SOME PROBABILITY MODELS ARISING FROM THE EXTENSION OF THE LACK OF MEMORY PROPERTY

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CERTIFICATE

Certified that the thesis entitled **"Some Probability Models Arising from the Extension of the Lack of Memory Property"** *is a bonafide record of work done by* Smt.K.G.GEETHA *under my guidance in the Division of Statistics, School of Mathematical Sciences, Cochin University of Science and Technology and that no part of it has been included any where previously for the award of any degree.*

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CONTENTS

Chapter I

INTRODUCTION

1.1 CONSTRUCTION OF MULTIVARIATE MODELS

The early developments in statistical distribution theory was dominated by the normal distribution due to various facts such as the pleasing solutions the assumptions of normality can produce, the belief of users that in many practical situations normal observations will result either naturally or as a very good approximations by invoking limit theorems and the large volume of theoretical results that justified normal approximations. A comprehensive study of multivariate distributions including mechanisms to generate them began only from the sixties of the century, and most of the developments in this connection are only in formative stages and is therefore a fertile area of research. One of the main problems associated with the evolution of multivariate models is that there is no unique way of extending a corresponding univariate law and this has

created a large body of distributions in the multivariate set up.

Of the several standard methods to generate multivariate distributions, one is to generalise a system or equation defining a univariate distribution into the multivariate case. This includes generalization of the differential equation representing the Pearson family by Van Uven (1925, 1926, 1929, 1947), Steyn (1960), Elderton and Johnson (1969) and the translation equation system by Johnson (1949). A second method is to construct a multivariate distribution by specifying the form of its marginal distributions. Work in this connection can be seen in the papers of Frechet (1951), Mongestern (1956), Farlie (1960), Mardia (1970). However, it is to be noted that there exist several multivariate forms with the same type of marginals and therefore, the basis of adopting a particular method in this category has to be accompanied by a proper justification based on physical considerations. When we replace marginal distributions by conditionals in the above specification, the picture is more rosy as in many cases it is possible to extract unique joint distribution, at least in the bivariate cases. Considerable interest has been aroused recently in deriving bivariate models which have pre-designated forms for their conditional distributions and the present study reviews the important work in this area and works out some new results in this direction.

Another approach results when one starts with the univariate density and then postulate a functional form for its multivariate analogue. For multivariate distributions generated in this fashion, we refer to Mardia (1962), Gumbel (1960), Bildikar and Patil (1968), Day (1969).

The main limitation of this approach is that, in many cases, it is difficult to give a physical interpretation to the model and its parameters. In contrast, the modelling approach takes into consideration the physical states of the system, the relationships between the variables and the parameters involved etc. so that when real world situations that conform to these properties, are encountered the model becomes the right choice. The distributions of Fruend (1961), Marshall and Olkin (1967), Block and Basu (1974) are typical examples. An amalgamation of these two types of model generation can be affected successfully, if the models belonging to the first category can be derived using the well-known system properties. The present study envisages some interpretations to some of the existing models.

A multivariate model that inherits the essential features of the corresponding univariate version is often a reasonable requirement for constructing the former. This can be easily accomplished by identifying a characteristic property that is of interest of the univariate law, which needs extension to higher dimension and then to find its multivariate analogue, wherever possible, uniquely. The construction of the desired multivariate distribution is complete, once the law is characterized by the newly identified property. Galambos and Kotz (1977) considers this as the soundest approach to develop multivariate distributions.

In the current investigation also, it is proposed to work along these lines in deriving certain classes of distributions, the main basis of which is an extension of the lack of memory property to the bivariate case. To prepare the background for the research problem in this study we briefly survey the important results in literature that is of concern.

1.2 SURVEY OF LITERATURE.

The importance of the exponential distribution among the class of continuous probability models is next only to that of the normal distribution. The reasons for the popularity of the exponential distribution in theoretical and applied investigations can be attributed mainly to the lack of memory property, its relation with the Poisson process and the properties enjoyed by the order statistics. Of these, the lack of memory property is perhaps the best studied and widely applied among the properties of the model, and the one that lends itself to extensions in various directions. From a theoretical point of view lack of memory property is justified from the fact that there are many real life situations where it holds good, while in applied work, it is extensively used as a characteristic property of the exponential distribution. In view of its implications in reliability and life testing, the property is best expressed in terms of the lifelength of a component or device, although the meaning conveyed by it can be shared among the class of other duration variables as well.

If X is a non-negative random variable possessing absolutely continuous distribution with respect to Lebesgue measure, we say that the random variable X or its distribution has lack of memory property if for all $x,y \geq 0$ such that $P(X \geq y) > 0$,

$$P(X \geq x+y \mid X \geq y) = P(X \geq x) \tag{1.1}$$

or equivalently, if $P(X = 0) \neq 1$ for all $x,y \geq 0$

$$P(X \geq x+y) = P(X \geq x)\, P(X \geq y). \tag{1.2}$$

In terms of the survival function of the random variable

$$R(x) = P(X \geq x),$$

(1.2) is restated as $R(x+y) = R(x)\,R(y)$. (1.3)

The characterization of the exponential distribution, using any one of the equivalent forms (1.1) to (1.3) arises from the functional equation explored by Cauchy (1821) and Darbaux (1875)

$$U(x+y) = U(x).U(y) \tag{1.4}$$

whose solution, is either $U(x) = 0$ for all x or $U(x) = e^{-\lambda x}$ for some constant λ, whenever $U(x)$ is a solution defined for $x > 0$.

For an absolutely continuous survival function $R(x)$, its failure rate $h(x)$ is defined as

$$h(x) = \frac{-d \log R(x)}{dx} \ . \tag{1.5}$$

The lack memory property is equivalent to the statement

$$h(x) = \text{a constant}.$$

Further, the truncated mean or mean residual life defined as

$$r(x) = E(X-x \mid X \geq x)$$

$$= [R(x)]^{-1} \int_{x}^{\infty} R(t)dt, \qquad (1.6)$$

often interpreted as the average life time remaining to a component at age z, is related to the failure rate through the equation

$$h(x) = [r(x)]^{-1}\left(1+ \frac{dr(x)}{dx}\right). \qquad (1.7)$$

It is given in Cox (1962) that for the exponential distribution r(x) = a constant Galambos and Kotz (1977) establishes the equivalence of lack of memory property, constancy of the failure rate, and constancy of the mean residual life.

The extension of the lack of memory property is often envisaged to serve one or both of the following objectives.

a. to extend the domain of the values of x and y for which equation (1.1) is true.

b. to provide a larger family of distributions that includes the exponential model as a special case.

By finite induction we obtain from (1.1) that

$$R(x_1 + x_2 + \dots + x_n) = R(x_1)\, R(x_2) \dots R(x_n) \qquad (1.8)$$

Setting $x_1 = x_2 = \dots = x_n = x \geq 0$ and requiring the resulting equation to hold for all integers $n \geq 1$ results in a characterization of the exponential model, in modification of the condition that (1.1) holds for all $x, y \geq 0$. Further Fortet (1977) considered the assumption that (1.7) is true almost everywhere with respect to Lebesgue measure for (x,z) in $[0,\infty)$, is sufficient to guarantee that the distribution is exponential. Another result in this direction due to Sethuraman (1965) shows that $x_1 = x_2 = \dots = x_n = x$, $x \geq 0$ in (1.8), together with $\frac{\log n_1}{\log n_2}$ is irrational, where n_1 and n_2 are integers satisfying (1.8) characterizes the distribution. Alternatively, a survival function satisfying (1.1) for two values y_1 and y_2 of y such the $y_1 | y_2$ is irrational and for all non-negative values of x, is equivalent to the lack of memory property.

Obviously, the values x and y in (1.1) can be replaced by random variables Y and Z with degenerate distributions to produce equivalent characterizations. Once the assumption of degenerate distribution is removed, the equality in (1.1) changes to the inequality

$$R(x+y) \geq R(x).\ R(y)$$

where x and y are random variables hailing from two families D_1 and D_2 respectively, with the following properties.

1. Every member of D_1 and D_2 is independent of X.
2. Every member of D_1 is independent of every member of D_2 and
3. $P(a \leq Y < b) > 0$ and $P(a \leq Z < b) > 0$ for every $[a,b)$.

A variant approach to extension of the lack of memory property takes advantages of the equivalence of lack of memory property and the constancy of residual life function for all $x \geq 0$. Often this approach provide a larger class of distributions than the exponential.

The choice in most cases will be a function g(x) for which either

$$E(h(x) \mid X \geq x) = g(x);\ x \geq 0 \qquad (1.9)$$

or

$$E(h(X-x) \mid X \geq x) = g(x) \qquad (1.10)$$

where h(.) and g(.) are known functions ending up with the solutions that are proper survival functions. For details we refer to Kotlarski (1972), Laurent (1972) Shanbhag and Rao (1975), Gupta (1976) and Dallas (1976).

An attempt along some what different lines by Muliere and Scarsini (1987) to extend (1.1) in generating a class of probability distribution, uses the extension of the LMP by the following equation,

$$P(X > x*y) = P(X > x)\, P(X > y). \qquad (1.11)$$

In equation (1.11) '*' is used to represent a binary operation that is associative, and reducible ($x * y = x*z \Rightarrow y = z$). When the last equation is read as

$$R(x*y) = R(x)\, R(y) \qquad (1.12)$$

its only continuous solution is

$$x * y = g^{-1}(g(x) + g(y)) \qquad (1.13)$$

with g(.) continuous and strictly monotonic. In this case, the only continuous solution of (1.12) is

$$R(x) = \exp(-\lambda\, g(x)),\ \lambda > 0 \qquad (1.14)$$

$u = g^{-1}(0) < x < g^{-1}(\infty)$. With appropriate choice for $g(x)$ the authors characterize class of probability distribution that includes the exponential, Pareto Type I, Weibull models. As a bivariate extension of the above functional equation Muliere and Scarsini (1987) also derives Marshall Olkin (1967) type class of distributions also.

