{i} = A^{a}_{i}(t) \tag{3.8}$$

In this case the equations of motion take the form

$$\ddot{A}_i^a - g^2 A_j^a A_j^b A_i^b + g^2 A_i^a A_j^b A_j^b = 0 \qquad (3.9)$$

with constraint (3.5b). Then equations follow from the Hamiltonian,

$$H_{YM} = \sum_{a,i} \frac{1}{2} \dot{A}_i^a \dot{A}_i^a + \frac{g^2}{4} [(A_i^a A_i^a)^2 - (A_i^a A_j^a)^2] \qquad (3.10)$$

Hence for spatially homogeneous YM fields the equations (3.3) reduces to a nonlinear mechanical system (YMCM). YMCM has been extensively studied by various authors. Matinyan et al (1981a) investigated a simplified model with n=2 given by

$$H = (\dot{x}^2 + \dot{y}^2)/2 + x^2y^2/2 \qquad (3.11)$$

obtained by taking $A_1^1 = \frac{1}{g} x(t)$, $A_2^2 = \frac{1}{g} y(t)$ and $A_2^1 = A_1^2 = 0$. They have shown that it is chaotic and nonintegrable. Study of the periodic orbits of the system shows that they are unstable. Further studies by Chirikov and Shepelyansky (1981), Avakyan et al (1982), Nikolaevskii and Schur (1982,1983), Steeb and Kunick (1985), and Steeb and Louw (1986c) have confirmed the nonintegrability by alternative techniques such as Poincaré surface of sections, Lyapunov exponents and Painlevé analysis. Moreover Savvidy (1983) has shown that this system is in fact a K-system. The model corresponding to n=3 has also been shown to be nonintegrable (Steeb et al 1986a). Higher dimensional cases have been investigated by Asatryan and Savvidy (1983), Froyland (1983) and Karkowski (1990,1991) and proved to be nonintegrable and chaotic.

What happens when the YM fields are space-time dependent? For simplicity let us consider the time dependent spherically symmetric ansatz,

$$A_o^a = 0, \qquad A_i^a = \frac{1}{g} \varepsilon_{ain} \frac{r_n}{r^2} (1-K(r,t)) \qquad (3.12)$$

which reduces the equations of motion, $D_\mu F^{\mu\nu a} = 0$ to

$$r^2(K_{rr} - K_{tt}) + K(1-K^2) = 0 \; . \qquad (3.13)$$

This is named as SSYM system. The static solutions of (3.13) given by K = 0 is the Wu-Yang monopole solution, K = -1 is the vacuum solution and K = 1 is the gauge equivalent to vacuum one. These static solutions are all unstable (Wu and Yang 1969). All solutions except the trivial ones $K = \pm 1$ are non self-dual.

A Lie symmetry analysis for a system including a Higgs field was carried out by Babu Joseph and Baby (1985). From that we infer that (3.13) admits a similarity variable,

$$\rho = r/\,(t^2-r^2\,) \qquad (3.14)$$

and on substituting (3.14) in (3.13) we get corresponding similarity reduced ODE

$$\rho^2 \frac{d^2K}{d\rho^2} = K(K^2-1) \qquad (3.15)$$

The singularity analysis of this equation is of significance in view of the conjecture by Ablowitz et al (1980), that a system is integrable if the corresponding similarity reduced system of ODE possesses the PP. It is also known that using an independent variable transformation (Arodz 1983)

$$\varkappa = \frac{t-to}{r} - 1 \qquad (3.16)$$

the nonlinear partial differential equation (3.13) can be reduced to a second order nonlinear ordinary differential equation,

$$(2+\varkappa)\varkappa \frac{d^2K}{d\varkappa^2} + 2(1+\varkappa)\frac{dK}{d\varkappa} + K(1-K^2) = 0 \qquad (3.17)$$

The domain of $\varkappa$ is $-1 \le \varkappa < \infty$. A family of regular solutions in this domain was obtained (Arodz 1983) with the property

$$K \longrightarrow 0 \qquad \text{as} \quad \tau \longrightarrow \infty$$

$$0 < |K| < 1 \quad \text{as} \quad \tau \longrightarrow -1.$$

3.2.2. YMHCM and SSYMH

Another system which we analyse is the SU(2) Yang-Mills-Higgs system which can exhibit spontaneous symmetry breakdown depending on the vacuum expectation value of the Higgs scalars. For a YMH model with N scalar fields transforming according to a N-dimensional representation of SU(2) group the Lagrangian density is

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu}^{a} F^{\mu\nu a} + \frac{1}{2} D_{\mu}\phi^{A} D^{\mu}\phi^{A} - V(\phi) \tag{3.18}$$

where
$$F_{\mu\nu}^{a} = \partial_{\mu}A_{\nu}^{a} - \partial_{\nu}A_{\mu}^{a} + g\,\varepsilon_{abc}A_{\mu}^{b}\,A_{\mu}^{c} \tag{3.19}$$

$$D_{\mu}\phi_{A} = \partial_{\mu}\phi_{A} - i\,g\,T_{AB}^{a}\,A_{\mu}^{a}\phi^{B} \tag{3.20}$$

$$V(\phi) = \frac{\lambda}{4}\left(\phi^{2} - \frac{m^{2}}{\lambda}\right)^{2} \tag{3.21}$$

$a,b,c = 1,2,3$; $A,B = 1,\dots,N$. T_{AB}^{a} is an N-dimensional matrix representation of the infinitesimal generators of SU(2) and $\phi^{2} = \phi_{A}\phi_{A}$. The equations of motion following from (3.18) are

$$D_{\nu}F^{\mu\nu A} = ig\,T_{AB}^{a}\,\phi^{B}D_{\mu}\phi^{A} \tag{3.22}$$

$$D_{\mu}D^{\mu}\,\phi_{A} = (\,m^{2} - \lambda\,\phi^{2}\,)\phi_{A} \tag{3.23}$$

Restricting oneself to the case of spatially homogeneous fields in the gauge $A_{0}^{a} = 0$ the equations get reduced to ODEs describing a finite dimensional mechanical system (YMHCM), as in the case with YM theory discussed above. For Higgs fields in the doublet representation ($T^{a} = \tau^{a}$, Pauli matrices) these equations can be derived from the

Hamiltonian (Matinyan et al 1981b)

$$H = H_{YM} + \frac{1}{2}\left(\dot{\sigma}^2 + \dot{B}_a^2\right) + \frac{1}{4} g^2 \left(A_i^a A_i^a\right) \left[\frac{1}{2} B_a^2 + \left(\frac{\sigma}{\sqrt{2}} + \eta\right)^2 \right]$$

$$+\lambda^2 \left[\frac{1}{2} B_a^2 + \left(\frac{\sigma}{\sqrt{2}} + \eta \right)^2 - \eta^2 \right]^2 \qquad (3.24)$$

The constraint to be satisfied is

$$\varepsilon_{abc} A_i^b \dot{A}_i^c - \sqrt{\tfrac{1}{2}}\, \eta \dot{B}_a + \frac{1}{2} \left[\sigma \dot{B}_a - B_a \dot{\sigma} - \varepsilon_{abc} B_b \dot{B}_c \right] = 0$$

where η is the vacuum expectation value of the scalar field ϕ.

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} iB_1 + B_2 \\ \sqrt{2}\eta + \sigma - iB_3 \end{pmatrix} \qquad (3.25)$$

λ is the self interaction constant of the scalar field ϕ. Detailed studies have been made for the special case of a two component gauge field with $A_1^1 = x(t)$ and $A_2^2 = y(t)$ and all other components of A zero interacting with the Higgs vacuum $B_a = 0$, $\sigma = 0$. The Hamiltonian for this case is

$$H \equiv \mu^4 = (\dot{x}^2 + \dot{y}^2)/2 + g^2 x^2 y^2/2 + g^2 \eta^2 (x^2 + y^2)/4 \qquad (3.26)$$

where μ^4 is the constant value of the energy of the system. This system is characterised by one parameter

$$\pi = (g/2)^2 \left(\eta/\mu \right)^4 \qquad (3.27)$$

and analysis reveal that the system can be integrable or nonintegrable depending on π. At $\pi_c \approx 0.15$ a phase transition like behaviour occurs from ordered motion to highly chaotic motion. For higher values of π the system is close to an integrable one. The Higgs field appears to subdue the chaos of the original YM fields (Chirikov and Shepelyansky 1982, Berman et al 1985).

We shall now turn to the field theory of YMH system with the Higgs field in the adjoint representation. Using the time dependent spherically symmetric 't Hooft-Polyakov ansatz

(Mecklenberg and O'Brein 1978)

$$A_o^a = 0 \quad , \quad A_i^a = \frac{1}{g}\varepsilon_{ain}\, r_n \frac{1 - K(r,t)}{r^2}$$
$$\phi_a = \frac{1}{g} r_a \frac{H(r,t)}{r^2} \tag{3.28}$$

where $r_n = x_n$ and r is the radial variable, the field equations of the SU(2) gauge theory are reduced to the form denoted as SSYMH

$$r^2(K_{rr} - K_{tt}) = K\,(K^2 - 1 + H^2)$$
$$r^2(H_{rr} - H_{tt}) = H\,(2K^2 - m^2r^2 + \frac{\lambda}{g^2} H^2) \tag{3.29}$$

In the Prasad-Sommerfeld (PS) limit they become

$$r^2(K_{rr} - K_{tt}) = K\,(K^2 - 1 + H^2)$$
$$r^2(H_{rr} - H_{tt}) = 2HK^2 \tag{3.30}$$

By using the similarity variable in (3.14), equations in (3.30) can be further reduced to the system of ODEs,

$$\rho^2 \frac{d^2K}{d\rho^2} = K\,(K^2-1+H^2)$$
$$\rho^2 \frac{d^2H}{d\rho^2} = 2HK^2 \tag{3.31}$$

Using the independent variable transformation (3.16) the system (3.31) yields the ODEs,

$$(2+\varkappa)\varkappa \frac{d^2K}{d\varkappa^2} + 2(1+\varkappa)\frac{dK}{d\varkappa} = K(K^2-1+H^2)$$
$$(2+\varkappa)\varkappa \frac{d^2H}{d\varkappa^2} + 2(1+\varkappa)\frac{dH}{d\varkappa} = 2HK^2 \tag{3.32}$$

It is not known whether the transformation (3.16) is related to any symmetry invariance of the system or whether there are other ODEs which may be obtained from (3.13) and (3.30).

With the intention of studying the integrability of Yang-Mills and Yang-Mills-Higgs field theories we shall now carry out a singular point analysis of the PDEs (3.13) and (3.29) as well as the ODEs (3.15),(3.17),(3.31) and (3.32).

3.3. Singular point analysis and Integrability

3.3.1. WTC Algorithm for Painleve analysis of PDE.

In Chapter 2 we have seen that the Painlevé property is a useful criterion for identifying integrable systems. For testing PP an algorithmic procedure has been devised by Ablowitz, Ramani and Segur (ARS) (Ablowitz et al 1980). The original idea of ARS was to attribute integrability to PDEs when their reductions to ODE all have the PP. They conjectured that every ordinary differential equation obtained by an exact reduction of a nonlinear PDE solvable by the inverse scattering transform (IST) method has the PP. In practice this is not useful very much in testing the integrability of PDE because it may not be possible to find all the possible similarity reductions of it or the reductions may be too trivial. It has been observed that this is not a sufficient condition for integrability of a PDE (Clarkson 1986). Weiss, Tabor and Carnevale (1983) introduced the concept of PP directly for PDEs. According to them a PDE possesses the PP if its solutions are single valued about a singularity manifold. In the case of PDEs, the singularities of the solutions can not be isolated as that for ODEs, which are analytic functions of only one complex variable. If $f=f(z_1,....,z_n)$ is a meromorphic function of N complex variables (2n variables), the singularities of f occur along analytic manifolds of real dimension 2n-2 These manifolds are determined by conditions of the form

$$\phi\ (z_1,\ldots\ldots,z_n) = 0 \qquad (3.33)$$

where ϕ ia an analytic function of $(z_1,\ldots,z_n)$ in a neighbourhood of the manifold. Ward (1984) has pointed out that the manifold must not be a characteristic : $\phi_z \neq 0$. For PP the solution of a PDE u_i can be expanded as a generalised Laurent series about the manifold (3.33) in the form,

$$u_i=\phi^{\alpha_i}\sum_{j=0}^{\infty} u_{ij}\phi^j \qquad (3.34)$$

where $u_{io}\neq 0$, $u_{ij} = u_{ij}(z_1,z_2,\ldots,z_n)$ and $\phi = \phi(z_1,z_2,\ldots,z_n)$ are analytic functions of the independent variables $z_1,z_2,\ldots,z_n$, in a neighbourhood of the manifold (3.33) and α_i is a negative integer. (3.34) must admit arbitrary functions equal to the order of the PDE. To test this we have an algorithm similar to that for ODEs. As a generalisation the concept of WPP can be introduced by allowing α_i to be rational. Substitution of (3.34) in the PDE provides us with recursion relations for u_{ij}. Kruskal (Ramani et al 1989) suggested a simplified algorithm in which $\phi = z_1+z_2+\ldots\ldots+\psi(z_n)$. The procedure of the algorithm is analogous to that of ODEs and there are three steps in it. They are (i) finding the dominant behaviours, (ii) finding the resonance values and (iii) checking whether arbitrary constants enter at the resonances without the introduction of movable critical manifolds. In the first step we find all the possible values of α_i and u_{i0} in the expansion (3.34). For this we substitute the $j=0$ term of the series (3.34) in the PDE and find α_i values at which two or more terms balance, these being known as leading order terms (or dominant terms). From this we can find corresponding u_{i0} value also. To find the resonances we extract the coefficient $\tilde{Q}(j) = Q(j)\, u_{ij}$ of the term $\phi^{j+\alpha-N}$, where N is the order of the PDE, from the recursion relations for u_{ij}. Resonances are roots of the equation $Q(j)=0$. We find always -1 to be a resonance which corresponds to the arbitrariness of ϕ. To avoid any movable critical manifolds, we require that the remaining roots be nonnegative integers. Correspondence of resonances and Kowalevskaya exponents may be invoked here also (Steeb and Euler 1988). In the third step we test whether positive resonances do indeed correspond to the arbitrary constants of the solution (3.34) without logarithmic singularities. This is done by expanding the solution up to the largest resonance. At each resonance we come across certain conditions on the preceding u_{ij} and ϕ, known as compatibility conditions, which must be satisfied in order to ensure that the corresponding u_{ij} is indeed arbitrary. If the system passes all the three steps we say that it is a P-case. Note that the possibility of movable essential singularities are not excluded and hence this test provides only a necessary condition to have PP. The WTC method described here can also be applied to ODEs. In the case of ODEs if we put $\phi = t - t_o$ and u_{ij} constants, we have the usual ARS Painlevé test.

