CHARACTERIZATION OF PROBABILITY DISTRIBUTIONS BY RELIABILITY CONCEPTS

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<u>CERTIFICATE</u>

This is to certify that the work reported in this thesis is based on the bona fide work done by Sri. P.G. Sankaran, under my guidance in the Division of Statistics, School of Mathematical Sciences, Cochin University of Science and Technology, Kochi 682022, and has not been included in any other thesis, submitted previously for the award of any degree.

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Chapter 1

PRELIMINARIES

1.1. Introduction

The term reliability is used in the present study to denote the probability of a device, component, material or structure, performing its intended function satisfactorily, for a given length of time in an environment for which it is designed and is often used as a yardstick of the capability of the device to operate without failure when put into service. Eventhough the above definition of reliability is explained with reference to the failure behaviour or length of life of an equipment, it is equally applicable in the analysis of any duration variable that describes a well defined population subject to decrementation due to the operation of forces of attrition over time. Accordingly, the concepts and tools in reliability have found applications in many areas of study such as biology, medicine, engineering, economics, epidemiology and demography. Reliability analysis, originally perceived as an area involving special class of problems inviting application of probability and statistics, has in recent times developed into a field that stimulates original research in new frontiers of statistics such as non-parametric character of distribution functions, ranking, identification and selection procedures and inference based on truncated, censored and post mortem data.,

A fundamental problem in reliability analysis, when the data on failure times is the only input, is to identify the underlying distribution that is supposed to generate the observations. Generally it is not easy to isolate all the physical causes that contribute individually or collectively to the failure mechanism and to mathematically account for each and as such the task of identifying the correct model representing the data becomes very difficult. In many situations, the fact is that the information content on the failure or ageing pattern available from the data is rarely specific enough to enable the analyst to narrow down his consideration to a particular model. When the data alone is the criteria for the model selection it is customary to start with a general system of distributions and then to select an appropriate member from the system that fits the data. The dilemma one has to face here is that many of the models used in this connection have markedly different right tail behaviour and the sample size may not be large enough to notice such differences.

The legitimacy of the model is ascertained by its consonance with the data as revealed through techniques, such as probability plots or tests of fits like Kolmogrov-Smirnov statistic, Cramer-Von Moses Statistic, Anderson-Darling statistic, or the classical χ^2 or likelihood ratio tests, which are at best only approximate evaluations. Some times a model may fit the data reasonably well, but the physical considerations of the system may point out to a failure pattern that is inconsistent with the best fitting model. An illustration of this point is the lognormal distribution, which has a failure rate that increases over time, then decreases, and finally approaches to zero for sufficiently large values of time; and yet it may fit to certain failure time data that does not follow the above pattern. All these point out to the need for some legitimate method of arriving at a model and some accurate criteria that assert the soundness of the method. A standard practice adopted in most modelling situations is to ascertain the physical properties of the process generating the observations, express them by means of equations or inequalities and then solve them to obtain the model. In reliability, analysts have developed certain basic concepts such as failure rate, mean residual life, vitality etc. through which the physical characteristics of the failure mechanism can be adequately described and therefore these concepts form the basis of specifying a probability distribution of failure times. The only exact method of determining a probability distribution is to use a characterization theorem, which in general terms say that under certain conditions a family of distributions $\mathcal{F}$ is the only one possessing a designated property $\mathcal{P}$. Thus if one can translate the characteristics of the failure mechanism in terms of the failure rate, mean residual life or an ageing criteria and if there exists a probability distribution characterised by such a property, the problem of model identification is satisfactorily resolved. This brings in the relevance and need of characterization of probability distributions by reliability concepts which is the focal theme of the present study. The endeavour here is to present characterization theorems of some important distributions or families that are potential lifetime models. As already mentioned, apart from the point of view of reliability theory, the results obtained here are of interest in their own right in statistical distribution theory and also in various applied studies where concepts in reliability theory are used with differing interpretations.

1.2. Basic Concepts in Reliability

It is evident from the foregoing discussions that most of the difficulties in reliability modelling can be substantially reduced by appealing to certain concepts associated with the failure process, that permit different distributions to be distinguished. In the present section we discuss these concepts and review the results that will be used in the sequel.

1.2.1. Univariate Case

Let X be a non-negative random variable on a probability space $(\Omega, \mathcal{F}, P)$ with distribution function $F(x) = P[X \leq x]$. In the reliability context, X generally represents the length of life of a device, measured in units of time and the function,

$$
\begin{aligned}
R(x) &= 1-F(x) \\
&= P[X>x],
\end{aligned}
$$

is called the survival or reliability function because it gives the probability that the device will operate without failure for a mission time x.

1.2.1.1. Failure rate

Defining the right extremity L of F(x) by

$$L = \inf \{x : F(x) = 1\},$$

the failure rate h(x) of X, when F(x) is absolutely continuous with respect to Lebesgue measure with probability density f(x), is defined for $x < L$ by

$$\begin{aligned} h(x) &= \lim_{u \to 0^+} \frac{P[x < X \leq x+u \mid X > x]}{u} \\ &= \frac{f(x)}{R(x)}, \\ &= \frac{d}{dx}(-\log R(x)). \end{aligned} \qquad (1.1)$$

In the general case, when X is a random variable on the entire real line, Kotz and Shanbhag (1980) define the failure rate as the Radon-Nikodym derivative with respect to Lebesgue measure on $\{x : F(x) < 1\}$, of the hazard measure,

$$H(B) = \int_B dF/[1-F(x)],$$

for every Borel set B of $(-\infty, L)$. Further the distribution of X is uniquely determined by the relationship

$$R(x) = \prod_{u<x} [1-H(u)]\exp[-H_c(-\infty, c)], \qquad (1.2)$$

where, H_c is the continuous part of H. When X is non-negative and has absolutely continuous distribution function, (1.2) reduces to

$$R(x) = \exp[-\int_0^x h(t)\,dt]. \qquad (1.3)$$

1.2.1.2. Mean Residual Life

The mean residual life (MRL), known in early literature in actuarial studies as expectation of life, was reintroduced in the reliability context by Knight in 1959 (see, Kupka and Loo, 1989). It represents the average life time remaining to a component which has survived time x. When X is defined on the real line with $E(X^+) < \infty$, the B-measurable function

$$r(x) = E[X-x|X\geq x], \qquad (1.4)$$

for all x such that $P[X\geq x] > 0$ is called the MRL function of X. In the case when X is non-negative with $E(X) < \infty$ and F(x) is absolutely continuous with respect to Lebesgue measure,

$$r(x) = \frac{1}{R(x)} \int_x^\infty R(t)dt \qquad (1.5)$$

Further, for every x in (O,L),

$$h(x) = \frac{1+r'(x)}{r(x)} \tag{1.6}$$

and

$$R(x) = \frac{r(o)}{r(x)} \exp[-\int_{o}^{x} \frac{dt}{r(t)}] \cdot \tag{1.7}$$

Watson and Wells (1961) have obtained general conditions on a life distribution so that the MRL operated for some fixed time period is greater than the original mean life and in this context they have examined the Weibull, gamma, lognormal and extreme value distributions. This work was extended by Weise and Dishan (1971) by assigning cost functions and by considering the economic aspects burn-in and replacements. Bryson and Siddiqui (1969) use the decreasing MRL function as a criterion for ageing and derive some implications with reference to other criteria based on failure rate, survival function etc. A test statistic for a decreasing MRL has been proposed by Hollander and Proschan (1975). In view of the pivotal role the exponential distribution has in life length studies, the conditions under which asymptotic exponentiality can be achieved is of some interest. Among other results, Meilijson (1972) shows that for a non-negative random variable X, the transformation

$Y = X-(x/r(x))$ that satisfy

$$\lim_{x \to \infty} r(x+yr(x)) \; / \; [r(x)] = c,$$

a constant, guarantee that Y has exponential distribution.

By virtue of equation (1.7) an MRL function uniquely determine a distribution and therefore, modelling can be done through an appropriate functional form for $r(x)$. However one cannot choose freely any real valued function $r(x)$ on (o,∞) as an MRL, as the function $g(x) = 1+x^2$ will testify. One set of necessary and sufficient condition for $r(x)$ to be an MRL given by Swartz (1973) is that along with (1.7),

(i) $r(x) \geq 0$,

(ii) $r(o) = E(X)$,

(iii) $r'(x) \geq -1$, and

(iv) $\int_0^\infty \frac{dx}{r(x)}$ should be divergent.

Although the failure rate, MRL and survival function are in one-to-one correspondence with each other, Muth (1977) consider the MRL to be a superior concept than the failure rate on the following grounds.

(a) Regarding the ageing phenomena the two concepts are not equivalent. A decreasing MRL does not imply an increasing failure rate, though the converse is true. Thus the DMRL class is more general in character.

(b) The failure rate accounts only for the immediate future in assessing failure phenomenon as described by the derivative of R(x), where as the latter is descriptive of the entire future implied through $\int_x^\infty R(t)dt$. A consequence of this is that a component may experience deterioration though its failure rate may be zero at a certain point.

(c) It is advantageous to use the MRL function as a decision making criterion for replacement or maintenance policies. The expected remaining life of the component gives an indication of whether to replace or to re-schedule and this could be more useful than the failure rate to formulate maintanance policies.

1.2.1.3. Vitality Function

The concept of vitality function is introduced by Kupka and Loo (1989) and they define it as the B-measurable function on the real line

$$m(x) = E[X|X \geq x]$$

$$= \frac{1}{R(x)} \int_x^\infty t\,dF(t) \cdot \qquad (1.8)$$

The vitality function satisfies the following properties,

(i) $m(x)$ is non-decreasing and right continuous on $(-\infty, L)$

(ii) $m(x) \geq x$ for all $x < L$.

(iii) $\lim_{x \to L^-} m(x) = L$.

(iv) $\lim_{x \to -\infty} m(x) = E(X) \cdot$

Moreover, $m(x)$ is related to $r(x)$ by

$$m(x) = x + r(x).$$

The main contribution of the paper is that it provides a tool to clarify the link between the decreasing nature of MRL and the increasing nature of the failure rate through the concept of normalised increasing vitality property in the interval $[A,B]$ defined by

$$\frac{m(a,b)}{r(b)} \leq \frac{m(a+x,\ b+x)}{r(b+x)},$$

where, a,b, $a \leq b$, are interior points of [A,B], $b+x<L$ and

$$m(a,b) = \int_a^b m(t)dt ;$$

Precisely, they proved that if F has increasing vitality property together with decreasing MRL, then F has increasing failure rate property on [A,B] with respect to x.

1.2.1.4. <u>Variance Residual Life</u>

The variance residual life (VRL) of a non-negative random variable with survival function R(x) is defined as

$$\begin{aligned} V(x) &= V(X-x \mid X \geq x) \\ &= E[(X-x)^2 \mid X \geq x] - r^2(x) . \qquad (1.9) \end{aligned}$$

The concept was introduced by Launer (1984) where he used it to define certain new classes of life distributions and to provide bounds for the reliability function for certain specified class of distributions. Gupta et al. (1987) proved that

$$V(x) = \frac{2}{R(x)} \int_x^\infty r(t)\, R(t)dt - r^2(x) \qquad (1.10)$$

and

$$\frac{dV(x)}{dx} = h(x)\ [V^2(x)-r^2(x)]. \tag{1.11}$$

He showed that the increasing (decreasing) VRL distributions have close relationship with increasing (decreasing) MRL models, but the former provides a more general class of distributions in comparison to many other criteria for discriminating life distributions. Several bounds that improve upon those given in Launer (1984) on the moments and reliability function are also developed. In another investigation, Gupta (1987) considers the residual life distribution and compares the behaviour of the reliability concepts of the original distribution with those of the former. Further the monotonic properties of the VRL are also characterized in terms of the residual coefficient of variation.

1.2.1.5. Memory

It is well known that a continuous non-negative random variable X possesses the lack of memory property (LMP) if the relationship

$$P[X>x+y|X>y] = P[X>x] \tag{1.12}$$

holds for all real $x,y > 0$,

which is a characteristic property of the exponential distribution. Galambos and Kotz (1978) establishe that (1.11) is equivalent to $r(x) = E(X)$ or to $h(x) = k$, a constant. From the definition in terms of $r(x)$ noted above, the meaning of the concept is well explained in the sense that the component possessing LMP is 'as good as new' as its MRL is the same as at age zero. But the classical definition (1.11) does not specify what one understands by the memory of a distribution. Muth (1977) defines the memory at a point x as

$$M(x) = -r'(x)$$

and for an interval (a,b) as

$$M(a,b) = (M(a)-M(b))/(b-a).$$

He classifies distributions according as they possess (a) positive memory whenever for $x > 0$, $r(o)-r(x) \geq x$ which includes perfect memory when the equality sign holds, (b) negative memory if $r(o)-r(x) < x$, and (c) lack of memory in case $r(x) = r(o)$. The global memory of a distribution on (o, ∞) is defined as

$$M = \int_0^\infty M(x)\, w(x)dx$$

where, w(x) is a weight function which Muth (1977) chooses as

$$w(x) = 2 \int_x^{\infty} R(t)dt \; / \; E(X)^2 .$$

In this way he shows that the memory can be measured by

$$M = 1 - \eta^2$$

where, η is the coefficient of variation of X and considers the nature of the distribution as of positive memory if $M > 0$, negative memory for $M < 0$ and no memory if $M = 0$. The formula for M works for the exponential distribution as in this case $\eta=1$ and $M=0$. However, in the discrete case, the counterpart of the exponential distribution is the geometric distribution for which M is not zero and therefore, a modified measure is called for. These aspects are discussed in Nair (1983).

1.2.2. Bivariate Case

The main problem in generalising the univariate concepts introduced in the previous section into higher dimensions is that it cannot be accomplished uniquely.

The definitions in the bivariate set up largely depend on how one visualises the physical situation in a manner comparable to the univariate case. As we shall see, most natural extensions of the univariate case fail to provide us a meaningful definition that takes care of the joint variation or dependency structure underlying the component variables. In the following, we shall assume that $X = (X_1, X_2)$ is a non-negative random vector admitting absolutely continuous distribution function $F(x_1, x_2)$ with respect to Lebesgue measure. The survival function of X denoted by

$$R(x_1, x_2) = P[X_1 > x_1,\ X_2 > x_2]$$

which is related to F as

$$R(x_1, x_2) = 1 - F_1(x_1) - F_2(x_2) + F(x_1, x_2)$$

where, $F_i(x_i)$ is the distribution function of X_i.
The density of X is

$$f(x_1, x_2) = \frac{\partial^2 R}{\partial x_1\, \partial x_2} .$$

where,

$$H = H(x_1,x_2) = -\log R(x_1,x_2). \qquad (1.15)$$

However, we note that from (1.15)

$$\frac{\partial H}{\partial x_1} = -\frac{1}{R} \frac{\partial R}{\partial x_1},$$

and

$$\frac{\partial^2 H}{\partial x_1 \partial x_2} = \frac{1}{R^2} \frac{\partial R}{\partial x_2} \frac{\partial R}{\partial x_1} - \frac{1}{R} \frac{\partial^2 R}{\partial x_1 \partial x_2}$$

which gives

$$a(x_1,x_2) = \frac{\partial H}{\partial x_1} \frac{\partial H}{\partial x_2} - \frac{\partial^2 H}{\partial x_1 \partial x_2}, \qquad (1.16)$$

as the differential equation connecting the failure rate and the survival function. It is easy to observe from (1.16) that being a second order partial differential equation, H and hence R need not be determined uniquely from $a(x_1,x_2)$. A simple proof of Basu's result that $a(x_1,x_2) = c$, a constant implies $R(x_1,x_2) = e^{-a_1x_1-a_2x_2}$ can be derived from (1.16).

A second approach to defining BVFR is provided by Johnson and Kotz (1975) who take it as the vector valued function,

$$h(x_1,x_2) = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right) H(x_1,x_2),$$

$$= \left(\frac{\partial H}{\partial x_1}, \frac{\partial H}{\partial x_2} \right),$$

$$= (h_1(x_1,x_2), h_2(x_1,x_2)). \qquad (1.17)$$

Observe that $h_i(x_1,x_2)$ is the analogue of the univariate failure rate in the sense

$$h_i(x_1,x_2) = \frac{-\partial R/\partial x_i}{R} .$$

When the components $h_i(x_1,x_2)$ exist and are continuous in an open set containing $R_2^+=\{(x_1,x_2)\,|\,x_i > 0,\ i=1,2\}$, by choosing a path orthogonal to the axis connecting (o,o) and (x_1,x_2) in R_2^+, we have the representation, from Galambos and Kotz (1978),

$$R(x_1,x_2) = \exp\left[- \int_o^{x_1} h_1(t_1,o)dt_1 - \int_o^{x_2} h_2(x_1,t_2)dt_2\right] \qquad (1.18)$$

or alternatively

$$R(x_1,x_2) = \exp\left[-\int_o^{x_2} h_2(o,t_2)dt_2 - \int_o^{x_1} h_1(t_1,x_2)dt_1\right], \qquad (1.19)$$

as an extension of the one-dimensional relationship (1.3). Thus the vector $h(x_1,x_2)$ uniquely determine the distribution of X through (1.18) and (1.19). Each of the components in $h(x_1,x_2)$ depends, in general, on x_1 and x_2 and to reduce this redundancy of variables in the structure of the failure rate, Shanbhag and Kotz (1987) have proposed some modifications. In the case of a non-negative continuous random vector X the modification in (1.17) is to take the vector

$$h^*(x_1,x_2) = (h_1(x_1|x_2),\ h_2(x_2)),$$

where,

$$h_1(x_1|x_2) = \frac{-\partial}{\partial x_1} \log P[X_1>x_1|X_2>x_2] \qquad (1.20)$$

and

$$h_2(x_2) = \frac{-\partial}{\partial x_2} \log P[X_2>x_2]. \qquad (1.21)$$

From (1.20) and (1.21),

$$P[X_1>x_1|X_2>x_2] = \exp\left[-\int_0^{x_1} h_1(t|x_2)dt\right]$$

and

$$P[X_2>x_2] = \exp\left[-\int_0^{x_2} h_2(t)dt\right],$$

so that

$$R(x_1,x_2) = \exp\left[-\int_0^{x_1} h_1(t|x_2)dt-\int_0^{x_2} h_2(t)dt\right]. \qquad (1.22)$$

Thus the new vector $h^*(x_1,x_2)$ also determines the distribution of X uniquely, subject to the order in which the variables are taken in the definition.

1.2.2.2. Bivariate Mean Residual Life

As a natural extension of the univariate definition in (1.5), Buchanan and Singpurwalla (1977) define the bivariate MRL function $g(x_1,x_2)$ by

$$g(x_1,x_2) = \frac{\int_0^\infty \int_0^\infty P(X_1>x_1+t_1,\ X_2>x_2+t_2)}{R(x_1,x_2)}, \quad x_i > 0,\ i = 1,2. \qquad (1.23)$$

Although $g(x_1,x_2)$ seems to be a reasonable and direct extension, nevertheless, it does not share the most essential property of the univariate MRL function, viz. that, it determines the corresponding distribution function uniquely.

A second definition for bivariate MRL function is provided in Shanbhag and Kotz (1987) and Arnold and Zahedi (1988) which proceed almost on similar lines. Let $X = (X_1,X_2)$ be a random vector on R_2 with joint distribution function $F(x_1,x_2)$ and $L = (L_1,L_2)$ be a vector of extended real numbers such that $L_i = \inf \{x | F_i(x_i) = 1\}$ where $F_i(x_i)$ is the distribution function of X_i. Further let $E(X_i^+) < \infty$, for $i=1,2$. The vector valued Borel measurable function $r(x_1,x_2)$ on R_2 defined by

$$r(x_1,x_2) = (r_1(x_1,x_2),\ r_2(x_1,x_2)), = E(X-x|X\geq x) \quad (1.24)$$

for all $x = (x_1,x_2) \in R_2$, $x_i < L_i$, $i=1,2$, such that $P(X>x) > 0$ and $X \geq x$ implies $X_i \geq x_i$, $i = 1,2$, is called the bivariate mean residual life function (BVMRLF). When (X_1,X_2) is continuous and non negative (1.24) is equal to

$$r(x_1,x_2) = E[X-x|X\geq x]. \quad (1.25)$$

In this case, the components of BVMRLF are given as

$$r_1(x_1,x_2) = E\,[X_1-x_1|X\geq x]$$

$$= \frac{1}{R(x_1,x_2)} \int_{x_1}^{\infty} R(t,x_2)dt \quad (1.26)$$

and

$$r_2(x_1,x_2) = E[X_2-x_2|X \geq x]$$

$$= \frac{1}{R(x_1,x_2)} \int_{x_2}^{\infty} R(x_1,t)dt. \qquad (1.27)$$

It is shown by the above authors that $r(x_1,x_2)$ determine the distribution of X uniquely. The unique representation of the survival function in terms of $r(x_1,x_2)$ is provided as, (Nair and Nair (1988))

$$R(x_1,x_2) = \frac{R(x_1,o)\; r_2(x_1,o)}{r_2(x_1,x_2)} \quad \exp[-\int_o^{x_2} \frac{dt}{r_2(x_1,t)}] \qquad (1.28)$$

$$= \frac{R(o,x_2)\; r_1(o,x_2)}{r_1(x_1,x_2)} \quad \exp[-\int_o^{x_1} \frac{dt}{r_1(t,x_2)}]. \qquad (1.29)$$

From the last two equations, it follows that one can arrive at the following

$$R(x_1,x_2) = \frac{r_1(o,o)\; r_2(x_1,o)}{r_1(x_1,o) r_2(x_1,x_2)} \exp[-\int_o^{x_1} \frac{dt}{r_1(t,o)} - \int_o^{x_2} \frac{dt}{r_2(x_1,t)}] \qquad (1.30)$$

$$= \frac{r_2(o,o)}{r_2(o,x_2)} \frac{r_1(o,x_2)}{r_1(x_1,x_2)} \exp[-\int_o^{x_2} \frac{dt}{r_2(o,t)} - \int_o^{x_1} \frac{dt}{r_1(t,x_2)}] \qquad (1.31)$$

The BVMRL in (1.24) and the bivariate failure rate in (1.17) are related by

$$h_i(x_1,x_2) = \frac{1+\dfrac{\partial r_i(x_1,x_2)}{\partial x_i}}{r_i(x_1,x_2)}, \quad i = 1,2. \qquad (1.32)$$

They also showed that

(i) $r_i(x_1,x_2)$ will be the same as the univariate MRL's of the component variables X_i, i=1,2, if and only if X_1 and X_2 are independent,

(ii) $r_i(x_1,x_2) = \alpha_i$; i=1,2 a constant, independent of both x_1 and x_2, if and only if X_1 and X_2 are independent and exponentially distributed,

(iii) $r_i(x_1,x_2) = b_i(x_{3-i})$, i=1,2, if and only if (X_1,X_2) follows the Gumbel's bivariate exponential distribution,
and

(iv) A necessary and sufficient condition for a vector valued function $r(x_1,x_2)$ to be a BVMRLF are

(a) $r_i(x_1,x_2) \geq 0$,

(b) $r_i(o,o) = E(X_i)$,

and hence

$$\frac{\partial r_1^*}{\partial x_1} R(x_1,x_2) + \frac{\partial R(x_1,x_2)}{\partial x_1} r_1^*(x_1,x_2) = -R(x_1,x_2) .$$

This gives

$$\frac{1 + \dfrac{\partial r_1^*(x_1,x_2)}{\partial x_1}}{r_1^*(x_1,x_2)} = \frac{-\partial \log R(x_1,x_2)}{\partial x_1} ,$$

or

$$R(x_1,x_2) = C(x_2) \exp[- \int_o^{x_1} \frac{1+\partial r_1^*/\partial t}{r_1^*(t,x_2)} dt] \tag{1.35}$$

$$= C(x_2) \frac{r_1^*(o,x_2)}{r_1^*(x_1,x_2)} \exp[- \int_o^{x_1} \frac{1}{r_1^*(t,x_2)} dt]. \tag{1.36}$$

Assuming the continuity $r_1^*(x_1,x_2)$ at zero,

$$R(o,x_2) = C(x_2) . \tag{1.37}$$

Likewise, from (1.34),

$$R(o,x_2) = \frac{r_2^*(o)}{r_2^*(x_2)} \exp[- \int_o^{x_2} \frac{dt}{r_2^*(t)}]. \tag{1.38}$$

Combining (1.36), (1.37) and (1.38),

$$R(x_1,x_2) = \frac{r_1^*(o,x_2)\ r_2^*(o)}{r_1^*(x_1,x_2)\ r_2^*(x_2)} \exp\left[-\int_o^{x_1} \frac{dt}{r_1^*(t,x_2)} - \int_o^{x_2} \frac{dt}{r_2^*(t)}\right] . \tag{1.39}$$

We notice that

$$r_1^*(x_1,x_2) = r_1(x_1,x_2)$$

and

$$r_2^*(x_2) = r_2(o,x_2)$$

so that the expression (1.39) is identical with that obtained in Nair and Nair (1989) already cited in equation (1.31).

