STUDIES ON FERMION - SOLITON SYSTEMS

M. RAVENDRANADHAN

THESIS SUBMITTED IN
PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF PHYSICS
COCHIN UNIVERSITY OF SCIENCE AND TECHNOLOGY
COCHIN - 682 022

1993

CERTIFICATE

Certified that the work reported in the present thesis is based on the bonafied work done by Shri M. Ravendranadhan, under my guidance in the Department of Physics, Cochin University of Science and Technology, and has not been included in any other thesis submitted previously for the award of any degree

Cochin- 22 M. Sabir
April 7, 1993 Supervising teacher

DECLARATION

Certified that the work reported in this thesis is based on the original work done by me under the guidance of Prof. M. Sabir in the Department of Physics, Cochin University of Science and Technology, and has not been included in any other thesis submitted previously for the award any other degree.

Cochin - 22 M. Ravendranadhan
April 7, 1993.

PREFACE

In quantum field theory vacuum is usually characterized by zero fermion number . Every state accessible from vacuum by local operator has integer quantum numbers . Jackiw and Rebbi were the first to observe that in presence of nonlocal nontrivial deformations such as topological solitons , vacuum acquires fractional values for the fermion number . This effect is known as fermion number fractionisation. One can cite many examples in which fractional fermion number play an importand role. In polyacetylene the conductivity is found to be enhanced by the effect of fermion number fractionisation. Fermion number fractionisation is important in understanding certain features of Skyrme model of baryons and its modification as well as in chiral bag models. The fractional charges of 't Hooft-Polyakov monopole is an importand ingredient in the study of monopole catalysis of proton decay. It is also speculated that the fractional electric charge of quarks may be explained through this phenomenon.

Jackiw and Rebbi studied the interaction of massless fermions with solitons in 1+1 dimensions and with a Julia Zee dyon in 3+1 dimensions. In these models the C-invariance and the existence of zero energy modes lead to a vacuum charge $\pm 1/2$. When a theory is not C-invariant calculation of the vacuum charge is not straight forward. In such cases Goldstone and Wilczek calculated the vacuum charge from vacuum polarization diagram.

In this thesis we present a method of calculating vacuum charge in models which are not C-invariant. Instead of evolving the whole background field adiabatically from vacuum we start from a vacuum having a background field which leads to C-invariance and spectral symmetry. Now other fields are evolved adiabatically so that the theory loses the C-invariance. During the adiabatic evolution of the fields some energy levels cross the centre of the mass gap (spectral flow). The spectral flow is evaluated by analysing the Dirac equation in the soliton background. The induced charge is calculated by the method of Goldstone and Wilczek. The ground state charge is independent of the way one arrives at the final configuration. The induced charge and spectral flow may, however, depend on the way one reaches the final configuration

The first chapter of this thesis is of an introductory nature. It opens with a concise account of gauge theories and spontaneous symmetry breaking. A brief review is then given of solitons and their properties. This is followed by a discussion of monopoles and dyons in gauge theories. Fermion number fractionisation is then introduced and most important results are summarised. The chapter ends with a discussion of several unusual properties of fermion monopole system.

Chapter 2 illustrate our technique for the evaluation of ground state charge of fermion soliton system in 1+1 dimensions. The ground state charge in specific models are obtained by evaluating the spectral flow by analyzing the bound state spectrum and induced charge from the vacuum polarization diagrams .It is shown that ground state charge is discontinuous at the fermion mass and is independent of the soliton width

In Chapter 3 ground state charge is evaluated in 2+1 dimension .Here we consider fermion number induced by solitons in O(3) nonlinear σ model .At present this model is of considerable importance since it provides a field theoretic description for high temperature superconductivity . In the model that we are considering there is a scalar triplet characterized by a nonzero winding number. Instead of evolving three fields simultaneously,we start from the fields say ϕ_3 as the background and allow other fields ϕ_1 and ϕ_2 to evolve adiabatically .It is found that ground state charge gets no contribution from the bound states . That is , there is no spectral flow .On the other hand if ϕ_3 is evolved adiabatically ,with ϕ_1 and ϕ_2 as the background induced charge is zero and ground state charge gets contribution only from the spectral flow as is evident from the analysis of bound state spectrum .In both cases , we get the same ground state charge . It is found that ground state charge take values 1 , 1/2 , and zero depending on the parameters in the theory

In the Chapter 4 we consider the interaction of fermions with a regular ′t Hooft-Polyakov monopole. We present a detailed study of bound state spectrum of this system including the effect of monopole core. The theory is C-invariant and the Dirac equation possess zero energy state . Hence ground state charge is ±1/2. It is shown that contribution to ground state charge is made only by the lowest angular momentum state. It is shown that there is a discontinuity in the ground state charge at the fermion mass. Number of bound states is found to depend on the fermion - Higgs coupling. The results of a closely related boson monopole system are presented as an appendix to this chapter.

In the Chapter 5 we analyze the interaction of fermions with a Julia-Zee dyon. Extending the technique developed earlier for 1+1 and 2+1 dimensional models we calculate the induced charge by starting with a C-invariant configuration. The induced charge depends on the dyon's electric charge.

We also consider the interaction of fermions with nonself dual monopole which is limiting case of a dyon. The induced charge and ground state charge is found to be 1/2 which is same as that of a self dual monopole . In this case higher angular momentum contribution is found to be zero and ground state charge is discontinuous at the fermion mass

Study of Dirac equation in the dyon background shows that the bound state spectrum is symmetric in the massless case and asymmetric in the massive case. In this case the necessity of imposing certain additional conditions does not allow us to carry out an analysis of spectral flow .

The material reported in this thesis have appeared in the form of following papers:

1. Bound states of fermions and bosons with a 't Hooft-Polyakov monopole, J.Phys.G: Nul.phys **14** (1988) 433
2. Fermion dyon bound states and fermion number fractionisation, J.Phys.G: Nucl.Part.Phys **15** (1989) 433
3. Ground state charge of solitons in 1+1 and 3+1 dimensions, Int.J.Mod.Phys.A8 (1993) 705
4. Ground state charge in O(3) nonlinear σ model in 2+1 dimensions, Int.J.Mod.Phys A (Communicated)

CONTENTS

CHAPTER 1

INTRODUCTION

1.1 Gauge field theories

Gauge theories provide a theoretical framework for our current understanding of the fundamental interactions of particle physics[1-6]. The idea of gauge invariance of electromagnetism was generalized by Yang and Mills[7] in 1954 to nonabelian internal symmetries. The emergence of nonabelian gauge theories paved the way for the unified electro-weak theory[8-10], the quantum chromodynamic approach to strong interactions and grand unification schemes. As an illustration of the method of constructing a nonabelian gauge theory let us consider the lagrangian for a scalar n-tuplet $\phi = (\phi_1 \ldots\ldots\ \phi_n)$[11]

$$\mathcal{L} = (\partial_\mu \phi)^+ (\partial^\mu \phi) + V(\phi^+ \ \phi) \tag{1.1}$$

where

$$V(\phi \ \phi) = - \ \mu^2 \ (\phi^+ \ \phi) - \lambda \ (\phi^+ \phi)^2 \tag{1.2}$$

The above lagrangian has a global U(N) symmetry. That is, it is invariant under the transformation

$$U : \quad \phi \longrightarrow \phi' = U \phi = \ e^{i\theta . T} \phi \tag{1.3}$$

where $\theta.T = \theta_a T_a$, $a = 1 \ldots\ldots N$ and $\theta_1 \ldots . \ \theta_N$ are the group parameters independent of x . T^a are the (n x n) matrices representing the generators of the group which satisfy the Lie algebra

$$[\; T^{a} \;,\; T^{b}] \;=\; \imath\; C^{abc}\; T^{c}$$
$$\mathrm{Tr}\;[\; T^{a} \;,\; T^{b} \;] \;=\; \delta^{ab} \qquad (1.4)$$

C^{abc} are the structure constants of the group

Noether's theorem give a connection between symmetries of a lagrangian and conservation laws . According to this theorem any continuous symmetry transformation which leaves the action invariant implies the existence of conserved currents J^{μ}_{a} . That is

$$\partial_{\mu} J^{\mu}_{a} = 0 \qquad (1.5)$$

The charges

$$Q_{a} = \int d^{3}x \; J^{0}_{a}(x) \qquad (1.6)$$

are constants of motion. The infinitesimal transformations corresponding to Eq (1.3) are

$$\delta\phi_{i}(x) = \phi'_{i}(x) - \phi_{i}(x) = \imath\; \vartheta^{a}\; T^{a}_{ij}\phi_{j}(x) \qquad (1.7)$$

and the conserved currents can be expressed as [4]

$$J^{a}_{\mu} = -\, \imath \frac{\delta \mathcal{L}}{\delta(\partial^{\mu}\phi_{i})}\; T^{a}_{ij}\; \phi_{j}(x) \qquad (1.8)$$

Now let us investigate the possibility of having a symmetry transformation in which ϑ^{a} are space time dependant.

$$\phi \longrightarrow \phi' = U\,\phi = e^{-i\vartheta(x).T}\,\phi \qquad (1.9)$$

Such a gauge transformation is called local gauge transformation. Under this, $\partial_\mu\phi$ get transformed by

$$\partial_\mu\phi \longrightarrow \partial_\mu\phi' = U\,(\partial_\mu\phi\,) + (\partial_\mu U)\,\phi \qquad (1.10)$$

Consequently the lagrangian (1.1) is not invariant under the transformation because of the extra term proportional to $\partial_\mu\vartheta(x)$. To have local gauge invariance one must introduce additional terms which can compensate for the noninvariant term. Equivalently , one should find a modified derivative $D_\mu\phi$ which transforms like ϕ

$$[\;D_\mu\phi\;]' = U\,(D_\mu\phi) \quad , \qquad (1.11)$$

and replace ∂_μ in the lagrangian (1.1) by D_μ .The derivative D_μ is called covariant derivative since it varies in the same way as ϕ . The covariant derivative can be constructed by introducing vector fields A_μ by defining

$$D_\mu\phi = (\partial_\mu - i\,g\,A_\mu.T\,)\,\phi = (\partial_\mu + B_\mu\,)\,\phi \qquad (1.12)$$

where

$$B_\mu = -\,i\,g\,A_\mu.T \qquad (1.13)$$

Evidently the number gauge fields are equal to the number of generators of the group. By substituting (1.12) in (1.11) we get

$$(\partial_\mu + B'_\mu)(U\phi) = U(\partial_\mu + B_\mu)\phi$$

with solution

$$B'_\mu = U(x)\,B_\mu\,U^{-1}(x) - [\,\partial_\mu U(x)\,]\,U(x)^{-1} \qquad (1.14)$$

Thus by introducing the gauge fields the lagrangian (1.1) is rendered invariant under local nonabelian gauge transformation. The required lagrangian can be written as

$$\mathcal{L} = (D_\mu\phi)^\dagger.(D^\mu\phi) + V(\phi^\dagger.\phi) \qquad (1.15)$$

This lagrangian , however , does not contain dynamical terms for the gauge fields . The simplest choice of such term gives the gauge invariant lagrangian

$$\mathcal{L} = (D_\mu\phi)^\dagger.(D^\mu\phi) + V(\phi^\dagger.\phi) - \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} \qquad (1.16)$$

where

$$F^a_{\mu\nu} = (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) + g\,C^{abc}A^b_\mu A^c_\nu \qquad (1.17)$$

It may be noted that we cannot add quadratic terms in gauge fields and preserve the local gauge invariance. Consequently gauge fields are massless.

The above formalism can be easily generalised for any other symmetry G and to include other types of matter fields as well. An important feature of gauge theories is that the interaction between the matter field and gauge field is uniquely fixed by the symmetry requirements

1.2 Spontaneous symmetry breaking

So far our consideration have been purely classical. Coming to quantum theory, where the fields are operators, an interesting phenomenon that occurs is spontaneous symmetry breaking (SSB)[12,13]. If vacuum or any other state does not respect the symmetry of the lagrangian, the symmetry is said to be spontaneously broken. This may happen when the vacuum or the ground state is degenerate.

As an example of SSB in quantum field theory let us consider the potential for a real scalar multiplet ϕ^a with O(n) symmetry (Fig. 1)

$$V(\phi.\phi) = -\frac{1}{2} m^2 (\phi.\phi) - \frac{\lambda}{4} (\phi.\phi)^2$$

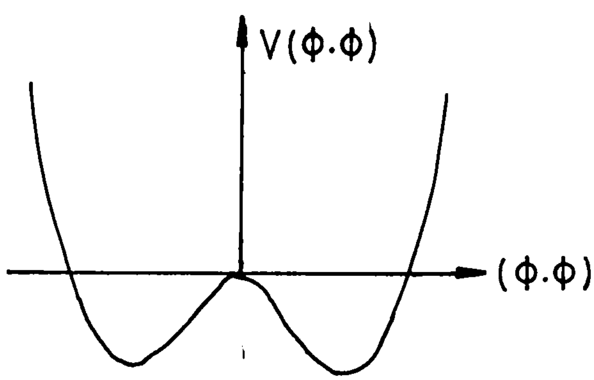

Fig. 1

When $m^2 < 0$ this potential has minima at $(\phi.\phi) = -\frac{m^2}{\lambda} = v^2$. Field theory vacuum corresponds to the minimum of the hamiltonian and hence to a minimum of the potential. In this case there are infinite number of potential minima and hence vacuum is infinite fold degenerate . All the vacua are connected by elements of O(n). However, choosing one of the vacua as physical vacuum results in spontaneous breaking of symmetry. One of the consequences of SSB is the production of massless excitations called Goldstone bosons.

For convenience let us choose the vacuum as the one where the field has expectation value $\langle 0 \,|\phi|\, 0\rangle = \langle\phi\rangle_o$ given by

$$\langle\phi\rangle_o = \begin{pmatrix} 0 \\ 0 \\ . \\ . \\ v \end{pmatrix} \tag{1.18}$$

This vacuum is not invariant under the full group O(n) but is invariant under the subgroup O(n-1). Let $L_1 \ldots\ldots L_{n-1}$ represents the n-1 broken generators which satisfy $L_i\langle\phi\rangle_o \neq 0$ .In terms of $L_1 \ldots L_{n-1}$ we can parameterize the field ϕ as

$$\phi(x) = e^{\frac{i}{v}(\alpha_1(x) L_1 + \ldots\ldots\ldots \alpha_{n-1}(x) L_{n-1})} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ v+\eta \end{pmatrix} \tag{1.19}$$

When the above equation is substituted in the lagrangian (1.1), α_1 , $\alpha_2 \ldots\ldots \eta(x)$ appear in the derivative term. But in the potential , there are no quadratic terms in $\alpha_1 \ldots \alpha_{n-1}$ etc . The only quadratic term is η^2 since $(\phi.\phi) = (v + \eta)^2$. Therefore there is only one massive field $\eta(x)$ and all other fields $\alpha_1 \ldots \alpha_{n-1}$ are massless . This means that when O(n) symmetry is broken to O(n-1) symmetry there are (n-1) massless particles and one massive particle . This is an instance of Goldstone's theorem which states that " For every broken continuous symmetry there is a massless particle" . Such a particle is called Goldstone boson[12,13]

The emergence of Goldstone boson is associated with the breakdown of a global symmetry . What happens if the symmetry is local ?. Higgs discovered a peculiar phenomena in this case . To explain this we consider the gauged O(n) model with lagrangian

$$\mathcal{L} = (D_\mu\phi).(D^\mu\phi) + V(\phi.\phi) - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} \qquad (1.20)$$

Let us perform a gauge transformation with $\phi \longrightarrow \phi' = U \phi$ with $U = . e^{\frac{i}{v}(\alpha_1(x)T^1+.........\alpha_{n-1}(x)T^{n-1})}$. Eq (1.18) now yields

$$\phi(x) \longrightarrow \phi'(x) = e^{\frac{-i}{v}(\alpha_1(x)T^1+.........\alpha_{n-1}(x)T^{n-1})}\phi(x) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ v+\eta \end{bmatrix} \qquad (1.21)$$

Under this gauge transformation gauge fields are transformed to

$$B'_\mu = U(x)\, B_\mu\, U^{-1}(X) - [\ \partial_\mu U(x)\ ]\ U^{-1}(x)$$

Here

$$U = e^{-\frac{i}{v}(\alpha_1(x)T^1+.........\alpha_{n-1}(x)T^{n-1})} \qquad (1.22)$$

Substitution of (1.22) in (1.20) gives

$$\mathcal{L} = (\partial_\mu\eta)(\partial^\mu\eta) - \mu^2\eta^2 - \frac{1}{4} F^a_{\mu\nu}F^{\mu\nu a} +$$

$$g^2v^2\ (A^1_\mu A^{\mu 1}.....A^{N-1}_\mu A^{\mu N-1}) + ... \qquad (1.23)$$

From the above equation it is evident that the gauge field corresponding to the broken gauge symmetries acquire $(\text{mass})^2 = g^2v^2$ while the other gauge boson remain massless. There are no massless scalar particles in the theory. This phenomenon in which the gauge fields acquire mass with the disappearance of Goldstone bosons is known as Higgs mechanism[14-17]. The idea of Higgs mechanism is crucial to the construction of the unified electro-weak theory and grand unified models.

We have found that existence of degenerate minima leads to SSB and the gauge fields acquiring mass . In the next section we shall see that the existence of degenerate minima is responsible for the appearance of soliton and monopole solutions.

1.3 Solitons

Solitary waves are the localized nondissipative solutions of classical field equations. In some nonlinear dispersive systems nonlinear and dissipative effects balance each other and there can exist solutions with following properties :

1). A wave packet travels without any dispersion .

2). After a collision of two such solutions they continue to travel with out any distortion[18-20] .

Solutions satisfying (1) are known as solitory waves (or kinks or lumps) . Solitons are defined as the solutions satisfying the properties (1) and (2) . Loosely speaking we may refer to a solitory wave as a soliton

As an example from field theory let us consider the lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2\phi^2}{2} - \frac{\lambda}{4}\phi^4 \qquad (1.24)$$

in 1+1 dimension . The equation of motion is

$$\Box \phi = -m^2\phi - \lambda\phi^3 \qquad (1.25)$$

where $\Box = \partial_t^2 - \nabla^2$. This equation has time independent (static) solutions

$$\begin{aligned} \phi_+ &= a \tanh(mx/\sqrt{2}) \\ \phi_- &= -a \tanh(mx/\sqrt{2}) \end{aligned} \qquad (1.26)$$

where $a = \sqrt{-m^2/\lambda}$. When $x \longrightarrow \pm\infty$ this solutions assume the values $\pm\sqrt{-m^2/\lambda} = \pm a$ corresponding to the minimum of the above potential

The energy of the system is given by

$$H = \frac{1}{2}\int_{-\infty}^{\infty} dx \left[(\partial_x\phi)^2 + m^2\phi^2 + \frac{\lambda}{2}\phi^4 \right] \qquad (1.27)$$

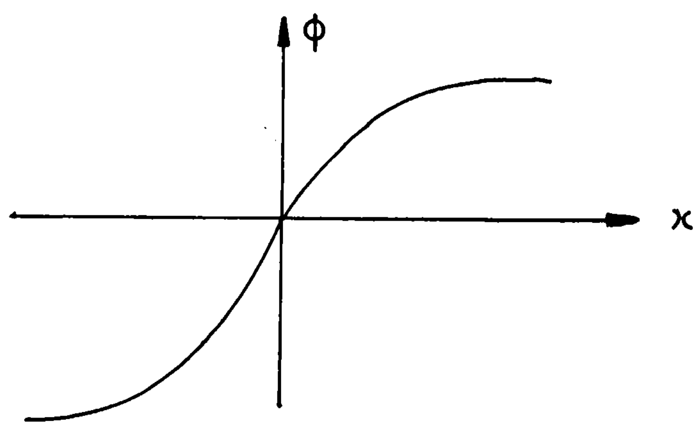

Fig. 2

When $x \longrightarrow \pm\infty$, ϕ take the values $\pm$ a and then energy is localized in space (Fig 2)

The solution given by Eq(1.26) is a soliton or kink.

Depending on the asymptotic value of ϕ at $x \longrightarrow \pm\infty$ we have the four sectors , namely

$$\begin{array}{llll} \phi = a & \text{as } x \longrightarrow \infty & \text{and } \phi = -a & \text{as } x \longrightarrow -\infty \\ \phi = -a & \text{as } x \longrightarrow \infty & \text{and } \phi = a & \text{as } x \longrightarrow -\infty \\ \phi = a & \text{as } x \longrightarrow \infty & \text{and } \phi = -a & \text{as } x \longrightarrow -\infty \\ \phi = -a & \text{as } x \longrightarrow \infty & \text{and } \phi = a & \text{as } x \longrightarrow -\infty \end{array} \tag{1.28}$$

These can be considered as mappings from spatial infinities to the potential minima . These mappings fall in to distinct topological classes and cannot be deformed into one another

Stability of soliton solutions can be related to their topological properties. The finite energy condition requires that, at spatial infinities

$$\phi(\infty) - \phi(-\infty) = n\ (2a) \tag{1.29}$$

$n = 0$ corresponds to a non topological soliton and $n = \pm 1$ is a soliton with winding number ± 1 (kink or antikink) . It is easy to see that

$$J^{\mu} = \varepsilon_{\mu\nu}\ \partial^{\nu}\phi \tag{1.30}$$

is a conserved current and the corresponding conserved charge

$$Q = \int_{-\infty}^{\infty} \partial^{x}\phi\ dx \tag{1.31}$$

is related to the soliton number n in (1.29). This is called the topological quantum number. It should be noted that topological current is not a Noether current arising from a symmetry of the lagrangian. The existence of the kind of topologically stable, finite energy solutions seen here in 1 + 1 dimensional field theories is possible only with degenerate vacua (SSB) in the theory. Such solutions also exist in higher dimensional field theories with SSB.

1.4 Magnetic monopoles

Finite energy solutions can also be found in more realistic model in 3 spatial dimensions . As a specific case we take the following lagrangian for a scalar triplet with SU(2) global symmetry :

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi).(\partial^\mu \phi) - V(\phi.\phi) \qquad (1.32)$$

where $V(\phi.\phi) = \frac{\lambda}{4}(\phi.\phi - \eta^2)^2$.