The interesting thing about the WTC approach is that the PP is directly connected to the linearisation properties, Lax pairs, Bäcklund transformations, Hirota bilinearisation and soliton solutions.

We obtain Bäcklund transformations (Weiss 1983,1989) by truncating the expansion (3.34) at the constant level term. That is we take

$$u_i = u_{i0}\phi^{-N} + u_{i1}\phi^{-N+1} + \ldots + u_{iN} \tag{3.35}$$

From the recursion relations for u_{ij}, we find an over determined system of equations for ϕ and $u_{ij}(j=0,1,\ldots,N)$, where u_{iN} will satisfy the original PDE. In many integrable PDEs by solving the overdetermined system, we obtain an equation satisfied by ϕ, involving Schwarzian derivative,

$$\{\phi,x\} = \frac{\partial}{\partial x}\left(\frac{\phi_{xx}}{\phi_x}\right) - \frac{1}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2 . \tag{3.36}$$

(3.36) is invariant under the Moebius group transformations

$$\phi = \frac{a\psi + b}{c\psi + d} \quad , \quad \{\phi,x\} = \{\psi,x\} . \tag{3.37}$$

This motivates the substitution $\phi = \frac{v_1}{v_2}$ by which Lax pairs may be found (Weiss 1983,1984).

It has been observed that there are connections with Hirota's bilinear transformation method to obtain N-soliton solutions and WTC approach (Gibbon and Tabor 1985, Hirota et al 1986).

In some systems arbitrary constants enter at the resonances, for some special choices of ϕ only. There will be consistency conditions to be satisfied by ϕ. Such systems are said to have conditional PP (Weiss 1984). Special solutions of such systems can be obtained using the truncated Laurent series expansion. Even if the system possesses neither PP nor conditional PP much useful information can be extracted from the WTC expansions (Newell et al 1987, Conte 1988, Cariello and Tabor 1989,1991).

3.3.2. Nonintegrability of SSYM and SSYMH systems

To investigate the integrability property of spherically symmetric YM (SSYM) fields we consider the system (3.13). We try to find solutions of the form

$$K = \phi^{\alpha}\sum_{j=0}^{\infty} u_j \phi^j \tag{3.38}$$

To obtain the leading order behaviour we put $K = u_o\phi^{\alpha}$. From that we get $\alpha = -1$ and $u_o^2 = 2r^2(\phi_r^2-\phi_t^2)$. The recursion relation is

$$r^2\Big[(j-1)(j-2)(\phi_r^2-\phi_t^2)u_j+(j-2)u_{j-1}(\phi_{rr}-\phi_{tt})+$$
$$2(j-2)(\phi_r u_{j-1,r}-\phi_t u_{j-1,t})+u_{j-2,rr}-u_{j-2,tt}\Big]$$
$$= \sum_{n=0}^{j}\sum_{s=0}^{n} u_{j-n}u_{n-s}u_s - u_{j-2} \tag{3.39}$$

Resonances are found to be -1 and 4. -1 corresponds to the arbitrariness of ϕ. Expansions of (3.39) upto j=4 are given in § 3.A.1 of Appendix. For the system to be integrable, at the resonance value 4 the expansion coefficient must be arbitrary. From the analysis of the recursion relations up to j=4 we find that u_4 is not arbitrary and therefore the system does not possess PP. The conclusion is that spherically symmetric time dependent Yang-Mills equations are non-integrable in the sense of WTC. To see whether it is integrable in the sense of ARS we shall do P-analysis of the ODE (3.15) and (3.17) obtained from (3.13). We find that even though resonances are rational, a sufficient number of arbitrary expansion coefficients does not exist and hence these systems are also nonintegrable.

Next we consider the spherically symmetric time dependent Yang-Mills-Higgs (SSYMH) system (3.29). We seek solutions of the form

$$K = \phi^{\alpha}\sum_{j=0}^{\infty} u_j\phi^j, \qquad H = \phi^{\beta}\sum_{j=0}^{\infty} v_j\phi^j \tag{3.40}$$

From the leading order analysis we obtain $\alpha = \beta = -1$,

$$u_o^2 = \left(1 - \frac{\lambda}{g^2} \right)v_o^2 \text{ and } \left(2 - \frac{\lambda}{g^2}\right)v_o^2 = 2r^2(\phi_r^2 - \phi_t^2) \tag{3.41}$$

Recursion relations for u_j and v_j are,

$$r^2\Big[(j-1)(j-2)(\phi_r^2-\phi_t^2)u_j+(j-2)u_{j-1}(\phi_{rr}-\phi_{tt})+ 2(j-2)(\phi_r u_{j-1,r}-\phi_t u_{j-1,t})+u_{j-2,rr}-u_{j-2,tt}\Big]$$

$$= \sum_{n=0}^{j}\sum_{s=0}^{n} u_{j-n}\left(u_{n-s}u_s + v_{n-s}v_s\right) - u_{j-2} \tag{3.42}$$

$$r^2\Big[(j-1)(j-2)(\phi_r^2-\phi_t^2)v_j+(j-2)v_{j-1}(\phi_{rr}-\phi_{tt})+ 2(j-2)(\phi_r v_{j-1,r}-\phi_t v_{j-1,t})+v_{j-2,rr}-v_{j-2,tt}\Big]$$

$$= \sum_{n=0}^{j}\sum_{s=0}^{n} v_{j-n}\left(2u_{n-s}u_s + \frac{\lambda}{g^2}v_{n-s}v_s\right) - m^2r^2v_{j-2} \tag{3.43}$$

Resonances are found to be real if $-\frac{2}{7} \leq \frac{\lambda}{g^2} \leq 2$. The resonances are found to be integers when $\frac{\lambda}{g^2} = 0$ or 1. But $\frac{\lambda}{g^2} = 1$ is not allowed by the assumption that $u_o \neq 0$, $v_o \neq 0$. The resonance values for $\frac{\lambda}{g^2} = 0$ are $-1,1,2$ and 4. Expansion of (3.42 and 3.43) upto $j=4$ are given in § 3.A.2. of Appendix. Arbitrary expansion coefficients do not exist at the resonance values. Hence the system is nonintegrable. When $\frac{\lambda}{g^2} = 0$ the leading order terms of the system (3.29) are equal to its PS limit (3.30). It is also non-Painlevé type and hence nonintegrable. The similarity reduced system (3.31) and (3.32) of the PS limit are also found to be nonintegrable by the same analysis.

It may be mentioned that the ARS method is not suitable for equations (3.15) and (3.31) but can be applied to (3.17) and (3.32) after they are converted to corresponding autonomous systems. In these cases we find that the resonances, which also happen to be the KE, are irrational and hence these systems are also algebraically nonintegrable in the sense of Yoshida (1983).

3.4. Conclusion

In this work we showed that spherically symmetric time dependent Yang-Mills equations as well as Yang-Mills-Higgs equations do not possess PP in the sense of WTC and the ODEs obtained from them are algebraically nonintegrable. These conclusions are in general agreement with those obtained by Matinyan et al (1986,1988) and by Furusawa (1987) for SU(2) Yang-Mills system. The noteworthy part is that we have been able to arrive at these results without introducing discretisation at any stage.

3.A. Appendix

3.A.1. Expansion of recursion relation for SSYM (3.39).

$j = 0 : 2r^2(\phi_r^2 - \phi_t^2) = u_0^2$

$j = 1 : r^2[-u_0(\phi_{rr} - \phi_{tt}) - 2(\phi_r u_{0,r} - \phi_t u_{0,t})] = 3u_1 u_0^2$

$j = 2 : r^2(u_{0,rr} - u_{0,tt}) = 3u_2 u_0^2 + 3u_1^2 u_0 - u_0$

$$j = 3 : r^2[u_2(\phi_{rr} - \phi_{tt}) + 2(\phi_r u_{2,r} - \phi_t u_{2,t}) + u_{1,rr} - u_{1,tt}] = 2u_3 u_0^2 + 6u_2 u_1 u_0 + u_1^3 - u_1$$

$$j = 4 : r^2[2u_3(\phi_{rr} - \phi_{tt}) + 4(\phi_r u_{3,r} - \phi_t u_{3,t}) + u_{2,rr} - u_{2,tt}] = 6u_3 u_1 u_0 + 3u_2^2 u_0 + 3u_2 u_1^2 - u_2$$

3.A.2. Expansion of recursion relation for SSYMH (3.42,3.43).

$$j = 0 : 2r^2(\phi_r^2 - \phi_t^2) = u_0^2 + v_0^2$$

$$2r^2(\phi_r^2 - \phi_t^2) = 2u_0^2 + (\lambda/g^2)v_0^2$$

$$j = 1 : r^2[-u_0(\phi_{rr} - \phi_{tt}) - 2(\phi_r u_{0,r} - \phi_t u_{0,t})] = 3u_1 u_0^2 + u_1 v_0^2 + 2u_0 v_0 v_1$$

$$r^2[-v_0(\phi_{rr} - \phi_{tt}) - 2(\phi_r v_{0,r} - \phi_t v_{0,t})] = 2v_1 u_0^2 + 4u_0 v_0 u_1 + 3(\lambda/g^2)v_1 v_0^2$$

$$j = 2 : r^2(u_{0,rr} - u_{0,tt}) = 3u_2 u_0^2 + 3u_1^2 u_0 + u_2 v_0^2 + 2u_1 v_1 + 2u_0 v_0 v_2 + u_0 v_1^2 - u_0$$

$$r^2(v_{0,rr} - v_{0,tt}) = 2v_2 u_0^2 + 4u_1 v_1 + 4u_0 v_0 u_2 + 2v_0 u_1^2 + 3(\lambda/g^2)(v_2 v_0^2 + v_1^2 v_0) - m^2 r^2 v_0$$

$$j = 3 : r^2[u_2(\phi_{rr} - \phi_{tt}) + 2(\phi_r u_{2,r} - \phi_t u_{2,t}) + u_{1,rr} - u_{1,tt}]$$
$$= 2u_3 u_0^2 + 6u_2 u_1 u_0 + u_1^3 + 2u_0 v_0 u_3 + 2u_2 v_1 v_0 + u_1 v_1^2 + 2u_0 v_2 v_1 + 2u_1 v_2 v_0 - u_1$$

$$r^2[v_2(\phi_{rr} - \phi_{tt}) + 2(\phi_r v_{2,r} - \phi_t v_{2,t}) + v_{1,rr} - v_{1,tt}]$$
$$= 2(2u_0 v_0 v_3 + 2v_2 u_1 u_0 + v_1 u_1^2 + 2v_0 u_2 u_1 + 2v_1 u_2 u_0) + (\lambda/g^2)(2v_3 v_0^2 + 6v_2 v_1 v_0 + v_1^3) - m^2 r^2 v_1$$

$$j = 4 : r^2[4(\phi_r^2 - \phi_t^2)u_4 + 2u_3(\phi_{rr} - \phi_{tt}) + 4(\phi_r u_{3,r} - \phi_t u_{3,t}) + u_{2,rr} - u_{2,tt}] = 2u_0^2 u_4 + 6u_3 u_1 u_0 + 3u_2^2 u_0 + 3u_2 u_1^2$$
$$+ 2u_3 v_1 v_0 + 2u_2 v_2 v_0 + 2u_1 v_3 v_0 + 2u_1 v_1 v_2 + 2u_0 v_0 v_4 + 2u_0 v_1 v_3 + u_2 v_1^2 + u_0 v_2^2 - u_2$$

$$r^2[4(\phi_r^2 - \phi_t^2)v_4 + 2v_3(\phi_{rr} - \phi_{tt}) + 4(\phi_r v_{3,r} - \phi_t v_{3,t}) + v_{2,rr} - v_{2,tt}] = 2(2v_3 u_1 u_0 + 2v_2 u_2 u_0 + 2v_1 u_3 u_0$$
$$+ 2v_1 u_1 u_2 + 2v_0 u_0 u_4 + 2v_0 u_1 u_3 + v_2 u_1^2 + v_0 u_2^2) + (\lambda/g^2)(2v_0^2 v_4 + 6v_3 v_1 v_0 + 3v_2^2 v_0 + 3v_2 v_1^2) - m^2 r^2 v_2.$$

CHAPTER 4

CHAOS AND CURVATURE IN A QUARTIC HAMILTONIAN SYSTEM

4.1. Introduction

The properties of a system that are responsible for the regular or chaotic behaviour are not known clearly. We have already seen that properties such as singularity structure of the solutions and stability of particular solutions have a definite role in the dynamics of the system. It has been shown that symmetry of the potential contours is related to chaotic behaviour (Ankiewicz and Pask 1984). Chaos is also related to the Riemannian curvature of the manifold in which the Hamiltonian flow can be considered as a geodesic flow (Arnold and Avez 1968). Negative curvature implies chaos. Implication of positive curvature is not clear.