1.3. Characterizations

The fact that a probability distribution can be uniquely determined by the failure rate, MRL function or vitality function makes it apparent that these basic concepts in reliability are tools for characterizing lifetime distributions. It is well known that for a non-negative random variable X with F(o+) = 0, the condition $h(x) = \lambda$, where λ is a constant, characterizes the exponential distribution with density function

$$f(x) = e^{-\lambda x} ; \quad x, \lambda > 0 .$$

Cox (1962) observed that the MRL function $r(x)$ is constant, for the exponential distribution. Later Guerrieri (1965) and Cundy (1966) have studied the MRL function for various distributions. They established that for a non-negative random variable X with finite mean, the MRL function $r(x) = c$, a constant, is a characterizing property of the exponential model. Shanbhag (1970) has shown that the vitality function $m(x) = x+k$, $x > 0$, with $P(X < 0) = 0$, $E(X) < \infty$ and k is constant, characterizes the exponential law. It is evident from the discussions we have had on $h(x)$ and $r(x)$ that the constancy in one of them implies and is implied by the other and therefore, all these results through independently established over looks this fact. At the same time, in terms of physical interpretation of the failure phenomenon they describe different characteristics. Reinhardt (1968) has established that for a non-negative strictly increasing function $h(x)$ with $h(A) = 0$ and $h(B) = +\infty$ for some $A < B$, $E[h(X)|X>y] = h(y)+h(b)$ for all $A < y < B$, where b is a constant, implies that $h(X)$ is exponential. Under the assumptions $\{X>y\} = \{h(X)>h(y)\}$ and taking $Z = h(X)$ and $z = h(y)$, Hamdan (1972) has proved that $E[Z|Z>z] = z+h(b)$ for all $z > 0$ is equivalent to the constancy of the MRL (see also, Swartz (1973)). Galambos and Kotz (1978) however point out the use of functions h(.) that are strictly monotone presents no new mathematical problems as always in this case $\{X>y\} = \{h(X) > h(y)\}$ and therefore theorems with monotone h(.) are reiteration of the respective linear case proved earlier. A general result in this direction given by Laurent (1974) and Vartak (1974) is the following.

Theorem 1.1.

Let $r(x) \geq 0$ be a decreasing and differentiable function for $0 \leq x < \infty$ with $r'(x) \geq -1$ and $r(o) \neq -1$. Assume that $\int_0^A \frac{1+r'(x)}{r(x)}\,dx$ is finite or infinite according as A is finite or infinite. If X is a non-negative random variable with continuous distribution function F(x) such that $r(x) = E(X-x|X>x)$, then the survival function R(x) will be given by

$$R(x) = \exp\left[-\int_0^x \frac{1+r'(y)}{r(y)}\,dy\right];\quad x \geq 0.$$

Sahobov and Geshev (1974) have considered the above problem in more general way. Let $X > 0$ be a random variable with survival function R(x). Assume that

$E(X^k)$ is finite, where $k \geq 2$ is a given integer. If

$$E[(X-x)^k | X>x] = E[X^k]$$

for $x > 0$, then $R(x) = e^{-\theta x}$, $x > 0$ and $\theta > 0$. The case when $k = 2$ was treated by several authors (see Laurent (1974), Dallas (1975)). Nagaraja (1975) has characterized the exponential distribution using the conditional variance. For a nonnegative random variable X, with $Var(X) < \infty$, the relationship $V(X|X>x) = c$, a constant, holds if and only if X is exponential.

In this connection, we observe that a unification of all these results proved independently by various authors, can be brought about under a single framework. To do so, we treat the residual life defined earlier as a random vector Y and note that it has survival function

$$S(y;x) = \frac{R(x+y)}{R(x)}, \tag{1.40}$$

for all y and x such that $R(x) > 0$. When X is distributed exponentially,

$$R(x) = \exp[-\lambda x], \quad x, \lambda > 0$$

and therefore,

$$S(y;x) = e^{-\lambda y}. \tag{1.41}$$

Conversely, when S(y;x) has the form (1.41)

$$R(x+y) = R(x)\, e^{-\lambda y}$$

so that as x tends to zero $R(y) = e^{-\lambda y}$. Therefore X is exponential as in (1.41). Thus the distribution of X and that of the residual life Y are identical in the exponential case. Whence, the corresponding expected values must be equal and most of the characterizations belong to this category. This points out to the fact that some general conclusions can be drawn if one looks at the residual life distributions as a whole, instead of probing the properties of summary measures derived from it. However, when used as tools of model identification, it is far more convenient to ascertain the behaviour of such measures as the mean, variance etc. rather than studying the whole distribution itself. The utility of the characterizations so far reviewed is to be assessed in this sense.

Most of the characterizations given above, in terms of the moments of residual life are shaped so as to cover exponential distributions. In an attempt to study the structure of moments of residual life, which they call truncated moments, Gupta and Gupta (1983) derived a recurrence relation satisfied by them. They show that in general one higher moment does not determine a distribution uniquely and that the ratio of two higher moments will be required to do so. The paper includes characterization of the exponential distribution by the property

$$\frac{\phi_2(x)}{\phi_1^2(x)} = 2$$

and the power distribution through

$$\frac{\phi_r(x)}{\phi_{r-1}^2(x)} = c,\ c \neq 2, \text{ a constant},$$

where,

$$\phi_r(x) = E[(X-x)^r | X>x] \quad r=1,2,\ldots$$

Eventhough $\phi_r(x)$ does not generally determine a distribution uniquely, in the exponential case $\phi_r(x)=k$, a constant, is a characterizing property as shown by Dallas (1981) and again by Gupta and Gupta (1983). Mukherjee and Roy (1986) have studied some special relationships between the failure rate h(x) and the MRL function r(x) that can provide unique distributions. According to them if E(X) is finite, then

$$h(x)\ r(x) = k, \text{ a constant,}$$

characterize exponential distribution when k=1, Pareto type II with

$$R(x) = (1+ x/a)^{-q}\ ; \quad q > 2$$

when $k > 1$ and finite range distribution with

$$R(x) = (1- x/R)^{\alpha},\ \alpha > 0,\ 0 \leq x \leq R$$

when $0 < k < 1$. The Pareto case is also discussed in Sullo and Rutherford (1977). Also if the coefficient variation of residual life $c(x) = k$, a constant, characterize the above class of distributions for the different ranges of values of k in the order just described.

A brief discussion of these results seems to be in order. In view of (1.6)

$$r(x)\ h(x) = 1 + \frac{dr(x)}{dx},$$

so that the condition $r(x)\ h(x) = k$ simplifies to

$$r(x) = (k-1)x+c \tag{1.42}$$

and conversely, where $c = r(o) = E(X)$. Thus the given condition can be linked with $\frac{dr(x)}{dx} = (k-1)$ which for $k \geq 1$ (≤ 1) describes the class of distributions with increasing (decreasing) MRL with the exponential distribution as the boundary. The property $r(x)h(x)=k$ can be viewed from another angle also. From (1.6) and (1.8),

$$m'(x) = h(x)\, r(x)$$

and according to the characterising property, this would mean decreasing vitality rate (positive ageing) for the finite range distributions, increasing vitality rate (negative ageing) for the Pareto distribution and a constant vitality rate (no ageing) for the exponential distribution.

From (1.42) it is evident that a linear MRL function of the form

$$r(x) = ax+b$$

holds if and only if X is exponential for $a = 0$, X is Pareto II for $a > 0$ and X is finite range distribution for $a < 0$. The result for the Pareto model has a long history, which dates back from Hangstroem (1925), the details of which are available in Arnold (1983).

The most general result in this connection seems to be given by Kotz and Shanbhag (1980) which states that the MRL will be a polynomial or a reciprocal polynomial (failure rate is a reciprocal polynomial or a polynomial) if and only if

$$G(x) = \exp[-a(x-\alpha)] \quad a > 0$$

or

$$G(x) = [1-c(x-\alpha)]^n, \; c > 0, \; n \leq -2$$

or

$$G(x) = [1+c(x-\alpha)]^r, \; c, r > 0$$

where,

$$G(x) = R(x)/R(\alpha).$$

A generalisation of these results in two dimensional random variables will be taken up in Chapter [illegible] .

The attainment of increasing MRL for the Pareto II distributions can be accomplished in a different manner also. When the distribution of X is exponential with parameter λ and the uncertainity in λ is summarised by a gamma distribution, the compound distribution so arrived is Pareto II. Morrison (1978) and Gupta (1981) have exploited this fact to arrive at a characterization by saying that the MRL is linearly increasing if and only if the distribution of λ is gamma. This compounding method via the exponential parameter allows itself an extremely useful physical interpretation in reliability analysis and the procedure can be easily extended to cover exponential distribution in higher dimensions. These aspects will be investigated in the present thesis in a subsequent chapter.

Osaki and Li (1988) established the following theorem for the gamma distribution using the relationship between the failure rate and the vitality function.

Theorem 1.2.

Let X be a non-negative continuous random variable with distribution function F(x) and mean μ; then X has a gamma distribution with F(x) as

$$F(x) = \int_0^x \frac{\alpha(\alpha t)^{\beta-1}}{\Gamma\beta} e^{-\alpha t}\, dt$$

if and only if

$$m(x) = \mu + \frac{xh(x)}{\alpha} \quad \text{for all } x \geq 0 \qquad (1.43)$$

where $\mu = \beta/\alpha$ and $m(x)$ and $h(x)$ are the vitality function and failure rate of X defined in (1.8) and (1.1) respectively.

Other than the standard distributions so far discussed, certain derived models are also of significance in reliability, especially for comparison purposes. Two major contributions in this context are those of the equilibrium distribution specified by the density function

$$f_1(x) = \frac{R(x)}{\mu} \qquad x > 0 \tag{1.44}$$

and the length biased distribution with density function

$$f_2(x) = \frac{xf(x)}{\mu} \qquad x > 0 \tag{1.45}$$

associated with a random variable X with finite mean μ. Since equilibrium distributions are not directly discussed in the present study we refer to Gupta (1976), Gupta and Kirmani (1987, 90), Gupta et al. (1987) for the literature on the subject. Length biased distributions form a particular case of what are called weighted distributions, introduced by Rao (1965) and later studied extensively by various authors in differing contexts. The important results relevant to the current investigation are presented in Chapter 2 to ensure the continuity of discussion there.

Unlike in the univariate case, the characterisations based on the multivariate concepts in reliability are of recent origin and are still in the formative stages. As remarked in section 2, there is no unique extension of univariate concepts into higher dimensions and often the choice depends upon the actual dependency relation one expects between the concerned variables. Much effort has not been spared to link the various definitions and to evaluate the relative merits of the different approaches. In the remainder of this section, we summarise a few results that have appeared in this area.

Basu (1971) has shown that no absolutely continuous bivariate distribution with constant failure rate exists, except in the special case when the marginals are independent exponentials. Precisely, he shows that for $R(x_1,x_2)$ satisfying

$$\frac{\partial^2 R}{\partial x_1 \, \partial x_2} - \lambda R(x_1,x_2) = 0 \qquad (1.46)$$

under the assumptions

$$R(x_1,o) = e^{-\lambda_1 x_1},\ R(o,x_2) = e^{-\lambda_2 x_2} \text{ and } R(o,o)=1,$$

$$R(x_1,x_2) = e^{-\lambda_1 x_1 - \lambda_2 x_2}$$ is the only solution to the above second order partial differential equation (1.46). Consequently, we cannot have an absolutely continuous bivariate exponential distribution exhibiting dependence between the constituent variables. Puri and Rubin (1974) generalised this result as follows by removing the condition on the marginals to be exponential. For a given $\beta>0$, the only absolutely continuous distribution with $a(x_1,x_2)=\beta$, is the one which are mixtures of exponential distributions with density function given by

$$f(x_1,x_2) = \beta^{-1} \int_0^\infty \int_0^\infty \exp[-u_1x_1-u_2x_2]\ G(du_1du_2)$$

for $x_1,x_2 > 0$ where the probability measure G is concentrated on the set $A = \{u_1u_2 = \beta^{-1},\ u_1,u_2 > 0\}$. Later, Marshall(1975) has shown that the distribution of (X_1,X_2) is Marshall and Olkin (1967) bivariate exponential if and only if

$$\frac{R(x_1+t,x_2+t)}{R(t,t)}$$ is both increasing and decreasing in t .

Galambos and Kotz (1978) have identified the situation when the failure rate defined in equation (1.17) is strictly constant and locally constant vectors. The only multivariate distribution for which the multivariate failure rate is a globally constant vector is the exponential model with independent exponential marginals. Also, the failure rate $h(x) = (h_1(x), h_2(x))$ where $h_i(x) = c_i(x_j)$, i,j $i \neq j$ ($h(x)$ is locally constant and continuous) if and only if the joint distribution is Gumbel's (1960) bivariate exponential specified by the survival function

$$R(x_1,x_2) = e^{-\alpha_1 x_1 - \alpha_2 x_2 - \theta x_1 x_2} \qquad (1.47)$$

$$x_1, x_2 > 0$$

$$\alpha_1, \alpha_2 > 0;\ 0 \leq \theta \leq \alpha_1 \alpha_2 .$$

Jupp and Mardia (1982) have established that every multivariate distribution whose mean exists can be determined by its MRL function. For the random vector X on X>b, the MRL $r(x) = Ax+k$ for some constant matrix A and constant vector k if and only if X can be partitioned into independent random vectors which have shifted multivariate exponential distribution. Later, Nair and Nair (1988) have shown that the vector valued MRL function in equation (1.25) is of the form $(a_1(x_2), a_2(x_1))$ if and only if the distribution is

Gumbel's bivariate exponential with parameter $\alpha_1 = a_1(0^+)^{-1}$, $\alpha_2 = a_2(0^+)^{-1}$ and $0 \leq \theta \leq \alpha_1\alpha_2$. Further, the MRL function is strictly constant if and only if the distribution is bivariate exponential with independent marginals. The result of Mukherjee and Roy (1986) has been generalized by Roy (1989) to show that the product $h_i(x)\ r_i(x) = c$; $i = 1,2$ characterizes Gumbel's bivariate exponential when c=1, bivariate Pareto with survival function

$$R(x_1,x_2) = (1+a_1x_1+a_2x_2+bx_1x_2)^{-c} \qquad (1.48)$$

$$x_1,x_2 > 0,\ a_1,a_2 > 0$$

$$0 \leq b \leq (c+1)\ a_1a_2$$

when c>1, and bivariate finite range model with

$$R(x_1,x_2) = (1-p_1x_1-p_2x_2+qx_1x_2)^{d} \qquad (1.49)$$

$$0 < x_1 < p_1^{-1}$$

$$0 < x_2 < 1-p_1x_1/p_2-qx_1$$

$$p_1,p_2,d>0,\ 1-d \leq q/p_1p_2 \leq 1$$

when $0<c<1$. A detailed discussion of the properties of the models (1.47), (1.48) and (1.49), which will be denoted respectively in the sequel by $E(\alpha_1,\alpha_2,\theta)$, $P(a_1,a_2,b,c)$, $F(p_1,p_2,q,d)$ and several new characterizations of these models will be presented in the subsequent chapters.

1.4. The Present Study

The present work is organised into six chapters. After the present introductory chapter, where we have pointed out the relevance and scope of the study along with a review of the definitions of basic concepts used in reliability analysis and the main results associated with them, the remaining chapters are addressed to some new results.

Several characterizations of specific models such as exponential, Pareto, gamma etc. based on the functional forms of the failure rate, MRL function and vitality function were presented in the previous section. A unification of many of these results is achieved in Chapter 2 where we prove a general theorem that characterizes the entire Pearson family of distribution by means of the relationship.

$$m(x) = \mu + (a_0+a_1x+a_2x^2)\ h(x)$$

where, m(x) is the vitality function, h(x) is the failure rate, $\mu = E(X)$ is the mean and a_0, a_1, a_2 are constants. An analogous result in the discrete case for the Ord family is also presented and several existing results are deduced as particular cases. The ageing patterns of equipments can be studied by comparing the structural properties of their life lengths with those from the corresponding length biased distributions. In this context, we derive certain necessary and sufficient conditions under which the members of the Pearson family are form-invariant (that is having the same form for the density) with respect to the formation of their length biased distributions. For this subclass of the Pearson family, some characterizations based on reliability concepts are established.

In chapter 3, the class of bivariate models having the property that the components of their MRL function are linear in the respective variables are derived. This extends the results for the class of distributions consisting of exponential, Pareto and finite range. The corresponding bivariate case, includes the exponential model $E(\alpha_1,\alpha_2,\theta)$, bivariate Pareto distribution $P(a_1,a_2,b,c)$ and the bivariate finite range distribution $F(p_1,p_2,q,d)$, already mentioned in equations (1.47), (1.48) and (1.49). The distributional properties along with the dependence structure among the component variables, some characterizations and certain specific reliability problems where the mode can be applied, are investigated. Various results obtained in Lindley and Singpurwalla (1986) and Nayak (1987) are deduced as special cases.

Characterization based on properties of mean, higher moments and the median of the residual life become apparent when the whole distribution of residual life is looked into. With this purpose, in chapter 4 the concepts of residual life distributions (RLD) in the bivariate set up is introduced and form of the RLD in many specific situations are examined. It is shown that for the class of distributions investigated in chapter 3, the RLD is of the same form as the parent distribution. Further, a necessary and sufficient condition under which they become invariant are investigated. The ageing behaviour when life times follow the parent distribution is inferred from that of the RLD.

The concept of bivariate variance residual life (VRL) is introduced in chapter 5, some results that relate VRL with MRL function are proved and the properties of life distributions with monotone VRL function are discussed. With a view to ascertain the significance of the VRL concept, a few theorems concerning the implication between monotone VRL and monotone failure rate (MRL function),are proved. The properties of VRL function and its relation to the conditional coefficient of variation are exploited to characterize certain probability distributions.

Chapter 6 is devoted to study of bivariate vitality functions. The definition and properties of bivariate vitality function are presented. We define the concept of local memory of bivariate distributions and examine its relationship with bivariate vitality functions. It is shown that a measure of local memory can be provided in terms of the conditional coefficient of variation. The prospect of characterizing probability distributions in a bivariate Pearson set up using a relationship between the vitality function and the failure rate that extends the result of chapter 2 is also investigated.

Chapter 2

CHARACTERIZATIONS OF THE PEARSON FAMILY OF DISTRIBUTIONS

2.1. Introduction

The normal distribution enjoyed a pivotal role in all kinds of statistical analysis till the end of the ninteenth century and most theoretical developments took place on the assumption that the population is normal or at least approximately so. However, when the interest was focussed on describing natural phenomena by finding statistical distributions that fit the data, it became apparent that the samples from many sources show characteristics that are markedly different from normal. The then prevailing practice of attributing departure from normality to errors in measurement or to imprecise methods of collection, began slowly giving way to the belief that such departures depicted certain inherent features of the population that required alternative models. By the turn of the twentieth century non-normal curves became an accepted fact and efforts were under way to generate systems of curves which include the normal only as a particular case. Of the different approaches initiated to meet this end, Karl Pearson's contribution to describe a system of distributions by the differential equations

$$\frac{1}{f(x)} \frac{df(x)}{dx} = \frac{-(x+d)}{b_0+b_1x+b_2x^2} \tag{2.1}$$

still stands out as a convenient family that includes many important probability models. Among them the normal, exponential, gamma, beta, Pareto, finite range etc. are used extensively as lifetime distributions and we have reviewed characterizations of these distributions by reliability concepts. Instead of looking at characterizing individual members of the family, in the present chapter we present some results that hold good for the entire system, and then verifying that many of the existing results can be obtained as particular cases of the general theorem.

2.2. Characterization by relation between failure rate and vitality function*

Let $(\Omega, \mathcal{F}, P)$ be a probability space and X be a random variable thereon such that the range of X is $H = (a,b)$ for some real $a < b$; where a can be $-\infty$ and

* The results in sections 2.2 and 2.3 have appeared in IEEE Trans. Rel. Vol. 39(1991) (reference 63)

and b can be $+\infty$. Assume that the distribution function $F(x)$ of X is absolutely continuous with respect to Lebesgue measure and that $f(x)$ is its density. Then the distribution of $X: \Omega \longrightarrow H$ belongs to the Pearson system, if the density function is differentiable and satisfies the equation (2.1).

Theorem 2.1.

Let $\mu = E(X) < \infty$ and $\lim_{x \to b} (b_0+b_1x+b_2x^2)xf(x) \to 0$ if $b = +\infty$ and $\lim_{x \to b} (b_0+b_1x+b_2x^2)f(x)=0$ if $b < \infty$. A necessary and sufficient condition for the distribution of X to belong to the Pearson family is that for all x in (a,b)

$$m(x) = \mu + (a_0+a_1x+a_2x^2)\ h(x) \tag{2.2}$$

Proof:

Suppose that X is a member of the Pearson family. Then, from the equation (2.1),

$$\int_x^b (b_0+b_1t+b_2t^2)\ f'(t)dt = \int_x^b (t+d)\ dR(t) \tag{2.3}$$

where, $R(x)$ is the survival function of X.

Integrating by parts and using the assumptions of the theorem, (2.3) becomes

$$-(b_o+b_1x+b_2x^2)f(x)-\int_x^b (b_1+2b_2t)\ f(t)dt$$

$$= -(x+d)\ R(x) - \int_x^b R(t)dt\ .$$

That is,

$$-f(x)(b_o+b_1x+b_2x^2)-(b_1-d)R(x) = (2b_2-1)m(x)R(x). \qquad (2.4)$$

The equation (2.4) can be written as

$$m(x) = \mu + (a_o+a_1x+a_2x^2)\ h(x),$$

in which,

$$\mu = \frac{b_1-d}{1-2a_2}$$

and

$$a_i = \frac{b_i}{1-2b_2} \qquad i = 0,1,2.$$

Conversely, if the relationship (2.2) is satisfied, then

$$\int_x^b tf(t)dt = (a_o+a_1x+a_2x^2)f(x) + \mu R(x)\ . \qquad (2.5)$$

Differentiating (2.5) with respect to x,

$$-xf(x) = (a_o+a_1x+a_2x^2)f'(x)+(a_1+2a_2x)f(x) \tag{2.6}$$

and finally,

$$f'(x) = \frac{-(x+d)\ f(x)}{b_o+b_1x+b_2x^2}, \tag{2.7}$$

which completes the proof.

Deductions.

1. For the gamma distribution,

$$f(x) = \alpha(\alpha x)^{\beta-1}\ e^{-\alpha x} / \Gamma(\beta),$$

direct calculations show that

$$a_o = 0, \quad a_1 = \alpha^{-1}, a_2 = 0 \text{ and } \mu = \beta\alpha^{-1}.$$

Accordingly,

$$m(x) = \mu + \alpha^{-1}x\ h(x),$$

which is the result of Osaki and Li (1988).

2. When X is normal $N(a,\sigma)$,

$$a_o = \sigma^2, \ a_1 = 0, \ a_2 = 0 \text{ and } \mu = a$$

so that

$$m(x) = \sigma^2 h(x) + a$$

as observed in Kotz and Shanbhag (1980).

