SOME CHARACTERIZATION PROBLEMS ASSOCIATED WITH THE BIVARIATE EXPONENTIAL AND GEOMETRIC DISTRIBUTIONS

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CERTIFICATE

Certified that the thesis entitled Some characterization problems associated with the bivariate exponential and geometric distributions" is a bonafide record of work done by Sri. K.R.Muraleedharan Nair under my guidance in the Department of Mathematics and Statistics, Cochin University of Science and Technology and that no part of it has been included anywhere previously for the award of any degree or title.

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of the appearance of non-normal models. Bivariate exponential distributions, the properties of one such model form the focal theme of the present investigation, were investigated by Gumbel in 1960 in which he pointed out several fundamental differences of his models with that of the multinormal. For example, the regression lines in his model were neither linear nor intersect at the mean and the dependence mechanism is not solely dependent on the coefficient of correlation. Based on physical characteristics that warranted non-normal models and also by way of extending desirable properties of univariate models into higher dimensions several types of multivariate exponential models were constructed by researchers like Freund, Marshall and Olkin, Moran, Downton, Block and Basu, Raftery, Sarkar, etc. The properties of these models vis a vis their ability to explain real world phenomena are taken up elsewhere in the present study.

The preliminary and perhaps most important step in analysing statistical data is to locate an appropriate model followed by it. This enables the analysis of the original problem into a comparatively simpler one as one can make use of the specific properties of special distributions in reaching at reasonable inferences. The natural question as to how fully such a reduction in effort can be attempted by utilising the special nature of a distribution has paved the way for detailed investigation on the inherent properties of probability distributions in mathematical statistics. Thus if one knows the experimental situations or properties of stochastic variables that can uniquely determine or characterize a distribution, such information is highly useful in checking whether a particular distribution is a good choice for explaining a real situation.

The search in this direction has necessitated the formulation of characterization theorems concerning all important distributions that could hope to be realistic models. As in the case of the evolution of distribution theory, in the area of characterization also the normal law attracted early attention with the work of Polya in 1926. Though the importance of the exponential distribution in present day literature is next only to that of the normal, the versatality of the latter was recognised a little late, perhaps with the pioneering worksof Sukhatme (1937), Epstein and Sobel (1953) and Renyi (1956). These authors laid the foundation for the lion share of characterizationsthat appeared subsequently on the three basic characteristics of the exponential distribution namely lack of memory property and its variants, its relation with the Poisson process and the properties of its order statistics.

Compared to the voluminous literature available on the characterization of the univariate exponential law, very little work has been done relating to their multivariate forms. In view of this, the present investigation is an attempt to study the following problems.

It is highly desirable that any multivariate distribution possesses characteristic properties that are generalisation in some sense of the corresponding results in the univariate case. Therefore it is of interest to examine whether a multivariate distribution can admit such characterizations. In the exponential context, the question to be answered is, in what meaningful way can one extend the unique properties in the univariate case in a bivariate set up? Since the lack of memory property is the best studied and most useful property of the exponential law, our first endeavour in the present thesis, is to suitably extend this property and its equivalent forms so as to characterize the Gumbel's bivariate exponential distribution. Though there are many forms of bivariate exponential distributions, a matching interest has not been shown in developing corresponding discrete versions in the form of bivariate geometric distributions. Accordingly, attempt is also made to introduce the geometric version of the Gumbel distribution and examine several of its characteristic properties. A major area where exponential models are successfully applied being reliability theory, we also look into the role of these bivariate laws in that context.

The present thesis is organised into five chapters of which after the present one, a review of literature on bivariate exponential and geometric distributions is taken up in chapter II. We devote chapter III to present some new results on the general properties of the Gumbel's type I bivariate exponential distribution. After introducing a slightly more general model than that prescribed by Gumbel and several of its characteristics in Section 2, the problem of finding new characterizations is attempted in the subsequent sections. The characterizations include those based on local lack of memory, truncated moments, measures of dispersion and geometric compounding. These are essentially results that are extension of the corresponding univariate properties in a specific bivariate set up. Some characteristic properties that has relevance only to the bivariate models, along with some general results such as recurrence relation for truncated bivariate moments and necessary and sufficient conditions for the unique determination of a bivariate models by conditional distributions also form part of the same chapter. In chapter IV the discrete analogue of the Gumbel's bivariate exponential distribution is developed. After pointing out the general properties of this bivariate geometric law we turn to prove some characterization theorems concerning the model, some of them extensions of the exponential results to the discrete sample space. One area of scientific activity where the exponential and geometric laws are of potential activity is reliability and life testing. A concept that can be used in assessing the role of these models is that of bivariate mean residual life. Mention has been made of the definition of this concept in earlier literature, but a systematic study relating to its properties has not yet been made. Accordingly, we present in chapter V the concept of bivariate mean residual life, its properties and role in the choice of reasonable models for bivariate failure time data; and some characterization using its properties.

The present study ends with section 5.5 in which some of its limitations and problems for future work are pointed out.

Chapter-2

REVIEW OF LITERATURE

2.1 Introduction

The history of multivariate exponential distributions is confined to a period of only three decades, mainly due to the rather late recognition of the univariate exponential model as a credible alternative in non-normal situations. Although the ideas of multivariate exponential distributions are explicit in the works of Gumbel on the extension of the extreme value distributions and also in some earlier developments concerning multivariate gamma distributions, a real thrust in this area seems to have begun with the 1960 paper of Gumbel on bivariate exponential distributions. In this paper he gave three forms of bivariate exponential distributions and their role in situations where the parent population is non-normal. This was followed by several papers describing different varieties of multivariate exponential distributions based on the physical characteristics that warrent their introduction. A brief review of the major developments in this area constitute the theme of the present chapter.

2.2 Gumbel's bivariate exponential distribution

Gumbel (1960) has introduced three bivariate models each with exponential marginals. The first of those is specified by the probability density function

$$f(x_1,x_2) = [(1+ \theta x_1)(1+ \theta x_2)-\theta] \exp[-x_1-x_2-\theta x_1 x_2] \quad (2.1)$$

$$x_1,x_2 > 0; \quad 0 \leqslant \theta \leqslant 1.$$

The marginal distributions of X_1 and X_2 are standard exponential, while the conditional distribution of X_i given X_j is

$$f(x_i|x_j) = [(1+\theta x_1)(1+\theta x_2)-\theta] \exp[-x_i(1+\theta x_j)]$$

$$i,j = 1,2; \; i \neq j \; ; \; x_i > 0, \qquad (2.2)$$

with moments

$$\mu_r'(X_2|X_1=x_1) = \frac{r!(1+ \theta x_1+ r\theta)}{(1+\theta x_1)^{r+1}} \; , \; r = 1,2,3 \ldots$$

The co-variance between X_1 and X_2 is obtained from

$$E(X_1,X_2) = - \theta^{-1} e^{\theta^{-1}} E_i(\theta^{-1})$$

where

$$E_i(\theta^{-1}) = \int_{\theta^{-1}}^{\infty} e^{-z} z^{-1} dz$$

The coefficient of correlation between X_1 and X_2 is $- \theta^{-1}e^{\theta^{-1}} E_i(\theta^{-1})-1$. When $\theta = 1$, the variables are independent. Unlike the normal case, the curves of equal probability density are not ellipses nor the regression curves linear that intersect at the common means With increasing values of one of the variables, the conditional expectation of the other remains within finite limits. As regards the coefficient of correlation, it tends to zero as θ tends to zero and as θ increases, the correlation decreases reaching a minimum value of -0.40365 at $\theta = 1$.

Although this distribution has appeared in 1960 and has a simple mathematical form, there has been only very few attempts to investigate its properties including characterizations. Seshadri and Patil (1964) was the first to offer characterizations based on this distribution. Their results may be summarised as follows.

Theorem 2.1

If (X_1,X_2) is a bivariate random vector such that $f(x_2|x_1)$ has the form in equation (2.2), then

$$f_{X_1}(x_1) = \exp(-x_1),\ x_1 > 0$$

if and only if

$$f_{X_2}(x_2) = \exp(-x_2),\ x_2 > 0$$

It is to be noted that a characterization of the bivariate distribution itself can be obtained if a conditional density of the above form is assumed along with the marginal distribution of X_2. The conditional distributions considered here are not exponential or even conform to a well known standard distributional form, to be able to make a worthwhile use of the last remark. Seshadri and Patil (1964) also show that a similar result does not hold for the Gumbel's second bivariate form specified by (2.10).

Another characterization of the Gumbel distribution is based on a property of bivariate failure rates. In the univariate case if the random variable possesses a density f(.), the failure rate function is defined as

$$r(x) = \frac{f(x)}{R(x)} \tag{2.3}$$

where $R(x) = P[X > x]$

It is common knowledge that r(x) is constant if and only if the distribution is exponential. A generalisation of this concept to higher dimensions can be effected in more than one way. Basu (1971) used the scalar quantity

$$r(\underline{x}) = \frac{f(\underline{x})}{R(\underline{x})} \tag{2.4}$$

to define the multivariate failure rate where $\underline{x} = (x_1, x_2, \ldots, x_n)$, $f(\underline{x})$ the joint probability density function of $(\underline{x})$ and $R(\cdot)$ is the survival function defined as $R(\underline{x}) = P[X_i > x_i,\ i=1,2,\ldots,n]$. He proved that there is no absolutely continuous distribution possessing exponential marginals with constant failure rate other than the one with independent (exponential) marginals.

Another interesting result in this connection is that of Puri and Rubin (1974) which states that the only absolutely continuous distribution satisfying

$$r(\underline{x}) = \lambda$$

are mixtures of exponential distributions given by

$$f(x_1, x_2, \ldots, x_n) = \lambda \int_0^\infty \cdots \int_0^\infty \exp[-\sum_1^n \lambda_j x_j] \, D(d\lambda_1 \ldots d\lambda_n)$$

$x_j \geqslant 0$, $j = 1, 2, \ldots, n$.

and D is a probability measure located on the set

$$A = [\prod_{i=1}^{n} \lambda_j = \lambda,\ \lambda_j > 0,\ j=1,2,\ldots,n]$$

An alternative vector valued multivariate failure rate proposed independently by Block (1973), Esary and Marshall [in Marshall (1975)] and Johnson and Kotz (1975) is defined as

$$\underset{\sim}{h}(\underset{\sim}{x}) = (h_1(\underset{\sim}{x}), h_2(\underset{\sim}{x}) \dots h_n(\underset{\sim}{x})) \tag{2.5}$$

where $h_r(\underset{\sim}{x}) = -\frac{\partial \log R(\underset{\sim}{x})}{\partial x_r}$, $r = 1,2,3,\dots,n$

It follows immediately that a constant failure rate of the form

$$\underset{\sim}{h}(\underset{\sim}{x}) = c$$

where $\underset{\sim}{c} = (c_1, c_2, \dots, c_n)$ is an absolute constant with respect to the variables if and only if $\underset{\sim}{X}$ has multivariate exponential distribution with independent exponential marginals.

The two definitions are indicative of the fact that absolute constancy of failure rate in higher dimensions will provide only trivial multivariate exponential distributions. On the other hand if the absolute constancy is relaxed to local constancy, a meaningful bivariate exponential distribution can be arrived at. The relevant results are stated in the following theorem, proved in Galambos and Kotz (1978).

Theorem 2.2.

The multivariate hazard rate $\underline{h}(\underline{x})$ is continuous and of the form (2.4) with $h_r(\underline{x})$ independent of x_r if and only if

$$R(\underline{x}) = \exp[-\{\sum_{i=1}^{n}\theta_i x_i + \sum_{i<j} \theta_{ij}x_i x_j + \dots + \theta_{1\dots n}\, x_1 x_2 \dots x_n\}] \qquad (2.6)$$

Equation (2.6) is easily recognised as the multivariate extension of the Gumbel's form.

In a recent paper Zahedi (1985) introduces the concepts of increasing mean residual life (IMRL) and decreasing mean residual life (DMRL) class of multivariate survival distributions. Among the different types of such classes he calls a p dimensional probability density function to belong to the decreasing (increasing) multivariate mean residual life DMMRL (2) [IMMRL (2)] class if for i=1,2,3,...,n.

$$r_i(t_1, t_2 \dots t_{i-1}, t_{i+\Delta}, t_{i+1} \dots t_n) \leqslant (\geqslant) \; r_i(t)$$

for all $\Delta \geqslant 0$, $\underline{t}$, $t_i > 0$

where $r_i(t)$ is the i^{th} element in the vector

$$\underline{r}(t) = E\,[\underline{X} - t \mid \underline{X} \geqslant \underline{t}\,] \qquad (2.7)$$

Based on this concept the following characterization theorem is proved.

Theorem 2.3

An n-dimensional continuously differentiable probability density function f is DMMRL (2) and IMMRL (2) if and only if f corresponds to the distribution in (2.6).

Nair and Nair (1988) considered the bivariate form of (2.6) to arrive at the following result.

Theorem 2.4

The joint distribution of $X = (X_1, X_2)$ admitting probability density function in R_2^+ has bivariate exponential distribution with

$$P[X_1 > x_1, X_2 > x_2] = e^{-\lambda_1 x_1 - \lambda_2 x_2 - \theta x_1 x_2} \tag{2.8}$$

if and only if

$$E\,[\underline{X}-\underline{t} \mid \underline{X} > \underline{t}] = [a_1(t_2),\ a_2(t_1)] \tag{2.9}$$

where a_i's are non-increasing functions, such that $a_i(t_j)$ is independent of t_i for $i,j=1,2$ and $j \neq i$ for every $t_1, t_2 > 0$, satisfying $a_1(o) = \lambda_1^{-1}$ and $a_2(o) = \lambda_2^{-1}$.

Gumbel (1960) has introduced two more bivariate exponential distributions of which one is derivable as a special case of the Morgenstern (1956) model with given exponential marginals. Here the joint distribution function of (X_1,X_2) is given by

$$F(x_1,x_2) = (1-e^{-x_1})(1-e^{-x_2})[1+\alpha e^{-(x_1+x_2)}] \qquad (2.10)$$

where $x_1,x_2 > 0$ and $|\alpha| < 1$.

It may be noticed that X_1 and X_2 are independent when $\alpha = 0$.

The third model is specified by the distribution function

$$F(x_1,x_2) = 1-e^{-x_1}-e^{-x_2} + \exp[-(x_1^m + x_2^m)^{1/m}] \qquad (2.11)$$

The case $m=1$ leads to the independence of the component variables.

The main motivation in introducing the above bivariate distributions seems to be towards providing an alternate model that could be used in situations where the properties of the multivariate normal distribution are not true. Gumbel does not suggest any specific problems where these models can arise in a natural way except mentioning some areas where the univariate models had been applied in the past. Further, the parameters contained in each of the models is not interpretable in terms of a population parameter in the same way as the parameter ρ in the bivariate normal is deducible as the correlation coefficient. The marginal distributions are standard exponentials which considerably reduces the flexibility of the model in modelling. In reliability studies it is the assumption of constancy of failure rates that brings about the exponential distribution. In all the above models, the failure rates of the variables X_1 and X_2 has to be unity for them to be usable and this turns out to be a serious handicap. To impart more meaning to the model, the present study centres around a revised form in which the component variables have means α_1^{-1} and α_2^{-1} instead of unity in (2.1).

2.3. Freund's distribution

Freund (1961) was the first to offer a bivariate model that has a meaningful physical interpretation. Let X_1 and X_2 represent the life times of two components A and B. Assume X_1 and X_2 to be independent exponential random variables with parameters α and β such that the failure of either component changes the parameters of the life distribution of the other on account of the fact that whenever A fails extra stress is placed on B resulting in the life distribution of B changed to exponential distribution with parameter β'. Similarly if B fails the life distribution of A is changed to exponential distribution with parameter α'. Under these assumptions the joint density of X_1 and X_2 is shown to be

$$f(x_1,x_2) = \begin{cases} \alpha\beta' \exp\left[-\beta' x_2-(\alpha+\beta-\beta')x_1\right] & \text{if } 0\leqslant x_1<x_2 \\ \beta\alpha' \exp\left[-\alpha' x_1-(\alpha+\beta-\alpha')x_2\right] & \text{if } 0\leqslant x_2<x_1 \end{cases} \qquad (2.12)$$

For this distribution the marginal distributions are not exponential, in the special case where $\alpha = \alpha'$ and $\beta = \beta'$, X_1 and X_2 are independent exponential variates and the regression of Y on X and X on Y are generally non-linear although the conditional expectation as α' tends to infinity is

$$E(Y/x) = x + \frac{\alpha}{\beta'(\alpha+\beta)}$$

The expression for correlation coefficient obtained by direct calculation is

$$\rho = \frac{\beta'}{\sqrt{\alpha^2+2\alpha\beta + \beta'^2}}$$

In spite of the physical meaning attached to the model, it has evoked only limited response in subsequent investigations. One can attribute the primary reason for this to the fact that the marginal densities are not exponential which is generally taken to be a primary requisite for a bivariate form. Further the distribution fails to accommodate most of the properties of the univariate exponential distribution when extended to higher dimensions as will be evident from the future discussions.

2.4. Marshall and Olkin distribution

Among bivariate exponential distributions that one which has induced considerable interest is the model suggested in Marshall and Olkin (1967). The primary reason for its popularity lies in the fact that it can be related to a suitably defined Poisson process and also that it satisfies a bivariate version of the lack of memory property, characteristic of the exponential model.

In the univariate case, a non-negative random variable X has the lack of memory property if and only if for all $s,t \geqslant 0$ for which $P(X > t) > 0$, the condition

$$P(X \geqslant t+s \mid X \geqslant t) = P(X \geqslant s) \tag{2.13}$$

or equivalently, in terms of survival function,

$$R(t+s) = R(t).\ R(s) \tag{2.14}$$

holds. An obvious extension of (2.14) to the bivariate case is that (X_1,X_2) satisfies the condition

$$R(t_1+s_1,\ t_2+s_2) = R(t_1,t_2).\ R(s_1,s_2) \tag{2.15}$$

for all $t_1,t_2,s_1,s_2 \geqslant 0$.

This is a too strong generalisation to be of any practical use as the solution of (2.15) leads to the trivial bivariate exponential distribution which is the product of two exponential marginals. In view of this Marshall and Olkin (1967) relaxed (2.15) in the form

$$R(s_1+t,\ s_2+t) = R(s_1,s_2)\ R(t,t) \tag{2.16}$$

for all $s_1,s_2,t \geqslant 0$ to represent bivariate lack of memory. They arrived at the unique solution of (2.16) under the assumption that marginal distributions of

X_1 and X_2 are exponential as

$$R(x_1,x_2) = \exp\left[-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_{12}\max(x_1,x_2)\right] \quad (2.17)$$

where λ_1, λ_2 and λ_{12} are non-negative.

The marginals of (2.17) are exponential with mean $(\lambda_1 + \lambda_{12})^{-1}$ and $(\lambda_2 + \lambda_{12})^{-1}$ and variances $(\lambda_1 + \lambda_{12})^{-2}$ and $(\lambda_2 + \lambda_{12})^{-2}$ respectively. Further the correlation coefficient between X_1 and X_2 is

$$\rho(X_1,X_2) = \frac{\lambda_{12}}{\lambda_1 + \lambda_2 + \lambda_{12}}$$

which lies between 0 and 1, both inclusive.

The Marshall-Olkin distribution can be viewed in the context of shock models as follows. Suppose that the components in a two component system die after receiving a shock which is always fatal. Independent Poisson processes $Z_1(t;\lambda_1)$, $Z_2(t;\lambda_2)$, $Z_{12}(t;\lambda_{12})$ govern the occurrance of shocks. Events in the process $Z_1(t;\lambda_1)$ are shocks to component 1, events in the process $Z_2(t;\lambda_2)$ are shocks to component 2 and $Z_{12}(t;\lambda_{12})$ are shocks to both components. If

X_1 and X_2 denote the life of the first and second components

$$\begin{aligned} R(s,t) &= P\,[X_1 > s,\ X_2 > t\,] \\ &= P\,[Z_1(s;\lambda_1) = 0,\ Z_2(t;\lambda_2) = 0, \\ &\qquad Z_{12}(\max(s,t);\lambda_{12}) = 0 \\ &= \exp[-\lambda_1 s - \lambda_2 t - \lambda_{12}\max(s,t)] \end{aligned}$$

The distribution can also be arrived at considering a non-fatal shock model for which occurrance of shocks are governed by independent Poisson processes with the modification that shocks are non-fatal.

The main characterization theorems of the distribution are summarised in the following theorems.

Theorem 2.5.

$X = (X_1, X_2)$ has a bivariate Marshall-Olkin distribution if and only if there exists random variables U_1, U_2 and U_{12} such that $X_1 = \min(U_1, U_{12})$ and $X_2 = \min(U_2, U_{12})$.

Theorem 2.6 [Block 1977]

A bivariate random vector (X_1, X_2) has Marshall-Olkin distribution if and only if

(a) (X_1,X_2) has exponential marginals

(b) $U = \min(X_1,X_2)$ is exponential

(c) $U = \min(X_1,X_2)$ is independent of $V=X_1-X_2$.

For an extension of (2.17) to n dimensions and characterizations thereof along the lines of the above theorems we refer to Galambos and Kotz (1978).

Inspite of the above interesting properties, the model has a limitation in that it is not absolutely continuous and involves a singular component. In fact $P[X_1=X_2] = (\lambda_1+\lambda_2+\lambda_{12})^{-1}$. The authors justify the appearance of this singular term by arguing that in reliability context the event $X_1=X_2$ can occur when failure is caused by a shock simultaneously felt by both components. However, this is to be seen as a serious drawback since the distribution is usable only when this particular requirement is met with by the system. It is for no other reason that several modifications have been offered to this model in subsequent research.

2.5 Moran's distribution

Moran (1967) has constructed a bivariate exponential distribution with exponential marginals as the joint distribution of

$$X_1 = \tfrac{1}{2}\left(u_1^2 + u_2^2 \right) \text{ and } X_2 = \tfrac{1}{2}\left(u_3^2 + u_4^2 \right)$$

where u_1, u_2, u_3 and u_4 are unit normal variables. (u_1, u_3) and (u_2, u_4) are mutually independent, but each pair has a bivariate normal distribution with correlation coefficient ω. The density function is

$$f(x_1,x_2) = \sum_{j=0}^{\infty} \omega^{2j}\left[\prod_{k=1}^{2}\left\{\sum_{g=0}^{j} (-1)^g \binom{j}{g} (g!)^{-1} x_k^{\,g}\, e^{-x_k}\right\}\right] \tag{2.18}$$

The means and variances of the component variables in (2.18) are unity and the correlation coefficient is ω^2. We observe that Moran's model is a particular case of the multivariate gamma distribution discussed in Johnson and Kotz (1972 p.220) and consequently inherits the properties mentioned therein. The distribution is of limited application in reliability context because of its connection with normal distributions while its form is not encouraging from the point of view of characterizations. Further, the coefficient of correlation is always non-negative, a feature that is not always realised in reliability studies of two component systems.