The concepts and methods so far reviewed extends to a multivariate setup. Since our interest in the present investigation concerns only bivariate distributions, the important developments in this area are now presented. An obvious extension of the LMP in the bivariate case is defined by the relationship

$$R(x_1 + y_1,\ x_2 + y_2) = R(x_1,x_2)\ R(y_1,y_2) \qquad (1.15)$$

for all $x_1,x_2,y_1,y_2 > 0$, where

$$R(x_1,x_2) = P(X_1 > x_1,\ X_2 > x_2)$$

is the survival function of the random vector $X = (x_1,x_2)$ in the support of $R_2^+ = \{(x_1,x_2) \mid x_1,x_2 > 0\}$.

A serious limitation of defining bivariate LMP, by equation (1.15) is that its unique solution turns out to be

$$R(x_1,x_2) = e^{-\lambda_1 x_1 - \lambda_2 x_2} \quad (1.16)$$

which is the trivial bivariate exponential distribution which is the product of its marginals. In the reliability context this amounts to the distribution of the life times of a two component system, in which the life time of each component is independent of the other. This severe restriction that prevents consideration of two-component systems where there is dependency among individual components, has led to a relaxation of the requirements on the values of y_1 and y_2. One way of doing this is to consider the equation,

$$R(x_1+t,x_2+t) = R(x_1,x_2).R(t,t) \quad (1.17)$$

for all $x_1,x_2,t>0$. Marshall and Olkin (1967) derived a solution of (1.17) by requiring the marginals of X_1 and X_2 to be exponential distribution with parameters λ_1 and λ_2 in the form,

$$R(x_1,x_2) = \exp[-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_{12}\max(x_1,x_2)]. \quad (1.18)$$

In this setup P(X=Y) is positive so that the simultaneous failure of the two components is a physical characteristic concerning the system, although this produces a singular component in the distribution. This is unavoidable in the sense that the assumptions of LMP, absolute continuity and exponential margianls can result only bivariate distribution with independent exponential marginals. It is therefore apparent that to arrive at a meaningful bivariate distribution, one has to abandon anyone of the three conditions mentioned just above. By preserving LMP and absolute continuity, Block and Basu (1974) derived bivariate exponential distribution, in which the marginals are mixture of exponentials. In spite of other extensions proposed by Friday and Patil (1977), Arnold (1975). Esary and Marshall (1974) based on reliability considerations a detailed assessment of these distributions vis-a-vis their usefulness through characterization by reliability concepts are yet to be established.

A fruitful alternative way of looking at the equipment behaviour can be accomplished by investigating the behaviour of one of the components, when life time of the other is pre-assigned. The first work in this direction concerning the bivariate system, appears to be that of Johnson and Kotz (1975) who defined the vector valued failure rate of a device with component lifetimes (X_1, X_2) as

$$h(x_1,x_2) = (h_1(x_1,x_2), h_2(x_1,x_2)) \tag{1.19}$$

where

$$h_i(x_1,x_2) = \frac{-\partial \log R(x_1,x_2)}{\partial x_i}, \; i = 1,2 \tag{1.20}$$

Drawing parallels from the univariate theory, they considered the situation when

$$h(x_1,x_2) = (c_1,c_2) \tag{1.21}$$

where c_i's are constants independent of x_1 and x_2, and established that such a case exist only when the joint distribution has independent exponential marginals. Accordingly, they considered the situation where the components of the failure rate are locally constant, in the form,

$$h(x_1,x_2) = (A_1(x_2),\; A_2(x_1)) \tag{1.22}$$

and characterized the Gumbel's (1960) bivariate exponential distribution with survival function,

$$R(x_1,x_2) = \exp(-\lambda_1 x_1 - \lambda_2 x_2 - \theta x_1 x_2), \tag{1.23}$$

$\lambda_1, \lambda_2 > 0$, $0 \le \theta \le \lambda_1 \lambda_2$ with such a property.
In proving this result, they used the representation

$$R(x_1,x_2) = \exp\left[- \int_0^{x_1} h_1(t,0)dt - \int_0^{x_2} h_2(x_1,t)dt \right] \tag{1.24}$$

that connects uniquely, the failure rate and the survival function. The fact that, unlike in the univariate case, the failure rate can be defined in a multiplicity of ways, leaves open the question of constructing multivariate distributions that can serve us models of specific equipment behaviour. This will be further discussed in the sequel.

Taking specific clues from the definition of bivariate failure rate, it is possible to look for representation for the mean residual life in higher dimensions. Defining the mean residual life function as the vector,

$$r(x_1,x_2) = (r_1(x_1,x_2), r_2(x_1,x_2)) \tag{1.25}$$

where

$$r_i(x_1,x_2) = E(X_i - x_i \mid X_1 > x_1,\ X_2 > x_2) \tag{1.26}$$

$$i = 1,2$$

and using the unique representation

$$R(x_1,x_2) = \frac{r_1(0,0).r_2(x_1,0)}{r_1(x_1,0).\ r_2(x_1,x_2)}$$

$$x \exp\left\{ -\int_0^{x_1} \frac{dt_1}{r_1(t,0)} - \int_0^{x_2} \frac{dt_2}{r_2(x_1,t_2)} \right\} \tag{1.27}$$

Nair and Nair (1988 a) established a characterization of the Gumbel's distribution (1.23) using the local constancy of

$$r(x_1,x_2) = (B_1(x_2), B_2(x_1))$$

This result was further extended by Nair and Nair (1988b) by showing that the local constancy of the truncated moments,

$$E((X_i - x_i)^r \mid X_1 > x_1,\ X_2 > x_2) = B_i(x_j) \tag{1.28}$$

$$i,j = 1,2,\ i \neq j$$

for every $r = 1,2,3,\ldots$ is a characteristic property of the same distribution. Nair and Nair (1991) further defined the local lack of memory property of the random vector X by the relations,

$$P(X_i > x_i+y_i \mid X_1 > x_1, X_2 > x_2) = P(X_i>x_i \mid X_j>x_j),$$

$$i,j = 1,2 \quad i \neq j \qquad (1.29)$$

and established the equivalence of (1.29) and (1.22).

Analogous to the univariate definition of LMP, one can think about a similar property for the conditional distribution arising from a bivariate distribution. In this way Nair and Nair (1991) defined the notion of conditional lack of memory for a random vector in the support of R_2^+ by the relationship

$$P(X_i \geq t_i+s_i \mid X_i \geq s_i, X_j=x_j) = P(X_i \geq t_i \mid X_j = x_j),$$

$$i,j=1,2, \quad i \neq j \qquad (1.30)$$

for all x_j, t_i, $s_i > 0$.

The equations (1.30) are satisfied if and only if the distribution of X_i given $X_j=x_j$ has density,

$$f(x_i | x_j) = \lambda_i(x_j) \exp(-\lambda_i(x_j)x_i)$$

which corresponds to the exponential form. Accordingly, the bivariate distribution that possesses conditional LMP is one for which the conditional densities are exponential. Arnold and Strauss (1988) has shown that there exist a unique bivariate distribution with exponential conditionals and obtained its density function as

$$f(x_1,x_2) = \lambda_1\lambda_2\theta \exp[-\lambda_1 x_1 - \lambda_2 x_2 - \delta\lambda_1\lambda_2 x_1 x_2]$$

$$\lambda_1,\lambda_2 > 0,\ \delta \geq 0,\ x_1,x_2 > 0 \qquad (1.31)$$

where $\theta = \theta(\delta) = \delta\ e^{-1/\delta} / [\ -E_i(1/\delta)]$

and E_i is the well known exponential integral function

$$-E_i(u) = \int_u^\infty e^{-w}\ w^{-1} dw$$

The above discussions provide the main results that are required to identify our research problem References needed to supplement the specific problems will be provided in the appropriate chapters.

1.3 RESERACH PROBLEM AND SUMMARY OF THE THESIS.

It is already shown that basically, there are two approaches to extend the definition of the LMP to the bivariate case; One given in Marshall and Olkin (1967) and the other designated as local LMP in Nair and Nair (1990). The generalised version of LMP proposed by Muliere and Scarsini (1987) in the univariate case, naturally allows extension to the bivariate case. In the same manner as the LMP of the univariate exponential distribution was extended to generate bivariate exponential model of Gumbel (1960) through the local LMP, there is scope for attempting a localized bivariate version of the relation specified by equation (1.16). Accordingly in Chapter II we consider the equations,

$$P(X_i > G(x_i, y_i) \mid X_i > x_i,\ i=1,2) = P(X_i > y_i \mid X_j > x_j)$$

$$i,j=1,2, i \neq j$$

Where G(.,.) is a function satisfying certain algebraic properties. The family of continuous distributions characterized by such a property is derived. It is shown that by appropriate choices of G(.,.) the bivariate exponential, Pareto, Weibull etc. can be reached as special cases. It is well known that by monotone transformations of the exponential variable, it is possible to obtain several other continuous distributions as Weibull, Pareto, Logistic extreme value, Uniform etc. but there exist no meaningful single defining property that embraces all these models. The above mentioned defining property forms a basis from which all such related distributions can be brought under a uniform framework. After deriving the basic model giving rise to what is called the bivariate exponential type family, some of its important members are identified. The fundamental objective in studying a family of distributions is to derive the global properties enjoyed by it, so that individual investigation of each of its members can be avoided. Thus we look at the properties of the family, like, the constituent marginals, conditional distributions, moments regression and correlation etc. Since the major application is envisaged in the analysis of reliability, the basic quantities required in such cases for modelling equipment behaviour such as failure rate, mean residual life and residual life distributions are derived.

Although the Bivariate exponential type (BET) family itself originated from a characteristic property, we look into other possible characterizations in Chapter III. The main objective of this investigation is to find out interesting properties by which specific members of the family can be identified in a practical situation. Characterizations through conditional distributions, functional forms of failure rates, conditional expectation and mean failure rates are presented.

Following the logic involved in the extension of local LMP, we consider extension of the conditional LMP defined in equation (1.42) in Chapter IV. In effect, this provides bivariate distributions that are determined uniquely, by their conditional distributions. The results in the Chapter supplements the efforts recently made by researchers in addressing the problem of finding bivariate distributions that are compatible with conditional distributions with pre-designated forms. The members of the BET family of Chapter II are viewed to represent the distribution of life lengths in two-component systems assessed in the tests in laboratory where the components are built. When they work outside the laboratory, the operating condition might be different. A new system of models that can accommodate such changes are also discussed in Chapter IV. This approach provides several new additions to the class of continuous bivariate distributions.