2.3. Discrete Case

The literature on reliability analysis is heavily biased towards continuous models and the use of discrete distributions in this context has not been properly investigated or exploited. Frequently, in reliability analysis, we need to know the probability that a specific number of events will occur or to calculate the average number of events that are taking place. For example, suppose that p is the probability that light bulb will fail during the first 100 hours of service. Then on string of 25 lights, what is the probability that there will be 'n' failures during this 100 hours period. To answer this question, we have to consider a discrete model representing by probability mass function. Xekalaki (1983) points out that the discrete models are more appropriate in a variety of applied problems due to the limitations in measuring equipments and to the fact that many continuous life length distributions can be very well approximated by the corresponding discrete counterparts. Gupta (1985) has given an example of discrete random variables that occur naturally, such as the case with the time to failure in fatigue studies measured in terms of the number of cycles to failure. Moreover, in the type I censoring, the number of failured units upto a certain time period can be represented by a discrete distribution and this may be used to study the failure process of the system. These considerations have opened up a spurt in characterization of discrete models using reliability concepts. For details we refer to Xekalaki (1983), Gupta (1984), Nair and Hitha (1989), and Hitha and Nair (1989).

Let X be a random variable in the support of the set of non-negative integers with the probability mass function f(x) and the survival function $R(x) = P(X \geq x)$. Then the failure rate h(x) of X is defined (Kalbfleisch and Prentice (1980)) as

$$h(x) = \frac{f(x)}{R(x)}, \quad x = 0,1,2,... \tag{2.8}$$

Further, R(x) can be written in terms of h(x) as

$$R(x) = \prod_{t=o}^{x-1} (1-h(t)) \quad . \tag{2.9}$$

As in the continuous case, here also the failure rate h(x) uniquely determines the survival function R(x) or the distribution of X.

The discrete analogue of MRL function r(x) of X given in equation (1.4) is defined in the same fashion as

$$\begin{aligned} r(x) &= E[X-x|X>x] \\ &= \frac{\sum_{t=x+1}^{\infty} R(t)}{R(x+1)} \end{aligned} \tag{2.10}$$

and the vitality function m(x) of X is

$$m(x) = E[X|X>x] \quad . \tag{2.11}$$

Further, h(x), r(x) and m(x) are related to one another by the following identities (Hitha and Nair, (1989))

$$m(x) = x + r(x) \tag{2.12}$$

and

$$1-h(x+1) = \frac{r(x)-1}{r(x+1)} . \tag{2.13}$$

With the aid of the above definitions we establish the following characterization theorem for the family of discrete distributions described by the difference equation

$$f(x+1)-f(x) = \frac{-(x+d)\ f(x)}{b_o+b_1x+b_2x^2} \tag{2.14}$$

which is the extension of the Pearson system to the discrete case given in Ord (1967). This theorem includes the result of Osaki and Li (1988) concerning the negative binomial distribution as a particular case.

Theorem 2.2.

Let X be a random variable with support as the set of non-negative integers with finite mean. Then the distribution of X belongs to the family mentioned in (2.14) if and only if the relationship

$$m(x) = \mu + (a_o+a_1x+a_2x^2)\ h(x+1) \tag{2.15}$$

where, $\mu = E(X)$, is satisfied for all non-negative integer values of X.

Proof:

We first prove the necessary part of the theorem. When(2.15) holds

$$\sum_{x+1}^{n} tf(t) = (a_o+a_1x+a_2x^2)\ f(x+1)+\ \mu R(x+1), \qquad (2.16)$$

where n > x+1, can be finite or infinite. The last equation reduces to

$$(x+1)R(x+1)+\sum_{x+1}^{n} R(t+1) = (a_o+a_1x+a_2x^2)f(x+1) + \mu R(x+1) \cdot \qquad (2.17)$$

Now, changing the variable x to (x-1), in equation (2.17) and subtracting (2.17) from it, we get

$$(f(x+1)-f(x))(a_o+a_1x+a_2x^2) = ((1+2a_2)x+a_1-a_2-\mu)f(x) \cdot \qquad (2.18)$$

On simplification, the equation (2.18) leads to (2.14) with the constants b_o, b_1, b_2 and d as

$$b_i = \frac{a_i}{1+2a_2} \qquad i = 0,1,2$$

and

$$d = \frac{a_1 - a_2 - \mu}{1 + 2a_2} \quad .$$

The converse part of the Theorem follows by retracing the above steps.

Corollary 2.1.

Taking $a_o = (1-r)p^{-1}$, $a_1 = p^{-1}$, $a_2 = 0$ and $\mu = rp^{-1}$ we find that

$$m(x) = \frac{r}{p} + \frac{[(1-r)+x]}{p}\, h(x)$$

characterizes the negative binomial distribution with probability mass function

$$f(x) = \binom{x-1}{r-1} p^r (1-p)^{n-r} \qquad x \geq r$$

as proved in Osaki and Li (1988).

The result of the last two theorems are operational in a practical situation once we know the value of d, a_o, a_1 and a_2 for various members of the respective families. The a_i's are related to the b_i's inthe systems and expressions for the latter in terms of the moments of the distribution are well known (see Johnson and Kotz (1969) and Ord (1967)). For easy reference, however, we present in Tables 1 and 2 the values of b_i for some popular models that are members of the two families.

Table 2.1.

Values a_o, a_1, a_2, μ for some continuous distributions

Model	a_o	a_1	a_2	μ
Gamma $\frac{\alpha^{\beta} e^{-\alpha x} x^{\beta-1}}{\Gamma\beta}$ (exponential for $\beta=1$)	0	α^{-1}	0	$\beta\alpha^{-1}$
Beta $\frac{x^{p-1}(1-x)^{q-1}}{B(p,q)}$	0	$(p+q)^{-1}$	$-(p+q)^{-1}$	$p(p+q)^{-1}$
Lomax $c\alpha^{c}(x+\alpha)^{-(c+1)}$	0	$\alpha(c-1)^{-1}$	$(c-1)^{-1}$	$\alpha(c-1)^{-1}$
Pareto $ak^{a}x^{-a-1}, x\geq k$	0	$-k(a-1)^{-1}$	$(a-1)^{-1}$	$ak(a-1)^{-1}$
Finite range $\frac{d}{R}(1-\frac{x}{R})^{d-1}; 0<x<R$	0	$R(d+1)^{-1}$	$-(d+1)^{-1}$	$R(d+1)^{-1}$
Normal $\frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$	σ^2	0	0	μ
Student's t $\frac{1}{\sqrt{v}\, B(\frac{v}{2},\frac{v}{2})} (1+\frac{x^2}{v})^{-(\frac{v+1}{2})}$	$v(v-1)^{-1}$	0	$(v-1)^{-1}$	0

Table 2.2.

Values of a_o, a_1, a_2 and μ for discrete distributions

Model	a_o	a_1	a_2	μ
Poisson $\frac{e^{-p}p^x}{x!}$	1	1	0	p
Binomial $\binom{n}{x}p^x q^{n-x}$	q	q	0	np
Negative Binomial $\binom{x-1}{r-1}p^r q^{x-r}$ (Geometric r=1)	$(1-r)p^{-1}$	p^{-1}	0	np^{-1}
Hyper geometric $\frac{\binom{D}{x}\binom{N-D}{n-x}}{\binom{N}{n}}$	$(N-D-n+1)N^{-1}$	$(N-D-n+2)N^{-1}$	N^{-1}	nDN^{-1}
Waring $\frac{(a-b)(b)_x}{(a)_{x+1}}$	$(a+1)(a-b-1)^{-1}$	$(a+2)(a-b-1)^{-1}$	$(a-b-1)^{-1}$	$b(a-b-1)^{-1}$
Negative hyper geometric $\frac{\binom{M}{a-1}\binom{N-M}{x+1-a}}{\binom{N}{x}}\frac{M-a+1}{N-x}$	$\frac{N(1-a)}{M+1}$	$\frac{N+a-1}{M+1}$	$\frac{-1}{M+1}$	$\frac{a(N+1)}{M+1}$
Beta Pascal $\left(\frac{A}{A+k}\right)\frac{\binom{k+x-1}{x}\binom{A+B-1}{A}}{\binom{k+A+B+x-1}{k+A}}$	$\frac{A+B+k}{A-1}$	$\frac{A+B+k+1}{A-1}$	$\frac{1}{A-1}$	$\frac{AB}{A-1}$

It appears to be in order to observe that the above theorems are of importance in reliability modelling in consideration of the following aspects.

1. Many distributions belonging to the two families such as gamma, beta, normal, hypergeometric and binomial do not have simple closed form expressions either for the failure rate or for the MRL function to extract a useful identity connecting the two. The theorems provides such expressions.

2. Extra flexibility is imparted in the choice of the model, as one can use the system as the basis of the model and then select a particular member depending on the values of the constants d, a_o, a_1, a_2 dictated by physical considerations or empirical evidence.

3. Not only many models that are extensively used in reliability analysis belong to the systems, but as an inherent property of the system, their truncated versions also are members. This helps when the data is truncated from the left or right.

2.4 Length biased models

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X: \Omega \rightarrow H$ be a random variable where $H = (a,b)$ is the subset of the real line with $a \geq 0$ and $b > a$ can be finite or infinite. The distribution function $F(x)$ is assumed to be absolutely continuous with respect to Lebesgue measure with probability density function $f(x)$ and $w(x)$ is a non-negative function of X such that $\mu = Ew(X) < \infty$. The random variable Y with probability density function

$$g(x) = \frac{w(x)\, f(x)}{\mu}, \quad x > 0 \tag{2.19}$$

is said to have a weighted distribution associated with X. When the counting measure is employed instead of the Lebesgue measure, the same equation (2.19) holds for the weighted distribution in the discrete case for $x = 1,2,\ldots,$ where $g(x)$ and $f(x)$ obviously are interpreted as the respective probability mass functions.

While, different weight functions such as $x^{\alpha} (\alpha > 0)$, $e^{\alpha x}$ etc in the continuous case and again x^{α}, $1-(1-\alpha)^{x}$ $(0<\alpha<1)$, $x+1$, t^{x} etc. when X is discrete have been used by various researchers, the simplest and most extensively studied form appears to be x. In this situation (2.19) specialises to

$$g(x) = \frac{xf(x)}{\mu}, \quad x > 0 \tag{2.20}$$

and $\mu = E(X)$. The version (2.20), often called the length biased distribution corresponding to X, will be the focus of attention in the rest of this chapter.

Although Rao (1965) introduced distributions of type (2.19) and cited practical examples where $w(x) = x$ or x^{α} are appropriate, the form (2.20) has found a place much earlier in the discussions given in Cox (1962) relating to renewal theory. Instead of the usual practice in random sampling of selecting units from the population, with probability of selection of each unit the same, regardless of the values of x it carries, Cox (1962) perceived the idea that from a population of failure times distributed according to f(x), the selection of any unit in the population is proportional to its length (or size), the random variable Y which is the failure time of the component whose life falls in the sample, has probability density function (2.20). This explains the terminology length biased distribution to such a model. It seems however that the same idea has originally been conceived much before as evidenced from Daniels (1942) who discusses length biased sampling in the analysis of the distribution of fibre lengths in wool. Practical problems where length biased models arise in a natural way include the analysis of a family size and in the study of albinism (Rao, 1965), family data in human heridity (Neel and Schull, 1966),aerial survey and visibility bias (Cook and Martin, 1974), forest disease and line transcend sampling (Patil and Rao, 1977), renewal theory (Cox, 1962), cell cycle analysis and pulse labelling (Takahashi, 1966) and efficacy of family screening for disease (Zelen, 1974). An exhaustive account of the research in this area is available in Patil and Rao (1977) and in Gupta and Kirmani (1990).

There are many situations when an investigator collects observations from real world phenomena and such data may not reproduce the original distribution believed to be true. Since the characteristics of the original distribution are the object of inference, one has to look into the structural relationship existing between the original model and the one that is realised in practice. This is especially important when length biased sampling is resorted to in drawing observations from the population. The paper by Gupta and Kirmani (1990) address this question and they develop several relationships that are of relevance to reliability analysis concerning the random variables X and Y. If $G(y)$, $k(y)$ and $s(y)$ represent respectively the survival function, failure rate and MRL of Y, they show that

$$G(x) = \frac{m(x)}{\mu} R(x) \qquad (2.21)$$

$$k(x) = \frac{x}{m(x)} h(x) \qquad (2.22)$$

and

$$s(x) = \frac{r(x)}{m(x)} \int_x^{\infty} \frac{t+r(t)}{r(t)} \exp\left[- \int_x^{t} \frac{du}{r(u)} \right] dt. \qquad (2.23)$$

The above identities along with some characterization theorems cited in Gupta and Kirmani (1990) show how length biased sampling affects the original distribution and how the corresponding reliability characteristics change under such a scheme of sampling. While comparing the distribution under length biased sampling with the parent model, it will be of some definite advantage if the original distribution keeps the same form under length biased sampling also, except possibly for a change in the parameters. In the next section, we prove a general theorem in this direction by identifying those distributions of X belonging to the Pearson family that retain the same form of the distribution of Y. Since the parameters of the distribution can often be interpreted in terms of the population characteristics this would mean that, a theorem of this kind provides a tool to ascertain the changes in such characteristics as a result of length biased sampling.

2.4.1. A Closure Property

When the distribution of Y is of the same form as that of X, we say that the distribution of X is closed with respect to length biased sampling. Retaining the notations of Section 2.2, we investigate the conditions under which the family in equation (2.1) induce the closure property. A major distinction made in the present section from the previous discussion of the Pearson family is that, the discussion is confined now only to distributions of non-negative random variables.

Theorem 2.3.

Among the members of (2.1) with $b_2 \neq 1$, in the support of the real line having the subset (a,b), $a \geq 0$ and $b > a$ can be finite or infinite, X and Y have the same type of distribution if and only if $b_0 = 0$ and the probability density function of Y satisfies

$$\frac{1}{g(x)}\frac{dg}{dx} = \frac{-(x+d_1)}{c_1x+c_2x^2} \qquad (2.24)$$

where,

$$c_i = \frac{b_i}{1-b_2} \qquad i = 1,2.$$

and

$$d_1 = \frac{d-b_1}{1-b_2} \ .$$

Proof.

From (2.20), we have

$$\frac{1}{g(x)} \frac{dg(x)}{dx} = \frac{1}{f(x)} \frac{df(x)}{dx} + \frac{1}{x} \ . \qquad (2.25)$$

First, suppose that X belongs to the family (2.1) and that X and Y have the same type of distributions. Since Y also must belong to the Pearson family, the equation (2.25) leads to

$$-\frac{(x+d_1)}{c_o+c_1x+c_2x^2} = \frac{1}{x} - \frac{x+d}{b_o+b_1x+b_2x^2}$$

$$= \frac{b_o+(b_1-d)x+(b_2-1)x^2}{x(b_o+b_1x+b_2x^2)} \ .$$

This identity is satisfied if and only if the following conditions hold.

$$c_o b_o = 0. \qquad (2.26)$$

$$b_2 = c_2(1-b_2) . \qquad (2.27)$$

$$(d-b_1)c_o = b_o(c_1+d_1) . \qquad (2.28)$$

$$b_o+b_1d_1 = c_o(1-b_2)+(d-b_1)c_1-b_oc_2 . \qquad (2.29)$$

$$b_1+b_2d_1 = c_1(1-b_2)+(d-b_1)c_2 . \qquad (2.30)$$

The three different possibilities arising out of condition (2.26) are

(i) $b_o \neq 0$ and $c_o = 0$

(ii) $b_o = 0$ and $c_o \neq 0$

(iii) $b_o = 0$ and $c_o = 0$.

When $b_o \neq 0$ and $c_o = 0$, from (2.28)

$$c_1 = -d_1 = b_o/d(1-b_2),$$

while,

$$b_o(1+c_2)+dd_1 = 0,$$

from (2.29) along with (2.27), leave the equation,

$$b_o-b_1d + b_2d^2 = 0.$$

Thus $(x+d)$ is a factor of $(b_o+b_1x+b_2x^2)$ and therefore from (2.1)

$$\frac{1}{f(x)} \frac{df(x)}{dx} = \frac{-1}{b_2(x+\alpha)} , \qquad (2.31)$$

where,

$$\alpha = \frac{b_1 - b_2 d}{b_2} .$$

Thus the distribution of X will be of the form

$$f(x) = c\ \alpha^c (x+\alpha)^{-(c+1)} \qquad (2.32)$$

with

$$c = \frac{1-b_2}{b_2} .$$

For $c > 0$, $\alpha > 0$ (2.32) represents the Pareto II model. From (2.24) we find in this case that,

$$g(x) = c(c-1)\alpha^{c-1}\ x(x+\alpha)^{-(c+1)} ,$$

which does not have the same form of density as X.

In the second case, when $b_o = 0$ and $c_o \neq 0$, we have a situation parallel to case (i) and the densities of X and Y do not have identical forms. Thus the first two cases lead to inadmissible solutions. This leaves us to examine the third possibility $c_o = 0$ and $b_o = 0$.

These values when inserted to equations (2.29) and (2.30) result in

$$b_1 d_1 = (d-b_1)c_1$$

and

$$b_1+b_2 d_1 = c_1(1-b_2)+(d-b_1)c_2 \quad .$$

Solving the last two equations for c_1 and d_1 after using (2.27) in (2.31), the resulting values are

$$d_1 = \frac{d-b_1}{1-b_2} \tag{2.33}$$

and

$$c_i = \frac{b_i}{1-b_2}\ , \ i = 1,2. \tag{2.34}$$

The various models generated by the above two equations depend on the nature of the roots of the equations $c_1x + c_2x^2 = 0$ and $b_1x + b_2x^2 = 0$. The roots of the first equation are 0 and $-c_1/c_2$, while that of the second are 0 and $-b_1/b_2$. However, since $c_1/c_2 = b_1/b_2$ from (2.34), the two roots of these equations have identical nature and therefore X and Y have the same type of distribution with possibly different parameters. The change in parameters are governed by equation (2.33) and (2.34).

Conversely, suppose that $b_o = 0$ in (2.1). Then from (2.24) and (2.25)

$$\frac{1}{f(x)} \frac{df(x)}{dx} = - \frac{x+d_1}{c_1 x + c_2 x^2} - \frac{1}{x}$$

$$= - \frac{(1+c_2)x + d_1 + c_1}{c_1 x + c_2 x^2}$$

$$= - \frac{x+d}{b_1 x + b_2 x^2} \qquad (2.35)$$

where,

$$d = \frac{d_1 + c_1}{1+c_2} \quad \text{and} \quad b_i = \frac{c_i}{1+c_2}, \; i = 1,2.$$

This completes the proof.

The idea of form-invariant length biased distributions have been discussed earlier in an investigation by Patil and Ord who shows that a necessary and sufficient condition for a distribution to be closed with respect to formation of length biased distribution is that it must belong to the log exponential family. The result in Theorem 2.3 does not provide any new model that is not presented in the log exponential family. The major difference between the two investigation is that our study is confined to the Pearson family and utilises a different approach with application pointed towards reliability analysis. The result will be used in the next section to characterize some important distributions belonging to the sub-class of the Pearson family defined in the differential equation (2.35). This sub-class, to be denoted by C, contains the beta distribution of the first kind and second kind and their translations (by the transformation $Z = \alpha X$), the gamma, the inverted gamma and the Pareto type I. The exponential, Pareto type II and finite range models discussed earlier have no form-invariant structure on their own, but when regarded as special cases of the above mentioned families, the same property can be attributed to them.

2.4.2. Characterization of form-invariant length biased distributions by reliability concepts.

In the light of Theorem 2.1 reflecting the relationship between vitality function and failure rate for the Pearson family, it is possible to achieve several characterizations of the class of models in C.

Theorem 2.4.

If $\lim_{x \to b} (b_1x+b_2x^2)\ f(x) = 0$, the probability density function f(x) belongs to C if and only if

$$k(x) = \frac{xh(x)}{\mu+(a_1x+a_2x^2)h(x)} \tag{2.36}$$

where,

$$a_i = b_i/1-2b_i, \quad i = 1,2.$$

Proof:

When X belongs to C, from the Theorem 2.1, m(x) and h(x) can be related as

$$m(x) = \mu + (a_1x + a_2x^2)h(x)\,. \tag{2.37}$$

From (2.22) and (2.37) we recover (2.36). The only if part follows from the equations (2.22) and (2.37) and the Theorem 2.1.

Theorem 2.5.

Under the conditions of Theorem 2.4, f(x) belongs to C if and only if

$$v(x)-m_2 = \frac{(m(x)-\mu)(1-2b_2)x}{m(x)(1-3b_2)} \tag{2.38}$$

where,

$$v(x) = E(Y|Y>x) \text{ and } m_2 = E(Y).$$

Proof:

Suppose that f(x) belongs to C. Then from (2.37)

$$m(x)-\mu = (a_1x+a_2x^2)h(x) \cdot \qquad (2.39)$$

And similarly,

$$v(x)-m_2 = (q_1x + q_2x^2)k(x) \qquad (2.40)$$

where,

$$q_i = b_i/(1-2b_2)$$

$$= a_i/(1-3a_2),\ i=1,2.$$

Eliminating k(x) and h(x) using the equations (2.39) and (2.40) we obtain (2.38). The only if part results from retracing the above steps.

There is an elegant relationship that characterize the Pareto I law among the class of all absolutely continuous distributions with non-negative support. In the following, we denote that MRL function of X by r(x) and that of Y by s(x).

Theorem 2.6.

For a continuous non-negative random variable with $E(X^2) < \infty$, $s(x) = kr(x)$ for $k>1$ if and only if X has probability density function

$$f(x) = ak^a x^{-(a+1)} \quad x \geq k > 0 \cdot \tag{2.41}$$

Proof:

We find from Gupta and Kirmani (1990) that

$$k(x) = xh(x)/(x+r(x))$$

and hence

$$\frac{1+s'(x)}{s(x)} = \frac{x(1+r'(x))}{r(x)[r(x)+x]} \tag{2.42}$$

where, the primes denote differentiation.

Substituting $s(x) = kr(x)$ in the equation (2.42), we get,

$$r(x) + kr'(x)\, r(x) = x(k-1).$$

Accordingly $r(x)$ must be linear and the only solution is $r(x) = x/(a-1)$ with $a = (2k-1)/(k-1)$. In the Pareto case, $r(x) = x/a-1$ and $s(x) = x/a-2$, and therefore, the condition of Theorem 2.6 is verified.

Corollary:

For a continuous non-negative random variable with $E(X) < \infty$, $k(x)\ r(x) = 1$ if and only if X has probability density function (2.41).

2.5. Discrete Length Biased Distribution

Analogous to the continuous case, the length biased distribution of a discrete random variable X with the set of non-negative integers as the support is defined as (Gupta, 1979),

$$g(x) = \frac{xf(x)}{\mu} \quad x = 1, 2, \ldots \tag{2.43}$$

where, $\mu = E(X) < \infty$. Clearly, the above random variable, Y will have no zero in its support. Applying a displacement of Y to the left, by taking $Z = Y-1$, Z would be realized by length biased sampling on X with the above dispacement and the support becomes the set of non-negative integers (see Patil and Ord, 1976). The resulting probability mass function of Z is

$$p(x) = g(x+1) \text{ for } x = 0, 1, 2, \ldots$$

With the notations of Section 2.3, we investigate the conditions under which the family in equation (2.14) induce the closure property.

Theorem 2.7.