2.6. Downton's distribution

In contrast to Moran's model, the one proposed by Downton (1970) introduces more stress on applications in reliability. Eventhough the Marshall-Olkin distribution possesses the lack of memory property in two dimensions, it appears to be disadvantageous in the sense that in a two component system correlation could often arise because one component possesses in some sense a memory of time to failure of the other. The model relies on the theory that successive damages of the components leads to its ultimate failure instead of the usual assumption in the renewal process that successive damages accumulate until it reaches a level to cause failure. Assuming that the interval between successive shocks received, T_i, are independent and identically distributed exponential random variables with parameter λ and the number of shocks to failure N is geometric with generating function

$$\pi(z) = \frac{(1-p)z}{1-pz}$$

it follows that

$$\pi[\emptyset(s)] = \frac{(1-p)\lambda}{(1-p)\lambda + s}$$

where $\emptyset(s)$ is the Laplace transform of T_i. When generalised to higher dimensions, the two components behave in such a way that the number of shocks to failure follow a bivariate geometric distribution with probability generating functions

$$\pi(z_1, 1) = \frac{(1-p_1)z_1}{1-p_1 z_1}$$

$$\pi(1, z_2) = \frac{(1-p_2)z_2}{1-p_2 z_2}$$

and the interval between successive shocks are independent and exponentially distributed with scale parameters λ_1 and λ_2 for each component and the joint distribution function of life times $F(t_1, t_2)$ with generating function

$$\Psi(s_1, s_2) = \pi\left[\frac{\lambda_1}{\lambda_1 + s_1}, \frac{\lambda_2}{\lambda_2 + s_2}\right]$$

Downton used this generating function to derive the Laplace transform of F as

$$\Psi(s_1, s_2) = \frac{\mu_1\mu_2}{(\mu_1+s_1)(\mu_2+s_2)-\rho s_1 s_2}$$

where

$$\mu_1 = \frac{\lambda_1}{1+\alpha+\gamma} \;; \quad \mu_2 = \frac{\lambda_2}{1+\beta+\gamma} \quad \text{and}$$

$$\rho = \frac{\alpha\beta + \beta\gamma + \gamma\alpha + \gamma + \gamma^2}{(1+\alpha+\gamma)\,(1+\beta+\gamma)}$$

This gives the joint density function as

$$f(t_1, t_2) = \frac{\mu_1\mu_2}{1-\rho} \exp\left(-\frac{\mu_1 t_1+\mu_2 t_2}{1-\rho}\right) I_o\left(\frac{2\sqrt{\rho\mu_1\mu_2 t_1 t_2}}{1-\rho}\right) \tag{2.19}$$

with $\mu_1, \mu_2 > 0$, $0 \leq \rho \leq 1$ and I_o is the modified Bessel function of the first kind of order 0. Distribution (2.19) happens to be a particular case of an infinitely divisible gamma distribution. The marginal distributions are exponential and the regression is linear with

$$E(\pi | t_2) = \frac{1-\rho}{\mu_1} + \rho\frac{\mu_2}{\mu_1} t_2$$

and conditional variance

$$V(T_1 | t_2) = \frac{1-\rho}{\mu_1} \left(\frac{1-\rho}{\mu_1} - 2\rho \frac{\mu_2}{\mu_1} t_2 \right)$$

The Downton distribution describes adequately a realistic physical situation and offers an alternative to the Marshall-Olkin distribution when the bivariate lack of memory property fails to hold. However, further exploration of the model in terms of the pattern of failure it represent in the reliability context is necessary in order to assess its potential as a worthy competitor to the other models.

2.7. Paulson's distribution

Paulson (1973) proposes a bivariate exponential model by adopting the standard technique of suitably extending a univariate characterizing property to higher dimensions. Let $\psi(t)$ be the characteristic function of a random variable u satisfying $P(u \geq 0) = 1$ and $P(u=0) < 1$ and v be another random variable such that $P(v \geq 0)=1$, $P(v=0) < 1$. Using the fact that the functional equation

$$\phi(t) = \psi(t) = E[\phi(tv)] \qquad (2.20)$$

where expectation is with respect to v, characterizes the exponential model, he took

$$T = (t_1, t_2)$$

$$\psi(t) = E[\exp(it_1u_1 + it_2u_2)] \quad \text{and}$$

$$\phi(t) = E[\exp(itx + ity)]$$

to form the multivariate analogue of (2.20) as

$$\phi(T) = \psi(T)\, E[\phi(TV)]$$

The choice of

$$v = \left[\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right]$$

with probabilities a,b,c,d respectively ($a+b+c+d=1$, $b+d < 1$, $c+d < 1$) yields

$$\Psi(t) = (1-i\theta_1 t_1)^{-1}\,(1-i\theta_2 t_2)^{-1}$$

$$= \Psi_1(t,o) \times \Psi_2(o,t_2)[a+b\,\phi(t_1,o) + c\,\phi(o,t_2) + d\,\phi(T)] \qquad (2.21)$$

Setting $t_i=0$, $i=1,2$ and solving for $\Psi(t_1,o)$ and $\Psi(o,t_2)$

leads to marginal distributions that are exponential. The bivariate exponential distribution is identified as that one with characteristic function (2.21) with parameters $(a,b,c,d,\theta_1,\theta_2)$.

The random variables X and Y with bivariate exponential distribution $(a,b,c,d,\theta_1,\theta_2)$ are independent if and only if ad-bc = 0. Applying inversion theorem the probability density function is

$$f(x_1,x_2) = [a+ \frac{b}{\mu_1} e^{-x_1/\mu_1} x_* + \frac{c}{\mu_2} e^{-x_2/\mu_2} y_*]$$

$$\frac{1}{\theta_1\theta_2} e^{-\frac{x}{\theta_2}-\frac{y}{\theta_1}} I_o\left(2\sqrt{\frac{dx_1x_2}{\theta_1\theta_2}} \right) \qquad (2.22)$$

where $x \geq 0$, $y \geq 0$ and

$$I_v(z) = \frac{(\frac{z}{2})^{v+2k}}{k! \; \Gamma(v+k+1)}$$

is the modified Bessel function of the first kind and of order v. The sumbols x_* and y_* are to be interpreted as operating on the function of x and y postmultiplying the term in brackets and represent the operation of convolution over x and y respectively, which is defined for two functions $h_1(x,y)$ and $h_2(x,y)$ by

$$h_1 * h_2 = \int_0^\infty h_1(\xi,y)\; h_2(x-\xi,y)d\xi .$$

2.8. Block and Basu distribution

Block and Basu (1974) observed that the assumption of absolute continuity in addition to the assumptions of lack of memory property yields a bivariate exponential distribution with independent (exponential) marginals only. Accordingly they relaxed the assumption of exponential marginals to that of mixtures of exponential distributions to arrive at a bivariate exponential distribution specified by the survival function

$$R(x_1,x_2) = \frac{\lambda}{\lambda_1+\lambda_2} \exp[-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_{12} \max(x_1,x_2)] - \frac{\lambda_{12}}{\lambda_1+\lambda_2} \exp[-\lambda \max(x_1,x_2)] \quad (2.23)$$

where $\lambda_1, \lambda_2, \lambda_{12} > 0$; $\lambda = \lambda_1 + \lambda_2 + \lambda_{12}$; $x_1, x_2 > 0$.

This distribution, referred to in literature as the ACBVE, turns out to be the absolutely continuous component of the Marshall-Olkin distribution.

Considering the fact that $\exp(-\lambda z)$ for $z > 0$ is the singular component of the bivariate exponential distribution (2.17) and that the marginals are respectively $\exp[-(\lambda_1 + \lambda_{12})x_1]$ and $\exp[-(\lambda_2 + \lambda_{12})x_2]$, the marginals considered in the derivation of the ACBVE are given by

$$P[X_1 > x_1] = (1+\alpha)\ \exp[-(\lambda_1 + \lambda_{12})x_1] - \alpha\ \exp[-\lambda x_1]$$
$$\text{for } x_1 > 0$$

and $$P[X_2 > x_2] = (1+\alpha)\ \exp[-(\lambda_2 + \lambda_{12})x_2] - \alpha\ \exp[-\lambda x_2]$$
$$\text{for } x_2 > 0$$

This assumption of the lack of memory property implies that these marginals have the form

$$R(x_1) = \frac{\lambda}{\lambda_1 + \lambda_2}\ \exp[-\ (\lambda_1 + \lambda_{12})x_1] - \frac{\lambda_{12}}{\lambda_1 + \lambda_2}\ \exp(-\lambda x_1)$$
$$\text{for } x > 0$$

$$R(x_2) = \frac{\lambda}{\lambda_1 + \lambda_2}\ \exp[-(\lambda_2 + \lambda_{12})x_2] - \frac{\lambda_{12}}{\lambda_1 + \lambda_2}\exp(-\lambda x_2)$$
$$\text{for } x_2 > 0$$

Using the condition

$$R(x_1, x_2) = \begin{cases} \exp(-\ \theta x_2)\ R_1(x_1 - x_2) & \text{if } x_1 \geqslant x_2 \geqslant 0 \\ \exp(-\ \theta x_1)\ R_2(x_2 - x_1) & \text{if } x_2 \geqslant x_1 \geqslant 0 \end{cases}$$

which is equivalent to the lack of memory property, the form of $R(x_1, x_2)$ is realised as in (2.23).

The model can also be derived using the Freund's approach discussed in Section 2.3. Assuming (X_1, X_2) to denote the life length of a two component system with X_1 and X_2 individually having exponential distributions with means α^{-1} and β^{-1} and that if component 1 fails extra stress is placed on component 2, reducing its mean life time to β'^{-1} when $\beta' > \beta$ and similarly if component 2 fails, using Freund's derivation for $\alpha = \lambda_1 + \lambda_{12}\,[\lambda_1/(\lambda_1 + \lambda_2)]$, $\alpha' = \lambda_1 + \lambda_{12}$, $\beta = \lambda_2 + \lambda_{12}[\lambda_2/(\lambda_1 + \lambda_2)]$, $\beta' = \lambda_2 + \lambda_{12}$, where $\lambda_1, \lambda_2, \lambda_{12} > 0$ it follows that $\alpha < \alpha'$, $\beta < \beta'$ and (X_1, X_2) has density as given in (2.23).

If (X_1, X_2) has distribution function given by (2.23), the corresponding density function is

$$f(x_1, x_2) = \begin{cases} \dfrac{\lambda_1 \lambda(\lambda_2 + \lambda_{12})}{\lambda_1 + \lambda_2}\, e^{-\lambda_1 x_1 - (\lambda_2 + \lambda_{12}) x_2} & \text{if } x_1 < x_2 \\[2ex] \dfrac{\lambda_2 \lambda(\lambda_1 + \lambda_{12})}{\lambda_1 + \lambda_2}\, e^{-(\lambda_1 + \lambda_{12}) x_1 - \lambda_2 x_2} & \text{if } x_1 > x_2 \end{cases}$$

with means, variances and co-variance given by

$$E(X_1) = \frac{1}{\lambda_1 + \lambda_{12}} + \frac{\lambda_{12}\ \lambda_2}{\lambda(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_{12})}$$

$$E(X_2) = \frac{1}{\lambda_2+\lambda_{12}} + \frac{\lambda_{12}\ \lambda_1}{(\lambda_1+\lambda_2)(\lambda_2+\lambda_{12})}$$

$$Var(X_1) = \frac{1}{(\lambda_1+\lambda_{12})^2} + \frac{\lambda_{12}\ \lambda_2[2\lambda_1\lambda + \lambda_{12}\lambda_2]}{\lambda^2(\lambda_1+\lambda_2)^2(\lambda_1+\lambda_{12})^2}$$

$$Var(X_2) = \frac{1}{(\lambda_2+\lambda_{12})^2} + \frac{\lambda_{12}\ \lambda_1[2\lambda_2\lambda + \lambda_{12}\lambda_1]}{\lambda^2(\lambda_1+\lambda_2)^2(\lambda_2+\lambda_{12})^2}$$

$$Cov(X_1,X_2) = \frac{(\lambda_1^2+\lambda_2^2)\lambda_{12}\lambda + \lambda_1\lambda_2\lambda_{12}^2}{\lambda^2(\lambda_1+\lambda_2)(\lambda_1+\lambda_{12})(\lambda_2+\lambda_{12})}$$

The moment generating function of the ACBVE is given by

$$m(s,t) = \frac{1}{\lambda_1+\lambda_2}\ \frac{\lambda}{\lambda-(s+t)}$$

$$\frac{\lambda_1(\lambda_2+\lambda_{12})}{\lambda_2+\lambda_{12}-t} + \frac{(\lambda_1+\lambda_{12})\lambda_2}{\lambda_1+\lambda_{12}-s}$$

Further the ACBVE $(\lambda_1, \lambda_2, \lambda_{12})$ has the following interesting properties

1. $\min(X_1,X_2)$ follows exponential distribution with parameter $\lambda = \lambda_1+\lambda_2+\lambda_{12}$.

2. (X_1-X_2) has distribution function

$$F(z)=\begin{cases} \frac{\lambda_1}{\lambda_1+\lambda_2} \exp\left[(\lambda_2+\lambda_{12})z\right] & \text{if } z \leq 0 \\ 1-[\lambda_2/\lambda_1+\lambda_2]\exp[-(\lambda_1+\lambda_{12})z] & \text{if } z > 0 \end{cases}$$

3. $Min(X_1,X_2)$ is independent of X_1-X_2 (and also $|X_1-X_2|$)

4. $$R(x_1,x_2) = \begin{cases} \exp(-\lambda x_2).R_1(x_1-x_2) & \text{if } 0 < x_2 < x_1 \\ \exp(-\lambda x_1).R_2(x_2-x_1) & \text{if } 0 < x_1 < x_2 \end{cases}$$

where $R(x,y) = P(X > x; Y > y)$

5. $$P(X_1 > x_1) = \frac{\lambda_1+\lambda_2}{\lambda} R_1(x) + \frac{\lambda_{12}}{\lambda} \exp(-\lambda x_1) \text{ if } x_1 > 0$$

$$P(X_2 > x_2) = \frac{\lambda_1+\lambda_2}{\lambda} R_2(x_2)+ \frac{\lambda_{12}}{\lambda} \exp(-\lambda x_2) \text{ if } x_2 > 0$$

A characterization of the distribution, is stated in the following theorem.

<u>Theorem</u>

Let (X_1,X_2) have a non-negative bivariate distribution which is absolutely continuous. Then the lack of memory property holds if and only if for $U = \min(X_1,X_2)$ and $V = X_1-X_2$ there is $\theta > 0$ such that

(1) U and V are independent

(2) U follows exponential distribution with parameter Θ

(3) $$P[V \leq t] = \begin{cases} F_1(t) + \Theta^{-1} f_1(t) & \text{if } t \geq 0 \\ 1-F_2(-t) - \Theta^{-1} f_2(-t) & \text{if } t < 0 \end{cases}$$

The bivariate failure rate of Basu is calculated as

$$r(x_1,x_2) = \begin{cases} \dfrac{\lambda_1(\lambda_2 + \lambda_{12})}{\lambda - \lambda_{12} \exp[-\lambda_1(x_2-x_1)]} & \text{if } x_1 < x_2 \\[2ex] \dfrac{(\lambda_1 + \lambda_{12})\lambda_2 \lambda}{\lambda - \lambda_{12} \exp[-\lambda_2(x_1-x_2)]} & \text{if } x_2 < x_1 \end{cases}$$

$r(x_1,x_2)$ is constant along the line $x_2 = x_1 + a$ for any a and that if one of x_1 and x_2, say x, is held fixed, then for values of $x_2 < x_1$, $r(x_1,x_2)$ is increasing and for values of $x_2 > x_1$, $r(x_1,x_2)$ is decreasing.

Block (1974) has extended the distribution to the multivariate case as well.

From the above discussions it is obvious that the bivariate extension of Block and Basu (1974) preserves in higher dimensions certain interesting aspects of the univariate exponential distribution. It satisfies the lack of memory property and has a meaningful reliability interpretation. However, a major limitation is the non-exponentiality of the marginals. This prevents one from assuming the lack of memory property for the marginals and we have a situation where there is lack of memory for the two variables together without either components having the same. The expression for marginal densities as mixtures has implied in it the fact that each variable is composed of two different types of behaviour. This hidden assumption had to be justified in a practical situation for one to be able to choose the Block and Basu model.

2.9. Raftery's Model

Raftery (1984) has introduced a multivariate exponential distribution which can model a full range of correlation structures that attains the Frechet's bounds in the bivariate case. Further he claims that it arises as a model that is easy to simulate and useful in studies on reliability and failure due to shocks. It is analogous to the multivariate normal distribution by being based on linear combinations of independent exponential random variables with marginals having the same form as the population.

To describe the model, let Y_1, Y_2 and Z be independent exponential random variables with parameter λ and I_1 and I_2 binary 0-1 random variables with joint distribution

$$p_{jk} = P[\ I_1=j,\ I_2=k\ ],\ j,k = 0,1$$

where $\quad P\,[I_i=1] = \pi_i,\ i = 1,2.$

The bivariate exponential model for (X_1,X_2) is

$$X_i = (1-\pi_i)\ Y_i + I_i Z,\ i = 1,2 \qquad (2.24)$$

A further extension is possible by replacing Z in the equation defining X_2 by a random variable Z' having exponential distribution and maximum negative correlation with Z to yield

$$\begin{aligned} X_1 &= (\ 1-\pi_1\ )\ Y_1 + I_1 Z \\ X_2 &= (\ 1-\pi_2\ )\ Y_2 + I_2 Z' \end{aligned} \qquad (2.25)$$

Different versions of the model can be arrived at by assigning suitable values for the parameters π_1, π_2 and p_{11}.

For this distribution, the marginal distributions of X_1 and X_2 are exponential with parameter λ and the correlation is

$$\rho = \begin{cases} 2p_{11} - \pi_1\pi_2 & \text{for} \quad (2.24) \\ (1-c)p_{11} - \pi_1\pi_2 & \text{for} \quad (2.25) \end{cases}$$

where $c = -\operatorname{corr}(Z, Z') = \frac{\pi^2}{6} - 1.$

The form of the probability density function given by the author is of a complicated form and is therefore not reproduced here.

2.10. <u>Sarkar's generalised model</u>

The latest member of the class of bivariate exponential distributions seems to be the one proposed in Sarkar (1987) which is a modification of the Block and Basu distribution. Instead of relaxing the assumption of exponential marginals as in Block and Basu (1974) in the derivation of the ACBVE, Sarkar abandoned the lack of memory property while retaining the condition of exponential marginals. The distribution which he calls the ACBVE$_2$ is specified by the survival function

$$R(x_1,x_2) = \exp\left\{-(\lambda_2+\lambda_{12})x_2\right\} \left\{1-[A(\lambda_1 x_2)]^{-\gamma}[A(\lambda_1 x_1)]^{1+\gamma}\right\} \quad \text{if } 0 < x_1 \leq x_2$$

$$= \exp\left\{-(\lambda_1+\lambda_{12})x_1\right\} \left\{1-[A(\lambda_2 x_1)]^{-\gamma}[A(\lambda_2 x_2)]^{1+\gamma}\right\} \quad \text{if } x_1 \geq x_2 > 0 \tag{2.27}$$

where $\gamma = \dfrac{\lambda_{12}}{\lambda_1+\lambda_2}$ and $A(Z) = 1-\exp(-Z)$ for $Z > 0$

when $\lambda_{12} = 0$, X_1 and X_2 are independent.

The derivation of the ACBVE$_2$ is based on the following characterizing properties

(i) The marginal densities of X_1 and X_2 are exponential with means $(\lambda_1+\lambda_{12})^{-1}$ and $(\lambda_2+\lambda_{12})^{-1}$ respectively.

(ii) Min (X_1,X_2) is exponential with mean $(\lambda_1+\lambda_2+\lambda_{12})^{-1}$

and (iii) $\text{Min}(X_1,X_2)$ is independent of $g(X_1,X_2)$ for some $g \in C$ where

$$C = \{g(x_1,x_2);\ g(x_1,x_1) = 0,\ g(x_1,x_2) \text{ is strictly increasing (decreasing) in } x_1(x_2) \text{ for fixed } x_2(x_1)\ .\}$$

Further it is assumed that the distribution function $F(.)$ of $g(X_1,X_2)$ satisfies

$$\begin{aligned} F(g(x_1,x_2)) &= \phi_1(x_1)\ \phi_2(x_2) \quad \text{if} \quad x_1 \le x_2 \\ &= 1 - \phi_3(x_1)\ \phi_4(x_2) \text{ if } x_1 > x_2 \end{aligned}$$

for some $\phi_i(.) \geq 0$, $i = 1,2,3,4.$

The density function of the ACBVE_2 is given by

$$\begin{aligned} f(x_1,x_2) = \frac{\lambda_1 \lambda}{(\lambda_1+\lambda_2)^2} \exp\left\{ -\lambda_1 x_1 - (\lambda_2+ \lambda_{12}) x_2 \right\} \\ \times \left\{(\lambda_1+\lambda_2)(\lambda_2+\lambda_{12}) - \lambda_2 \lambda \times \exp(-\lambda_1 x_2)\right\} \\ [A(\lambda_1 x_1)]^{\gamma}\ [A(\lambda_1 x_2)]^{-(1+\gamma)} \\ \text{if } 0 < x_1 < x_2 \end{aligned}$$

$$F(z) = \frac{\lambda_1}{\lambda_1+\lambda_2} \exp\{(\lambda_2+\lambda_{12})z\} \text{ if } z < 0$$

$$= 1 - \frac{\lambda_2}{\lambda_1+\lambda_2} \exp\{-(\lambda_1+\lambda_{12})z\} \text{ if } z \geq 0$$

Another property of the $ACBVE_2$ is stated as

Theorem

Let Z_1, Z_2, Z_3 and V be independent random variables distributed as exponential with parameters λ_1, λ_2 and λ_{12} respectively and U(o,1). Define

$$X = X_1 + \{1-I(z_1>z_2)\}\{\lambda_1^{-1}A^{-1}(V^{1/1+\gamma}A(\lambda_1X_1)-X_2\}$$

$$Y = X_2 + I(z_1 > z_2)\{\lambda_2^{-1} A^{-1}(V^{1/1+\gamma} A(\lambda_2X_1)-X_2\}$$

where $X_1 = \min(Z_1, Z_3)$ and
$X_2 = \min(Z_2, Z_3)$

Then (X,Y) in $ACBVE_2(\lambda_1, \lambda_2, \lambda_{12})$

It is to be noted that there is no closed form for the distribution function of this bivariate exponential distribution. The physical characteristics vis a vis its applications, that leads to the present model has also not been explained or easily deduced from the properties enjoyed by it. However, it rectifies one important limitation of the ACBVE of Block and Basu by bringing out marginals that are exponentials. In doing so the bivariate lack of memory property is sacrificed, but it is more than compensated by adhering to the same property for the components which is more meaningful. A more detailed study of the distribution is needed to ascertain its applicability. Some attempts in this direction will be made in a subsequent work.