The potential minima occur at $\phi.\phi = \eta^2$. Evidently these points are connected by the SU(2) symmetry operators and hence lie on the surface of a sphere S^2 in three dimensional internal space. It is easy to see that when we go to quantum theory there is SSB. The finite energy requirements means that as $\hat{r} \longrightarrow \infty$ ϕ should approach the value in S^2. Since the spatial infinities also form a two sphere S^2_∞ the finite energy configurations can be labeled by a map

$$\phi : \quad S^2_\infty \longrightarrow S^2 \qquad (1.33)$$

These maps can be classified into the homotopy classes and each one characterized by an integer called its winding number. The homotopy class form a group called the second homotopy group denoted by $\Pi_2(S^2)$. It can be shown that

$$\Pi_2(S^2) = Z \tag{1.34}$$

where Z is a set of integers. The nontrivial topology of these configuration will ensure the topological stability of the finite energy configurations if they exist. However , it is not difficult to see that with scalar fields alone topologically stable finite energy solutions will not exist . To show this let us consider the expression for energy of static solution:

$$H = \int d^3x \, [\frac{1}{2} (\nabla\phi).(\nabla\phi) + V(\phi.\phi)] \tag{1.35}$$

Expansion of $\nabla\phi$ in radial coordinates gives

$$(\nabla\phi)^2 = (\partial_r\phi)^2 + (\hat{r} \times \vec{\nabla}\phi)^2 \tag{1.36}$$

Then if ϕ is expressed in radial coordinates, the second term in the above expression contribute $1/r^2$ and hence energy integral diverges. Hence with scalar fields alone there is no hope to get the finite energy solutions. One can prove the same result in any scalar theory in space time dimension $\geq$ 2. However, the discouraging result due to the Derrick[21] is no longer valid if we enlarge the theory by adding gauge fields. Also, in two dimensions this result is not valid if $V(\phi) = 0$ as is the case with O(n) non-linear σ model[18]. To illustrate what happens with the gauge fields let us study the gauged version of the example discussed above . The lagrangian for the model is

$$\mathcal{L} = \frac{1}{2}(D_\mu\phi).(D^\mu\phi) - \frac{\lambda}{4}(\phi.\phi - \eta^2) - \frac{1}{4}F^{\ a}_{\mu\nu}F^{\mu\nu a} \tag{1.37}$$

This is essentially the Georgi-Glashow[22,23] model with the fermion fields ignored. Choosing a vacuum where the scalar fields have v.e.v

$$\langle\phi\rangle_o = \begin{bmatrix} 0 \\ 0 \\ v \end{bmatrix} \tag{1.38}$$

the SU(2) symmetry is spontaneously broken to U(1). In the spherically symmetric ´t Hooft-Polyakov[24-27] ansatz one looks for solutions of the form

$$\phi^a = \frac{r^a}{gr^2}H(r) \quad ; \; A_i^b = -\varepsilon_{bij}\frac{r^j}{gr^2}[1 - K(r)] \quad ; A_o^b = 0 \tag{1.39}$$

where $r^i = x^i$ is the radial variable and H and K are dimensionless functions which are to be determined from the equation of motion . This ansatz defines a mapping of winding number 1 . The equations of motion following from the lagrangian (1.37) are

$$\begin{aligned} (D_\nu F^{\mu\nu})_a &= -g\,\varepsilon_{abc}\phi^b(D^\mu\phi)_c \\ (D^\mu D_\mu\,\phi)_a &= -\lambda\,\phi_a(\phi.\phi - \eta^2) \end{aligned} \tag{1.40}$$

Substitution of the ansatz (1.39) yields the following equations for H and K.

$$\begin{aligned} r^2H'' &= H\,(2k^2 - m^2r^2 + \frac{\lambda}{g^2}H^2) \\ r^2K'' &= K\,(K^2 + H^2 - 1) \end{aligned} \tag{1.41}$$

where $m^2 = \eta^2\lambda$. The total energy of a stable solution is given by

$$E = \int \mathcal{H} d^3x = - \int \mathcal{L}\, d^3x$$

$$= \int d^3x \Big[1/4\, F^a_{ij}\, F^a_{ij} + 1/2\, D_i\phi^a\, D_i\phi^a + V(\phi) \qquad (1.42)$$

$$+ 1/2\, F^a_{oi} F^a_{oi} + 1/2\, D_o\phi^a\, D_o\phi^a \Big]$$

By substituting Eq.(1.39) in Eq.(1.42) we get the energy integral as

$$H = \frac{4\pi}{g^2} \int_0^\infty dr \left\{ (k')^2 + \frac{(rH' - H)^2}{2r^2} + \frac{(K^2 - 1\,)^2}{2r^2} + \frac{K^2H^2}{2r^2} + \right.$$

$$\left. \frac{\lambda r^2}{4g^2} (H/r^2 - g^2m^2/\lambda\,)^2 \right\} \qquad (1.43)$$

In order that this integral be finite the functions H and K defined in EQ. (1.39) should satisfy the following conditions.

$$\begin{aligned} H(r) &\longrightarrow 0 \quad ; \quad K \longrightarrow 1 \text{ as } r \longrightarrow 0 \\ H(r) &\longrightarrow \frac{g\, m}{\sqrt{\lambda}}\, r \; ; \quad K \longrightarrow 0 \text{ as } r \longrightarrow \infty \end{aligned} \qquad (1.44)$$

As stated earlier, corresponding to the unbroken U(1) symmetry along ϕ_3 direction there should be a massless gauge field, the electromagnetic field. There is however no unique way to identify this U(1) gauge field through out the space. 't Hooft proposed gauge invariant definition [24,26]

$$F_{\mu\nu} = \frac{1}{|\phi|} F_{\mu\nu}.\phi - \frac{1}{g\phi^3} \varepsilon_{abc}\phi_a\, D_\mu\phi_b\, D_\nu\phi_c \qquad (1.45)$$

This can also be written as[28]

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{1}{g} \varepsilon_{abc}\hat{\phi}_a\, D_\mu\hat{\phi}_b\, D_\nu\hat{\phi}_c \qquad (1.46)$$

where $A_\mu = A_\mu^a \hat{\phi}_a$ and $\hat{\phi}_a = \dfrac{\phi_a}{|\phi|}$. From Eq.(1.39) we can find that for $r > 0$, $A_\mu = 0$, and $\hat{\phi}_a = \hat{r}_a$. Then Eq.(1.46) gives

$$F_{0i} = E_i = 0$$
$$\frac{1}{2}\varepsilon_{ijk}F_{jk} = B_i = \frac{1}{g}\frac{\hat{r}_i}{r^2} \tag{1.47}$$

This is the electromagnetic field of a point magnetic monopole at rest with magnetic charge $1/g$.

The 't Hooft's definition of electromagnetic field tensor is singular at the origin[24,25]. Another equivalent nonsingular definition due to Fadeev is [24,29]

$$F_{\mu\nu} = F_{\mu\nu}^a \hat{\phi}_a \tag{1.48}$$

The ansatz in Eq.(1.39) corresponds to a map of winding number 1 and magnetic charge one unit of $1/g$. The general form of the relation between winding number and magnetic charge is [28]

$$Q_m = \frac{n}{g} \tag{1.49}$$

Unlike electric charge magnetic charge has a topological origin.

Making use of the asymptotic condition of K and H it is straight forward to deduce from Eq.(1.43) that for large r[25]

$$K(r) \simeq O(\exp(-M_v r))$$
$$H(r) \simeq \frac{gm}{\sqrt{\lambda}} r + O(\exp(-\mu r)) \tag{1.50}$$

where $\mu = \sqrt{2}\, m$ is the mass of the Higg's particle and $M_v = gm/\sqrt{\lambda}$ is mass of scalar gauge boson. Each field approach the asymptotic form that is determined by the corresponding mass. Hence we can think of 't Hooft - Polyakov monopole having a definite size determined by these masses. For distances larger than this size the field is essentially is that of a Dirac monopole.

So far our concern was with the asymptotic form of the ansatz function which will ensure finiteness of energy. Let us now consider the nature of the exact solution of Eq.(1.41). For non zero values of m^2 and λ , analytic solution are not known. However in the limit $m^2 \longrightarrow 0$, $\lambda \longrightarrow 0$ but m^2/λ finite (Prasad-Sommerfield limit)[30] analytic solution have been found. The solutions are

$$K(r) = \frac{\beta r}{\sinh \beta r} \quad ; \quad H(r) = \beta r \operatorname{Coth} \beta r - 1 \qquad (1.51)$$

Where $\beta = gm/\sqrt{\lambda}$. Asymptotically ($r \longrightarrow \infty$) these solutions become

$$H(r) = ar + b \quad ; \quad K(r) = 0 \qquad (1.52)$$

Comparison of (1.52) with Eq (1.49) shows that the constant a can be identified as the inverse diameter of the monopole.

The 't Hooft-Polyakov monopole possesses only magnetic charge and does not carry electric charge. Julia and Zee showed that this is so because A_o^a is set equal to zero. For nonzero values of A_o^a there can be nonvanishing electric fields. Magnetic monopoles which carry electric charge as well are known as dyons[32]. Julia and Zee[24,25,31] obtained dyon solutions with the ansatz

$$\phi^a = \frac{r^a}{gr^2} H(r)$$

$$A_i^b = - \varepsilon_{bij} \frac{r^j}{gr^2} [1 - K(r)] \qquad (1.53)$$

$$A_o^a = \frac{r^a}{gr^2} J(r)$$

The equation of motion now become

$$r^2 J'' = J (2 K^2)$$

$$r^2 H'' = H (2k^2 - m^2 r^2 + \frac{\lambda}{g^2} H^2) \qquad (1.54)$$

$$r^2 K'' = K (K^2 + H^2 - J^2 - 1)$$

and the energy is

$$H = \frac{4\pi}{g^2} \int_0^\infty dr \left\{ (k')^2 + \frac{(rH' - H)^2}{2r^2} + \frac{(K^2 - 1)^2}{2r^2} + \frac{(K^2 - 1)^2}{2r^2} \right.$$

$$\left. + \frac{K^2 (H^2 - J^2)}{r^2} + \frac{(rJ' - J)^2}{2r^2} + \frac{\lambda r^2}{4g^2} (H/r^2 - g^2 m^2 / \lambda)^2 \right\} \qquad (1.55)$$

For the energy to be finite the ansatz functions must have the behavior

$$K(r) = O(\exp [-\sqrt{M_v^2 - M^2}\, r])$$

$$J(r) = M r + d + O(1/r) \qquad (1.56)$$

$$H(r) = \frac{g\, m}{\sqrt{\lambda}} r + O(e^{-\mu r})$$

when $r \longrightarrow \infty$ and

$$H(r) \longrightarrow 0$$

$$J(r) \longrightarrow 0 \qquad (1.57)$$

$$K(r) \longrightarrow 0$$

when $r \longrightarrow 0$. Here μ and d are parameters . $|M| < M_v$ but d is unrestricted.

In the ps limit the exact dyon solutions are obtained as

$$K(r) = \frac{\beta r}{\sinh \beta r}$$
$$H(r) = \cosh \eta(\beta r \operatorname{Coth} \beta r - 1) \qquad (1.58)$$
$$J(r) = \sinh \eta(\beta r \operatorname{Coth} \beta r - 1)$$

The asymptotic form ($r \longrightarrow \infty$) of these solutions are:

$$K(r) = 0$$
$$H(r) = ar + b \qquad (1.59)$$
$$J(r) = cr + d$$

By using Gauss's law, the electric charge of the dyon can be written as

$$Q = \frac{4\pi}{g} d = - \frac{4\pi}{g} \sinh \eta \qquad (1.60)$$

The original motivation of Dirac[61] in proposing the existence of magnetic monopoles was to explain the quantization of electric charge. It appears that magnetic monopoles present in almost all grand unified theories . The presence of monopoles and dyons leads to several interesting phenomena such as fermion number fractionisation , spin isospin mixing, and baryon number violation

1.5 Fermion number fractionisation

Usually in a quantum theory vacuum is characterized by zero charge or zero fermion number . Consequently any state obtained by the evolution of a local operator as excitation of vacuum will also have integer fermion number . But it has been found that in presence of solitons fermion number of vacuum becomes fractional . This effect is known as fermion number fractionisation and was first analyzed by Jackiw and Rebbi[33]. It occurs in a number of models of phenomenological importance, in presence of magnetic monopoles and other solitons[35].

The fermion number is the conserved charge corresponding to the abelian phase change of a lagrangian of a Dirac field . The conserved current is[34]

$$J^{\mu} = \frac{1}{2} [\bar{\psi} , \gamma^{\mu} \psi] \tag{1.61}$$

To determine the spectrum of conserved charge the standard method is to expand ψ in terms of the plane wave solutions of Dirac equation :

$$\psi(x) = \int dk \, [\; b_k \; \psi_k(x) \; e^{-iEt} \; - \; d_k^+ \; C \; \psi_k^*(x) \; e^{iEt}] \tag{1.62}$$

Where $\psi_k(x)$ and $\psi_k(x)$ are positive and negative energy solutions and C the charge conjugation matrix . Here we assume that the theory is C-invariant. Substitution of Eq(1.62) in (1.61) gives the conserved charge as

$$Q = \int d^3x \; J^0(x) \; = \int dk \; (b_k^+ \; b_k \; - \; d_k^+ \; d_k) \tag{1.63}$$

This leads to

$$Q|\ 0 \rangle = 0 \tag{1.64}$$

implying that fermion number of the vacuum is zero

However there are cases in which the Dirac equation possess zero energy solution . This generally occurs for fermions in the background of solitons . As an example let us consider the following lagrangian in 1+1 dimension

$$\mathcal{L} = \bar{\psi}(i\ \gamma^{\mu}\partial_{\mu} - \phi(x)\ )\ \psi \tag{1.65}$$

Where $\phi(x)$ is soliton field taken as the background

In this case

$$\gamma^{1} = \alpha = \sigma^{2} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}\ :\ \gamma^{0} = \beta = \sigma^{1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{1.66}$$

The Dirac equation can brought to the form

$$\begin{aligned} [\ \partial_{x} + \phi(x)\ ]\ U(x) &= E\ V(x) \\ [\ -\partial_{x} + \phi(x)\ ]\ V(x) &= E\ U(x) \end{aligned} \tag{1.67}$$

where $\psi = \begin{bmatrix} U \\ V \end{bmatrix}$. The zero energy solution of the above equation are obtained as

$$\begin{aligned} U(x) &= \alpha\ \exp(\ -\int_{0}^{x} \phi(y)\ dy\ ) \\ V(x) &= \beta\ \exp(\ \int_{0}^{x} \phi(y)\ dy\ ) \end{aligned} \tag{1.68}$$

These solutions are normalisable if $\phi(x)$ is topological soliton and either α or β is zero. With zero energy solution the eigen mode expansion become

$$\psi(x) = a\,\psi_o(x) + \int dk\,[\;b_k\,\psi_k(x)\,e^{-iEt} - d_k^+\,C\,\psi_k^*(x)\,e^{iEt}] \qquad (1.69)$$

The operators b_k^+ , b_k and d_k^+ , d_k are the creation and annihilation operators of fermion and anti fermion in the soliton sector (bound states of fermions and anti fermions with solitons). Here a is associated with the zero energy eigen mode and when operating on any other state it gives a ground state with same energy. Therefore soliton ground state must be doubly degenerate . Let us denote these states by $|\pm\,,S\rangle$. The operators b and d obey the anticommutation rules

$$\{\,d_{k'}\,,\,d_k^+\,\} = \{\,b_{k'}\,,\,b_k\,\} = \delta(k' - k) \qquad (1.70)$$

with all other anti commutators vanishes . If we assume the same algebra for a:

$$\{\,a\,,\,a^+\} = 1 \qquad (1.71)$$

we get

$$\begin{aligned} a^+|+\,,S\rangle &= |-\,,\,S\rangle \\ a|-\,,S\rangle &= |+\,,\,S\rangle \\ a^+|-\,,S\rangle &= a|+\,,S\rangle = 0 \end{aligned} \qquad (1.72)$$

The fermion number operator is obtained by substituting the expansion (1.69) in the expression for $Q = \int dx\; J^o(x)$ and with the aid of (1.72) we find

$$Q|\pm, S\rangle = \pm \frac{1}{2} |\pm, S\rangle \tag{1.73}$$

This means that the ground state charge of soliton is $\pm$ 1/2. Consequently all the other states will also have fractional fermion number

In the above example we assumed that the theory is C-invariant. But if the theory is not C- invariant, it is not possible to calculate the vacuum charge by the above method. In that case Goldstone and Wilczek developed a method in which fermion number fractionisation is considered as due to the polarization of vacuum by solitons . As an example let us consider the lagrangian

$$\mathcal{L} = \bar{\psi}\,(i\gamma^{\mu}\partial_{\mu} - \phi_1(x) + i\gamma^5\phi_2(x))\,\psi \tag{1.74}$$

The theory is not C- invariant. Another important feature is that the interaction term is invariant under chiral rotation. Here ϕ_1 and ϕ_2 are soliton profiles with

$$\begin{aligned} \phi_1(\pm\infty) &= \phi_1^{\pm} \\ \phi_2(\pm\infty) &= \phi_2^{\pm} \end{aligned} \tag{1.75}$$

For free fermions the ground state expectation value of current can be expressed as[34]

$$\langle 0|J^{\mu}(x)|0\rangle = \underset{x' \longrightarrow x}{\mathrm{Lt}}\; ie\,\mathrm{Tr}\,[\gamma^{\mu}\, S_F(x',x)] \tag{1.76}$$

where $S_F(x',x)$ is the Greens function satisfying

$$(i\,\partial\!\!\!/_x - m)\, S_F(x',x) = \delta(x' - x) \tag{1.77}$$

for free fermions. Here we take $m = |\phi| = |(\phi_1^2 + \phi_2^2)|^{1/2}$. Let S'_F be the Greens function in presence of solitons ϕ_1 and ϕ_2. Then

$$(i \not{\partial}_x - m + \phi_1 + i\gamma^5 \phi_2)\, S'_F(x',x) = \delta(x' - x) \qquad (1.78)$$

$$(i \not{\partial}_x - m)\, S'_F(x',x) = \delta(x' - x) - (\phi_1 + i\gamma^5 \phi_2)\, S'_F(x',x)$$

Then the two Greens functions are related by an integral equation[2,34]

$$S'_F(x',x) = S_F(x',x) - \int dy\, S_F(x',y)[\,\phi_1(y) + i\gamma^5\phi_2(y)\,]\, S'_F(y,x) \qquad (1.79)$$

the first iteration of which gives

$$S'_F(x',x) = S_F(x',x) - \int dy\, S_F(x',y)[\,\phi_1(y) + i\gamma^5\phi_2(y)\,]\, S_F(y,x) \qquad (1.80)$$

The current induced in the soliton ground state is therefore is given by

$$\langle 0|J^\mu|0\rangle = \operatorname*{Lt}_{x' \to x} \{\, [\mathrm{Tr}\gamma^\mu S_F(x',x)] - \int dy\, \mathrm{Tr}\,[\gamma^\mu\, S_F(x',y)(\phi_1(x) + i\gamma^5\phi_2(x))S_F(y,x)\} \qquad (1.81)$$

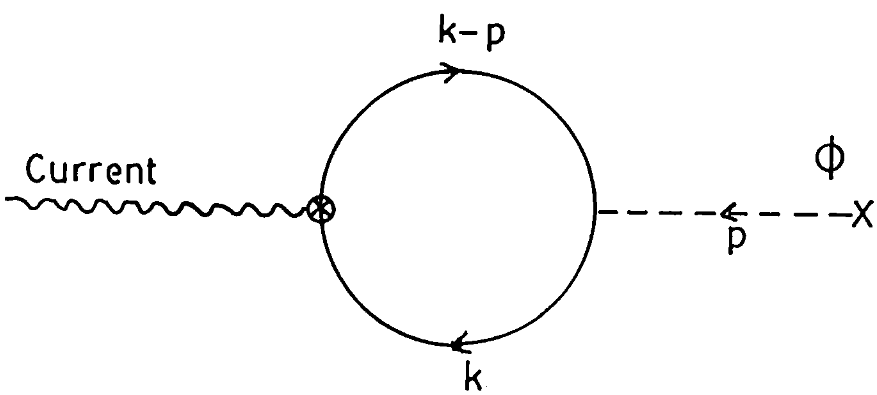

Fig. 3

The first term is vanishing since this corresponds to the vacuum charge without any perturbation . The second term is represented by digram (Fig 3). A straight forward calculation of the diagram gives[36]

$$\langle J^{\mu}(x)\rangle = (2\pi)^{-1} \frac{\varepsilon^{\mu\nu}\,\varepsilon_{ab}\,\phi^{a}\,\partial_{\nu}\phi_{b}}{|\phi|^{2}} \tag{1.82}$$

where $|\phi|^2 = \phi_1^2 + \phi_2^2$. Then the charge is

$$Q = \int J^{0}(x)\,dx = 1/2\pi\ \mathrm{Tan}^{-1}(b/a) \tag{1.83}$$

where $\phi_2(\pm) = \pm\, b$ and $\phi_1(\pm) = \mp\, a$. This is the expression for induced charge through vacuum polarization when soliton fields evolve adiabatically . In the process it may happen that some of the energy levels cross zero of the spectrum[38,39]. Then the ground state charge is given by[37]

$$Q_{ground} = (n_+ - n_-) + Q_{induced} \tag{1.84}$$

where n_+ is the number levels crossing to the positive side of the energy spectrum and n_- is the number levels crossing to the negative side

If the theory is not C-invariant ,several alternative methods has been developed for the calculation of induced charge. Bardeen, Elitzur, Frishman, Rabinovici[40,41] used the connection between fermion number fractionisation and chiral anomalies to show that some of the results of Goldstone and Wilczek can be derived using anomalous commutators. Roy and Singh[42] analyzed the problem by considering the problem in a finite dimensional box and then applying boundary conditions. A non perturbative technique has been developed by Niemi and Semenoff[43-47] for a particular class of field theory models. A mathematical technique for the computation of fermion numbers for arbitrary Dirac hamiltonians has also been introduced by Lott[48].