Among the members of the family (2.14) with $b_2 \neq 1$, in the support of non-negative integers, X and Z have the same type of distribution if and only if either

1) $b_o = d$ and the distribution of Z satisfies

$$\frac{g(x+1)-g(x)}{g(x)} = -\frac{x+d_1}{c_o+c_1x+c_2x^2}, \tag{2.44}$$

with

$$d_1 = \frac{1+d-b_1}{1-b_2} \quad \text{and} \quad c_i = \frac{b_i}{1-b_2}, \; i=0,1,2.;$$

or

2) $b_1 = b_o+b_2$ and the distribution of Z satisfies

$$\frac{g(x+1)-g(x)}{g(x)} = -\frac{(x+d_1)}{x(c_1+c_2x)} \tag{2.45}$$

with

$$c_1 = b_o/1-b_2, \quad c_2 = b_2/1-b_2 \text{ and } d_1 = \frac{d-b_o}{1-b_2}.$$

Proof:

From the definition,

$$\frac{g(x+1)-g(x)}{g(x)} = \frac{(x+1)\ f(x+1) - xf(x)}{xf(x)} \tag{2.46}$$

First, the necessary part will be proved. For this, suppose that Z satisfies

$$\frac{g(x+1)-g(x)}{g(x)} = -\frac{x+d_1}{c_o+c_1x+c_2x^2} \; . \tag{2.47}$$

From (2.14) and (2.47), using (2.46), we have

$$-\frac{(x+d_1)}{c_o+c_1x+c_2x^2} = \frac{(b_o-d)+(b_1-d-1)x+(b_2-1)x^2}{(b_o+b_1x+b_2x^2)x} \, .$$

That is,

$$-(x^2+dx)(b_o+b_1x+b_2x^2)=[(b_2-1)x^2+(b_1-1-d)x+(b_o-d)] (c_o+c_1x+c_2x^2).$$

Equating coefficients on either side, the following conditions hold.

$$b_2 = c_2(1-b_2) \, . \tag{2.48}$$

$$c_2(b_1-1-d)+c_1(b_2-1) = -b_2d_1-b_1 \, . \tag{2.49}$$

$$c_2(b_o-d)+c_1(b_1-1-d)+c_o(b_2-1) = -b_o-b_1d_1 \, . \tag{2.50}$$

$$c_1(b_o-d)+c_o(b_1-1-d) = -b_od_1 \, . \tag{2.51}$$

$$c_o(b_o-d) = 0. \tag{2.52}$$

The three different possibilities arising out of the last equation are

i) $b_o = d$ and $c_o \neq 0$,

ii) $b_o \neq d$ and $c_o = 0$,

and

iii) $b_o = d$ and $c_o = 0$.

When $b_o = d$ and $c_o \neq 0$,

$$\frac{f(x+1)-f(x)}{f(x)} = -\frac{x+d}{d+b_1x+b_2x^2}$$

and

$$\frac{g(x+1)-g(x)}{g(x)} = \frac{(b_2-1)x+b_1-1-d}{b_o+b_1x+b_2x^2}$$

$$= -\frac{x+d_1}{c_o+c_1x+c_2x^2}$$

where,

$$c_i = b_i/1-b_2, \quad i = 0,1,2.$$

and

$$d_1 = (1+d-b_1)/(1-b_2).$$

The roots of $c_o+c_1x+c_2x^2 = 0$ are,

$$-\frac{c_1 \pm\sqrt{c_1^2-4c_oc_2}}{2c_2} = -\frac{b_1 \pm\sqrt{b_1^2-4b_ob_2}}{2b_2}$$, equal to the

roots of $b_o+b_1x+b_2x^2 = 0$ by virtue of the relationship mentioned above between c_i and b_i and the two equations therefore produce roots of identical nature. Hence X and Z have form-invariant distributions.

In the second case, substituting $c_o=0$ in equations (2.48), (2.49) and (2.51), the following equalities result.

$$c_1 = \frac{b_o(b_2d+b_2-b_1)}{(b_2d-b_o)(1-b_2)} .$$

$$c_2 = b_2/(1-b_2) .$$

$$d_1 = \frac{(b_2d+b_2-b_1)(d-b_o)}{(b_2d-b_o)(1-b_2)} .$$

Substituting these values in the equation (2.48) we have,

$$(b_1-b_o-b_2)(b_2d^2-b_1d-b_1d+b_o) = 0 \cdot \tag{2.53}$$

Thus, whenever $b_1-b_o-b_2 \neq 0$, $(x+d)$ is a factor of $(b_o+b_1x+b_2x^2)$ and then

$$\frac{f(x+1)-f(x)}{f(x)} = - \frac{1}{b_2x+b_1-b_2d}$$

and

$$\frac{f(x+1)}{f(x)} = \frac{x+a}{x+b+1}$$

where,

$$a = \frac{(b_1-b_2d-1)}{b_2}$$

and

$$b = \frac{(b_1-b_2d-b_2)}{b_2} .$$

Solving the above equation ,

$$f(x) = k(a)_x/(b)_{x+1}$$

with k as the normalising constant, and

$$(a)_x = a(a+1)\ldots(a+x-1) .$$

When $b>a$ or $0<b_2<1$, the distribution of X is Waring with

$$f(x) = \frac{(b-a)\ (a)_x}{(b)_{x+1}} , \quad x = 0,1,2,\ldots \tag{2.54}$$

but, the distribution of Y is

$$g(x) = \frac{(b-a)(b-a-1)\ x(a)_x}{a(b)_{x+1}} \quad x = 1,2,\ldots$$

and that of Z is

$$p(x) = \frac{(b-a)(b-a-1)(x+1)(a)_{x+1}}{a(b)_{x+2}} \quad x = 0,1,2,\ldots \qquad (2.55)$$

Thus X and Z are not form-invariant. The remaining case arising out of equation (2.53) is, when $b_1=b_o+b_2$,

$$d_1 = \frac{d-b_o}{1-b_2},$$

$$c_2 = \frac{b_2}{1-b_2},$$

and

$$c_1 = \frac{b_o}{1-b_2}.$$

In this case,

$$\frac{f(x+1)-f(x)}{f(x)} = \frac{-(x+d)}{(x+1)(b_o+b_2x)} \qquad (2.56)$$

while,

$$\frac{g(x+1)-g(x)}{g(x)} = -\frac{(x+d_1)}{x(c_1+c_2x)} \qquad (2.57)$$

The distribution of Z satisfies

$$\frac{p(x+1)-p(x)}{p(x)} = \frac{-(x+d_1+1)}{(x+1)(c_1+c_2+c_2x)}. \qquad (2.58)$$

The roots of $(x+1)(b_o+b_2x)=0$ are $x=-1$ and $x = -b_o/b_2$ and those of $(x+1)(c_1'+c_2+c_2x)=0$ are $x=-1$ and

$$x = \frac{-(c_1+c_2)}{c_2} = -\frac{b_1}{b_2} .$$

Hence, when $b_1=b_o$, the distribution of X is closed with respect to the formation of the distribution of Z. On the other hand if $b_o=d$ and $c_o=0$ as in case (iii), we have again three different cases, namely

(a) $b_o = 0$ and $d_1 \neq 0$

(b) $b_o \neq 0$ and $d_1 = 0$

and (c) $b_o = 0$ and $d_1 = 0$.

Considering case (a), we must have

$$\frac{f(x+1)-f(x)}{f(x)} = -\frac{1}{b_1+b_2 x}$$

and

$$f(x) = k_1 \frac{(m)_x}{(n)_{x+1}}$$

with k_1 as the normalising constant, $m = \frac{b_1-1}{b_2}$ and $n = \frac{b_1-b_2}{b_2}$.

Notice that if $n>m$ or $0<b_2<1$, the distribution of X will be Waring with probability mass function

$$f(x) = \frac{(n-m)(m)_x}{(n)_{x+1}} , \quad x = 0,1,2,\ldots .$$

The corresponding length biased distribution, specified by

$$g(x) = \frac{(n-m)(n-m-1)}{m} \frac{x(a)_x}{(b)_{x+1}} , \quad x = 1,2\ldots$$

or

$$p(x) = \frac{(n-m)(n-m-1)}{m} \frac{(x+1)(a)_{x+1}}{(b)_{x+2}} , \quad x=0,1,2,\ldots$$

is not form-invariant.

When $b_o \neq 0$ and $d_1 = 0$, we have a parallel result that the distribution of Z is Waring, but that of X has form (2.55).

Finally, the values $b_o = 0$ and $d_1 = 0$ provide us

$$c_1 (b_1 - 1) = 0.$$

This implies either $c_1 = 0$ or $b_1 = 1$.

If $c_1 = 0$, from (2.48) and (2.49), we should have $b_1 = b_2$ or $b_1 = b_o + b_2$; the cases already discussed leading to form-invariance.

When $b_1 = 1$, from (2.48) and (2.49)

$$c_2 = b_2/(1-b_2)$$

and

$$c_1 = 1/(1-b_2).$$

In this case,

$$\frac{f(x+1)-f(x)}{f(x)} = -\frac{1}{b_2x+1}$$

and the distribution of X has the form (2.54). Accordingly, the distribution of Z is not identical with that of X.

Conversely, when b_o=d and g(x) satisfies (2.44). Then,

$$\frac{f(x+1)-f(x)}{f(x)} = -\frac{x+d}{b_2x^2+b_1x+b_o}$$

where,

$$b_i = \frac{c_i}{1+c_2}, \quad i = 0,1,2.$$

When $b_2=b_1+b_o$ and g(x) satisfies (2.45),

$$\frac{f(x+1)-f(x)}{f(x)} = -\frac{x+d}{(x+1)(b_o+b_2x)}$$

where,

$$b_o = c_1/1+c_2$$

$$b_2 = c_2/1+c_2$$

and $$d = (d_1+c_1)/(1+c_2).$$

This completes the proof.

We give two examples to illustrate the results of the theorem.

1. For the negative binomial distribution with probability mass function

$$f(x) = \binom{x-1}{r-1} p^r q^{x-r} \quad x=r,\ r+1,\ \ldots,$$

$$\frac{f(x+1)-f(x)}{f(x)} = - \frac{x+(1-r)/(1-q)}{(x/1-q)+(1-r)/(1-q)}$$

and therefore, $b_o = d = \frac{1-r}{1-q}$, $b_1 = \frac{1}{1-q}$ and $b_2 = 0$.

Thus X and Z have form-invariant and g(x) satisfies

$$\frac{g(x+1)-g(x)}{g(x)} = - \frac{x+(1-r-q)/(1-q)}{(x/1-q)+(1-r/1-q)}$$

2. When X is Poisson,

$$\frac{f(x+1)-f(x)}{f(x)} = - \frac{x+1-\lambda}{x+1} .$$

Since $b_1 = b_o + b_2$, with $b_2 = 0$, the distribution of X is closed with respect to the length biased distribution and probability mass function is given by

$$g(x+1) = \frac{e^{-\lambda} \ x}{x!} \quad x = 0,1,2,\ \ldots$$

Besides the above two models some others belonging to the family that satisfy the closure property are the binomial, the hypergeometric, the binomial beta and the beta Pascal.

2.5.1. Characterizations of discrete life distributions

The survival function G(x), corresponding to (2.43) is given by

$$G(x) = \frac{m(x-1)\,R(x)}{\mu} \tag{2.59}$$

where, m(x) is the vitality function of X. Then the survival function P(x) of Z is specified by

$$P(x) = \frac{m(x)\,R(x+1)}{\mu}\,. \tag{2.60}$$

Denoting by u(x) and e(x) the failure rate and the MRL function of Z, we have,

$$u(x) = \frac{(x+1)\,h(x+1)}{m(x)} \tag{2.61}$$

and

$$e(x-1) = \frac{\sum_{x}^{\infty} R(t+1)m(t)}{R(x+1)m(x)}\,. \tag{2.62}$$

In analogy with the continuous case, the above identities connecting reliability characteristics of X and Z can be employed in the characterization of the distribution of X.

Theorem 2.8.

The probability mass function f(x) satisfies (2.14) if and only if

$$u(x) = \frac{(x+1)h(x+1)}{\mu+(a_o+a_1x+a_2x^2)h(x+1)} \cdot \qquad (2.63)$$

Proof:

When f(x) satisfies (2.14), from Theorem 2.2

$$m(x) = \mu + (a_o+a_1x+a_2x^2)\ h(x+1) \ .$$

Substituting the above identity in the equation (2.61) we have (2.63). The only if part follows from (2.61) and (2.63) along with Theorem 2.2.

Theorem 2.9.

The probability mass function f(x) satisfies (2.14) if and only if

$$v(x)-m_2 = \frac{(x+1)(m(x)-\mu)(1-2b_2)}{m(x)\ (1-3b_2)} \qquad (2.64)$$

where,

$$v(x) = E[Y|Y>x]$$

and $$m_2 = E(Y)\,.$$

Proof:

When f(x) belongs to the family (2.14)

$$m(x)-\mu = (a_o+a_1x+a_2x^2)\ h(x+1)\,. \qquad (2.65)$$

Similarly,

$$v(x)-m_2 = (q_o+q_1x+q_2x^2)\ u(x), \qquad (2.66)$$

where,

$$q_i = b_i/1-3b_2, \quad i = 0,1,2,...$$

Dividing (2.65) by (2.66) and simplifying the resulting identity, using (2.61), we have (2.64). The converse part of the theorem follows by retracing the above steps.

As a point of departure from the family specified by Theorem 2.7, some characterization will be presented associated with distributions which are not its members. The geometric, Waring and negative hypergeometric distributions are shown to be unique models from the class of all discrete distributions with non-negative integers as support satisfying certain simple properties.

Theorem 2.10.

Let X be a non-negative integer valued random variable. Then a necessary and sufficient condition that

$$\frac{P(x-1)}{R(x)} = 1+dx \tag{2.67}$$

is that the distribution of X is

(i) geometric, G(p) with probability mass function

$$f(x) = pq^x, \ x = 0,1,2,..., \tag{2.68}$$

when $\mu d-1 = 0$

(ii) Waring, W(a,b) specified by

$$f(x) = (b-a)\ (a)_x/(b)_{x+1}, \ x = 0,1,2,... \tag{2.69}$$

when $\mu d-1 > 0$

(iii) negative hypergeometric distribution, H(k,n) with

$$f(x) = \binom{-1}{x}\binom{-k}{n-x} \Big/ \binom{-1-k}{n} \ x=0,1,2,... \tag{2.70}$$

when $\mu d-1 < 0$.

Proof:

When (2.67) holds, from the equation (2.60)

$$m(x-1) = \mu+\mu dx$$

and consequently

$$r(x-1) = (\mu d-1)x + \mu-1. \quad (2.71)$$

The MRL function of X of the form a_1+b_1x characterizes the distributions (2.68), (2.69) and (2.70) (see Nair and Hitha (1989)).

Conversely, when X is G(p) (W(a,b); H(k,n))

$$\frac{P(x-1)}{R(x)} = (1+ \frac{p}{q}x\ ),\ (1+ \frac{b-a}{a}\ x;\ 1 + \frac{k}{n+k}\ x\ )$$

This completes the proof.

Theorem 2.11.

For the random variable in Theorem 2.10 , the relationship

$$u(x) = \frac{(x+1)h(x+1)}{d(1+cx)} \quad d,c > 0 \quad (2.72)$$

holds if and only if X is G(p) (W(a,b) or H(k,n)) according as $dc-1 = 0$ ($dc-1 > 0$ or $dc-1 < 0$).

Proof:

When (2.72) holds, from the equation (2.61),

$$m(x) = d + dcx ,$$

which leads to

$$r(x) = d + (dc-1)x.$$

The remaining part of the proof will follow from Nair and Hitha (1989).

The results in sections 2.4 and 2.5 are reported in Sankaran and Nair (1992d).

Chapter 3

BIVARIATE MODELS WITH LINEAR MEAN RESIDUAL LIFE COMPONENTS

3.1. Introduction

The discussions in the previous chapter were confined to univariate life distributions and representations that enable their unique determination. In addition to these, there exist a large class of problems in reliability that necessitate the generation and use of multivariate distributions like the study of multi-component systems where a random variable has to be associated with the lifetime of each component. The system works in unision with the various components and therefore the life distributions of all of them, individually and collectively would be of interest in evaluating the performance of the system. Bivariate distributions are often singled out for detailed study in view of its special relevance to systems consisting of two components. The pre-occupation with multivariate normal distribution by most researchers of early days coupled with the widespread belief that normal law holds either exactly or approximately in many natural phenomena led to a slow progress in generating non-normal multivariate models. Consistent with this general trend, the literature on life distributions in higher dimensions is limited and even those models that have already surfaced needs detailed scrutiny.

Galambos and Kotz (1978) lists the different methods of constructing a multivariate distribution as

(a) extending a univariate system to the multivariate set up,

(b) deriving the model through the mathematical relations between the joint distribution and its marginals,

(c) postulating a multivariate form via extending the functional form of the corresponding univariate family, and

(d) extending a meaningful characterizing property of the univariate case to the multivariate case and then deriving the distribution characterized by such a property.

As already mentioned a desirable option in many modelling problem is to look out for the physical properties of the system and to extract a probability distribution consistent with them. The failure rate, MRL function etc. being the summary characteristics of the failure patterns, in the present chapter we postulate the form of such functions in developing the required models. A functional form that is simple in nature and at the same time can depict the different patterns of ageing via decreasing, increasing or constant mean residual life will serve our purpose. Motivated by the result of Kotz and Shanbhag (1980) in the univariate case that a linear MRL function or a reciprocal linear failure rate function characterizes the family consisting of the exponential, Pareto and finite range distributions, we concentrate on a generalization of this form to generate a class of corresponding bivariate distributions and examine its properties and applications in reliability analysis. The univariate exponential, Pareto and finite range models have been elaborately explored in literature in the context of characterization, reliability modelling and a variety of other applications. Extensions of these models by generalising different characteristic properties in the univariate case, may often lead to different bivariate versions. In other words the bivariate models, we are seeking, need not inherit extended versions of all the univariate characteristics. Thus it is our endeavour here to examine the nature of the characterizations brought out by the new models.

In the rest of this study an attempt is made to resolve some of these problems.

3.2. The Models

Let $X = (X_1, X_2)$ be a bivariate random vector admitting absolutely continuous distribution function with respect to Lebesgue measure in the positive octant $R_2^+ = \{(x,y) \mid x,y > 0\}$ of the two dimensional Euclidean space R_2. For $(x_1,x_2) \in R_2^+$, the MRL $r(x_1,x_2)$ defined in equation (1.24) is vector valued with components

$$r_i(x_1,x_2) = E[X_i - x_i \mid X_1 > x_1, X_2 > x_2] \ .$$

In the following theorem we identify the models that are uniquely determined by the fact that $r_i(x_1,x_2)$ is linear in x_i.

Theorem 3.1.

The random vector X defined above has an MRL function $r(x_1,x_2)$ with components of the form

$$r_i(x_1,x_2) = Ax_i + B_i(x_j) \quad i,j = 1,2,\ i \neq j, \tag{3.1}$$

where, $B_i(x_j) > 0$ for all $x_j > 0$ if and only if it is distributed as

(a) Gumbel's (1960) bivariate exponential distribution with survival function,

$$R(x_1,x_2) = e^{-\alpha_1 x_1 - \alpha_2 x_2 - \theta x_1 x_2} \qquad (3.2)$$

$$\alpha_1, \alpha_2 > 0, \; x_1, x_2 > 0$$

$$0 \leq \theta \leq \alpha_1 \alpha_2$$

when A = 0,

(b) bivariate Pareto distribution specified by

$$R(x_1,x_2) = (1+a_1x_1+a_2x_2+bx_1x_2)^{-c} \; ; \; x_1,x_2 >,0 \qquad (3.3)$$

$$a_1, a_2, \; c > 0$$

$$0 \leq b \leq (c+1)a_1a_2$$

when A > 0,

(c) bivariate finite range model with

$$R(x_1,x_2) = (1- p_1x_1 - p_2x_2 + qx_1x_2)^d ; 0 < x_1 < p_1^{-1} \qquad (3.4)$$

$$0 < x_2 < \frac{1-p_1x_1}{p_2-qx_1}$$

$$p_1, \; p_2 > 0, \; 1-d \leq qp_1^{-1}p_2^{-1} \leq 1.$$

when A < 0.

Proof:

The sufficiency part will be first established. When A = 0, (3.1) becomes,

$$r_i(x_1,x_2) = B_i(x_j),\ i,j = 1,2,\ i \neq j.$$

That this form characterizes the Gumbel's bivariate exponential distribution is proved in Nair and Nair (1988). We now turn to the other two cases. Suppose that A > 0. Using the formulas in equations (1.28) and (1.29), connecting the survival function and the components of the MRL,

$$R(x_1,x_2) = \left(1+\frac{Ax_1}{B_1(o)}\right)^{-c}\left(1+\frac{Ax_2}{B_2(x_1)}\right)^{-c} \tag{3.5}$$

and also,

$$R(x_1,x_2) = \left(1+\frac{Ax_2}{B_2(o)}\right)^{-c}\left(1+\frac{Ax_1}{B_1(x_2)}\right)^{-c} \tag{3.6}$$

where, $c = \frac{1}{A} > 0$.

On equating the expressions in (3.5) and (3.6),

$$\frac{x_1}{\mu_1} + \frac{x_2}{B_2(x_1)} + \frac{Ax_1x_2}{\mu_1 B_2(x_1)} = \frac{x_2}{\mu_2} + \frac{x_1}{B_1(x_2)} + \frac{Ax_1x_2}{\mu_2 B_1(x_2)} \tag{3.7}$$

where, $\mu_i = B_i(0+)$, $i = 1,2$.

Dividing the equation (3.7) by $x_1 x_2$ and simplifying yield,

$$-\frac{1}{\mu_2 x_1} + \frac{1}{x_1 B_2(x)} + \frac{A}{\mu_1 B_2(x_1)} = \frac{-1}{x_2 \mu_1} + \frac{1}{x_2 B_1(x_2)} + \frac{A}{\mu_2 B_1(x_2)} . \tag{3.8}$$

That the last equation holds for all $x_1, x_2 > 0$ would, however, mean that

$$\frac{A}{\mu_i B_j(x_i)} + \frac{1}{x_i B_j(x_i)} - \frac{1}{\mu_j x_i} = \Theta,$$ a constant independent of both x_1 and x_2.

Hence for i=1,

$$B_2(x_1) = \frac{(\mu_1 + A x_1)\mu_2}{\mu_1(1 + \Theta x_1 \mu_2)} . \tag{3.9}$$

Substituting (3.9) in (3.5),

$$R(x_1, x_2) = \left(1 + \frac{A x_1}{\mu_1}\right)^{-c} \left(1 + \frac{A x_2 (1 + \Theta x_1 \mu_2)\mu_1}{(\mu_1 + A x_1)\mu_2}\right)^{-c}$$

$$= (1 + a_1 x_1 + a_2 x_2 + b x_1 x_2)^{-c}$$

with $a_i = \frac{A}{\mu_i}$ $i = 1,2$, and $b = A\Theta$.

Now we derive the conditions on the parameters that render (3.3) the status of a survival function. When (3.3) is a joint survival function, $R(x_1,o)$ **and** $R(o,x_2)$ are the marginal survival functions of X_1 and X_2. This gives $a_1, a_2, c > 0$. From the condition $R(x_1,o) \geq R(x_1,x_2)$ for all x_1, x_2, we get $b \geq 0$. The probability density function corresponding to (3.3) is

$$f(x_1,x_2) = c[c(a_1+bx_2)(a_2+bx_1)+a_1a_2-b](1+a_1x_1+a_2x_2+bx_1x_2)^{-(c+2)}. \tag{3.10}$$

Since $f(o,o) \geq 0$, $b \leq (c+1)a_1a_2$.