2.10. Bivariate Geometric distributions

Compared to the literature on bivariate exponential distributions, only very little work seems to have been done in developing multivariate geometric distributions as is the case with discrete distributions in higher dimensions, in general. Since the discrete analogue of the exponential distribution is the geometric distribution, it is natural that the characteristic properties of the exponential holds good for a corresponding geometric as well.

Lukacs and Laha (1964) defines a p dimensional random vector X to follow the multivariate negative binomial distribution if the probability mass function is given by

$$p(x_1, x_2, \ldots, x_n) = \frac{\Gamma(\alpha+N)}{\Gamma(\alpha)} \left(1 + \sum_{j=1}^{p} \theta_j\right)^{-\alpha-N} \prod_{j=1}^{p} \frac{\theta_j^{x_j}}{x_j!} \qquad (2.27)$$

where $\alpha > 0$; $\theta_j > 0 (j=1,2,\ldots,p)$; $x_1, x_2, \ldots, x_p = 0,1,2,\ldots$;

$N = x_1 + x_2 + \ldots + x_p$; $N = 0,1,2, \ldots$

A special case of (2.27) obtained by taking $p=2$ and $\alpha=1$ gives rise to a bivariate geometric distribution specified by the probability mass function

$$p(x_1, x_2) = \binom{x_1+x_2}{x_1} \theta_1^{x_1} \theta_2^{x_2} (1+\theta_1+\theta_2)^{-(x_1+x_2+1)} \qquad (2.28)$$

$x_1, x_2 = 0,1,2,\ldots,$; $\theta_1, \theta_2 > 0$.

Paulson and Uppuluri (1972) has proposed another bivariate geometric distribution obtained by generalising to two dimensions a functional equation involving the characteristic function, that characterizes the univariate geometric law. The basic property under consideration is stated as follows.

Let $\phi(t)$ and $\Psi(t)$ be characteristic functions and V be a random variable with distribution function $G(v)$. The solution $\phi(t)$ of the equation

$$\phi(t) = \Psi(t)\, E[\, \phi(tv)\,] \qquad (2.29)$$

is the characteristic function of the geometric distribution if and only if $\Psi(t)$ is the characteristic function of a geometric distribution and G(v) is such that

$$P[V=o] = a;\ P[V=1] = b;\ a+b = 1;\ 0 < a \leq 1$$

Extending to the bivariate case, let v be a matrix valued random variable taking values in the set

$$\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

with probabilities a,b,c,d respectively,

$$a+b+c+d = 1,\ b+d < 1,\ c+d < 1 \text{ and let } T = (t_1, t_2)$$

Let $\Psi(t)$ and $\phi(t)$ be characteristic functions defined by

$$\Psi(t) = E\,[e^{it_1X_1 + it_2X_2}]$$

$$\phi(t) = E\,[e^{it_1Y_1 + it_2Y_2}]$$

The bivariate analogue of (2.29) is given by

$$\phi(T) = \Psi(T)\ E[\phi(TV)] \qquad (2.30)$$

Choose $\Psi(T)$ of the form

$$\Psi(T) = [\ 1+ \tfrac{p}{1-p}(1-e^{it_1})]^{-1} [1+ \tfrac{q}{1-q}(1-e^{it_2})]^{-1}$$

$$= \Psi_1(t_1, o) \times \Psi_2(o, t_2)$$

Now (2.30) can be written as

$$\phi(T) = \Psi_1(t_1,o)\,\Psi_2(o,t_2)[a+b\,\phi(t_1,o)+c\,\phi(o,t_2)+d\phi(T)] \tag{2.31}$$

The bivariate geometric distribution is identified as that one with characteristic function (2.31). Setting $t_1, t_2 = 0$ in (2.31) verifies the fact that the marginal distributions are geometric. Also

$$E(Y_i) = \theta_i$$

$$V(Y_i) = \theta_i(1+\theta_i);\ i = 1,2.$$

$$Cov(Y_1 Y_2) = \frac{ad-bc}{1-d}\ \theta_1\,\theta_2 \text{ where}$$

$$\theta_1 = p\,[(1-p)(1-b-d)]^{-1} \text{ and}$$

$$\theta_2 = q\,[(1-q)(1-c-d)]^{-1}.$$

Recently Pathak and Sreehari (1981) considered bivariate extensions of some of the properties of the univariate geometric distribution that could characterize a bivariate geometric distribution with probability mass function

$$p(x_1,x_2) = \binom{x_1+x_2}{x_1}\ p_1^{\ x_1}\ p_2^{\ x_2}(1-p_1-p_2) \tag{2.32}$$

$x_1, x_2 = 0,1,2, \ldots;\ 0 < p_1, p_2 < 1;\ p_1+p_2 < 1$

The univariate characterizing properties

$$P[X=x+1 \mid X \geq 1] = P[X=x]$$
$$P[Y \leq n] - P[X+Y \leq n] = \beta\, P[X+Y=n] \qquad (2.33)$$

$x,n = 0,1,2,\ldots$

when extended to the bivariate case reads

$$P[X_1=x_1+1,\ X_2=x_2+1 \mid X_1 \geq 1,\ X_2 \geq 1] = P[X_1=x_1, X_2=x_2] \qquad (2.34)$$

and $$P[Y \leq n] - P[X+Y \leq n] = \beta\, P[X+Y = n] \qquad (2.35)$$

where $X = (x_1,x_2)$; $Y = (y_1,y_2)$; $n = (n_1,n_2)$

$x_1,x_2 = 0,1,2,\ldots$

Observing that (2.32) does not satisfy (2.34), the author establishes that there does not exist a random vector satisfying (2.35).

In continuation of these investigations, Nagaraja (1983) extended (2.33) in the form

$$P[X=(x_1,x_2)] = c_1\, P[X=(x_1-1,x_2)]+c_2\, P[X=(x_1,x_2-1)]$$
$$x_1,x_2 = 0,1,2\ldots \quad (x_1,x_2) \neq (0,0) \qquad (2.36)$$

where

$c_1 > 0,\ c_2 > 0,\ c_1+c_2 < 1$

and

$$P[X = (x_1, x_2)] = 0$$

whenever $x_1 < 0$, $x_2 < 0$ and

$$P[Y=n] = (1+\beta)\ P[X+Y=n] - \theta\beta\ P[X+Y = n-I_1] - (1-\theta)\beta\ P[X+Y = n-I_2] \qquad (2.37)$$

and has shown that these properties also characterize (2.32). The rest of the paper involves the proof of the result that if $X = (X_1, X_2)$ is a bivariate random vector with support I_2 then any of the statements, equation (2.36), X_1 and X_2 are independent, X_1 and X_2 are geometric random variables imply the rest.

It is interesting to note that the above bivariate geometric distribution can be deduced from Lukacs and Laha (1964) by setting $p_1 = \theta_1(1+\theta_1+\theta_2)^{-1}$ and $p_2 = \theta_2(1+\theta_1+\theta_2)^{-1}$ in (2.28).

In this chapter we have presented a brief review of the literature on bivariate exponential and bivariate geometric distributions, obtained as a result of investigations in the past three decades. A striking feature of these investigations is that in most papers, interest is shown in generating new models than in assessing the various properties and implications of the existing models. In the succeeding chapters our endeavour will be towards this end, by investigating the properties of the Gumbel's bivariate exponential distribution and its corresponding geometric version.

Chapter-3

CHARACTERIZATIONS OF THE GUMBEL'S BIVARIATE EXPONENTIAL DISTRIBUTION

3.1 Modified Gumbel's form

Instead of utilising the original Gumbel distribution as given in (2.1), in the present study, a more flexible model obtained by introducing two additional parameters is investigated. The motivation for considering this modified version arises from the observation on the original model made in Section 2.2. Accordingly the distribution of the random vector (X_1,X_2) considered throughout the present study is specified by the probability density function

$$f(x_1,x_2) = [(\alpha_2+\theta x_1)(\alpha_1+\theta x_2)-\theta] \exp(-\alpha_1 x_1-\alpha_2 x_2-\theta x_1 x_2) \qquad (3.1)$$

$$x_1,x_2 > 0;\ \alpha_1,\alpha_2 > 0;\ \theta \geq 0.$$

The corresponding distribution function and survival functions are respectively

$$F(x_1,x_2) = 1-\exp(-\alpha_1 x_1) - \exp(-\alpha_2 x_2) +\exp(-\alpha_1 x_1-\alpha_2 x_2-\theta x_1 x_2) \qquad (3.2)$$

and

$$R(x_1,x_2) = \exp(-\alpha_1 x_1-\alpha_2 x_2-\theta x_1 x_2) \qquad (3.3)$$

Although the distribution in the above form has appeared in Galambos and Kotz (1978) in connection with bivariate failure rates, a detailed investigation of its properties does not appear to have been undertaken so far. Therefore as a prelude to the focal theme of the present investigation, namely characteristic properties of the model, we examine some of its basic properties that are of relevance in the sequel. Most of the characterization theorems we present subsequently are motivated by these properties.

3.1.1 Marginal and conditional distributions

When the random vector (X_1,X_2) has distribution (3.1), the marginal distributions are of the usual exponential form namely

$$f_i(x_i) = \alpha_i e^{-\alpha_i x_i}, \; x_i > 0, \; i = 1,2. \tag{3.4}$$

with means α_i^{-1}.

The conditional distributions of X_i given $X_j=t_j$ have density

$$f(x_i|X_j=t_j) = [(\alpha_i+\theta t_j)(\alpha_j+\theta x_i)-\theta]\alpha_j^{-1}\exp[-(\alpha_i+\theta t_j)x_i]$$
$$x_i > 0; \; i,j = 1,2; \; i \neq j \tag{3.5}$$

so that the means and variances are

$$E(X_i|X_j=t_j) = (\alpha_i+\Theta t_j)^{-1} + \Theta\alpha_j^{-1}(\alpha_i+\Theta t_j)^{-2} \qquad (3.6)$$

$$V(X_i|X_j=t_j) = (\alpha_i+\Theta t_j)^{-2} + 2\Theta\, \alpha_j^{-1}(\alpha_i+ \Theta t_j)^{-3} - \Theta^2\, \alpha_j^{-2}(\alpha_i+ \Theta t_j)^{-4} \qquad (3.7)$$

It is to be noted that throughout the present investigation the suffixes i and j will be used in the manner explained in equation (3.5).

It is of considerable interest in our future investigation to introduce conditional distributions of a different kind than (3.5) in which the exceedences $X_j > t_j$ is taken as the conditioning event. The conditional survival function of X_i given $X_j > t_j$ is

$$\begin{aligned} R(x_i|X_j > t_j) &= P(X_i > t_i \mid X_j > t_j) \\ &= \frac{R(t_1,t_2)}{R_j(t_j)} \\ &= \exp[-(\alpha_i + \Theta t_j)]x_i \qquad (3.8) \end{aligned}$$

From (3.8) the corresponding density is stated as

$$f(x_i|X_j>t_j) = (\alpha_i+ \Theta t_j)\ \exp-[(\alpha_i+\Theta t_j)x_i] \qquad (3.9)$$

which is in the univariate exponential form.

Thus the marginal and conditional distributions (in the above sense) of the bivariate exponential distribution are exponential. The result (3.9) plays an important role in the characterizations of (3.1) by giving scope to extend theorems in the univariate case to higher dimensions. Further from the application side also the same condition remains quite meaningful. When (X_1,X_2) represents the life time of a two component system, the exceedences $X_j > t_j$ denote the survival of the component after time t_j. Accordingly the condition enables to look at the life distribution of one of the components in a two-component system when the other is known to be performing adequately its intended function. In chapter 5 we examine in detail the implications of these observations, in connection with reliability analysis.

3.1.2 Local lack of memory

One of the most well studied property of the exponential law that forms the basis of many theoretical and applied researches is the lack of memory. It is therefore, important to investigate how our model accommodates this property in the bivariate set up.

We presently establish that for the model (3.1), the extended version of memorylessness in the form

$$P[X_i > t_i + s_i | X_i > s_i, X_j > t_j] = P[X_i > t_i \mid X_j > t_j] \qquad (3.10)$$
$$i,j = 1,2;\ i \neq j$$

holds.

To verify this, we take i=1 and note that the above statement is equivalent to

$$R_2(t_2)\, R(t_1+s_1, t_2) = R(s_1, t_2)\, R(t_1, t_2)$$

where $R(x_1, x_2)$ is as defined in (3.3) and

$$\begin{aligned} R_2(t_2) &= P\,[X_2 > t_2] \\ &= \exp\,(-\alpha_2 t_2) \end{aligned}$$

Substituting the relevant expressions from equation (3.3) the result is seen to hold for i=1. The proof for i=2 is similar.

The above result indicates that each of the components X_i lacks memory and depends only on the other component or in other words the residual life of each component depends on the life time of the other. This property will be referred to as the local lack of memory of bivariate distribution and it will be shown in a subsequent section that the only absolutely continuous bivariate model that exhibits this property is (3.1).

3.1.3 Moments

When the random vector (X_1,X_2) has distribution (3.1), the $(r,s)^{th}$ order raw moment

$$\mu'_{rs} = E(X_1^{\,r} X_2^{\,s})$$

simplifies to

$$\mu'_{rs} = s!\ [\alpha_1 J(r,s) + \Theta s\ J(r,s+1)]$$

where

$$J(r,s) = \int_0^\infty x_1^{\,r}(\alpha_2 + \Theta x_1)^{-s}\ e^{-\alpha_1 x_1}\ dx_1$$

In particular

$$\mu'_{ro} = E(X_1^{\,r}) = r!\ \alpha_1^{-r} \text{ and}$$

$$\mu'_{os} = E(X_2^{\,s}) = s!\ \alpha_2^{-s}$$

The characteristic function of the distribution is

$$\phi(t_1,t_2) = (1 - \frac{it_1}{\alpha_1})^{-1} + (1- \frac{it_2}{\alpha_2})^{-1} - 1$$

$$- t_1t_2\theta^{-1} \exp[(\alpha_1-it_1)(\alpha_2-it_2)\theta^{-1}]$$

$$E_1[(\alpha_1-it_1)(\alpha_2-it_2)\theta^{-1}] \qquad (3.11)$$

where

$$E_1(x) = \int_x^\infty e^{-z} z^{-1} dz \qquad (3.12)$$

Bivariate distributions are primarily intended to provide models when there is some kind of dependency between the underlying variables. It is therefore important to look at the correlation structure associated with (3.1) to be able to know the type of random phenomena it can represent reasonably well. We first notice that the regression equation of x_i given x_j is

$$E(x_i|x_j) = \frac{\alpha_j(\alpha_i + \theta x_j) + \theta}{\alpha_j(\alpha_i + \theta x_j)^2} \qquad (3.13)$$

which are non-linear. The above function is ever decreasing and crosses the x_i axis at $\alpha_i\alpha_j + \theta/\alpha_j\alpha_i^2$. The x_j axis becomes an asymptote when x_j increases indefinitely. The regression curves does not intersect at the means of the variables as in the normal case, except when $\theta = 0$ in which case the variables are independent. On the other hand, the correlation coefficient is

$$\mathrm{Cor}(X_1,X_2) = \alpha_1\alpha_2\theta^{-1} \exp(\alpha_1\alpha_2\theta^{-1})E_1(\alpha_1\alpha_2\theta^{-1})-1 \quad (3.14)$$

3.1.4. Truncated moments

Another type of moments that are of interest in practical applications is the truncated version defined as follows. For a random vector $X=(X_1,X_2)$ admitting absolutely continuous distribution in the support of the first quadrant

$$Q = \left\{(x_1,x_2);\ x_1 \geq 0,\ x_2 \geq 0\right\}$$

of the two dimensional space R_2, we define its $(r,s)^{th}$ bivariate truncated moment as

$$\phi^{r,s}(t_1,t_2) = E[(X_1-t_1)^r\ (X_2-t_2)^s \mid X > t]$$

or

$$R(t_1,t_2)\ \phi^{rs}(t_1,t_2) = \int_{t_1}^{\infty}\int_{t_2}^{\infty} (x_1-t_1)^r\ (x_2-t_2)^s dF \quad (3.15)$$

Taking s = 0, we get

$$\phi^{r,o}(t_1,t_2) = E[(X_1-t_1)^r \mid X > t] \quad (3.16)$$

Notice however that this is different from the r^{th} truncated moment of the component variable X_1, which is in fact

$$\phi_1^r(t_1) = E[(X_1-t_1)^r \mid X_1 > t_1] \tag{3.17}$$

$$= \phi^{r,o}(t_1,o)$$

When X has the Gumbel distribution $\phi^{r,o}(t_1,t_2)$ simplifies to

$$\phi^{r,o}(t_1,t_2) = r!(\alpha_1 + \theta t_2)^{-r} \tag{3.18}$$

which is independent of t_1.

A symmetric expression is available corresponding to $r = 0$ in (3.15).

A detailed discussion of some properties associated with bivariate truncated moments in general and also some features peculiar to the bivariate exponential distribution will be taken up in Section 3.3.

3.1.5. **Partial moments**

The $(r,s)^{th}$ partial moment of the random variable X defined in Section 3.1.4 is given by

$$\Psi^{r,s}(t_1,t_2) = E[(X_1-t_1)^+]^r\,[(X_2-t_2)^+]^s \tag{3.19}$$

where

$$(X_i-t_i)^+ = \max(X_i-t_i, 0) \text{ for } i=1,2.$$

From (3.19)

$$\psi^{r,s}(t_1,t_2) = \int_{t_1}^{\infty}\int_{t_2}^{\infty} (x_1-t_1)^r(x_2-t_2)^s \, dF \tag{3.20}$$

and from (3.15)

$$\psi^{r,s}(t_1,t_2) = R(t_1,t_2)\, \emptyset^{rs}(t_1,t_2) \tag{3.21}$$

In particular for s=0 in (3.19)

$$\psi^{r,o}(t_1,t_2) = E\,[(X_1-t_1)^+]^r \tag{3.22}$$

$$= \int_{t_1}^{\infty}\int_{t_2}^{\infty} (x_1-t_1)^r \, dF$$

We distinguish this with the r^{th} partial moment of the random variable X_1 which is

$$\psi^{r}(t_1) = \int_{t_1}^{\infty} (x_1-t_1)^r f_1(x_1)dx_1 \tag{3.23}$$

$$= \psi^{r,o}(t_1,0)$$

For the Gumbel model from (3.18) and (3.21)

$$\psi^{r,o}(t_1,t_2) = \exp\,[-\alpha_1 t_1-\alpha_2 t_2-\theta t_1 t_2](\alpha_1+\theta t_2)^{-r} \quad (3.24)$$

3.1.6 Distributions of maxima and minima

In life length studies of two component systems, certain systems operate on the condition that it fails when one of the components fails. In such situations, the random variable of interest is $Y = \min(X_1,X_2)$. The distribution of minimum is also useful in other contexts as well. From the formula

$$F_Y(y) = F_{X_1}(y) + F_{X_2}(y) - F_{X_1,X_2}(y,y)$$

the distribution of Y for the Gumbel distribution is represented by

$$f_Y(y) = (\alpha_1+\alpha_2+2\theta y)\ \exp(-\alpha_1 y-\alpha_2 y-\theta y^2) \quad (3.25)$$

The distribution (3.25) reduces to the standard Rayleigh distribution under the transformation

$$t = \alpha_1 y + \alpha_2 y + \theta y^2$$

However, when the system fails only when both components fail, its life time is $Z = \max(X_1,X_2)$.

For the particular distribution (3.1), Z has density function

$$f_Z(z) = \alpha_1 \exp(-\alpha_1 z) + \alpha_2 \exp(-\alpha_2 z) - (\alpha_1+\alpha_2+2\theta z) \exp(-\alpha_1 z-\alpha_2 z- \theta z^2) \qquad (3.26)$$

3.2 Characterization problems

It is evident from the review of literature, in the previous chapter, that most characterizations on bivariate exponential distributions revolve around suitable extension of the properties in the univariate case. Since such extensions can be achieved in a variety of ways our aim is to find meaningful definitions analogous to the concepts in one dimension that can characterize the Gumbel's form. The properties of the distribution discussed in the previous section form the basis of our investigation. In the following sections we identify those properties that are unique to the Gumbel's bivariate exponential distribution and which have meaningful physical interpretations related to real world phenomena.

The theorems that follow in the succeeding sections are broadly classified under three heads

(i) those based on properties of truncated moments, (ii) by geometric compounding and (iii) by form of conditional distributions.