Chapter II

BIVARIATE EXPONENTIAL TYPE DISTRIBUTIONS

2.1 INTRODUCTION

Consistent with the objectives of the present study setforth in Section 1.2, an extension of the bivariate local lack of memory property (LLMP) and the class of probability distributions characterized by such a property will be introduced in this Chapter. The extended version of the LLMP worked out here concerns a random vector $X=(X_1,X_2)$ admitting absolutely continuous survival function with respect to Lebsegue measure in the support of $R_2^+=\{(x,y)|x,y>0\}$ or its subspaces. The pre-requisite for the development of the subject is a function $G(.,.)$; defined from R_2^+ into (a,b) where (a,b) is an interval on the real line, satisfying the following properties.

P_1 $\quad G(G(x,y),z) = G(x,G(y,z))$ for all x,y,z in (a,b) (2.1)

P_2 $\quad$ There exists an element u in (a,b) satisfying

$$G(u,x)=x \qquad \text{for every } x \tag{2.2}$$

and

P_3 $G(x,r) = G(x,s)$ and $G(r,y) = G(s,y)$ (2.3)

both imply $r=s$.

Under these conditions, it is shown in Aczel (1966), that the only continuous solution of (2.1) in (u,c), where c is in (a,b), is

$$G(x,y) = g^{-1}(g(x)+g(y)) \tag{2.4}$$

with continuous and strictly increasing $g(.)$, provided that $x,y,z,G(x,y),G(x,z),G(G(x,y),z)$ all lie in (a,b). In this case the inverse function $g^{-1}(.)$ exists for every x in (u,c) and $G(x,y)$ is strictly increasing in each of the variables. We utilize this result to seek the class of bivariate distributions satisfying the extended version of the LLMP defined by

$$P(X_i>G(x_i,y_i)|X>x) = P(X_i>y_i|X_j>x_j) \tag{2.5}$$

$i,j = 1,2;\ i\neq j$ for every x_i,y_i in $(u,g^{-1}(\infty))$. The symbol $X>x$ in (2.5) stands for $X_i>x_i$, $i=1,2$.

The conditional probability statement on the left side of (2.5) for i=1, becomes

$$P[x_1>G(x_1,y_1),X_1>x_1,X_2>x_2]/P[X>x]$$

$$=P[X_1>G(x_1,y_1),X_1>G(x_1,u),X_2>x_2]/P[X>x] \quad (2.6)$$

using (2.2). Since $G(x_1,y_1)$ is strictly increasing in each of the variables, $G(x_1,y_1) > G(x_1,u)$ for every y_1 in $(u,g^{-1}(\infty))$. Thus (2.6) reduces to

$$P(X_1>G(x_1,y_1),X_2>x_2)/P[X>x] = R(G(x_1,y_1),x_2)/R(x_1,x_2). \quad (2.7)$$

A similar expression results in the case of i=2.

In section 2.2 we establish that the property (2.5) uniquely determines a class of bivariate models which will be designated in the sequel as "bivariate exponential type" (BET) probability distributions.

2.2 DERIVATION OF THE MODEL.

Theorem 2.1

Let $X=(X_1,X_2)$ be a random vector admitting absolutely continuous survival function $R(x_1,x_2)$. A necessary and sufficient condition for X to have the generalised LLMP defined by (2.5) with respect to a function G(.,.) satisfying P_1, P_2 and P_3 is that

$$R(x_1,x_2) = \exp[-\alpha_1 g(x_1) - \alpha_2 g(x_2) - \theta\, g(x_1)g(x_2)] \qquad (2.8)$$

$$u<x_i<g^{-1}(\infty),\ i=1,2;\ \alpha_1,\alpha_2>0\ \ 0\le\theta\le\alpha_1\alpha_2,$$

where g(.) is a continuous and strictly increasing function.

Proof:

Starting from (2.5) and using the arguments that led to (2.7) for a random vector X in the support of $\{u,g^{-1}(\infty)\} \times \{u,g^{-1}(\infty)\}$, the generalised LLMP can be stated as

$$R(G(x_1,y_1),x_2)\ R(u,x_2) = R(x_1,x_2)R(y_1,x_2) \qquad (2.9)$$

and

$$R(x_1,G(x_2,y_2))\ R(x_1,u) = R(x_1,x_2)R(x_1,y_2) \qquad (2.10)$$

As $x_2 \rightarrow u$ in (2.9),

$$R(G(x_1,y_1),u)\ R(u,u) = R(x_1,u)R(y_1,u),$$

or

$$R_1(G(x_1,y_1)) = R_1(x_1).\ R_1(y_1) \tag{2.11}$$

where

$$R_1(x_1) = R_1(x_1,u) = P(X_1>x_1)$$

is the marginal survival function of the component variable X_1. Equation (2.11) is of Cauchy form whose only continuous solution is (Muliere and Scarsini (1987))

$$R_1(x_1) = \exp[-\alpha_1 g(x_1)], \qquad \alpha_1>0. \tag{2.12}$$

A similar deduction using (2.10) leads to

$$R_2(x_2) = \exp[-\alpha_2 g(x_2)], \qquad \alpha_2>0. \tag{2.13}$$

On writing

$$H(x_1,x_2) = R(x_1,x_2)\ \exp\,[\alpha_2 g(x_2)], \tag{2.14}$$

(2.9) takes the form

$$H(G(x_1,y_1),x_2) = H(x_1,x_2)\ H(y_1,x_2) \tag{2.15}$$

which is again of the form (2.11) so that (2.15) has the unique solution

$$H(x_1,x_2) = \exp[-c_1(x_2)g(x_1)] \quad \text{for some } c_1(x_2)>0. \tag{2.16}$$

Inserting (2.16) into (2.14),

$$R(x_1,x_2) = \exp[-\alpha_2 g(x_2)-C_1(x_2)g(x_1)]. \tag{2.17}$$

Repeating the same type of reasoning with respect to (2.10), using (2.13), an alternative expression for $R(x_1,x_2)$ results as

$$R(x_1,x_2) = \exp[-\alpha_1 g(x_1)-C_2(x_1)g(x_2)]. \tag{2.18}$$

Identifying (2.17) and (2.18) we have

$$\alpha_2\ g(x_2) + C_1(x_2)\ g(x_1) = \alpha_1\ g(x_1) + C_2(x_1)\ g(x_2)$$

Dividing by $g(x_1)\ g(x_2)$ and re - arranging the terms lead to the variable separable form

$$\frac{c_2(x_1) - \alpha_2}{g(x_1)} = \frac{c_1(x_2) - \alpha_1}{g(x_2)}$$

The last equation holds for every x_1 and x_2 if and only if

$$\frac{c_i(x_j) - \alpha_i}{g(x_j)} = \theta, \text{ a constant}$$

independent of x_i and x_j. Thus

$$c_i(x_j) = (\alpha_i + \theta\, g(x_j)) \tag{2.19}$$

Substituting for $c_i(x_j)$ in either (2.17) or (2.18) we find that $R(x_1,x_2)$ is as stated in equation (2.8).

Conversely, assume that the survival function of the random vector X is of the form (2.8) and conditions P_1 to P_3 are satisfied then by direct calculations it can be shown that the LLMP is satisfied.

To complete the proof of our assertion it remains to show that the parameters of the distribution lie within the ranges specified in (2.8). For this we first note that for $R(x_1,u)\geq 0$ and $R(u,x_2)\geq 0$, for each value of the argument, $\alpha_1>0$ and $\alpha_2>0$.

Now consider the inequality

$$R(x_1,u) \geq R(x_1,x_2)$$

which implies

$$\exp\,[\alpha_2 + \theta\, g(x_1)]\; g(x_2) \geq 1$$

or

$$[\alpha_2 + \theta\, g(x_1)]\; g(x_2) \geq 0. \tag{2.20}$$

From $g(G(x_1,x_2)) = g(x_1) + g(x_2)$, setting $x_2 = u$ and using P_2, $g(x_1) = g(x_1) + g(u)$ showing that $g(u) = 0$. Since $g(.)$ is strictly increasing in $(u, g^{-1}(\infty))$ this would mean that for every x_1, x_2 in $(u, g^{-1}(\infty))$, $g(x_i)>0$ for $i=1,2$. Hence for the inequality (2.20) to hold for all x_1, x_2 one must have $\theta \geq 0$.

Finally, the density of X is

$$f(x_1,x_2) = \frac{\partial^2 R(x_1,x_2)}{\partial x_1 \, \partial x_2}$$

$$= [(\alpha_1+\theta g(x_2))(\alpha_2+\theta g(x_1)) - \theta] \; g'(x_1) \; g'(x_2)$$

$$\exp[-\alpha_1 g(x_1) - \alpha_2 g(x_2) - \theta g(x_1) g(x_2)] \quad (2.21)$$

The requirement $f(u,u) \geq 0$ leaves the condition $\alpha_1\alpha_2 - \theta \geq 0$ or $\theta \leq \alpha_1\alpha_2$. This completes the proof of Theorem 2.1.

2.3 SOME MEMBERS OF THE CLASS.

Obviously, Theorem 2.1 provides a class of probability distributions as appropriate choices of $G(x_1,x_2)$

will lead to the various members of the family. The function $G(x_1, x_2)$ can be thought of as a binary operator among the various points in the interval $(u, g^{-1}(\infty))$, that are algebraically and physically meaningful. We give below some useful models arising out of such considerations.

2.3.1 $G(x,y) = x+y$

Taking $G(x,y) = x+y$, we see that $g(.)$ satisfies the functional equation, $g(x+y) = g(x) + g(y)$ which has unique continuous solution $g(x)=cx$, $c>0$, further $g^{-1}(\infty)=\infty$ and from $G(x,u) = x$, we have $x>0$. This leads to the Gumbel's (1960) bivariate exponential distribution with survival function,

$$R(x_1, x_2) = \exp [-\alpha_1 x_1 - \alpha_2 x_2 - \theta x_1 x_2) \qquad (2.22)$$

$x_1, x_2 > 0$, $\alpha_1, \alpha_2 > 0$, $0 \le \theta \le \alpha_1 \alpha_2$. In this case the property (2.5) becomes the usual LLMP of Nair and Nair (1991), defined in equation (1.35).