This comples the proof of the sufficiency part when $A>0$. When $A<0$,

$$R(x_1,x_2) = \left(1-\frac{Ax_1}{B_1(o)}\right)^d \left(1-\frac{Ax_2}{B_2(x_1)}\right)^d \tag{3.11}$$

and

$$R(x_1,x_2) = \left(1-\frac{Ax_2}{B_2(o)}\right)^d \left(1-\frac{Ax_1}{B_1(x_2)}\right) \tag{3.12}$$

where,

$$d = \frac{-1}{A} > 0.$$

Identifying (3.11) and (3.12) and then simplifying the resulting equation as in the previous case renders

$$\frac{1}{x_i\mu_j} - \frac{1}{x_iB_j(x_i)} + \frac{A}{\mu_iB_j(x_i)} = \beta; \; i,j = 1,2. \; i \neq j$$

and

$$B_1(x_2) \;\; = \;\; \frac{(\mu_2-Ax_2)\mu_1}{\mu_2(1-\mu_1\beta x_2)} \;. \qquad (3.13)$$

Substituting (3.13) in (3.12), the survival function of X is

$$R(x_1,x_2) \;\; = \;\; (1-p_1x_1-p_2x_2+qx_1x_2)^d$$

where

$$p_i \;\; = \;\; \frac{A}{\mu_i} \quad i=1,2.$$

and

$$q \;\; = \;\; A\beta \;.$$

The marginal density of X_i is

$$f_i(x_i) \;\; = \;\; dp_i(1-p_ix_i)^{d-1}$$

since $f_i(\sigma) > 0$, $p_i > 0$ $i = 1,2$.

The probability density function corresponding to (3.4) is given by

$$f(x_1,x_2) = d[q-p_1p_2+d(qx_1-p_2)(qx_2-p_1)]$$
$$(1-p_1x_1-p_2x_2+qx_1x_2)^{d-2}. \quad (3.14)$$

The condition $f(o,o) \geq 0$ implies

$$1-d \leq \frac{q}{p_1p_2} .$$

Since $R(x_1,x_2) \leq R(x_1,o)$ for all $x_1,x_2 > 0$, one must have $\frac{q}{p_1p_2} \leq 1$. Conversely, the MRL functions of the Gumbel's bivariate exponential, bivariate Pareto and the bivariate finite range distributions are respectively

$$r_E(x_1,x_2) = \left(\frac{1}{\alpha_1+\theta x_2}, \frac{1}{\alpha_2+\theta x_1} \right) , \quad (3.15)$$

$$r_P(x_1,x_2) = \left(\frac{x_1}{c-1} + \frac{1+a_2x_2}{(a_1+bx_2)(c-1)}, \frac{x_2}{c-1} + \frac{1+a_1x_1}{(a_2+bx_1)(c-1)} \right) \quad (3.16)$$

and

$$r_F(x_1,x_2) = \left(-\frac{x_1}{d+1} + \frac{1-p_2x_2}{(p_1-qx_2)(d+1)}, \frac{-x_2}{d+1} + \frac{1-p_1x_1}{(d+1)(p_2-qx_1)} \right) . \quad (3.17)$$

It is easy to see that the conditions of the theorem are necessary.

Corollaries.

(i) When $\theta = 0$ in (3.2),

$$R(x_1,x_2) = e^{-\alpha_1 x_1 - \alpha_2 x_2}, \tag{3.18}$$

showing that X_1 and X_2 are independent and exponentially distributed. In this case,

$$r(x_1,x_2) = (\alpha_1^{-1}, \alpha_2^{-1}),$$

a constant vector, characterizes (3.18).

(ii) Setting $b = 0$, in (3.3)

$$R(x_1,x_2) = (1+a_1x_1+a_2x_2)^{-c}, \quad x_1,x_2>0, \; c>0 \tag{3.19}$$

which is the model obtained in Lindley and Singpurwalla (1986) under a different set of conditions. The MRL function, characteristic of the model, has form

$$r(x_1,x_2) = \left(\frac{x_1}{c-1} + \frac{1+a_2x_2}{a_1}, \; \frac{x_2}{c-1} + \frac{1+a_1x_1}{a_2} \right),$$

so that the components are linear in both x_1 and x_2 with positive coefficients.

(iii) For $b = a_1a_2$ in (3.3), X_1 and X_2 have independent Pareto II distributions with survival functions

$$R_i(x_i) = (1+a_ix_i)^{-c},\ i=1,2,$$

and i^{th} component of the MRL is linear in x_i alone.

(iv) Taking $q = 0$ in (3.4),

$$R(x_1,x_2) = (1-p_1x_1-p_2x_2)^d.$$

We have the bivariate finite range model given in Roy (1989). The components of the MRL function are again linear with negative coefficients.

(v) In the case $q=p_1p_2$, the survival function of $X=(X_1,X_2)$ is the product of survival functions of univariate finite range distributions and thus X_1 and X_2 are independent.

(vi) The bivariate failure rate defined in equation (1.17) for these models of theorem is of the form

$$h(x_1,x_2) = \left((Cx_1+D_1(x_2))^{-1},\ (Cx_2+D_2(x_1))^{-1}\right), \quad (3.20)$$

Since,

$$h_i(x_1,x_2) = \frac{1+\frac{\partial r_i}{\partial x_i}}{r_i(x_1,x_2)},$$

$$= Cx_i + D_i(x_j),$$

with

$$C = A(1+A)^{-1}$$

and

$$D_i(x_j) = B_i(x_j)\ (1+A)^{-1}. \qquad (3.21)$$

The forms of the failure rates in the different cases considered in corollaries i to v follow immediately from the transformations in (3.21).

Note:

The bivariate exponential distribution with $\alpha_1=\alpha_2=1$ and its properties are discussed in Gumbel (1960). The bivariate Pareto has appeared in a different context in Hutchinson (1979) and is a particular case of the bivariate Burr distribution cited in Johnson and Kotz (1972).

3.3. Bivariate Pareto Distribution

Inspite of the appearance of the form of the density function of the bivariate Pareto distribution described in section 3.2 in some earlier investigations as cited at the end of the last section, the distributional properties do not seem to have been discussed anywhere. Accordingly we explore this aspect in the rest of this section and list the salient properties.

(1) The marginal distribution of X_i is

$$f_i(x_i) = ca_i(1+a_ix_i)^{-(c+1)}, \quad i=1,2,$$

which is of univariate Pareto II form whose properties are discussed in Arnold (1983). In particular the mean and variance of X_i are

$$E(X_i) = \frac{1}{a_i(c-1)} \tag{3.22}$$

and

$$V(X_i) = \frac{c}{(c-1)^2(c-2)a_i^2}.$$

(2) The conditional distribution of X_i given $X_j=x_j$ $i,j=1,2,$ $i\neq j$ as

$$g_i(x_i|x_j) = \frac{[c'(a_1+bx_2)(a_2+bx_1)+a_1a_2-b]}{a_j(1+a_jx_j)^{-c-1}}(1+a_1x_1+a_2x_2+bx_1x_2)^{-c-2}.$$

Once this form of the conditional probability density function is assumed, it can be shown that X_1 is Pareto type II if and only if X_2 is of the same form (Hitha and Nair, 1991).

(3) By direct calculation from the conditional density given above,

$$E[X_i|X_j=x_j]=[a_j c(c-1)(a_i+bx_j)^2](1+a_j x_j)\,[a_j c(a_i+bx_j)+b-a_1 a_2]^{-1}. \qquad (3.23)$$

Accordingly, for $0<b\leq(c+1)a_1a_2$, the regression curves are non-linear. Specialising for $b=0$, equation (3.23) becomes,

$$E[X_i|X_j=x_j] = \frac{1+a_j x_j}{(c-1)a_i} ,$$

and the regression turns out to be linear . The two lines intersect at

$$(-(a_1+a_2)^{-1},\ c^{-1}(a_1+a_2)^{-1}),$$

which is not $(E(X_1), E(X_2))$ as in the bivariate normal case. Further, the coefficient of correlation in this situation is c^{-1} which is always positive.

Also,

$$E(X_1X_2) = \int_0^\infty \int_0^\infty x_1x_2 \; c[c(a_1+bx_2)(a_2+bx_1)+a_1a_2-b] (1+a_1x_1+a_2x_2+bx_1x_2)^{-c-2} \, dx_1dx_2,$$

$$= \frac{b-a_1a_2}{c-1} \int_0^\infty \frac{x_1}{(a_2+bx_1)^2(1+a_1x_1)^c} dx_1 + \frac{a_1c}{(c-1)} \int_0^\infty \frac{x_1}{(a_2+bx_1)(1+a_1x_1)^c} dx_1 \; .$$

From Erdelyi et.al. (1954)

$$\int_0^\infty \frac{x^{v-1}}{(a+x)^\mu (x+y)} dx = \frac{\Gamma v \, \Gamma \mu-v+\rho}{\Gamma \mu+\rho \;\; a^\mu} y^{v-\rho} F(\mu, v, \mu+\rho; \frac{1-y}{a}) \quad (3.24)$$

where, F(p,q,r;t) represent the hypergeometric function,

$$F(p,q,r;t) = \frac{\Gamma r}{\Gamma p \, \Gamma q} \sum_{n=0}^{\infty} \frac{\Gamma p+n \, \Gamma q+n}{\Gamma r+n} \frac{t^n}{n!} \; .$$

With the aid of (3.24),

$$\int_0^\infty \frac{x_1 \; dx_1}{(a_2+bx_1)^2(1+a_1x_1)^c} = \frac{F(2,2,2+c,y)}{a_1^2 \; a_2^2 \; c(c+1)}$$

and

$$\int_0^\infty \frac{x_1 dx_1}{(a_2+bx_1)(1+a_1x_1)^c} = \frac{F(1,2,c+1,y)}{a_1^2\, a_2 c(c-1)} .$$

Thus,

$$E(X_1X_2) = \frac{b-a_1a_2}{a_1^2\, a_2^2\, c(c+1)(c-1)} F(2,2,2+c,y) +$$

$$\frac{a_1 c}{(c-1)^2 c a_1^2 a_2} F(1,2,c+1,y) \cdot \qquad (3.25)$$

From (3.22) and (3.25), the coefficient of correlation between X_1 and X_2 for the general model turns out to be

$$= (c-2)c^{-1}[F(1,2,c+1,y)-1-y(c-1)(c+1)^{-1}c^{-1}F(2,2,c+2y)],$$

where, $y = \dfrac{a_1a_2-b}{a_1a_2}$.

(4) From the reliability point of view, the conditional distribution of X_i given $X_j=x_j$ is not of particular significance. It is of more interest to study the conditional distributions of X_i given $X_j>x_j$, which has the density function,

$$f_i(X_i|X_j>x_j) = ca_i(x_j)[1+a_i(x_j)x_i]^{-c-1} \qquad (3.26)$$

with, $a_i(x_j) = \frac{a_i+bx_j}{1+a_jx_j}$.

Equation (3.26) reveals that such conditional distributions again are of Pareto II form. This fact enables the characterization of the bivariate Pareto distribution (Hitha and Nair (1991)).

(5) From $h_i(x_1,x_2) = \frac{c(a_i+bx_j)}{1+a_1x_1+a_2x_2+bx_1x_2}$,

it is evident that $h_i(x_1,x_2)$ decreases on x_i and accordingly the distribution of $X=(X_1,X_2)$ is bivariate decreasing failure rate in the sense of Johnson and Kotz (1975).

Similarly, using the MRL components,

$$r_i(x_1,x_2) = (c-1)^{-1} \left(x_i + \frac{1+a_jx_j}{a_i+bx_j}\right),$$

one can see that $r_i(x_1,x_2)$ is an increasing function of x_i. Therefore X is IMRL (2) (Zahedi (1985)).

(6) The Basu's failure rate defined in equation (1.13)

$$a(x_1,x_2) = \frac{c[c(a_1+bx_2)(a_2+bx_1)+a_1a_2-b]}{(1+a_1x_1+a_2x_2+bx_1x_2)^2}.$$

(7) The distribution has modified MRL vector (equations (1.33) and (1.34))

$$\left(\frac{1+a_1x_1+a_2x_2+bx_1x_2}{(c-1)(a_1+bx_2)} , \frac{1+a_2x_2}{(c-1)a_2} \right) .$$

(8) If $(X_1^{(i)}, X_2^{(i)})$, $i=1,2,\ldots,n$ are independent and identically distributed random variables following bivariate Pareto, then $(\min X_1^{(i)}, \min X_2^{(i)})$ also has the same distribution.

(9) Generalised versions of several bivariate distributions that could be meaningful from the context of reliability such as Pareto of Mardia (1962), version of Burr model of Johnson and Kotz (1972), logistic of Satterthwaite and Hutchinson (1978) are obtainable through monotonic transformations, described in Nayak (1987), on (3.3). For example, setting $Y_i=X_i+a_i^{-1}$ will result in the generalised Mardia family specified by

$$R(y_1,y_2) = \left(1+ \frac{\theta}{a_1a_2}\right)^c [a_1y_1+a_2y_2+\theta y_1y_2-1]^{-c}, y_i \geq a_i^{-1}$$

after some obvious reparametrisations.

3.4. Physical Interpretation of the Bivariate Pareto Model

In the study of reliability of series and parallel systems assessment of their component reliabilities is an important practical problem. The component reliabilities assessed at the manufacturing stage or at the quality performance test stage are usually taken as the reliability expected of the components at all subsequent stages of its utility. In general, the environmental conditions under which a component is operating need not be the same as that in the laboratory where preliminary tests are performed to measure the reliability. Often a component may perform better or worse in an environment different from that of the test site. This brings in the problem of studying the effect of change in operating conditions in evaluating the reliability of a system.

On the ground that most studies on reliability do not take into consideration, the influence of the operating environment to the system, Lindley and Singpurwalla (1986), proposed a method of modelling the life lengths of the components of a system and derived a bivariate Pareto distribution that could accommodate such changes in the operating environment. The major assumptions used in developing the model were:

(i) the components life lengths are independent exponential variables, and

(ii) the influence of the operating conditions change the original failure rates α_1 and α_2 to $\beta\alpha_1$ and $\beta\alpha_2$, where the uncertainity in β is described by a gamma distribution.

The distribution obtained under (i) and (ii) is given in equation (3.19). In this formulation $\beta>1$, $\beta<1$ and $\beta=1$ respectively suggest a harsh, mild and same conditions of use as compared to the laboratory environments. The assumption of independence of the components made by Lindley and Singpurwalla (1986) is not always a reality as the life lengths of a system can depend on one another. Instead of assuming independent exponential laws for the component lives, Bandyopadhyay and Basu (1990) proposed a dependence structure among them by permitting the system to operate in a test environment consisting of shocks that lead to the Marshall-Olkin (1967) bivariate exponential distribution. In this way, they obtained a bivariate

Pareto law with survival function,

$$R(x_1,x_2) = (1+\alpha_1 x_1+\alpha_2 x_2+\alpha_{12}\max(x_1,x_2))^{-b}, \qquad (3.27)$$
$$x_1,x_2>0,\ \alpha_1,\alpha_2,b>0,\ \alpha_{12}\geq 0,$$

which included the Lindley-Singpurwalla model as a special case when $\alpha_{12}=0$.

Apart from making provision for dependent life lengths, the generalisation of Bandyopadhyay and Basu (1990) incorporates into the system the remarkable property of bivariate lack of memory defined as

$$P[X_1>x_1+t, X_2>x_2+t \mid X_1>t,\ X_2>t] = P[X_1>x_1, X_2>x_2]$$

inherent in the Marshall-Olkin distribution. However, in situations where the simultaneous failure of components in a two-component parallel system is not to be expected, the Marshall-Olkin model is not appropriate and an absolutely continuous bivariate exponential distribution is more realistic. Accordingly, we assume that the component life lengths X_1 and X_2 follow the Gumbel's bivariate exponential law with survival function (3.2), whose dependence structure is different from that of (3.27). In the process, the bivariate lack of memory property is sacrificed and in its place the bivariate local lack of memory defined as

$$P[X_i > t_i + s_i \mid X_i > s_i,\ X_j > t_j] = P[X_i > t_i \mid X_j > t_j],$$
$$i,j=1,2,\ i \neq j,$$

which is characteristic of distribution (3.2) (see Nair and Nair (1991)) is retained. Recalling that, when $X=(X_1,X_2)$ follows Gumbel's exponential distribution (3.2), the bivariate failure rate defined in equation (1.17) is

$$h(x_1,x_2) = (\alpha_1 + \theta x_2,\ \alpha_2 + \theta x_1). \tag{3.28}$$

The effect of the operating environment being to increase or decrease (3.28) by a quantity β, the new model is to be characterized by the failure rate,

$$h_1(x_1,x_2) = (\alpha_1\beta + \beta\theta x_2,\ \beta\alpha_2 + \beta\theta x_1),$$

which again leads to the Gumbel's exponential with survival function

$$R(x_1,x_2) = \exp[-\beta(\alpha_1 x_1+\alpha_2 x_2+\theta\ x_1 x_2)]. \tag{3.29}$$

In view of the uncertainity involved in β, we associate with β a gamma distribution with parameters m and c. After averaging (3.29) using this distribution of β, the joint survival function of X_1 and X_2 is arrived at as

$$R(x_1,x_2) = (1+a_1x_1+a_2x_2+bx_1x_2)^{-c},$$

where,

$$a_i = \alpha_i/m \quad \text{and} \quad b = \Theta/m, \; i=1,2.$$

Notice that the above survival function corresponds to the bivariate Pareto distribution analysed in the previous section and that the Lindley-Singpurwalla model is its particular case. The MRL function of the Gumbel distribution has components that are locally constant and the impact of the change in environment is that they become a linear function of the life-times. The physical interpretation we have now obtained leads some further analysis leading to the comparison of the effect of test and operating conditions on the reliability of the system.

3.5. Applications to Reliability of Series Systems

Consider a two-component series system in which the life times of the components have joint distribution

(3.2) and the environment effect is described as in the previous section. The system reliability at time t is then given by

$$R_b(t) = (1+a_1t+a_2t+bt^2)^{-c}. \qquad (3.30)$$

For such a system the failure rate is

$$h_b(t) = \frac{c(a_1+a_2+2bt)}{(1+a_1t+a_2t+bt^2)}$$

and the mean residual life is

$$r_b(t) = \frac{1+a_1t+a_2t+bt^2}{(c-1)(a_1+a_2+2bt)} .$$

The corresponding expressions for Lindley-Singpurwalla models are readily obtained by setting b = 0. Accordingly, the relative error in the reliability function as defined in Gupta and Gupta (1990) is

$$e_R(t) = \frac{R_b(t)-R_o(t)}{R_o(t)}$$

$$= \left(\frac{1+a_1t+a_2t}{1+a_1t+a_2t+bt^2}\right)^c - 1$$

which decreases from zero to -1 as a function of t.

The relative error in failure rate is

$$e_h(t) = \frac{h_b(t)-h_o(t)}{h_o(t)}$$

$$= \frac{(a_1+a_2+2bt)(1+a_1t+a_2t)}{(a_1+a_2)(1+a_1t+a_2t+bt^2)} - 1.$$

This is an increasing function of t with minimum value zero and maximum value unity. On the other hand, the error in mean residual life decreases in t from $\frac{1}{2}$ to zero, as seen from the expression,

$$e_r(t) = \frac{(a_1+a_2)(1+a_1t+a_2t+bt^2)}{(a_1+a_2+2bt)(1+a_1t+a_2t)} - 1$$

or from the relationship $[1+e_h(t)][1+e_r(t)] = 1$. The extreme values of the relative error in all the three cases do not depend on the parameters of the model.

A comparison of model (3.30) which incorporates the effect of environment and the counterpart

$$R(t) = \exp[-\alpha_1 t-\alpha_2 t- \theta t^2] \tag{3.31}$$

which corresponds to the laboratory environment (β=1) seems worthwhile. The failure rate of the former is

$$h_1(t) = \frac{c(\alpha_1+\alpha_2+ 2\theta t)}{(m+\alpha_1 t+\alpha_2 t+\theta t^2)}$$

and that of (3.31) is

$$h_2(t) = (\alpha_1+\alpha_2+ 2\theta t) \cdot$$

$h_1(t) \gtrless h_2(t)$ according as the expression

$$\theta t^2+(\alpha_1+\alpha_2)t + m-c \lessgtr 0.$$

If t_1 and t_2 are the roots of the equation obtained by setting $\theta t^2+(\alpha_1+\alpha_2)+m-c = 0$, it is easy to see that one root, say t_2, is always either negative or imaginary. The last case being of no interest, the sign of above equation depends on the other root

$$t_1 = (2\theta)^{-1}[-(\alpha_1+\alpha_2)+\sqrt{(\alpha_1+\alpha_2)^2- 4\theta(m-c)}] \cdot$$

Since $E(\beta) = c/m$, this shows that $h_1(t) \gtrless h_2(t)$ according as $E(\beta) \gtrless 1$, giving conditions for a harsher, same or milder operating environment, irrespective of the values of the parameters α_1,α_2 and θ.

These results in the last four sections are taken from Sankaran andNair (1992 b).

3.6. Bivariate Finite Range Distribution

The importance of the univariate finite range distribution in the analysis of life time data is given in Mukherjee and Islam (1983) who observe that " statistical theory does not restrict a finite upper limit to the life time of an equipment and many failure time distribution are defined over the range (o,∞), but the designed life time of equipment should only be finite".

Further, since life tests are usually censored or truncated, the observed life time of equipment varies over only a finite range. It is therefore, worthwhile to look at a generalisation of the finite range distribution in higher dimensions which is precisely the model (3.4) derived in Section 3.2. The properties of this model bears a very close resemblence to that of the bivariate Pareto distribution especially in the expressions of the various population characteristics. However, the reliability characteristics is seen to behave more or less in the opposite sense.

Properties.

1. The probability density function corresponding to (3.4) is

$$f(x_1,x_2) = d(1-p_1x_1-p_2x_2+qx_1x_2)^{d-2}[d(p_1-qx_2)(p_2-qx_1) + q-p_1p_2] \cdot$$

2. The marginal distributions of X_i are

$$f_i(x_i) = dp_i(1-p_ix_i)^{d-1} \quad d>0, \; 0<x_i<\frac{1}{p_i}, \; i=1,2$$

which is univariate finite range model.

Further,

$$E(X_i) = \frac{1}{p_i(d+1)}$$

and

$$V(X_i) = \frac{d}{(d+1)^2(d+2)p_i^2} \cdot$$

3. The conditional distribution of X_i given $X_j=x_j$ is

$$g_i(x_i|x_j) = \frac{(1-p_1x_1-p_2x_2+qx_1x_2)^{d-2}[d(p_1-qx_2)(p_2-qx_1)+q-p_1p_2]}{p_j(1-p_jx_j)^{d-1}}$$

$$i,j = 1,2 \quad i \neq j$$

which is not the finite range distribution.

4. The conditional expectation,

$$E[X_i|X_j=x_j] = [p_j d(d+1)(p_i-qx_j)^2]^{-1}(1-p_jx_j)$$
$$[p_j d(p_i-qx_j)+q-p_1p_2] \qquad (3.32)$$

and hence, the regression curves are non-linear.
When q=0, the equation (3.32) becomes,

$$E[X_i|X_j=x_j] = \frac{1-p_jx_j}{(d+1)p_i},$$

and the regression turns out to be linear. In this case, the correlation coefficient is $-\frac{1}{d}$, which is always negative.

5. The conditional distribution of X_i given $X_j>x_j$ is

$$h_i(x_i|X_j>x_j) = \frac{d(p_i-qx_j)}{(1-p_jx_j)}\left(1-\frac{x_i(p_i-qx_j)}{(1-p_jx_j)}\right)^{d-1}$$

and is of the finite range form. This fact enables the characterization of the bivariate finite range distribution (see Hitha and Nair (1991)).

6. The bivariate failure rate, defined in the equation (1.17) is

$$h(x_1,x_2) = \left(\frac{d(p_1-qx_2)}{(1-p_1x_1-p_2x_2+qx_1x_2)}, \frac{d(p_2-qx_1)}{(1-p_1x_1-p_2x_2+qx_1x_2)}\right).$$

Since the components $h_i(x_1,x_2)$ increases in x_i, the distribution (3.4) has increasing failure rate property.

7. The bivariate MRL vector, defined in the equation (1.24) is

$$r(x_1,x_2) = \left(\frac{1-p_1x_1-p_2x_2+qx_1x_2}{(1+d)(p_1-qx_2)} , \frac{1-p_1x_1-p_2x_2+qx_1x_2}{(d+1)(p_2-qx_1)} \right).$$

It is evident that $r_i(x_1,x_2)$ decreases as x_i increases, and therefore, the distribution of $X=(X,X_2)$ is having DMRL-(2) (see Zahedi (1985)).