3.3 Characterizations based on truncated moments *

Theorem 3.1

Let $X = (X_1, X_2)$ be a vector of non-negative random variables admitting probability density function with respect to Lebesgue measure given by $f(x_1, x_2)$ such that $E(X_i^k) < \infty$. Then X follows the Gumbel's bivariate exponential distribution specified by (3.1) if and only if for all positive integers k

$$E[(X_i - t_i)^k \mid X_1 > t_1,\ X_2 > t_2] = a_k^{(i)}(t_{3-i}) \qquad (3.27)$$

where

$$a_k^{(i)}(t_{3-i}) = E\ [X_i^k \mid X_{3-i} > t_{3-i}\ ] \qquad (3.28)$$

are non-increasing, $a_k^{(i)}$ is independent of t_i for all $t_i > 0$ with

$$a_1^{(i)}(o) = \alpha_i^{-1} \qquad (3.29)$$

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* Some results in this section have appeared in the Annals of the Inst. Stat. Math. Vol. 40(2) (1988) p.267-271. (Reference 36)

Proof

When the conditions of Theorem 3.1 are true, equation (3.27) can be written as

$$a_k^{(i)} \; (t_{3-i}) \int_{t_1}^{\infty} \int_{t_2}^{\infty} f(x_1,x_2)dx_1dx_2$$

$$= \int_{t_1}^{\infty} \int_{t_2}^{\infty} (x_i-t_i)^k \; f(x_1,x_2)dx_1 \; dx_2 \qquad (3.30)$$

Taking i=1

$$a_k^{(1)}(t_2) \; R(t_1,t_2) = \int_{t_1}^{\infty} \int_{t_2}^{\infty} (x_i-t_i)^k \, \frac{\partial^2 R}{\partial x_1 \partial x_2} \; dx_1 \; dx_2 \qquad (3.31)$$

where $R = R(t_1,t_2)$ is the survival function of (X_1,X_2) given by

$$R(t_1,t_2) \;\; = \;\; P[X_1 > t_1, \; X_2 > t_2] \qquad (3.32)$$

Integrating the right side of (3.31) with respect to x_2

$$a_k^{(1)}(t_2) \; R(t_1,t_2) = \int_{t_1}^{\infty} (x_1-t_1)^k \, \frac{\partial}{\partial x_1}[F_1(x_1)-F(x_1,t_2)]dx_1 \qquad (3.33)$$

where $F_1(.)$ and $F(.,.)$ are the distribution functions of

X_1 and X respectively. Differentiating (3.33) with respect to t_1

$$a_k^{(1)}(t_2) \frac{\partial R}{\partial t_1} = - \int_{t_1}^{\infty} k(x_1-t_1)^{k-1} \frac{\partial}{\partial x_1} [F_1(x_1)-F(x_1,t_2)]dx_1$$

and performing the same operation successively

$$a_k^{(1)}(t_2) \frac{\partial^{k-1} R}{\partial t_1^{k-1}} = (-1)^{k-1}\ k! \int_{t_1}^{\infty} (x_1-t_1) \frac{\partial}{\partial x_1} [F_1(x_1)-F(x_1,t_2)]\ dx_1$$

$$= (-1)^{k-1}\ k! \int_{t_1}^{\infty} [F_1(x_1)-F(x_1,t_2)]dx_1$$

or

$$a_k^{(1)}(t_2) \frac{\partial^k R}{\partial t_1^k} = (-1)^k\ k!\ R(t_1,t_2) \qquad (3.34)$$

For k=1, equation (3.34) reduces to

$$\frac{\partial \log R}{\partial t_1} = \frac{-1}{a_1^{(1)}(t_2)}$$

The solution of this equation is

$$R(t_1,t_2) = c_1(t_2) \exp \left[\frac{-t_1}{a_1^{(1)}(t_2)} \right] \qquad (3.35)$$

When t_1 tends to zero in the last expression,

$$1 - F_2(t_2) = c_1(t_2)$$

where $F_2(.)$ is the distribution function of X_2.
Substituting in (3.35)

$$R(t_1,t_2) = 1-F_1(t_1) - F_2(t_2) + F(t_1,t_2)$$

$$= [1-F_2(t_2)] \exp \left[\frac{-t_1}{a_1^{(1)}(t_2)} \right] \qquad (3.36)$$

To obtain a general solution of (3.34), notice that the equation may be rewritten in the form

$$\frac{a_k^{(1)}(t_2)}{(-1)^k k!} \quad \frac{\partial^k R(t_1,t_2)}{\partial t_1{}^k} = R(t_1,t_2) \qquad (3.37)$$

which is satisfied by the function

$$R(t_1,t_2) = \sum_{j=1}^{k} c_j(t_2)\, e^{b_j(t_2)t_1} \qquad (3.38)$$

where $c_j(t_2)$ are arbitrary functions independent of t_1 and $b_j(t_2)$ are the k solutions of the auxiliary equation

$$\frac{a_k^{(1)}(t_2)}{(-1)^k\ k!}\ m^k = 1 \qquad (3.39)$$

For k=3, the equation (3.39) takes the form

$$(am^3 + 6) = 0$$

whose solutions are a negative root with value $-(\frac{b}{a})^{1/3}$ and two complex roots

$$\frac{1}{2}\ \bar{a}^{\frac{1}{3}}\ 6^{\frac{1}{3}}\ (\ 1 \pm \sqrt{3}i\ )$$

where $a = a_3^{(1)}\ (t_2)$

Since a > 0, the real parts are positive and accordingly

$$|\ e^{b_j(t_2)t_1}\ | \longrightarrow \infty \text{ as } t_1 \longrightarrow \infty$$

Since $R(t_1,t_2)$ tends to zero as t_1 tends to infinity, we must have $c_2(t_2) = c_3(t_2) = 0$ in (3.38). Thus

$$R(t_1,t_2) = c_1(t_2)\ \exp[-b_1(t_2)t_1] \qquad (3.40)$$

When k=4, the equation to be considered is

$$(am^4 - 24) = 0$$

with real roots

$$m = \pm \left(\frac{24}{a} \right)^{\frac{1}{4}}$$

and imaginary roots

$$m = \pm i\left(\frac{24}{a} \right)^{\frac{1}{4}}$$

with real parts zero.

When $m = \left(\frac{24}{a} \right)^{\frac{1}{4}}$, arguing as before $c_4(t_2) = 0$.

For the two imaginary roots, the expression

$$c_2(t_2)e^{b_2(t_2)t_1} + c_3(t_2)e^{b_3(t_2)t_1}$$

has to decrease to zero for large t_1 and hence once again we have the form in (3.40) for $R(t_1,t_2)$. For $k = 5,6, \ldots$ the argument is similar and therefore the general solution of (3.39) is

$$R(t_1,t_2) = c_1(t_2) \exp-[b_1(t_2)]t_1$$

As $t_1 \longrightarrow 0$ in the above equation

$$c_1(t_2) = 1 - F_2(t_2)$$

Hence from (3.36) and (3.40) we find

$$b_1(t_2) = \frac{1}{a_1^{(1)}(t_2)}$$

for all values of k.

Equation (3.36) can also be written as

$$1-F_1(t_1)-F_2(t_2)+F(t_1,t_2) = [1-F_2(t_2)]\exp[\frac{-t_1}{a_1^{(1)}(t_2)}] \tag{3.41}$$

As t_2 tends to zero

$$1-F_1(t_1) = \exp\left[\frac{-t_1}{a^{(1)}(o)}\right]$$

$$= \exp(-\alpha_1 t_1), \text{ using (3.29)}$$

or

$$F_1(t_1) = 1-\exp(-\alpha_1 t_1)$$

Similarly

$$F_2(t_2) = 1-\exp(-\alpha_2 t_2)$$

Thus from (3.41)

$$R(t_1,t_2) = \exp\left[-\alpha_2 t_2 - \frac{t_1}{a_1^{(1)}(t_2)}\right] \tag{3.42}$$

On similar lines one can show by taking i=2 and k=1 in equation (3.30) that

$$R(t_1,t_2) = \exp\left[-\alpha_1 t_1 - \frac{t_2}{a_1^{(2)}(t_1)}\right] \tag{3.43}$$

Equating the expressions for $R(t_1,t_2)$ in (3.42) and (3.43) results in the functional equation

$$\alpha_2 t_2 + \frac{t_1}{a_1^{(1)}(t_2)} = \alpha_1 t_1 + \frac{t_2}{a_1^{(2)}(t_1)} \tag{3.44}$$

To solve (3.44) we write it in the form

$$\frac{1-\alpha_1 a_1^{(1)}(t_2)}{t_2 a_1^{(1)}(t_2)} = \frac{1-\alpha_2 a_1^{(2)}(t_1)}{t_1 a_1^{(2)}(t_1)} \tag{3.45}$$

Since (3.45) has to be true for all $t_1, t_2 > 0$

$$\frac{1-\alpha_i a_1^{(i)}(t_{3-i})}{t_{3-i}\, a_1^{(i)}(t_{3-i})} = \theta \tag{3.46}$$

a constant, independent of t_1 and t_2 for i=1,2.

Hence

$$a_1^{(i)}(t_{3-i}) = \frac{1}{\alpha_i + \theta t_{3-i}}$$

and therefore, the survival function of X is

$$R(t_1,t_2) = \exp(-\alpha_1 t_1 - \alpha_2 t_2 - \theta t_1 t_2) \qquad (3.47)$$

and this completes the proof of the necessity of the condition. From the monotonicity of $a_1(t_2)$ we have $\theta \geq 0$. Further for the marginals of the bivariate exponential distribution to be proper densities we should have $\alpha_1, \alpha_2 > 0$.

The sufficiency part follows from the actual expression for truncated moments of the bivariate exponential distribution namely

$$E[(X_i - t_i)^k | X > t] = k!\ (\alpha_i + \theta t_{3-i})^{-k},$$

from where, it is easy to verify that the conditions of the theorem are true.

Corollary-1

Taking k=1 in equation (3.27) we get the characterizing property

$$E[(X_i - t_i) | X_1 > t_1, X_2 > t_2] = E[X_i | X_j > t_j] \qquad (3.48)$$

of (3.1) which is proved in Nair and Nair (1988).

Corollary-2

Setting i=1 and allowing t_2 to tend to zero the relationship

$$E\,[(X_1-t_1)^k \mid X_1>t_1] = E(X_1)^k \tag{3.49}$$

for k=1,2,3,... characterizes the univariate exponential distribution with survival function

$$P\,[X_1 > x_1] = \exp(-\alpha_1 x_1)$$

the result due to Sahobov and Geshev cited in Galambos and Kotz (1978).

Corollary-3

When k=1 in (3.49), we have

$$E[X_1-t_1 \mid X_1 > t_1] = E(X_1) \tag{3.50}$$

which is the well known constancy property of the mean residual life function of the exponential distribution proved in several investigations as Reinhardt (1968), Shanbhag(1970), Gupta (1975) etc.

In the following theorem we prove that the local lack of memory property explained in Section 3.1.2 is characteristic of the distribution (3.1) in the class of absolutely continuous bivariate models.

Theorem 3.2

The random vector X in Theorem 3.1 has the Gumbel's bivariate exponential distribution if and only if for all $t_i, s_i \geq 0$, there holds the relations

$$P[X_i > t_i + s_i | X_1 > t_1,\ X_2 > t_2] = P[X_i > s_i | X_j > t_j] \qquad i=1,2,\ i \neq j \tag{3.51}$$

Proof

It is enough to establish the equivalence of (3.51) and (3.48). To prove this we note that when i=1 (3.51) is equivalent to

$$R(t_2)\ R(t_1+s_1, t_2) = R(s_1, t_2)\ R(t_1, t_2) \tag{3.52}$$

where

$$R(t_2) = P[X_2 > t_2]$$

Integrating (3.52) with respect to s_1

$$\int_0^\infty s_1\ f(t_1+s_1 | X > t) ds_1 = \int_0^\infty s_1\ f(s_1 | X_2 > t_2) ds_1$$

which is the same as

$$\int_{t_1}^{\infty}(s_1-t_1)\ f(s_1|X>t)ds_1 = \int_{0}^{\infty} s_1 f(s_1|X_2>t_2)ds_1$$

or

$$E[X_1-t_1 \mid X > t] = E[X_1|X_2 > t_2]$$

as stated in equation (3.48).

The proof for i=2 is similar. The converse follows by retracing the steps and this completes our assertion.

We notice that

(i) for i=1, and t_2, s_2 tending to zero (3.51) becomes

$$P[X_1 > t_1+s_1 \mid X_1 > t_1] = P[X_1 > s_1]$$

the lack of memory property of the random variable X_1.

(ii) The condition (3.48) is weaker than (3.51) as the former requires only the knowledge of the expected values while the latter requires the entire truncated distribution.

(iii) In (3.48) the existance of the mean is a necessity while in (3.50) only the distribution function need be known.

3.3.2 Properties of truncated moments.

The property (3.27) that characterizes the bivariate exponential distribution requires that it must be true for every positive integer k which appears to be a rather stringent condition when one wishes to verify the property to identify the distribution in a practical situation. It is therefore of some interest to enquire whether a relaxation of the requirement can be accomplished. An investigation in this direction necessitates a more detailed study of the properties of bivariate truncated moments. To begin with we establish a recurrence relation satisfied by truncated moments $\phi^{r,s}(t_1,t_2)$ defined in Section 3.1.3.

Theorem 3.3

The truncated moments $\phi^{r,s}$ satisfy the recurrence relation

$$(\phi^{r,s})^{-2}\,[D\phi^{rs} - rs\,E\,\phi^{r,s}] = (\phi^{r-1,s-1})^{-2}\,[D\phi^{r-1,s-1} - (r-1)(s-1)\,E\phi^{r-1,s-1}] \qquad (3.53)$$

$r,s \geq 2$.

where

$$D = \begin{vmatrix} 1 & \frac{\partial}{\partial t_1} \\ \frac{\partial}{\partial t_2} & \frac{\partial^2}{\partial t_1 \partial t_2} \end{vmatrix} \; ; \; E = \begin{vmatrix} E_{00} & E_{01} \\ E_{10} & E_{11} \end{vmatrix} \qquad (3.54)$$

and $E_{mn}\, \phi^{r,s} = \phi^{r-m,s-m}$

Proof

From the definition in equation (3.15)

$$R(t_1,t_2)\, \phi^{r,s} = \int_{t_1}^{\infty} \int_{t_2}^{\infty} (x_1-t_1)^r (x_2-t_2)^s \frac{\partial^2 R}{\partial x_1 \partial x_2}\, dx_1\, dx_2$$

$$= rs \int_{t_1}^{\infty} \int_{t_2}^{\infty} (x_1-t_1)^{r-1} (x_2-t_2)^{s-1} R(x_1,x_2)\, dx_1\, dx_2 \qquad (3.55)$$

by partial integration. Logarithmic differentiation in (3.55) with respect to t_1 and then differentiation of the resulting expression with respect to t_2 yields after some involved algebra to the equation

$$(\phi^{r,s})^{-2} \Big\{ [\phi^{r,s} \frac{\partial^2 \phi^{r,s}}{\partial t_1 \partial t_2} - \frac{\partial \phi^{r,s}}{\partial t_1} \frac{\partial \phi^{r,s}}{\partial t_2}] - rs[\, \phi^{r,s}\, \phi^{r-1,s-1} - \phi^{r-1,s}\, \phi^{r,s-1}] \Big\}$$

$$= R^{-2} [\, R \frac{\partial^2 R}{\partial t_1 \partial t_2} - \frac{\partial R}{\partial t_1} \frac{\partial R}{\partial t_2}] \qquad (3.56)$$

or

$$(\emptyset^{r,s})^{-2}\left\{ \begin{vmatrix} \emptyset^{r,s} & \frac{\partial \emptyset^{r,s}}{\partial t_1} \\ \frac{\partial \emptyset^{r,s}}{\partial t_2} & \frac{\partial^2 \emptyset^{r,s}}{\partial t_1 \partial t_2} \end{vmatrix} - rs \begin{vmatrix} \emptyset^{r,s} & \emptyset^{r-1,s} \\ \emptyset^{r,s-1} & \emptyset^{r-1,s-1} \end{vmatrix} \right\}$$

$$= R^{-2} \begin{vmatrix} R & \frac{\partial R}{\partial t_1} \\ \frac{\partial R}{\partial t_2} & \frac{\partial^2 R}{\partial t_1 \partial t_2} \end{vmatrix}$$

or

$$(\emptyset^{r,s})^{-2} \quad D\emptyset^{r,s} - rs\ E\ \emptyset^{r,s} = R^{-2}\ DR \qquad (3.57)$$

Since the right hand side is independent of both r and s, the recurrence relation (3.53) is immediate if we change r and s respectively to (r-1) and (s-1) and substract the resulting equation from (3.57).

Theorem 3.4

The truncated moments $\emptyset^{r,o}(t_1,t_2)$ satisfy the recurrence relation

$$\frac{\partial}{\partial t_1} \Psi_{r,o} - (r-1)\ (\Psi_{r,o})\ /\ \Psi_{r-1,o}) + r = 0 \qquad (3.58)$$

where $\Psi_{r,o} = \dfrac{\emptyset^{r,o}}{\emptyset^{r-1,o}}$

Proof:

By definition,

$$R(t_1,t_2)\ \phi^{r,o}(t_1,t_2) = \int_{t_1}^{\infty} \int_{t_2}^{\infty} (x_1-t_1)^r \frac{\partial^2 R}{\partial x_1 \partial x_2}\, dx_1 dx_2$$

$$= \int_{t_1}^{\infty} (x_1-t_1)^r \frac{\partial R}{\partial x_1}\, dx_1$$

$$= \int_{t_1}^{\infty} r(x_1-t_1)^{r-1} R(x_1,t_2) dx_1$$

Logarithmic differentiation yields

$$\left(\frac{\partial}{\partial t_1} \phi^{r,o} + r\, \phi^{r,o}\right)/\phi^{r,o} = -\left(\frac{\partial R}{\partial t_1}\right) / R \qquad (3.59)$$

Changing r to (r-1) in (3.59) and substracting the resulting equation from (3.59) leads to (3.58) after simplification.

In an identical fashion one can also prove that

$$\frac{\partial}{\partial t_2}\Upsilon_{o,s} - (s-1)\ (\Upsilon_{o,s} / \Upsilon_{o(s-1)}) + s = 0, \quad s \geq 2 \quad (3.60)$$

Theorem 3.5.

When X is a random vector which satisfies the conditions in theorem (3.1), X follow the Gumbel distribution if for two specific integers r and s ($r,s \geq 2$)

$$\frac{\phi^{r,o}(t_1,t_2)}{r\,\phi^{r-1,o}(t_1,t_2)} = a_1(t_2) \tag{3.61}$$

and

$$\frac{\phi^{o,s}(t_1,t_2)}{s\,\phi^{o,s-1}(t_1,t_2)} = a_2(t_1) \tag{3.62}$$

where $a_i(t_{3-i})$ are non-increasing in t_{3-i}, $i=1,2$.

Proof:

The recurrence relation in theorem 3.4 can be written as

$$\frac{\frac{\partial\phi^{r,o}}{\partial t_1} + r\phi^{r-1,o}}{\phi^{r,o}} = \frac{\frac{\partial\phi^{r-1,o}}{\partial t_1} + (r-1)\,\phi^{r-2,o}}{\phi^{r-1,o}} \tag{3.63}$$

Using (3.63) in (3.61)

$$\frac{\phi^{r-1,o}}{(r-1)\phi^{r-2,o}} = a_1(t_2)$$

and hence

$$\phi^{1,0}(t_1,t_2) = a_1(t_2)$$

Similarly we can show that

$$\phi^{0,1}(t_1,t_2) = a_2(t_1)$$

Thus by corollary 1 to theorem 3.1, the distribution of X is Gumbel's bivariate exponential distribution. By virtue of theorem 3.5 it becomes evident that the ratios of consecutive higher order truncated moments satisfy the specified functional form is sufficient to guarantee the bivariate exponential distribution, in relaxation to the conditions in theorem 3.1.

The constancy of the coefficient of variation of the residual life $X-t|X > t$ of a continuous non-negative random variable X is cited as a unique property exhibited by the univariate exponential distribution in several investigations eg. Nagaraja (1975), Mukherjee and Roy (1986) and Gupta and Gupta (1983). While this result focusses on a property of relative measure of dispersion of the residual life, similar characterizations exist if we consider various absolute measures of dispersion also. Johnson and Kotz (1970) points out the following results due to

Guerrieri.

(a) the variance of the conditional distribution, given that the variable takes values exceeding x does not depend on x.

(b) As for (a) "mean deviation" replacing "variance".

(c) As for (a) "mean difference" replacing "variance".

In the remainder of this section we establish some bivariate analogous of these results that characterize the bivariate exponential distribution.

Theorem 3.6.

Let $X = (X_1, X_2)$ be a random vector admitting a non-degenerate distribution function in R_2^+ and $t = (t_1, t_2)$ be a vector of non-negative real numbers. Then X follow the Gumbel distribution if and only if

$$\frac{\phi^{2,o}(t_1,t_2)}{[\phi^{1,o}(t_1,t_2)]^2} = \frac{\phi^{o,2}(t_1,t_2)}{[\phi^{o,1}(t_1,t_2)]^2} = 2 \qquad (3.64)$$

Proof:

When r=2, the recurrence relation in theorem 3.4 takes the form

$$\frac{\partial \phi^{1,o}}{\partial t_1} = -1 + \frac{\frac{\partial \phi^{2,o}}{\partial t_1}\phi^{1,o}}{\phi^{2,o}} + \frac{2[\phi^{1,o}]^2}{\phi^{2,o}} \tag{3.65}$$

Introducing (3.64) into (3.65)

$$\frac{\partial}{\partial t_1}\phi^{1,o} = -1 + \frac{\frac{\partial}{\partial t_1}[2(\phi^{1,o})^2]}{2\,\phi^{1,o}} + \frac{2[\phi^{1,o}]^2}{2[\phi^{1,o}]^2}$$

$$= 2\,\frac{\partial \phi^{1,o}}{\partial t_1} \tag{3.66}$$

For equation (3.66) to be true we must have

$$\frac{\partial \phi^{1,o}}{\partial t_1} = 0$$

Integrating

$$\phi^{1,o}(t_1,t_2) = c_1(t_2) \tag{3.67}$$

where $c_1(t_2)$ is independent of t_1. Proceeding on similar lines one can also show that

$$\phi^{o,1}(t_1,t_2) = c_2(t_1) \tag{3.68}$$

Noticing that (3.67) and (3.68) are essentially same as (3.27) with k=1, the sufficiency follows. Conversely when X has distribution (3.1), utilising the expression for truncated moments in (3.18) we arrive at (3.64).

Theorem 3.7

Let $X = (X_1, X_2)$ be a continuous non-negative random variable admitting absolutely continuous distribution with $V(X_i) < \infty$.

Denoting

$$V(X_i | X > t) = V_i(t_1, t_2) \tag{3.69}$$

X follows the Gumbel distribution if and only if

$$V_i(t_1, t_2) = V_i(t_{3-i}) \tag{3.70}$$

where $V_i(t_{3-i})$ are non-increasing functions, independent of t_i, with

$$V_i(o) = \alpha_i^{-2} \; ; \; i = 1,2.$$

Proof:

When (3.70) holds for i=1

$$V(X_1 - t_1 \mid X > t) = V_1(t_2)$$

implies

$$V_1(t_2)\, R^2(t_1,t_2) = R(t_1,t_2) \int_{t_1}^{\infty} \int_{t_2}^{\infty} (x_1-t_1)^2 \, dF - \left\{ \int_{t_1}^{\infty} \int_{t_2}^{\infty} (x_1-t_1) dF \right\}^2$$

$$= -\, 2R(t_1,t_2) \int_{t_1}^{\infty} (x_1-t_1)[F_1(x_1)-F(x_1,t_2)]dx_1$$

$$- \left\{ \int_{t_1}^{\infty} [F_1(x_1)-F(x_1,t_2)]dx_1 \right\}^2 \qquad (3.71)$$

Introducing the transformation

$$h(t_1) = \int_{t_1}^{\infty} (x_1-t_1)[F_1(x_1)-F(x_1,t_2)]dx_1 \qquad (3.72)$$

treating t_2 as a parameter and writing

$$g(h) = \left(\frac{dh}{dt_1} \right)^2 \qquad (3.73)$$

we get

$$\frac{d^2h}{dt_1^{\,2}} = R(t_1,t_2) \qquad (3.74)$$

and

$$\frac{dg}{dh} = 2\, \frac{d^2h}{dt_1^{\,2}}$$

Equation (3.71) after effecting the transformation reduces to

$$V_1(t_2) \left(\frac{d^2h}{dt_1{}^2} \right)^2 = 2 \left(\frac{d^2h}{dt_1{}^2} \right) h - \left(\frac{dh}{dt_1} \right)^2$$

or

$$\left(\frac{dg}{dh} \right)^2 - \frac{4}{V_1(t_2)} h(t_1) \left(\frac{dg}{dh} \right) + \frac{4}{V_1(t_2)} g = 0 \qquad (3.75)$$

This is Clairant's equation with solutions

$$h_1(t_1) = c \exp \left(\pm t_1 \sqrt{\frac{1}{V_1(t_2)}} \right)$$

and

$$h_2(t_1) = - \frac{4}{V_1(t_2)} (2t_1+c_1)^2 - c_2$$

Of these, the only solution which meets the requirements of a probability density function is

$$h(t_1) = c(t_2) \exp \left[-t_1[V(t_2)]^{-\frac{1}{2}} \right]$$

where $c(t_2)$ is a constant independent of t_1.