There are many circumstances in which soliton number is an observable. If it couples to the U(1) gauge field it may be observed as the electric charge of the soliton . An experimental realization of this phenomena is in linearly conjugated polymers, for example, polyacetylene[49-53]. Here the fractional charge can be detected through the enhancement of conductivity. In these systems neutral spin 1/2 solitons have also been observed through electron spin resonance experiments. It has also found that fermion number fractionisation plays an important role in the current algebra soliton model of hadrons (Skyrme model)[54,55]. In a related development, it has been found that the ground state charge of the chiral version of the MIT bag model also carries fractional fermion number[56,57]. The fractional fermion charges of 't Hooft-Polyakov magnetic monopoles are important in the study of monopole catalysis of proton decay[58-60,79].

1.6 Monopoles and Fermions

Fermion monopole systems have several interesting features . One of these is the existence of states with peculiar angular momentum properties . To understand this let us consider a classical charged particle moving in the magnetic field of a monopole given by $\vec{B} = \frac{g}{r^2}\hat{r}$. The charged particle will experience a Lorentz force $e(\vec{r} \times \vec{B})$ and change in angular momentum is

$$\partial_t(\vec{r} \times m\dot{\vec{r}}) = \vec{r} \times m\ddot{\vec{r}} = \frac{eg}{r^2}\,\vec{r} \times (\dot{\vec{r}} \times \hat{r})$$
$$= \partial_t(eg\,\hat{r}) \qquad (1.85)$$

This suggests that we can define the conserved total angular momentum of the charge-pole system as [61]

$$\vec{J} = \vec{r} \times \vec{p} - eg\,\hat{r} \qquad (1.86)$$

In the quantum mechanical[62] description of the system the angular momentum is generally integer or half integer in units of $h/2\pi$. This means that

$$2eg = 2q = \text{integer} \qquad (1.87)$$

This is the famous Dirac[63] quantisation condition . If 2q is odd, two bosons may be combined to give a fermion [64,65]. In (1.88) the peculiar angular momentum is associated with the electromagnetic field . In SU(2) gauge theories, electromagnetic field survives SSB and in presence of a monopole the peculiar angular momentum leads to the phenomena of " spin from isospin "[66,67]

For fermions the Eq (1.86) get modified to

$$\vec{J} = \vec{r} \times \vec{P} - eg\,\hat{r} + \vec{\sigma}/2 \qquad (1.88)$$

where $\vec{\sigma}/2$ is the intrinsic angular momentum. Therefore if we consider the scattering of an electron by a monopole in which $\hat{r}$

$\longrightarrow -\hat{r}$, the conservation of angular momentum requires either $\vec{\sigma} \longrightarrow -\vec{\sigma}$ or $e \longrightarrow -e$. That is , either there should be spin flip or some charge has to be deposited on the monopole core by the electron . However , it can be shown that for abelian Dirac monopole the second alternative is not available[69]

On the other hand , from the Dirac hamiltonian

$$H = \gamma_5 \, \vec{\sigma}.\vec{P} + \beta \, m \tag{1.89}$$

we find that

$$[\, H \, , \, \vec{\sigma}.\vec{P} \,] = 0 \tag{1.90}$$

This implies that helicity is conserved and spin can not flip . This paradox is resolved by means of a partial wave analysis of the Dirac equation[69]. For a point monopole one finds that the wave function diverges according to $\psi_0 \sim 1/r$ as $r \longrightarrow 0$ irrespective of the boundary condition . The origin of this problem can be traced to the simple fact that the hamiltonian of this system is not selfadjoint in the space of scattering solutions . The same thing happens with the bound state solutions and hence an electron cannot form bound states with Dirac monopole[70] .

In order to have a well defined self adjoint scattering problem we have to impose a special boundary condition at $r = 0$ which relates the positive and negative eigen functions of the operator $\hat{q} \, \gamma^5$. The most general condition is[71,72]

$$\psi_+(0) = e^{i\vartheta}\psi_-(0) \qquad (1.91)$$

where ϑ is an arbitrary phase angle

This yields a one parameter family of self adjoint extensions of the hamiltonian (1.89). It is the boundary condition (1.91) which is responsible for the helicity flipping .

An alternative to the boundary condition (1.91) is to modify the hamiltonian by including a dyon charge[73] e at the centre of the monopole. The hamiltonian now becomes selfadjoint and then results bound states of electrons and abelian point monopoles . These states are also parameterized by the phase angle ϑ .

We can attach a physical significance to the phase angle ϑ by regarding the Dirac monopole as the limiting case of a nonabelian monopole . The vacuum charge of the monopole due to the zero point fluctuations of fermion field around the monopole is given by [69]

$$\langle Q \rangle = - \frac{e\vartheta}{2\pi} \qquad (1.92)$$

This is analogous to the Witten effect[75] in which nonabelian monopoles acquires fractional dyon charge in an instanton background characterized by the vacuum angle ϑ

Let us now consider a system involving fermions and nonabelian monopoles described by the lagrangian for the spontaneously broken SU(2) model

$$\mathcal{L} = -\frac{1}{4} F^{a}_{\mu\nu} F^{\mu\nu a} + \frac{1}{4}|D_\mu\phi^a|^2 - V(\phi) + \bar{\psi}(iD - M)\psi \tag{1.93}$$

where M is simply the Dirac mass $M = m$ or represents a Yukawa coupling $M = \lambda\, \phi^a\tau^a$. When a charged fermion scatter of a 't Hooft - Polyakov monopole the total angular momentum is given by

$$\vec{J} = \vec{L} + \vec{S} + \vec{T} \tag{1.94}$$

where $\vec{S}$ is the ordinary spin and $\vec{T}$ is the extra spin coming from the charge field interaction . Both helicity flip and charge exchange are possible in this case as that for a point monopole. However it is to be noted that the nonabelian monopole is a nonsingular object and hence no special boundary conditions are necessary. Further , the nonabelian monopole can carry charge and hence may be transformed to a dyon state.

A quantum mechanical analysis of scattering of fermions from monopoles can be carried out by studying the Dirac equation with the field of nonabelian monopole as the background[76,77]:

$$\alpha.(\vec{p} - \frac{A(r)}{2}(\hat{r} \times \vec{\tau}\,))\, \psi_m = (E - \beta M)\, \psi_n \tag{1.95}$$

Defining

$$\psi = \begin{bmatrix} \chi^+ \\ \chi^- \end{bmatrix}$$

and by using the representation

$$\vec{\alpha} = \begin{bmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{bmatrix} \quad \text{and} \quad \beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

of the Dirac matrices the above equation reduces to

$$\left[i\,\vec{\sigma}.\vec{\nabla} + \frac{1}{2r}A(r)(\vec{\sigma}\times\hat{n}).\vec{\tau}\right]\chi^{\pm} = -(E \pm M)\,\chi^{\mp} \qquad (1.96)$$

Writing

$$\chi^{\pm}_{\alpha r} = \frac{1}{r}[\,g^{\pm}(r)\,\varepsilon_{\alpha,r} + h^{\pm}(r)\,(\vec{\sigma}.\vec{n})_{\alpha\beta}\,\varepsilon_{\beta r}] \qquad (1.97)$$

the above equations can be converted to the following equations for $g^{\pm}$ and $h^{\pm}$:

$$\begin{aligned}
&[\,\partial_r + r^{-1} - A(r)\,]\; g^{\pm} = (E \pm M)\; h^{\mp} \\
&[\,\partial_r + r^{-1} - A(r)\,]\; h^{\pm} = -\,(E \pm M)\; g^{\mp}
\end{aligned} \qquad (1.98)$$

Marciano and Muzinich[76,77] obtained the solution of the equation in the following form

$$\begin{aligned}
g^{\pm}(r) = C^{\pm}\Big\{&\left[\frac{2ik}{m} + \tanh\left(\frac{Mr}{2}\right)\right]e^{-ikr} - \\
&\left[\frac{2ik}{m} - \tanh\left(\frac{Mr}{2}\right)\right]\Big\}\,e^{ikr}
\end{aligned} \qquad (1.99)$$

$$\begin{aligned}
h^{\pm} = -\,C^{\pm}\frac{k}{E+M}\Big\{&\left[\frac{2ik}{m} + \coth\left(\frac{Mr}{2}\right)\right]e^{-ikr} + \\
&\left[\frac{2ik}{m} - \coth\left(\frac{Mr}{2}\right)\right]\Big\}\,e^{ikr}
\end{aligned} \qquad (1.100)$$

By choosing the normalization constant $C^{+} = 1/2$ and $C^{-} = i\left(\frac{2}{E+M}\right)$, Marciano and Muznich have shown that the above solution corresponds to incoming right handed spinor with charge $Q = -1/2$ scattering into a left handed spinor with charge $Q = 1/2$.

Similarly by reversing the sign of C^- we can find that right handed spinor with charge Q = 1/2 get scattered to left handed spinor with charge Q = -1/2. That is, in this process there is a charge exchange at the centre. A full understanding of this process is possible only with quantum field theoretic analysis[80,81]. It may be noted that in the above calculation the Yukawa coupling has been neglected . The inclusion of Yukawa coupling can lead to helicity flipping interaction . This also leads to the existence of Jackiw and Rebbi zero modes which can result in fractional fermion numbers for monopoles. The question of fractional fermion number of monopoles and dyons are discussed in in detail in Ch 4 and 5

Dokos and Tomaras[78] pointed out that magnetic monopoles can catalyze process which change baryon number. They noted that dyonic excitations of SU(5) monopole have baryon number violating couplings , and that a collision that excite the dyon degree of freedom need not conserve baryon number . Wilczek[79] and Rubakov[80,81] emphasized that because of axial anomaly[82,83] the monopole is not an eigen state of chirality or baryon number . This leads to the chirality violating or baryon number violating cross section in monopole-fermion scattering

Callan [59,60] and Besson[86] recognized that the baryon number violating interactions inside the core can induce baryon number changing scattering process with cross section unsuppressed by the exceedingly small core size . There exists an extensive literature on monopole catalysis of proton decay and its experimental consequences .

CHAPTER 2

GROUND STATE CHARGE OF SOLITONS IN 1+1 DIMENSIONS

2.1 Introduction

The fermion number fractionisation in 1+ 1 dimension has been investigated by several authors[33,35,36]. In their pioneering work, Jackiw and Rebbi[33] found that in presence of solitons in 1+1 dimension there is zero energy state and spectral symmetry . This leads to ground state charge ± 1/2. When the theory is not C-invariant the calculation of vacuum charge is not straight forward. In such cases Goldstone and Wilczek[36] introduced the method of adiabatic evolution of the fields from vacuum configuration outlined in §1.4. Later Mackenzie and Wilczek[38,39] investigated the problem and found that during the adiabatic evolution , there can be spectral flow and adiabatic calculation is not always comprehensive . Niemi and Semenoff[35,43-47] by analyzing the spectrum of Dirac hamiltonian formulated an index theorem for the Dirac hamiltonian and found that the continuum part of the ground state charge is related to the topology of the background field. A closer investigation of this problem was done by Blankenbecler and Boyanovsky[85] . They found that even though the induced charge is a topological invariant, the spectral flow may depend on the width of the solitons.

In this thesis we present an alternative approch for the calculation of ground state charge in non C-invariant models. Instead of evolving the whole background field adiabatically from vacuum we start from a vacuum having a background field which leads to a zero energy bound state and spectral symmetry .Under these conditions the vacuum charge is ±1/2. Now the other fields are evolved adiabatically so that theory loses the C-invariance and zero energy state disappears. The induced charge is calculated by the method of Goldstone and Wilczek[30]. If N is the induced charge,ground state charge is

$$Q_{ground} = N \pm 1/2 -(n_+ -n_-) \qquad (2.1)$$

where n_+ is the number of levels crossing to the positive side of energy spectrum and n_- is the number of levels crossing to the negative side. $n_\pm$ is obtained by analyzing the bound state spectrum of the theory. The ground state charge is independent of the way one arrives at the final configuration. The induced charge and spectral flow may, however, depend on the way one reaches the final configuration.

In this chapter we illustrate our technique by applying it to some simple models in 1+1 dimensions. Application to more important realistic models are given in Ch 4 and Ch 5. In § 2 the induced charge is calculated from vacuum polarization diagrams. Spectral flow is calculated by analyzing the bound state spectrum. In § 3 spectral flow is calculated in the case of solitons of finite width. It is shown that the ground state charge is independent of the soliton width .

2.2 Polarization of vacuum by solitons

Consider the Lagrangian for a massless Dirac field and soliton field in 1+1 dimensions

$$\mathcal{L} = \bar{\psi}\ (i\partial\!\!\!/ - \phi_1 + i\gamma^5\phi_2)\psi \tag{2.2}$$

The one dimensional Dirac algebra

$$\{\gamma^\mu,\gamma^\nu\} = 2\ g^{\mu\nu} \quad ; \quad \{\gamma^5,\gamma^\mu\} = 0 \quad ; \quad (\gamma^5)^2 = 1 \tag{2.3}$$

is represented by $\gamma^5 = \gamma^0\gamma^1 = \sigma_2$, $\gamma^0 = \sigma^1$, $\gamma^1 = i\sigma^3$ and ϕ_1 and ϕ_2 are soliton profiles with

$$\begin{aligned} \phi_1(\pm\infty\ ) &= \phi_1(\pm) \\ \phi_2(\pm\infty\ ) &= \phi_2(\pm) \end{aligned} \tag{2.4}$$

Let us start from the Lagrangian

$$\mathcal{L}_0 = \bar{\psi}\ (i\partial\!\!\!/ - \phi_1(x))\psi \tag{2.5}$$

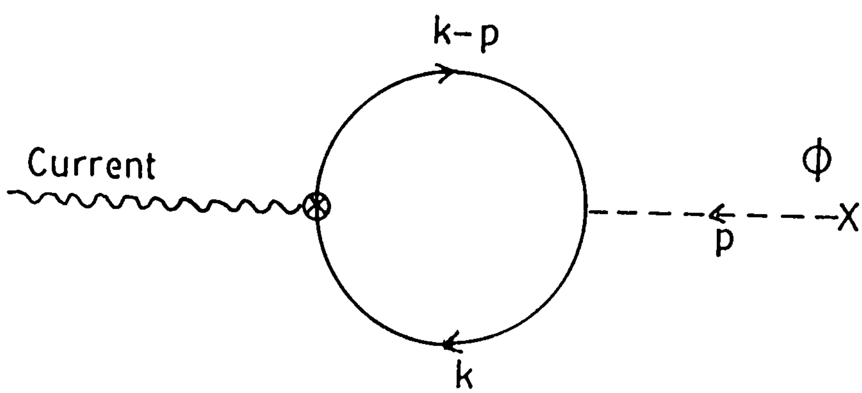

Fig.4

An analysis of the bound state spectrum (given in § 2.3)shows that the theory is C-invariant and that there is zero a energy bound state and hence ground state charge is ± 1/2. The second soliton profile ϕ_2 is now adiabatically evolved so that we reach the configuration (2.2). Induced current can be calculated from the Feynman diagram (Figure 4) as

$$\langle J^{\mu}(p)\rangle = \int \frac{d^2k}{(2\pi)^2}\, \mathrm{Tr}\left[\gamma_\mu \frac{1}{((\not k - \not p) - |\phi|)} \gamma_5 \frac{1}{(\not k - |\phi|)} \right] \tag{2.6}$$

where $|\phi|^2 = a^2 + \phi_2^2$. a is the average of the asymptotic values ϕ_1. The above integral is evaluated to give (Appendix A)

$$\langle J^{\mu}(x)\rangle = (2\pi)^{-1} \frac{a\, \varepsilon^{\mu\nu}\, \partial_\nu \phi_2}{|\phi|^2}$$

$$= (2\pi)^{-1} \varepsilon^{\mu\nu}\, \partial_\nu\, \mathrm{Tan}^{-1}\left(\phi_2/a\right)$$

Therefore the induced charge is

$$Q_{induced} = (2\pi)^{-1}\left[\mathrm{Tan}^{-1}\left(\phi_2^{(+)}/a\right) - \mathrm{Tan}^{-1}\left(\phi_2^{(-)}/a\right)\right]$$

In the case when $\phi_2(\pm\ ) = \mp b$, the induced charge becomes

$$Q_{induced} = -(\pi)^{-1}\, \mathrm{Tan}^{-1}(\, b/a) \tag{2.8}$$

2.3 Spectral flow and ground state charge in presence of solitons

The spectral flow can be calculated by analyzing the bound state spectrum. The Dirac equation following from (2.2) can be written as

$$\begin{bmatrix} \phi_2(x) & \partial_x - \phi_1(x) \\ -\partial_x - \phi_1(x) & \phi_2(x) \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix} = E \begin{bmatrix} U \\ V \end{bmatrix} \tag{2.9}$$

This equation is to solved for $x < 0$ and for $X > 0$. On matching the solution at $x = 0$ we get the energy spectrum . For solving the above equation we assume

$$\left.\begin{aligned} \phi_2(x) &= \phi_2(+\infty) = -b \\ \phi_1(x) &= \phi_1(+\infty) = \phi_1(+) \end{aligned}\right\} \quad \text{for } x > 0$$

and (2.10)

$$\left.\begin{aligned} \phi_2(x) &= \phi_2(-\infty) = b \\ \phi_1(x) &= \phi_1(-\infty) = \phi_1(-) \end{aligned}\right\} \quad \text{for } x < 0$$

The solutions are readily found to be

$$U = \alpha \exp(-k_+ x) \quad ; \quad V = -\alpha\left[\frac{\phi_1(+) - k_+}{E - b}\right] \exp(-k_+ x) \tag{2.11}$$

for $x > 0$, and

$$U = \beta \exp(k_- x) \quad ; \quad V = -\beta\left[\frac{\phi_1(-) - k_+}{E - b}\right] \exp(k_- x) \tag{2.12}$$

for $x < 0$. α and β are integration constants and $k_\pm = [\phi_1^2(\pm) - (E^2 - b^2)]^{\pm 1/2}$ .On matching (2.11) and (2.12) at $x = 0$ we get bound states as the zeros of the function

$$f(E) = E[\phi_1(-) - \phi_1(+) + k_- + k_+] - b[\phi_1(-) + \phi_1(+) + k_- - k_+] \tag{2.13}$$

Evidently the zero energy solution when $b = 0$ is shifted to $E =$

b. Therefore the sign of the ground state charge when b = 0 is negative. As b changes from its initial zero value there will be a spectral flow and for each level crossing there must be a zero energy level for some value of b. To find the spectral flow it is then enough to find the number of values of b for which E = 0 is solution of (2.13). This is given by the zeros of the function

$$f(0) = -b\ \left[\ \phi_1(-) + \phi_1(+) + k_- - k_+\right] \tag{2.14}$$

$f(0)$ is symmetric about b = 0 and by calculating $\frac{\partial f(0)}{\partial\ b}$ it is easy to find that $f(0)$ is a monotonic function in b . Therefore energy level cross E = 0 only for one value of b , that is for b = 0 and hence spectral flow is zero. Then from (2.1) and (2.8) ground state charge is

$$Q_{ground} = -\ \frac{1}{\pi}\ \mathrm{Tan}^{-1}\left(\ b/a\ \right)\ -\ 1/2 \tag{2.15}$$

When b = 0 , that is with the lagrangian (2.5) $f(E) = -f(-E)$. Then there is zero energy state and spectral symmetry. This leads to ground state charge ± 1/2 as found by Jackiw and Rebbi . This is also obtained from (2.15) on taking b = 0 .
That is

$$Q_{ground} = -\ 1/2 \tag{2.16}$$

If ϕ_1 is a nontopological soliton or if ϕ_1 is taken as the fermion mass ,then , even though zero is a solution of (2.13), the wave function is not well behaved at $x = \pm\infty$. Consequently there is no zero energy state initially and hence

$$Q_{ground} = - \frac{1}{\pi} \mathrm{Tan}^{-1}\left(b/a \right) \tag{2.17}$$

as found in Ref.(36) and (44) . When $\phi_1 = 0$

$$Q_{ground} = -1/2 \tag{2.18}$$

This is also evident from (2.13) . Since ,when ϕ_1 is zero $k_+ = k_-$. Then equation (2.13) is reduced to $f(E) = E\ k_+$

2.4. Ground state charge of solitons of finite width in 1+1 dimensions

Let us consider background solitons of finite width with

$$\begin{aligned}
&\left.\begin{aligned} \phi_2(x) &= \phi_2(-\infty) = b \\ \phi_1(x) &= \phi_1(-\infty) = \phi_1(-) \end{aligned}\right\} \quad \text{for } x < -d/2 \\
&\left.\begin{aligned} \phi_2(x) \\ \phi_1(x) \end{aligned}\right\} = 0 \qquad \text{for } -d/2 < x < d/2 \\
&\left.\begin{aligned} \phi_2(x) &= \phi_2(+\infty) = -b \\ \phi_1(x) &= \phi_1(+\infty) = \phi_1(+) \end{aligned}\right\} \quad \text{for } x > d/2
\end{aligned} \tag{2.19}$$

In this case expression for induced charge is same as that given by (2.8). To find spectral flow we solve equation (2.9) in the background of soliton having the configuration (2.19). For $x < -d/2$ solutions are given by (2.12) and for $x > d/2$ solutions are given by (2.11). For $-d/2 < x < d/2$ equation (2.9) gives

$$\partial_x V = E\ U \quad \text{and} \quad \partial_x U = E\ V \tag{2.20}$$

with solutions

$$\left.\begin{aligned} U &= \left(\gamma \exp(iEx) + \delta \exp(-i\,Ex) \right) \\ V &= -\, i \left(\gamma \exp(iEx) - \delta \exp(-iEx)\right) \end{aligned}\right\} \qquad (2.21)$$