In this chapter we study a quartic Hamiltonian system with two degrees of freedom and explore the connection between the chaotic behaviour and the Riemannian curvature. We find that there is a direct link between the chaotic behaviour as measured by Lyapunov exponents and the negative curvature of the potential boundary which is not considered in the Riemannian curvature calculation. In section 2 we briefly describe the system under study and in section 3 we give details of the calculation of Lyapunov exponents. An account of the relation between dynamics and Riemannian geometry and Riemannian curvature of the associated manifold is given in section 4. In section 5 potential boundary curvature is calculated and its connection with the LEs established. Section 6 contains our conclusions.

4.2. The Hamiltonian system

We shall study a system whose Hamiltonian is given by

$$H = 1/2\ (\ p_1^2 + p_2^2\ ) + V(q)\ , \tag{4.1}$$

where $$V(q) = \frac{(\ 1-\alpha\ )}{12}\ (\ q_1^4 + q_2^4\ ) + 1/2\ q_1^2 q_2^2 \tag{4.2}$$

and α is a parameter, $0 \le \alpha \le 1$. Potential V(q) for different α values are plotted in figure 4.1, for V = 1. Chaos of this system has been studied in detail by Carnegie and Percival (1984) using the techniques of Poincaré surface of section and by studying the properties of periodic orbits. The system has got $\pi/4$ symmetry. At $\alpha=0$ it is integrable and the corresponding second integral of motion is given by

$$I = 3p_1p_2 + q_1q_2(\ q_1^2+q_2^2\ )\quad . \tag{4.3}$$

Phase space motion is regular and all trajectories lie on an invariant tori. As α increases regular regions break up and irregular regions appear. When $\alpha=1$ system become highly chaotic and has shown to be equivalent to a K-system by Savvidy (1983). It has been used as a simplified model of the spatially homogeneous classical Yang-Mills field (Savvidy 1984).

The system (4.1) is scale invariant and we can study the chaotic behaviour at a fixed value of energy H=E. By scaling we may obtain the behaviour at any other energy value. Using singular point analysis Steeb et al (1986b) have shown that the system is nonintegrable except when $\alpha=0$. The resonances

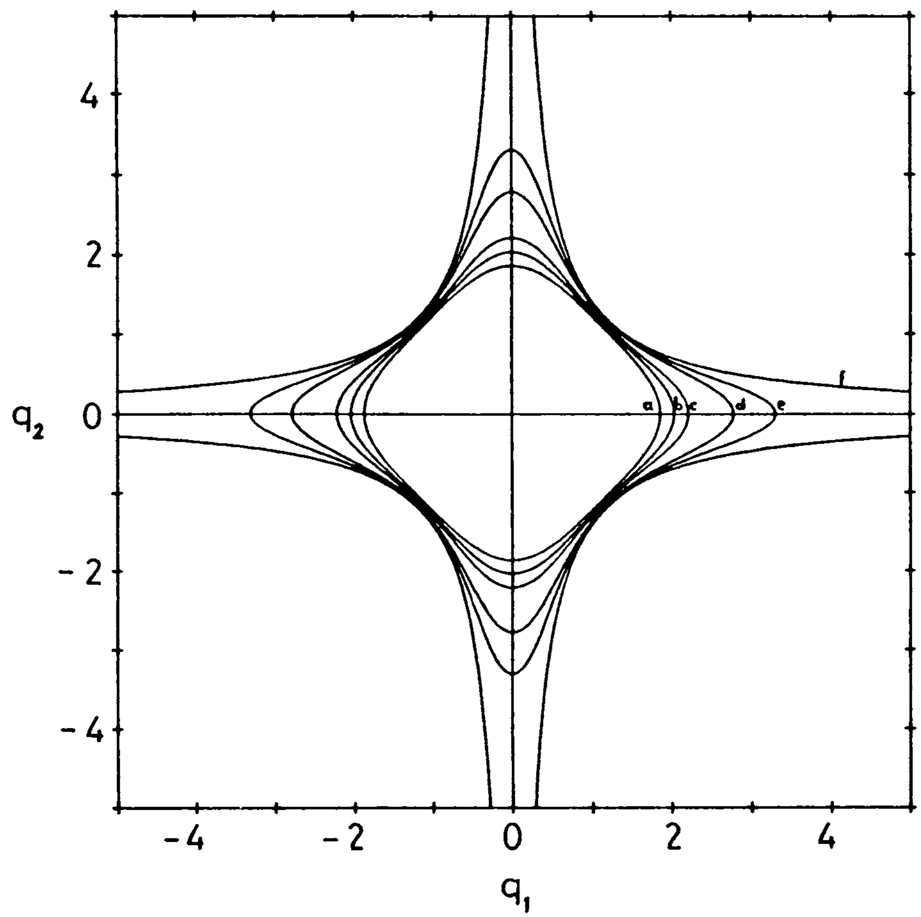

Figure 4.1. Curve of potential V = 1 for α values equal to (a) 0.0, (b) 0.3, (c) 0.5, (d) 0.8, (e) 0.9 and (f) 1.0.

or Kowalevskaya exponents are $r_1 = -1$, $r_2 = 4$, $r_{3,4} = 3/2 \pm (9/4 + 4(\alpha+2)/(\alpha-4))^{1/2}$. When $\alpha > 4/25$ resonances are complex. Integrability of such systems in general have been studied by Joy and Sabir (1988) using singular point analysis and stability analysis. Quantum chaos of this system has been studied by Steeb and Louw (1986a) and Kotze (1988). Studies on the effect of quantum fluctuations on this system by calculating the Gaussian effective potential is given in Chapter 6.

4.3. Lyapunov exponents

We shall investigate the possibility of chaotic behaviour of the system by computing the maximal Lyapunov exponent (LE). Lyapunov exponents provide a quantitative measure of the degree of chaos for both Hamiltonian and dissipative dynamical systems. It is easily computable and is a reliable quantity to characterise a chaotic system. Another important aspect is that it is related to other measures of chaos such as Kolmogorov entropy and capacity dimension. LE of a given trajectory characterise the mean exponential rate of divergence of nearby trajectories.

Consider an autonomous first order system

$$\dot{x}_i = F_i(x) \quad , \quad i = 1,\dots n, \tag{4.4}$$

where n is the dimension of the system. Consider a trajectory in the n-dimensional phase space and a nearby trajectory with initial conditions x_0 and $x_0 + \Delta x_0$, respectively. Time evolution of the variation $y = \Delta x$ is given by the linearised variational equations

$$\dot{y}_i = \sum_j (\partial F_i/\partial x_j)\, y_j,\ i,j=1,\ldots,n. \tag{4.5}$$

Mean exponential rate of divergence of two initially close trajectories is given by

$$\lambda(x_0,y_0) = \lim_{\substack{t\to\infty \\ d(0)\to 0}} \frac{1}{t} \ln \frac{d(x_0,t)}{d(x_0,0)}\ , \tag{4.6}$$

where $d(x_0,t) \equiv d(t) = |y(x_0,t)|$ is the Euclidean norm of y. It has been shown that λ exists and is finite. There is an n-dimensional basis $\{\hat{e}_i\}$ of y such that for any y, λ takes on one of the n values $\lambda_i(x_0) = \lambda(x_0,\hat{e}_i)$ which are the Lyapunov characteristic exponents (LE) of order one. They can be ordered by size, $\lambda_1 \geq \lambda_2 \geq \ldots\ldots\ldots \geq \lambda_n$. For almost all y, $\lambda = \lambda_1$ (Contopoulos et al 1978). λ_1 is known as the maximal LE.

One of the LEs will be always zero because along the direction of the flow y grows only linearly with time. In the case of a Hamiltonian system 1-dimensional LE are symmetric about zero. $\lambda_i = -\lambda_{2N-i+1}$, where $2N = n$, N the number of degrees of freedom. Therefore here at least two LEs are zero. Sum of LEs will be zero. Hence for a chaotic Hamiltonian system with two degrees of freedom there will be only one positive LE.

Higher order LE can be defined by generalising the concept to describe the mean exponential divergence rate of a p-dimensional volume in the tangent space $p \leq n$. Using the wedge operator notation

$$V_p = y_1 \wedge y_2 \wedge \ldots \wedge y_p$$

for the volume V_p of a p-dimensional parallelepiped whose edges are the vectors $y_1, y_2, \ldots, y_p$. Then,

$$\lambda^{(p)}(x_0, V_p) = \lim_{t\to\infty} \frac{1}{t} \ln \frac{|V_p(x_0,t)|}{|V_p(x_0,0)|} \tag{4.7}$$

defines a LE of order p. λ is given as the sum of p LEs of order 1.

$$\lambda^{(p)} = \lambda_1^{(p)} = \lambda_1 + \lambda_2 + \ldots\ldots + \lambda_p,$$

for almost all V_ps.

For a Hamiltonian system sum of LEs $\sum_{i=1}^{n} \lambda_i(x_0) = 0$, while for a dissipative system, it is negative. Pesin has obtained a relation between KS entropy and LE.

$$h_k = \int_{\mathcal{M}} [\sum_{\lambda_i>0} \lambda_i(x)] \, d\mu,$$

where the sum is over all positive LE and the integral is over a specified region of phase space. For a two degree of freedom Hamiltonian system only λ_1 is greater than zero and $h_k = \lambda_1$ when we consider only the connected chaotic regions. When $\lambda_1 > 0$ the system is said to be chaotic. In the numerical calculation we obtain the maximal LE λ_1, if we take the initial variations at random. To calculate λ_1 we chose an initial y_0 and then integrate the system (4.5) for y alongwith (4.4) for x. From that we obtain the quantity $d(t)=|y(t)|$ where for convenience d_0 is usually chose to be unity. If the system is chaotic d increases exponentially with t and this will lead to overflow and other numerical errors. To avoid this we chose a small time interval τ and normalise y to a norm of unity at every interval τ. Thus we iteratively compute

$$d_k = |\, y_{k-1}(\tau)|, \quad y_k(0) = y_{k-1}(\tau)/d_k. \tag{4.8}$$

$y_k(\tau)$ is obtained by integrating (4.5) with initial value

$y_k(0)$ along the trajectory from $x(k\tau)$ to $x((k+1)\tau)$. Now we define LE as

$$\lambda_1 = \frac{1}{n\tau} \sum_{i=1}^{n} \ln d_i. \qquad (4.9)$$

For small τ and large n, the above definition is valid and λ_1 is independent of the choice of τ. In a connected chaotic region λ_1 is independent of x also.

Details of calculating higher order LE are described by Nagashima and Shimada (1979), Benettin et al (1980) and Wolf et al (1985)

Equations of motion of our system (4.1) can be written as,

$$\begin{aligned} \dot{x}_1 &= x_3 \\ \dot{x}_2 &= x_4 \\ \dot{x}_3 &= (\alpha-1)x_1^3/3-x_1x_2^2 \\ \dot{x}_4 &= (\alpha-1)x_2^3/3-x_1^2x_2 \end{aligned} \qquad (4.10)$$

Corresponding variational system is given by,

$$\begin{aligned} \dot{y}_1 &= y_3 \\ \dot{y}_2 &= y_4 \\ \dot{y}_3 &= ((\alpha-1)x_1^2-x_2^2)y_1-2x_1x_2y_2 \\ \dot{y}_4 &= ((\alpha-1)x_2^2-x_1^2)y_2-2x_1x_2y_1 \end{aligned} \qquad (4.11)$$

We numerically solve the system (4.10) and (4.11) together. LE is calculated for different values of the parameter α, with different sets of initial variations. In figure 4.2 maximal LE vs α is plotted. We take the energy E=1 for our calculations. As α increases, one can see from the value of LE that the chaos in the system also increases.