Using (3.74)

$$R(t_1,t_2) = c(t_2) [V_1(t_2)]^{-1} \exp[-t_1[V_1(t_2)]^{-\frac{1}{2}}] \qquad (3.76)$$

As t_1 tends to zero

$$c(t_2) = [1-F_2(t_2)] V_1(t_2)$$

Equation (3.76), therefore turns out to be

$$R(t_1,t_2) = [1-F_2(t_2)] \exp [-t_1(V_1(t_2)]^{-\frac{1}{2}}] \tag{3.77}$$

Similarly for i=2,

$$R(t_1,t_2) = [1-F_1(t_1)] \exp [-t_2[V_2(t_1)]^{-\frac{1}{2}}] \tag{3.78}$$

In (3.77) and (3.78) as t_2 and t_1 respectively tends to zero

$$F_i(t_i) = 1-\exp(-\alpha_i t_i) \tag{3.79}$$

leading to the functional equation

$$\alpha_1 t_1-\alpha_2 t_2 = t_1[V_1(t_2)]^{-\frac{1}{2}} - t_2[V_2(t_1)]^{-\frac{1}{2}}$$

or

$$\frac{t_1[V_2(t_1)]^{\frac{1}{2}}}{1-\alpha_2[V_2(t_1)]^{\frac{1}{2}}} = \frac{t_2[V_1(t_2)]^{\frac{1}{2}}}{1-\alpha_1[V_1(t_2)]^{\frac{1}{2}}} \tag{3.80}$$

for all t_1 and t_2. Proceeding as in (3.45), the only solutions that satisfy (3.80) are

$$V_i(t_{3-i}) = \frac{1}{(\alpha_i+ \theta t_{3-i})^2} \tag{3.81}$$

Substituting in (3.77) or (3.78) we get

$$R(t_1,t_2) = \exp(-\alpha_1 t_1-\alpha_2 t_2-\theta t_1 t_2).$$

The sufficiency follows from the fact that when X has the proposed distribution

$$V(X_i|X>t) = \frac{1}{(\alpha_i+ \theta t_{3-i})^2}$$

which satisfy the conditions of the theorem.

3.4 Characterizations based on Geometric Compounding

Let $X_1,X_2 \ldots$ be independent and identically distributed random variables with common distribution function $F(x)$ and N be a random variable following the geometric law

$$P(N=n) = pq^{n-1}, \quad n=1,2,3\ldots \tag{3.82}$$

independently of the X_i's. If $F^*(x)$ is the distribution function of S^* defined by

$$pS^* = X_1 + X_2 + \ldots + X_N, \tag{3.83}$$

the point of interest in geometric compounding models is the relation between $F^*(x)$ and $F(x)$. When the common distribution of the X_i's is exponential, Arnold (1973) has established that the distribution of pS^* is identical with that of X_1. A detailed exposition of the geometric compounding model and its relationship with the rarefaction models of Renyi (1956) in renewal processes and the damage models introduced by Rao and Rubin(1964) is discussed in Galambos and Kotz (1978). Although these models are of wide applicability in biology, analysis of incomes, under reporting of accidents etc, there have been only a few investigations (see Talwalker (1970) and Patil and Ratnaparkhi (1975)) that extend such ideas to higher dimensions. Our aim in the present section is to generalise the concept of geometric compounding to two dimensions by using the bivariate exponential distribution.

Theorem 3.8*

Let (X_k) be a sequence of non-degenerate, independent and identically distributed random variables admitting probability density function with respect to Lebesgue measure, with components $X_k = (X_{1k}, X_{2k})$ and support $R_2^+ = \{(x,y) \mid x,y > 0\}$ such that the conditional

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* forms part of the paper "Characterizations of the Gumbel's bivariate exponential distribution " Statistics, Vol. 21 (1990)(Reference 39)

expectations

$$m_j(t_{3-j}) = E[X_{jk} \mid X_{3-j,k} > t_{3-j}],\ j=1,2 \qquad (3.84)$$

exist for all real $t_1, t_2 \geq 0$ and are non-increasing.

The relations,

$$P[\ pS_{jN} > x \mid X_{3-j,k} > t_{3-j},\ 1 \leq k \leq N\ ]$$

$$= P\ [X_{jk} > x \mid X_{3-j,k} > t_{3-j}] \qquad (3.85)$$

where $S_{jN} = \sum_{k=1}^{N} X_{jk}$ and N is a random variable following the geometric law

$$P\ [N=n] = p(1-p)^{n-1},\ n=1,2,3\ \ldots$$

independently of X_{jk}, are satisfied for all real $x > 0$ and $t_1, t_2 \geq 0$ if and only if, X_k has bivariate exponential distribution (3.1) with $\alpha_j = [\ m_j(o)\ ]^{-1}$.

Proof:

The logic in the proof is the same as in the univariate case (see Azlarov and Volodin (1986)) appropriately adapted to the bivariate situation.

When X_k has the bivariate exponential distribution, the density function of X_{1k} given $X_{2k} > t_2$ is

$$f(x|X_{2k} > t_2) = (\alpha_1 + \Theta t_2)\exp[-(\alpha_1 + \Theta t_2)x], x > 0$$

with characteristic function

$$A(s,t_2) = [\ 1 - is(\alpha_1 + \Theta t_2)\ ]^{-1} \qquad (3.86)$$

Now, if $B(s,t_2)$ is the characteristic function of S_{jN} given $X_{2k} > t_2$ for $1 \leq k \leq N$, we have

$$B(s,t_2) = \sum_{n=1}^{\infty} P(N=n)\ [A(ps,t_2)]^n$$

$$= \sum_{n=1}^{\infty} p(1-p)^{n-1}\ [A(ps,t_2)]^n$$

$$= pA(ps,t_2)\ [1-(1-p)\ A(ps,t_2)]^{-1} \qquad (3.87)$$

Substituting (3.86) into (3.87) we find

$$A(s,t_2) = B(s,t_2) \text{ for all } s,t_2 \geq 0 \qquad (3.88)$$

proving (3.85) for j=1. The proof for j=2 follows by symmetry.

Conversely if (3.85) holds for j=1, (3.88) is true and therefore from (3.87)

$$A(s,t_2) = p^n A(p^n s,t_2)[1-(1-p^n)A(p^n s,t_2)]^{-1} \qquad (3.89)$$

for all n = 1,2,3, ... Following the proof in the univariate case given in Azlarov and Volodin (1986) we write (3.89) in the form

$$[1-A(s,t_2)]\ A(s,t_2)^{-1} = [1-A(p^n s,t_2)]p^{-n}[A(p^n s,t_2)]^{-1}$$

Taking limits as n tends to infinity

$$\frac{1-A(s,t_2)}{A(s,t_2)} = \lim_{p^n s \to 0} \frac{1-A(p^n s,t_2)}{p^n s\ A(p^n s,t_2)}$$

$$= - \lim_{p^n s \to 0} \frac{1}{A(p^n s,t_2\ )} \frac{A(p^n s,t2)-1}{p^n s}$$

$$= -A'(o,t_2),\ \text{since } A(o,t_2) = 1$$

$$= im_1(t_2)$$

Thus $A(s,t_2) = [\ 1 - ism_1(t_2)\ ]^{-1}$

and whence the density function of X_{1k} given $X_{2k} > x_2$ is

$$f(x_1|X_{2k}>x_2) = [m_1(x_2)]^{-1} \exp-[m_1(x_2)]^{-1}x_1 \qquad (3.90)$$

If we consider the relation (3.85) for j=2 arguments similar to the above leads to

$$f(x_2|X_{1k}>x_1) = [m_2(x_1)]^{-1} \exp [m_2(x_1)]^{-1} x_2 \qquad (3.91)$$

As x_1 tends to zero in (3.91)

$$P [X_{2k} > x_2] = \exp [-\alpha_2 x_2]$$

Accordingly

$$R(x_1,x_2) = \int_{x_1}^{\infty} [m_1(x_2)]^{-1} \exp[-m_1(x_2)]^{-1}x_1 \exp[-\alpha_2 x_2]\, dx_1$$

$$= \exp\left\{-\alpha_2 x_2 - [m_1(x_2)]^{-1} x_2 \right\} \qquad (3.92)$$

Similarly

$$R(x_1,x_2) = \exp\left\{- \alpha_1 x_1 - [m_2(x_1)]^{-1} x_1 \right\} \qquad (3.93)$$

leading to the functional equation

$$(\alpha_1 x_1-\alpha_2 x_2)\, m_1(x_2)m_2(x_1) = x_1 m_2(x_1)-x_2 m_1(x_2) \qquad (3.94)$$

Under conditions imposed on the m's in the theorem, proceeding as in theorem 3.1, the unique form of the solution of (3.94) is

$$m_j(x_{3-j}) = (\alpha_j + \theta x_{3-j})^{-1}, \; j=1,2 \tag{3.95}$$

Using (3.95) in (3.92) or (3.93) we get the desired form of the bivariate exponential distribution. The result of Theorem 3.8 concerns the geometric sum of the components of (X_k) that are independent and identically distributed satisfying condition (3.85). The question that arises now is, what can we say about the distribution of (X_k) if we have two such partial geometric sums that are identically distributed? The only answer to the problem turns out to be that each (X_k) has Gumbel's bivariate exponential distribution. This we establish in

Theorem 3.9.

If (X_k) be the sequence of random variables in theorem 3.8, then the conditions

$$P[p_1 S_{jN_1} > x_j \mid X_{3-j,k} > t_{3-j};\ 1\le k\le N_1]$$

$$= P[p_2 S_{jN_2} > x_j \mid X_{3-j,k} > t_{3-j},\ 1\le k\le N_2] \tag{3.96}$$

$$j = 1,2.$$

holds for all $t_j, x_j > 0$ if and only if the common distribution of (X_k) is the bivariate exponential distribution (3.1) where N_j, $j = 1,2$ are geometric variables with

$$P(N_j=n_j) = p_j(1-p_j)^{n_j-1}, \quad n_j = 1,2,3... \tag{3.97}$$

independently of X_{jk}.

Proof:

To prove the necessary part, following the notation in the proof of theorem 3.8, the characteristic function $A(s,t_2)$ of X_{1k} given $X_{2k} > t_2$, when X_k has distribution (3.1) is by (3.8)

$$A(s,t_2) = [\ 1-i(\alpha_1 + \theta t_2)s]^{-1}$$

The characteristic function $B_1(s,t_2)$ of $p_1\ S_{1N}$ given $X_{2k} > t_2$ for $1 \le k \le N$ is given by (3.87) as

$$B_1(s,t_2) = p_1\ A(p_1 s,t_2)\ [1-(1-p_1)\ A(p_1 s,t_2)]^{-1}$$

We see that

$$B_1(s,t_2) = A(s,t_2) \text{ for all } t_2.$$

Likewise, the characteristic function of $p_2 S_{1N_2}$ given $X_{2k} > t_2$, $B_2(s,t_2)$, reduces to $A(s,t_2)$, proving (3.96) for j=1. Since the proof for j=2 similar, the necessity of the condition follows.

Conversely condition (3.96) implies that for j=1

$$B_1(s,t_2) = B_2(s,t_2)$$

That is

$$p_1 A(p_1 s,t_2)[1-(1-p_1)A(p_1 s,t_2)]^{-1} = p_2 A(p_2 s,t_2)\,[1-(1-p_2)A(p_2 s,t_2)]^{-1}$$

or

$$\frac{p_1 s\, A(p_1 s,t_2)}{1-(1-p_1)A(p_1 s,t_2)} = \frac{p_2 s\, A(p_2 s,t_2)}{1-(1-p_2)A(p_2 s,t_2)} \qquad (3.98)$$

Equation (3.98) can be rearranged as

$$\frac{1}{ip_2 s} - \frac{1}{ip_2 s\, A(p_2 s,t_2)} = \frac{1}{ip_1 s} - \frac{1}{ip_1 s\, A(p_1 s,t_2)}$$

for all $0 < p_1, p_2 < 1$ so that

$$\frac{1}{ip_j s} - \frac{1}{ip_j s\ A(p_j s, t_2)} = k_1(t_2)$$

where $k_1(.)$ is independent of p_j, $j=1,2$. This gives

$$A(ps, t_2) = \frac{1}{1-k_1(t_2)ips}$$

or

$$A(s, t_2) = \frac{1}{1-isk_1(t_2)}$$

This leads to

$$P[X_{1k} > t_1 \mid X_{2k} > t_2] = \exp\left(-k_1(t_2)t_1\right) \qquad (3.99)$$

and

$$P[X_{1k} > t_1] = \exp\left(-\alpha_1 t_1\right)$$

where $\alpha_1 = k_1(o)$.

Similarly the analysis for $j=2$ leaves

$$P[X_{2k} > t_2 \mid X_{1k} > t_1] = \exp(-k_2(t_1)t_2) \qquad (3.100)$$

and

$$P[X_{2k} > t_2] = \exp\left(-\alpha_2 t_2\right),\ \alpha_2 = k_2(o)$$

The rest of the proof is similar to that in the previous theorem and the result is established.

In the last two theorems, while taking the geometric sum of random variables, the parameter was confined to a fixed value in the interval (0,1). We presently examine the possibility of relaxing this assumption by permitting p to be the value of a random variable in (0,1) when the X_k's follow distribution (3.1).

Theorem 3.10.

Let the sequence (X_k) and the random variable N be as in Theorem 3.8. If p is a random variable with distribution function G(p) in (0,1), then the random variables X_{jk} and p S_{jN} have the same distribution if (X_k) follow bivariate exponential distribution (3.1).

Proof:

Assuming X_k has bivariate exponential distribution, in the notations used in theorem 3.8,

$$B(s,t_2) = \int_0^1 p\, A(ps,t_2)\, [1-(1-p)\, A(ps,t_2)]^{-1}\, dG(p)$$

$$= \int_0^1 \frac{p[1-ips(\alpha_1+ \theta t_2)]^{-1}}{1-(1-p)[1-ips(\alpha_1+ \theta t_2)]^{-1}}\, dG(p)$$

$$= \int_0^1 \frac{1}{1-is(\alpha_1+ \theta t_2)}\, dG(p)$$

$$= A(s,t_2)$$

3.5 Characterization by form of the conditional distributions

It is well known that a bivariate distribution is not uniquely determined by its marginal distributions. The best illustration of this fact is provided by the bivariate exponential models reviewed in the previous chapter. However, the form of marginal distributions can be taken as the basis of constructing bivariate versions as seen in the works of Morgenstern (1956) and Farlie(1960). However if we turn attention from marginals to the conditional distributions, there is possibility of uniquely determining the bivariate model with specified conditional densities. Abrahams and Thomas (1984) has shown that the conditional densities $f_1(x|y)$ and $f_2(y|x)$ determines uniquely a bivariate density $f(x,y)$ if and only if

$$\frac{f_1(x|y)}{f_2(x|y)} = \frac{g(x)}{h(y)} \tag{3.101}$$

where g(.) and h(.) are non-negative integrable functions with equal marginals. In the case of distribution (3.1) we see that

$$\frac{f_1(x_1|x_2)}{f_2(x_2|x_1)} = \frac{[(\alpha_1+\theta x_2)(\alpha_2+\theta x_1)-\theta]\exp[-(\alpha_1+\theta x_2)x_1]}{[(\alpha_1+\theta x_2)(\alpha_2+\theta x_1)-\theta]\exp[-(\alpha_2+\theta x_1)x_2]}$$

$$= \frac{\exp(-\alpha_1 x_1)}{\exp(-\alpha_2 x_2)} \qquad (3.102)$$

with the terms on the right side satisfy the required conditions. Thus the two conditional densities confirm to the model (3.1). However, we note that the form of the conditionals are neither exponential nor reducible to any well known standard model to be of any practical interest. On the other hand if we consider the conditional densities $f(x_i|X_j>x_j)$ presented in equation (3.9) which are exponential, a characterization in terms of them could be more useful. In this section we present a general result on a necessary and sufficient condition that enable the determination of the joint density $f(x_1,x_2)$ in terms of the conditional densities $f(x_i|X_j>x_j)$ and then use it to characterize (3.1).

Theorem 3.11

Let $X = (X_1,X_2)$ be a random vector possessing absolutely continuous distribution with respect to Lebesgue measure in the support of $Q=\{(x_1,x_2)|x_i>0, i=1,2\}$

$t = (t_1, t_2)$ a vector of non-negative reals and

$$R_i(t_i | t_j) = P[X_i > t_i | X_j > t_j] \tag{3.103}$$

$i, j = 1,2$; $i \neq j$. The density function of X is uniquely determined by the survival functions $R_1(t_1|t_2)$ and $R_2(t_2|t_1)$ at those points for which these functions are non-zero if and only if

$$\frac{R_1(t_1|t_2)}{R_2(t_2|t_1)} = \frac{g(t_1)}{h(t_2)} \tag{3.104}$$

where g(.) and h(.) are non-negative real functions with continuous derivatives in the subspace $Q_1 = \{x | x > 0\}$ satisfying

$$g(o+) = h(o+)$$

Proof:

To prove the sufficiency we note that under the conditions of the theorem, there exist functions u and v in Q_1 such that

$$g(t_1) = \int_{t_1}^{\infty} u(y)dy$$

and

$$h(t_2) = \int_{t_2}^{\infty} v(y)dy$$

Equation (3.104) is equivalent to

$$\frac{R_1(t_1|t_2)}{R_2(t_2|t_1)} = \frac{\int_{t_1}^{\infty} u(y)dy}{\int_{t_2}^{\infty} v(y)dy} \tag{3.105}$$

Since g(o+) = h(o+), (3.105) can be written as

$$\frac{R_1(t_1|t_2)}{R_2(t_2|t_1)} = \frac{\int_{t_1}^{\infty} u(y)dy \;/\; \int_{o}^{\infty} u(y)dy}{\int_{t_2}^{\infty} v(y)dy \;/\; \int_{o}^{\infty} v(y)dy} \tag{3.106}$$

If we write

$$S(t_1) = \frac{\int_{t_1}^{\infty} u(y)dy}{\int_{o}^{\infty} u(y)dy}$$

We see that S is non-increasing, $S(+\infty) = 0$ and $S(o) = 1$. Further

$$S(t_1+h) - S(t_1) = \int_{t_1}^{t_1+h} u(y)dy \;/\; \int_{o}^{\infty} u(y)dy$$

so that $S(t_1+o) = S(t_1)$, proving the right continuity of S. Thus S is the survival function $R_1(t_1)$ of the random variable X_1 and similarly the denominator on the right hand side of (3.106) is the survival function

$R_2(t_2)$ of X_2. Thus the survival function of X is uniquely obtained as

$$R(t_1,t_2) = R_1(t_1|t_2)\, R_2(t_2)$$

and hence the corresponding density $f(x_1,x_2)$.

The necessary part is obtained by writing

$$R(t_1|t_2) = \frac{R(t_1,t_2)}{R_2(t_2)}$$

$$R(t_2|t_1) = \frac{R(t_1,t_2)}{R_1(t_1)}$$

taking their ratios and cancelling out common terms, if any, to $R_1(t_1)$ and $R_2(t_2)$ to arrive at g(.) and h(.)

Corollary.

X follows the bivariate exponential distribution (3.1) if and only if the conditional distributions of X_i given $X_j > t_j$ are exponential.

Proof:

We see from equation (3.8) that

$$\frac{R(t_1|t_2)}{R(t_2|t_1)} = \frac{\exp[-(\alpha_1 + \theta t_2)t_1]}{\exp[-(\alpha_2 + \theta t_1)t_2]}$$

$$= \frac{\exp(-\alpha_1 t_1)}{\exp(-\alpha_2 t_2)}$$

$$= \frac{g(t_1)}{h(t_2)}$$

where g and h satisfy the conditions of the Theorem.

Observations.

1. Theorem 3.11 is quite general in character and therefore applies to any bivariate distribution. It can be used to characterize other bivariate distributions as Pareto [Mardia, 1962], Lomax [Lindley and Singpurwalla, 1986] and Burr [Durling 1975]. Since these results do not come under the scope of the present thesis, they are not discussed here.

2. Unlike other characterizations presented in Sections 3.3 and 3.4 which are extensions of the corresponding univariate property in some sense, the results in this section apply solely to bivariate distributions.

Chapter 4

BIVARIATE GEOMETRIC DISTRIBUTION

The role of the geometric distribution as the discrete counterpart of the exponential distribution in varied fields of theoretical and applied statistics is well known. Accordingly several characterization theorems of the univariate exponential distribution have been extended to the discrete sample space to prove similar results for the geometric distribution. Even though several forms of bivariate exponential distributions are discussed in literature, there has not been a matching effort to evolve bivariate geometric forms corresponding to them. One of the main reasons for the lack of interest in evolving multivariate geometric distributions is the fact that continuous models are of general appeal in reliability studies, the major field of application of the exponential model and also the connection of this model with the Poisson process. However, in recent times a lot of interest is generated in the application of reliability concepts in discrete time domain. Xekalaki (1983) advocates the use of discrete models in life length studies on the ground that inadequacies of measuring devices often warrant measurement of survival times in discrete time while Gupta (1985) cites example of discrete random variable that occur naturally such as the case with the time to failure in fatigue studies measured in terms of the number of cycles to failure. Further when data on life times are in grouped forms discrete models becomes very much handy in their analysis. These considerations have opened up investigations towards characterizing discrete models via reliability concepts. The importance of the geometric distribution in such contexts as a basic model cannot be over emphasised and therefore there is enough scope for looking into possible generalisations of that law to higher dimensions.

In the present chapter our endeavour will be to generalise the one dimensional geometric distribution by considering a proper extension of the lack of memory property enjoyed by that model to the bivariate case so that the bivariate geometric distribution so derived becomes the discrete analogue of the Gumbel's bivariate exponential distribution discussed in the previous chapter. Further, the characterizations in the continuous sample space, established, are translated into appropriate results in the discrete domain.

4.1. Extension of the lack of memory property*

The well known lack of memory property that characterizes the geometric distribution with probability mass function

$$f(y) = p^{y}(1-p);\ 0 < p < 1 \tag{4.1}$$

is that for a random variable Y with support $I^{+} = \{0,1,2\ldots\}$ the relationship

$$P\,[\,Y \geq t+s \mid Y \geq t\,] = P\,[Y \geq s] \tag{4.2}$$

holds good for all non-negative integer values of t and s.