8. The relationship,

$$h_i(x_1,x_2)\ r_i(x_1,x_2) = k < 1$$

is a characteristic property of the model.

9. The Basu's failure rate defined in (1.13) is

$$a(x_1,x_2) = \frac{d[d(p_1-qx_2)(p_2-qx_1)+q-p_1p_2]}{(1-p_1x_1-p_2x_2+qx_1x_2)^2} .$$

10. The distribution has modified MRL (equations (1.33) and (1.34))

$$\left(\frac{1-p_1x_1-p_2x_2+qx_1x_2}{(d+1)(p_1-qx_2)}, \frac{1-p_2x_2}{p_2(d+1)} \right).$$

The three different models arising from the characterization theorem of Section 3.2 from a class that enjoy several interesting properties. Various distributional properties discussed in the previous sections form the basis of our future investigation in this direction. Regarding the Gumbel's exponential distribution, since a full discussion is available in Nair (1990), we mention only such properties that are of interest to the present study at the appropriate places.

Chapter 4

BIVARIATE RESIDUAL LIFE DISTRIBUTION

4.1. Introduction

While reviewing the literature on characterization of life distributions in Section 1.3 reference was made to the notion of residual life distribution (RLD) and it was pointed out there that the basis of many characterizations can be traced to the form of the RLD. However, this point has only been recognized very recently. A comparision of the RLD and the parent distribution is informative in the study of the reliability characteristics. Some results using this approach in the univariate case have been reported in Gupta and Kirmani (1990). Since this concept does not appear to have been introduced in the bivariate case, the primary concern of the present chapter is to define bivariate RLD. The failure rate (MRL) determines a distribution uniquely and therefore, it follows that the failure rate (MRL) of the RLD will enable us to identify the form of RLD. In situations where the failure rate (MRL) of the basic distribution has the same form as the corresponding characteristic of the RLD, we can conclude that the parent distribution is form-invariant with respect to the construction of the RLD. This argument will be used in the forthcoming discussions to identify a certain class of distributions that possess the closure property just described. We also infer the structure of the basic life pattern from that of the distribution of residual life.

4.2. Definition and Examples

As in the previous chapter we suppose that $X = (X_1, X_2)$ is a random vector admitting absolutely continuous survival function $R(x_1, x_2)$ with respect to Lebesgue measure in the support of R_2^+. For $x=(x_1,x_2)$ and $y = (y_1, y_2)$ in R_2^+, the bivariate RLD is defined by the survival function

$$G(y;x) = P[X>x+y|X>x]$$

$$= \frac{R(x+y)}{R(x)} \qquad (4.1)$$

for all $y \in R_2^+$ and those values of X for which $R(x) > 0$. In the above definition the ordering $X>x$ implies $X_1>x_1$ and $X_2>x_2$.

Examples

Here, we give examples of some bivariate distributions and their residual life distributions.

1. When X follows Gumbel's exponential distribution with survival function (3.2), the bivariate RLD of X is given by

$$G(y;x) = e^{-(\alpha_1+\theta x_2)y_1-(\alpha_2+\theta x_1)y_2-\theta y_1 y_2} \qquad y_1, y_2 > 0,$$

which is again Gumbel's exponential model with parameters $(\alpha_1+ \theta x_2)$, $(\alpha_2+ \theta x_1)$ and θ.

2. For the Marshall-Olkin distribution specified by

$$R(x_1,x_2) = e^{-a_1x_1-a_2x_2-a_{12}\max(x_1,x_2)}, \qquad x_1,x_2>0;\ a_1,a_2>0,\ a_{12}\geq 0,$$

the RLD is of the form,

$$R(y;x) = e^{-a_1y_1-a_2y_2-a_{12}[\max(x_1+y_1,x_2+y_2)+\max(x_1,x_2)]}$$

3. In the case of bivariate Pareto distribution with survival function (3.3), the RLD is given by

$$G(y;x) = \left[1 + \frac{(a_1+bx_2)y_1 + (a_2+bx_1)y_2 + by_1y_2}{(1+a_1x_1+a_2x_2+bx_1x_2)}\right]^{-c}$$

which is again bivariate Pareto.

4. When the distribution of X is bivariate finite range given in (3.4), the RLD of X follows again finite range, specified by

$$G(y;x) = \left[1- \frac{(p_1-qx_2)y_1+(p_2-qx_1)y_2-qy_1y_2}{(1-p_1x_1-p_2x_2+qx_1x_2)}\right]^{d}$$

4.3. <u>Properties</u>

1. $E\,Y(x) = r(x),$ (4.2)

where, $Y(x) = (Y_1(x),Y_2(x))$ is the random variable with survival function (4.1).

Proof:

$$E\,Y_1(x) = \int_0^\infty \int_0^\infty Y_1[R(x_1,x_2)]^{-1} \frac{\partial^2 R}{\partial y_1 \partial y_2}\, dy_1 dy_2$$

$$= (R(x_1,x_2))^{-1} \int_0^\infty y_1 \frac{\partial R(x_1+y_1,x_2)}{\partial y_1} dy_1$$

$$= (R(x_1,x_2))^{-1} \int_0^\infty R(x_1+y_1,x_2)dy_1$$

$$= r_1(x).$$

Similarly,

$$E\, Y_2(x) = r_2(x).$$

Thus, the mean of the RLD, is the MRL vector we have encountered in Chapter 1.

Further, the marginal survival functions of (2.1) are given by

$$G_i(y_i;x) = \frac{R(x_i+y_i,x_j)}{R(x_1,x_2)}, \quad y_i > 0 \qquad (4.3)$$
$$i,j=1,2,\ i \neq j.$$

Since, the MRL determines the corresponding distribution uniquely, the functional forms of $E\, Y(x)$ characterize the probability law of X.

2. In the following discussion, we denote by $h(x) = (h_1(x),h_2(x))$ and $k(x) = (k_1(x), k_2(x))$, the vector valued failure rates of X and Y respectively and

by $r(x) = (r_1(x), r_2(x))$ and $s(x) = (s_1(x), s_2(x))$ the corresponding MRL's, we then have

$$k_i(y) = h_i(x+y) \qquad (4.4)$$

and

$$s_i(y) = r_i(x+y), \text{ for } i=1,2. \qquad (4.5)$$

Proof:

For i=1,

$$k_1(y_1,y_2) = \frac{-\partial \log G}{\partial y_1}$$

$$= \frac{-\partial \log R(x_1+y_1, x_2+y_2)}{\partial y_1},$$

$$= h_1(x_1+y_1, x_2+y_2).$$

Similarly, $k_2(y_1,y_2) = h_2(x_1+y_1, x_2+y_2)$.

The MRL, for i=1,

$$s_1(y_1,y_2) = G(y;x)^{-1} \int_{y_1}^{\infty} G(y;x)dy.$$

$$= [R(x+y)]^{-1} \int_{y_1}^{\infty} R(x_1+t,\ x_2+y_2)dt$$

$$= [R(x+y)]^{-1} \int_{x_1+y_1}^{\infty} R(t, x_2+y_2)dt$$

$$= r_1(x_1+y_1,\ x_2+y_2).$$

On similar lines,

$$s_2(y_1,y_2) = r_2(x_1+y_1,\ x_2+y_2).$$

From the above result, when $r_i(x)$ is linear in x_i, the MRL corresponding to Y, $s_i(x)$, is also linear in y_i. This shows that X follows the distributions defined in the equations (3.2), (3.3) and (3.4), if and only if the distribution of Y(x) is of the same form.

3. The distribution of X is bivariate IFR(1) (DFR(1)) if and only if

$$k_i(y) \geq (\leq)\ h_i(y), \qquad \text{for all } y_i>0,\ i=1,2. \tag{4.6}$$

Proof:

A bivariate random vector X is said to have IFR(1) property if and only if

$$h_i(x_1+y_1, x_2+y_2) \geq (\leq) \; h_i(y_1, y_2)$$

for all $x_i, y_i > 0$, $i = 1,2$.

Therefore, from (4.4),

$$X \text{ is IFR(1) (DFR(1))} \Longleftrightarrow k_i(y) \geq (\leq) \; h_i(y).$$

4. X is bivariate DMRL (1) (IMRL(1)) if and only if

$$s_i(y) \leq (\geq) \; r_i(y) \tag{4.7}$$

for all $y_i > 0$, $i = 1,2$.

Proof:

From Zahedi (1985),

$$X \text{ is DMRL(1) (IMRL(1))} \Longleftrightarrow r_i(x_1+y_1, x_2+y_2) \leq (\geq) \; r_i(y_1, y_2)$$

for all $x_i, y_i > 0$, $i = 1,2$.

or

$$s_i(y) \leq (\geq) \; r_i(y).$$

5. X is bivariate NBU(VS) (NWU(VS)) if and only if

$$G(y) \leq (\geq) \quad R(y) \tag{4.8}$$

for all $y_1, y_2 > 0$.

Proof:

From the definition (Buchanan and Singpurwalla (1977)),

X is NBU(VS) (NWU(VS)) $\Longleftrightarrow R(x_1+y_1, x_2+y_2) \leq (\geq) R(x_1, x_2)R(y_1, y_2)$

for all $x_i, y_i > 0$, $i=1,2$, or

$$G(y_1, y_2) \leq (\geq) R(y_1, y_2).$$

6. For $i=1,2$, $h_i(y) \leq k_i(y)$ if and only if $\frac{R(y)}{G(y)}$ is non-decreasing in y_i, for all $y_1, y_2 > 0$.

Proof:

$$h_i(y_1, y_2) - k_i(y_1, y_2) \leq 0$$

$$\Longleftrightarrow - \frac{\partial \log \frac{R(y)}{G(y)}}{\partial y_i} \leq 0$$

$$\Longleftrightarrow \frac{\partial \log \frac{R(y)}{G(y)}}{\partial y_i} \geq 0$$

$$\Longleftrightarrow \log \frac{R(y)}{G(y)} \text{ is non-decreasing in } y_i.$$

$$\Longleftrightarrow \frac{R(y)}{G(y)} \text{ is non-decreasing in } y_i$$

for all $y_1, y_2 > 0$.

7. If $s_i(y) \leq r_i(y)$ and $\frac{s_i(y)}{r_i(y)}$ is non-decreasing in y_i, then $h_i(y) \leq k_i(y)$, $i=1,2$.

Proof:

$\frac{s_i(y)}{r_i(y)}$ is non-decreasing **implies**

$\log \frac{s_i(y)}{r_i(y)}$ is non-decreasing **and therefore,**

$$\frac{s_i'(y)}{s_i(y)} - \frac{r_i'(y)}{r_i(y)} \geq 0 , \tag{4.9}$$

where, primes denote the derivative with respect to y_i. Since $s_i(y) \leq r_i(y)$ and the inequality (4.9) leads to

$$\frac{1+s_i'(y)}{s_i(y)} - \frac{1+r_i'(y)}{r_i(y)} \geq 0$$

or

$$k_i(y) \geq h_i(y).$$

8. X is IFR if and only if Y is NBU(VS)

Proof:

By definition (Buchanan and Singpurwalla (1977)) Y is

Y is NBU (VS) $\Longleftrightarrow$ $G(y_1+t_1, y_2+t_2) \leq G(y_1,y_2)\, G(t_1,t_2)$
for all $y_1, y_2, t_1, t_2 > 0$

$$\Longleftrightarrow \frac{R(x_1+y_1+t_1, x_2+y_2+t_2)}{R(x_1,x_2)} \leq \frac{R(x_1+y_1, x_2+y_2)}{R(x_1,x_2)} \frac{R(x_1+t_1, x_2+t_2)}{R(x_1,x_2)}$$

$$\Longleftrightarrow \frac{R(x_1+y_1+t_1, x_2+y_2+t_2)}{R(x_1+y_1, x_2+y_2)} \leq \frac{R(x_1+t_1, x_2+t_2)}{R(x_1,x_2)} .$$

Since the above inequality is true for all x_i, y_i and $t_i > 0$, we have

$$\frac{R(x_1+t_1, x_2+t_2)}{R(x_1,x_2)} \text{ is decreasing in } x_1, x_2.$$

Therefore, from Marshall (1975), X is IFR.

The converse part follows by retracing the above steps.

9. X is IFR if and only if Y is NBUFR.

Proof:

A bivariate random vector X is said to have bivariate new better than used in failure rate (NBUFR) property if and only if

$$h_1(x_1,o) \geq h_1(o,o)$$

and

$$h_2(o,x_2) \geq h_2(o,o) \quad \text{for all } x_1,x_2 > 0.$$

Y is NBUFR $\Longleftrightarrow$ $k_1(y_1,o) \geq k_1(o,o)$, and

$$k_2(o,y_2) \geq k_2(o,o)$$

$$\Longleftrightarrow h_1(x_1+y_1,x_2) \geq h_1(x_1,x_2), \text{ and}$$
$$h_2(x_1,x_2+y_2) \geq h_2(x_1,x_2).$$

Thus , X is IFR (Johnson and Kotz (1975)).

10. X is DMRL (1) and $\frac{s_i(y)}{r_i(y)}$ is non-decreasing in y_i, $i=1,2$ together implies X is IFR.

Proof:

X is DMRL(1) $\Longleftrightarrow$ $s_i(x) \leq r_i(x)$ from (4.7)

The inequality (4.7) and $s_i(y)/r_i(y)$ is non-decreasing in y_i together leads,

$$h_i(y) \leq k_i(y) \quad \text{from property 7 .}$$

Thus , X is IFR, from (4.6).

4.4 Characterizations by properties of RLD.

In this section, we establish certain transformations under which the RLD's of some bivariate distributions are identical with the original distributions. The result in this section is to appear in Sankaran and Nair (1992 a).

Theorem 4.1.

The necessary and sufficient condition that G(y;x) satisfies the relationship

$$G(u(x)y;\ x) \quad = \quad R(y) \tag{4.10}$$

for all $x,y > 0$, where

$$u(x) \quad = \quad \frac{r_1(x)}{r_1(o)} = \frac{r_2(x)}{r_2(o)} \tag{4.11}$$

is that X is distributed either as $P(a_1,a_2,o,c)$ or $F\ (p_1,p_2,o,d)$ or $E(\alpha_1,\alpha_2,o)$.

Proof:

Assume that condition (4.10) is satisfied. Then from (4.1) we can write,

$$R(x_1+y_1u(x),x_2+y_2u(x)) = R(y_1,y_2)R(x_1,x_2). \tag{4.12}$$

Denoting $x_1+y_1u(x) = s$, $x_2+y_2u(x) = t$, we have on differentiating (4.12),

$$\frac{\partial R}{\partial s}\left[1+y_1\frac{\partial x}{\partial x_1}\right] + \frac{\partial R}{\partial t}\, y_2\frac{\partial u}{\partial x_1} = R(y_1,y_2)\,\frac{\partial R(x_1,x_2)}{\partial x_1}, \quad (4.13)$$

$$\frac{\partial R}{\partial s}\, u(x) = \frac{\partial R(y_1,y_2)}{\partial y_1}\, R(x_1,x_2), \quad (4.14)$$

$$\frac{\partial R}{\partial s}\, y_1\frac{\partial u}{\partial x_2} + \frac{\partial R}{\partial t}\left[1+y_2\frac{\partial u}{\partial x_2}\right] = R(y_1,y_2)\frac{\partial R(x_1,x_2)}{\partial x_2}, \quad (4.15)$$

and

$$\frac{\partial R}{\partial t}\, u(x) = \frac{\partial R(y_1,y_2)}{\partial y_2}\, R(x_1,x_2) \quad (4.16)$$

for all $y_1, y_2 > 0$.

Because of the absolute continuity of R, the partial derivatives mentioned above exist.

When y_2 tends to zero in (4.13),

$$\frac{\partial R}{\partial s}\left[1+ y_1\frac{\partial u}{\partial x_1}\right] = R(y_1,o)\,\frac{\partial R(x_1,x_2)}{\partial x_1}. \quad (4.17)$$

Eliminating $\frac{\partial R}{\partial s}$ between (4.14) and (4.17), we have

$$[u(x)]^{-1}\left(1+y_1\frac{\partial u}{\partial x_1}\right) = R(y_1,o)\,\frac{\partial R}{\partial x_1}\Big/\left[R(x_1,x_2)\,\frac{\partial R}{\partial y_1}\right]$$

and whence, by making y_1 tend to zero,

$$[u(x)]^{-1} = \frac{\partial R}{\partial x_1} / [R(x_1,x_2)(\frac{\partial R}{\partial y_1})_{y_1=o}].$$

The last two equations lead to

$$1+y_1 \frac{\partial u}{\partial x_1} = R(y_1,o)(\frac{\partial R}{\partial y_1})^{-1} [(\frac{\partial R}{\partial y_1})_{y_1=o}]. \qquad (4.18)$$

The right side of (4.18) is independent of both x_1 and x_2 and therefore,

$$\frac{\partial u}{\partial x_1} = a_1, \quad \text{a constant.}$$

Similarly,

$$\frac{\partial u}{\partial x_2} = a_2.$$

The above two conditions hold good if and only if for some k,

$$u(x) = a_1x_1 + a_2x_2 + k \cdot$$

Since $u(o,o) = 1$, $k=1$.

Thus,

$$r_i(x_1,x_2) = (1+a_1x_1+a_2x_2)\ r_i(o),\ i=1,2.$$

The three admissible cases about the values of a_1 and a_2 are that either $a_1=a_2=0$, or $a_1>0$, $a_2>0$ or $a_1<0$, $a_2<0$. In the first case $r(x_1,x_2)=(r_1,r_2)$ where both r_1 and r_2 are independent of X_1 and X_2. This is true if and only if X is distributed as the product of two independent exponentials (Nair and Nair, 1988) . When both a_1 and a_2 are positive reals, we find from the equation (1.28) that

$$R(x_1,x_2) = (1+a_1x_1)^{a_2r_2-a_1r_1}\,(1+a_1x_1+a_2x_2)^{-r_2a_2}. \quad (4.19)$$

For $R(x_1,x_2)$ given above to be a proper survival function, one should have $a_2r_2 \leq a_1r_1$. Now, the roles of $r_1(x_1,x_2)$ and $r_2(x_1,x_2)$ in (4.19) can be interchanged and this leads to the condition $a_1r_1 \leq a_2r_2$ and hence $a_1r_1 = a_2r_2 = c$, with $c > 0$. Thus X follows $P(a_1,a_2,o,c)$. When $a_1,a_2 < 0$, taking $a_1 = -p_1$ and $a_2 = -p_2$, $p_1,p_2 > 0$ and repeating the above steps,

$$R(x_1,x_2) = (1-p_1x_1-p_2x_2)^d, \; d > 0$$

so that X is $F(p_1,p_2,o,d)$.

The converse part follow from examples 1,3 and 4.

As a consequence of Theorem 4.1, we get the following characterization theorem connected with the proportionality of the two components of vector valued failure rate $h(x_1,x_2)$.

Theorem 4.2.

The relationship,

$$a_2 h_1(x_1,x_2) = a_1 h(x_1,x_2) \tag{4.20}$$

is satisfied for all $x_1,x_2 > 0$ if and only if X follow one of the distributions in Theorem 4.1.

Proof:

Eliminating $\frac{\partial R}{\partial s}$ and $\frac{\partial R}{\partial t}$ from equations (4.13) through (4.16) we have for $i,j=1,2,\ i\neq j$

$$\frac{1}{R(y_1,y_2)} \frac{\partial R}{\partial y_j} \frac{R(x_1,x_2)}{u(x_1,x_2)}[1+ \frac{\partial u}{\partial x_1} y_1+ \frac{\partial u}{\partial x_2} y_2]$$

$$= \frac{\partial R}{\partial x_j}[1+ \frac{\partial u}{\partial x_i} y_i] - \frac{\partial R}{\partial x_i} \frac{\partial u}{\partial x_j} y_i$$

and whence

$$(\frac{\partial R}{\partial x_2} \frac{\partial u}{\partial x_1} - \frac{\partial R}{\partial x_1} \frac{\partial u}{\partial x_2})(\frac{\partial R}{\partial y_2} y_2 + \frac{\partial R}{\partial y_1} y_1) = 0.$$

Since the second expression in the above product non-zero, we find

$$\frac{\partial R}{\partial x_1}\frac{\partial u}{\partial x_2} = \frac{\partial R}{\partial x_2}\frac{\partial u}{\partial x_1}$$

or

$$a_2\, h_1(x_1,x_2) = a_1 h_2(x_1,x_2)\ ;$$

Since the equations (4.13), (4.14), (4.15) and (4.16) taken together characterize the three distributions given in Theorem 4.1, the relationship (4.20) is satisfied for those three distributions.

Conversely, when X follows $E(\alpha_1,\alpha_2,o)$ $(P(a_1,a_2,o,c);\ \ F(p_1,p_2,p,d))$, the ratio $\frac{h_1(x_1,x_2)}{h_2(x_1,x_2)}$ is $1(\ \frac{a_1}{a_2}\ ;\ \frac{p_1}{p_2}\ )$. **This completes the proof.**

In the following theorem we address to a more general question, by appealing to the functions $u_1(x) = \frac{r_1(x)}{r_1(o)}$ and $u_2(x) = \frac{r_2(x)}{r_2(o)}$ to provide some characterizations.

Theorem 4.3.

The RLD admits the conditions

$$G(y_i u_i(x);x) = R_i(y_i) \quad i=1,2 , \tag{4.21}$$

where, $R_i(y_i)$ is the marginal survival function of y_i, if and only if X follows one of the distributions specified in equations (3.2), (3.3) and (3.4).

Proof:

When X follows exponential, $E(\alpha_1,\alpha_2,\Theta)$, for i=1

$$u_1(x) = \alpha_1(\alpha_1 + \Theta x_2)^{-1}$$

and

$$G_1(y_1;x) = e^{-(\alpha_1+\Theta x_2)x_1} ,$$

so that

$$G_1(y_1 u_1(x);x) = e^{-\alpha_1 y_1}$$

$$= R_1(y_1) ,$$

Where as for Pareto $P(a_1,a_2,b,c)$

$$u_1(x) = \frac{(1+a_1x_1+a_2x_2+bx_1x_2)a_1}{a_1+bx_2}$$

and

$$G_1(y_1;x) = [1+(a_1+bx_2)y_1/(1+a_1x_1+a_2x_2+bx_1x_2)]^{-c}$$

giving

$$G_1(y_1 u_1(x);x) = (1+a_1y_1)^{-c}$$

$$= R_1(y_1) .$$

In the finite range case $F(p_1,p_2,q,d)$

$$u_1(x) = \frac{(1-p_1x_1-p_2x_2+qx_1x_2)p_1}{p_1-qx_2}$$

and

$$G_1(y_1;x) = [1-(p_1-qx_2)y_1/(1-p_1x_1-p_2x_2+qx_1x_2)]^d.$$

Thus,

$$G_1(y_1u_1(x);x) = (1-p_1y_1)^d$$

$$= R_1(y_1).$$

To prove the only if part, we proceed along the lines of Theorem 4.1. The equation (4.21) is equivalent, for i=1, to

$$R(x_1+y_1u_1(x),x_2) = R(x_1,x_2)\,R_1(y_1). \qquad (4.22)$$

From (4.17) and (4.14)

$$\frac{\partial R}{\partial s}\left(1+y_1\frac{\partial u_1}{\partial x_1}\right) = \frac{\partial R}{\partial x_1}R_1(y_1) \qquad (4.23)$$

or

$$\frac{\partial R}{\partial s}u_1(x_1,x_2) = R(x_1,x_2)\frac{\partial R}{\partial y_1}. \qquad (4.24)$$

Dividing the equation (4.23) by (4.24), we get

$$1+y_1 \frac{\partial u_1}{\delta x_1} = \frac{R_1(y_1)}{\frac{\delta R}{\delta y_1}(\frac{\delta R}{\delta y_1})_{y_1=0}}$$

which implies

$$\frac{\partial u_1}{\delta x_1} = a_1$$

or

$$u_1(x) = a_1 x_1 + B_1(x_2).$$

Similarly,

$$u_2(x) = a_2 x_2 + B_2(x_1).$$

Using the same arguments in Theorem 4.1, we have

$$a_1 \, r_1(o) = a_2 \, r_2(o) = \ell \, .$$

Thus the MRL function is of the form

$$r(x) = (\ell \, x_1 + m_1(x_2), \ell \, x_2 + m_2(x_1))$$

which is a characteristic property of the family of distributions mentioned in Theorem 4.3 and the proof is complete.