4.4. Riemannian curvature

Any Hamiltonian flow can locally be considered as a geodesic

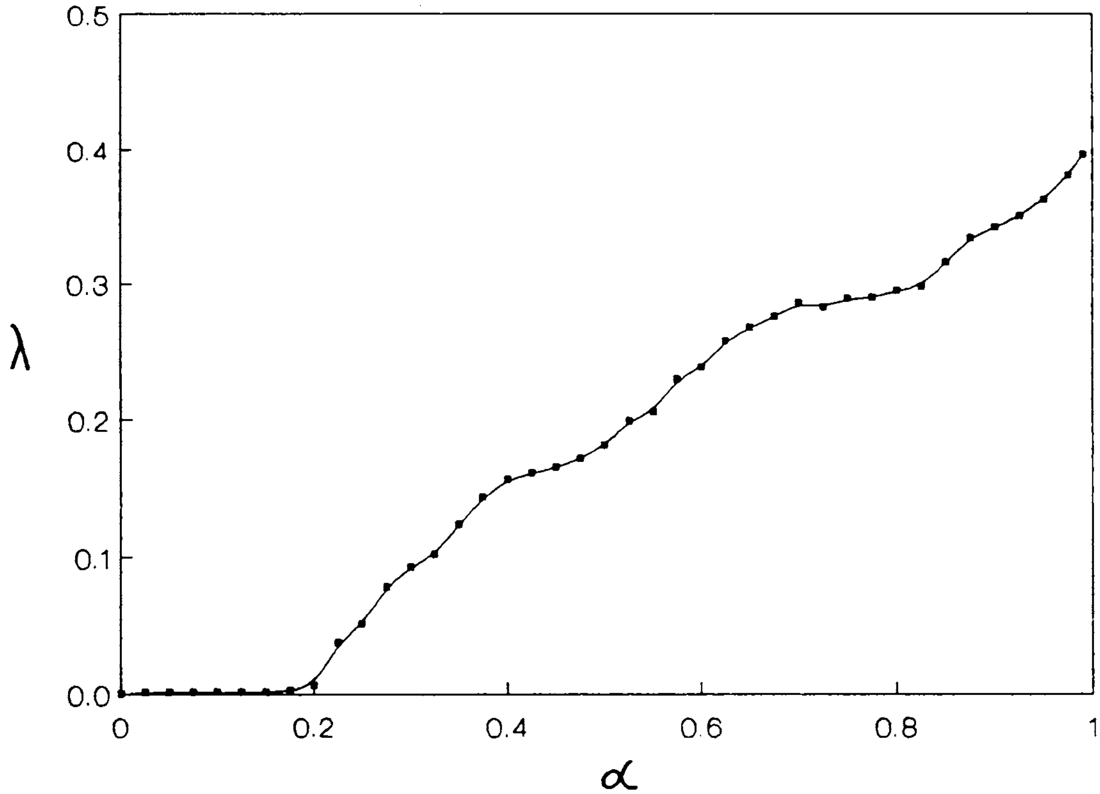

Figure 4.2. Plot of maximal LE (λ) vs α.

flow on a Riemannian manifold (Arnold and Avez 1968). Consider a system with Hamiltonian of the form

$$H(p,q) = 1/2 \sum_{i,j=1}^{N} a_{ij}(q)\, \dot{q}_i \dot{q}_j + V(q) \qquad (4.12)$$

The solutions $q_i(t)$ are extremes of Euler-Maupertuis principle,

$$\delta \int_{M_0}^{M_1} (2\{E-V(q)\} \sum_{i,j=1}^{N} a_{ij}(q) dq_i dq_j)^{1/2} = 0 \qquad (4.13)$$

where M_0 and M_1 are the end points of the trajectory. This may be considered as the variational equation for geodesics in a Riemannian space with a line element

$$ds^2 = \sum_{i,j=1}^{N} g_{ij} dq_i dq_j \quad , \qquad (4.14)$$

and metric coefficients

$$g_{ij} = 2\{E-V(q)\} a_{ij}(q) \qquad (4.15)$$

Evolution of the separation ρ between the nearby geodesics obey (to the lowest order in ρ) the Jacobi equation

$$D^2\rho/dt^2 = - K(q,t)\, \rho \quad , \qquad (4.16)$$

where D/dt is the covariant derivative in the Riemannian geometry defined by the metric g_{ij}. If we restrict ourselves to initial separations perpendicular to an orbit, the covariant derivative can be replaced by ordinary one and we can write the Jacobi equation (4.16) as

$$d^2\rho/dt^2 = - K(q,t)\, \rho \quad , \qquad (4.17)$$

where K(q,t) is the Riemannian curvature calculated along the orbit.

When $E-V = 1/2 \sum P_i^2$, so that $a_{ij} = \delta_{ij}$, Riemannian curvature K(q,t) is given by (Van Velsen 1978),

$$K = (N-1)/8(E-V)^3 \{ 2 \,\mathrm{Tr}\, (\mu_{ij}) - N\, V_m V_m \} \qquad (4.18)$$

where $\mu_{ij} = 3V_i V_j + 2(E-V)V_{ij}$,

$V_i = \partial V/\partial q^i$ and $V_{ij} = \partial^2 V/\partial q^i \partial q^j$.

In two dimensions K is same as Gaussian curvature. Sign of K indicates the stability of the orbit. Positive curvature implies local stability, whereas negative curvature means instability. Hadamard-Lobachevsky theorem suggests that if the Riemannian curvature is negative, the system behaves chaotically; there is exponential divergence of nearby trajectories (Arnold and Avez 1968). Surfaces of constant negative curvature are chaotic. It may be noted that while negative R-curvature everywhere is a sufficient condition for chaotic behaviour the converse is not true. Positive curvature does not mean that the system is integrable. Local instability everywhere implies global instability but local stability everywhere does not imply global stability (Eckhardt et al 1985). Association of the system with the geodesic flow on R-manifold is not valid at the boundary of the manifold, ie., at E = V, where the metric tensor becomes singular.

In 2 dimensions R-curvature (Gaussian curvature) is given by,

$$K = 1/2\ (E-V)^2\ \{\ \partial^2 V/\partial q_1^2 + \partial^2 V/\partial q_2^2 + 1/(E-V)\ [\ (\partial V/\partial q_1)^2 + (\partial V/\partial q_2)^2]\}\ ,\ E > V. \quad (4.19)$$

For the system (4.1),

$$K = 1/2(E-V)^2\{(2-\alpha)(q_1^2+q_2^2) + 1/(E-V)[\ (\partial V/\partial q_1)^2 + (\partial V/\partial q_2)^2]\} \quad (4.20)$$

One can see from (4.20) that K is always positive implying local stability. But we know that the system is nonintegrable except for $\alpha = 0$.

4.5. Potential boundary

In the Riemannian curvature calculation we did not include the potential boundary given by E = V. Now let us consider the potential boundary given by

$$\frac{(1-\alpha)}{12}(q_1^4 + q_2^4) + 1/2\, q_1^2 q_2^2 = E \qquad (4.21)$$

Extrinsic curvature of the curve (4.21) is given by

$$R = \frac{4E}{3} \frac{[(1-\alpha)^3-3]q_1^2q_2^2 + (1-\alpha)(q_1^4+q_2^4)}{[(1-\alpha)^2/8(q_1^6+q_2^6) + (5-2\alpha)/3(q_1^2+q_2^2)q_1^2q_2^2\;]^{3/2}} \qquad (4.22)$$

When $\alpha = 0$, R is positive for all values of q_1 and q_2. R is negative in between the points of the boundary (q_1, q_2) and (q_1', q_2'). Because of symmetry we consider only the first quadrant.

$$q_1 = \frac{E^{1/4}}{[(1-\alpha)/12(1+p^4)+1/2p^2]^{1/4}} \qquad (4.23)$$

$$q_2 = p\, q_1$$

$$q_1' = p\, q_2' \qquad (4.24)$$

$$q_2' = \frac{E^{1/4}}{[(1-\alpha)/12(1+p^4)+1/2p^2]^{1/4}}$$

where, $p^2 = \{3-(1-\alpha)^2 - \sqrt{[(1-\alpha)^2-3]^2-4(1-\alpha)^2}\;\}/2(1-\alpha)$ (4.25)

When $q_1 = q_2$, R is the maximum and it is given by

$$R_m = \frac{-\alpha}{(4-\alpha)^{3/4}}\left(\frac{3}{2E}\right)^{1/4} \qquad (4.26)$$

In figure 4.3 $|R_m|$ versus α is plotted for energy E = 1.

Comparing figures 4.2 and 4.3 we can see that the chaos in the system is directly correlated to the negative curvature of the potential boundary.

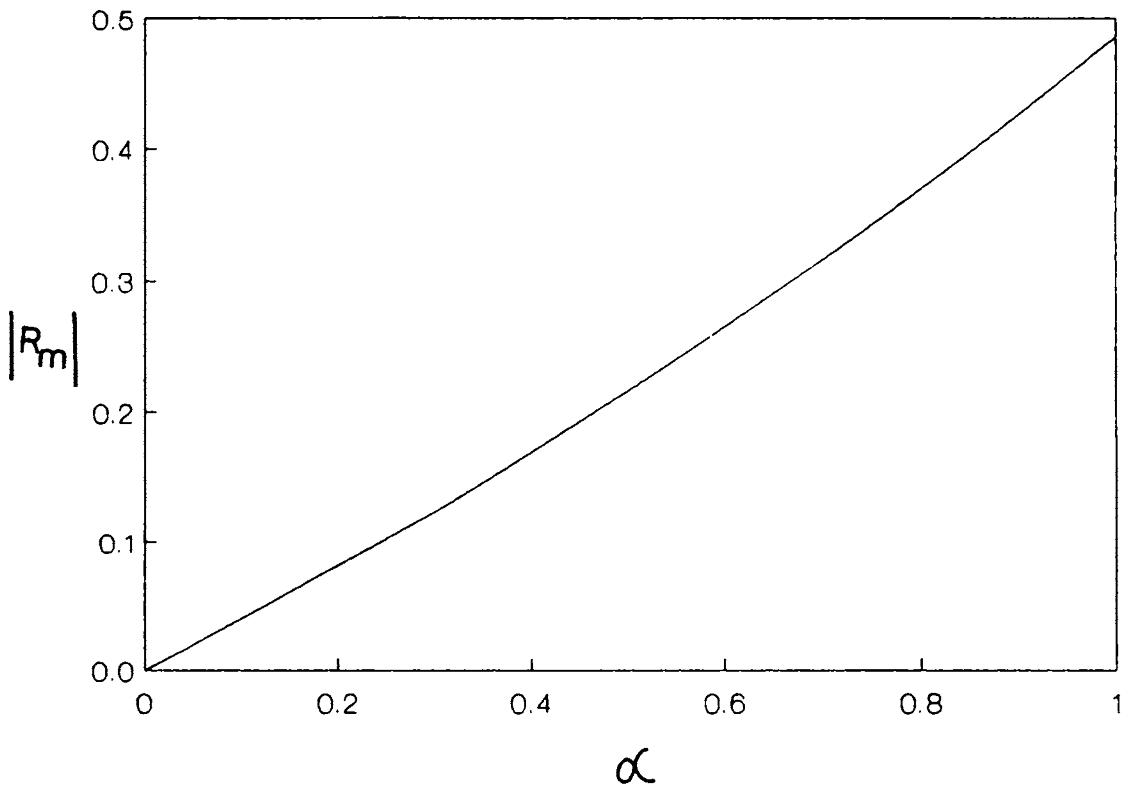

Figure 4.3. Plot of maximum curvature $|R_m|$ vs α.

4.6. Conclusion

In this chapter we have presented a simple model in which there is connection between chaos and the curvature of the Riemannian manifold in which the evolution can be considered as a geodesic flow. Negative curvature implies chaos but positive curvature does not give rise to integrability. A chaotic quartic system is investigated which has strictly positive curvature. We calculate the LE and show that these are directly correlated with negative curvature of the potential boundary. As the negative boundary curvature increases chaos also increases in the system. Exponential instability of trajectories occurs by scattering at the negatively curved potential boundary. Such systems may be considered as billiards with boundary as the potential boundary. The connection between billiards and Hamiltonian systems have been observed in some particular cases (Savvidy 1984, Kawabe and Ohta 1989). Further investigations are necessary to establish general connections.

CHAPTER 5

CHAOTIC BEHAVIOUR IN YANG-MILLS-HIGGS SYSTEM

5.1. Introduction

Recently much interest has been focused on the question of non-integrability and chaos in classical non-Abelian gauge theories. As we have seen in Chapter 3 spatially homogeneous Yang-Mills system (YMCM) is non-integrable and shows strong chaotic properties in general. This has been established by many authors using various analytical and numerical techniques. Studies on the more important and more realistic space-time dependent systems are however much less in number. Studies on such non-Abelian field theoretic systems are of relevance in understanding quark confinement in QCD, monopole stability, etc. Study of spatio-temporal chaos in itself is also very interesting. Matinyan et al (1986,1988) showed that space-time dependent Yang-Mills system can also exhibit dynamical chaos. They studied time-dependent spherically symmetric solutions of SU(2) Yang-Mills system, in particular the Wu-Yang monopole solution. Exponential instability of trajectories was found using Fermi-Pasta-Ulam technique of studying the distribution of energy among different harmonic modes. Kawabe and Ohta (1990) studied the system further by calculating the induction period, the equal time correlation and the maximal Lyapunov exponents and showed the existence of chaos in the YM system. Using the technique of Painlevè analysis we (Joy and Sabir 1989) have recently shown that time-dependent spherically symmetric SU(2) Yang-Mills and Yang-Mills-Higgs systems are non-integrable. (See chapter 3.)

Chaotic behaviour of classical systems with spontaneous symmetry breaking is also very interesting and investigations on such systems were made by Matinyan et al (1981b). They found an order to chaos transition in spatially uniform Yang-Mills system with Higgs scalar fields (YMHCM), as the vacuum expectation value of Higgs field is changed. Recently Matinyan et al (1989) performed some preliminary numerical calculations on time dependent spherically symmetric SU(2) Yang-Mills-Higgs system (SSYMH) and showed that there can be chaos. Details of chaotic behaviour of SSYMH is unclear and whether there is an order to chaos transition similar to YMHCM is an open question.

In this chapter we present the results of a numerical study on the chaotic behaviour of SSYMH system. It is more complicated than the spatially homogeneous cases because of the presence of a singular potential and space-time dependence. We consider specifically the 't Hooft-Polyakov monopole solution. Because of the large mass of monopole quantum fluctuations are reduced and classical system may be a good approximation to the real quantum case. We find a phase-transition like behaviour from order to chaos as we tune the parameter which depends on the self interaction constant of scalar fields. For our study we discretise the system into a collection of interacting coupled nonlinear oscillators and calculate the maximal

Lyapunov exponents for various parameter values and different number of oscillators. Calculation of maximal Lyapunov exponents is a reliable criterion to determine whether a system is chaotic or not.

In the next section we briefly describe the studies on the chaotic behaviour of spherically symmetric time dependent SU(2) Yang-Mills system (SSYM). The Yang-Mills-Higgs system (SSYMH) under investigation is also presented there. We present the numerical techniques applied and the results in section 3. Section 4 contains our conclusions.