An equivalent form of (4.3) in terms of the expected values in [Nair (1983)]

$$E[\,Y-t \mid Y \geq t\,] = E(Y) \tag{4.3}$$

In order to generalise (4.3) to the bivariate case we consider a random vector $X = (X_1,X_2)$ in the support of $I_2^{+} = \{(x_1,x_2);\ x_1,x_2 = 0,1,2\ \ldots\}$ having distribution function $F(x_1,x_2)$ and joint probability mass function

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* The results in Sections 4.1 and 4.2 have appeared in the Journal of the Indian Statistical Association (1988) Vol. 26, p. 45-49. (Reference 37)

$f(x_1,x_2)$. We define a vector valued function

$$\underline{r}(\underline{t}) = E\,[\,X-t \mid X \geq t\,]$$

with components

$$r_i(t_1,t_2) = E\,[X_i-t_i \mid X \geq t] \qquad (4.4)$$

where $\underline{t} = (t_1,t_2)$ is a vector of non-negative integers, $X \geq t$ stands for $X_i \geq t_i$ and $i = 1,2$. Notice that

$$r_i(o,o) = E(X_i)$$

Writing

$$R(t_1,t_2) = P[X_1 \geq t_1,\ X_2 \geq t_2]$$

we find

$$r_i(t_1,t_2)\,R(t_1,t_2) = \sum_{t_1}^{\infty} \sum_{t_2}^{\infty} (x_i-t_i)\,f(x_1,x_2) \qquad (4.5)$$

$$= \sum_{t_1}^{\infty} (x_i-t_i)\,P[X_1=x_1, X_2 \geq t_2]$$

$$= \sum_{s=1}^{\infty} s\,P[X_1=t_1+s,\ X_2 \geq t_2]$$

$$= \sum_{s=1}^{\infty} R(t_1+s,\ t_2) \qquad (4.6)$$

Changing t_1 to (t_1+1) in equation (4.6) and substracting the resulting equation from (4.6) we arrive at the recurrence formula

$$R(t_1,t_2)r_1(t_1,t_2)-R(t_1+1,t_2)[1+r_1(t_1+1,t_2)] = 0 \quad (4.7)$$

A similar consideration of $r_2(t_1,t_2)$ leads to

$$R(t_1,t_2)\ r_2(t_1,t_2)-R(t_1,t_2+1)[1+r_2(t_1,t_2+1)]=0 \quad (4.8)$$

A straight forward generalisation of (4.3) to the bivariate case provides us with the definition

$$\begin{aligned} \underline{r}(\underline{t}) &= (r_1(o,o),\ r_2(o,o)) \\ &= (r_1,r_2) \end{aligned} \quad (4.9)$$

with both r_1 and r_2 independent of t_1 and t_2 for every $t_1,t_2 \geq 0$. If we adopt (4.9) as our definition of bivariate lack of memory, the implication is as follows:

Theorem 4.1.

The condition (4.9) is satisfied by a random vector with support I_2^+ if and only if X_1 and X_2 are independent and geometrically distributed.

Proof:

When (4.9) is true, equation (4.7) can be written as

$$R(t_1+1,t_2) = \frac{r_1}{1+ r_1} R(t_1,t_2)$$

Iteration for decreasing values of t_1 provides

$$R(t_1,t_2) = \left(\frac{r_1}{1 + r_1} \right)^{t_1} R(o,t_2) \qquad (4.10)$$

Putting $t_2 = 0$ in (4.10)

$$R(t_1,o) = \left(\frac{r_1}{1+r_1} \right)^{t_1} \qquad (4.11)$$

Similarly using (4.8)

$$R(o,t_2) = \left(\frac{r_2}{1+r_2} \right)^{t_2} \qquad (4.12)$$

It follows from equations (4.10), (4.11) and (4.12) that

$$R(t_1,t_2) = \left(\frac{r_1}{1+r_1} \right)^{t_1} \left(\frac{r_2}{1+r_2} \right)^{t_2}$$

where

$$r_i = E(X_i)$$

which is equivalent to

$$R(t_1,t_2) \;=\; R(t_1,o)\,R(o,t_2)$$

and

$$P[X_1 \geq t_1,\; X_2 \geq t_2] = P[X_1 \geq t_1]\; P[X_2 \geq t_2] \qquad (4.13)$$

for every (t_1,t_2) in I_2^+. Hence X_1 and X_2 are independent and further X_i are geometrically distributed with mean r_i.

Conversely when X_1 and X_2 are independent geometric variables with parameters p_1 and p_2 respectively direct calculations concurrs with equation (4.9) with $r_i = p_i(1-p_i)^{-1}$, i=1,2 which does not involve t_1 and t_2.

An immediate consequence of Theorem 4.1 is that our search for the generalisation of (4.2) through the definition given in (4.9) ends up with the constraint that the variables must be independent for it to be operative. Needless to say that this is highly restrictive and does not contribute to models in which the component life times have joint variation. Accordingly, in our attempt to have a meaningful extension of lack of property, analogue to (3.48), we introduce the concept of local lack of memory property in the discrete domain by defining it as

$$E[X_i - t_i \mid X \geq t] = E[X_i \mid X_j \geq t_j] \qquad (4.14)$$
$$i,j = 1,2;\; i \neq j$$

consistent with our objective spelt out at the beginning of this chapter. The bivariate geometric distribution we seek is that one which is characterized by the property (4.14). The two equations given in (4.14) can be restated as

$$r_i(t_1,t_2) = a_i(t_j);\; i,j = 1,2,\; i \neq j \qquad (4.15)$$

where $a_i(t_j)$ are functions independent of t_i.

4.2. Bivariate Geometric Distribution

The proposed characterization based on property (4.15) which leads to a bivariate geometric distribution is given in the following theorem.

Theorem 4.2.

For all non-negative integers t_1, t_2, the conditions (4.15) are satisfied by a random vector $X = (X_1, X_2)$ with support I_2^+, where $a_1(t_2)$ and $a_2(t_1)$ are non-increasing in the respective variables with

$$a_i(o) = p_i(1-p_i)^{-1},\; i=1,2. \qquad (4.16)$$

if and only if X has a bivariate geometric distribution specified by the survival function

$$R(x_1,x_2) = p_1^{x_1} p_2^{x_2} \theta^{x_1,x_2}; \quad x_1,x_2 = 0,1,2\ldots \tag{4.17}$$

$0 \le p_1,\ p_2 \le 1;\ 0 \le \theta \le 1;\ 1 - \theta \le (1-p_1\theta)(1-p_2\theta)$

Proof:

The first part of the theorem follows by computing the required expectations in (4.15). When the survival function of X is as in (4.17), the distribution function of X is

$$F(x_1,x_2) = 1-p_1^{x_1+1} - p_2^{x_2+1} + p_1^{x_1+1} p_2^{x_2+1} \theta^{(x_1+1)(x_2+1)} \tag{4.18}$$

with joint probability mass function

$$f(x_1,x_2) = p_1^{x_1} p_2^{x_2} \theta^{x_1 x_2 - 1} [(1-p_1\theta^{x_2+1})(1-p_2\theta^{x_1+1}) + \theta - 1] \tag{4.19}$$

From (4.5) with i=1 and $f(x_1,x_2)$ as in (4.19)

$$r_1(t_1,t_2)\ R(t_1,t_2) = \sum_{t_1}^{\infty} \sum_{t_2}^{\infty} (x_1-t_1)\ f(x_1,x_2)$$

$$= \sum_{t_1}^{\infty} (x_1-t_1)\ p_1^{x_1}[(1-p_2\theta^{x_1}) \sum_{t_2}^{\infty} (p_2\theta^{x_1})^{x_2}$$

$$- p_1(1-p_2\theta^{x_1+1}) \sum_{t_2}^{\infty} (p_2\theta^{x_1+1})^{x_2}]$$

$$= p_2^{t_2}(1-p_1\theta^{t_2}) \sum_{t_1}^{\infty} (x_1-t_1)\ (p_1\theta^{t_2})^{x_1}$$

$$= p_1^{t_1}\ p_2^{t_2}\ \theta^{t_1 t_2}\ p_1\theta^{t_2}(1-p_1\theta^{t_2})^{-1}$$

Substituting for $R(t_1,t_2)$ from (4.17)

$$r_1(t_1,t_2) = \frac{p_1\theta^{t_2}}{1-p_1\theta^{t_2}} \tag{4.20}$$

Proceeding on similar lines one can also show that

$$r_2(t_1,t_2) = \frac{p_2\theta^{t_1}}{1-p_2\theta^{t_1}}$$

It is easy to verify that (4.15) holds with the functions $a_i(t_{3-i})$ satisfying the conditions specified in the theorem.

Conversely, let (4.15) hold. From (4.7),

$$R(t_1,t_2)\ a_1(t_2) - R(t_1+1,t_2)[1+a_1(t_2)] = 0$$

or

$$R(t_1+1,t_2) = \frac{a_1(t_2)}{1+a_1(t_2)} R(t_1,t_2)$$

Changing t_1 to (t_1-1)

$$R(t_1,t_2) = \frac{a_1(t_2)}{1+a_1(t_2)} R(t_1-1,t_2)$$

Iteration for decreasing values of t_1 gives

$$R(t_1,t_2) = \left(\frac{a_1(t_2)}{1+a_1(t_2)}\right)^{t_1} R(o,t_2) \tag{4.21}$$

(4.20) with $t_2 = 0$ yields

$$R(t_1,o) = p_1^{\,t_1}$$

where p_1 is as defined in (4.16). A similar consideration of (4.8) gives

$$R(o,t_2) = p_2^{\,t_2}$$

Substituting for $R(o,t_2)$ in (4.21) we get

$$R(t_1,t_2) = \left(\frac{a_1(t_2)}{1+a_1(t_2)}\right)^{t_1} p_2^{\,t_2} \tag{4.22}$$

Similarly

$$R(t_1,t_2) = \left(\frac{a_2(t_1)}{1+a_2(t_1)} \right)^{t_2} p_1^{\,t_1} \tag{4.23}$$

Equating (4.22) and (4.23) we have the functional equation,

$$\left(\frac{a_1(t_2)}{p_1[1+a_1(t_2)]} \right)^{1/t_2} = \left(\frac{a_2(t_1)}{p_2[1+a_2(t_1)]} \right)^{1/t_1} \tag{4.24}$$

which is true for all t_1 and t_2. Clearly

$$\left(\frac{a_i(x)}{p_i[1+a_i(x)\ ]} \right)^{\frac{1}{x}}$$

is a constant, say θ, for all integral values of x. Thus

$$\frac{a_i(x)}{1+\ a_i(x)} = p_i\ \theta^x$$

or

$$a_i(x) = \frac{p_i\theta^x}{1-p_i\ \theta^x} \tag{4.25}$$

From (4.22) we get

$$R(t_1,t_2) = p_1^{\,t_1}\ p_2^{\,t_2}\ \theta^{t_1 t_2}$$

as required. The conditions on the parameters can be obtained as follows. From the monotonicity of $a_1(t_2)$

$$\frac{p_1\theta}{1-p_1\theta} \leq \frac{p_1}{1- p_1}$$

or

$$\theta \leq 1$$

For $R(x_1,x_2)$ to be non-negative, $\theta \geq 0$. Also from (4.19)

$$f(o,o) = \theta^{-1}[(1-p_1\theta)(1-p_2\theta) + \theta -1]$$

which is non-negative if and only if

$$1 - \theta \leq (1-p_1\theta)(1-p_2\theta)$$

The proof of the theorem is thus completed.

It is informative to investigate what would be the equivalent of (4.15) in terms of conditional probabilities as the lack of memory is popularly expressed in that form. This is established in the next theorem.

Theorem 4.3.

For all non-negative integers t_1, t_2 and s the conditional probability statements

$$P[X_i \geq t_i + s | X \geq t] = P[X_i \geq s | X_j \geq t_j] \qquad (4.26)$$

$$i,j=1,2;\ i \neq j$$

hold good if and only if X has the bivariate geometric distribution (4.17) with support I_2^+.

Proof:

When the survival function of X is as in (4.17)

$$\frac{R(t_1+s, t_2)}{R(t_1, t_2)} = \frac{R(s, t_2)}{R(o, t_2)} = p_1{}^s \, \theta^{st_2} \qquad (4.27)$$

which is independent of t_1. A similar expression independent of t_2 can be obtained for $R(t_1, t_2+s)/R(t_1, t_2)$. Thus condition (4.26) which reads

$$R(t_i+s, t_j)\ R_j(t_j) = R_i(s)\ R(t_1, t_2),\ i=1,2$$

with $R(t_i+s, t_j)$ as in (4.27) and $R_i(s) = p_i{}^s$ is satisfied.

Conversely when (4.26) is true, for i=1

$$R(t_1+s,t_2)\ R(o,t_2) = R(t_1,t_2)\ R(s,t_2) \tag{4.28}$$

Summation over s from 1 to ∞ gives

$$R(o,t_2) \sum_{s=1}^{\infty} R(t_1+s,t_2) = R(t_1,t_2) \sum_{s=1}^{\infty} R(s,t_2)$$

Using (4.26), the equation can be written as

$$R(o,t_2)\ R(t_1,t_2)\ r_1(t_1,t_2) = R(t_1,t_2)R(o,t_2)r_1(o,t_2)$$

or

$$r_1(t_1,t_2) = r_1(o,t_2)$$

Taking i=2 in (4.26) one can also show that

$$r_2(t_1,t_2) = r_2(t_1,o)$$

The necessary part now follows from theorem 4.2.

4.3 Properties of the distribution

Having obtained the required bivariate model, in the present section, we investigate some of its basic properties. The motivation in this regard is two fold; one is the importance of these results in their own right and the second is their possible role in providing further characterizations.

4.3.1. Marginal and conditional distributions

The marginal distributions corresponding to the probability mass function (4.19) are

$$f_i(x_i) = (1-p_i)p_i^{x_i}; \; x_i=0,1,2,\ldots; i = 1,2. \quad 0 < p_i < 1. \tag{4.29}$$

which are of standard geometric form. It is in this sense that (4.19) was referred to in the earlier discussions as a bivariate geometric distribution.

The conditional distribution of X_i given $X_j=t_j$ is

$$f(X_i|X_j=t_j)=(1-p_j)^{-1} p_i^{x_i}\theta^{x_i t_j-1} [(1-p_j\theta^{x_i+1})(1-p_i\theta^{t_j+1}) + \theta - 1] \tag{4.30}$$

with means

$$E[X_1|X_2=t_2] = \alpha_2 r_1(t_1,t_2) + (1-\alpha_2)r_1(t_1,t_2+1)$$

and

$$E(X_2|X_1=t_1) = \alpha_1 r_2(t_1,t_2) + (1-\alpha_1)\ r_2(t_1+1,t_2) \tag{4.31}$$

where $\alpha_j = 1 + E(X_j)$ and $r_j(t_1,t_2)$; j=1,2 is as defined earlier.

Further,

$$E(X_1\ X_2) = \sum_{x_2=1}^{\infty} \frac{p_1(p_2\theta)^{x_2}}{1-p_1\theta^{x_2}} \tag{4.32}$$

The coefficient of correlation between X_1 and X_2 is

$$r_{X_1,X_2} = \frac{A(\theta;p_1,p_2) - \dfrac{p_1}{1-p_1}\dfrac{p_2}{1-p_2}}{\left[\dfrac{p_1}{(1-p_1)^2}\quad\dfrac{p_2}{(1-p_2)^2}\right]^{1/2}}$$

$$= [(1-p_1)(1-p_2)\ A(\theta;p_1,p_2)-p_1p_2](p_1p_2)^{-\frac{1}{2}}$$

where $A(\theta;p_1,p_2)$ is the expression in (4.32). For fixed p_1 and p_2, r is a non-decreasing function of θ as evidenced from the sign of $\frac{dr}{d\theta}$. This shows that the coefficient of correlation of the distribution ranges from 0 to $-\sqrt{p_1p_2}$, the value zero, that corresponds to $\theta=1$, being attained when and only when X_1 and X_2 are independent univariate geometric. Thus the correlation always remains negative which is to some extent a handicap when the distribution is to be thought of as a model of life lengths in a two component system. The physical constraint imposed by the correlation structure is that the increased life length of one component should necessitate the inevitable decrease in the life time of the other. Thus the bivariate geometric distribution can model only life length data that exhibit this mode of behaviour.

The random variables X_1 and X_2 are independent when $\Theta = 1$.

The conditional survival function of X_i given $X_j \geq t_j$ is

$$P[X_i \geq x_i \mid X_j \geq t_j] = (p_i \, \Theta^{t_j})^{x_i}, \quad i=1,2 \tag{4.33}$$

with corresponding probability mass function

$$f(X_i \mid X_j \geq t_j) = (p_i \Theta^{t_j})^{x_i} (1-p_i \Theta^{t_j})$$

$$x_i = 0,1,2, \ldots$$

which is again geometric with parameter $(p_i \, \Theta^{t_j})$.

It may be noticed from (4.29) and (4.33) that the marginal and conditional distributions of the bivariate geometric distribution are geometric, analogous to corresponding results in the Gumbel's bivariate exponential distribution.

4.3.2 Moments

The expression for the $(r,s)^{th}$ order row moment can be written as

$$\mu'_{rs} = \sum_{x_1} x_1^r \left[A(x_1,s) - A(x_1+1,s) \right]$$

where

$$A(s_1,s) = p_1^{x_1}(1-p_2\theta^{x_1}) \sum_{x_2} x_2^s \, (p_2\theta^{x_1})^{x_2} \tag{4.34}$$

The probability generating function of the distribution is

$$E(s_1^{x_1} s_2^{x_2}) = \sum_{x_1} s_1^{x_1}[B(x_1,s_2)-B(x_1+1,s_2)] \quad (4.35]$$

where

$$B(x_1,s_2) = p_1^{x_1}(1-p_2\theta^{x_1})(1-s_2p_2\,\theta^{x_1})^{-1}$$

The r^{th} order truncated factorial moment of X_i is defined as

$$\phi_i^{(r)}(t_1,t_2) = E\,[(X_i-t_i)^{(r)} \mid X_1 \geq t_1, X_2 \geq t_2],$$
$$i = 1,2. \qquad (4.36)$$

where

$$(X_i-t_i)^{(r)} = (X_i-t_i)(X_i-t_i-1)\ \ldots\ (X_i-t_i-r+1)$$

Equation (4.36) with i=1 can be written as

$$\phi_1^{(r)}(t_1,t_2)R(t_1,t_2) = \sum_{t_1}^{\infty}\sum_{t_2}^{\infty}(x_1-t_1)^{(r)}\ f(x_1,x_2)$$

$$= \sum_{t_1}^{\infty}(x_1-t_1)^{(r)}\ \sum_{t_2}^{\infty}\ P[X_1=x_1,X_2=x_2]$$

$$= \sum_{t_1}^{\infty}(x_1-t_1)^{(r)}\ P[X_1=x_1,X_2 \geq t_2]$$

$$= \sum_{t_1}^{\infty}(x_1-t_1)^{(r)}\ [R(x_1,t_2)-R(x_1+1,t_2)]$$

$$= \sum_{t_1+1}^{\infty}[(x_1-t_1)^{(r)}-(x_1-t_1-1)^{(r)}]R(x_1,t_2)$$

$$= r\sum_{t_1+1}^{\infty}(x_1-t_1-1)^{(r-1)}\ R(x_1,t_2) \qquad (4.37)$$

Changing t_1 to (t_1+1) in (4.37) we get

$$\phi_1^{(r)}(t_1+1,t_2)\ R(t_1+1,t_2)= r \sum_{t_1+2}^{\infty}(x_1-t_1-2)^{(r-1)}R(x_1,t_2) \tag{4.38}$$

Subtracting (4.38) from (4.37) we arrive at the equation

$$\phi_1^{(r)}(t_1,t_2)\ R(t_1,t_2) - \phi_1^{(r)}(t_1+1,t_2)\ R(t_1+1,t_2)$$

$$= \sum_{t_1+1}^{\infty} \sum_{t_2}^{\infty} (x_1-t_1-1)^{(r-1)}\ f(x_1,x_2)$$

$$= R(t_1+1,t_2)\ \phi_1^{(r-1)}(t_1,t_2)$$

or

$$\frac{R(t_1+1,t_2)}{R(t_1,t_2)} = \frac{\phi_1^{(r)}\ (t_1,t_2)}{r\phi_1^{(r-1)}(t_1+1,t_2) + \phi_1^{(r)}(t_1+1,t_2)} \tag{4.39}$$

When the survival function of X is specified by (4.17)

$$\frac{R(t_1+1,t_2)}{R(t_1,t_2\ )} = p_1\ \theta^{t_2}$$

so that $\phi_1^{(r)}(t_1,t_2)$ is given by the relation

$$\frac{\phi_1^{(r)}\ (t_1,t_2)}{r\ \phi_1^{(r-1)}(t_1+1,t_2)+ \phi_1^{(r)}(t_1+1,t_2)} = p_1\ \theta^{t_2} \tag{4.40}$$

It may be observed from (4.40) that $\phi_1^{(r)}(t_1,t_2)$ is independent of t_1.

Setting r = 0 in (4.40)

$$\frac{\phi_1^{(1)}}{1 + \phi_1^{(1)}} = p_1\,\theta^{t_2}$$

or

$$\phi_1^{(1)} = \frac{p_1\,\theta^{t_2}}{1-p_1\theta^{t_2}}$$

When r=2,

$$\frac{\phi_1^{(2)}}{2\phi_1^{(1)} + \phi_1^{(2)}} = p_1\,\theta^{t_2}$$

or

$$\phi_1^{(2)} = \frac{2(p_1\theta^{t_2})^2}{(1-p_1\theta^{t_2})^2}$$

Proceeding like this

$$\phi_1^{(r)}(t_1,t_2) = r!\left(\frac{p_1\theta^{t_2}}{1-p_1\theta^{t_2}}\right)^r \tag{4.41}$$

Similarly

$$\phi_2^{(r)}(t_1,t_2) = r!\left(\frac{p_2\,\theta^{t_1}}{1-p_2\theta^{t_1}}\right)^r \tag{4.42}$$

4.3.3. Limiting form

A limiting form of our bivariate geometric distribution can be obtained by setting $p_i=1-\alpha_i n_i$; $i=1,2$; $\theta = 1-\alpha_n$ where $n = n_1 n_2$ and considering $R(\frac{x_1}{n_1}, \frac{x_2}{n_2})$ as n_1 and n_2 tend to zero. The expression for the survival function under the above mentioned transformations is from (4.17)

$$R(\frac{x_1}{n_1}, \frac{x_2}{n_2}) = (1-\alpha_1 n_1)^{x_1/n_1}(1-\alpha_2 n_2)^{x_2/n_2}(1-\alpha_n)^{x_1 x_2/n}$$

Taking limits as n_1 and n_2 tends to zero we get

$$R(x_1,x_2) = \exp\,[-\alpha_1 x_1-\alpha_2 x_2-\alpha x_1 x_2]$$

$$x_1,x_2 > 0;\ \alpha_1,\alpha_2 > 0;\ \alpha \geq 0$$

which is the survival function of the Gumbel's bivariate exponential distribution discussed in the previous chapter.

4.4 Characterizations based on properties of truncated moments

In this section we discuss characterization problems associated with the bivariate geometric model based on properties of truncated moments.

Theorem 4.4.