It is in this sense that (2.5) was designated as the extended version of the LLMP and the family derived on that basis are called the family of bivarite exponential type distributions.

The properties of the standard form of this distribution (when $\alpha_i=1$) are discussed in Gumbel (1960). while those of (2.2) along with various characterizations are available in Nair (1990). The potential of this distribution in the context of reliability model is evident from the functional form of its failure rate and mean residual life given in Section 1.2.

2.3.2 $G(x,y) = xy$

This results when the binary operation is ordinary multiplication and the equation to be satisfied by g(.) is

$$g(xy) = g(x)\ g(y),$$

with unique continuous solution $g(x) = c \log x$. Also $G(x,u)=x=xu$ gives $u=1$ and $g(\infty)=\infty$, so that each of X_1 and X_2 has support $(1,\infty)$. The form of the survival function is

$$R(x_1,x_2) = x_1^{-\alpha_1} x_2^{-\alpha_2} x_1^{-\theta \log x_2}, \quad x_1,x_2>1. \tag{2.23}$$

The marginal distribution of X_i, specified by

$$R_i(x_i) = x_i^{-\alpha_i}, \qquad x_i \geq 1,\ i=1,2$$

is easily recognised as that of Pareto type I (The numbering of the Pareto classes is as in Arnold (1983)), so that (2.23) is a bivariate Pareto type I distribution. Reading (2.23) along with (2.5), the characteristic property of the model becomes,

$$P[X_i > x_i y_j | X > x] = P[X_i > y_i | X_j > x_j]. \tag{2.24}$$

This generalises the "dullness" property defined by Talwalker (1980) in the univariate case, which says that an absolutely continuous random variable X is totally dull at a point X on its support, if

$$P[X \geq xy | X \geq x] = P[X \geq y], \qquad \text{for all } y \geq 1. \tag{2.25}$$

Thus the result established in the present section becomes a bivariate analogue of Talwalker's result.

2.3.3 $G(x_i, y_i) = (x_i^p + y_i^p)^{1/p}$

Considering the choice $G(x_i, y_i) = (x_i^p + y_i^p)^{1/p}$, we have all conditions P_1 to P_3 imposed on G are satisfied and

$$g\,[(x_i^p + y_i^p)^{1/p}] = g(x_i)\, g(y_i). \qquad (2.26)$$

The unique solution of this functional equation that provides a survival function is

$$g(x_i) = c x_i^a, \qquad c, a > 0$$

Further,

$x_i = G(x_i, u) = (x_i^p + u^p)^{1/p}$ holds if and only if $u = 0$ and further $g(\infty) = \infty$. Thus we arrive at the survival function,

$$R(x_1, x_2) = \exp\,[\,-\alpha_1 x_1^{\beta} - \alpha_2 x_2^{\beta} - \theta x_1^{\beta} x_2^{\beta}). \qquad (2.27)$$

This represents a bivariate Weibull distribution with marginal density functions

$$f_i(x_i) = \alpha\beta\, x_i^{\beta - 1} \exp\,\{-\alpha x_i^{\beta}\}, \quad \alpha, \beta > 0, x_i > 0,\ i = 1, 2 \qquad (2.28)$$

being univariate Weibull with scale parameter $\alpha^{(-1/\beta)}$ and shape parameter β. Obviously for $\beta=1$ we have the distribution in section 2.3.1 and for $\beta=2$, it becomes the bivariate Rayleigh distribution discussed in Roy and Mukherjee (1989) and Roy (1993).

2.3.4 $G(x,y) = x+y+axy$

Setting $G(x,y) = x+y+axy$ gives,

$$g(x+y+axy) = g(x)+g(y)$$

which is equivalent to

$$h(x+y+axy) = h(x)h(y) \tag{2.29}$$

where

$$g(x) = \log h(x)$$

The unique solution of (2.29) is, however,

$$h(x) = (1+ax)^c$$

or

$$g(x) = c \log (1+ax)$$

This would give the survival function of X as

$$R(x_1,x_2) = \exp[\ -\lambda_1 c \log(1+ax_1) - \lambda_2 c \log(1+ax_2)$$

$$-\delta\ c^2 \log(1+ax_1) \log(1+ax_2)]$$

$$= (1+ax_1)^{-(\alpha_1+\theta \log(1+ax_2))} (1+ax_2)^{-\alpha_2} \quad (2.30)$$

with $\alpha_i = \lambda_i c$ and $\theta = \delta c^2$. The support of the distribution is found from

$$x=G(x,u)=x+u+axu$$

giving u=0 and $g^{-1}(\infty)=\infty$. For R to be a proper survival function, the parameters must satisfy the conditions $\alpha_i>0$ and $\theta \geq -\alpha_i^2$. The marginal distributions of (2.30) are Pareto type II distribution with survival functions,

$$R_i(x_i) = (1+a_i x_i)^{-c}, \quad a_i,x_i,c>0. \quad (2.31)$$

2.3.5 $G(x,y) = (x+y)(1+(xy/c^2))^{-1}$

In this case, the functional equation to be solved is

$$g((x+y)/(1+(xy/c^2))) = g(x) + g(y)$$

As before, setting $g(x) = \log h(x)$, we arrive at

$$h((x+y)/(1+(xy/c^2)) = h(x)h(y)$$

with unique solution in the form $h(x) = \left(\frac{c+x}{c-x} \right)^a$. Thus

$$g(x) = \log h(x) = a \log \left(\frac{c+x}{c-x} \right)$$

The survival function of X reduces to

$$R(x_1,x_2) = \exp \left[-\lambda_1 a \log \left(\frac{c+x_1}{c-x_1}\right) - \lambda_2 a \log \left(\frac{c+x_2}{c-x_2}\right) - \delta a \log \left(\frac{c+x_1}{c-x_1}\right) \log \left(\frac{c+x_2}{c-x_2}\right) \right]$$

$$= \left(\frac{c-x_1}{c+x_1}\right)^{\alpha_1} \left(\frac{c-x_2}{c+x_2}\right)^{\alpha_2} \left(\frac{c-x_1}{c+x_1}\right)^{\theta \log [(c-x_2)/(c+x_2)]} \tag{2.32}$$

with $\alpha_i = \lambda_i a$ and $\theta = \delta a^2$. To complete the derivation of the distribution, the support of the variables and the ranges of the values of the parameters are required. As usual appealing to $G(x,u)=x$, the resulting condition is

$$\mho(x^2-c^2) = 0.$$

Since at $x=\pm C$ the function $g(x)$ will explode and therefore, $u=0$. Being a logarithmic function, $g(x)$ has to remain positive which forbids the consideration of $g^{-1}(\infty)$ as the right end of the support. Instead we restrict $g(x)$ to be positive for all values of x. This gives $x<c$ and hence the support of the bivariate distribution settles as $(0,c) \times (0,c)$.

For $R(x_1,x_2)$ to be a proper survival function, the parameters must satisfy the conditions, $\alpha_i>0$ and $\theta\geq-\alpha_1\alpha_2$, i=1,2. The marginal distributions of (2.32) are finite range distribution with survival functions

$$R_i(x_i) = \left(\frac{c+x_1}{c-x_1}\right)^{-\alpha_1}, \qquad \alpha_i, c>0, \ i=1,2.$$

Note: The calculations for model 5 is given in Appendix

2.4. GENERAL PROPERTIES OF THE BET FAMILY

In this section some general properties of the family with probability density function (2.21) are investigated. The marginal density function of X_i are directly obtained from the joint survival function (2.8) by allowing x_i to tend to u. Thus

$$R(x_1,u) = \exp[-\alpha_1 g(x_1)]$$

gives the marginal density of X_1 as

$$f_1(x_1) = -\frac{\partial}{\partial x_1} R(x_1,u)$$

$$= \alpha_1 g'(x_1) \exp[-\alpha_1 g(x_1)]; \quad x_1>u, \alpha_1>0. \qquad (2.33)$$

Similarly,

$$f_2(x_2) = \alpha_2 g'(x_2) \exp[-\alpha_2 g(x_2)]; \quad x_2>u, \alpha_2>0. \qquad (2.34)$$

The conditional probability density functions are

$$f_i(x_i|x_j) = f(x_1,x_2)/f_j(x_j), \quad i,j=1,2, \; i\neq j$$

$$= \alpha_j^{-1}[(\alpha_i+\theta g(x_j))(\alpha_j+\theta g(x_i))-\theta]\, g'(x_i)$$

$$\exp[-(\alpha_i+\theta g(x_j))g(x_i)]. \qquad (2.35)$$

2.4.1 Moments

The r^{th} raw moment of X_1, whenever it exists is given by

$$E(X_1^r) = \int_u^\infty x_1^r \alpha_1 g'(x_1) e^{-\alpha_1 g(x_1)} dx_1$$

$$= \alpha_1 \int_0^\infty [g^{-1}(t)]^r e^{-\alpha_1 t} dt$$

$$= \alpha_1 L(g^{-1}(t))^r \tag{2.36}$$

From (2.36) it is easy to see that the mean and variance of X_1 are

$$E(X_1) = \alpha_1 L(g^{-1}(t))$$

$$Var(X_1) = \alpha_1 \left[L(g^{-1}(t))^2 - L^2(g^{-1}(t))\right].$$

where L(.) denotes the Laplace transform. The values corresponding to the various members of the family presented in Section 2.3 are given in Table 2.2.