5.2. Chaos in SSYM and the 't Hooft Polyakov monopole in SSYMH.

In Chapter 3 we discussed some of the spatially homogeneous models of Yang-Mills theory which are nonintegrable and chaotic. We shall now consider some aspects of chaotic behaviour in space-time dependent YM systems. Matinyan et al (1986,1988) were the first to investigate the chaotic behaviour in SSYM system given by the equation (3.13).

$$(\partial_r^2 - \partial_t^2) K = K (K^2 - 1)/ r^2 \qquad (5.1)$$

Details of this system have been given in Chapter 3. Space-time dependence and singular potential complicate the analysis of the system which is also devoid of any control parameter. Matinyan et al used the Fermi-Pasta-Ulam technique for their study. The continuous system is discretised to obtain N coupled anharmonic oscillators. Corresponding equations of motion are,

$$\ddot{K}(i,t) = \frac{K(i+1,t) - 2K(i,t) + K(i-1,t)}{h^2} - \frac{K(i,t)\,[K(i,t)^2 - 1\,]}{(ih)^2} \qquad (5.2)$$

where h is the space discretisation step. The solution K(i,t) of (5.2) is expanded in harmonics :

$$K(i,t) = \sqrt{2/N} \sum_{j=1}^{N-1} \psi(j,t) \sin(\pi ij/N) \qquad (5.3)$$

Then the total energy of the discrete analog of (5.1) is given by the expression,

$$E = E_0 + 1/4 \sum_{i=1}^{N-1} \frac{[1 - K^2(i,t)]^2}{(ih)^2}\, h \qquad (5.4)$$

where,

$$E_0 = 1/2\ h \sum_{j=1}^{N-1} [\ \dot{\psi}^2 + \phi_j^2 \psi^2\ ], \qquad (5.5)$$

$$\phi_j = 2/h \sin \pi j/2N.$$

Dynamics of the system near the Wu-Yang monopole solution $K(r) = 0$ has been investigated. Boundary conditions taken were $K(0,t) = K(N,t) = 0$, $\dot{K}(i,0) = 0$, with some modes excited at t=0. This corresponds to a deformed but initially resting string. Boundary conditions for non-deformed string at t=0 are $K(i,0)=0$, $K(0,t) = K(N,t) = 0$, $\dot{K}(i,t) \neq 0$. Matinyan et al found that the energy is shared uniformly among different modes indicating the ergodic nature of the system. Kawabe and Ohta (1990) investigated the system in more detail by evaluating the induction period, equal time correlation and maximal Lyapunov exponents. These studies confirmed that the SSYM system is always chaotic. The induction period never becomes infinite indicating the absence of quasiperiodic behaviour even for small perturbations. Moreover the maximal LE is always positive confirming the chaotic nature of the system.

Now let us consider the spherically symmetric time dependent SU(2) Yang-Mills-Higgs (SSYMH) system. 't Hooft (1974) and Polyakov (1974) discovered magnetic monopoles as finite energy solutions of non-Abelian gauge theories, in the Georgi-Glashow model with the gauge group SU(2) is broken down to U(1) by Higgs triplets. More details on this model are contained in sections 1.6 and 3.2. The field equations with time dependent 't Hooft-Polyakov ansatz (Mecklenberg and O'Brien 1978) are,

$$r^2 (\partial_r^2 - \partial_t^2) K = K (K^2 + H^2 - 1)$$
$$r^2 (\partial_r^2 - \partial_t^2) H = H (2K^2 - m^2 r^2 + \frac{\lambda}{g^2} H^2). \quad (5.6)$$

The vacuum expectation value of the scalar field and Higgs boson mass are $< \phi^2 > = F^2 = m^2 / \lambda$ and $M_H = \sqrt{2\lambda}\ F$ respectively. Mass of the gauge boson is $M_w = gF$. With $\beta = \frac{\lambda}{g^2} = \frac{M_H^2}{2M_M^2}$, introducing the variables $\xi = M_w r$ and $\tau = M_w t$, the equations (5.6) become

$$(\partial_\xi^2 - \partial_\tau^2) K = K (K^2 + H^2 - 1) / \xi^2$$
$$(\partial_\xi^2 - \partial_\tau^2) H = H (2K^2 + \beta (H^2 - \xi^2)) / \xi^2 . \quad (5.7)$$

Total energy of the system E is given by

$$C(\beta) = \frac{g^2 E}{4\pi M_w} = \int_0^\infty \left\{ K_\tau^2 + \frac{H_\tau^2}{2} + K_\xi^2 + \frac{1}{2} (H_\xi - \frac{H}{\xi})^2 + \frac{1}{2\xi^2} (K^2 - 1)^2 + \frac{K^2 H^2}{\xi^2} + \frac{\beta}{4\xi^2} (H^2 - \xi^2)^2 \right\} d\xi \quad (5.8)$$

Time independent ansatz (1.61) gives the 't Hooft-Polyakov monopole solution with winding number 1 the details regarding which are given in Chapter 1. In the limit $\beta \longrightarrow 0$ known as the Prasad-Sommerfeld (PS) limit, we have the static solutions,

$$K(\xi) = \xi / \operatorname{Sinh} \xi$$

$$H(\xi) = \xi \operatorname{Coth} \xi - 1 \; . \tag{5.9}$$

It has not been possible to find exact nontrivial solutions for $\beta \neq 0$ analytically.

Matinyan et al (1989) investigated the possibility of chaos in SSYMH near the ´t Hooft-Polyakov monopole solutions. They found that it can be chaotic by calculating the Lyapunov exponents (LE). Their calculations are not either exhaustive or satisfactory to arrive at a definite conclusion. They calculated LE for a time of t = 3 which is not sufficient for obtaining the asymptotic value of LE. Dependence of chaos on the parameter β has also not been investigated. We present the details of our numerical study of these aspects in the next section.

5.3. Lyapunov exponents and Order to Chaos transition

As has been discussed in Chapters 1 and 4 calculation of LE is a reliable and convenient way to study chaos. If the maximal LE is greater than zero the system is said to be chaotic.

For our study we discretise the original infinite dimensional system (5.7) to obtain a set of N coupled anharmonic oscillators. The discrete model is given by

$$\ddot{K}(i,t) = \frac{K(i+1,t) - 2K(i,t) + K(i-1,t)}{h^2} - \frac{K(i,t)\,[K(i,t)^2 + H(i,t)^2 - 1\,]}{(ih)^2} \tag{5.10}$$

$$\ddot{H}(i,t) = \frac{H(i+1,t) - 2H(i,t) + H(i-1,t)}{h^2}$$
$$- \frac{2H(i,t)\ K(i,t)^2 - \beta H(i,t)[H(i,t)^2 - (ih)^2]}{(ih)^2}$$
$$i = 1, \ldots\ldots N-1,$$

where h is the space discretisation step. Corresponding variational system is obtained by discretising the following equations :

$$\begin{aligned} (\partial_\xi^2 - \partial_\tau^2)\delta K &= \frac{(3K^2 + H^2 - 1)\delta K + 2HK\ \delta H}{\xi^2} \\ (\partial_\xi^2 - \partial_\tau^2)\delta H &= \frac{(2K^2 + 3\beta H^2 - \beta\xi^2)\delta H + 4KH\ \delta K.}{\xi^2} \end{aligned} \qquad (5.11)$$

For calculating LE we have to solve system (5.10) along with the variational system obtained from (5.11). In the system, there exist two parameters, the energy and the value of β. For the numerical integration we can start from arbitrary values of K and H. But we are interested in the evolution of 't Hooft-Polyakov monopole solutions. Static monopole solutions occur at the minimum of energy functional $C(\beta)$ for a fixed β. So we choose $K(i,0)$ and $H(i,0)$ as the static solution of the YMH system which we find using a finite difference method for solving boundary value problems. We use the asymptotic form of the solutions for fixing the boundary values. $C(\beta)$ for different β values are given in table 5.1. Static solutions of SSYMH for some β values are shown in figures 5.1a and 5.1b.

We use fixed boundary conditions and numerically solve the system (5.10) with static solutions as initial conditions along with the discretised system obtained from

Table 5.1. $C(\beta)$ and Maximal LE for different β values.

β	$C(\beta)$	LE
0.0	1.000	0.00
0.1	1.006	0.00
0.5	1.193	0.00
1.0	1.243	0.00
2.0	1.302	0.00
5.0	1.386	0.00
10.0	1.451	0.00
50.0	1.600	0.00
75.0	1.641	1.54E-3
100.0	1.671	2.32E-3
200.0	1.762	1.13E-2
500.0	1.971	2.54E-2
1000.0	2.301	7.01E-2
5000.0	4.641	1.00E-1

(5.11). For our calculations we take N=100 and the discretisation step h=0.1. In figures 5.2a and 5.2b plot of LE versus time is given for some values of β. We calculate up to t=1000.0 which is sufficient for obtaining asymptotic values of LE. We use an IMSL routine for Bulirsh-Stoer algorithm for the numerical integration of the differential equations with a tolerance value 10^{-3}. Calculations are done in double precision in a CYBER 180/830 computer. In the case