Let $X = (X_1, X_2)$ be a discrete random vector in the support of $I_2 = \left\{(x_1,x_2) \mid x_1,x_2 = 0,1,2,\ldots\right\}$ with $E(X_i^k) < +\infty$. X follows the bivariate geometric distribution (4.17) if and only if for all positive integers k

$$E[(X_i-t_i)^{(k)} \mid X_1 \geq t_1, X_2 \geq t_2] = a_i^k(t_j), \quad i,j = 1,2, \; i \neq j \tag{4.43}$$

where

$$a_i^{(k)}(t_j) = E(X_i^{(k)} \mid X_j \geq t_j)$$

are non-increasing in t_j and are independent of t_i.

Proof:

When X follow the bivariate geometric distribution from equation (4.41) we get

$$E[(X_i-t_i)^{(k)} \mid X \geq t] = k!\left(\frac{p_i\theta^{t_j}}{1-p_i\theta^{t_j}}\right)^k$$

Further differentiating the identity

$$\sum_{x_i} (p_i\theta^{t_j})^{x_i} = (1-p_i\theta^{t_j})^{-1}$$

k times with respect to $(p_i\theta^{t_j})$ and then multiplying both sides by $(1-p_i\theta^{t_j})(p_i\theta^{t_j})^k$, we find

$$E\,[\,X_i^{(k)} | X_j \geq t_j] = k!\,\left(\frac{p_i\,\theta^{t_j}}{1-p_i\,\theta^{t_j}}\right)^k$$

$$= a_i^{(k)}\,(t_j)$$

that satisfy the conditions of the theorem. The only if part now follows. For proving the if part, we see that when (4.43) holds, it can be written using (4.39) as

$$\frac{R(t_1+1,t_2)}{R(t_1,\,t_2\,)} = \frac{a_1^{(k)}(t_2)}{k\,a_1^{(k-1)}(t_2)+\,a_1^{(k)}(t_2)}$$

$$= B_1(t_2),\ \text{say}$$

or

$$R(t_1+1,t_2) = B_1(t_2)\,R(t_1,t_2)$$

Iteration for decreasing values of t_1 gives

$$R(t_1,t_2) = [\,B_1(t_2)\,]^{t_1}\,R(o,t_2) \qquad (4.44)$$

Setting $t_2 = 0$ in (4.44)

$$R(t_1,o) = p_1^{t_1} \tag{4.45}$$

with

$$p_1 = B_1(o)$$

Similarly

$$R(o,t_2) = p_2^{t_2} \tag{4.46}$$

Substituting for $R(o,t_2)$ from (4.46) in (4.44),

$$R(t_1,t_2) = [B_1(t_2)]^{t_1} p_2^{t_2} \tag{4.47}$$

Proceeding on similar lines one can also show that

$$R(t_1,t_2) = [B_2(t_1)]^{t_2} p_1^{t_1} \tag{4.48}$$

From (4.47) and (4.48), the usual arguments for such functional equations give

$$B_i(x) = p_i\, \theta^x \tag{4.49}$$

where θ is a constant. It follows from (4.47) or (4.48) that

$$R(t_1,t_2) = p_1^{t_1} p_2^{t_2} \theta^{t_1 t_2}$$

and this completes the proof.

Corollary-1

Using the relation between row moments and the ascending factorial moments it follows that the condition

$$E[(X_i-t_i)^k|X_1\geq t_1,\ X_2\geq t_2] = E[X_i^k|X_i\geq t_1, X_2\geq t_2] \quad (4.50)$$

also characterizes the bivariate geometric distribution analogous to theorem 3.1 for the Gumbel distribution.

Corollary-2

Taking k=1 in (4.50) we get the condition given in theorem 4.2.

Corollary-3

Setting $t_2 = 0$ in (4.43) it follows that the condition

$$E[(X_1-t_1)^{(k)}|X_1 > t_1] = c \quad \text{for all non-negative integers } t$$

where $c = E(X_1^{(k)})$ characterizes the univariate geometric distribution

$$P\ [X_1=x] = p_1^x(1-p_1)$$

In particular for k=1,

$$E[X_1 - t_1 \mid X_1 \geq t_1] = E(X_1)$$

which is the well known property of the geometric distribution.

We now prove another related characterization of the bivariate geometric distribution based on the variance of the truncated variables $X_i - t_i | X_1 \geq t_1,\ X_2 \geq t_2$. This result is intitutively clear from theorem 4.4; but we give an independent proof.

Theorem 4.5

The random vector $X = (X_1, X_2)$ of theorem 4.4 with $E(X_i^2) < +\infty$ follow the bivariate geometric distribution (4.17) if and only if

$$V(X_i | X \geq t) = V_i(t_j);\ i,j = 1,2,\ i \neq j \qquad (4.51)$$

where $V_i(t_j)$ are non-increasing with

$$V_i(o) = p_i(1-p_i)^{-2} \qquad (4.52)$$

Proof:

We have

$$V\ (X_1 \mid X \geq t) = V(X_1 - t_1 \mid X \geq t)$$

$$= \frac{\sum_{t_1}^{\infty}\sum_{t_2}^{\infty} (x_1-t_1)^2 f(x_1,x_2)}{\sum_{t_1}^{\infty}\sum_{t_2}^{\infty} f(x_1,x_2)} - \left[\frac{\sum_{t_1}^{\infty}\sum_{t_2}^{\infty} (x_1-t_1) f(x_1,x_2)}{\sum_{t_1}^{\infty}\sum_{t_2}^{\infty} f(x_1,x_2)}\right]^2$$

$$= A_1(t_1,t_2) + [r_1(t_1,t_2)]^2 \qquad (4.53)$$

where $r_1(t_1,t_2)$ is as defined in (4.4) and $A_1(t_1,t_2)$ is such that

$$A_1(t_1,t_2)\ R(t_1,t_2) = \sum_{t_1}^{\infty}\sum_{t_2}^{\infty}(x_1-t_1)^2\ f(x_1,x_2) \qquad (4.54)$$

When X follow the bivariate geometric distributions, using (4.19)

$$A_1(t_1,t_2)\ R(t_1,t_2) = \sum_{t_1}^{\infty} (x_1-t_1)^2\ p_1^{x_1}$$

$$\left\{(1-p_2\theta^{x_1}) \sum_{t_2}^{\infty} (p_2\theta^{x_1})^{x_2}\right.$$

$$\left.-p_1(1-p_2\theta^{x_1+1}) \sum_{t_2}^{\infty} (p_2\theta^{x_1})^{x_2+1}\right\}$$

$$= p_2^{t_2}(1-p_1\theta^{t_2}) \sum_{t_1}^{\infty} (x_1-t_1)^2\ (p_1\theta^{t_2})^{x_1}$$

$$= p_1^{t_1} p_2^{t_2} \theta^{t_1 t_2} \sum_{y=1}^{\infty} [y^2-(y-1)^2] (p_1\theta^{t_2})^y$$

$$= p_1^{t_1} p_2^{t_2} \theta^{t_1 t_2} \sum_{y=1}^{\infty} (2y-1)(p_1\theta^{t_2})^y$$

$$= p_1^{t_1} p_2^{t_2} \theta^{t_1 t_2} \left[\frac{2(p_1\theta^{t_2})}{(1-p_1\theta^{t_2})^2} - \frac{p_1\theta^{t_2}}{(1-p_1\theta^{t_2})} \right]$$

$$= p_1^{t_1} p_2^{t_2} \theta^{t_1 t_2} \frac{p_1\theta^{t_2}(1+ p_1\theta^{t_2})}{(1-p_1\theta^{t_2})^2}$$

Substituting for $R(t_1,t_2)$ from (4.17) we get

$$A_1(t_1,t_2) = \frac{p_1\theta^{t_2} (1+ p_1\theta^{t_2})}{(1-p_1\theta^{t_2})^2}$$

Also by (4.20)

$$r_1(t_1,t_2) = \frac{p_1\theta^{t_2}}{1- p_1 \theta^{t_2}}$$

Equation (4.53) now takes the form

$$V(X_1|X\geq t) = \frac{p_1\theta^{t_2}}{(1-p_1\theta^{t_2})^2} \qquad (4.55)$$

On similar lines one can also show that

$$V(X_2 \mid X \geq t) = \frac{p_2\theta^{t_1}}{(1-p_2\theta^{t_1})^2}$$

so that the conditions of the theorem are sufficient.

Conversely (4.54) can be written as

$$A_1(t_1,t_2)\ R(t_1,t_2) = 1^2 \sum_{t_1}^{\infty} f(t+1,x_2) +$$

$$2^2 \sum_{t_2}^{\infty} f(t+2,x_2) +$$

$$3^2 \sum_{t_2}^{\infty} f(t+3,x_3) + \ldots \quad (4.56)$$

Changing t_1 to (t_1+1) and subtracting the resulting equation from (4.56)

$$A_1(t_1,t_2)\ R(t_1,t_2) - A_1(t_1+1,t_2)\ R(t_1+1,t_2)$$

$$= \sum_{x_1=o}^{\infty} [(x_1+1)^2-x_1^2] \sum_{x_2=t_2}^{\infty} f(t_1+x_1+1,t_2)$$

$$= \sum_{x_1=o}^{\infty} \sum_{x_2=t_2}^{\infty} (2x_1+1)\ f(t_1+x_1+1,t_2)$$

$$= \sum_{x_1=t_1+1}^{\infty} \sum_{x_2=t_2}^{\infty} [2(x_1-t_1)-1] \; f(x_1,x_2)$$

$$= 2r_1(t_1+1,t_2) \; R(t_1+1,t_2) + R(t_1+1,t_2) \qquad (4.57)$$

When (4.5.1) holds (4.53) takes the form

$$A_1(t_1,t_2) = V_1(t_2) + r_1^{\,2}(t_1,t_2)$$

Substituting in (4.57)

$$[V_1(t_2) + r_1^{\,2}(t_1,t_2)]R(t_1,t_2)$$

$$= [V_1(t_2) + r_1^{\,2}(t_1+1,t_2) + 2r_1(t_1+1,t_2)+1]R(t_1+1,t_2)$$

$$= [V_1(t_2) + (r_1(t_1+1,t_2)+1)^2] \; R(t_1+1,t_2) \qquad (4.58)$$

From (4.57)

$$R(t_1,t_2) = \frac{1+r_1(t_1+1,t_2)}{r_1(t_1,t_2)} \; R(t_1+1,t_2) \qquad (4.59)$$

(4.58) can now be written as

$$[V_1(t_2) + r_1^{\,2}(t_1,t_2)] \; [1+r_1(t_1+1,t_2)]$$

$$= r_1(t_1,t_2)[V_1(t_2)+(1+r_1(t_1+1,t_2))^2]$$

or

$$V_1(t_2)\left\{\frac{1+r_1(t_1+1,t_2)-r_1(t_1,t_2)}{r_1(t_1,t_2)[1+r_1(t_1+1,t_2)]}\right\}=1+r_1(t_1+1,t_2)-r_1(t_1,t_2) \tag{4.60}$$

The solutions of (4.60) are

$$1 + r_1(t_1+1,t_2) = r_1(t_1,t_2) \tag{4.61}$$

and

$$r_1(t_1,t_2)\ [1+r_1(t_1+1,t_2)] = V_1(t_2) \tag{4.62}$$

(4.61) gives

$$R(t_1,t_2) = R(t_1+1,t_2)$$

producing the result that $R(t_1,t_2)$ is independent of t_1 which is clearly inadmissible.

From

$$r_1(t_1+1,t_2) = \frac{V_1(t_2) - r_1(t_1,t_2)}{r_1(t_1,t_2)}$$

$$= \frac{V_1(t_2) - \dfrac{V_1(t_2)-r_1(t_1-1,t_2)}{r_1(t_1-1,t_2)}}{\dfrac{V_1(t_2) - r_1(t_1-1,t_2)}{r_1(t_1-1,t_2)}}$$

$$= \frac{[\,1+V_1(t_2)\,]\, r_1(t_1-1,t_2) - V_1(t_2)}{V_1(t_2) - r_1(t_1-1,t_2)}$$

$$= \frac{[1+ 2V_1(t_2)]\, r_1(t_1-2,t_2)-V_1(t_2)[V_1(t_2)+1]}{V_1(t_2)-(1+V_1(t_2))\, r_1(t_1-2,t_2)}$$

$$= \frac{[V_1(t_2)(1+V_1(t_2)+2V_1(t_2))+1]r_1(t_1-3,t_2)-V_1(t_2)[1+2V_1(t_2)]}{[1+V_1(t_2)]V_1(t_2) - [1+2V_1(t_2)]r_1(t_1-3,t_2)}$$

etc.

producing the result that $r_1(t_1+1,t_2)$ is a function of t_2 only. That is,

$$r_1(t_1,t_2) = r_1(o,t_2)$$

Now from (4.59)

$$R(t_1+1,t_2) = \left(\frac{r_1(o,t_2)}{1+r_1(o,t_2)}\right) R(t_1,t_2)$$

$$= g_1(t_2)\, R(t_1,t_2) \qquad (4.63)$$

where

$$g_1(t_2) = \frac{r_1(o,t_2)}{1+ r_1(o,t_2)}$$

$$= \frac{r_1^{\,2}(o,t_2)}{V_1(t_2)}\text{ , by (4.62)}$$

Iteration for successive values of t_1 in (4.63) leads to

$$R(t_1,t_2) = [\ g_1(t_2)\ ]^{t_1} R(o,t_2) \tag{4.64}$$

As t_2 tends to zero in (4.64)

$$R_1(t_1) = [\ g_1(o)\ ]^{t_1}$$

$$= p_1^{\ t_1}$$

where

$$p_1 = \frac{r_1(o,o)}{1+\ r_1(o,o)}$$

$$= \frac{r_1^{\ 2}(o,o)}{V_1(o)}$$

Similarly

$$R_2(t_2) = p_2^{\ t_2}$$

Substitution in (4.64) gives

$$R(t_1,t_2) = [g_1(t_2)]^{t_1}\ p_2^{\ t_2} \tag{4.65}$$

Similarly

$$R(t_1,t_2) = [g_2(t_1)]^{t_2}\ p_1^{\ t_1} \tag{4.66}$$

From (4.65) and (4.66) we get

$$\left(\frac{g_1(t_2)}{p_1}\right)^{\frac{1}{t_2}} = \left(\frac{g_2(t_1)}{p_2}\right)^{\frac{1}{t_1}} \tag{4.67}$$

whose only solution is

$$g_i(t_j) \quad = \quad p_i\, \theta^{t_j}$$

From (4.67)

$$R(t_1,t_2) \quad = \quad p_1^{\ t_1}\, p_2^{\ t_2}\, \theta^{t_1 t_2}$$

which is the survival function of the bivariate geometric distribution. Further

$$V_1(t_2) \quad = \quad r_1(t_1,t_2)\ [1+r_1(t_1+1,t_2)]$$

$$= \quad r_1(o,t_2)\ [1+r_1(o,t_2)]$$

or

$$1 \quad = \quad \frac{r_1(o,t_2)}{V_1(t_2)} + \frac{r_1^2(o,t_2)}{V_1(t_2)}$$

It follows that

$$0 \quad \leq \quad \frac{r_1^{\ 2}(o,t_2)}{V_1(t_2)} \quad \leq \quad 1$$

That is

$$0 \leq g_1(t_2) \leq 1$$

or

$$0 \leq p_1 \leq 1$$

The value of θ is deduced from the monotonicity of $V_i(t_j)$ and the proof is completed.

4.5 Characterization based on distributional properties *

Apart from the results that are extended versions of the univariate case, there are certain properties that arise from the bivariate set up itself. The three theorems presented below belong to this category. First, we establish a characterization of the bivariate geometric distribution by the form of conditional densities of X_i given $X_j \geq t_j$; $i,j = 1,2$, $i \neq j$.

Theorem 4.6.

Let $X = (X_1,X_2)$ be a random vector with support I_2^+. X has bivariate geometric distribution specified by (4.17) if and only if the conditional distribution of X_i given $X_j \geq t_j$; $i,j = 1,2$, $i \neq j$ is geometric with parameters $p_i(t_j)$ which are non increasing functions of t_j.

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* The results in Section 4.5 along with some other results concerning another Bivariate geometric distribution is to appear in STATISTICA (1990). (Reference 40)

Proof

When the conditional distributions of X_i given $X_j \geq t_j$ are geometric, the conditional survival function of X_1 given X_2 is of the form

$$R(t_1|t_2) = P[X_1 \geq t_1 \mid X_2 \geq t_2]$$

$$= [\, p_1(t_2)\,]^{t_1};\ 0<p_1(t_2)<1,\ t_1=0,1,2... \tag{4.68}$$

Setting $t_2 = 0$

$$R_1(t_1) = P[X_1 \geq t_1]$$

$$= p_1^{\,t_1}$$

where $p_1 = p_1(o)$

Now

$$R(t_1,t_2) = R(t_1|t_2)\, R_2(t_2)$$

$$= [p_1(t_2)]^{t_1}\, p_2^{\,t_2} \tag{4.69}$$

By a similar argument concerning X_2 given X_1, we can also get

$$R(t_1,t_2) = [p_2(t_1)]^{t_2}\, p_1^{\,t_1} \tag{4.70}$$

From (4.69) and (4.70) we get

$$[\ p_1(t_2)\ ]^{t_1}\ p_2^{\ t_2} = [p_2(t_1)]^{t_2}\ p_1^{\ t_1} \tag{4.71}$$

Since (4.71) is true for all t_1 and t_2, its solution can only be such that

$$\left(\frac{p_1(t_2)}{p_1} \right)^{1/t_2} = \left(\frac{p_2(t_1)}{p_1} \right)^{1/t_1} = \theta \tag{4.72}$$

a constant independent of both t_1 and t_2. This gives

$$p_i(t_j) = p_i\ \theta^{t_j}$$

which when substituted in (4.70) or (4.71) leads to (4.17).

On the other hand if we assume

$$R(t_1,t_2) = p_1^{\ t_1}\ p_2^{\ t_2}\ \theta^{t_1 t_2}$$

we have

$$R(t_1|t_2) = \frac{R(t_1,t_2)}{R_2(t_2)}$$

$$= p_1^{\ t_1}\ \theta^{t_1 t_2}$$

The conditional probability mass function of X_1 given $X_2 > t_2$ is

$$f(x_1|X_2 \geq x_2) = R(t_1|t_2) - R(t_1+1|t_2)$$
$$= (p_1\theta^{t_2})^{t_1} (1-p_1\theta^{t_2})$$

which is geometric with parameter $p_1 \theta^{t_2}$. Also since $0 \leq \theta \leq 1$, $0 < p_1\theta^{t_2} < 1$ for all t_2. Hence the conditional distribution of X_1 given $X_2 > t_2$ is geometric. Similarly the conditional distribution of X_2 given $X_1 \geq t_2$ is also geometric.

The conditional densities discussed above are of application in modelling reliability data. The matter will be taken up in the succeeding chapter.

When the random vector X follows the bivariate geometric distribution, we recall that the conditional probability mass function of X_1 given $X_2 = x_2$ is given by (4.30) and the marginal probability mass functions by (4.29). The form of the conditional mass function (4.30) can be utilised to arrive at characterizations of the univariate geometric model as well as bivariate form (4.17).

Theorem 4.7.

If $X = (X_1, X_2)$ is a random vector with support I_2^+ such that the conditional distribution of X_1 given X_2 is of the form (4.30), then a geometric distribution for X_1 implies and is implied by a geometric form for X_2.

Proof:

Assume that X_2 is geometric. Then the probability mass function of X_2 is of the form

$$f_2(x_2) = p_2^{x_2}(1-p_2), \quad x_2 = 0,1,2, \ldots \tag{4.73}$$

Using the expression for $f(x_1|x_2)$ from (4.30) and that of $f_2(x_2)$ given above, the joint probability mass function of X turns out to be

$$f(x_1,x_2) = p_1^{x_1} p_2^{x_2} \theta^{x_1x_2-1}[(1-p_1\theta^{x_2+1})$$
$$(1-p_2\theta^{x_1+1}) + \theta - 1]$$

Summing over the support of X_2,

$$f_1(x_1) = p_1^{x_1}(1-p_1), \quad x_1 = 0,1,2,\ldots \tag{4.74}$$

At the same time if the component X_1 is taken to be geometric of the form (4.74), the identity

$$f_1(x_1) = \sum_{x_2=0}^{\infty} f(x_1|x_2)f_2(x_2)$$

provides

$$p_1^{x_1}(1-p_1) = \sum_{x_2=0}^{\infty} \frac{p_1^{x_1}\theta^{x_1x_2-1}}{1-p_2}[(1-p_1\theta^{x_2+1})(1-p_1\theta^{x_1+1})+\theta-1]f_2(x_2)$$

or

$$(1-p_1)(1-p_2) = \sum_{x_2=0}^{\infty} \theta^{x_1x_2-1}[\theta-p_2\theta^{x_1+1}-p_1\theta^{x_2+1}+p_1p_2\theta^{x_1+x_2-2}]f_2(x_2)$$

$$= \sum_{x_2=0}^{\infty} [\theta^{x_1x_2}(1-p_1\theta^{x_2})-p_2\theta^{x_1(x_2+1)}(1-p_1\theta^{x_2+1})]f_2(x_2)$$

Equating coefficients of θ^{rx_1}, $r = 0,1,2,\ldots$ on both sides,

$$f_2(x_2) = p_2^{x_2}(1-p_2), \quad x_2 = 0,1,2,\ldots$$

and our result is proved.

Corollary

The conditional distribution of X_i given $X_j=x_j$ is of the form (4.30) and X_i is geometric is a necessary and sufficient condition for (X_1,X_2) to be bivariate geometric.

Theorem 4.8.

Let $X = (X_1, X_2)$ be a random vector with support I_2^+. X has bivariate geometric distribution (4.17) if and only if the following conditions hold good.

(i) The marginal distribution of X_1 is geometric, and

(ii) The conditional mean of X_2 given $X_1 \geq t$, namely $E(X_2 - t_2 | X \geq t)$ is $p_2 \theta^{t_2} (1 - p_2 \theta^{t_1})^{-1}$

Proof:

Condition (ii) in the theorem can also be written as

$$r_2(t_1)\ R(t_1, t_2) = \sum_{t_1}^{\infty} \sum_{t_2}^{\infty} (x_2 - t_2) f(x_1, x_2)$$

$$= \sum_{s=1}^{\infty} R(t_1, t_2 + s) \qquad (4.75)$$

Changing t_2 to (t_2+1) and subtracting from (4.75)

$$r_2(t_1)[R(t_1, t_2) - R(t_1, t_2+1)] = R(t_1, t_2+1)$$

or

$$R(t_1, t_2+1) = \left[\frac{r_2(t_1)}{1 + r_2(t_1)} \right] R(t_1, t_2) \qquad (4.76)$$

Successive reduction for t_2 gives

$$R(t_1,t_2) = [\frac{r_2(t_1)}{1+r_2(t_1)}]^{t_2} R(t_1,o) \qquad (4.77)$$

When (2) holds,

$$\frac{r_2(t_1)}{1+r_2(t_1)} = p_2 \theta^{t_1}.$$

From condition (i) of the theorem

$$R(t_1,o) = p_1^{t_1}$$

(4.77) now reads as

$$R(t_1,t_2) = p_1^{t_1} p_2^{t_2} \theta^{t_1 t_2}$$

Conversely when X follow the bivariate geometric distribution, (i) and (ii) follows from (4.20) and (4.29) and the proof is completed.