On matching the solutions at x = -d/2 and x =d/2 we get the condition

$$f(E) = \mathrm{Tan}(Ed) - \frac{\left[\dfrac{k_- + \phi_1(-)}{E + b}\right] - \left[\dfrac{k_+ + \phi_1(+)}{E - b}\right]}{1 + \left[\dfrac{k_- + \phi_1(+)}{E + b}\right]\left[\dfrac{k_+ + \phi_1(+)}{E - b}\right]} = 0 \qquad (2.22)$$

Evidently $f(E) = f(-E)$ and there is no zero energy state. When $\phi_2 = b = 0$, $f(E) = f(-E)$ and there is zero energy solution. As in the previous case spectral flow is calculated by analyzing $f(0)$.

$$f(0) = \frac{-\,b\left[\left(\phi_1^2(-) + b^2\right)^{\pm 1/2} + \phi_1(-) + \phi_1(+) + \left(\phi_1^2(+) + b^2\right)^{\pm 1/2}\right]}{b^2 - \left[\left(\phi_1^2(-) + b^2\right)^{\pm 1/2} + \phi_1(-)\right]\left[\phi_1(+) + \left(\phi_1^2(+) + b^2\right)^{\pm 1/2}\right]} \qquad (2.23)$$

Evidently $f(0) = 0$ when b = 0. It is easy to show that $f(0)$ is monotonic function of b. Then there is no spectral flow when the ϕ_2 field is evolved adiabatically. Then the ground state charge is given by

$$Q_{\text{ground}} = -\frac{1}{\pi}\,\mathrm{Tan}^{-1}\left(b/a \right) - 1/2 \qquad (2.24)$$

If we take massive fermions with mass M , equation (2.2) gets modified with $\phi_1 \longrightarrow \phi_1 + M$. If $\phi_1(\pm) = \pm a$, $k_\pm$ in (2.11) and (2.12) gets replaced by $k_\pm = [m_\pm^2 - (E^2 - b^2)]^{1/2}$ with $m_\pm = M \pm a$. Then the initial zero energy state when $b = 0$ exists only when $m_\pm > 0$ and hence ground state charge is discontinuous at $M = a$ as found in reference (89)

2.5 Conclusion

In this chapter we have calculated the ground state charge by a combination of adiabatic and spectral flow calculations . Our results are in agreement with that given by Blankenbeoler and Boyonovsky . In our model ground state charge is found to be independent of the soliton width . With massless fermions we reproduce the value ± 1/2 for ground state charge . It turns out that the ground state charge has a discontinuity at the fermion mass

2.A Appendix

In this appendix we present the details of the calculation of induced charge from the diagram given by Fig 4. Eq (2.6) can be written as

$$= \int \frac{d^2k}{(2\pi)^2} \frac{\mathrm{Tr}\, \gamma_\mu \left[(\not{k} - p) + |\phi|\right] \gamma_5 \left[\not{k} + |\phi|\right]}{\left[(k - p)^2 - |\phi|^2\right]\left[k^2 - |\phi|^2\right]}$$

By using Feynman variable α, we get

$$\langle J^\mu \rangle = ie \int_0^1 d\alpha \int \frac{d^2k}{(2\pi)^2} \frac{\mathrm{Tr}\, \gamma^\mu [\, (\not{k} - p) + |\phi|]\, \gamma^5 [\not{k} + |\phi|]}{[\, (k - \alpha p)^2 + \alpha(1-\alpha)p^2 - \phi^2]^2}$$

$$= ie\int^{1} d\alpha \int \frac{d^2k}{(2\pi)^2} \frac{\mathrm{Tr}\ \gamma^{\mu}[\not k + (\alpha - 1)\not p + |\phi|]\gamma^5 [\not k + \alpha \not p + |\phi|]}{[k^2 + \alpha(1-\alpha)p^2 - \phi^2]^2}$$

In 1+1 dimension $\mathrm{Tr}\ \gamma^{\mu} \not k \gamma^5 \not k = 0$ and $\mathrm{Tr}\ \gamma^{\mu} \gamma^5 = 0$. After dropping linear terms in k , and after Wick rotation we get

$$\langle J^{\mu} \rangle = e \int_0^1 d\alpha \int \frac{d^2k}{(2\pi)^2} \frac{2\pi \varepsilon^{\mu\nu} p_{\nu}}{[k^2 + |\phi|^2]^2}$$

$$= 1/2\pi\ \varepsilon^{\mu\nu} p_{\nu} / |\phi|^2$$

here we have neglected p^2 terms with respect to $|\phi|^2$.

CHAPTER 3

GROUND STATE CHARGE OF FERMION SOLITON SYSTEM IN 2 + 1 DIMENSIONS

3.1. Introduction

Recently there has been a renewal of interest in 2+1 dimensional field theory models. In particular much attention has been paid to O(3) nonlinear σ model[18,20]. With inclusion of a topological term (Hopf term) the solitons of this model become objects obeying fractional statistics[21] (anyons) which appear to have a role in high T_c super conductivity[22]. The σ model is also of relevance in explaining some magnetic properties of solids[23]. Though certain aspects of the model remain speculative the nonlinear σ model has many interesting physical properties that render it a worthwhile object for study

Investigations have been reported of various aspects of fermion number fractionisation in presence of O(3) nonlinear σ model solitons[24-26]. A recent study has been made by Carena[27,28] et al on the ground state charge of fermions in the background of nonlinear σ model solitons including parity breaking mass term. In this chapter we investigate the same model on the basis of the technique developed in the previous chapter. We evaluate the induced charge by studying the adiabatic evolution of the solitons not from the vacuum but from two configurations which have C-invariance. Spectral flow in each case is studied by solving the Dirac equation. While the induced charge and spectral flow depends on the initial conditions the total vacuum charge remains the same. Our calculations confirms some of the results obtained by Carena et al[87]

In § 2 discusses some of the preliminary matters concerning nonlinear σ model and fermions in 2+1 dimensions. Sec 3 contain the study of spectral flow and in § 4 the adiabatic calculation is given. Summary and conclusion are given in § 5.

3.2. O(3) nonlinear σ model and fermions in 2+1 dimensions

The O(3) nonlinear σ model is described by the lagrangian

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^a (\partial^\mu \phi)_a, \qquad \mu = 0,1,2 \quad : a = 1,2,3 \qquad (3.1)$$

where ϕ is a three component field $\phi = (\bar{\phi}, \phi_3) = (\phi_1, \phi_2, \phi_3)$ which obeys the nonlinear constraint $|\phi|^2 = \phi_a \phi_a = v^2$. This model has topological solitons which are charecterised by the homotophy group $\Pi_2(S^2) = Z$ (assuming the boundary condition $|\phi|$ = constant when $x \longrightarrow \infty$). The topological charge (winding number) of the soliton is given by

$$Q = \int d^2x \, J_o$$

where

$$J_\mu = \frac{1}{8 \pi v^3} \varepsilon_{\mu\nu\lambda} \, \varepsilon_{abc} \, \phi_a \, \partial^\nu \phi_b \, \partial^\lambda \phi_c \qquad (3.2)$$

The Lagrangian for the Dirac field with scalar field as background is chosen to be

$$\mathcal{L} = \bar{\psi} (i \gamma_\mu \partial^\mu - \phi.\tau - \eta) \psi \qquad (3.3)$$

Our conventions for the γ matrices in 2+1 dimensions are $\gamma^0 = S_3$ and $\gamma^i = i S^i$ obeying the algebra $[\gamma^\mu, \gamma^\nu] = 2 g^{\mu\nu}$ and $\gamma^\mu \gamma^\nu = g^{\mu\nu} - i\varepsilon^{\mu\nu\lambda}\gamma_\lambda$. S_a and τ_a (a = 1,2,3) are Pauli spin matrices, and η is the fermion mass . The inclusion of fermion mass term will make the lagrangian odd under parity[87,90].

The Dirac equation following from (3.3) is

$$(i \gamma_\mu \partial^\mu - \phi.\tau - \eta) \psi = 0 \qquad (3.4)$$

3.3 Dirac equation in the scalar background

In this section we evaluate spectral flow from the Dirac equation by using the technique developed in the previous chapter for 1+1 dimensional models. We start with a fixed configuration for the component ϕ_3 ($\phi_{1,2}$) and other field component are then varied. With ϕ_3 ($\phi_{1,2}$) zero and other fields (field) nonzero, the hamiltonian is C-invariant and there is no zero energy state[94].

By defining $\psi(x) = \psi(\bar{x}) \exp(iEt)$ we get, from (3.4), the eigenvalue equation

$$- S_i \partial^i (S_3 \psi) + (\phi.\tau)(S_3\psi) + \eta (S_3 \psi) = E \psi \qquad (3.5)$$

Choosing a background field configuration of the form $\phi_3 = \phi_3(r)$ and $\phi_i = \phi(r) \hat{r}_i$ (the explicit form is given in the next section) it easy to observe that $M_3 = (-i\partial_\vartheta + S_3/2 + I_3)$ is a constant of motion and hence commutes with the hamiltonian . Therefore the eigenfunction takes the form

$$\psi_m = e^{i\theta m} \begin{bmatrix} g_1(r)\ e^{-i\theta} \\ g_2(r) \\ g_3(r) \\ g_4(r)e^{i\theta} \end{bmatrix} \tag{3.6}$$

where $M_3\psi_m = m\ \psi_m$

In terms of the radial functions g , Eq(3.5) reduces to

$$\begin{aligned} \partial_r g_2 &= \frac{m}{r}\ g_2 - \phi\ g_3 + (E - \eta - \phi_3)\ g_1 \\ \partial_r g_3 &= -\ \frac{m}{r}\ g_3 - \phi\ g_2 + (E + \eta - \phi_3)\ g_4 \\ \partial_r g_1 &= -\ \frac{(1 + m\ )}{r} g_1 - \phi\ g_4 - (E + \eta + \phi_3)\ g_2 \\ \partial_r g_4 &= -\ \frac{(1 - m\ )}{r} g_4 - \phi\ g_1 + (E - \eta + \phi_3)\ g_3 \end{aligned} \tag{3.7}$$

by changing $m \longrightarrow -m$ and $E \longrightarrow -E$ we can find that if there is a solution

$$\psi_m = e^{i\theta m} \begin{bmatrix} g_1(r)\ e^{-i\theta} \\ g_2(r) \\ g_3(r) \\ g_4(r)e^{i\theta} \end{bmatrix} \tag{3.8}$$

with energy E and angular momentum m , there is a solution

$$\psi_m = e^{-i\theta m} \begin{bmatrix} g_1(r)\ e^{-i\theta} \\ g_2(r) \\ g_3(r) \\ g_4(r)e^{i\theta} \end{bmatrix} \tag{3.9}$$

with energy -E and angular momentum -m .Therefore during the adiabatic evolution of the scalar fields if a solution with energy E and angular momentum m crosses zero , another with energy -E and angular momentum -m crosses zero in opposite direction . Therefore we can say that during the adiabatic evolution there is no spectral flow of nonzero angular momentum states. Or in other words the ground state charge get contribution only from the lowest angular momentum state. Therefore for the analysis of zero energy solution of Dirac equation we have to consider only the zero angular momentum states . For m = 0 and E = 0 Eq(3.7) reduces to

$$
\begin{aligned}
\partial_r g_2 &= -\phi\, g_3 - (\eta + \phi_3)\, g_1 \\
\partial_r g_3 &= -\phi\, g_2 - (\eta - \phi_3)\, g_4 \\
\partial_r g_1 &= -\frac{1}{r}\, g_1 - \phi\, g_4 - (\eta + \phi_3)\, g_2 \\
\partial_r g_4 &= -\frac{1}{r}\, g_4 - \phi\, g_1 - (\eta - \phi_3)\, g_3
\end{aligned}
\qquad (3.10)
$$

Let us consider the scalar background with the explicit form[20]

$$
\begin{aligned}
\phi_3 &= v \cos f(r) \\
\phi_1 &= v \cos\theta \sin f(r) \\
\phi_2 &= v \sin\theta \sin f(r)
\end{aligned}
\qquad (3.11)
$$

where $f(r)$ is a function with asymptotic properties

$$
\begin{aligned}
f(r) &= 0 \qquad \text{when } r \longrightarrow 0 \\
f(r) &= \pi \qquad \text{when } r \longrightarrow \infty
\end{aligned}
\qquad (3.12)
$$

so that in the limit $r \longrightarrow 0$ and $r \longrightarrow \infty$, $\phi_{1,2} = 0$. When $r \longrightarrow 0$, $\phi_3 = v$ and when $r \longrightarrow \infty$, $\phi_3 = -v$. To solve Eq (3.10) we divide the space in to three regions

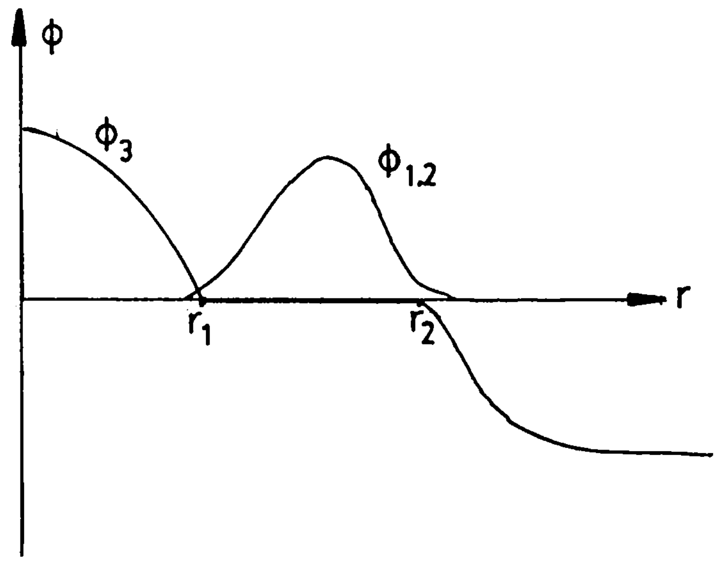

Fig. 5

For $0 < r < r_1$ we assume

$$\phi_3 = v = a \ : \quad \phi_{1,2} = 0 \tag{3.13}$$

Then Eq (3.10) reduces to

$$\begin{aligned} \partial_r g_2 &= -(\eta + a)\, g_1 \\ \partial_r g_3 &= -(\eta - a)\, g_4 \\ \partial_r g_1 &= -\frac{1}{r} g_1 - (\eta + a)\, g_2 \\ \partial_r g_4 &= -\frac{1}{r} g_4 - (\eta - a)\, g_3 \end{aligned} \tag{3.14}$$

By using first and third equation we get

$$\partial_r^2 g_2 + \frac{1}{r} \partial_r g_2 - (a - \eta)^2 g_2 = 0 \tag{3.15}$$

with solutions

$$g_2 = \alpha_1\, I_0(kr)$$

$$g_1 = -\frac{1}{(a+\eta)}\, \partial_r g_2 = -\alpha_1\, I_1(kr) \qquad (3.16)$$

where α_1 is the integration constant, $I_\nu(\eta)$ is the modified Bessel function[104] and $k = (a - \eta)$. In a similar manner we get

$$g_3 = \beta\, I_0(k'r) \quad \text{and } g_4 = -\beta\, I_1(k'r) \qquad (3.17)$$

with $k' = (a + \eta)$

For $r > r_2$ we assume

$$\phi_3 = -a \qquad \text{and } \phi_{1,2} = 0 \qquad (3.18)$$

Then we get from Eq (3.10) , equations similar to (3.14) with $a \longrightarrow -a$. But in this case solutions have to vanish at infinity and are given by

$$g_2 = \gamma\, K_0(k'r) \quad : \quad g_1 = \gamma\, K_1(k'r)$$
$$g_3 = \delta\, K_0(kr) \quad : \quad g_4 = \delta\, K_1(kr) \qquad (3.19)$$

where γ and δ are integration constants and $K_\nu(\eta)$ is the modified Bessel functions .

In the region $r_1 < r < r_2$ we assume

$$\phi_3 = 0 \quad \text{and } \phi_{1,2} = v = b \qquad (3.20)$$

Then Eq (3.10) become

$$\partial_r g_2 = - b\ g_3 - \eta\ g_1$$

$$\partial_r g_3 = - b\ g_2 - \eta\ g_4$$

$$\partial_r g_1 = - \frac{1}{r} g_1 - b\ g_4 - \eta\ g_2 \qquad (3.21)$$

$$\partial_r g_4 = - \frac{1}{r} g_4 - b\ g_1 - \eta\ g_3$$

By identifying $g_1^{(-)} = g_1 = - g_4$ and $g_2^{(-)} = g_2 = - g_3$ the set of of four coupled equations get transformed to

$$\partial_r g_2^{(-)} = b\ g_2^{(-)} - \eta\ g_1^{(-)} \qquad (3.22)$$

$$\partial_r g_1^{(-)} = \frac{1}{r} g_1^{(-)} + b\ g_1^{(-)} - \eta\ g_2^{(-)}$$

Decoupling of these equations gives

$$\partial_r^2 g_2^{(-)} - (2b - \frac{1}{r})\ \partial_r g_2^{(-)} - \frac{b}{r}\ g_2^{(-)} + (b^2-\eta^2) g_2^{(-)} = 0$$

$$(3.23)$$

The solutions are

$$g_2^{(-)} = e^{br} [\ \alpha_-\ I_o(\rho) + \beta_-\ K_o(\rho)]$$

$$g_1^{(-)} = \frac{1}{\eta} (- \partial_r + b)\ g_2^{(-)} \qquad (3.24)$$

$$= - e^{br} [\ \alpha_-\ I_1(\rho) - \beta_-\ K_1(\rho)]$$

where $\rho = kr$ and $k^2 = b^2-\eta^2$. I_ν and K_ν are modified Bessel functions. Here use has been made of the recursion relations $\partial_r I_o = I_1$ and $\partial_r K_o = K_1$

Another possibility is to define

$$g_1^{(-)} = g_1 = g_4$$
$$g_2^{(+)} = g_2 = g_3 \tag{3.25}$$

Corresponding to this the solutions are

$$g_2^{(+)} = e^{-br} [\ \alpha_+ \ I_0(\rho) \ + \ \beta_+ \ K_0(\rho)]$$
$$g_1^{(+)} = - \ e^{-br} [\ \alpha_+ \ I_1(\rho) \ - \ \beta_+ \ K_1(\rho)] \tag{3.26}$$

Since these solutions need not satisfy any normalisability condition in this region, the general solutions are the linear combination of (3.24) and (3.26):

$$g_1 = g_1^{(+)} + g_1^{(-)} = e^{-br} [\ \alpha_+ \ K_1(\rho) \ - \ \beta_+ \ I_1(\rho)] \ + \ e^{br} [\ \beta_- \ K_1(\rho) \ - \ \alpha_- \ I_1(\rho)]$$

$$g_2 = g_2^{(+)} + g_2^{(-)} = e^{-br} [\ \alpha_+ \ K_0(\rho) \ + \ \beta_+ \ I_0(\rho)] \ + \ e^{br} [\ \beta_- \ K_0(\rho) \ - \ \alpha_- \ I_0(\rho)] \tag{3.27}$$

$$g_1 = g_2^{(+)} - g_2^{(-)} = e^{-br} [\ \alpha_+ \ K_0(\rho) \ + \ \beta_+ \ I_0(\rho)] \ - \ e^{br} [\ \beta_- \ K_0(\rho) \ - \ \alpha_- \ I_0(\rho)]$$

$$g_1 = g_1^{(+)} - g_1^{(-)} = e^{-br} [\ \alpha_+ \ K_1(\rho) \ - \ \beta_+ \ I_1(\rho)] \ - \ e^{br} [\ \beta_- \ K_1(\rho) \ - \ \alpha_- \ I_1(\rho)]$$

On matching the solution at $r = r_1$ and at $r = r_2$ we get

$$A \, . \, R = 0$$

where **A** is a (4x4) matrix with elements

$$A_{11} = (K_1 - E\ K_o)\ e^{-br} \quad ; \quad A_{12} = (I_1 + E\ I_o)\ e^{-br}$$
$$A_{13} = (K_1 - E\ K_o)\ e^{br} \quad ; \quad A_{14} = (I_1 + E\ I_o)\ e^{br}$$
$$A_{21} = -(K_1 - F\ K_o)\ e^{-br} \quad ; \quad A_{22} = (I_1 + F\ I_o)\ e^{-br}$$
$$A_{23} = (K_1 - FK_o)\ e^{br} \quad ; \quad A_{24} = (I_1 + F\ I_o)\ e^{br}$$

at r_1 and

$$A_{31} = (K_1 - G\ K_o)\ e^{-br} \quad ; \quad A_{32} = (I_1 + G\ I_o)\ e^{-br}$$
$$A_{33} = (K_1 - G\ K_o)\ e^{br} \quad ; \quad A_{34} = (I_1 + G\ I_o)\ e^{br}$$
$$A_{41} = -(K_1 - H\ K_o)\ e^{-br} \quad ; \quad A_{42} = (I_1 + H\ I_o)\ e^{-br}$$
$$A_{43} = (K_1 - H\ K_o)\ e^{br} \quad ; \quad A_{44} = (I_1 + H\ I_o)\ e^{br}$$

at r_2 . Here

$$E = -\frac{I_1(|a+\eta|r)}{I_o(|a+\eta|r)} \quad : \quad F = \pm\frac{I_1(|a-\eta|r)}{I_o(|a-\eta|r)} \qquad (3.28)$$

at r_1. The negative sign corresponds to $a < \eta$

$$G = \mp\frac{K_1(|a-\eta|r)}{K_o(|a-\eta|r)} \quad : \quad H = \frac{K_1(|a+\eta|r)}{K_o(|a+\eta|r)} \qquad (3.29)$$

at r_2 . The positive sign corresponds to $a < \eta$

R is a vector:

$$\mathbf{R} = \begin{bmatrix} \alpha_+ \\ \alpha_- \\ \beta_+ \\ \beta_- \end{bmatrix}$$

Since $\alpha_\pm$ and $\beta_\pm$ are linearly independent for the existence of nontrivial solutions it require that

$$\det \mathbf{A} = 0 \tag{3.30}$$

The det **A** is evaluated numerically and it is found that by starting with ϕ_3, when $\phi_{1,2}$ are evolved , there is no zero energy level crossing hence there is no spectral flow . As already mentioned when $\phi_{1,2}$ is zero, there is no zero energy state. It is also evident by taking b=0 in (3.30),then det **A** = 0. On the other hand , by starting with $\phi_{1,2}$, when ϕ_3 is evolved there is one level crossing when $a > \eta$ and this leads to spectral asymmetry two. It may also be noted that for $a < \eta$ there is no crossing of the zero level. The zero energy state corresponds to $|\phi| = \eta$.