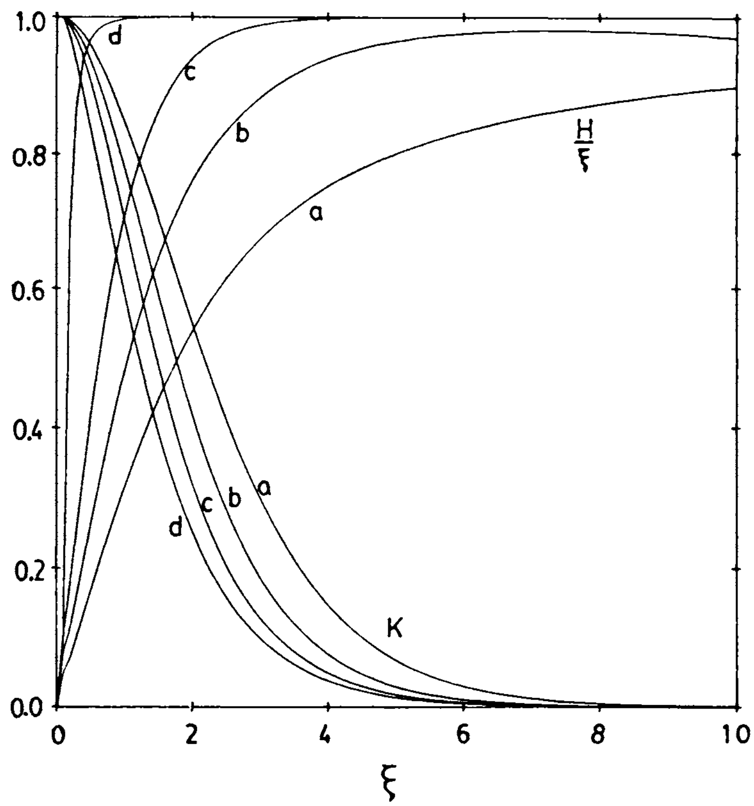

Figure 5.1a. K and H/ξ vs ξ for (a) β=0.0, (b) β=0.1, (c) β=1.0 and (d) β=100.0

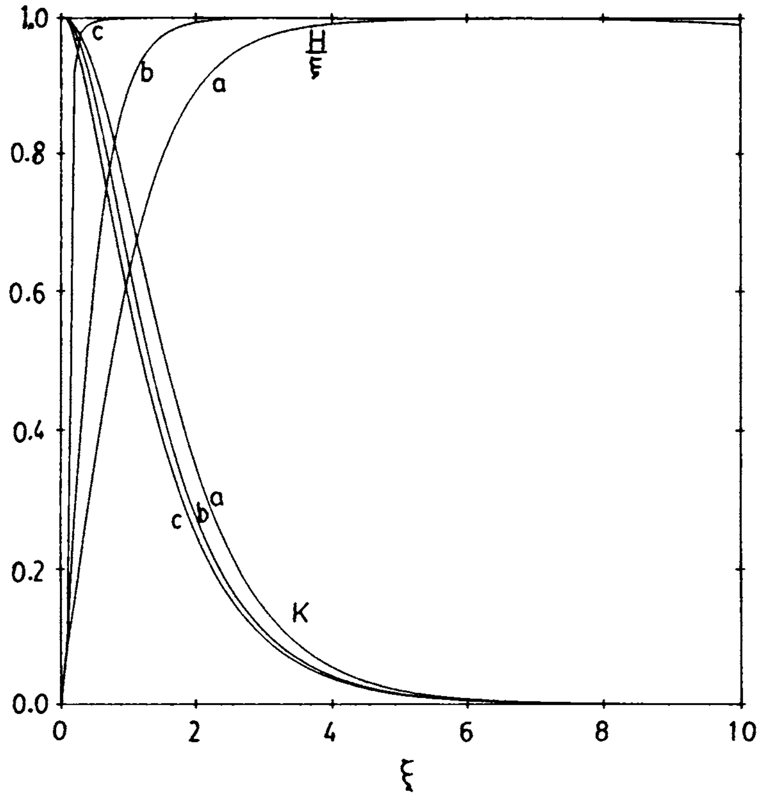

Figure 5.1b. K and H/ξ vs ξ for (a) β=0.5, (b) β=5.0 and (c) β=500.0

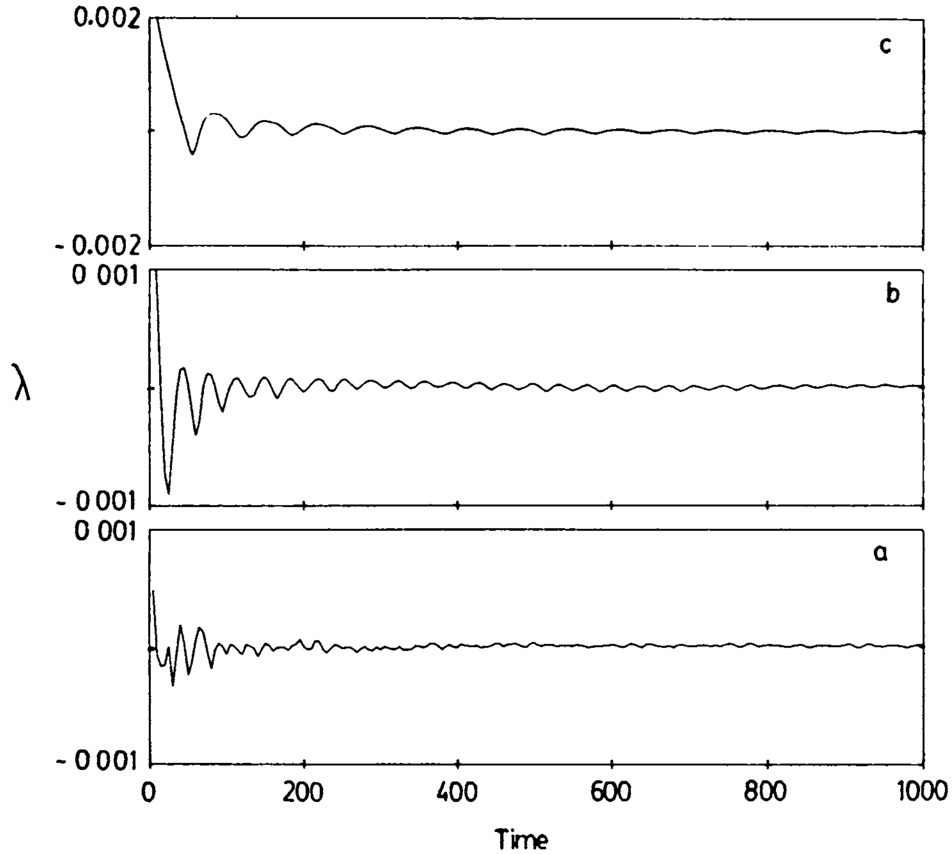

Figure 5.2a. Maximal LE (λ) vs time for (a) β=0.0, (b) β=10.0 and (c) β=50.0

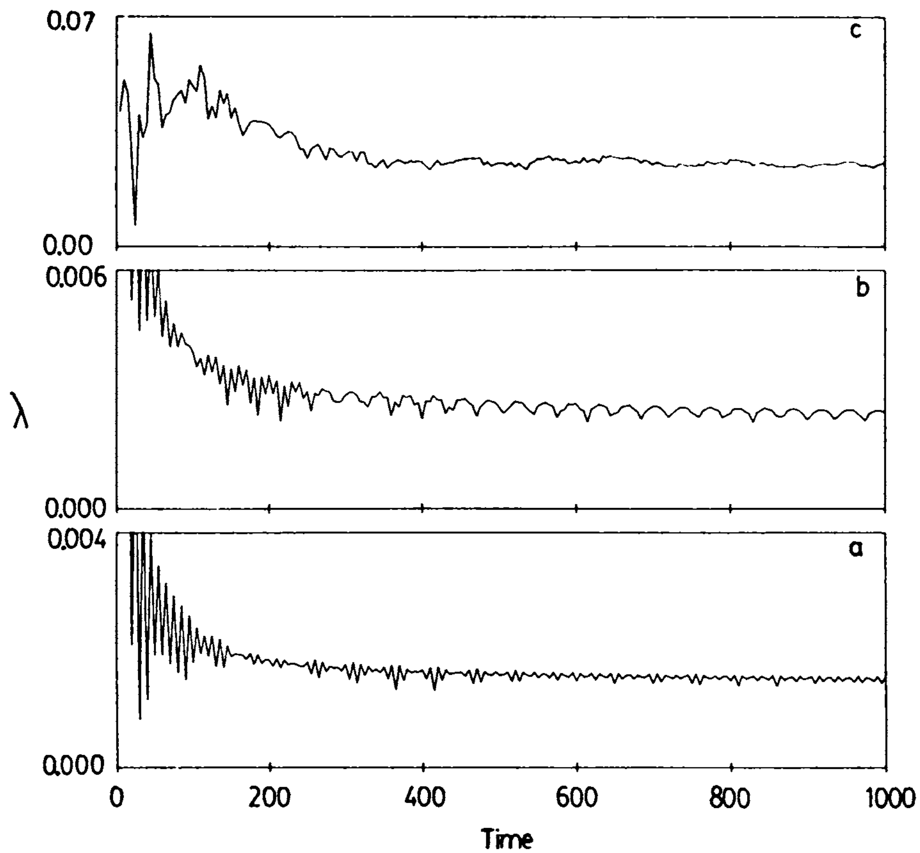

Figure 5.2b. Maximal LE (λ) vs time for (a) β=75.0, (b) β=100.0, and (c) β=500.0

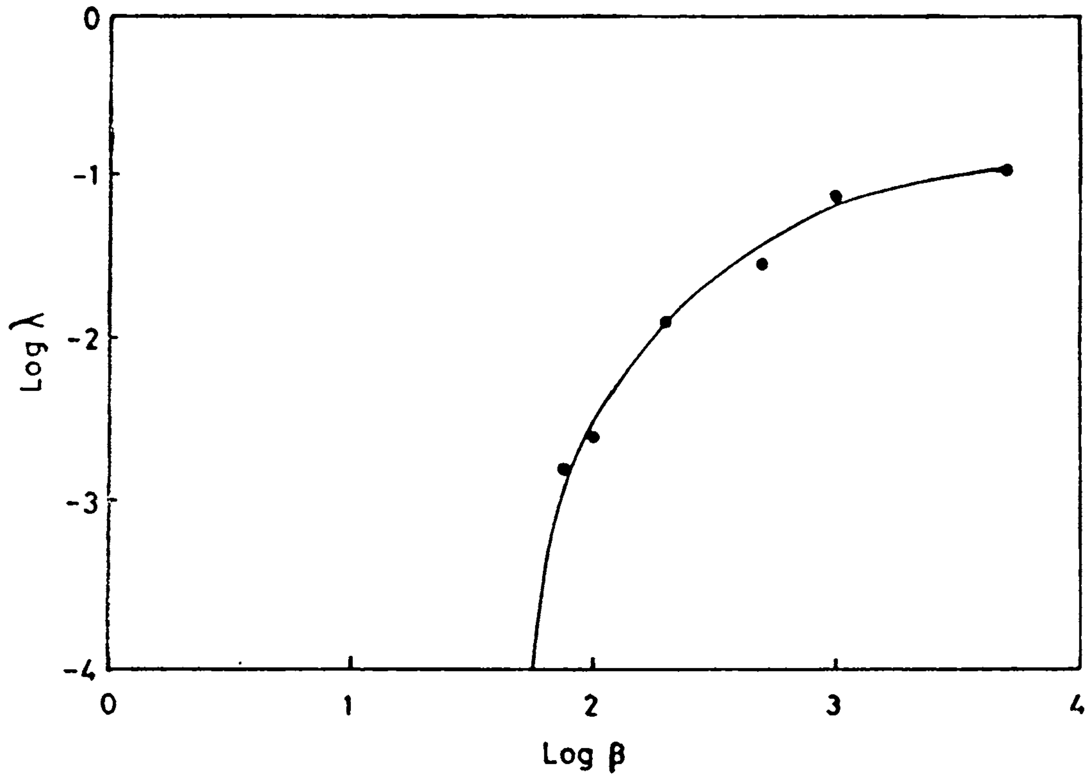

Figure 5.3. Log (λ) vs Log(β).

of β=1000 and β=5000 we used a higher tolerance value because of the enormous amount of computer time required otherwise. We did the calculations with high accuracy such that change in energy is less than 1%. Lyapunov exponents for different values of β are also given in table 5.1. From figure 5.3, where log(LE) vs log(β) is plotted, we can see that there is a transition from order to chaos near β=75.0. Up to β=50.0 LE is zero within the limits of numerical accuracy. For β=75.0 LE becomes positive and reaches an asymptotic value 1.54×10^{-3}. For higher β values we get higher and higher positive LEs. However LE is not seen increasing indefinitely with β. At the transition region the increase is rapid but as β increases further the rate of increase in LE falls. As $\beta \longrightarrow \infty$, LE appears to attain an asymptotic value.

Table 5.2. Maximal LE for different values of N and β.

β \ N	LE			
	N = 16	32	64	100
50.0	0.00	0.00	0.00	0.00
75.0	0.006	0.0012	0.0013	0.0015
100.0	0.0015	0.0021	0.0021	0.0023
200.0	0.027	0.0091	0.0105	0.011

We have repeated the calculations with different values of N also. Results are qualitatively the same as that of N=100. LE for N = 16,32,64,100 are given for various β

values in table 5.2. Increasing N does not have much effect after N=32. This indicates that the results obtained are good approximations to the original infinite dimensional system. This behaviour can be compared to the results obtained by Livi et al. (1986) for a Fermi-Pasta-Ulam chain of anharmonic oscillators. For FPU β-model LE reaches an asymptotic value when the number of oscillators is N = 20-40. For small N values boundary values also have effects on the dynamics. Asymptotic values of K and H are reached only after ξ = 3 - 4. The quartic oscillator system corresponding to N = 1 is nonintegrable and chaotic for all β values.

5.4. Conclusion

Our calculations show that there is a phase-transition-like behaviour from order to chaos in SU(2) SSYMH system. This result is in agreement with that obtained in the case of spatially homogeneous YMH system, where Higgs field manifests only as the vacuum expectation value F. As F increases there is an order to chaos transition and in that case there are no terms dependent on the self interaction constant. There is only one parameter for YMHCM, namely $g^2E/4\pi M_w$. On the other hand here we consider the time evolution of both gauge and scalar fields and there exist two parameters $C(\beta)$ and β. β depends on the self interaction constant λ. Since we are interested in monopole solutions we took the minimum value of energy functional $C(\beta)$ for a specific β value. It is known that as β increases the effect of Higgs field decreases and when $\beta \longrightarrow \infty$ system becomes purely Yang-Mills, which is highly chaotic. The effect of Higgs scalar fields is to reduce the stochasticity of the YM system. In the central part of the monopole the scalar field is approximately equal to zero and the YM field which dominates this region displays chaotic behaviour. Outside the monopole core the Higgs field approaches its mean value and the YM field behaves in regular manner. From our study one can see that 't Hooft-Polyakov monopole solutions show irregular behaviour in time, and they are exponentially unstable. Our results can be compared with that of Brandt and Neri (1979) in the context of Wu-Yang monopoles. They have shown that negative modes exist in the spectrum of the operator describing small perturbations of monopole solutions, implying exponential growth of perturbations with time. Solutions with magnetic charge $q \geq 1$ are unstable. Arbitrary continuous deformations of the field configurations do not change the topological charge, during the evolution of the fields with time. The evolution of the fields in the central part of the monopole can be arbitrarily complicated, may oscillate or vary ergodically. Though in the case of the monopole classical description may be a meaningful approximation to the quantum case the implications of the result in the exact quantum field theory of this object is a separate issue requiring detailed study.

CHAPTER 6

CHAOS AND QUANTUM FLUCTUATIONS IN A QUARTIC HAMILTONIAN SYSTEM

6.1. Introduction

Different approaches to the characterisation of quantum chaos and the variety of techniques for its investigation have been discussed in section 5 of Chapter 1. Most of the studies so far made are semiclassical and/or numerical. Recently an application of the Gaussian Effective Potential (GEP) method in quantum mechanics has been made for an approximate but analytical study of the effect of quantum fluctuations on chaotic systems (Carlson and Schieve 1989). GEP is an approximate potential describing the quantum effects on a classical potential and it is not a semiclassical quantity (Stevenson 1984). Carlson and Schieve used this method to study the effect of quantum fluctuations in Henon-Heiles potential and four leg potential (GEP calculations given in that paper contain inaccuracies which do not change their conclusions. Correct GEP calculation for Henon-Heiles system is given in the Appendix 6.A.). They studied the variation in the nature of the GEP as $\hbar$, the Planck's constant is varied and found that as $\hbar \longrightarrow 1$ (large values of $\hbar$) the GEP reduces to an integrable potential even though the classical potential is nonintegrable and chaotic. They also tested the conditions for hyperbolicity of periodic orbits (Churchill et al 1975,1977) when quantum fluctuations are present. Their conclusion was that the quantum fluctuations destroy the chaotic behaviour in the Henon-Heiles potential and in the four leg potential.

In this chapter we apply the GEP method for studying the quantum chaos in a generalised quartic Hamiltonian system,

$$H = \frac{1}{2}\,(p_1^2+p_2^2) + \frac{1-\alpha}{12}\,(q_1^4+q_2^4) + \frac{1}{2}\,q_1^2q_2^2 \qquad (6.1)$$

where $0 \le \alpha \le 1$. This is a system which can exhibit chaotic behaviour of various degrees depending on the parameter α. Our aim is to study the change in quantum chaos as α is varied and compare the classical and quantum behaviour. System (6.1) is integrable and shows regular behaviour when $\alpha=0$. But when $\alpha=1$ system is highly chaotic and can be used as simplified model for classical Yang-Mills system (Savvidy 1984). For value of α in between these extremes the behaviour interpolates between order and chaos. Many investigations have been made on the classical dynamics of the system in detail (Carnegie and Percival 1984, Steeb et al 1986b). Recently we (Joy and Sabir 1992) studied the relation between curvature and chaos in the system the details of which have been discussed in Chapter 4. The system has positive Riemannian curvature implying local stability for all values of α. But the potential boundary of the system has negative curvature except when $\alpha=0$, and chaos of the system, as measured by Lyapunov exponents, increases with α as the negative curvature of the potential boundary is increased. Quantum chaos in the system has been investigated by means of spectral properties (Steeb and Louw 1986b, Kotze 1988). Nearest neighbour spacing of energy eigenvalues obey Poisson distribution when $\alpha=0$ and a Wigner distribution when $\alpha=1$. In between there is an intermediate behaviour.