Chapter-5

CHARACTERIZATION BY RELIABILITY CONCEPTS

A considerable amount of results dealing with characterization of the exponential distribution has their origin in the mathematical theory of reliability. The role of the univariate exponential distribution as a model for life time data was brought to light in a series of papers by Epstein and Sobel (1953, 54) who popularised its role in reliability studies. It is evident from the review of literature that a number of bivariate exponential distributions have been proposed to describe failure patterns in two-component systems.

The basic concepts used to model life time distributions are failure rate, mean residual life and some other notions of ageing. While most works on modelling consider the failure rate, defined as

$$h(x) = \frac{f(x)}{R(x)} \tag{5.1}$$

where f(x) is the probability density function and

$R(x) = P[X>x]$, as the basic concept there is a strong case for using the mean residual life function (MRLF) to meet the same purpose. For the random variable X representing the length of life of a component, (X-t) given $X > t$ is called the residual life and its expected value

$$r(t) = E(X-t \mid X>t) \tag{5.2}$$

is the mean residual life function, which represents the average life time remaining for the component given that it has survived upto time t.

Muth (1977) points out the important differences while inferring ageing characteristics using MRLF's and failure rate functions, although both are used for the same purpose by researchers. Though positive ageing corresponds to decreasing MRL (DMRL) and increasing hazard function (IHF), the class of IHF distributions forms a proper subset of DMRL family as DMRL implies IHF but not conversely. The failure rate function cannot have a unique inverse and consequently the age cannot be inferred from the failure rate function. The essential difference, however, is that while failure rate accounts only for the immediate future in assessing the event component failure as evident from the equation

$$h(t)\,R(t) = -\frac{d}{dt}R(t),$$

the MRL accounts for the complete failure in view of the equation

$$r(t)\,R(t) = \int_t^\infty R(x)dx$$

When decisions are to be made on replacement policies, the expected remaining life will be a more suitable criterion to decide upon whether to replace or not than the failure rate from which it is difficult to earn even a suitable yardstick for the purpose. Apart from these considerations arising out of reliability context, the MRL concept when considered as a truncated mean has enough potential in characterizing probability distributions and also in modelling data in other areas such as income analysis [Kakwani (1980)],manpower planning [Bartholomew and Forbes (1980)], bio-statistics and actuarial science to name a few.

Although the concept of MRLF has been extensively discussed in univariate theory vis a vis its ability to decide upon a useful choice of the model to represent the life data, in the bivariate context this notion does not seem to have been studied in depth although some references to it are available in literature (see p. 13). In this chapter the main focus of attention is on bivariate mean residual life and its potential in life length studies.

In a simple life testing experiment, a number of items are subjected to test and the data consists of the recorded lives of all or some of the items. Since our interest is on bivariate distributions, we consider a two-component system the life time of the i^{th} component is X_i, $i=1,2$. If $X = (X_1,X_2)$ denotes the failure time of the device, the basic concepts associated with X under review here are

(i) the reliability function

(ii) the failure rate, and

(iii) the mean residual life

The survival function $R(t_1,t_2)$ has already been defined as

$$R(t_1,t_2) = P\,[X_1>t_1,\ X_2>t_2]$$

which denotes the probability of failure free operation of the device until $\underline{t} = (t_1, t_2)$. The function $R(t_1, t_2)$ can also be expressed as

$$R(t_1,t_2) = 1-F_1(t_1) - F_2(t_2) + F(t_1,t_2)$$

where $F_1(.)$, $F_2(.)$ and $F(.,.)$ are the distribution functions of X_1, X_2 and X respectively. This quantity is a numerical measure of how far the system is reliable and is referred to as the reliability function. One aspect of interest in reliability studies is the precise functional form of $R(t_1, t_2)$.

The problem of extending the concept of failure rate into higher dimensions has been considered by Basu (1971) and Johnson and Kotz (1975). While the former has given a single component for the failure rate, the latter views it as a vector valued function. The definitions have been given earlier in equations (2.3) and (2.5). A characterization of the Gumbel's bivariate exponential distribution by the local constancy of failure rate due to Johnson and Kotz (1975) have been reproduced in theorem 2.2.

5.1 Bivariate mean residual life *

Let $\underline{X} = (X_1, X_2)$ be a random vector admitting absolutely continuous distribution function in the support of the octant $Q = \{(x_1, x_2) \mid x_i \geq 0\}$ in the two dimensional space R_2. With $\underline{X} \geq \underline{x}$ denoting $X_i \geq x_i$, $i = 1,2$, the bivariate mean residual life function of $\underline{X}$ is defined as the vector function

$$\underline{r}(\underline{t}) = E(\underline{X} - \underline{t} \mid \underline{X} > \underline{t}) \tag{5.3}$$

where $\underline{t} = (t_1, t_2)$ and t_i are non-negative real numbers. The survival functions of X_1, X_2 and $\underline{X}$ are

$$R_1(t_1) = 1 - F_1(t_1)$$

$$R_2(t_2) = 1 - F_2(t_2)$$

$$R(\underline{t}) = 1 - F_1(t_1) - F_2(t_2) + F(t_1, t_2)$$

In the first place, a relationship enabling the determination of the survival functions in terms of $\underline{r}(t)$ will be established. To do so we notice that $\underline{r}(t)$ has two components, say $r_1(t)$ and $r_2(t)$, of which the

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* Part of the material in this section has appeared in I.E.E.E Trans.Rel.Vol.38, No.3,(1989) (Reference 38)

former can be written as

$$r_1(\underline{t}) = E(X_1-t_1 \mid \underline{X} > \underline{t})$$

or

$$R(\underline{t})\, r_1(\underline{t}) = \int_{t_1}^{\infty}\int_{t_2}^{\infty} (x_1-t_1)\; f(x_1,x_2)dx_1\, dx_2$$

$$= \int_{t_1}^{\infty}\int_{t_2}^{\infty} (x_1-t_1)\, \frac{\partial^2 R}{\partial x_1 \partial x_2}\, dx_1\, dx_2$$

$$= \int_{t_1}^{\infty} (x_1-t_1)\; \left[-\frac{\partial R}{\partial x_1}\right]_{t_2}^{\infty} dx_1$$

$$= \int_{t_1}^{\infty} R(x_1,t_2)\; dx_1 \qquad (5.4)$$

Taking logarithm of both sides and then differentiating with respect to t_1

$$\frac{\partial \log R(\underline{t})}{\partial t_1} + \frac{1}{r_1(\underline{t})}\; \frac{\partial r_1(\underline{t})}{\partial t_1} = \frac{-1}{r_1(\underline{t})}$$

or

$$-\frac{\partial \log R(\underline{t})}{\partial t_1} = \frac{1}{r_1(\underline{t})}\left[\, 1+ \frac{\partial r_1(\underline{t})}{\partial t_1}\,\right] \qquad (5.5)$$

A similar consideration on $r_2(\underline{t})$ leads to

$$-\frac{\partial \log R(\underline{t})}{\partial t_2} = \frac{1}{r_2(\underline{t})}\left[1+ \frac{\partial r_2(\underline{t})}{\partial t_2} \right] \qquad (5.6)$$

To solve the differential equations (5.5) and (5.6) we use the gradient operator

$$\nabla = \left(\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2} \right)$$

Equations (5.5) and (5.6) can be together written as

$$\nabla \, (-\log R) = \left[r_1^{-1}\left(1+ \frac{\partial r_1}{\partial t_1}\right), \; r_2^{-1}\left(1+ \frac{\partial r_2}{\partial t_2}\right)\right] \qquad (5.7)$$

Since $-\log R$ is continuous in an open set containing Q, for an arbitrary path which is orthogonal to the axis connecting $\underline{o}$ and $\underline{t}$ in Q, we can write

$$\int_{\underline{o}}^{\underline{t}} \nabla \, (-\log R)d\underline{t} = -\log R(\underline{t}) + \log R(\underline{o})$$

$$= -\log R(\underline{t}) \qquad (5.8)$$

From (5.7) and (5.8) we get

$$-\log R(\underline{t}) = \int_o^{t_1} \left(1+ \frac{\partial r_1(x_1,o)}{\partial x_1}\right)(r_1(x_1,o))^{-1}\, dx_1$$

$$+ \int_o^{t_2} \left(1+ \frac{\partial r_2(t_1,x_1)}{\partial x_2}\right)(r_2(t_2,x_2))^{-1}\, dx_2$$

or

$$R(\underline{t}) = \frac{r_1(\underline{o})\ r_2(t_1,o)}{r_1(t_1,o) r_2(\underline{t})} \exp\left[- \int_o^{t_1} \frac{dx_1}{r_1(x_1,o)} - \int_o^{t_2} \frac{dx_2}{r_2(t_1,x_2)}\right] \tag{5.9}$$

We can also obtain the two equivalent forms for the relationship as

$$R(\underline{t}) = \frac{R_2(t_2)\ r_1(o,t_2)}{r_1(\underline{t})} \quad \exp\left[- \int_o^{t_1} \frac{dx_1}{r_1(x_1,t_2)}\right] \tag{5.10}$$

$$= \frac{R_1(t_1)\ r_2(t_1,o)}{r_2(\underline{t})} \quad \exp\left[- \int_o^{t_2} \frac{dx_2}{r_2(t_1,x_2)}\right] \tag{5.11}$$

Observations:

(i) From (5.9) it is evident that $r(\underline{t})$ determines the distribution uniquely. By postulating a functional form of $r_1(\underline{t})$ and $r_2(\underline{t})$ on the basis of known behaviour one can obtain the failure distribution and the reliability function.

For instance, if we take $r_1(\underline{t})$ and $r_2(t)$ as constants independent of t_1 and t_2, relationship (5.9) leaves $R(\underline{t})$ as a bivariate exponential distribution with independent components. On the other hand taking $r_1(t_1,t_2)$ as a function of t_2 alone and $r_2(t_1,t_2)$ as a function of t_1 alone in the form

$$r_1(\underline{t}) = (\alpha_1 + \theta t_2)^{-1} \text{ and } r_2 = (\alpha_2 + \theta t_1)^{-1}$$

(5.9) gives $R(\underline{t})$ as the survival function of the Gumbel's bivariate exponential distribution.

(ii) The bivariate MRLF is related to the vector valued failure rate function

$$h(\underline{t}) = \nabla [- \log R(\underline{t})]$$

If $h_j(\underline{t})$ are the components of $h(\underline{t})$ we have

$$h_j(\underline{t}) = [r_j(\underline{t})]^{-1} [1+ \frac{\partial r_j(\underline{t})}{\partial t_j}],\ j=1,2 \qquad (5.12)$$

(iii) The MRLF's of the component variables are

$$m_j(t_j) = E[X_j-t_j \mid X_j > t_j]$$

The conditions

$$m_j(t_j) \;=\; r_j(\underline{t}),\; j=1,2 \tag{5.13}$$

holds if and only if X_1 and X_2 are independent.

Proof:

When X_1 and X_2 are independent, (5.4) can be written as

$$R_1(t_1)\; R_2(t_2)\; r_1(\underline{t}) = \int_{t_1}^{\infty} [R_1(x_1)\; R_2(t_2)]\; dx_1$$

giving

$$r_1(\underline{t}) = \frac{1}{R_1(t_1)} \int_{t_1}^{\infty} R_1(x)dx$$

$$= \; m_1(t_1)$$

Similarly

$$r_2(\underline{t}) \;=\; m_2(t_2)$$

Conversely when (5.13) holds, from (5.10)

$$R(\underline{t}) \;= R_2(t_2)\; \frac{m_1(o)}{m_1(t_1)}\; \exp[-\int_{o}^{t_1} \frac{dy}{m_1(y)}\;]$$

$$= R_2(t_2)\; R_1(t_1)$$

from (5.10). Thus X_1 and X_2 are independent.

(iv) From the bivariate MRLF, one can obtain the MRLF's of the component variables. As t_2 tends to 0 in (5.4)

$$R_1(t_1)\ r_1(t_1,o) = \int_{t_1}^{\infty} R_1(x_1)\ dx_1$$

$$= \int_{t_1}^{\infty} (x_1-t_1)\ dF_1$$

$$= R_1(t_1)\ m_1(t_1)$$

which shows that

$$m_1(t_1) = r_1(t_1,o)$$

Similarly one can also show that

$$m_2(t_2) = r_2(o,t_2)$$

(v) In the univariate set up, the condition that the MRLF is constant is a characteristic property of the exponential distribution. Referring to the bivariate case, the constancy of the components of $r(\underline{t})$ can be achieved globally and locally specified respectively by the conditions $r_j(\underline{t}) = \alpha_j$ where α_j is independent of both t_1 and t_2 and $r_j(\underline{t}) = \alpha_j(t_{3-j})$ with α_j independent of t_j. It is shown in Nair and Nair (1988) that the former case is true if and only if X_1 and X_2 are independent with means α_1 and α_2 and the second representation with $\alpha_j(t_{3-j})$ monotone increasing is a characteristic property of the Gumbel's bivariate exponential distribution (see also theorem 3.1).

5.2 Conditions for bivariate MRLF

Extending the results in the univariate case in Swartz (1973), we can obtain a set of necessary and sufficient conditions for a vector valued function $\underline{r}(\underline{t})$ to be a bivariate MRLF. These are equation (5.9) along with

(i) $\underline{r}(\underline{t}) \geq 0$ implying $r_i(\underline{t}) \geq 0$, $i = 1,2$.

(ii) $\underline{r}(\underline{o}) = E(X)$

(iii) $\dfrac{\partial r_j(\underline{t})}{\partial t_j} \geq -1$, $j = 1,2$

(iv) $\int_0^\infty \dfrac{dy}{r_1(y,t_2)}$ and $\int_0^\infty \dfrac{dy}{r_2(t_1,y)}$ diverges

5.3 Asymptotic exponentiality

In view of the simple form and other interesting properties that have relevance in reliability analysis, attempts are seen in literature to realise exponential models either exactly or approximately. In terms of the bivariate MRLF one can obtain transformations that renders the conditional distribution of X_i given $\underline{X} > \underline{t}$ asymptotically exponential. The reliability of one component, known that the survival times of both upto given instants, can often be assessed by virtue of such approximations.

Considering the reduced variables

$Y_i = \frac{X_i - t_i}{r_i(\underline{t})}$ which satisfy the conditions

$$\lim_{t_1 \to \infty} \frac{r_1(t_1 + r_1 y_1, t_2)}{r_1(\underline{t})} = a_1(t_2)$$

$$\lim_{t_2 \to \infty} \frac{r_2(t_1, t_2 + r_2 y_2)}{r_2(t)} = a(t_1)$$

We can write

$$\frac{R(t_1 + y_1 r_1, t_2)}{R(t)} = \frac{r_1(\underline{t})}{r_1(t_1 + y_1 r_1, t_2)} \exp[- \int_{t_1}^{t_1 + y_1 r_1} \frac{dx_1}{r_1(x, t_2)}]$$

$$= \frac{r_1(\underline{t})}{r_1(t_1 + y_1 r_1, t_2)} \exp[- \int_{o}^{y_1} \frac{r_1(t) du}{r_1(t_1 + u r_1, t_2)}]$$

As t_1 tends to infinity

$$P[X_1 > t_1+y_1r_1|\underline{X} > \underline{t}] = [a_1(t_2)]^{-1} \exp[-a_1(t_2)]^{-1}y_1$$

A similar result holds for the variable X_2.

5.4 Failure rate and MRL in the discrete case

Although continuous distributions are generally explored and used in the context of modelling life time data, some interest has been evoked recently in using discrete models for the purpose as pointed out in the introduction of chapter-4. In this section we introduce the failure rate concept and the mean residual life function in the discrete domain and use it as a tool to characterize the bivariate geometric distribution.

Analogous to the definition in the continuous case proposed by Galambos and Kotz (1978) we define the bivariate failure rate as a two component vector, $\underline{h}(t) = (h_1(\underline{t}), h_2(\underline{t}))$ where

$$h_i(t) = \frac{P(X_i=t_i,\ X_j\geq t_j)}{P(X_1\geq t_1,\ X_2\geq t_2)}\ ,\quad i,j = 1,2;\ i \neq j$$

$$= \frac{R_2(t_2)\ f_i(t_i|X_j \geq t_j)}{R(t_1,t_2)} \tag{5.14}$$

Notice that the conditional distribution on the numerator is the one we have encountered in Section 4.2.

In the succeeding theorem we prove that local constancy of $h_i(t_1,t_2)$ is a characteristic property of the bivariate geometric distribution (4.17).

Theorem 5.1.

A random vector $X = (X_1,X_2)$ with support I_2^+ has bivariate geometric distribution (4.17) if and only if it has a failure rate function of the form $h(t_1,t_2) = [h_1(t_2),\ h_2(t_1)]$ where h_1 and h_2 are non-increasing functions in the respective variables, such that $h_i(o) = 1-p_i$, $i = 1,2$.

Proof:

To prove the necessary part, we have from (5.14)

$$R(t_1,t_2)\ h_1(t_1,t_2) = R(t_1,t_2)-R(t_1+1,t_2)$$

giving

$$R(t_1+1,t_2) = [\ 1-h_1(t_1,t_2)]\ R(t_1,t_2) \qquad (5.15)$$

Similarly

$$R(t_1,t_2+1) = [\ 1-h_2(t_1,t_2]\ R(t_1,t_2) \qquad (5.16)$$

Setting $t_2 = 0$ in (5.15)

$$R_1(t_1) = [\ 1-h_1(t_1-1,o)]\ R_1(t_1-1)$$

Iterating this equation for decreasing values of t_1 yields

$$R_1(t_1) = \prod_{r=1}^{t_1} [1-h_1(t_1-r,o)]$$

Also

$$R_2(t_2) = \prod_{r=1}^{t_2} [1-h_2(o,t_2-r)] \tag{5.17}$$

Further iterating (5.15) for decreasing values of t_1 gives

$$R(t_1,t_2) = \prod_{r=1}^{t_1} [1-h_1(t_1-r,t_2)]\ R(o,t_2) \tag{5.18}$$

Substituting for $R(o,t_2)$ from (5.17)

$$R(t_1,t_2) = \prod_{r=1}^{t_1} [1-h_1(t_1-r,t_2) \prod_{r=1}^{t_2} [1-h_2(o,t_2-r)] \tag{5.19}$$

(5.19), in fact, gives a general formula, that is outside the framework of the present theorem, that relate the failure rate function and the survival function. It expresses that the failure rate function determines the corresponding distribution uniquely.

Under conditions of the theorem, (5.18) reads as

$$R(t_1,t_2) = [1-h_1(t_2)]^{t_1} R_2(t_2) \tag{5.20}$$

When $t_2 = 0$ in (5.20) we get

$$R(t_1,t_2) = p_1^{t_1}$$

Employing the same technique, the relationship

$$R(t_1) = [1-h_2(t_1)]^{t_2} R_1(t_1) \tag{5.21}$$

gives

$$R_2(t_2) = p_2^{t_2}$$

Equations (5.20) and (5.21) leads to the functional equation

$$[1-h_1(t_2)]^{t_1} p_2^{t_2} = [1-h_2(t_1)]^{t_2} p_1^{t_1}$$

whose solution is

$$\left\{\frac{1-h_1(t_2)}{p_1}\right\}^{\frac{1}{t_2}} = \left\{\frac{1-h_2(t_1)}{p_2}\right\}^{\frac{1}{t_1}} = \theta \tag{5.22}$$

where Θ is independent of both t_1 and t_2. (5.22) provides us with the representation

$$h_i(t_j) = 1-p_i \, \Theta^{t_j} \tag{5.23}$$

Introducing (5.23) into (5.20) or (5.21) we get

$$R(t_1,t_2) = p_1^{t_1} \, p_2^{t_2} \, \Theta^{t_1 t_2}$$

and accordingly (X_1,X_2) has the bivariate geometric distribution. Conversely, when X has bivariate geometric distribution we have

$$R_i(t_i) = p_i^{t_i}$$

and $f(t_i|X_j \geq t_j)$ is given by (4.33) so that from (5.14)

$$h_i(t_1,t_2) = 1-p_i \, \Theta^{t_j}$$

which is independent of t_i and the proof is completed.

In the discrete case, we define the bivariate MRLF, $\underline{r}(\underline{t})$, as a vector with components

$$r_i(t_1,t_2) = E[X_i-t_i|X_i \geq t_i, \ i=1,2 \ ] \tag{5.24}$$

so that

$$r_i(t_1,t_2)\ R(t_1,t_2) = \sum_{t_1}^{\infty} \sum_{t_2}^{\infty} (x_i-t_i)\ f(x_1,x_2)$$

Using (4.7) namely

$$r_1(t_1,t_2)\ R(t_1,t_2) = R(t_1+1,t_2)[1+r_1(t_1+1,t_2)]$$

and (5.15) we see that h_1 and r_1 are connected by the relation

$$1-h_1(t_1,t_2) = \frac{r_1(t_1,t_2)}{1+r_1(t_1+1,t_2)} \qquad (5.25)$$

In terms of bivariate MRLF, theorem 5.1 can be stated as follows.

Theorem 5.2.

The discrete random vector X in theorem 5.1 has bivariate geometric distribution (4.17) if and only if its MRLF is of the form

$$\underline{r}(\underline{t}) = (r_1(t_2),\ r_2(t_1))$$

with both r_1 and r_2 non-increasing in the respective variables with $r_i(o) = p_i/1-p_i$.

The proof of the theorem follows directly from (5.25) and theorem 5.1.

There is a simple and interesting relationship between the product of the components of $\underline{h}(\underline{t})$ and $\underline{r}(\underline{t})$ that characterizes the bivariate exponential distribution.

Theorem 5.3.

A necessary and sufficient condition for the relationship

$$r_j(\underline{t})\ h_j(\underline{t}) = 1,\ j = 1,2 \tag{5.26}$$

hold for all $\underline{t} > 0$ is that the random vector X is distributed as bivariate exponential with density (3.1).

Proof:

From (5.12)

$$r_j(\underline{t})\ h_j(\underline{t}) = 1 + \frac{\partial r_j(\underline{t})}{\partial t_j}$$

It follows that $r(\underline{t}) = (a_1(t_2),\ a_2(t_1))$ which is characteristic of the bivariate exponential distribution. We further note that there exist an analogous result in the discrete case that characterizes our bivariate geometric model.

5.5 Conclusion

The characterizations so far established in the present study are based on the localised lack of memory and its variants viewed as a sort of extension of the corresponding properties in the univariate case. The results are specialised to the Gumbel's type I model and its discrete version. The properties of these two distribution in terms of order statistics and the development of a bivariate Poisson Process that corresponds to the bivariate exponential distribution are two main problems that await solutions. Further, there is the larger problem of examining the characteristic properties of the other bivariate exponential distributions reviewed in Chapter-2. Some progress has already been made in this direction and will be presented in a future work.

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