3.4 The induced charge

To calculate the ground state charge, we have to calculate the induced charge through vacuum polarization. In the first case when the field $\bar{\phi}$ is evolved adiabatically induced charge is calculated from the diagram (Fig.6)

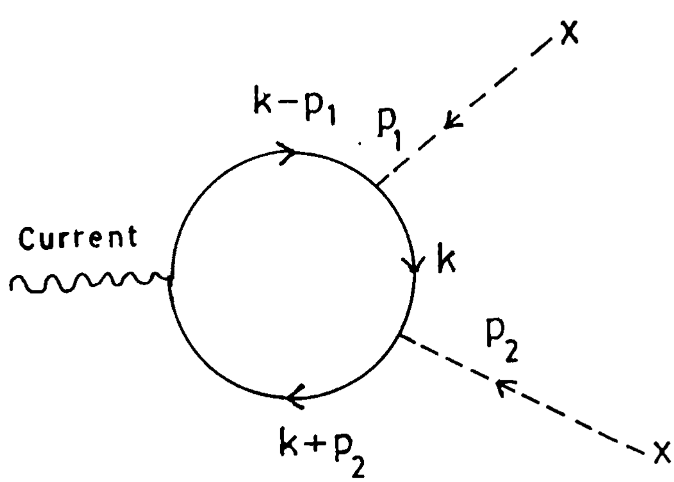

Fig. 6

$$\langle \tilde{J}^{\mu} \rangle = 2 \int \frac{d^3k}{(2\pi)^3} \, \mathrm{Tr}\left[\gamma^{\mu} \frac{1}{(\not{k} - \not{p}_1 - \Delta)} \tau_1 \frac{1}{(\not{k} - \Delta)} \tau_2 \frac{1}{(\not{k} + \not{p}_2 - \Delta)} \right]$$

where $\Delta = |\phi| \, \tau_3 + \eta$

$$= - 2 \int \frac{d^3k}{(2\pi)^3} \, \mathrm{Tr}\left[\gamma^{\mu}_{i} \, \tau_3 \frac{(\not{k} - \not{p}_1 + \Delta)(\not{k} + \tilde{\Delta})(\not{k} + \not{p}_2 + \Delta)}{[(k - p_1)^2 + \Delta^2][k^2 + \tilde{\Delta}^2][(k - p_2)^2 + \Delta^2]} \right]$$

where $\tilde{\Delta} = -|\phi|\tau_3 + \eta$. By using Feynman parameters and after dropping p_1^2 and p_2^2 with respect to Δ^2 and $\tilde{\Delta}^2$ we get

$$\langle \tilde{J}^{\mu} \rangle = \frac{1}{4\pi} \, \varepsilon^{\mu\beta\gamma} p^1_{\beta} \, p^2_{\gamma} \, \mathrm{Tr} \left[\int_0^1 dx \, \frac{[\eta + |\phi|(2x - 1)\tau_3] \, x}{\sqrt{\phi^2 + \eta^2 + 2|\phi||\eta|(2x-1)}^{\,2}} \right]$$

$$= \frac{C}{8\pi v^2} \, \varepsilon^{\mu\beta\gamma} \, p^1_{\beta} \, p^2_{\gamma}$$

where C is a constant depending on the fermion mass:

$$\begin{aligned} C &= 1 && \text{for } |\phi| > \eta \\ &= 1/2 && \text{for } |\phi| = \eta \\ &= 0 && \text{for } |\phi| < \eta \end{aligned}$$

Therefore

$$\langle J^{\mu}(x) \rangle = \frac{C}{8\pi} \, \varepsilon^{\mu\beta\gamma} \, \partial_{\beta}\hat{\phi}^1 \, \partial_{\gamma}\hat{\phi}^2 \qquad (3.31)$$

where $\hat{\phi}^i = \phi^i/|\phi|$. By substituting ϕ_1 and ϕ_2 from (3.11) we get the induced charge as

$$Q = \int \langle J^0(x)\rangle \, d^2x = \int_0^\infty \int_0^{2\pi} r \, dr \, d\vartheta \, \langle J^0(r,\vartheta)\rangle$$

$$= \frac{C}{8\pi} \int_0^\infty 4\pi \, dr \left[\partial_r f \ \mathrm{Sin}\, f \right]$$

$$= \frac{C}{8\pi} \int_0^\infty 4\pi \, d(\mathrm{Cos}\, f(r)) = C \qquad (3.32)$$

In this case since there is no spectral flow the ground state charge is the induced charge itself. Another important point is that, when $|\phi| > \eta$, the ground state charge is the winding number of the soliton field.

Now, instead of starting from ϕ_3, let $\bar{\phi}$ be switched first. Now ϕ_3 is evolved and then the induced current can be calculated from the diagram (Fig. 7)

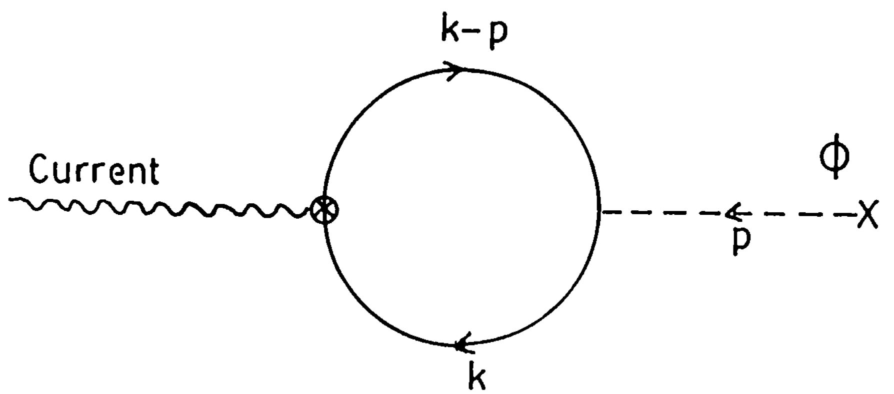

Fig.7

$$\langle \tilde{J}^\mu \rangle = \int \frac{d^3k}{(2\pi)^3} \, \mathrm{Tr}\left[\gamma^\mu \frac{1}{(\not{k} - \not{p} - \Delta)} \tau_3 \frac{1}{(\not{k} - \Delta)} \right]$$

where $\Delta = |\phi|(\tau_1 + \tau_2) + \eta$. By using the property of γ matrices and by simple algebra we find that $\langle J^\mu\rangle = 0$

Therefore in this case also induced charge is 1, 1/2, or zero . In this case ground state charge is contributed only by the bound states

3.5 Conclusion

In this chapter we evaluated the ground state charge of fermions in presence of solitons in nonlinear σ model. The adiabatic evolution is done in two different ways and in each case even though the induced charge and spectral flow are different the ground state charge is found to be the same. In one case ground state charge get contribution only from the induced charge and in the second case ground state charge is contributed only by the bound states. Another observation is that the ground state charge is contributed only by the zero angular momentum states. It is found that ground state charge depends on the fermion mass acquired through Yukawa coupling. The ground state charge can be 1, 1/2, or zero depending on the fermion mass.

CHAPTER 4

BOUND STATES OF FERMIONS AND BOSONS WITH A ´t HOOFT – POLYAKOV MONOPOLE

4.1 Introduction

The study of bound states of fermions and bosons with nonabelian monopoles has been of interest ever since ´t Hooft[26] and Polyakov[27] discovered magnetic monopoles in nonabelian gauge theories . A general analysis of the Dirac equation or the Klein - Gorden equation in the background of the ´t Hooft -Polyakov monopole is, however, not possible because the regular monopole solution is not cast in the closed form . In the Prasad-Sommerfield (PS)limit[30] were a closed solution is available for monopole solution , scattering solutions were constructed by Marciano and Muzinich[76,77]. Bound state solution were not obtained by these authors probably because of the neglect of the Higgs-Fermi coupling . Tang[101,102] obtained bound states of fermions and bosons with a PS monopole ignoring the core effects . Cox and Yildiz[97] also performed a similar work . In the PS limit there exists point singular monopoles in addition to the regular PS solution[98]. Din and Roy[99] considered such a field configuration and obtained bound states of fermions with a point monopole . Ajithkumar and Sabir[100] constructed bound states of fermion and bosons with a general point dyon . The dyon configuration used there may be interpreted either as point - singular dyon or as the asymptotic form of the regular PS solution

In all the works mentioned above the effect of monopole core is neglected . Moreover these studies were done in the PS limit. Whether such a limit exists in nature or not remains to be verified. In this limit there is a 1/r term (this will be explained in the next section) in the asymptotic form of the Higgs field and it is this term which is responsible for the existence of bound states . A term of this type is absent in the general case . In this case Callias[72] analyzed the corresponding Dirac equation and arrived at the general result that there can be only a finite number of bound states of a fermion with a regular monopole .

In this chapter we study bound states fermions and bosons with a regular 't Hooft - Polyakov monopole . Ground state charge of fermion monopole system is also calculated . We incorporate the effect of monopole core as well . In order to make it a solvable problem we assume the the monopole core to be a spherical region of radius $r_o = 1/M_v$, where M_w is the vector boson mass in the theory . We represent the field inside the core by its value as $r \longrightarrow 0$ and outside by the asymptotic form as $r \longrightarrow \infty$. The problem thus reduces to the three dimensional potential well problem which has to be solved by finding solutions inside and outside the monopole core and matching the solutions at the boundary . This is the method adopted by Besson[84] to investigate the structure of the fermi vacuum in the field of a magnetic monopole

The matching problem to obtain the energy levels requires the solution of transcendental equations involving bessel functions . This has been done numerically by assigning arbitrary numerical values to fermion and boson masses , vector boson mass and Higgs coupling . We have studied the bound state spectrum by varying this parameters. For massive as well as massless fermions, there is C-invariance. The zero energy state and hence ground state charge is found to depend on the monopole radius. For fermions it has been found that the number of bound states depend on the Higgs coupling as observed by Callias[72] . A similar result is obtained for the bound states of bosons . In the case of massive fermions the number of bound states depends on the size of the monopole core , the number being reduced to zero when this exceeds a limiting value. In § 2 we review SU(2) monopole theory and discuss the fields inside and outside the monopole core used in our calculation . In § 3 the Dirac equation is setup and the corresponding radial equation are obtained .

4.2 The background potential

For studying the bound state spectrum we has to solve Dirac equation in the monopole background given by Eq (1.51) which has an asymptotic value given by Eq (1.52). The corresponding Higgs field is,

$$\phi_a = (1/g)\, r_a\, (a + b/r) \qquad (4.1)$$

It is the b/r term that is responsible for the existence of bound states in the PS limit

In this chapter we shall divide the space surrounding the monopole into two regions separated by a monopole core boundary which we assume as spherical surface with radius $r_o = 1/a = 1/M_v$. In the interior we approximate the fields by the values $r \longrightarrow 0$ as given by Eq (1.44) and out side by the field configuration at $r >> r_o$ as given by Eq (1.52). In other words we approximate the monopole field by

$$K(r) = \theta(r-r_o) \quad \text{and} \quad H(r) = ar\, \theta(r-r_o) \tag{4.2}$$

where $\theta(x)$ is the step function. Here we have taken $b = 0$ in (1.52)

4.3 Dirac equation in the monopole background

To study the bound states of fermions with monopoles we have to consider the relevant Dirac equation . After separating the angular parts of these equation the radial equation are obtained in this section . We shall consider isodoublet fermions moving in the potential (4.2)

The fermionic lagrangian in the background of the monopole is

$$\mathcal{L} = \bar{\psi}_n(i\not{D} - M)\psi_n - \frac{ig}{2} G\, \tau^a_{nm}\psi_n\psi_m\phi^a \tag{4.3}$$

where

$$D_\mu\psi_n = \partial_\mu\psi_n - \frac{i\, g}{2} \tau^a_{nm}A_\mu{}^a\psi_m \tag{4.4}$$

G is the Higgs coupling and M , the fermion mass . The corresponding Dirac equation to be solved is

$$i\not{D}\psi_n - 1/2\; g\; G\; \tau^a_{nm}\; \phi^a\; \psi_m = M\; \psi_n \tag{4.5}$$

By taking $\psi_n(x) = \psi_n(\vec{x})\exp(-iEt)$ and by using ansatz (1.39) we get

$$\left\{\vec{\alpha}.\left[\vec{P}(r)-\frac{A(r)}{2}(\hat{r}\times\vec{\tau})\right] - \beta G\frac{H(r)}{2r}(\vec{\tau}.\vec{r})\right\} \psi(\vec{x})= (E-\beta M)\psi(\vec{x}) \tag{4.6}$$

were $\vec{\alpha}$ and β are Dirac matrices

To solve (4.6) let us define

$$\psi_n(\vec{x})= \begin{bmatrix} \chi_n^+ \\ \chi_n^- \end{bmatrix} \tag{4.7}$$

By using

$$\vec{\alpha} = \begin{bmatrix} 0 & i\vec{\sigma} \\ -i\vec{\sigma} & 0 \end{bmatrix} \qquad \beta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{4.8}$$

Equation (4.6) becomes

$$\left[\vec{\sigma}_{ij}.(\vec{\nabla}\, \delta_{nm} - \frac{1}{2}\, A(r)(\hat{r}\times\vec{\tau}_{nm})) \pm \delta_{ij}(\frac{G\, H(r)}{2r}\, \hat{r}.\vec{\tau}_{nm} - M\, \delta_{nm})\right] \chi_{jm}^{\pm}$$

$$= \pm\, E\, \delta_{nm}\quad \chi_{im}^{\mp} \tag{4.9}$$

Here $\chi_{im}^{\pm}$ is defined as in Ref. 33 .The first index of χ refers to the spin part and and second index to isospin part:

$$\chi_{im}^{\pm} = \left\{ G_j^{\pm}(r)\; Y_j^m(\Omega)\; \delta_{im} + \left[P_j^{\pm}(r)\; Y_j^m(\Omega) + B_j^{\pm}(r)\frac{r}{j}\; \partial_a Y_j^m(\Omega)\right.\right.$$

$$\left.\left. + C_j^{\pm}(r)\,\frac{1}{i\,j}\;\epsilon_{abc}\; r_a\; \partial_c Y_j^m(\Omega)\;\right]\; \sigma_{im}^a \right\} \tau_{nm}^z \tag{4.10}$$

with $j = \sqrt{J(J+1)}$, J being the total angular momentum . Total angular momentum is obtained by combining orbital and spin angular momentum and isospin . In this case it takes values 0,1,...etc . $B_o = C_o = 0$ by definition . By substituting (4.10) in (4.9) we get eight coupled differential equations .

$$\left.\begin{aligned}
&\left(\partial_r + \frac{1}{r} \mp \frac{G\,H(r)}{2r}\right) P_j^{\pm} - \frac{j}{r} B_j^{\pm} \mp M\, G_j^{\pm} = \mp\; E\, G_j^{\mp} && j \geq 0 \\
&\left(\partial_r + \frac{1}{r} \mp \frac{G\,H(r)}{2r}\right) G_j^{\pm} - \frac{j}{r} C_j^{\pm} \mp M\, P_j^{\pm} = \mp\; E\, P_j^{\mp} && j \geq 0 \\
&\left(\partial_r + \frac{1}{r} \pm \frac{G\,H(r)}{2r}\right) B_j^{\pm} - \frac{j}{r} P_j^{\pm} \pm M\, C_j^{\pm} = \pm\; E\, C_j^{\mp} && j > 0 \\
&\left(\partial_r + \frac{1}{r} \pm \frac{G\,H(r)}{2r}\right) C_j^{\pm} - \frac{j}{r} G_j^{\pm} \pm M\, B_j^{\pm} = \pm\; E\, B_j^{\mp} && j > 0
\end{aligned}\right\} \quad (4.11)$$

4.4 Zero angular momentum fermions in presence of monopole

For zero angular momentum equation (4.11) inside the monopole core can be written as

$$\begin{aligned}
&\partial_r G^{\pm} \mp M\, P^{\pm} = \mp E\, P^{\mp} \\
&\partial_r P^{\pm} + 2r^{-1} \mp M\, G^{\pm} = \mp E\, G^{\mp}
\end{aligned} \qquad (4.12)$$

where we have suppressed the subscripts on $P^{\pm}$ and $G^{\pm}$. By defining $X^{\pm} = P^{+} \pm P^{-}$ and $Y^{\pm} = G^{+} \pm G^{-}$ equation (4.12) can be written as

$$\begin{aligned}
&\partial_r Y^{\pm} = (M \pm E)\, X^{\pm} \\
&(\partial_r + 2\, r^{-1}) = (M \pm E)\, Y^{\pm}
\end{aligned} \qquad (4.13)$$

The solutions to (4.13) which are regular at $r = 0$ are

$$Y^{\pm} = \alpha_{\pm} \operatorname{Sinh}(kr) / (kr)$$
$$X^{\pm} = \alpha_{\mp} \frac{k}{M \mp E} \left[\frac{\operatorname{Cosh}(kr)}{kr} - \frac{\operatorname{Sinh}(kr)}{(kr)^2}\right] \tag{4.14}$$

for $|E| < M$ with $k = (M^2 - E^2)^{1/2}$, and

$$Y^{\pm} = \alpha_{\pm} \operatorname{Sin}(k' r) / k' r)$$
$$X^{\pm} = \alpha_{\mp} \frac{k'}{M \mp E} \left[\frac{\operatorname{Cos}(k' r)}{k' r} - \frac{\operatorname{Sin}(k' r)}{(k' r)^2}\right] \tag{4.15}$$

for $|E| > M$ with $k' = (E^2 - M^2)^{1/2}$. Here $\alpha_{\pm}$ are integration constants .

Outside the core equations (4.11) become

$$(\partial_r + \frac{1}{r} \mp \frac{G\, H(r)}{2r})\, P^{\pm} \mp M\, G^{\pm} = \mp E\, G^{\mp}$$
$$(\partial_r + \frac{1}{r} \mp \frac{G\, H(r)}{2r})\, G^{\pm} \mp M\, P^{\pm} = \mp E\, P^{\mp} \tag{4.16}$$

By substituting H(r) from (4.2) and by defining $R^{\pm} = P^{\pm}+G^{\pm}$ and $S^{\pm} = P^{\pm}-G^{\pm}$ equation (4.16) can be transformed to

$$(\partial_r + r^{-1} \mp m_+) R^{\pm} = \mp E \quad R^{\mp}$$
$$(\partial_r + r^{-1} \mp m_-) S^{\pm} = \pm E \quad S^{\mp} \tag{4.17}$$

where $m_{\pm} = aG/2 \pm M$. Solutions of (4.17) regular at $r \longrightarrow \infty$ are

$$R^{+} = \beta_1 \exp(-k_+ r)\,/(k_+ r)$$

$$R^{-} = \frac{\beta_1}{E}\left(\frac{k_+ + m_+}{k_+ r}\right)\exp(-k_+ r) \tag{4.18}$$

$$S^{+} = \beta_2 \exp(-k_- r)\,/(k_- r)$$

$$S^{-} = \frac{-\beta_2}{E}\left(\frac{k_- + m_-}{k_- r}\right)\exp(-k_- r) \tag{4.19}$$

Here $k_\pm = (m_\pm^2 - E^2)^{1/2}$ and β_1 and β_2 are integration constants. For bound states $E < m_\pm$ and for scattering states $E > m_\pm$ and since $m_- < m_+$ the possible range of bound state energies is

$$-(Ga/2 - M) < E < (Ga/2 - M) \tag{4.20}$$

The solutions inside and outside the core are matched at the core boundary $r = r_o$. As in 2+1 dimension we get the matching condition,

$$\det \mathbf{A} = 0 \tag{4.21}$$

where $\mathbf{A}$ is a (2×2) matrix given by

$$\begin{pmatrix} (f + F_+) - \frac{E}{m_+ + k_+}(f - F_+) & (f + F_-) + \frac{E}{m_+ + k_+}(f - F_-) \\ (f - F_+) + \frac{E}{m_- + k_-}(f + F_+) & (f - F_-) - \frac{E}{m_- + k_-}(f + F_-) \end{pmatrix} \tag{4.22}$$

with

$$f = \mathrm{Sinh}(kr_o) / (kr_o)$$

$$F_{\pm} = \frac{k}{M \mp E}\left[\frac{\mathrm{Cosh}(kr_o)}{kr_o} - \frac{\mathrm{Sinh}(kr_o)}{(kr_o)^2}\right] \tag{4.23}$$

for E < M and

$$f = \mathrm{Sin}\,(k'r_o) / (k'r_o)$$

$$F_{\pm} = \frac{k'}{M \mp E}\left[\frac{\mathrm{Cos}\,(k'r_o)}{k'r_o} - \frac{\mathrm{Sin}\,(k'r_o)}{(k'r_o)^2}\right] \tag{4.24}$$

for E > M

it is trivial to show that

$$\det A(E) = - \det A(-E) \tag{4.25}$$

This implies charge conjugation symmetry and existence of zero energy bound state . In this case ground state charge is contributed only from the zero energy bound state . In the case of massless fermions the zero energy solution inside and outside the core exists for all values of the parameters and hence the ground state charge is $\pm$ 1/2 and is a constant . However for massive fermions for the zero energy solutions to vanish at ∞ ,$m_{\pm}$ > 0 . That is Ga/2 > M , from the definition of m_- . Then the ground state charge is $\pm$ 1/2 only when Ga/2 > M , otherwise it is zero. Therefore ground state is discontinuous at the fermion mass, as reported in the literature[89].