2.4.2 Conditional means

The conditional mean of X_1 given $X_2=x_2$ is

$$E(X_1|X_2=x_2) = \int x_1 f(x_1|x_2)dx_1$$

$$= \int_u^\infty x_1 \alpha_2^{-1}[(\alpha_1+\theta g(x_2))(\alpha_2+\theta g(x_1))-\theta] g'(x_1)$$

$$\exp[-(\alpha_1+\theta g(x_2))g(x_1)] dx_1$$

$$= \int_0^\infty g^{-1}(t)[(\alpha_1+\theta g(x_2))(\alpha_2+\theta t)-\theta]\,\alpha_2^{-1} \exp[-(\alpha_1+\theta g(x_2)t]\,dt$$

$$= \alpha_2^{-1}\Big[-g^{-1}(t)(\alpha_2+\theta t)G_1(t)\Big]_0^\infty + \int_0^\infty \Big[(\alpha_2+\theta t)\frac{dg^{-1}(t)}{dt}+\theta g^{-1}(t)\Big]G_1(t)dt - \theta\int_0^\infty g^{-1}(t)G_1(t)dt$$

where $G_1(t) = \exp[-(\alpha_1+\theta g(x_2))t]$

Thus

$$E(X_1|X_2=x_2) = u + \alpha_2^{-1}\int_0^\infty (\alpha_2+\theta t)\frac{dg^{-1}(t)}{dt}G_1(t)dt \qquad (2.37)$$

and

$$E(X_2|X_1=x_1) = u + \alpha_1^{-1}\int_0^\infty (\alpha_1+\theta t)\frac{dg^{-1}(t)}{dt}G_2(t)dt \qquad (2.38)$$

where $G_2(t) = \exp[-(\alpha_2+\theta g(x_1))t]$.

Model 1.

Specialising for Model 1, $g(x)=x$, $u=0$

$$E(X_1|X_2=x_2) = \alpha_2^{-1}\int_0^\infty (\alpha_2+\theta t)\, e^{-(\alpha_1+\theta x_2)t}\,dt$$

$$= \frac{\alpha_2(\alpha_1+\theta x_2)+\theta}{\alpha_2(\alpha_1+\theta x_2)^2}\,. \qquad (2.39)$$

$$E(X_i|X_j=x_j) = \frac{\alpha_j(\alpha_i+\theta x_j)+\theta}{\alpha_j(\alpha_i+\theta x_j)^2}, \quad i,j=1,2,i\neq j. \tag{2.40}$$

The regression of X_i on X_j is non-linear and decreasing in X_j.

Model 2

The regression functions for Model 2 can be obtained as follows. Here $g(x) = \log x$, $u=1$

$$E(X_1|X_2=x_2) = 1 + \alpha_2^{-1}\int_0^{\infty}(\alpha_2+\theta t)\, e^{-(\alpha_1+\theta\log x_2-1)t}\, dt$$

$$=1+ \frac{\alpha_2(\alpha_1+\theta\log x_2-1)+\theta}{\alpha_2(\alpha_1+\theta\log x_2-1)^2}. \tag{2.41}$$

Similarly,

$$E(X_1|X_2=x_2) = 1 + \frac{\alpha_1(\alpha_2+\theta\log x_1-1)+\theta}{\alpha_1(\alpha_2+\theta\log x_1-1)^2}. \tag{2.42}$$

Model 3

For model 3, $g(x)=x^{\beta}$, $u=0$

$$E(X_1|X_2=x_2) = \alpha_2^{-1}\int_0^{\infty}(\alpha_2+\theta t)\frac{1}{\beta}\, t^{(1/\beta)-1}\, e^{-(\alpha_1+\theta x_2^{\beta})t}\, dt$$

$$= \alpha_2^{-1}\left[\frac{\alpha_2}{\beta}\,\frac{\Gamma(\frac{1}{\beta})}{(\alpha_1+\theta x_2^{\beta})^{1/\beta}} + \frac{\theta}{\beta}\,\frac{\Gamma(\frac{1}{\beta}+1)}{(\alpha_1+\theta x_2^{\beta})^{(1/\beta)+1}}\right]$$

$$= \alpha_2^{-1}\left[\frac{\alpha_2}{\beta}\frac{\Gamma(\frac{1}{\beta})}{(\alpha_1+\theta x_2^{\beta})^{1/\beta}}+\frac{\theta}{\beta}\frac{\Gamma(\frac{1}{\beta})\ \frac{1}{\beta}}{(\alpha_1+\theta x_2^{\beta})^{(1/\beta)+1}}\right]$$

$$= (\alpha_2\beta)^{-1}\Gamma(\tfrac{1}{\beta})\left[\frac{\alpha_2(\alpha_1+\theta x_2^{\beta})+\frac{\theta}{\beta}}{(\alpha_1+\theta x_2^{\beta})^{(1/\beta)+1}}\right]$$

$$= \frac{\alpha_2^{-1}\Gamma(\frac{1}{\beta})}{\beta^2}\ \frac{[\alpha_2\beta(\alpha_1+\theta x_2^{\beta})+\theta]}{(\alpha_1+\theta x_2^{\beta})^{(1/\beta)+1}}$$

$$= \frac{\Gamma(\frac{1}{\beta})}{\alpha_2\beta^2}\ \frac{[\alpha_2\beta(\alpha_1+\theta x_2^{\beta})+\theta]}{(\alpha_1+\theta x_2^{\beta})^{(1/\beta)+1}} \tag{2.43}$$

$$E(X_2|X_1=x_1) = \frac{\Gamma(\frac{1}{\beta})}{\alpha_1\beta^2}\ \frac{[\alpha_1\beta(\alpha_2+\theta x_1^{\beta})+\theta]}{(\alpha_2+\theta x_1^{\beta})^{(1/\beta)+1}}\ . \tag{2.44}$$

Model 4

In respect of model 4, $g(x)=\log(1+ax)$, $u=0$.

$$E(X_1|X_2=x_2) = \alpha_2^{-1}\int_0^{\infty}(\alpha_2+\theta t)\frac{1}{a}\ e^{-(\alpha_1+\theta\log(1+ax_2))t}\ dt$$

$$= \frac{1}{\alpha_2 a}\int_0^{\infty}(\alpha_2+\theta t)\ e^{-(\alpha_1+\theta\log(1+ax_2)-1)}\ dt$$

$$= \frac{\alpha_2(\alpha_1+\theta\log(1+ax_2)-1)+\theta}{\alpha_2 a(\alpha_1+\theta\log(1+ax_2)-1)^2}\ . \tag{2.45}$$

Thus,

$$E(X_i|X_j=x_j) = \frac{\alpha_j(\alpha_i+\theta\log(1+ax_j)-1)+\theta}{\alpha_j a(\alpha_i+\theta\log(1+ax_j)-1)^2}\ ,\ i,j=1,2,\ i\neq j. \qquad (2.46)$$

$$E(X_1X_2) = \int_u^\infty x_1x_2f(x_1,x_2)dx_1dx_2$$

$$= \int_u^\infty x_2f_2(x_2)E(X_1|X_2=x_2)dx_2$$

$$= \int_u^\infty \left[u+\alpha_2^{-1}\int_0^\infty(\alpha_2+\theta t)\frac{dg^{-1}(t)}{dt}G_1(t)dt\right]x_2f_2(x_2)dx_2$$

$$= u\int_u^\infty \alpha_2\, e^{-\alpha_2 g(x_2)}x_2g'(x_2)dx_2$$

$$+\int_u^\infty\int_0^\infty(\alpha_2+\theta t)\frac{dg^{-1}(t)}{dt}G_1(t)e^{-\alpha_2 g(x_2)}x_2g'(x_2)dtdx_2$$

$$=u^2+u\int_0^\infty e^{-\alpha_2 g(x_2)}dx_2 + \int_0^\infty(\alpha_2+\theta t)\frac{dg^{-1}(t)}{dt}e^{-\alpha_1 t}$$

$$\left(\int_0^\infty x_2\, e^{-(\alpha_2+\theta t)g(x_2)}g'(x_2)dx_2\right)dt$$

$$=u^2+u\int_0^\infty \frac{dg^{-1}(y)}{dy}e^{-\alpha_2 y}dy + u\int_0^\infty \frac{dg^{-1}(t)}{dt}e^{-\alpha_1 t}dt$$

$$+\int_0^\infty\int_0^\infty e^{-\alpha_1 t-\alpha_2 y-\theta ty}\frac{dg^{-1}(t)}{dt}\frac{dg^{-1}(y)}{dy}dt\ dy$$

$$=u^2+u\left(L_{\alpha_2}\frac{dg^{-1}(y)}{dy} + L_{\alpha_1}\frac{dg^{-1}(t)}{dt}\right)$$

$$+L_{\alpha_1,\alpha_2}\left(e^{-\theta ty}\ \frac{dg^{-1}(t)}{dt}\ \frac{dg^{-1}(y)}{dy}\right). \qquad (2.47)$$

Specialising the above expression for the different models we have the following expressions.

Model 1 has

$$E(X_1X_2) = \theta^{-1}\exp[\alpha_1\alpha_2\theta^{-1}]E_1[\alpha_1\alpha_2\theta^{-1}] \qquad (2.48)$$

For model 2, with $g(x_i) = x_i^\beta$ and $u=0$, $i=1,2$

$$E(X_1X_2) = \int_0^\infty\int_0^\infty e^{-\alpha_1 t-\alpha_2 y-\theta ty}\ \frac{1}{\beta}\ t^{(1/\beta)-1}\frac{1}{\beta}\ y^{(1/\beta)-1}\ dy\ dt$$

$$=\int_0^\infty e^{-\alpha_1 t}\frac{1}{\beta^2}\ t^{(1/\beta)-1}\int_0^\infty e^{-(\alpha_2+\theta t)y}\ y^{(1/\beta)-1}\ dy\ dt$$

$$=\frac{\Gamma(\frac{1}{\beta})}{\beta^2}\int_0^\infty (\alpha_2+\theta t)^{-(1/\beta)}\ e^{-\alpha_1 t}\ t^{(1/\beta)-1}\ dt$$

$$= \frac{\Gamma(\frac{1}{\beta})}{\beta^2} \int_0^\infty (\alpha_2+z)^{-(1/\beta)} e^{-(\alpha_1/\theta)z} (z/\theta)^{(1/\beta)-1} dz/\theta$$

$$= \frac{\Gamma(\frac{1}{\beta})}{\beta^2\theta^{(1/\beta)}} \int_0^\infty \frac{z^{(1/\beta)-1}}{(\alpha_2+z)^{1/\beta}} e^{-(\alpha_1/\theta)z} dz \qquad (2.49)$$