Corollaries:

1. Taking i=1 and allowing x_2 to tend to zero, the conditions stated in Oakes and Dasu (1990) that

characterizes the univariate exponential, Pareto and finite range distributions result.

2. $u_1(x) = u_2(x)$ in Theorem 4.3 characterizes the reduced models of Theorem 4.1.

Chapter 5

BIVARIATE VARIANCE RESIDUAL LIFE

5.1. Introduction

It is evident from the earlier discussions that the three basic reliability concepts namely, the survival function, the failure rate and the mean residual life are equivalent, in the sense that knowing any one of them, the other two can be uniquely determined. Another concept which has generated interest in life length studies in recent years is the variance residual life (VRL) which was briefly reviewed in Section 1.2.1.4. A special feature of the VRL is that it does not determine the corresponding life distribution uniquely. This and the fact that monotone behaviour of the VRL generate classes of distribution that are not covered by similar behaviour of the other concepts makes it an interesting proposition to extend the concept from the univariate to the bivariate case. As in the case of bivariate MRL, the bivariate VRL is useful in modelling and analyzing failure data of a two-component system which does not satisfy the assumption of independence among the components life times. Apart from looking at several properties of this function, in the present chapter we develop some characterization theorems of the life distribution associated in the previous chapter. Further, we introduce four new classes of life distributions that are determined by the monotone behaviour of the VRL. Also, the monotonic properties of the VRL are characterized in terms of the residual coefficient of variation and is used to study the behaviour of VRL for certain family of bivariate distributions. It is demonstrated that using these different ageing criteria derived from the VRL, one can make comparison among various **lifetime** models generated in different physical situations.

5.2. Definition and Properties

Let $X = (X_1,X_2)$ be a bivariate random vector admitting absolutely continuous distribution function with respect to Lebesgue measure in the positive octant $R_2^+ = \{(x_1,x_2) | x_1,x_2 > 0\}$ of the two dimensional Euclidean space R_2 and having survival function $R(x_1,x_2)$. Assume that $E(X_i^2) < \infty$, $i = 1,2$. Then the random vector

$$V(x_1,x_2) = (V_1(x_1,x_2), V_2(x_1,x_2)) \tag{5.1}$$

where,

$$V_i(x_1,x_2) = E[(X_i-x_i)^2 | X_1 \geq x_1,\ X_2 \geq x_2] - r_i^2(x_1,x_2) \tag{5.2}$$

is defined as the bivariate variance residual life and $V_i(x_1,x_2)$ as its components. The properties of VRL are the following,

1. $$V_1(x_1,x_2) = 2[R(x_1,x_2)]^{-1} \int_{x_1}^{\infty} r_i(y_1,x_2)R(y_1,x_2)dy_1 - r_1^2(x_1,x_2). \tag{5.3}$$

Proof:

$$E[(X_1-x_1)^2 | X_1 \geq x_1, X_2 \geq x_2] = [R(x_1,x_2)]^{-1} \int_{x_1}^{\infty}\int_{x_2}^{\infty} (t_1-x_1)^2 \frac{\partial^2 R}{\partial t_1 \partial t_2} dt_1 dt_2$$

$$= 2[R(x_1,x_2)]^{-1} \int_{x_1}^{\infty} (t_1-x_1)\, R(t_1,x_2)dt_1$$

$$= 2[R(x_1,x_2)]^{-1} \int_{x_1}^{\infty} \left(\int_{t_1}^{\infty} R(y_1,x_2)dy_1 \right) dt_1$$

$$= 2[R(x_1,x_2)]^{-1} \int_{x_1}^{\infty} r_1(y_1,x_2)R(y_1,x_2)dy_1. \tag{5.4}$$

Substituting (5.4) on (5.2), we recover (5.3).

On similar lines,

$$V_2(x_1,x_2) = 2[R(x_1,x_2)]^{-1} \int_{x_2}^{\infty} r_2(x_1,y_2)R(x_1,y_2)dy_2 - r_2^2(x_1,x_2). \tag{5.5}$$

Formulas (5.3) and (5.5) express the VRL vector in terms of the survival function and the mean residual life of (X_1,X_2).

2. $$\frac{\partial V_i}{\partial x_i} = h_i(x_1,x_2)\,[V_i(x_1,x_2) - r_i^2(x_1,x_2)] \quad . \tag{5.6}$$

Proof:

Differentiating (5.3) with respect to x_i, we have

$$\frac{\partial V_i}{\partial x_i} = -2r_i(x_1,x_2) - 2\frac{1}{R(x_1,x_2)^2}\frac{\partial R}{\partial x_i}\int_{x_i}^{\infty} r_i(x_1,x_2) R(x_1,x_2)dx_i - 2r_i(x_1,x_2)\frac{\partial r_i}{\partial x_i}$$

$$= -2r_i(x_1,x_2)\,[1+\frac{\partial r_i}{\partial x_i}] + 2h_i(x_1,x_2)[V_i(x_1,x_2)+r_i^2(x_1,x_2)] \tag{5.7}$$

Using the relationship,

$$h_i(x_1,x_2) = \frac{1+\frac{\partial r_i}{\partial x_i}}{r_i(x_1,x_2)} ,$$

the equation (5.7) becomes

$$\frac{\partial V_i}{\partial x_i} = -2r_i^2(x_1,x_2)h_i(x_1,x_2)+h_i(x_1,x_2)[V_i(x_1,x_2)+r_i^2(x_1,x_2)]$$

$$= h_i(x_1,x_2)\ [V_i(x_1,x_2)-r_i^2(x_1,x_2)] \cdot$$

The relationship (5.6) will be used in the sequel to explore the monotone behaviour of VRL.

3. When X_1 and X_2 are independent random variables

$$V(x_1,x_2) = (V_1(x_1,o),\ V_2(o,x_2)) \cdot \tag{5.8}$$

Proof:

When X_1 and X_2 are independent

$$r_1(x_1,x_2) = r_1(x_1,o)$$

and

$$R(x_1,x_2) = R(x_1,o)\ R(o,x_2) \cdot$$

Thus,

$$V_1(x_1,x_2) = \frac{2}{R(x_1,o)R(o,x_2)} \int_{x_1}^{\infty} r_1(y,o)\ R(y,o)\ R(o,x_2)dy - r_1^2(x_1,o)$$

$$= V_1(x_1,o)\ .$$

Similarly

$$V_2(x_1,x_2) \;=\; V_2(o,x_2).$$

It is easy to see that $V_1(x_1,o)$ and $V_2(x_1,o)$ are the univariate VRL's of the component variables X_1 and X_2.

4. The VRL in general does not determine a distribution uniquely .

Proof:

To prove the assertion, consider the bivariate Pareto model of Lindley and Singpurwalla (1986) with

$$R(x_1,x_2) = (1+a_1x_1+a_2x_2)^{-c}, \; x_1,x_2>0 . \tag{5.9}$$

For this distribution, by direct computation we get

$$E[(X_1-x_1)^2|X_1\geq x_1,X_2\geq x_2] = 2(1+a_1x_1+a_2x_2)^2/a_1^2(c-1)(c-2) .$$

Substituting in (5.4)

$$\frac{(1+a_1x_1+a_2x_2)^2}{a_1^2(c-1)(c-2)} R(x_1,x_2) = \int_{x_1}^{\infty} r_1(y_1,x_2)R(y_1,x_2)dy_1 . \tag{5.10}$$

Differentiating both sides of (5.10) with respect to x_1, we have

$$\frac{(1+a_1x_1+a_2x_2)^2}{a_1^2(c-1)(c-2)}\frac{\partial R}{\partial x_1}+\frac{2(1+a_1x_1+a_2x_2)}{a_1(c-1)(c-2)}R(x_1,x_2)$$

$$= -r_1(x_1,x_2)\ R(x_1,x_2)\ . \qquad (5.11)$$

Dividing by $R(x_1,x_2)$ and using the definition of bivariate failure, (5.11) becomes

$$-\frac{(1+a_1x_1+a_2x_2)^2}{a_1^2(c-1)(c-2)}h_1(x_1,x_2)+2\frac{(1+a_1x_1+a_2x_2)}{a_1(c-1)(c-2)}=-r_1(x_1,x_2).$$

Further simplification is achieved by using the relationship between $h_1(x_1,x_2)$ and $r_1(x_1,x_2)$. This leaves the differential equation

$$\frac{\partial r_1}{\partial x_1}=A(x_1)\ r_1^2(x_1,x_2)+B(x_1)\ r_1(x_1,x_2)+c(x_1) \qquad (5.12)$$

where,

$$A(x_1) = a_1^2(c-1)(c-2)/(1+a_1x_1+a_2x_2)^2,$$

$$B(x_1) = 2a_1/(1+a_1x_1+a_2x_2)$$

and

$$C(x_1) = -1.$$

Treating x_2 as a constant, (5.12) is Riccati's equation, which can be solved by the method described in Simmons (1974, p. 63).

Since $r_1^0 = \dfrac{(1+a_1x_1+a_2x_2)}{a_1(c-1)}$ satisfies (5.12), we choose it as a particular solution to write the general solution of (5.12) as

$$r_1(x_1,x_2) = \frac{1+a_1x_1+a_2x_2}{a_1(c-1)} + Z(x_1,x_2)$$

where, Z satisfies the equation,

$$\frac{\partial Z}{\partial x_1} - [B(x_1)+2r_1(x_1,x_2)A(x_1)]Z = A(x_1)Z^2. \qquad (5.13)$$

The next step is to solve (5.13). Towards this end, we set $y = \frac{1}{Z}$ in (5.13) to find

$$\frac{\partial y}{\partial x_1} + [B(x_1)+2r_1(x_1,x_2)A(x_1)]y = -A(x_1) \qquad (5.14)$$

which is of the well-known linear form.

Substituting the expressions for $A(x_1)$ and $B(x_1)$ in (5.14), we have

$$\frac{\partial y}{\partial x_1} + \left(\frac{2a_1(c-1)}{1+a_1x_1+a_2x_2}\right) y = \frac{-a_1^{\,2}(c-1)(c-2)}{(1+a_1x_1+a_2x_2)^2} \quad . \qquad (5.15)$$

The solution of the above equation will be

$$y.e^{\int P(x_1)dx_1} = \int Q(x_1)\, e^{\int P(x_1)}\, dx_1 + K(x_2)$$

where,

$$P(x_1) = \frac{2a_1(c-1)}{1+a_1x_1+a_2x_2}$$

and

$$Q(x_1) = \frac{-a_1^2(c-1)(c-2)}{(1+a_1x_1+a_2x_2)^2} \ .$$

By direct integration,

$$e^{\int P(x_1)\,dx_1} = (1+a_1x_1+a_2x_2)^{2(c-1)}$$

and

$$\int Q(x_1)\,e^{\int P(x_1)}\,dx_1 = \frac{-(c-1)(c-2)a_1(1+a_1x_1+a_2x_2)^{2c-3}}{2c-3} \ .$$

Therefore the solution of the differential equation (5.15) is

$$y = \frac{K(x_2)(2c-3)-a_1(c-1)(c-2)(1+a_1x_1+a_2x_2)^{2c-3}}{(2c-3)(1+a_1x_1+a_2x_2)^{2(c-1)}} \ .$$

Thus the general solution of (5.12) is prescribed as

$$r_1(x_1,x_2) = \frac{1+a_1x_1+a_2x_2}{a_1(c-1)} + \frac{1}{y} \ .$$

A similar result can be obtained in the case of $r_2(x_1,x_2)$. We have therefore different sequences of truncated moments arising from (5.12) and each such sequence must correspond to a particular distribution and our assertion is completely proved.

5.3. Characterizations

Eventhough, VRL in general does not determine a distribution uniquely, we can use the relationship among VRL and MRL to characterize certain bivariate life models. Further, in certain cases VRL itself can be employed to characterize some bivariate distribution.

Theorem 5.1.

The VRL vector

$$V(x_1,x_2) = (c_1,c_2) \tag{5.16}$$

where, c_1 and c_2 are independent of both x_1 and x_2 if and only if $X=(X_1,X_2)$ is distributed as $E(\alpha_1,\alpha_2,0)$.

Proof:

To prove the assertion we directly from (5.6) that

$$h_i(x_1,x_2)\,[c_i-r_i^2(x_1,x_2)] = 0, \; i=1,2.$$

This implies $r_i(x_1,x_2) = \ell_i$, a constant independent of both x_1 and x_2.

Therefore, from Nair and Nair (1989), X_1 and X_2 follows $E(\alpha_1,\alpha_2,0)$. Conversely, when $X = (X_1,X_2)$ follows $E(\alpha_1,\alpha_2,0)$, the VRL

$$V(x_1,x_2) = \left(\frac{1}{\alpha_1^2}, \frac{1}{\alpha_2^2}\right),$$

is a constant vector and the proof is complete.

Theorem 5.2.

The random vector $X=(X_1,X_2)$ follows Gumbel's bivariate exponential model $E(\alpha_1,\alpha_2,\Theta)$ if and only if

$$V(x_1,x_2) = (B_1(x_2),\, B_2(x_1)). \tag{5.17}$$

Proof:

For the Gumbel's bivariate exponential distribution

$$V(x_1,x_2) = \left(\frac{1}{(\alpha_1+\Theta x_2)^2}, \frac{1}{(\alpha_2+\Theta x_1)^2}\right)$$

so that the if part is true.

Conversely, from (5.6) and the stipulated form for $V(x_1,x_2)$,

$$h_i(x_1,x_2)\ [V_i(x_1,x_2)-r_i^2(x_1,x_2)] = 0$$

or

$$V_i(x_1,x_2) = r_i^2(x_1,x_2).$$

Therefore,

$$r_i(x_1,x_2) = A_i(x_j),\ i,j = 1,2,\ i\neq j,$$

which is a characteristic property of the Gumbel's bivariate exponential distribution established in Nair and Nair (1988).

Sometimes it is more convenient to deal with the coefficient of variation of residual life rather than the VRL. We now prove a characterization theorem based on the values assumed by the coefficient of variation

$$C(x_1,x_2) = (C_1(x_1,x_2),C_2(x_1,x_2)),$$

where,

$$C_i^2(x_1,x_2) = \frac{V_i(x_1,x_2)}{r_i^2(x_1,x_2)}\ .$$

Theorem 5.3.

Let $X=(X_1,X_2)$ be a non-negative random vector admitting absolutely continuous distribution function with respect to Lebesgue measure, such that $E(X_i^2) < \infty$. Then $C_i(x_1,x_2) = k$, a constant if and only if X is

(i) Gumbel's exponential distribution $E(\alpha_1,\alpha_2,\theta)$ for $k=1$,

(ii) bivariate Pareto Model $P(a_1,a_2,b,c)$ for $k>1$

and

(iii) bivariate finite range distribution $F(p_1,p_2,q,d)$ for $0<k<1$.

Proof:

We first prove the necessary part. The given condition can be stated as

$$V_1(x_1,x_2) = \ell r_1^2(x_1,x_2),$$

where,

$$\ell = k^2.$$

Then from (5.6),

$$h_1(x_1,x_2)(V_1'(x_1,x_2)-r_1^2(x_1,x_2) = \ell 2r_1(x_1,x_2)\frac{\partial r_1}{\partial x_1}.$$

Since,

$$h_1(x_1,x_2) = \frac{1 + \frac{\partial r_1}{\delta x_1}}{r_1(x_1,x_2)} ,$$

we have

$$\frac{\partial r_1}{\delta x_1} = m,$$

where,

$$m = \frac{\ell - 1}{\ell + 1} .$$

The general solution to the last equation is

$$r_1(x_1,x_2) = mx_1 + B_1(x_2).$$

Similarly one can obtain,

$$r_2(x_1,x_2) = mx_2 + B_2(x_1).$$

We conclude that the distribution of X is as stated in Theorem 5.3 from Theorem 3.1.

In seeing the converse is true, note that

for $E(\alpha_1,\alpha_2,\theta)$,

$$C_i^2(x_1,x_2) = 1,$$

for $P(a_1,a_2,b,c)$,

$$C_i^2(x_1,x_2) = \frac{c}{c-2}$$

and for $F(p_1,p_2,q,d)$,

$$C_i^2(x_1,x_2) = \frac{d}{d+2} .$$

Corollary

When $X_2 \longrightarrow 0$ in $V_1(x_1,x_2)$, the univariate property

$$C_1(x_1) = k$$

holds if and only X is exponential, Pareto and finite range as proved in Mukherjee and Roy (1986).

5.4. Monotone behaviour of VRL

The reliability concepts like failure rate, MRL etc. are used to describe the pattern of functioning of the systems of components. However, in order to have a fuller understanding of the importance of various distributions in reliability theory, various notions of ageing are helpful (see Barlow and Proschan (1975)). Traditionally ageing is conveniently discussed and various life distributions are assessed in terms of the monotonic behaviour of failure rates or MRLFs. The aim of the present section is to introduce new classifications of bivariate distributions based on the monotone behaviour of VRL. The relationship of the classes so defined with other classes existing in literature and the chain of implications among them are also examined. Various classes and their interpretation in terms of ageing behaviour are detailed below. In defining the various classes we have kept in mind that, the definitions should be based upon conditions imposed upon the joint survival function and not on the constituent random variables, they should be valid for the established definitions in the univariate case and that the arguments that generate bivariate definitions should be natural extensions in some sense of the corresponding univariate definition of VRL.

Definition 5.1.

A bivariate life distribution or random vector is decreasing (increasing) variance residual life-1 (D(I) VRL - 1) if

$$V_i(x_1+y_1, x_2+y_2) \leq (\geq)\ V_i(x_1, x_2),\ i=1,2 \qquad (5.18)$$

for all (x_1, y_1) and (x_2, y_2) in R_2^+.

The condition (5.18) is appropriate when the components in a system with different ages x_i are required to survive different times y_1, y_2 and implies that for DVRL-(1) (IVRL-(1)) distribution, the VRL at various ages decreases (increases) as the component ages.

The ages x_1 and x_2 are chosen to be distinct by anticipating a replacement policy or by considering new components at the same time origin after which time moves at different rates for the two components which is true of accelerate life test situations where the stresses operating on the two components are different. Obviously, the boundary of the two classes satisfy

$$V_i(x_1+y_1, x_2+y_2) = V_i(x_1, x_2), \quad i=1,2 \qquad (5.19)$$

in which case X is both DVRL-1 and IVRL-1. The following theorem explains the situation when (5.19) holds good.

Theorem 5.4.

The bivariate random variable X is both DVRL-(1) and IVRL (1) if and only if X_1 and X_2 are independent and exponentially distributed.

Proof.

Condition (5.19) is equivalent to

$$V_i(x_1, x_2) = c_i$$

a constant independent of x_1 and x_2. The result now follows from Theorem 5.1.

Definition 5.2.

A bivariate distribution or random variable is said to be DVRL-(2) (IVRL-(2)) if

$$V_1(x_1+y,x_2) \leq (\geq) V_1(x_1,x_2)$$

and

$$V_2(x_1,x_2+y) \leq (\geq) V_2(x_1,x_2) \qquad (5.20)$$

for all (x_1,x_2) in R_2^+ and $y > 0$.

The physical situation when condition (5.20) is of interest occurs when the individual lives of the components when the other has survived a specific lifetime are subject to study. In fact (5.20) means that given a two-component system with ages x_1 and x_2, the VRL of the i^{th} component can be decreased(increased) on replacing it by a similar component of larger age. Using differential calculus (X_1,X_2) is DVRL (2) (IVRL(2)) according as

$$\frac{\partial V_1}{\partial x_1} \leq (\geq) 0 \qquad (5.21)$$

and

$$\frac{\partial V_2}{\partial x_2} \leq (\geq) 0 .$$

The boundary of the class is obviously the one satisfying the equality in (5.20). More precisely we have,

Theorem 5.5.

A bivariate distribution will be DVRL-(2) and IVRL-(2) if and only if it is the Gumbel's bivariate exponential model.

Proof:

From (5.21), the distribution satisfies the conditions of the theorem if and only if

$$\frac{\partial V_1}{\partial x_1} = 0$$

and

$$\frac{\partial V_2}{\partial x_2} = 0$$

or when

$$V(x_1,x_2) = (B_1(x_2),\ B_2(x_1))$$

which is the characteristic property of Gumbel's exponential model by Theorem 5.2.

There exists an implication between the DVRL-(2) (IVRL-(2)) class and the DMRL-(2) (IMRL-(2)) class, where the latter is defined by the condition

$$r_1(x_1+y,\ x_2) \ \leq\ (\geq)\ \ r_1(x_1,x_2)$$

and

$$r_2(x_1,x_2+y) \ \leq\ (\geq)\ \ r_2(x_1,x_2) \qquad (5.22)$$

for all (x_1,x_2) in R_2^+ and $y > 0$.

Theorem 5.6.

Let $E(X_i^2)$ be finite and the life distribution is DMRL (2) (IMRL (2)). Then $X = (X_1, X_2)$ is DVRL (2) (IVRL (2)) at all points for which $R(x_1,x_2)>0$.

Proof:

We prove the result only in the DMRL case. The proof for the dual case will follow by reversing the inequality signs. Since the random vector has DMRL (2) distribution, one can write from (5.22) that

$$r_1(t_1,x_2) \leq r_1(x_1,x_2)$$

and (5.23)

$$r_2(x_1,t_2) \leq r_2(x_1,x_2)$$

for all $t_1 \geq x_1 > 0$ and $t_2 \geq x_2 > 0$. Since $R(x_1,x_2)$ is positive for all $x_1,x_2>0$, it is true from (5.23) that

$$2[R(x_1,x_2)]^{-1} \int_{x_1}^{\infty} R(t_1,x_2)[r_1(t_1,x_2)-r_1(x_1,x_2)]dt_1 \leq 0 \cdot \tag{5.24}$$

The first term on the left of the equation (5.24) is

$$E[(X_1-x_1)^2 | X_1>x_1,\ X_2>x_2]$$

from equation (5.4) and using the relationship

$$R(x_1,x_2)\ r_1(x_1,x_2) = \int_{x_1}^{\infty} R(t_1,x_2)dt_1 ,$$

the second term simplifies to $-\ 2r_1^{\,2}(x_1,x_2)$. Thus (5.24) reduces to

$$V_1(x_1,x_2) - r_1^{\,2}(x_1,x_2) \leq 0.$$

Hence from (5.6), $\frac{\partial V_1}{\partial x_1} \leq 0$. Similarly there holds the inequality $\frac{\partial V_2}{\partial x_2} \leq 0$ and thus the distribution is DVRL (2).

There exists a characterization of the DVRL-(2) (IVRL-(2)) models in terms of the coefficient of variation of residual life, which is presented in the following theorem.

Theorem 5.7.

The distribution of $X=(X_1,X_2)$ has DVRL-(2) (IVRL-(2)) if and only if $C_i(x_1,x_2) \leq (\geq)\ 1$, where $C_i(x_1,x_2)$ is defined in section 5.3.

Proof:

The result follows from the identity,

$$\frac{\partial V_i}{\partial x_i} = \frac{h_i(x_1,x_2)}{r_i^2(x_1,x_2)}[c_i(x_1,x_2)-1]$$

derived from (5.6).

Definition 5.3.

A bivariate random variable X on its distribution is said to have DVRL-(3) (IVRL-(3)) if

$$V_i(x_1+y,\ x_2+y) \leq (\geq)\ V_i(x_1,x_2) \tag{5.25}$$

for all (x_1,x_2) in R_2^+ and $y > 0$.