The energy levels are given by the roots of the equation (4.21), a transcendental equation which can be solved by numerical methods . The values of E must be searched for in the range given by (4.20)

We have obtained the number of energy levels for a range of values of G and r_o. For the case of massless fermions the results are given in Table 1 . The number of energy levels is independent of the core radius but strongly depends on the Higgs coupling G . For G < 1 the only bound state is the zero energy bound state .

Table 1 . Number of bound states (including zero)for massless zero angular momentum fermions

$r_o \backslash G$	1	4	6	8	10	12	14	20
10^{-16}	1	1	2	3	3	4	4	6
10^{-10}	1	1	2	3	3	4	4	6
10^{-5}	1	1	2	3	3	4	4	6
10^{-3}	1	1	2	3	3	4	4	6
1	1	1	2	3	3	4	4	6

Table 2 presents the number of bound states for massive fermions . As in the case of massless fermions the number of bound states increases with G . However , for massive fermions the number of bound states also depends on the core radius . When $r_o \simeq$ G/2M the number of bound states is reduced . This corresponds to a fermion mass $M = G\, M_w/2$

Table 2 . Number of bound states (including zero)for massive zero angular momentum fermions

$r_o \backslash G$	1	4	6	8	10	12	14	20
10^{-16}	1	1	2	3	3	4	4	6
10^{-10}	1	1	2	3	3	4	4	6
10^{-5}	1	1	2	3	3	4	4	6
10^{-3}	1	1	2	3	3	4	4	6
1	1	1	2	2	2	3	3	5

4.5 Higher angular momentum fermion bound states

In this case Dirac equation inside the core can be written as

$$
\begin{aligned}
\partial_r G^{\pm} - r^{-1} j\, C^{\pm} \mp M\, P^{\pm} &= \mp E\, P^{\pm} \\
(\partial_r + 2r^{-1}) P^{\pm} - r^{-1} j\, B^{\pm} \mp M G^{\pm} &= \mp E\, G^{\mp} \\
(\partial_r + r^{-1}) B^{\pm} - r^{-1} j\, P^{\pm} \pm M G^{\pm} &= \pm E\, C^{\mp} \\
(\partial_r + r^{-1}) C^{\pm} - r^{-1} j\, G^{\pm} \pm M B^{\pm} &= \pm E\, B^{\mp}
\end{aligned}
\tag{4.26}
$$

By defining

$$
\begin{aligned}
W_{\pm} &= P^{+} \pm P^{-} \quad ; \quad Z_{\pm} = G^{+} \pm G^{-} \\
X_{\pm} &= B^{+} \pm B^{-} \quad ; \quad Y_{\pm} = C^{+} \pm C^{-}
\end{aligned}
\tag{4.27}
$$

the above equations can be transformed to the following set

$$
\begin{aligned}
\partial_r Z_{\pm} - r^{-1} j\, Y_{\pm} &= (M \pm E)\, W_{\mp} \\
(\partial_r + 2 r^{-1}) W_{\pm} - r^{-1} j\, X_{\pm} &= (M \pm E)\, Z_{\mp} \\
(\partial_r + r^{-1}) X_{\pm} - r^{-1} j\, W_{\pm} &= -(M \pm E)\, Y_{\mp} \\
(\partial_r + r^{-1}) Y_{\pm} - r^{-1} j\, Z_{\pm} &= -(M \pm E)\, X_{\mp}
\end{aligned}
\tag{4.28}
$$

These can be decoupled to yield

$$Z_{\pm} = \alpha_{\pm} (k\, r)^{-1/2} I_{j+1/2}(kr)$$

$$Y_{\pm} = \beta_{\pm} (k\, r)^{-1/2}\, I_{j+1/2}(kr)$$

(4.29)

$$W_{\pm}=\frac{k}{M\mp E}(kr)^{-1/2}\left[\frac{\alpha_{\mp}}{2}(I_{j-1/2}(kr)+I_{j+3/2}(kr)\ -\frac{1}{kr}\ I_{j+1/2}(kr))\right.$$

$$\left.-\frac{\beta_{\mp}}{kr}\sqrt{j(j+1)}\ I_{j+1/2}(kr)\right]$$

$$X_{\pm}=\frac{-k}{M\mp E}(kr)^{-1/2}\left[\frac{\beta_{\mp}}{2}(I_{j-1/2}(kr)+I_{j+3/2}(kr)\ +\frac{1}{kr}\ I_{j+1/2}(kr))\right.$$

$$\left.-\frac{\alpha_{\mp}}{kr}\sqrt{j(j+1)}\ I_{j+1/2}(kr)\right]$$

for $|\ E\ |\ <\ M$ and

$$Z_{\pm}=\alpha_{\pm}\ (k'r)^{1/2}J_{j+1/2}(k'r)$$

$$Y_{\pm}\ =\beta_{\pm}\ (k'r)^{1/2}\ J_{j+1/2}(k'r)$$

$$W_{\pm}=\frac{k'}{M\mp E}(k'r)^{-1/2}\left[\frac{\alpha_{\mp}}{2}(J_{j-1/2}(k'r)-J_{j+3/2}(k'r)-\frac{1}{kr}\ J_{j+1/2}(k'r))\right.$$

$$\left.-\frac{\beta_{\mp}}{k'r}\sqrt{j(j+1)}\ J_{j+1/2}(k'r)\right]$$

$$X_{\pm}=\frac{-k'}{M\mp E}(k'r)^{-1/2}\left[\beta_{\mp}(J_{j-1/2}(k'r)-J_{j+3/2}(k'r)\ +\frac{1}{k'r}\ J_{j+1/2}(k'r))\right.$$

$$\left.-\frac{\alpha_{\mp}}{k'r}\sqrt{j(j+1)}\ J_{j+1/2}(k'r)\ \right]$$

(4.30)

for $E > M$. Here $I_{\nu}(\eta)$ is the modified Bessel function and $J_{\nu}(\eta)$ is the spherical Bessel function[104]. $\alpha_{\pm}$ and $\beta_{\pm}$ are integration constants and $k = (M^2 - E^2)^{1/2}$ and $k' = (E^2 - M^2)^{1/2}$

In order to solve the equation outside the core , we define

$$
\begin{aligned}
X^{\pm} &= P^{\pm} + G^{\pm} + B^{\mp} + C^{\mp} \\
y^{\pm} &= P^{\pm} + G^{\pm} - B^{\mp} - C^{\mp} \\
Z^{\pm} &= P^{\pm} - G^{\pm} + B^{\mp} - C^{\mp} \\
W^{\pm} &= P^{\pm} - G^{\pm} - B^{\mp} + C^{\mp}
\end{aligned}
\tag{4.31}
$$

In terms of these variables , the equation outside the core can be written as

$$
\begin{aligned}
(\partial_r + r^{-1} \mp m_+)\, X^{\pm} &= (r^{-1} j \mp E)\, X^{\mp} \\
(\partial_r + r^{-1} \mp m_+)\, Y^{\pm} &= (-r^{-1} j \mp E)\, Y^{\mp} \\
(\partial_r + r^{-1} \mp m_-)\, Z^{\pm} &= (r^{-1} j \pm E)\, Z^{\mp} \\
(\partial_r + r^{-1} \mp m_-)\, W^{\pm} &= (-r^{-1} j \pm E)\, W^{\mp}
\end{aligned}
\tag{4.32}
$$

By defining

$$
\begin{aligned}
R^{\pm} &= X^{+} \pm X^{-} \;,\; S^{\pm} = Y^{+} \pm Y^{-} \\
T^{\pm} &= Z^{+} \pm Z^{-} \;,\; U^{+} = W^{+} \pm W^{-}
\end{aligned}
\tag{4.33}
$$

we can decouple and solve (4.32) to get

$$R^{-} = \gamma_1 (k_+ r)^{-1/2} K_{j+1/2}(k_+ r)$$

$$R^{+} = -\gamma_1 k_+ (k_+ r)^{-1/2} (m_+ - E)^{-1} K_{j-1/2}(k_+ r)$$

$$S^{-} = \gamma_2 (k_+ r)^{-1/2} K_{j-1/2}(k_+ r)$$

$$S^{+} = -\gamma_2 k_+ (k_+ r)^{-1/2} (m_+ - E)^{-1} K_{j+1/2}(k_+ r)$$

$$T^{-}=\gamma_{3}(k_{-}r)^{-1/2}\,K_{j+1/2}(k_{-}r)$$

$$T^{+}=-\gamma_{3}k_{-}(k_{-}r)^{-1/2}\,(m_{-}+E)^{-1}K_{j-1/2}(k_{-}r)$$

$$U^{-}=\gamma_{4}(k_{-}r)^{-1/2}\,K_{j-1/2}(k_{-}r)$$

$$U^{+}=-\gamma_{4}k_{-}(k_{-}r)^{-1/2}\,(m_{-}+E)^{-1}K_{j+1/2}(k_{-}r)$$

where $K_{\nu}(\mu)$ is modified Bessel function and γ_1 , γ_2 , γ_3 and γ_4 are integration constants

On matching the solutions at the core boundary of radius r_o we get the condition

$$\det A(E) = 0 \tag{4.36}$$

where **A** is a 4×4 matrix whose elements are

$$
\begin{aligned}
A_{11} &= R\,(f_1^{+} + g^{+}) - f \;; & A_{12} &= R\,f - (f_1^{-} - g^{-}) \\
A_{13} &= R\,(-f_2^{+} + g^{+}) - f \;; & A_{14} &= -R\,f - (f_2^{-} + g^{-}) \\
A_{21} &= S\,(f_1^{+} - g^{+}) - f \;; & A_{22} &= S\,f - (f_1^{-} + g^{-}) \\
A_{23} &= S\,(f_2^{+} + g^{+}) + f \;; & A_{24} &= S\,f - (-f_2^{-} + g^{-}) \\
A_{31} &= T\,(f_1^{+} + g^{+}) + f \;; & A_{32} &= -T\,f - (f_2^{-} + g^{-}) \\
A_{33} &= T\,(-f_2^{+} + g^{+}) + f \;; & A_{34} &= T\,f - (f_2^{-} + g^{-}) \\
A_{41} &= U\,(f_2^{+} - g^{+}) + f \;; & A_{42} &= -U\,f - (f_1^{-} + g^{-}) \\
A_{43} &= U\,(f_2^{+} - g^{+}) - f \;; & A_{44} &= -U\,f - (-f_2^{-} + g^{-})
\end{aligned}
$$

where

$$R = R^{+}/R^{-} = -k_{+}(m_{+}-E)^{-1} K_{j-1/2}(k_{+}r_{o})/K_{j+1/2}(k_{+}r_{o})$$

$$S = S^{+}/S^{-} = -k_{+}(m_{+}-E)^{-1} K_{j+1/2}(k_{+}r_{o})/K_{j-1/2}(k_{+}r_{o})$$

$$T = T^{+}/T^{-} = -k_{-}(m_{-}-E)^{-1} K_{j-1/2}(k_{-}r_{o})/K_{j+1/2}(k_{-}r_{o}) \qquad (4.37)$$

$$U = U^{+}/U^{-} = -k_{-}(m_{-}-E)^{-1} K_{j+1/2}(k_{-}r_{o})/K_{j-1/2}(k_{-}r_{o})$$

Also

$$f = I_{j+1/2}(kr_{o}) \quad ; \quad g^{\pm} = - \frac{k}{M \pm E} \frac{\sqrt{J(J+1)}}{kr_{o}} I_{j+1/2}(kr_{o})$$

$$f_{1}^{\pm} = \frac{k}{M \pm E} \frac{1}{2} (I_{j-1/2}(kr_{o}) + I_{j+3/2}(kr_{o}) - \frac{1}{k r_{o}} I_{j+1/2}(kr_{o}))$$

$$f_{2}^{\pm} = \frac{-k}{M \pm E} \frac{1}{2} (I_{j-1/2}(kr_{o}) + I_{j+3/2}(kr_{o}) + \frac{1}{k r_{o}} I_{j+1/2}(kr_{o})) \qquad (4.38)$$

for $E < M$ and

$$f = J_{j+1/2}(k' r_{o}) \quad ; \quad g^{\pm} = - \frac{k'}{M \pm E} \frac{\sqrt{J(J+1)}}{kr_{o}} J_{j+1/2}(k' r_{o})$$

$$f_{1}^{\pm} = \frac{k'}{M \pm E} \frac{1}{2} (J_{j-1/2}(k' r_{o}) - J_{j+3/2}(k' r_{o}) - \frac{1}{k' r_{o}} J_{j+1/2}(k' r_{o}))$$

$$(4.39)$$

$$f_{2}^{\pm} = \frac{-k'}{M \pm E} \frac{1}{2} (J_{j-1/2}(k' r_{o}) - J_{j+3/2}(k' r_{o}) + \frac{1}{k' r_{o}} J_{j+1/2}(k' r_{o}))$$

for $E > M$

The energy levels are determined numerically by searching for values of E satisfying the equation (4.36) in the range given in (4.20) . Numerically it is trivial to show that there is no zero energy bound state . The number of energy levels for a range of values of G and r_o with j = 1 massless fermions are given in Table 3 .

Table 3 . Number of bound states for massless fermions with angular momentum j = 1

$r_o \backslash G$	1	4	6	8	10	12	14	20
10^{-16}	0	1	2	2	2	2	2	5
10^{-10}	0	1	2	2	2	2	2	5
10^{-5}	0	1	2	2	2	2	2	5
10^{-3}	0	1	2	2	2	2	2	5
1	0	1	2	2	2	2	2	5

In this case number of bound states is independent of the core radius but increases with G as in the case of zero angular momentum fermions . This is true in the case of massive fermions as given in Table 4 . However , the number of bound states is reduced for a fermion mass $M \simeq GM_W/2$ as in the case of zero angular momentum

Table 4 . Number of bound states for massive fermions with angular momentum j = 1

$r_o \backslash G$	1	4	6	8	10	12	14	20
10^{-16}	0	2	2	3	3	3	3	7
10^{-10}	0	2	2	3	3	3	3	7
10^{-5}	0	2	2	3	3	3	3	7
10^{-3}	0	2	2	3	3	3	3	7
1	0	0	1	2	3	3	3	7

4.6 Conclusion

In this chapter we have calculated ground state charge as well as the number of bound states of fermion soliton system . Since the system is C -invariant and due to the presence of zero energy bound state ground state charge is ± 1/2 . It has been found that ground state charge is discontinuous at the fermion mass and the monopole radius . Since there is no zero energy bound states at the higher angular momentum , we can say that ground state charge is contributed only from the lowest angular momentum . The number of bound states is found to depend on monopole radius and Higgs coupling . It is also found that there is an upper bound on the number of bound states as found by Callias[72].Studies similar to which has been done for fermions can also be done for bosons . A summary of this calculation is given in the appendix

4. A Appendix

Bound state of bosons

In case of bosons , the Klein Gorden equation

$$D_\mu D^\mu U(x) = - (M^2 + g^2 h^2 \phi^2 + g\ G\ \tau^a \phi^a /2)\ U(x) \qquad \text{(A.1)}$$

can be simplified to

$$\left[\nabla^2 - \frac{A(r)(L.\tau)}{r} - G\ H(r)\ (\vec{\tau}.\hat{r}) - \frac{A^2(r)}{2} - \left(\frac{h\ H(r)}{2}\right)^2 + E^2 - M^2\right] U(\vec{x}) = 0 \qquad \text{(A.2)}$$

were $U(x) = \exp(\ iEt\ )\ U(\vec{x})$, G and h are Higgs coupling and M is the boson mass . The angular part can be separated by using spinor harmonics :

$$U(\vec{x}) = F_+(r)\ y_{Jm}(\Omega) + F_-(r)\ y'_{Jm}(\Omega) \qquad \text{(A.3)}$$

where

$$y_{Jm}(\Omega) = \begin{pmatrix} \sqrt{\frac{J+m}{2J}}\, Y^{m-1/2}_{J-1/2} \\ \\ \sqrt{\frac{J-m}{2J}}\, Y^{m+1/2}_{J-1/2} \end{pmatrix}$$

and

$$y'_{Jm}(\Omega) = \tau \,.\, \hat{r}\, y_{Jm}(\Omega)$$

Here J is the total angular momentum of the state which take values 1/2 , 3/2 ,.....The radial equations are obtained by substituting (A.3) in (A.2) :

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{A^2}{2} - \left(\frac{h\,H}{r}\right)^2 + E^2 - M^2 - (J - 1/2)\,\frac{A}{r} - \frac{(J - 1/2)(J + 1/2)}{r^2}\right] F_+ = - \frac{G\,H}{2r}\, F_- \qquad \text{(A.4)}$$

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{A^2}{2} - \left(\frac{h\,H}{r}\right)^2 + E^2 - M^2 + (J + 3/2)\,\frac{A}{r} - \frac{(J + 1/2)(J + 3/2)}{r^2}\right] F_- = - \frac{G\,H}{2r}\, F_+ \qquad \text{(A.5)}$$

Outside the monopole these equations can be written as

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\,\frac{d}{dr} + \frac{1/4 - J(J + 1)}{r^2}\, E^2 - (\,M^2 + h^2a^2)\right]\begin{bmatrix} F_+ \\ F_- \end{bmatrix} = - \frac{a\,G}{2}\begin{bmatrix} F_- \\ F_+ \end{bmatrix} \qquad \text{(A.6)}$$

Solutions to the above equation can be obtained by defining new functions $R^{\pm} = F^{+} \pm F^{-}$. Equation (A.6) in terms of $R^{\pm}$ becomes

$$\left[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+\frac{1/4-J(J+1)}{r^2}E^2-(M^2+h^2a^2\mp Ga/2)\right] R^{\pm} = 0$$

(A.7)

The solutions , which is regular for large r is given by

$$R^{\pm} = \alpha^{\pm} \frac{1}{\sqrt{k_{\pm} r}} K_n(k_{\pm}r) \qquad (A.8)$$

with $n = \sqrt{J(J+1)}$ and $k_{\pm} = (M^2 + h^2a^2 \mp Ga/2 - E^2)^{1/2}$. Bound states are possible in the energy range

$$| E | < (M^2 + h^2a^2 - aG/2) \qquad (A.9)$$

Inside the core the equations (A.4) and (A.5) reduces to

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{(J-1/2)(J+1/2)}{r^2} + E^2-M^2\right] F_+ = 0$$

(A.10)

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{(J+3/2)(J+1/2)}{r^2} + E^2-M^2\right] F_- = 0$$

(A.11)

and the regular solutions for the bound states are

$$F_+ = \alpha_+ (kr)^{-1/2} I_J(kr) \quad : \quad F_- = \alpha_- (kr)^{-1/2} I_{J+1}(kr) \qquad (A.12)$$

where $k = (M^2 - E^2)^{1/2}$. The regular solution for $E > M$ are

$$F_+ = \alpha_+ \,(k'r)^{-1/2} J_J(k'r) \quad : \quad F_- = \alpha_- \,(k'r)^{-1/2} J_{J+1}(k'r) \qquad (A.13)$$

where $k' = (E^2 - M^2)^{1/2}$

The solutions inside and outside can be matched at the boundary . In this case one has to match the first derivative as well . The resulting transcendental equation is solved to obtain the energy levels . Here there are two Higgs couplings G and h. For the lowest angular momentum J = 1/2 and for h = 5 , M = 0.1, G = 10 four bound states are obtained . The number of bound states increases rapidly when h is increased but only slowly when G is increased.

CHAPTER 5

GROUND STATE CHARGE OF FERMION DYON SYSTEM

5.1 Introduction

The ground state charge of a fermion in the background of a 't Hooft - Polyakov monopole is purely topological and will depend only on the asymptotic values of the fields. Niemi and Semenoff[35,47], employing a mathematical technique they had developed and which can be applied to a generic class of Hamiltonians, obtained an expression for the fermion number of a Dirac fermion coupled to a monopole background. However, in the case of dyons a general formula applicable to massive as well as massless fermions is not available, mainly on account of its greater complexity. The Hamiltonian for the fermion dyon system does not fit into the class of Hamiltonians studied by Niemi and Semenoff. It is also not known whether the dyon core will have any influence on charge fractionisation as is the case with the monopole fermion system discussed in the previous chapter.

In this chapter we investigate the problem of vacuum polarization by dyons . Here the theory is not C invariant and hence to calculate the ground state charge we adopt the method used in chapter 2 and 3 . We start from a pure monopole potential . Then the ground state charge is ± 1/2 as has been noted in the previous chapter . Now the other fields are evolved adiabatically so that the theory loses the C invariance and zero energy state disappears . In § 2 induced charge is calculated from vacuum polarization diagram .In § 3 Dirac equation is setup in the background of a dyon. A detailed analysis of spectral flow is possible only for a special case of dyon solutions which describes a nonselfdual monopole[24]. In § 4 spectral flow is calculated for zero angular momentum fermions in presence of nonselfdual monopole . In § 5 spectral flow is calculated for higher angular momentum fermions in presence of nonselfdual monopole. In the case of a nonselfdual monopole, even though there is spectral asymmetry, the ground state charge is found to be same as that of a selfdual monopole discussed in the previous chapter. In § 6 and § 7 the Dirac equation is solved in presence of Julia-Zee dyons taking into account the dyon core. It is shown that the spectrum is symmetric in the massless case and asymmetric in the massive case. The existence of zero energy state in the massless case depends on the parameters in the dyon structure. However, an explicit calculation of spectral flow turns out to be difficult.