Model 3

$$E(X_1X_2) = 1+ \frac{1}{\alpha_1-1} + \frac{1}{\alpha_2-1} + \theta^{-1} \exp[(\alpha_1-1)(\alpha_2-1)\theta^{-1}] E_1[(\alpha_1-1)(\alpha_2-1)\theta^{-1}]$$

Model 4

$$g(x_i)= \log(1+ax_i),\ u=0,\ i=1,2. \qquad (2.50)$$

It has $E(X_1X_2)$ as

$$E(X_1X_2) = \int_0^\infty \int_0^\infty e^{-\alpha_1 t-\alpha_2 y-\theta t y} \frac{1}{a^2} e^t e^y \, dy \, dt$$

$$= \frac{1}{a^2} \int_0^\infty e^{-(\alpha_1-1)t} \int_0^\infty e^{-[(\alpha_2-1)+\theta t]y} dy \, dt$$

$$= \frac{1}{a^2} \int_0^\infty (\alpha_2-1+\theta t)^{-1} e^{-(\alpha_1-1)t} dt$$

$$= a^{-2}\theta^{-1} \exp[(\alpha_1-1)(\alpha_2-1)\theta^{-1}]E_1[(\alpha_1-1)(\alpha_2-1)\theta^{-1}] \qquad (2.51)$$

2.4.3 The conditional distribution of X_i given $X_j > x_j$.

The density function of X_1 given $X_2 > x_2$ is

$$f(x_1 | X_2 > x_2) = \frac{1}{R_2(x_2)} \int_{x_2}^{\infty} f(x_1, t_2) dt_2$$

$$= e^{\alpha_2 g(x_2) - \alpha_1 g(x_1)} g'(x_1) \left[\int_{x_2}^{\infty} [(\alpha_1 + \theta g(t_2))(\alpha_2 + \theta g(x_1)) - \theta] \right.$$

$$\left. e^{-(\alpha_2 + \theta g(x_1)) g(t_2)} g'(t_2) dt_2 \right]$$

$$= [(\alpha_1 + \theta g(x_2)] g'(x_1) e^{-(\alpha_1 + \theta g(x_2)) g(x_1)} \qquad (2.56)$$

Similarly,

$$f(x_2 | X_1 > x_1) = [(\alpha_2 + \theta g(x_1)] g'(x_2) e^{-(\alpha_2 + \theta g(x_1)) g(x_2)} \qquad (2.57)$$

2.5 RELIABILITY MEASURES

2.5.1 Failure Rates

The bivariate vector valued failure rate defined by Johnson and Kotz (1975) is given in equation (1.24) is given as

$$h(x_1,x_2) = (h_1(x_1,x_2),h_2(x_1,x_2))$$

where

$$h_i(x_1,x_2) = (\alpha_i+\theta g(x_j))g'(x_i); \quad i,j=1,2, \ i\neq j. \qquad (2.58)$$

Basu's (1971) failure rate is

$$b(x_1,x_2) = f(x_1,x_2)/R(x_1,x_2)$$

$$= [(\alpha_1+\theta g(x_2))(\alpha_2+\theta g(x_1))-\theta]g'(x_1)g'(x_2) \qquad (2.59)$$

The forms of the failure rate function in respect of the standard models are presented in Table 2.3 and 2.4. The typical vector failure rate function in each of the models in Table 2.3 determines the corresponding distribution uniquely by virtue of the representation (1.29). However, some more general functional forms that can be postulated without identifying these exact expressions that characterize the distributions exist and these are investigated in Chapter III.

The knowledge of the form of $h(x_1,x_2)$ is helpful in informing whether it is monotone increasing or decreasing, thereby classifying the distributions according as IFR or DFR. Buchanan and Singpurwalla (1977) defines a multivariate distribution as belonging to IFR (DFR) class in the very strong sense if $R(x_1+t_1,x_2+t_2)/R(x_1,x_2)$ is a decreasing (increasing) of (x_1,x_2) for every $x_1,x_2,t_1,t_2>0$. It is easy to translate the above definition in terms of the behaviour of the failure rate components $h_1(x_1,x_2)$ and $h_2(x_1,x_2)$, since

$$\frac{R(x_1+t_1,x_2+t_2)}{R(x_1,x_2)} \uparrow \text{ in } x_1,x_2, \text{ for every } t_1,t_2,x_1,x_2>0.$$

$$\leftrightarrow \log \frac{R(x_1+t_1,x_2+t_2)}{R(x_1,x_2)} \uparrow \text{ in } x_1,x_2, \text{ for every } t_1,t_2,x_1,x_2>0.$$

$$\leftrightarrow \frac{\partial}{\partial x_1} \log \frac{R(x_1+t_1,x_2+t_2)}{R(x_1,x_2)} >0 \text{ for every } t_1,t_2,x_1,x_2>0.$$

$$\leftrightarrow -h_1(x_1+t_1,x_2+t_2)+h(x_1,x_2)>0$$

$$\leftrightarrow h_1(x_1+t_1,x_2+t_2)<h_1(x_1,x_2) \qquad \forall\ x_1,t_1,t_2>0$$

$$\leftrightarrow h_1(x_1,x_2) \text{ is } \downarrow \text{ in } x_1,x_2 \text{ for every } t_1,t_2>0.$$

Thus the bivariate distribution is IFR (DFR) in the very strong (VS) sense if $h_i(x_1,x_2)$, i=1,2 is a decreasing (increasing) function in both the arguments x_1 and x_2. In a similar manner we see that following the definitions of Buchanan and Singpurwalla (1977) that a bivariate life distribution is IFR (DFR) in the strong sense (S) if $h_i(x_1+t,x_2+t)$ is $\downarrow(\uparrow)$ in (x_1,x_2). IFR (DFR) in the weak (W) if $h_i(x_1+t_1,x_2+t_2)$ is $\uparrow(\downarrow)$ in x and IFR(DFR) in the very weak (VW) if $h_i(x_1+t,x_2+t)$ is $\downarrow(\uparrow)$ in x and IFR(DFR) in the very weak (VW) if $h_i(x+t,x+t)$ $\downarrow(\uparrow)$ in x.

In the light of these definitions, we see that the BET family consists of distributions that belongs to most of the IFR and DFR classes of models.The bivariate exponential distribution with independent marginals is both IFR-VS and DFR-VS, Model 1 is IFR-S, model 2 is DFR-S, model 3 can be IFR-S or DFR-S depending on the values of β and so on. Thus we have a flexible family of distributions from which a required model that satisfies the physical condition of the system can be chosen.

On the other hand, the scalar failure rates do not determine the distribution of (x_1,x_2) uniquely as survival function $R(x_1,x_2)$ is determined as solution of the second order differential equation,

$$\frac{\partial^2 R(x_1,x_2)}{\partial x_1 \partial x_2} = R(x_1,x_2)\, b(x_1,x_2).$$

2.5.2. The Mean Residual Life Function

In general for the BET family, the mean residual life function defined by (1.31) does not have a closed form expression. However, for certain specific models, this expression allows simplification.

Model 1 $\qquad r_i(x_1,x_2) = (\alpha_i+\theta x_j)^{-1}$

Model 2 $\qquad r_i(x_1,x_2) = (\alpha_i+\theta \log x_j-1)^{-1}\, x_i$

Model 4 $\qquad r_i(x_1,x_2) = a^{-1}(1+ax_i)\,(\alpha_i+\theta \log(1+ax_j)-1)^{-1}$

$i,j=1,2, \quad i\neq j.$

2.5.3 Residual Life Distributions

The mean residual life, variance residual life etc: often used in reliability modelling can be used as summary measures derived from the distribution of residual lives. Often it is more informative to look at the residual life distribution itself rather than these measures to have a detailed understanding of the failure process. The survival function of the residual lives corresponding to (x_1,x_2) is

$$S(y;x) = P[X>x+y \mid X>x]$$

where $X = (X_1,X_2)$, $x = (x_1,x_2)$ and $y = (y_1,y_2)$ and the ordering assumed to be componentwise. It is easy to see that

$$S(y;x) = R(x+y)/R(x)$$

Thus for the family (2.8)

$$S(y;x) = \exp[-\alpha_1(g(x_1+y_1)-g(x_1)) - \alpha_2(g(x_2+y_2)-g(x_2)) - \theta(g(x_1+y_1)g(x_2+y_2) - g(x_1)g(x_2))]$$

where $g(x)$ is continuous and has continuous derivatives in $(u,g^{-1}(\infty))$, it follows that

$$R(y,x) = \exp[\, -\alpha_1(y_1 g'(x_1)+\ldots) - \alpha_2(y_2 g'(x_2)+\ldots)$$
$$-\theta(y_1 g'(x_1)g(x_2)+y_2 g(x_1)g'(x_2)+y_1 y_2 g'(x_1)g(x_2)+\ldots)]$$

Thus when g(.) is a linear increasing function,

$$R(y;x) = \exp[\, -(\alpha_1+g(x_2))g'(x_1)y_1-(\alpha_2+g(x_1))g'(x_2)y_2$$
$$-\theta g'(x_1)g'(x_2)y_1 y_2]$$

the residual life distribution has the same form as the original life distribution, except for the parameters.

2.5.4 Distribution of Minimum and Maximum of (X_1,X_2)

The distribution of $T=\mathrm{Min}(X_1,X_2)$ and $Z=\mathrm{Max}(X_1,X_2)$ are specified by their survival functions,

$$P(T>t) = \exp[\, -\alpha_1 g(t) - \alpha_2 g(t) - \theta g^2(t)]$$

$$P(Z>z) = \exp[\, -\alpha_1 g(z)] + \exp[\, -\alpha_2 g(z)]$$
$$- \exp[\, -\alpha_1 g(z) - \alpha_2 g(z)-\theta g^2(z)]$$

These distributions arise in reliability when the components are connected in parallel or series.