This condition is of interest when the components have different ages x_1 and x_2 and our concern is to a common time horizon y. The difference in ages contemplated here can arise out of a replacement policy and the common time y is of particular significance when we look at a series system. The condition (5.25), on the other hand, can be interpreted in the following way. As the system ages, the VRL of all components decrease (increase). Again the distribution separating the DVRL-(3) class and IVRL-(3) class should satisfy the property

$$V_i(x_1+y,\ x_2+y) = V_i(x_1,x_2) \tag{5.26}$$

for all (x_1,x_2) in R_2^+ and $y > 0$.

Theorem 5.8.

A bivariate random vector X in the support of R_2^+ with $E(X_i^2) < \infty$, is both DVRL-(3) and IVRL-(3) if it is distributed in the Marshall-Olkin bivariate exponential form.

Proof:

When X follows Marshall and Olkin exponential model, it satisfies the bivariate lack of memory property

$$R(x_1+y,\ x_2+y) = R(x_1,x_2)\ R(y,y) \qquad (5.27)$$

and from Zahedi (1985),

$$r_i(x_1+y,\ x_2+y) = r_i(x_1,x_2),\ i=1,2. \qquad (5.28)$$

Using (5.27) and (5.28) in the equation (5.3), we have

$$V_i(x_1+y,\ x_2+y) = V_i(x_1,x_2).$$

Thus X has both DVRL-3 and IVRL-3

Definition 5.4.

The random vector X or its distribution is said to possess the DVRL-(4) (IVRL-(4)) property if

$$V_i(x+y,\ x+y) \leq (\geq)\ V_i(x,y),\ i=1,2 \qquad (5.29)$$

for all $x,y > 0$.

The physical situation contemplated here is that a two-component system starts working at the same time and our interest lies in observing its performance after a time y. It follows that the condition (5.29) has the interpretation that the VRL of each component decreases (increases) as the components age.

The following relationship hold among the various calsses of DVRL (IVRL) distributions

$$\begin{array}{ccccc} \text{DVRL-(1)} & \Longrightarrow & \text{DVRL-(3)} & \Longrightarrow & \text{DVRL-(4)} \\ \Downarrow & & & & \\ \text{DVRL-(2)} & & & & \end{array}$$

and

$$\begin{array}{ccccc} \text{IVRL-(1)} & \Longrightarrow & \text{IVRL-(3)} & \Longrightarrow & \text{IVRL-(4)} \\ \Downarrow & & & & \\ \text{IVRL-(2)} & & & & \end{array}$$

These implications follow directly from the respective definitions of the various classes. At the same, the question of reverse implications in each case is also of considerable importance. Some results in this connection are presented in the following theorems.

Theorem 5.9.

IVRL-(2) does not imply IVRL-(3) and IVRL-(2) does not imply IVRL-(4).

Proof:

Consider the Gumbel's exponential distribution with

$$R(x_1,x_2) = e^{-\alpha_1 x_1 - \alpha_2 x_2 - \theta x_1 x_2}; \quad x_1,x_2 > 0.$$

From direct calculations

$$V_1(x_1,x_2) = \frac{1}{(\alpha_1+\theta x_2)^2}$$

and

$$V_2(x_1,x_2) = \frac{1}{(\alpha_2+\theta x_1)^2}.$$

Clearly,

$$V_1(x_1+y,x_2) = V_1(x_1,x_2)$$

and

$$V_2(x_1,x_2+y) = V_2(x_1,x_2).$$

Thus X has IVRL-(2) property.

Since,

$$V_1(x_1+y,x_2+y) \not> V_1(x_1,x_2)$$

and

$$V_2(x_1+y,x_2+y) \not> V_2(x_1,x_2),$$

X does not satisfy the property IVRL-(3). Similarly, X does not have the property IVRL-(4) and the proof is complete.

Theorem 5.10.

IVRL-(1) does not imply DVRL-(2) and

IVRL-(1) does not imply DVRL-(4).

Proof:

Taking

$$R(x_1,x_2) = (1+a_1x_1+a_2x_2)^{-c}, \quad x_1,x_2 > 0,$$

the VRL vector is

$$V(x_1,x_2) = \left(\frac{c(1+a_1x_1+a_2x_2)^2}{a_1^2(c-1)(c-2)}, \quad \frac{c(1+a_1x_1+a_2x_2)^2}{a_2^2(c-1)(c-2)}\right).$$

Obviously,

$$V_i(x_1+y_1,x_2+y_2) > V_i(x_1,x_2), \quad i=1,2,$$

but

$$V_1(x_1+y,x_2) \nless V_1(x_1,x_2)$$

and

$$V_2(x,x_2+y) \nless V_2(x_1,x_2).$$

Thus IVRL-(1) does not imply DVRL-(2).

Similarly,

$$V_i(x+y,x+y) \nless V_i(x,x).$$

Hence, IVRL-(1) does not imply DVRL-(4).

Theorem 5.11.

DVRL-(1) does not imply IVRL-(2) and also DVRL-(1) does not imply IVRL-(4).

Proof:

From,

$$R(x_1,x_2) = (1-p_1x_1-p_2x_2)^d, \quad 0<x_i< \frac{1}{p_i}$$

the VRL,

$$V_i(x_1,x_2) = \frac{d(1-p_1x_1-p_2x_2)^2}{p_i^2(d+1)(d+2)}, \quad i=1,2.$$

It is evident that

$$V_i(x_1+y_1,x_2+y_2) \leq V_i(x_1,x_2), \quad i=1,2.$$

Since,

$$V_1(x_1+y,x_2) \not\geq V_1(x_1,x_2)$$

and

$$V_2(x_1,x_2+y) \not\geq V_2(x_1,x_2)$$

DVRL-(1) does not imply IVRL-(2).

On similar lines, we have

$$V_i(x+y,x+y) \not\geq V_i(x,x), \quad i=1,2.$$

Thus DVRL-(1) does not imply IVRL-(4).

The results in sections 5.2 and 5.3 of the present chapter form part of Sankaran and Nair (1992 c).

Chapter 6

BIVARIATE VITALITY FUNCTION

6.1. Introduction

The manner in which ageing affects various components and devices is of primary concern in reliability analysis and life distributions are often classified according to different criteria for ageing. Such criteria uses the monotone behaviour of the basic characteristics such as failure rate, mean residual life etc. Recently, Kupka and Loo(1989) have employed a new method of measuring the phenomenon of ageing with the aid of vitality function which is the expectation of a random variable X conditioned on $X>x$. The properties of vitality function and its relationship to the other ageing concepts were discussed in Section 1.2.1.3. In the following sections we extend the notion of vitality function to the bivariate case and point out some of its applications in the analysis of lifetime data. The contents of the present chapter are due to appear in Sankaran and Nair (1991).

6.2. Definition and Properties

Let $X = (X_1, X_2)$ be a random vector in the support of $\{(x_1, x_2) | a_i \leq x_i \leq b_i,\ i = 1,2\}$ for $a_i \geq -\infty$ and $b_i \leq +\infty$, with survival function $R(x_1, x_2)$. For values of $x_i < b_i$ such that $P[X \geq x] > 0$ and $X_i^+ = \max(0, X_i)$ satisfying $E(X_i^+) < \infty$, the vector valued function,

$$m(x_1,x_2) = (m_1(x_1,x_2),\ m_2(x_1,x_2)), \tag{6.1}$$

where,

$$m_i(x_1,x_2) = E[X_i | X_1 \geq x_1,\ X_2 \geq x_2] \tag{6.2}$$

is called the bivariate vitality function of X. In a two-component system, where the life lengths of the components are X_1 and X_2 (which are non-negative), $m_1(x_1, x_2)$ measures the expected age at failure of the first component as the sum of the present age x_1 and the average lifetime remaining to it, given the survival of the second at age x_2. A similar interpretation can be given to $m_2(x_1, x_2)$. By straight forward integration, we have

$$m_1(x_1,x_2) = x_1 + [R(x_1,x_2)]^{-1} \int_{x_1}^{b_1} R(t_1,x_2)\,dt_1 \tag{6.3}$$

and

$$m_2(x_1,x_2) = x_2 + [R(x_1,x_2)]^{-1} \int_{x_2}^{b_2} R(x_1,t_2)dt_2 \cdot \qquad (6.4)$$

The m_i's are, in general, different from the univariate vitality functions of the component variables X_i specified by

$$M_i(x_i) = E[X_i | X_i \geq x_i] \quad i=1,2$$

with

$$m(x_1,x_2) = (M_1(x_1),\ M_2(x_2))$$

if and only if X_1 and X_2 are independent.

Further,

$$(M_1(x_1),\ M_2(x_2)) = (m_1(x_1,a_2), m_2(a_1,x_2)) \qquad (6.5)$$

where,

the expressions on the right hand side are evaluated as limits. Moreover,

$$m_i(x_1,x_2) \geq x_i \quad \text{for all } a_i \leq x_i \leq b_i.$$

In view of the relationship,

$$m_i(x_1,x_2) = x_i + r_i(x_1,x_2)\ , \qquad (6.6)$$

$R(x_1,x_2)$ is uniquely determined from bivariate vitality function. Also, the bivariate failure rate $h(x_1,x_2)$, given in (1.17), is related to $m(x_1,x_2)$ by

$$h_i(x_1,x_2) = [m_i(x_1,x_2)-x_i]^{-1} \frac{\partial m_i(x_1,x_2)}{\partial x_i} \tag{6.7}$$

and hence

$$\frac{\partial m_i(x_1,x_2)}{\partial x_i} = h_i(x_1,x_2)\ r_i(x_1,x_2). \tag{6.8}$$

All the above properties are direct implications of the corresponding results concerning bivariate MRL reviewed in Nair and Nair (1989).

6.3. Measure of Local Memory

Traditionally, the influence of age on equipment behaviour is manifested through, either positive ageing in which the equipment gradually deteriorates in its functioning as time progresses, or negative ageing indicating a beneficial effect due to increase in age or no-ageing where it continue to have some capacity to perform regardless of age. Life distributions that are characterized by no-ageing or lack of memory property (LMP) as it is called in probability calculus, form the boundary of the class of distributions belonging to the other two categories. In the univariate case it is well known that the only continuous distributions that possess the LMP is the exponential distribution and therefore it forms the dividing line between distributions that represent positive and negative ageing behaviour. However, in the bivariate case, notion of LMP can be spelt in more than one way (see Section 3.4) and we choose in our present discussion, the definition based on the local behaviour of the component variables. The random vector (X_1,X_2) is said to possess the local lack of memory (LLMP) if there holds the relationship

$$P[X_1>t_1+x_1|X_1>x_1,X_2>x_2] = P[X_1>t_1|X_2>x_2] \quad (6.9)$$

and

$$P[X_2>t_2+x_2|X_1>x_1, X_2>x_2] = P[X_2>t_2|X_1>x_1]. \quad (6.10)$$

From (6.9) we have,

$$\frac{R(x_1+t_1,x_2)}{R(x_1,x_2)} = \frac{R(t_1,x_2)}{R(o, x_2)} .$$

Integrating the above equation and using the definition of bivariate MRL given in (1.26), one can see that the above equation is equivalent to

$$r_1(x_1,x_2) \quad = \quad r_1(o,x_2)$$

or in terms of vitality function

$$m_1(x_1,x_2) \quad = \quad x_1+m_1(o,x_2)\ .$$

Defining

$$D_1(x_1,x_2) \quad = \quad m_1(x_1,x_2)-m_1(o,x_2)$$

and

$$D_2(x_1,x_2) \quad = \quad m_2(x_1,x_2)-m_2(x_1,o),$$

it is clear that $D_i(x_1,x_2)$ indicates the gain in the conditional mean life of the i^{th} component in the interval (o,x_i) when the second has survived age x_j. Since x_i is the actual age attained by the i^{th} component, $D_i(x_1,x_2) = x_i$ represents the situation when it did not age in (o,x_i) while $D_i(x_1,x_2)<x_i(>x_i)$ corresponds to its positive (negative) ageing. Noticing that $\frac{\partial D_i}{\partial x_i} = \frac{\partial m_i}{\partial x_i}$, we propose the following definition of local memory at a point.

Definition 6.1.

At a point $x = (x_1, x_2)$, the random vector X has positive local memory (PLM) if and only if $\frac{\partial D_i}{\partial x_i} < 1$ for $i = 1,2$, negative local memory (NLM) if and only if $\frac{\partial D_i}{\partial x_i} > 1$ for $i=1,2$ and local lack of memory (LLM) if and only if $\frac{\partial D_i}{\partial x_i} = 1$ for $i=1,2$.

Since $\frac{\partial D_i}{\partial x_i} \gtreqless 1$ is equivalent to $\frac{\partial}{\partial x_i}(D_i - x_i) \gtreqless 0$, one can use the latter expression also in defining local memory.

Examples:

(i) For the Gumbel's bivariate exponential law with survival function (3.2)

$$m_i(x_1, x_2) = x_i + (\alpha_i + \theta x_j)^{-1}, \quad i,j=1,2; \ i \neq j$$

for every $x_1, x_2 > 0$ so that (3.2) possesses LLM.

(ii) The bivariate Pareto distribution, with survival function (3.3), has NLM at each point of its support since,

$$m_i(x_1,x_2) \quad = \quad c(c-1)^{-1}x_i + \frac{(1+a_jx_j)}{(c-1)(a_i+bx_j)}$$

and

$$\frac{\partial D_i}{\delta x_i} \quad = \quad c(c-1)^{-1} > 1.$$

(iii) The bivariate finite range model, specified by the survival function (3.4), satisfies the condition for PLM at each point in its support. This follows from

$$D_i(x_1,x_2) \quad = \quad q(q+1)^{-1}\; x_i.$$

It is obvious that a bivariate distribution need not have the same type of memory at various points and accordingly it is of some interest to have a consolidated measure of local memory that spreads over the entire support. Such a measure will be useful in (a) comparing the overall ageing behaviour in different populations by a single index, and (b) in selecting models that conform to the objectives of the experimenter.

One way of arriving at such an overall measure is to consider a weighted average of the local memories at various points in support of the distribution as done by Muth (1980) in the univariate case. Thus a measure of local memory of a bivariate distribution can be prescribed as $(\alpha_1(x_2),\alpha_2(x_1))$, where,

$$\alpha_i(x_j) = \int_0^\infty \frac{\partial}{\partial x_i}(x_i-D_i)\ W_i(x_1,x_2)dx_i \qquad (6.11)$$

$$i,j = 1,2,\ i\neq j,$$

for a suitably chosen weight function W_i. In the present investigation our choice is

$$W_1(x_1,x_2) = 2[m_1(o,x_2)]^{-2}\ [R(o,x_2)]^{-1}\int_{x_1}^\infty R(t,x_2)dt \qquad (6.12)$$

and

$$W_2(x_1,x_2) = 2[m_1(x_1,o)]^{-2}\ [R(x,o)]^{-1}\int_{x_2}^\infty R(x_1,t)dt. \qquad (6.13)$$

This choice of the weight function is motivated by the desire to obtain a measure that reduces to the corresponding quantity in the univariate case discussed in Muth (1980). Specialising for i=1 in (6.11) and noting that as x_1 tends to infinity $W_1(x_1, x_2)$ and $m_1(x_1, x_2) - x_1$ by virtue of equation (6.3) tend to zero and $\lim(D_1 - x_1) = 0$ as x_1 tend to zero, we have by partial integration of (6.11)

$$\alpha_1(x_2) = \int_0^\infty (D_1 - x_1)\left(\frac{\partial W_1}{\delta x_1}\right) dx_1 \qquad (6.14)$$

where,

$$\frac{\partial W_1}{\delta x_1} = -2[m_1(o, x_2)]^{-2} R(x_1, x_2)[R(o, x_2)]^{-1}. \qquad (6.15).$$

Now, under the assumption of $E(X_1^2) < \infty$,

$$R(o, x_2)\ E[X_1^2 | X_2 \geq x_2] = \int_0^\infty \int_{x_2}^\infty t_1^2 \frac{\partial^2 R}{\delta t_1 \delta t_2} dt_1\, dt_2$$

$$= -\int_0^\infty t_1^2 \left(\frac{\partial R}{\delta t_1}\right) dt_1$$

$$= 2 \int_0^\infty t_1 R(t_1, x_2) dt_1 \cdot \qquad (6.16)$$

From equation (6.3),

$$\frac{\partial}{\delta x_1}[(m_1 - x_1) R(x_1, x_2)] = -R(x_1, x_2)$$

and therefore,

$$R(o,x_2)\ E[X_1^2|X_2\geq x_2] = -2\int_o^\infty t_1 \frac{\partial}{\partial t_1}[(m_1(t_1,x_2)-t_1)R(t_1,x_2)]dt_1$$

$$= 2\int_o^\infty [m_1(x_1,x_2)-x_1]R(x_1,x_2)dx_1.$$

(6.17)

Equations (6.14) and (6.15) lead to

$$\alpha_1(x_2) = \int_o^\infty [m_1(x_1,x_2)-x_1-m_1(o,x_2)]\ (\frac{\partial W_1}{\partial x_1})dx_1$$

$$= 2[m_1(o,x_2)R(o,x_2)]^{-1}\int_o^\infty R(x_1,x_2)dx_1$$

$$-E(X_1^2|X_2\geq x_2)\ [m_1(o,x_2)]^{-2}.$$

One can write

$$\int_o^\infty R(x_1,x_2)dx_1 = -\int_o^\infty\int_{x_2}^\infty (\frac{\partial R}{\partial x_2})dx_1dx_2$$

$$= \int_o^\infty\int_{x_2}^\infty x_1(\frac{\partial^2 R}{\partial x_1 dx_2})dx_1dx_2$$

$$= R(o,x_2)\ m_1(o,x_2).$$

Thus,

$$\alpha_1(x_2) = 2-[m_1(o,x_2)]^{-2} E[X_1^2 | X_2 \geq x_2]$$

$$= 1-C_1(x_2)^2, \qquad (6.18)$$

where, $C_1(x_2)$ is the coefficient of variation of X_1 conditioned on $X_2 \geq x_2$. Thus the measure of overall memory of a distribution over the entire support is

$$(1-C_1(x_2)^2, 1-C_2(x_1)^2). \qquad (6.19)$$

Using the measure (6.19) one can classify distribution according to the type of memory they possess.

Definition 6.2.

A continuous bivariate random vector X has PLM(NLM, LLM) according as $C_i(x_j)^2$ is $< 1 (>1, =1)$ for $i,j=1,2;\ i \neq j$.

The proposed measure is unit free and is directly related to the absolute dispersion in the variable. Apart from the application to life length studies, the measure can also be used to characterize bivariate distributions.

6.4. Characterizations using Bivariate Vitality Function

Although the general relationship between bivariate vitality function and bivariate failure rates governed by (6.7), there exist some explicit expressions connecting the two that characterize several bivariate distributions. First we prove a general theorem and then obtain some useful deductions.

Let X be the random vector specified in Section 6.2 possessing a probability density function $f(x_1,x_2)$. Writing,

$$L_i = (\ell_i, d_i, n_i); \quad Y_i = (x_i, x_j, 1); \quad Z_i = (x_i, 1)$$

$$A_i = \begin{pmatrix} p_i & k_i & g_i \\ k_i & o & f_i \\ g_i & f_i & c_i \end{pmatrix}; \quad B = \begin{pmatrix} p_i & g_i \\ g_i & c_i \end{pmatrix} \quad ; \quad C_i = (k_i, f_i),$$

where, the elements of L_i and A_i are real and $i,j=1,2$ with $i \neq j$.

Theorem 6.1.

A necessary and sufficient condition that $f(x_1, x_2)$ satisfies the differential equations

$$\frac{\partial \log f}{\partial x_i} = L_i Y_i' \,/\, Y_i A_i Y_i', \quad i=1,2 \tag{6.20}$$

is that

$$P_i m_i + Q_i m_j + R_i = -(Z_i B_i Z_i')\, h_i(x_1, x_2) + 2C_i Z_i' \left(\frac{\partial m_j}{\partial x_i} - m_j h_i(x_1, x_2) \right) \tag{6.21}$$

provided that $\lim x_i^2 f(x_1, x_2) = 0$ as x_i tends to b_i.

Here,

$$P_i = 2p_i + \ell_i, \quad Q_i = 2k_i + d_i \quad \text{and} \quad R_i = 2g_i + n_i .$$

Proof:

When i=1, (6.20) reads

$$(L_1 Y_1')f = (Y_1 A Y_1') \frac{\partial f}{\partial x_1} . \tag{6.22}$$

We have,

$$Y_1 A Y_1' = p_1 x_1^2 + 2k_1 x_1 x_2 + 2g_1 x_1 + 2f_1 x_2 + c_1$$

and

$$\frac{\partial}{\partial x_1} Y_1 A Y_1' = 2p_1 x_1 + 2k_1 x_2 + 2g_1.$$

Hence by applying integration by parts on (6.22)

$$\int_{x_1}^{b_1} L_1 Y_1' f dx_1 = \left[(Y_1 A Y_1') f\right]_{x_1}^{b_1} - 2 \int_{x_1}^{b_1} (p_1 x_1 + k_1 x_2 + g_1) f dx_1.$$

and

$$\int_{x_1}^{b_1} \int_{x_2}^{b_2} (p_1 x_1 + Q_1 x_2 + R_1) f dx_1 dx_2$$

$$= Y_1 A Y_1' \frac{\partial R}{\partial x_1} + 2C_1 Z' \int_{x_2}^{b_2} \frac{\partial R}{\partial x_1} dx_2 \quad (6.23)$$

Substituting,

$$\int_{x_2}^{b_2} \frac{\partial R}{\partial x_1} dx_2 = \frac{\partial}{\partial x_1} R(m_2 - x_2)$$

$$= (m_2 - x_2) \frac{\partial R}{\partial x_1} + R \frac{\partial m_2}{\partial x_1},$$

into (6.23), we have

$$\int_{x_1}^{b_1} \int_{x_2}^{b_2} (p_1 x_1 + Q_1 x_2 + R_1) f dx_1 dx_2$$

$$= Y_1 A Y_1' \frac{\partial R}{\partial x_1} + 2C_1 Z' \left(R \frac{\partial m_2}{\partial x_1} + (m_2 - x_2) \frac{\partial R}{\partial x_1} \right)$$

$$= (Y_1 A Y_1' - 2C_1 Z' . x_2) \frac{\partial R}{\partial x_1} + 2C_1 Z' \left(R \frac{\partial m_2}{\partial x_1} + m_2 \frac{\partial R}{\partial x_1} \right).$$

(6.24)

Using the definition of $h_1(x_1,x_2)$ in (6.24), and then simplifying, we recover (6.21), specialised for i=1. For i=2, the procedure is similar. As the converse follows by retracing the above steps, our proof is complete.

Though the expression (6.21) looks some what lengthy, when we adhere to specific models simple expressions will result, as the following corollaries show.

We illustrate this by some examples.

Corollary 6.1.

The relationship $m(x)B = - H$

where,

$$B = \begin{pmatrix} 2p_1 & 2k \\ 2k & 2p_2 \end{pmatrix}$$

and

$$H = (h_1(x), h_2(x)),$$

characterizes the bivariate normal law,

$$f(x_1,x_2) = C \exp[-p_1x_1^2-2kx_1x_2-p_2x_2^2] \quad -\infty<x_i<\infty$$

(or its truncated form in the appropriate support) for all p_1, p_2, k such that $p_1, p_2 > 0$ and $p_1 p_2 - k^2 \geq 0$.

Corollary 6.2.

The random vector (X_1, X_2) follows the bivariate law with exponential conditionals (Arnold and Strauss (1988)) with probability density function

$$f(x_1, x_2) = C\, e^{-a_1 x_1 - a_2 x_2 - b x_1 x_2} \qquad \begin{array}{l} a_1, a_2 > 0,\ b \geq 0 \\ x_1, x_2 > 0 \end{array}$$

if and only if

$$a_i + b m_j(x_1, x_2) = -h_i(x_1, x_2);\ i=1,2.$$

The models described by (6.20) is a sub-class of the bivariate Pearson family discussed in Johnson and Kotz (1972) and includes several models useful in reliability analysis. The above results form a bivariate extension of the characterization of the univariate Pearson family given in Theorem 2.1.

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