5.2 The background potential

In this case the Dirac equation is setup in the potential given by (1.53). We divide the space surrounding the dyon into two regions separated by a spherical surface. In side the region we assume the values for the potential as given by Eq(1.57) and out side by the values as given by Eq (1.59):

$$H(r) = ar + b \quad ; \quad J(r) = cr + d \quad ; \quad A(r) = 1/r$$

for $r > r_o$ and $\qquad (5.1)$

$$H(r) = J(r) = A(r) = 0$$

for $r < r_o$

For a nonselfdual monopole, d = 0 in the above equation. (It is discussed in § 5.4)

5.3 The induced charge

The fermionic lagrangian in the background of dyon is

$$\mathcal{L} = \bar{\psi}_n(i\not{D} - M)\psi_n - \frac{ig}{2} G \tau^a_{nm}\psi_n\psi_m\phi^a \tag{5.2}$$

where

$$D_\mu\psi_n = \partial_\mu\psi_n - \frac{i g}{2} \tau^a_{nm}A_\mu^a\psi_m \tag{5.3}$$

G is the Higgs coupling and M , the fermion mass

We start from the Lagrangian

$$\mathcal{L}_o = \bar{\psi}_n(i\gamma^\mu\partial_\mu - M)\psi_n - \frac{i g}{2} G \tau^a_{nm}\psi_n\psi_m\phi^a \tag{5.4}$$

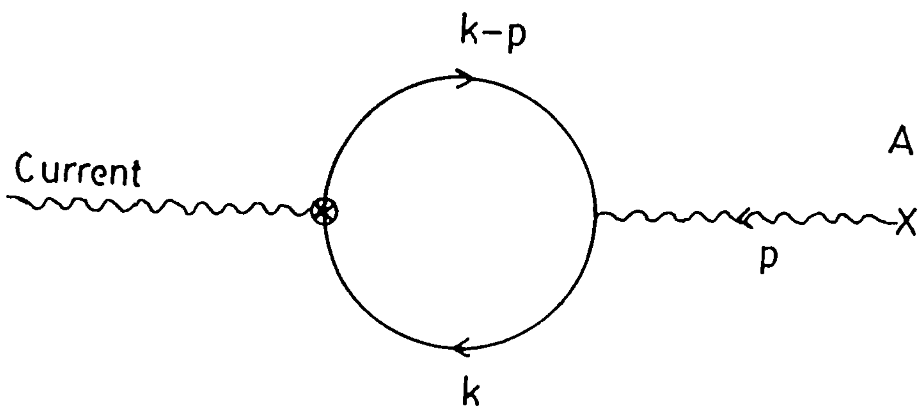

Fig. 8

In this situation a zero energy bound state exists and spectrum is symmetric as shown in the previous chapter. Now the gauge field is adiabatically evolved to get the Lagrangian (5.2) .To calculate the abelian induced charge we shall define the abelian gauge potential associated with the dyon as $A^\mu = A^\mu_a\hat{\phi}^a$. It may be noted that by ansatz (1.53) this is in agreement with the Fadeev's definition[24,29] of the abelian field strength of a dyon as $F = F_a^{\mu\nu}\phi^{\mu\nu\hat{}a}$. The abelian induced current can be calculated from the diagram (Figure 8) as in the case of electrodynamics[34].

$$< J^{\mu} > = -eg\int\frac{d^4p}{(2\pi)^4}\int\frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left[\gamma^{\mu}\frac{1}{(\not{k}-\not{p})-M}\gamma^{\nu}\frac{1}{\not{k}-M}\right]\tilde{A}_{\nu}\ e^{-ipx}$$

$$= - eg\int\frac{d^4p}{(2\pi)^4}\,\Pi(p^2)\left[g^{\mu\nu}p^2-p^{\mu}p^{\nu}\right]\tilde{A}_{\nu}\ e^{-ipx} \qquad (5.5)$$

where $\tilde{A}_{\nu}$ is the Fourier transform of A_{ν}. In (5.5)

$$\Pi(p^2) = p^2/60\pi^2(M^2+p^2) \qquad (5.6)$$

By spontaneous symmetry breaking gauge bosons become massive and hence $p^2 = 2\,m_v^2$. Therefore $\Pi(p^2) = 2\,m_v^2/60\pi^2(M^2 + m_v^2)$ and

$$\langle J^{\mu}\rangle = -eg\,\frac{2\,m_v^2}{60\pi^2(M^2+m_v^2)}\int\frac{d^4p}{(2\pi)^4}\left[g^{\mu\nu}p^2-p^{\mu}p^{\nu}\right]\tilde{A}_{\nu}\ e^{-ipx}$$

In momentum space $F^{\mu\nu}$ can be written as

$$ip^{\nu}\,\tilde{F}^{\mu\nu} = (p^{\mu}p^{\nu} - g^{\mu\nu}p^2)\,\tilde{A}^{\nu}$$

Consequently

$$\langle J^{\mu}\rangle = \frac{i2egm_v^2}{60\pi^2(M^2+m_v^2)}\int\frac{d^4p}{(2\pi)^4}\left[p^{\nu}\,\tilde{F}^{\mu\nu}\right]e^{-ipx}$$

The charge density is hence given by

$$\langle J^{o}\rangle = \frac{i\,2eg\, m_{\nu}^{2}}{60\pi^{2}(M^{2}+ m_{\nu}^{2})}\int\frac{d^{4}p}{(2\pi)^{4}}\left[p^{i}\ \tilde{F}^{oi}\right]e^{-ipx}$$

$$= \frac{2egm_{\nu}^{2}}{60\pi^{2}(M^{2}+ m_{\nu}^{2})}\,\partial_{i} E^{i}$$

Corresponding to this the induced electric charge is

$$Q_{I} = \int \langle J^{o}\rangle\, d^{3}x = \frac{2egm^{2}_{\nu}}{60\pi^{2}(M^{2}+ m^{2})}\left(\frac{4\pi d}{g}\right)$$

$$= \frac{-2em_{\nu}^{2}d}{15\pi\ (M^{2}+ m_{\nu}^{2})} \tag{5.7}$$

by applying Gauss theorem . Thus the induced charge depends on the constant d in (5.1) but is independent of the coupling constant g.

5.4. Dirac equation in the dyon background

As in chapter 2 and 3 the spectral flow is calculated by analysing the bound state spectrum. Here we follow a method adopted by us in the previous chapter for determining the bound states of monopoles with fermions and bosons . In the case of isospinor fermions Dirac equation to be solved is given by equation (4.5) and corresponding to equation (4.6) we get

$$\left\{\vec{\alpha}.\left[\vec{P}(r)-\frac{A(r)}{2}(\hat{r}\times\vec{\tau})\right]+ \frac{J(r)}{2r}(\vec{\tau}\,.\,\hat{r}) - \frac{\beta H(r)}{2r}(\vec{\tau}\,.\,\vec{r})\right\}\psi(\vec{x})= (E-\beta M)\psi(\vec{x}) \tag{5.8}$$

Equation (5.8) is solved inside the core by assuming the fields with their value at $r \longrightarrow 0$ as given by (1.57) and solved outside the core by assigning to the fields their asymptotic values as $r \longrightarrow \infty$, as

$$H(r) = ar \quad ; \quad J(r) = cr \quad ; \quad A(r) = 1/r \tag{5.9}$$

We have taken $d = 0$; otherwise our procedure fails to give analytic results . This configuration is essentially a monopole since its electric charge is zero by equation (1.60) . The induced charge is also zero as given by (5.7). An important feature of the above solution is that it does not satisfy the Bogomolny condition[24,103]

$$\left.\begin{aligned} F^{a}_{ij} &= \mathrm{Cos}\theta \; \varepsilon_{ijn} \; D_n \phi_a \\ F^{a}_{on} &= \mathrm{Sin}\theta \; D_n \phi^{a} \end{aligned}\right\} \tag{5.10}$$

for selfduality . The angle θ is related to the magnetic charge g_m and electric charge Q by $\mathrm{Tan}\,\theta = Q/g_m$.

As in the previous chapter equation (5.8) can be written as

$$\left[\vec{\sigma}_{ij} \cdot (\vec{\nabla}\,\delta_{nm} - \frac{1}{2}\, A(r)(\hat{r} \times \vec{\tau}_{nm})) \pm \delta_{ij}(\frac{G\; H(r)}{2r}\, \hat{r}.\vec{\tau}_{nm} - M\,\delta_{nm})\right] \chi^{\pm}_{jm}$$
$$= \pm \left[\frac{J(r)}{2r} \,(\, \vec{\tau}_{nm} . \hat{r}\,) - E\, \delta_{nm} \right] \chi^{\mp}_{im} \tag{5.11}$$

By substituting (4.10) in (5.11) we get the coupled differential equations as :

$$\left.\begin{array}{ll}
(\partial_r + \frac{1}{r} \mp \frac{G\,H(r)}{2r})\, P_j^{\pm} - \frac{j}{r} B_j^{\pm} \mp M\, G_j^{\pm} = \mp (\frac{J(r)}{2r} P_j^{\mp} + E\, G_j^{\mp}) & j \geq 0 \\
(\partial_r + \frac{1}{r} \mp \frac{G\,H(r)}{2r})\, G_j^{\pm} - \frac{j}{r} C_j^{\pm} \mp M\, P_j^{\pm} = \mp (\frac{J(r)}{2r} G_j^{\mp} + E\, P_j^{\mp}) & j \geq 0 \\
(\partial_r + \frac{1}{r} \pm \frac{G\,H(r)}{2r})\, B_j^{\pm} - \frac{j}{r} P_j^{\pm} \pm M\, C_j^{\pm} = \pm (\frac{J(r)}{2r} B_j^{\mp} + E\, C_j^{\mp}) & j > 0 \\
(\partial_r + \frac{1}{r} \pm \frac{G\,H(r)}{2r})\, C_j^{\pm} - \frac{j}{r} G_j^{\pm} \pm M\, B_j^{\pm} = \pm (\frac{J(r)}{2r} C_j^{\mp} + E\, B_j^{\mp}) & j > 0
\end{array}\right\} \quad (5.12)$$

5.5 Zero angular momentum fermions in presence of nonselfdual monopole

For zero angular momentum equation (5.12) inside the monopole core is same as that given by (4.12) with solutions

$$\begin{aligned}
Y^{\pm} &= \alpha_{\pm} \, \mathrm{Sinh}(kr) \,/\, (kr) \\
X^{\pm} &= \alpha_{\mp} \frac{k}{M \mp E} \left[\frac{\mathrm{Cosh}(kr)}{kr} - \frac{\mathrm{Sinh}(kr)}{(kr)^2}\right]
\end{aligned} \qquad (5.13)$$

for $E < M$ with $k = (M^2 - E^2)^{1/2}$, and

$$\begin{aligned}
Y^{\pm} &= \alpha_{\pm} \, \mathrm{Sin}(k'r) \,/\, k'r \\
X^{\pm} &= \alpha_{\mp} \frac{k'}{M \mp E} \left[\frac{\mathrm{Cos}(k'r)}{k'r} - \frac{\mathrm{Sin}(k'r)}{(k'r)^2}\right]
\end{aligned} \qquad (5.14)$$

for $E > M$ with $k' = (E^2 - M^2)^{1/2}$. Here $\alpha_{\pm}$ are integration constants .

Outside the core equations (5.12) become

$$\begin{aligned}
(\partial_r + \frac{1}{r} \mp \frac{G\,H(r)}{2r})\, P^{\pm} \mp M\, G^{\pm} &= \mp (\frac{J(r)}{2r} P^{\mp} + E\, G^{\mp}) \\
(\partial_r + \frac{1}{r} \mp \frac{G\,H(r)}{2r})\, G^{\pm} \mp M\, P^{\pm} &= \mp (\frac{J(r)}{2r} G^{\mp} + E\, P^{\mp})
\end{aligned} \qquad (5.15)$$

By substituting J(r) and H(r) from (5.1) and by defining $R^{\pm} = P^{\pm} + G^{\pm}$ and $S^{\pm} = P^{\pm} - G^{\pm}$ equation (5.15) can be transformed to

$$
\begin{aligned}
(\partial_r + r^{-1} \mp m_+)R^{\pm} &= \mp \varepsilon_+ R^{\mp} \\
(\partial_r + r^{-1} \mp m_-)S^{\pm} &= \pm \varepsilon_- S^{\mp}
\end{aligned}
\tag{5.16}
$$

where $m_{\pm} = aG/2 \pm M$ and $\varepsilon_{\pm} = c/2 \pm E$. Solutions of (5.16) regular at $r \longrightarrow \infty$ are

$$
\begin{aligned}
R^+ &= \beta_1 \exp(-k_+ r) /(k_+ r) \\
R^- &= \frac{\beta_1}{\varepsilon_+}\left(\frac{k_+ + m_+}{k_+ r}\right)\exp(-k_+ r)
\end{aligned}
\tag{5.17}
$$

$$
\begin{aligned}
S^+ &= \beta_2 \exp(-k_- r) /(k_- r) \\
S^- &= -\frac{\beta_2}{\varepsilon_-}\left(\frac{k_- + m_-}{k_- r}\right)\exp(-k_- r)
\end{aligned}
\tag{5.18}
$$

Here $k_{\pm} = (m_{\pm}^2 - \varepsilon_{\pm}^2)^{1/2}$ and β_1 and β_2 are integration constants. For bound states $\varepsilon_{\pm} < m_{\pm}$ and for scattering states $\varepsilon_{\pm} > m_{\pm}$

The solutions inside and outside the core are matched at the core boundary $r = r_o$ and the matching condition is deduced in the form

$$
\det \mathbf{A}(E) = 0 \tag{5.19}
$$

where $\mathbf{A}$ is a (2×2) matrix given by

$$\begin{bmatrix} (f+F_+) - \dfrac{\varepsilon_+}{m_+ + k_+}(f-F_+) & (f+F_-) + \dfrac{\varepsilon_+}{m_+ + k_+}(f-F_-) \\ (f-F_+) - \dfrac{\varepsilon_-}{m_- + k_-}(f+F_+) & (f-F_-) + \dfrac{\varepsilon_-}{m_- + k_-}(f+F_-) \end{bmatrix} \tag{5.20}$$

with

$$f = \mathrm{Sinh}(kr_o) / (kr_o)$$

$$F_\pm = \frac{k}{M \mp E}\left[\frac{\mathrm{Cosh}(kr_o)}{kr_o} - \frac{\mathrm{Sinh}(kr_o)}{(kr_o)^2}\right]$$

for $E < M$ and

$$f = \mathrm{Sin}(k'r_o) / (k'r_o)$$

$$F_\pm = \frac{k'}{M \mp E}\left[\frac{\mathrm{Cos}(k'r_o)}{k'r_o} - \frac{\mathrm{Sin}(k'r_o)}{(k'r_o)^2}\right]$$

for $E > M$.

Evidently $\det A(E) \neq \det A(-E)$. This shows the spectral asymmetry of the system . However when $c = 0$ from (5.20) we can show that

$$\det A(E) = - \det A(-E) \tag{5.21}$$

This implies charge conjugation symmetry and existence of zero energy bound state . In this case vacuum charge is $\pm$ 1/2 .Therefore it is enough to calculate the spectral flow when the constant c is varied

As in the case of solitons in 1+1 dimension the number of values of c for which $\det A(0)$ becomes zero gives the spectral asymmetry . When $E = 0$,$F_+=F_-=F$ in (5.20)

$$\det A(0) = \begin{vmatrix} 2(f + F) & (f + F) + \dfrac{c}{m_+ + k_+}(f - F) \\ 2(f - F) & (f - F) + \dfrac{c}{m_- + k_-}(f + F) \end{vmatrix} \tag{5.22}$$

where $k_\pm = (m_\pm - c^2)^{1/2}$ .It is straightforward to show that

$$\partial_c[\ \det A(0)\ ] > 0 \tag{5.23}$$

and when $c= 0$, $\det A(0) = 0$ .Therefore $\det A(0)$ is a monotonic function in c passing through $c = 0$ and so there is zero energy state only when $c = 0$. Conclusion is that there is no spectral flow .

In this case the induced charge is zero by (5.7) .Therefore the ground state charge according to equation (2.1) is

$$Q_{\text{ground}} = \pm 1/2 \tag{5.24}$$

From (5.17) and (5.18) the condition for the existence zero energy state is $M < ag/2$. Therefore the ground state charge is discontinuous at $M = ag/2$ as already noted in literature[89] .

5.6 Higher angular momentum fermions in the nonselfdual monopole background

In this case Dirac equation inside the core is given by (4.26) and the solutions are

$$Z_{\pm} = \alpha_{\pm}\ (k\ r)^{-1/2} I_{j+1/2}(kr)$$

$$Y_{\pm} = \beta_{\pm}\ (k\ r)^{-1/2}\ I_{j+1/2}(kr)$$

$$W_{\pm} = \frac{k}{M\mp E}(kr)^{-1/2}\left[\frac{\alpha_{\mp}}{2}(I_{j-1/2}(kr)+I_{j+3/2}(kr) - \frac{1}{kr}\ I_{j+1/2}(kr)) - \frac{\beta_{\mp}}{kr}\sqrt{j(j+1)}\ I_{j+1/2}(kr)\right]$$

$$X_{\pm} = \frac{-k}{M\mp E}(kr)^{-1/2}\left[\frac{\beta_{\mp}}{2}(I_{j-1/2}(kr)+I_{j+3/2}(kr) + \frac{1}{kr}\ I_{j+1/2}(kr)) - \frac{\alpha_{\mp}}{kr}\sqrt{j(j+1)}\ I_{j+1/2}(kr)\right]$$

for E < M and

$$Z_{\pm} = \alpha_{\pm}\ (k'\ r)^{1/2} J_{j+1/2}(k'r)$$

$$Y_{\pm} = \beta_{\pm}\ (k'\ r)^{1/2}\ J_{j+1/2}(k'r)$$

$$W_{\pm} = \frac{k'}{M\mp E}(k'\ r)^{-1/2}\left[\frac{\alpha_{\mp}}{2}(J_{j-1/2}(k'r)-J_{j+3/2}(k'r)-\frac{1}{kr}\ J_{j+1/2}(k'r)) - \frac{\beta_{\mp}}{k'r}\sqrt{j(j+1)}\ J_{j+1/2}(k'r)\right]$$

$$X_{\pm} = \frac{-k'}{M\mp E}(k'\ r)^{-1/2}\left[\beta_{\mp}(J_{j-1/2}(k'r)-J_{j+3/2}(k'r) + \frac{1}{k'r}\ J_{j+1/2}(k'r)) - \frac{\alpha_{\mp}}{k'r}\sqrt{j(j+1)}\ J_{j+1/2}(k'r)\right] \tag{5.25}$$

for E > M .

In order to solve the equation outside the core , we define

$$
\begin{aligned}
X^{\pm} &= P^{\pm} + G^{\pm} + B^{\mp} + C^{\mp} \\
y^{\pm} &= P^{\pm} + G^{\pm} - B^{\mp} - C^{\mp} \\
Z^{\pm} &= P^{\pm} - G^{\pm} + B^{\mp} - C^{\mp} \\
W^{\pm} &= P^{\pm} - G^{\pm} - B^{\mp} + C^{\mp}
\end{aligned} \qquad (5.26)
$$

In terms of these variables , the equation outside the core can be written as

$$
\begin{aligned}
(\partial_r + r^{-1} \mp m_+)\, X^{\pm} &= (r^{-1} j \mp \varepsilon_+)\, X^{\mp} \\
(\partial_r + r^{-1} \mp m_+)\, Y^{\pm} &= (-r^{-1} j \mp \varepsilon_+)\, Y^{\mp} \\
(\partial_r + r^{-1} \mp m_-)\, Z^{\pm} &= (r^{-1} j \pm \varepsilon_-)\, Z^{\mp} \\
(\partial_r + r^{-1} \mp m_-)\, W^{\pm} &= (-r^{-1} j \pm \varepsilon_-)\, W^{\mp}
\end{aligned} \qquad (5.27)
$$

By defining

$$
R^{\pm} = X^{+} \pm X^{-} \ , \ S^{\pm} = Y^{+} \pm Y^{-} \ , \ T^{\pm} = Z^{+} \pm Z^{-} \ , \ U^{+} = W^{+} \pm W^{-} \qquad (5.28)
$$

we can decouple and solve (5.27) to get

$$
R^{-} = \gamma_1 (k_+ r)^{-1/2} K_{j+1/2}(k_+ r)
$$

$$
R^{+} = -\gamma_1 k_+ (k_+ r)^{-1/2} (m_+ - \varepsilon_+)^{-1} K_{j-1/2}(k_+ r)
$$

$$
S^{-} = \gamma_2 (k_+ r)^{-1/2} K_{j-1/2}(k_+ r)
$$

$$
S^{+} = -\gamma_2 k_+ (k_+ r)^{-1/2} (m_+ - \varepsilon_+)^{-1} K_{j+1/2}(k_+ r)
$$

(5.29)

$$T^- = \gamma_3 (k_- r)^{-1/2} K_{j+1/2}(k_- r)$$

$$T^+ = -\gamma_3 k_- (k_- r)^{-1/2} (m_- - \varepsilon_-)^{-1} K_{j-1/2}(k_- r)$$

$$U^- = \gamma_4 (k_- r)^{-1/2} K_{j-1/2}(k_- r)$$

$$U^+ = -\gamma_4 k_- (k_- r)^{-1/2} (m_- - \varepsilon_-)^{-1} K_{j+1/2}(k_- r)$$

where $K_\nu(\mu)$ is modified Bessel function and γ_1 , γ_2 , γ_3 and γ_4 are integration constants

On matching the solutions at the core boundary of radius r_0 we get the condition

$$\det \mathbf{A}(E) = 0 \tag{5.30}$$

where $\mathbf{A}$ is a 4×4 matrix whose elements are

$$\begin{aligned}
A_{11} &= R (f_1^+ + g^+) - f \quad ; \quad & A_{12} &= R f - (f_1^- - g^-) \\
A_{13} &= R (-f_2^+ + g^+) - f \quad ; \quad & A_{14} &= -R f - (f_2^- + g^-) \\
A_{21} &= S (f_1^+ - g^+) - f \quad ; \quad & A_{22} &= S f - (f_1^- + g^-) \\
A_{23} &= S (f_2^+ + g^+) + f \quad ; \quad & A_{24} &= S f - (-f_2^- + g^-) \\
A_{31} &= T (f_1^+ + g^+) + f \quad ; \quad & A_{32} &= -T f - (f_2^- + g^-) \\
A_{33} &= T (-f_2^+ + g^+) + f \quad ; \quad & A_{34} &= T f - (f_2^- + g^-) \\
A_{41} &= U (f_2^+ - g^+) + f \quad ; \quad & A_{42} &= -U f - (f_1^- + g^-) \\
A_{43} &= U (f_2^+ - g^+) - f \quad ; \quad & A_{44} &= -U f - (-f_2^- + g^-)
\end{aligned} \tag{5.31}$$

where

$$R = R^{+}/R^{-} = -k_{+}(m_{+}-\varepsilon_{+})^{-1} K_{j-1/2}(k_{+}r_{o}) / K_{j+1/2}(k_{+}r_{o})$$

$$S = S^{+}/S^{-} = -k_{+}(m_{+}-\varepsilon_{+})^{-1} K_{j+1/2}(k_{+}r_{o}) / K_{j-1/2}(k_{+}r_{o})$$

$$T = T^{+}/T^{-} = -k_{-}(m_{-}-\varepsilon_{-})^{-1} K_{j-1/2}(k_{-}r_{o}) / K_{j+1/2}(k_{-}r_{o}) \tag{5.32}$$

$$U = U^{+}/U^{-} = -k_{-}(m_{-}-\varepsilon_{-})^{-1} K_{j+1/2}(k_{-}r_{o}) / K_{j-1/2}(k_{-}r_{o})$$