Table 2.1

Model No.	G(x,y)	g(x)	Survival function	support
1 (exponential)	$x+y$	x	$\exp[-\alpha_1 x_1-\alpha_2 x_2-\theta x_1 x_2]$	$(0,\infty)x(0,\infty)$
2 (Pareto I)	xy	$\log x$	$x_1^{-(\alpha_1+\theta\log x_2)}\; x_2^{-\alpha_2}$	$(1,\infty)x(1,\infty)$
3 (Weibull)	$(x^p+y^p)^{\frac{1}{p}}$	x^a	$\exp[-\alpha_1 x_1^a-\alpha_2 x_2^a-\theta x_1^a x_2^a]$	$(0,\infty)x(0,\infty)$
4 (Pareto II)	$x+y+axy$	$\log(1+ax)$	$(1+ax_1)^{-(\alpha_1+\theta\log(1+ax_2))}\; (1+ax_2)^{-\alpha_2}$	$(0,\infty)x(0,\infty)$
5 (Finite Range)	$\dfrac{(x+y)}{\left(1+\dfrac{xy}{c^2}\right)}$	$\log\left(\dfrac{c+x}{c-x}\right)$	$\left(\dfrac{c-x_1}{c+x_1}\right)^{\left(\alpha_1+\theta\log\frac{c-x_2}{c+x_2}\right)} \left(\dfrac{c-x_2}{c+x_2}\right)^{\alpha_2}$	$(0,c)x(o,c)$

Table 2.2

Model No	$E(X_i)$	$Var(X_i)$
1	α_i^{-1}	α_i^{-2}
2	$\alpha_i(\alpha_i-1)^{-1}$	$\alpha_i(\alpha_i-2)^{-1}(\alpha_i-1)^{-2}$
3	$\alpha_i^{-1/\beta}\Gamma(1+\beta^{-1})$	$\alpha_i^{-2/\beta}[\Gamma(1+2\beta^{-1})-\Gamma^2(1+\beta^{-1})]$
4	$a^{-1}(\alpha_i-1)^{-1}$	$\alpha_i a^2(\alpha_i-2)^{-1}(\alpha_i-1)^{-2}$
5	$c\left[1-\alpha_i\left(\psi\left(\frac{\alpha_i+2}{2}\right)-\psi\left(\frac{\alpha_i+1}{2}\right)\right)\right]$	$\alpha_i c^2\left[2-2(\alpha_i+1)\left(\psi\left(\frac{\alpha_i+3}{2}\right)-\psi\left(\frac{\alpha_i+2}{2}\right)\right)-\alpha_i\left(\psi\left(\frac{\alpha_i+2}{2}\right)-\psi\left(\frac{\alpha_i+1}{2}\right)\right)^2\right]$

Table 2.3

Vector failure rate function of the BET family

Model	failure rate
1	$(\alpha_1+\theta x_2, \alpha_2+\theta x_1)$
2	$\left[(\alpha_1+\theta \log x_2)x_1^{-1}, (\alpha_2+\theta \log x_1)x_2^{-1}\right]$
3	$\left[\beta(\alpha_1+\theta x_2^{\beta})x_1^{\beta-1}, \beta(\alpha_2+\theta x_1^{\beta})x_2^{\beta-1}\right]$
4	$\left[a(\alpha_1+\theta \log(1+ax_2))(1+ax_1)^{-1},\right.$ $\left. a(\alpha_2+\theta \log(1+ax_1))(1+ax_2)^{-1}\right]$
5	$\left[2c(\alpha_1+\theta \log\left(\frac{c-x_2}{c+x_2}\right)(c^2-x_1^2)^{-1},\right.$ $\left. 2c(\alpha_2+\theta \log\left(\frac{c+x_1}{c-x_1}\right)(c^2-x_2^2)^{-1}\right]$
6	
Bivariate exponential with independent marginals	(α_1, α_2)

Table 2.4

Scalar failure rate

Model	Scalar failure rate
1	$[(\alpha_1+\theta x_2)(\alpha_2+\theta x_1)-\theta]$
2	$[(\alpha_1+\theta \log x_2)(\alpha_2+\theta \log x_1)-\theta]x_1^{-1}x_2^{-1}$
3	$\beta^2[(\alpha_1+\theta x_2^{\beta})(\alpha_2+\theta x_1^{\beta})-\theta]x_1^{\beta-1}x_2^{\beta-1}$
4	$\dfrac{a^2[(\alpha_1+\theta \log(1+ax_2))(\alpha_2+\theta \log(1+ax_1))-\theta]}{[1+ax_1][1+ax_2]}$
5	$4c^2\ (c^2-x_1^2)^{-1}(c^2-x_2^2)^{-1}$
6	$\left[\left(\alpha_1+\theta \log\ \dfrac{c+x_2}{c-x_2}\right)\left(\alpha_2+\theta \log\ \dfrac{c+x_1}{c-x_1}\right)-\theta\right]$
Bivariate exponential with independent marginals	$\alpha_1\alpha_2$

Appendix

Mean and variance of Model 5

$$E(X_1) = \alpha_1 \int_0^c x_1 \left(\frac{c-x_1}{c+x_1}\right)^{\alpha_1 -1} \frac{2c}{(c+x_1)^2}\, dx_1$$

$$= \alpha_1 c \int_0^1 \left(\frac{1-t}{1+t}\right) t^{\alpha_1 -1} dt$$

$$= c - 2\alpha_1 c \int_0^1 t^{\alpha_1} (1+t)^{-1}\, dt$$

$$= c - 2\alpha_1 c(\alpha_1+1)^{-1} {}_2F_1(1, \alpha_1+1, \alpha_1+2 : -1)$$

$$= c\left[1 - 2\alpha_1(\alpha_1+1)^{-1} \frac{(\alpha_1+1)}{2} \left\{\psi\left(\frac{(\alpha_1+2)}{2}\right) - \psi\left(\frac{(\alpha_1+1)}{2}\right)\right\}\right]$$

using the relationship ${}_2F_1(1,a,a+1:-1) = \frac{a}{2}\left[\psi\left(\frac{(a+1)}{2}\right) - \psi\left(\frac{a}{2}\right)\right]$ between the hypergeometric function and the digamma function $\psi(.)$ in Abramowitz and Stegun (1972, p557).

$$E(X_1) = c\left[1 - \alpha_1 \left\{\psi\left(\frac{(\alpha_1+2)}{2}\right) - \psi\left(\frac{(\alpha_1+1)}{2}\right)\right\}\right]$$

Similarly,

$$E(X_1^2) = \alpha_1 c^2 \int_0^1 \left(1 - \frac{2t}{1+t}\right)^2 t^{\alpha_1 - 1} dt$$

$$= \alpha_1 c^2 \int_0^1 \left[t^{\alpha_1 - 1} - 4t^{\alpha_1}(1+t)^{-1} + 4t^{\alpha_1 + 1}(1+t)^{-2}\right] dt$$

$$= \alpha_1 c^2 \Big\{ \alpha_1^{-1} - 4(\alpha_1+1)^{-1} {}_2F_1(1, \alpha_1+1, \alpha_1+2 : -1)$$

$$+ 4(\alpha_1+2)^{-1} {}_2F_1(2, \alpha_1+2, \alpha_1+3 : -1) \Big\}$$

$$= \alpha_1 c^2 \Big[\alpha_1^{-1} - 2\left\{\psi\left(\frac{(\alpha_1+2)}{2}\right) - \psi\left(\frac{(\alpha_1+1)}{2}\right)\right\}$$

$$+2 - 2(\alpha_1+1)\left\{\psi\left(\frac{(\alpha_1+3)}{2}\right) - \psi\left(\frac{(\alpha_1+2)}{2}\right)\right\}\Big]$$

$$[E(X_1)]^2 = c^2 \Big[1 + \alpha_1^2 \left\{\psi\left(\frac{(\alpha_1+2)}{2}\right) - \psi\left(\frac{(\alpha_1+1)}{2}\right)\right\}^2$$

$$- 2\alpha_1 \left\{\psi\left(\frac{(\alpha_1+2)}{2}\right) - \psi\left(\frac{(\alpha_1+1)}{2}\right)\right\}\Big]$$

$$\mathrm{Var}(X_1) = E(X_1^2) - [E(X_1)]^2$$

$$= \alpha_1 c^2 \Big[2 - 2(\alpha_1+1) \left\{\psi\left(\frac{(\alpha_1+3)}{2}\right) - \psi\left(\frac{(\alpha_1+1)}{2}\right)\right\}$$

$$- \alpha_1\left\{\psi\left(\frac{(\alpha_1+2)}{2}\right) - \psi\left(\frac{(\alpha_1+1)}{2}\right)\right\}^2\Bigg].$$

Note 1. ${}_2F_1(a,b,c:x)$ is the hypergeomtric function defined by ${}_2F_1(a,b,c:x) = F(a,b,c:x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{x^n}{n!}$

with $(a)_n = a(a+1)\ldots(a+n-1)$. (Abramowitz and Stegun, page 556)

2. The value for model 1 through 4 corresponds to univariate distrbutions available in Johnson and Kotz (1972)

Chapter III

CHARACTERIZATIONS OF BET FAMILY

3.1 INTRODUCTION

A characterization theorem is the only exact method to identify uniquely the form of distribution function, that generated the observations. These theorems also help us to have a better understanding of the structure and the implication of the physical system that is subjected to investigation. Accordingly, in this Chapter, we establish some characterizations related to BET family, as a whole and several unique properties that pertain to some of its members.

3.2 CHARACTERIZATION BY CONDITIONAL DISTRIBUTIONS

Events of the form $X_j > x_j$, where X_j represents a non-negative, continuous random variable are of specific interest in various fields of applied research such as reliability, survival analysis, extreme value theory etc., with appropriate interpretations for X_j. In the bivariate set up it is possible to have access on information about the behaviour of the variable X_i given that $X_j>x_j$ $i,j = 1,2;$ $i\neq j$ and therefore there is some apparent interest in characterizing the joint distribution of (X_1,X_2) given the forms of the conditional distribution of the X_i given $X_j>x_j$. We first establish a theorem that characterizes the BET family using such conditional distributions.

Theorem 3.1

Let $X = (X_1,X_2)$ be a bivariate random vector admitting absolutely continuous distribution with respect to Lebesgue measure in the support of $(u,g^{-1}(\infty)) \times (u,g^{-1}(\infty))$. The survival function of X is of the form (2.8) with continuous and strictly increasing g(.) if and only if for every $x_i>u$, the conditional densities $f(x_i|X_j>x_j)$, $i,j=1,2;$ $i\neq j$ are of univariate exponential type.