Studying the GEP as function of $\hbar$ for various values of α we find that even though there is quantum suppression of chaos there is a correspondence between classical behaviour and quantum behaviour. The value of $\hbar$ above which the GEP become an integrable one increases with the degree of chaos in the classical system. In the next section we briefly describe the computational details of the Gaussian Effective Potential. Section 6.3 contains the results and the conclusions.

6.2. The Gaussian Effective Potential

The Gaussian Effective Potential (GEP) method as formulated by Stevenson (1984) is a very convenient technique for estimating the quantum effects on a classical potential. It gives us a picture of how quantum fluctuations modify the classical potential. For a system with Hamiltonian H the GEP is defined as $\overline{V}_G(q_0) \equiv \min\limits_{\Omega} \langle\psi|H|\psi\rangle$ where $|\psi\rangle$ is Gaussian state localised around q_0 and Ω denotes a set of parameters governing its width. Compared with the exact effective potential which is obtained by minimising over all states localised around q_0 and the one loop effective potential it has several distinct advantages and gives a more realistic description of quantum phenomena. Though approximate the GEP is neither perturbative nor is semiclassical.

Let us consider the GEP estimation of a Hamiltonian system with two degrees of freedom with a Hamiltonian of the form,

$$H = \frac{1}{2} (p_1^2 + p_2^2) + V(q_1,q_2).$$

To evaluate the GEP we first compute the expectation value of the energy $V_G \equiv \langle\psi|H|\psi\rangle$, where the normalized wave functions ψ are two dimensional Gaussians of the form $\exp(-q_i\Omega_{ij}q_j/2)$, and Ω_{ij} is a symmetric matrix that in general depends on the position variables. GEP is obtained by minimising this with respect to three variational parameters : two principal frequencies Ω and ω and an angle θ specifying the orientation of the principal axes of the wave function with respect to the q_1,q_2 axes. Using the creation and annihilation operator formalism one can make the calculation purely algebraic. The annihilation and creation operators a_1,a_2 and $a_1^\dagger,a_2^\dagger$ are defined through

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} q_{10} \\ q_{20} \end{pmatrix} + \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \hbar(2\hbar\Omega)^{-\frac{1}{2}} (a_1+a_1^\dagger) \\ \hbar(2\hbar\omega)^{-\frac{1}{2}} (a_2+a_2^\dagger) \end{pmatrix} \qquad (6.2)$$

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} -\frac{i}{2}(2\hbar\Omega)^{\frac{1}{2}} (a_1-a_1^\dagger) \\ -\frac{i}{2}(2\hbar\omega)^{\frac{1}{2}} (a_2-a_2^\dagger) \end{pmatrix} \qquad (6.3)$$

where $[a_1,a_1^\dagger] = 1$ and $[a_2,a_2^\dagger] = 1$. The expectation value of H is evaluated in the state $|0\rangle_\Omega$ defined by $a_1|0\rangle_\Omega = 0$ and $a_2|0\rangle_\Omega = 0$.

For the system (6.1) GEP is given by the minimum of V_G which is obtained by a straight forward calculation :

$$V_G = \frac{1-\alpha}{12}(q_1^4 + q_2^4) + \frac{1}{2} q_1^2 q_2^2 + \frac{\hbar\Omega^{-1}}{2}\Big[\frac{1-\alpha}{2}(q_1^2\cos^2\theta + q_2^2\sin^2\theta)$$
$$+ \frac{1}{2}(q_1^2\sin^2\theta + q_2^2\cos^2\theta) + q_1 q_2 \sin 2\theta \Big]$$
$$+ \frac{\hbar\omega^{-1}}{2}\Big[\frac{1-\alpha}{4}(q_1^2\sin^2\theta + q_2^2\cos^2\theta) \tag{6.4}$$
$$+ \frac{1}{2}(q_1^2\cos^2\theta + q_2^2\sin^2\theta) - q_1 q_2 \sin 2\theta \Big]$$
$$+ \frac{\hbar^2}{2}(\Omega^{-2}+\omega^{-2})\Big(\frac{1-\alpha}{8}(\cos^4\theta + \sin^4\theta) + \frac{3}{16}\sin^2 2\theta \Big)$$
$$+ \frac{\hbar^2}{8}(\Omega\omega)^{-1}\Big(\sin^4\theta + \cos^4\theta - \frac{1+\alpha}{2}\sin^2 2\theta \Big) + \frac{\hbar}{4}(\Omega + \omega)$$

To find the minimum we have to solve the following set of equations.

$$\partial V_G/\partial\theta = 0 \equiv (\Omega^{-1}+ \omega^{-1})\Big(\frac{\alpha}{2}(q_1^2 - q_2^2) \sin 2\theta + 2q_1 q_2 \cos 2\theta \Big)$$
$$+ \frac{(\alpha+2)\hbar}{8}(\Omega^{-1} - \omega^{-1})^2 \sin 4\theta \tag{6.5a}$$

$$\partial V/\partial\Omega = 0 \equiv - \Omega^{-2}\Big[\frac{1-\alpha}{2}(q_1^2\cos^2\theta + q_2^2\sin^2\theta)$$
$$+ \frac{1}{2}(q_1^2\sin^2\theta + q_2^2\cos^2\theta) + q_1 q_2 \sin 2\theta \Big]$$
$$- 2\hbar\,\Omega^{-3}\Big(\frac{1-\alpha}{8}(\cos^4\theta + \sin^4\theta) + \frac{3}{16}\sin^2 2\theta \Big)$$
$$- \frac{\hbar}{4}\Omega^{-2}\omega^{-1}\Big(\sin^4\theta + \cos^4\theta - \frac{1+\alpha}{2}\sin^2 2\theta \Big) + \frac{1}{2} \tag{6.5b}$$

$$\partial V/\partial\omega = 0 \equiv - \omega^{-2}\Big[\frac{1-\alpha}{2}(q_1^2\sin^2\theta + q_2^2\cos^2\theta)$$
$$+ \frac{1}{2}(q_1^2\cos^2\theta + q_2^2\sin^2\theta) - q_1 q_2 \sin 2\theta \Big]$$
$$- 2\hbar\,\omega^{-3}\Big(\frac{1-\alpha}{8}(\cos^4\theta + \sin^4\theta) + \frac{3}{16}\sin^2 2\theta \Big)$$
$$- \frac{\hbar}{4}\Omega^{-1}\omega^{-2}\Big(\sin^4\theta + \cos^4\theta - \frac{1+\alpha}{2}\sin^2 2\theta \Big) + \frac{1}{2} \tag{6.5c}$$

The solutions θ, Ω and ω should be substituted in V_G to obtain the GEP $\overline{V}_G$. Even then we can find only a local minimum. But these coupled set of equations are not of the form where we can calculate the solutions analytically. Hence we resort to numerical minimisation of the potential V_G at each q_1 and q_2.

Since our system is scale invariant we evaluate $\overline{V}_G$ for a fixed value of energy; we fix $\overline{V}_G = 1$. The level curves of $\overline{V}_G$ for different values of Planck's constant $\hbar$ are plotted in figures 6.1 - 6.6. Figure 6.1 shows the $\overline{V}_G$ for $\alpha=0$, when the system is integrable classically. For $\hbar=0$ we have the classical potential boundary which has no negative curvature. As we increase the value of $\hbar$ we see that the GEP becomes more and more circular. In the other figures for nonzero α with negatively curved classical potential boundary we observe that the increase in the value of $\hbar$ leads to a reduction of the negative curvature of the boundary and ultimately the boundary becomes a circle around the origin. This, as noted by Carlson and Schieve, is the manifestation of quantum suppression of chaos. As $\hbar$ is increased the value at which the level curve first becomes a circle is a measure of the amount of quantum fluctuations needed to suppress classical chaos. From the figures 6.1-6.6 we can make an important observation that the value of $\hbar$ at which $\overline{V}_G$ becomes a circle increases monotonically with α. Denoting this value of $\hbar$ by $\hbar_c$ in table 6.1 we give approximate $\hbar_c$ for various values of α. In the most chaotic regime in α any magnitude of strong quantum fluctuations chaos will be completely wiped off. In other words the quantum system will exhibit chaotic features if quantum fluctuations are small. On the other hand for weakly chaotic systems (α small) even slight quantum

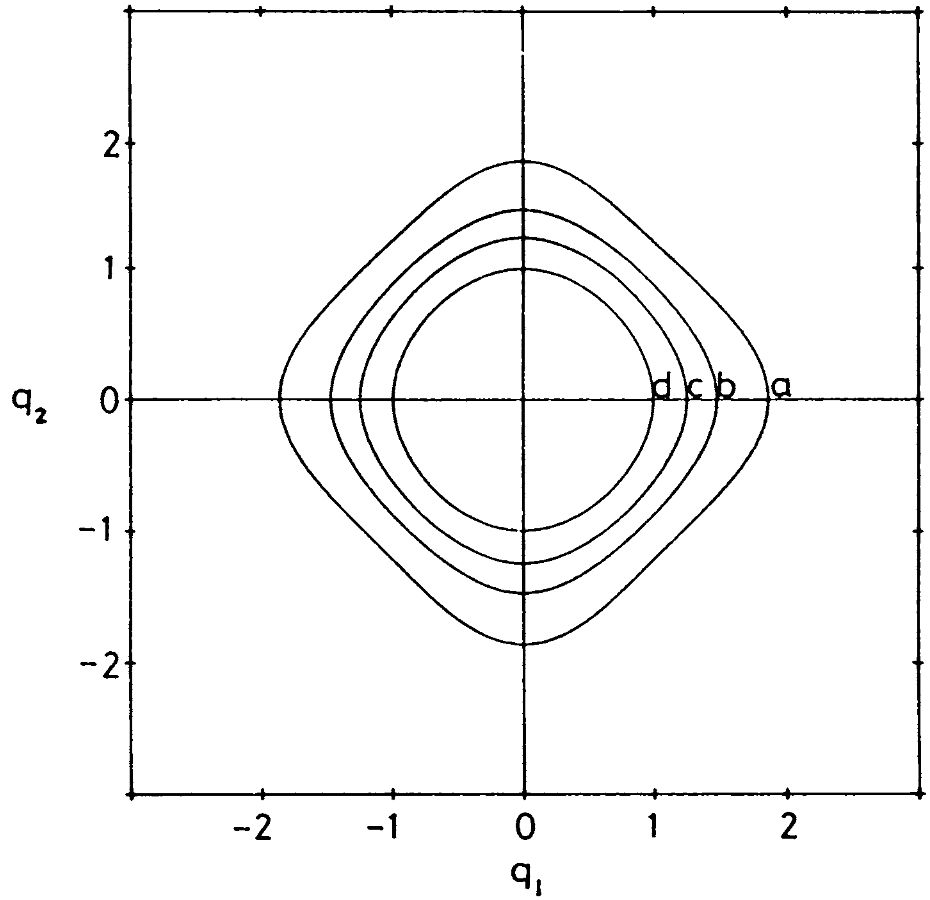

Figure 6.1. $\bar{V}_a$ for α = 0.0 with $\hbar$ values (a) 0.0, (b) 0.4, (c) 0.6 and (d) 0.8.

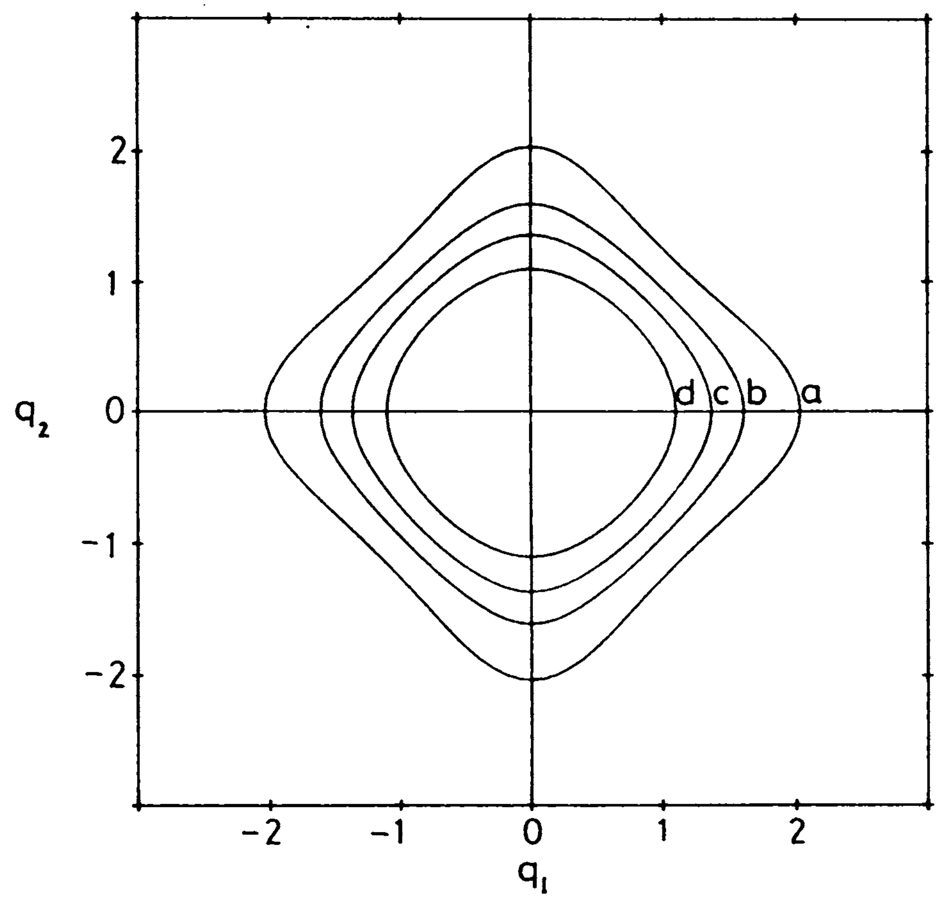

Figure 6.2. $\overline{V}_\alpha$ for α = 0.3 with $\hbar$ values (a) 0.0, (b) 0.4, (c) 0.6 and (d) 0.8.