Also

$$f = I_{j+1/2}(kr_{o}) \quad ; \quad g^{\pm} = -\frac{k}{M\pm E}\frac{\sqrt{J(J+1)}}{kr_{o}} I_{j+1/2}(kr_{o})$$

$$f_{1}^{\pm} = \frac{k}{M \pm E}\,\frac{1}{2}\left(I_{j-1/2}(kr_{o}) + I_{j+3/2}(kr_{o}) - \frac{1}{k\,r_{o}}\, I_{j+1/2}(kr_{o})\right)$$

$$f_{2}^{\pm} = \frac{-k}{M \pm E}\,\frac{1}{2}\left(I_{j-1/2}(kr_{o}) + I_{j+3/2}(kr_{o}) + \frac{1}{k\,r_{o}}\, I_{j+1/2}(kr_{o})\right)$$

for $E < M$ and

$$f = J_{j+1/2}(k'\,r_{o}) \quad ; \quad g^{\pm} = -\frac{k'}{M\pm E}\frac{\sqrt{J(J+1)}}{kr_{o}} J_{j+1/2}(k'\,r_{o})$$

$$f_{1}^{\pm} = \frac{k'}{M \pm E}\,\frac{1}{2}\left(J_{j-1/2}(k'\,r_{o}) - J_{j+3/2}(k'\,r_{o}) - \frac{1}{k'\,r_{o}}\, J_{j+1/2}(k'\,r_{o})\right)$$

$$f_{2}^{\pm} = \frac{-k'}{M \pm E}\,\frac{1}{2}\left(J_{j-1/2}(k'\,r_{o}) - J_{j+3/2}(k'\,r_{o}) + \frac{1}{k'\,r_{o}}\, J_{j+1/2}(k'\,r_{o})\right)$$

for $E > M$

When $c = 0$, $\det A(E) = \det A(-E)$. This guarantees the charge conjugation symmetry . From direct substitution in (5.30) shows $\det A(0) \neq 0$ .That is there is no zero energy state . When $c \neq 0$, $\det A(E) \neq \det A(-E)$ and so there is no spectral symmetry . Due to lengthy algebra involved the analysis of spectral flow as has been done in zero angular momentum state is not easy . Numerical study shows that det A(0) as a function of c has no zeros and consequently there is no spectral flow. In this case induced charge vanishes and consequently the ground state charge is zero by (2.1).

5.7 Zero angular momentum fermions in presence of dyons

In this case Dirac equation and its solution inside the core is given by equation (4.12) , (4.18) and (4.19). out side the core equation (5.12) can be written as

$$\left[\partial_r + r^{-1} \mp (B/r + m_+)\right] R^{\pm} = \mp (D/r + \varepsilon_+) R^{\mp} \qquad (5.33)$$

$$\left[\partial_r + r^{-1} \mp (B/r + m_-)\right] S^{\pm} = \mp (D/r + \varepsilon_-) S^{\mp} \qquad (5.34)$$

where we have substituted for J(r) and H(r) from (1.59). Also $B = Gb/2$, $D = d/2$. To solve the above equation we define $X_{\pm} = R^{+} \pm R^{-}$ and then we can write from (5.33)

$$\left[\partial_r + r^{-1}\right] X_{\pm} = \left[\frac{B \pm D}{r} + m_+ \pm \varepsilon_+\right] X_{\mp} \qquad (5.35)$$

For bound states $\varepsilon_+ < m_+$ and with the ansatz

$$X_{\pm} = (m_+ \pm \varepsilon_+)^{1/2} \exp(-\rho_+/2)\, \rho_+^{\gamma-1} (Q_2^+ \pm Q_1^+) \qquad (5.36)$$

equation (5.35) can be decoupled and solved to give

$$Q_1^+ = {}_1F_1\left(\gamma + \frac{B\, m_+ - D\varepsilon_+}{\lambda_+} , 2\gamma + 1 , \rho_+\right) \qquad (5.37)$$

$$Q_2^+ = \frac{\gamma + (B\, m_+ - D\varepsilon_+)/\lambda_+}{(Dm_+ - B\varepsilon_+)/\lambda_+}\, {}_1F_1\left(1 + \gamma + \frac{B\, m_+ - D\varepsilon_+}{\lambda_+} , 2\gamma + 1 , \rho_+\right)$$

where ${}_1F_1(a\ ,b\ ,c\ )$ is the Kummer function , and

$$\rho_+ = 2\,\lambda_+ r\ , \quad \lambda_+ = (m^2_+ - \varepsilon^2_+)^{1/2} \quad , \quad \gamma = (B^2 - D^2)^{1/2} \tag{5.38}$$

In a similar manner , by defining $Y_\pm = S^+ \pm S^-$ the solution of equation (5.34) can be written as

$$Y_\pm = (\ m_- \pm \varepsilon_-)^{1/2} \exp(-\rho_-/2)\ \rho_-^{\gamma-1}\ (\ Q_2^- \pm Q_1^-\ ) \tag{5.39}$$

where

$$\rho_- = 2\,\lambda_- r\ , \quad \lambda_- = (m^2_- - \varepsilon^2_-)^{1/2}$$

$$Q_1^- = {}_1F_1\left[\ \gamma + \frac{B\ m_- - D\varepsilon_-}{\lambda_-}\ ,2\gamma + 1\ ,\ \rho_-\right] \tag{5.40}$$

$$Q_2^- = \frac{\gamma + (B\ m_- - D\varepsilon_-)/\lambda_-}{(Dm_- - B\varepsilon_-)/\lambda_-}\ {}_1F_1\left[1 + \gamma + \frac{B\ m_- - D\varepsilon_-}{\lambda_-}\ ,2\gamma + 1\ ,\ \rho_-\right]$$

From (5.36) and (5.39)solutions of (5.33) and (5.34) can be written as

$$R^\pm = \alpha\ \exp(-\rho_+/2)\ \rho_+^{\gamma-1}\left[\sqrt{m_+ + \varepsilon_+}\ (Q_2^+ + Q_1^+\ ) \pm \right. \tag{5.41}$$
$$\left. \sqrt{m_+ - \varepsilon_+}\ (Q_2^+ - Q_1^+\ )\right]$$

$$S^\pm = \beta\ \exp(-\rho_-/2)\ \rho_-^{\gamma-1}\left[\sqrt{m_- + \varepsilon_-}\ (Q_2^- + Q_1^-\ ) \pm \right. \tag{5.42}$$
$$\left. \sqrt{m_- - \varepsilon_-}\ (Q_2^- - Q_1^-\ )\right]$$

By matching the solutions at the boundary $r = r_0$ we arrive at a condition

$$\det \mathbf{A} = 0 \tag{5.43}$$

where $\mathbf{A}$ is a (2x2) matrix given by

$$\begin{bmatrix} (f+F_+) - R(f-F_+) & (f+F_-) + R(f-F_-) \\ (f-F_+) - S(f+F_+) & (f-F_-) + S(f+F_-) \end{bmatrix} \tag{5.44}$$

with $R = R^+/R^-$ and $S = S^+/S^-$ and

$$\begin{aligned} f &= \mathrm{Sinh}(kr_o)/(kr_o) \\ F_\pm &= \frac{k}{M \mp E}\left[\frac{\mathrm{Cosh}(kr_o)}{kr_o} - \frac{\mathrm{Sinh}(kr_o)}{(kr_o)^2}\right] \end{aligned} \tag{5.45}$$

for $E < M$ and

$$\begin{aligned} f &= \mathrm{Sin}(k'r_o)/(k'r_o) \\ F_\pm &= \frac{k'}{M \mp E}\left[\frac{\mathrm{Cos}(k'r_o)}{k'r_o} - \frac{\mathrm{Sin}(k'r_o)}{(k'r_o)^2}\right] \end{aligned} \tag{5.46}$$

for $E > M$. Further for the convergence of normalization integral the Kummer function in (5.36) and (5.39) should reduce to polynomials . This will be true only if we impose the conditions

$$\gamma + \frac{B\, m_+ - D\varepsilon_+}{\lambda_+} = -\, n_1 \tag{5.47}$$

$$\gamma + \frac{B\, m_- - D\varepsilon_-}{\lambda_-} = -\, n_2 \tag{5.48}$$

were n_1 and n_2 are non zero positive integers . If n_1 and n_2 are zero Q_1 and Q_2 are not reduced to polynomials

During adiabatic evolution we cannot guarantee the existence of energy levels since they has to satisfy the conditions (5.47) and (5.48). Therefore we cannot evaluate the spectral flow directly as done in the case of nonselfdual monopole. The same problem arises in the higher angular momentum case also. This is demonstrated in the next section.

In the massless case we have

$$f(E) = f(-E) \quad ; \quad F_{\pm}(E) = -F_{\pm}(-E)$$
$$Q_1^{\pm}(E) = Q_1^{\mp}(-E) \quad ; \quad Q_2^{\pm}(E) = Q_2^{\mp}(-E)$$

From this it follows that det $A(E)$ = det $A(-E)$. There is spectral symmetry and hence charge conjugation symmetry . Zero energy is a trivial solution of (5.43) as pointed out by Jackiw and Rebbi for point dyons . However , in this case when the core effects are included , for the existence of zero energy state conditions (5.47) and (5.48) are to be satisfied . Therefore fermion number fractionisation depend on dyon parameters .

5.8 Higher angular momentum bound states

In this case the Dirac equation inside the core is same as given in the equation (4.26) and its solution is given in equation (4.29) and (4.30). By defining

$$\begin{aligned} X^{\pm} &= P^{\pm} + G^{\pm} + B^{\mp} + C^{\mp} \\ y^{\pm} &= P^{\pm} + G^{\pm} - B^{\mp} - C^{\mp} \\ Z^{\pm} &= P^{\pm} - G^{\pm} + B^{\mp} - C^{\mp} \\ W^{\pm} &= P^{\pm} - G^{\pm} - B^{\mp} + C^{\mp} \end{aligned} \tag{5.49}$$

the Dirac equation out side the core becomes

$$
\begin{aligned}
\mathcal{D}^{\pm}_{m_+} X^{\pm} &= \left[\frac{j}{r} - (D/r + \varepsilon_+)\right] X^{\mp} \\
\mathcal{D}^{\pm}_{m_+} Y^{\pm} &= \left[-\frac{j}{r} - (D/r + \varepsilon_+)\right] Y^{\mp} \\
\mathcal{D}^{\pm}_{m_-} Z^{\pm} &= \left[\frac{j}{r} - (D/r + \varepsilon_-)\right] Z^{\mp} \\
\mathcal{D}^{\pm}_{m_-} W^{\pm} &= \left[-\frac{j}{r} - (D/r + \varepsilon_-)\right] W^{\mp}
\end{aligned}
\tag{5.50}
$$

where

$$\mathcal{D}^{\pm}_{m} = \partial_r + r^{-1} \mp (B\, r^{-1} + m)$$

As in the previous case bound state solution ($\varepsilon < m$) can be obtained as

$$X^{\pm} = \alpha \exp(-\rho_+/2)\, \rho_+^{\gamma-1} \Big[\sqrt{m_+ + \varepsilon_+}\, (Q_2^+(j) + Q_1^+) \pm \sqrt{m_+ - \varepsilon_+}\, (Q_2^+(j) - Q_1^+) \Big]$$

$$Y^{\pm} = \beta \exp(-\rho_+/2)\, \rho_+^{\gamma-1} \Big[\sqrt{m_+ + \varepsilon_+}\, (Q_2^+(-j) + Q_1^+) \pm \sqrt{m_+ - \varepsilon_+}\, (Q_2^+(-j) - Q_1^+) \Big]$$

$$Z^{\pm} = \eta \exp(-\rho_-/2)\, \rho_-^{\gamma-1} \Big[\sqrt{m_- + \varepsilon_-}\, (Q_2^-(j) + Q_1^-) \pm \sqrt{m_- - \varepsilon_-}\, (Q_2^-(j) - Q_1^-) \Big]$$

$$W^{\pm} = \delta \exp(-\rho_-/2)\, \rho_-^{\gamma-1} \left[\sqrt{m_- + \varepsilon_-}\,(Q_2^-(-j) + Q_1^-) \pm \sqrt{m_- - \varepsilon_-}\,(Q_2^-(-j) - Q_1^-) \right]$$

where

$$\gamma = \sqrt{B^2 + j^2 + - D^2}$$

$$Q_1^{\pm} = {}_1F_1\left(\gamma + \frac{B\, m_{\pm} - D\varepsilon_{\pm}}{\lambda_{\pm}}\ , 2\gamma + 1\ , \rho_{\pm} \right)$$

$$Q_2^{\pm}(j) = \frac{\gamma + (B\, m_{\pm} - D\varepsilon_{\pm})/\lambda_{\pm}}{-j + (Dm_{\pm} - B\varepsilon_{\pm})/\lambda_{\pm}}\ {}_1F_1\left(1 + \gamma + \frac{B\, m_{\pm} - D\varepsilon_{\pm}}{\lambda_{\pm}}\ , 2\gamma + 1\ , \rho_{\pm}\right)$$

$Q_2^{\pm}(-j)$ is obtained by changing sign of j in $Q_2^{\pm}(j)$. On matching the solutions at the core boundary $r = r_c$ we get the condition

$$\det A(E) = 0 \tag{5.51}$$

where A is a (4×4) matrix with

$$A_{11} = (1 - x)\left[F_-' + (j/r)\, F_- \right] + (1 + x)\, f$$

$$A_{13} = (1 + x)\left[F_+' - (j/r)\, F_+ \right] + (1 - x)\, f$$

$$A_{21} = (1 - Y)\left[F_-' - (j/r)\, F_- \right] + (1 + Y)\, f$$

$$A_{23} = (1 + Y)\left[F_+' + (j/r)\, F_+ \right] + (1 - Y)\, f$$

$$A_{31} = (1 - Z)\left[F_-' + (j/r)\, F_- \right] - (1 + Z)\, f$$

$$A_{33} = (1 + Z) \left[F_+' + (j / r) F_+ \right] - (1 - Z) f$$

$$A_{41} = (1 - W) \left[F_-' - (j / r) F_- \right] - (1 + W) f$$

$$A_{43} = (1 + W) \left[F_+' + (j / r) F_+ \right] - (1 - W) f$$

$$A_{12} = (x - 1) \left\{ F_-' + \left[(j + 1) / r \right] F_- \right\} - (1 + x) f$$

$$A_{14} = (x + 1) \left\{ F_+' - \left[(j - 1) / r \right] F_+ \right\} + (1 - x) f$$

$$A_{22} = (1 - Y) \left\{ F_-' - \left[(j - 1) / r \right] F_- \right\} + (1 + Y) f$$

$$A_{24} = - (1 + Y) \left\{ F_+' + \left[(j + 1) / r \right] F_+ \right\} - (1 - Y) f$$

$$A_{32} = (Z - 1) \left\{ F_-' + \left[(j +1) / r \right] F_- \right\} + (1 + Z) f$$

$$A_{34} = (Z + 1) \left\{ F_+' - \left[(j - 1) / r \right] F_+ \right\} - (1 - Z) f$$

$$A_{42} = (1 - W) \left\{ F_-' - \left[(j - 1) / r \right] F_- \right\} - (1 + W) f$$

$$A_{44} = - (1 + W) \left\{ F_+' + \left[(j + 1) / r \right] F_+ \right\} + (1 - W) f$$

Here $X = X^+/ X^-$, $Y = Y^+/Y^-$, $Z = Z^+/Z^-$, $W = W^+/W^-$ and

$$f = \sqrt{kr_o}\; I_{J+1/2}(kr_o) \quad : \qquad F_\pm = f/(M \pm E)$$

for $E < M$ and

$$f = \sqrt{k' r_o}\, J_{J+1/2}(k' r_o) \ : \qquad F_{\pm} = f/(M \pm E)$$

$$F'_{\pm} = \frac{1}{(M \pm E)} \frac{d}{d(k'r)} \left[\frac{J_{J+1/2}(k'r)}{\sqrt{k'r}} \right]_{r=r_o}$$

for $E > M$.

As in the zero angular momentum case , for the convergence of the normalization integral , we should impose conditions

$$\gamma + \frac{B\, m_+ - D\varepsilon_+}{\lambda_+} = - n_1 \qquad (5.52)$$

$$\gamma + \frac{B\, m_- - D\varepsilon_-}{\lambda_-} = - n_2 \qquad (5.53)$$

In this case also the existence of the levels crossing from one side of the spectrum to the other side is not guaranteed and hence we can not say anything about the spectral flow

For the massless fermions we have

$$\begin{aligned} &f(E) = f(-E) \quad ; \quad F_{\pm}(E) = - F_{\pm}(-E) \\ &F'_{\pm}(E) = - F_{\pm}(-E) \\ &Q_1^{\pm}(E) = Q_1^{\mp}(E) \quad ; \quad Q_2^{\pm}(E) = Q_2^{\mp}(-E) \end{aligned} \qquad (5.54)$$

from which it follows that det $A(E)$ = det $A(-E)$. Therefore there is spectral symmetry. If E_1 satisfy (5.52) and (5.53) $-E_1$ also satisfy the same equations and hence the spectral symmetry is guaranteed . In this case by direct substitution it is easy to verify that there is no zero energy state

5.9 Conclusion

In this chapter we investigated the problem of fermion number fractionisation in presence of nonselfdual monopoles and dyons. The induced charge is calculated and was found to depend on dyons electric charge. In presence of nonselfdual monopole the ground state charge is found to be same as that in the presence of a selfdual monopole. In this case also ground state charge is contributed only by the lower angular momentum states and is discontinuous at the fermion mass. With massless fermions in presence of dyons, it is found that there is spectral symmetry and charge conjugation symmetry but the ground state charge need not be ± 1/2 because of nonzero induced charge. For massive fermions the spectrum is asymmetric. However, a direct study of spectral flow is hindered by the occurence of certain conditions to be satisfied

5.A Appendix

In this appendix we solve the first order coupled differential equations

$$\left[\partial_r + \frac{1 \pm j}{r}\right] X_{\pm} = \left[\frac{B \pm D}{r} + m \pm \varepsilon\right] X_{\mp} \qquad (A.1)$$

This equation is similar to the hydrogen atom problem in relativistic theory if the B/r term is absent. On dividing through by $2\lambda = 2\,(m^2 - \varepsilon^2)^{1/2}$, we get

$$\left[\partial_\rho + \frac{1 + j}{\rho}\right] X_{+} = \left[\frac{B + D}{\rho} + \sqrt{\frac{m + \varepsilon}{m - \varepsilon}}\right] X_{-} \qquad (A.2)$$

$$\left[\partial_\rho + \frac{1-j}{\rho}\right] X_- = \left[\frac{B-D}{\rho} + \sqrt{\frac{m-\varepsilon}{m+\varepsilon}}\right] X_+ \qquad (A.3)$$

where $\rho = 2\lambda r$. For the above equation we assume solutions of the form

$$X_\pm = \sqrt{m \pm \varepsilon}\; e^{-\rho/2} \rho^{\gamma-1} [\; Q_2(\rho) \pm Q_1(\rho)\;] \qquad (A.4)$$

$Q_{1,2}$ and γ can be determined by substituting (A.4) in (A.3). Then we get

$$\rho\,(\partial_\rho + \gamma + j\,)(\,Q_1 + Q_2\,) - \rho\, Q_2 = (\,B + D\,)\sqrt{\frac{m-\varepsilon}{m+\varepsilon}}\,(\,Q_1 - Q_2)$$

$$\rho\,(\partial_\rho + \gamma - j\,)(\,Q_1 - Q_2\,) + \rho\, Q_2 = -(\,B - D\,)\sqrt{\frac{m+\varepsilon}{m-\varepsilon}}\,(\,Q_1 + Q_2) \qquad (A.5)$$

These can be decoupled to get

$$\rho\,\partial^2_\rho\, Q_1 + (\,2\gamma + 1 - \rho\,)\,\partial_\rho Q_1 - (\,\gamma + \frac{Bm - D\varepsilon}{\lambda}\,)\,Q_1 = 0$$

$$\rho\,\partial^2_\rho\, Q_2 + (\,2\gamma + 1 - \rho\,)\,\partial_\rho Q_2 - (\,\gamma + 1 + \frac{Bm - D\varepsilon}{\lambda}\,)\,Q_2 = 0 \qquad (A.6)$$

There we assume that

$$\gamma^2 - \left(\frac{Bm - D\varepsilon}{\lambda}\right)^2 = j^2 - \left(\frac{Dm - B\varepsilon}{\lambda}\right)^2$$

Then $\gamma = (\,B^2 + j^2 - D^2)^{1/2}$

The solutions of the equations (A.6) are

$$Q_1 = \alpha\,{}_1F_1\,(\,\gamma + \frac{Bm - D\varepsilon}{\lambda}\,,\; 2\gamma + 1\;,\; \rho)$$
$$Q_2 = \beta\,{}_1F_1\,(1 + \gamma + \frac{Bm - D\varepsilon}{\lambda}\,,\; 2\gamma + 1\;,\; \rho)$$

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