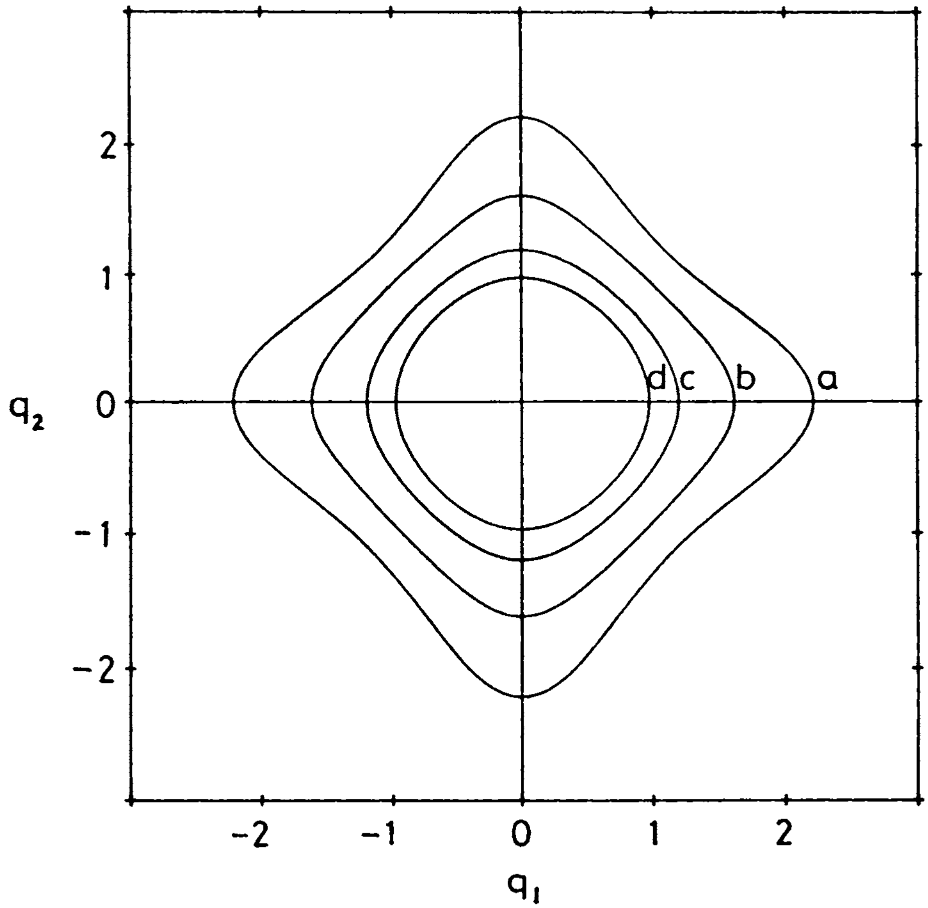

Figure 6.3. $\bar{V}_a$ for α = 0.5 with $\hbar$ values (a) 0.0, (b) 0.5, (c) 0.8 and (d) 0.95.

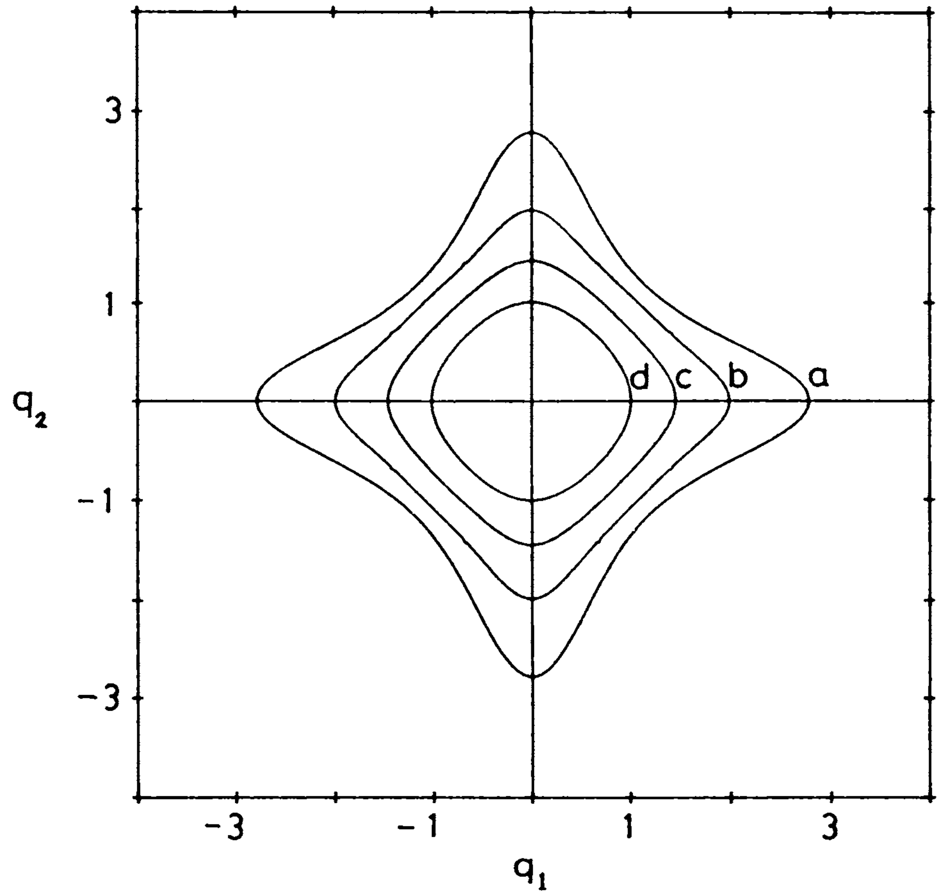

Figure 6.4. $\bar{V}_a$ for α = 0.8 with $\hbar$ values (a) 0.0, (b) 0.5, (c) 0.8 and (d) 1.05.

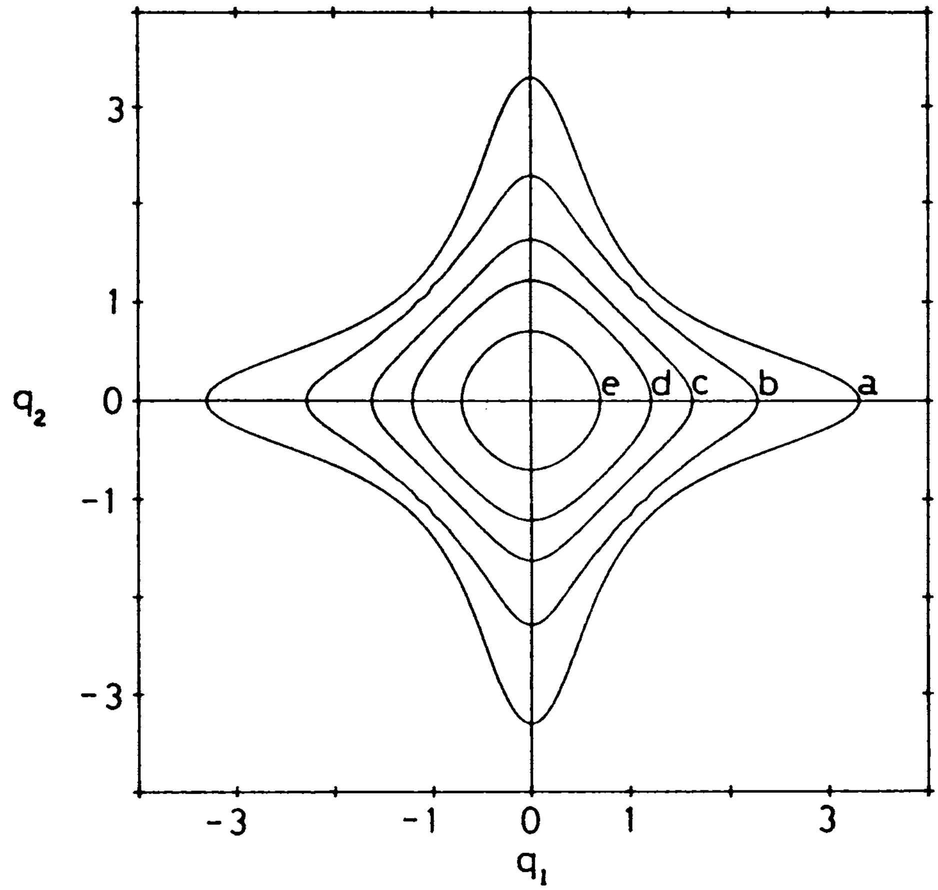

Figure 6.5. $\bar{V}_a$ for $\alpha = 0.9$ with $\hbar$ values (a) 0.0, (b) 0.5, (c) 0.8, (d) 1.0 and (e) 1.25.

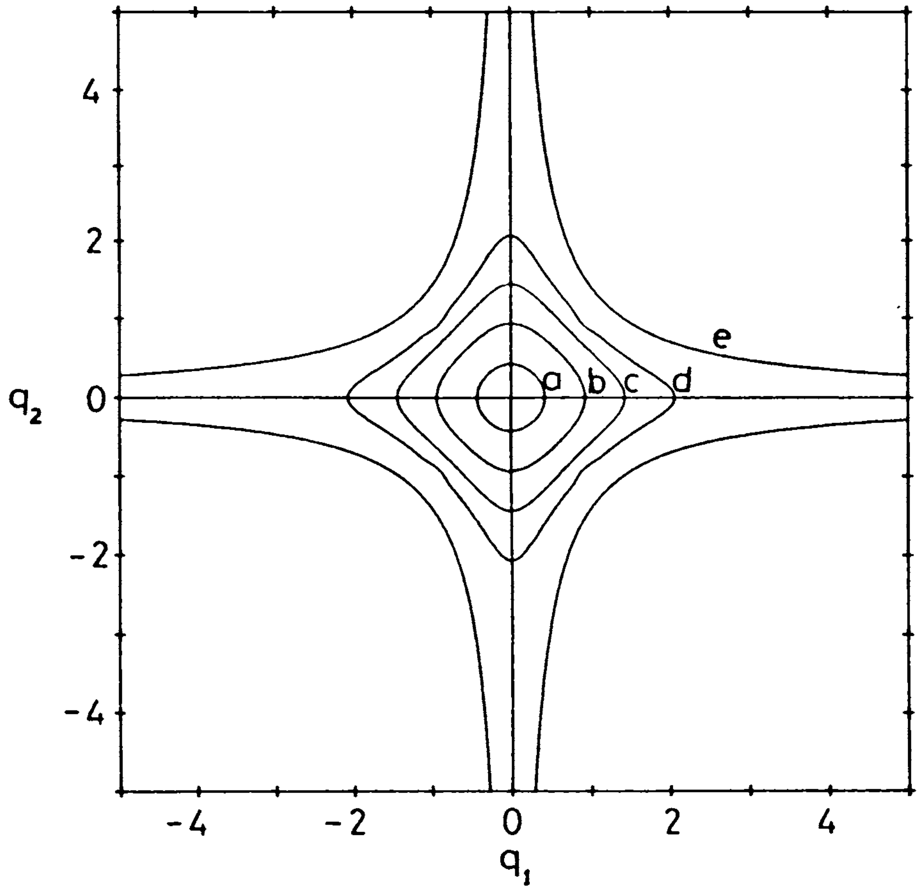

Figure 6.6. $\overline{V}_\sigma$ for α = 1.0 with $\hbar$ values (a) 1.4, (b) 1.2, (c) 1.0, (d) 0.8 and (e) 0.0.

Table 6.1. $\hbar_c$ values for various α values

α	0.0	0.3	0.5	0.8	0.9	1.0
$\hbar_c$	0.8	0.8	0.95	1.05	1.25	1.4

fluctuations will destroy chaos. We can thus see from the GEP study a correlation between the classical and quantum system with regard to integrability and chaos.

6.3. Conclusion

Our GEP calculation for one particular Hamiltonian system shows that even though quantum fluctuations reduce chaos, it exists for small values of $\hbar$. For sufficiently large value of $\hbar$ all of these become regular, but the value at which they become regular increases with the chaoticity of the original classical system. Other conclusion is that though quantum fluctuations diminishes chaos there exist remnants of classical chaos in the quantum regime.

6.A. Appendix

Here we give the correct GEP for the Henon-Heiles potential studied by Carlson and Schieve (1989)

$$V(q_1,q_2) = (q_1^2+q_2^2)/2 - q_1q_2^2 + q_1^3/3. \qquad (6.6)$$

Using (6.2) and (6.3) we obtain,

$$V_G = \langle\psi|H|\psi\rangle = V(q_1,q_2) + \hbar(\Omega+\omega)/4 + \hbar(1/\Omega+1/\omega)/4 + (1/\Omega-1/\omega)(q_1\cos 2\beta - q_2\sin 2\beta). \qquad (6.7)$$

On minimizing V_G we get

$$\bar{V}_G = V(q_1,q_2) + \hbar\ (\Omega+\omega)/2, \tag{6.8}$$

where,

$$\begin{aligned} \Omega &= [\ 1 + 2q_1\cos 2\beta - 2q_2\sin 2\beta\ ]^{1/2} \\ \omega &= [\ 1 - 2q_1\cos 2\beta + 2q_2\sin 2\beta\ ]^{1/2} \end{aligned} \tag{6.9}$$

and $\tan 2\beta = -q_2/q_1$.

In the case of four leg potential discussed in Carlson and Schieve (1989) given by

$$V(q_1,q_2) = (q_1^2+q_2^2)/2 - q_1^2q_2^2/2 \tag{6.10}$$

one can not find the GEP because the expectation value V_G is not bounded from below.

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