Proof:

If the survival function of X is of the form (2.8), we have

$$P(X_i>x_i|X_j>x_j) = \exp\{-\alpha_i g(x_i)-\theta g(x_i)g(x_j)\} \tag{3.1}$$

So that differentiation with respect to x_i yields

$$f(x_i|X_j>x_j)=(\alpha_i+\theta g(x_j))g'(x_i)\ \exp\{-(\alpha_i+\theta g(x_j))g(x_i)\} \tag{3.2}$$

which is of exponential type. Conversely, assuming $f(x_i|X_j>x_j)$ are of univariate exponential type, it should be of the form,

$$f(x_i|X_j>x_j)=\alpha_i(x_j)\ \exp\{-\alpha_i(x_j)g(x_i)\}g'(x_i) \tag{3.3}$$

and therefore,

$$P[X_i>x_i|X_j>x_j] = \exp\{-\alpha_i(x_j)g(x_i)\}. \tag{3.4}$$

Setting $X_j= u$ in (3.4) we have,

$$P[X_i>x_i] = \exp\{-\alpha_i g(x_i)\},$$

where $\alpha_i = \alpha_i(u)$. Thus the joint survival function can be written either as

$$R(x_1,x_2) = P(X_1>x_1|X_2>x_2)P(X_2>x_2)$$

$$= \exp\{-\alpha_1(x_2)g(x_1)-\alpha_2 g(x_2)\} \tag{3.5}$$

or as

$$R(x_1,x_2) = P(X_2>x_2|X_1>x_1)P(X_1>x_1)$$

$$= \exp\{-\alpha_1 g(x_1)-\alpha_2(x_1)g(x_2)\}. \qquad (3.6)$$

Equating (3.5) and (3.6)

$$\alpha_1(x_2)g(x_1)+\alpha_2 g(x_2) = \alpha_1 g(x_1)+\alpha_2(x_1)g(x_2)$$

which after rearrangement yields,

$$\frac{\alpha_1(x_2)-\alpha_1}{g(x_2)} = \frac{\alpha_2(x_1)-\alpha_2}{g(x_1)} \qquad (3.7)$$

for all $x_1,x_2>u$. Using the arguments similar to that led to the solution in equation (2.19) We find that

$$\alpha_i(x_j) = \alpha_i+\theta g(x_j), \quad i,j=1,2,\ i\neq j \qquad (3.8)$$

where θ is a constant. Substituting (3.8) in (3.5) or in (3.6) we arrive at (2.8) and this completes the proof.

3.3 CHARACTERIZATION BY RELIABILITY CONCEPTS.

A glance at some important members of BET family reveals that most of them are useful in the context of modelling equipment behaviour in two-component systems where the random variable of interest are the life times X_1 and X_2 of the components. This is in view of the popularity of the exponential, Weibull and Pareto enjoyed as life distributions in the context of reliability and life testing. It is therefore of interest to look at the nature of the failure rate of the system of distributions, under consideration. Using vector valued failure rate given in equation (2.58) and the result that $h(x_1,x_2)$ can uniquely determine a distribution, we already have a characterization of the BET family. With a more general form of the failure rate than the specific expression in (2.58), it is possible to have a characterization and this we give in the following result.

Theorem 3.2

A continuous random vector in the support of $(u,g^{-1}(\infty)) \times (u,g^{-1}(\infty))$ has distribution specified by (2.8) if and only if the failure rate function is of the form

$$h(x_1,x_2)=(b_1(x_2)g'(x_1),b_2(x_1)g'(x_2)) \tag{3.9}$$

where $b_i(u) = \alpha_i$, $i=1,2$ and the function $g(.)$ is as stipulated in theorem 3.1 with $g(u)=0$.

Proof:

Given a failure rate vector of the form (3.9), using relationships in equation (1.26) one can write,

$$R(x_1,x_2) = \exp\left\{-\int_u^{x_1} b_1(x_2)g'(t_1)dt_1 - \int_u^{x_2} b_2(u)g'(t_2)dt_2\right\}$$

$$= \exp\{-b_1(x_2)g(x_1)-\alpha_2 g(x_2)\} \tag{3.10}$$

and

$$R(x_1,x_2) = \exp\{-\alpha_1 g(x_1)-b_2(x_1)g(x_2)\}. \tag{3.11}$$

Equating (3.10) and (3.11) we get the functional equation

$$\frac{b_1(x_2)-\alpha_1}{g(x_2)} = \frac{b_2(x_1)-\alpha_2}{g(x_1)}$$

holding for all x_1,x_2 which is infact true if and only if,

$$b_i(x_j) = (\alpha_i+\theta g(x_j)), \qquad i,j=1,2;i\neq j$$

where θ is independent of x_1 and x_2. Substituting this in (3.10) we have the survival function of the required form. The converse is obvious from equation (2.58).

It has already been pointed out in section 2.5.2 in general, the BET family does not have a closed-form expression for its mean residual life and therefore, there does not exist characterization of the entire family in terms of simple functional forms of the mean residual life. However, it is worthwhile to investigate the existence of characterization theorems comprising certain subclass of the BET family. We now present a result in this direction.

Theorem 3.3

If $g(x)$ is differentiable in $(u, g^{-1}(\infty))$ and $[g^{-1}(u)]^{-1} < \infty$, then a necessary and sufficient condition that a bivariate density has a mean residual life vector of the form

$$\left(\frac{a_1(x_2)}{g'(x_1)} , \frac{a_2(x_1)}{g'(x_2)} \right)$$

where $a_2(x_1)$ and $a_1(x_2)$ are non-negative continuous functions, is that the corresponding survival function is,

$$R(x_1, x_2) = \frac{g'(x_1)g'(x_2)}{[g'(u)]^2} \exp\{-\alpha_1 g(x_1) - \alpha_2 g(x_2) - \theta g(x_1)g(x_2)\} \tag{3.12}$$

Proof:

Given that

$$r_i(x_1,x_2) = \frac{a_i(x_j)}{g'(x_i)} .$$

From relation (1.32),

$$R(x_1,x_2) = \frac{\dfrac{a_1(u)}{g'(u)}\,\dfrac{a_2(u)}{g'(u)}}{\dfrac{a_1(u)}{g'(x_1)}\,\dfrac{a_2(u)}{g'(x_2)}} \quad \exp\left[- \frac{g(x_1)}{a_1(u)} - \frac{g(x_2)}{a_2(x_1)} \right]$$

$$= \frac{g'(x_1)g'(x_2)}{(g'(u))^2} \exp\left[- \frac{g(x_1)}{a_1(u)} - \frac{g(x_2)}{a_2(x_1)} \right] \quad (3.13)$$

and

$$R(x_1,x_2) = \frac{g'(x_1)g'(x_2)}{(g'(u))^2} \exp\left[- \frac{g(x_1)}{a_1(x_2)} - \frac{g(x_2)}{a_2(u)} \right]. \quad (3.14)$$

Equating the two forms

$$g(x_1) \left[\frac{1}{a_1(x_2)} - \frac{1}{a_1(u)} \right] = g(x_2) \left[\frac{1}{a_2(x_1)} - \frac{1}{a_2(u)} \right].$$

By the usual arguments, this yields

$$a_i(x_j) = [\alpha_i + \theta g(x_j)]^{-1}, \quad i,j=1,2;\ i \neq j$$

where $\alpha_i = [a_i(u)]^{-1}$.

Substituting this value of $a_i(x_j)^{-1}$ in $R(x_1,x_2)$ we get the distribution specified by (3.12). The converse is obvious.

Corollary

(i) Taking $g(x_i) = x_i$, (3.12) becomes model 1 in Table 2.1.

(ii) Taking $g(x_i) = \log x_i$

$$R(x_1,x_2) = x_1^{-(\alpha_1+1)} x_2^{-(\alpha_2+1)} x_1^{-\theta \log x_2}$$

which is bivariate Pareto type I distribution.

(iii) $g(x_i) = \log(1+ax_i)$, yields

$$R(x_1,x_2)=(1+ax_1)^{-\alpha_1} (1+ax_2)^{-\alpha_2} (1+ax_1)^{-\theta \log(1+ax_2)}$$

Remark:

In general the form of the distribution (3.12) is not the same as that in the BET family. For example, if we set $g(x_i) = x_i^a$, a>1 in (3.12)

$$g'(x_i) = ax_i^{a-1} \quad \text{and } g'(0)=0$$

So that no density exists for (X_1,X_2). Further for model 5, (3.12) is of different form.

From the expressions for the vector valued failure rates and mean residual lives given earlier, we notice that the product $r_i(x_1,x_2)h_i(x_1,x_2)$ has the value 1 for the bivariate exponential distribution, the value $(\alpha_i+\theta\log(1+ax_j))(\alpha_i+\theta\log(1+ax_j)-1)^{-1}$ which is greater than unity for the bivariate Pareto Type II.

3.4 MEAN FAILURE RATES

Considering the vector valued failure rate and mean residual life, we can arrive at certain derived measures viz. arithmetic mean failure rate (AFR), geometric mean failure rate (GFR), harmonic mean failure rate (HFR) and similiar measures involving mean residual life. These measures were introduced in the univariate case by Roy and Mukherjee (1992). We now extend these definitions to the bivariate case. Accordingly, the bivariate AFR is a vector of two components defined as,

$$A(x_1,x_2) = (A_1(x_1,x_2),A_2(x_1,x_2))$$

where

$$A_i(x_1,x_2) = \frac{1}{x_i}\int_0^{x_i} h_i(x_1,x_2)\, dx_i, \quad i=1,2;\ x_i>0. \qquad (3.15)$$

The bivariate GFR is defined as

$$G(x_1,x_2) = (G_1(x_1,x_2),G_2(x_1,x_2))$$

where

$$G_i(x_1,x_2) = \exp\left[\frac{1}{x_i}\int_0^{x_i} \log h_i(x_1,x_2)\, dx_i\right] \qquad (3.16)$$

and

$$H(x_1,x_2) = (H_1(x_1,x_2),H_2(x_1,x_2))$$

with

$$H_i(x_1,x_2) = \left[\frac{1}{x_i}\int_0^{x_i} \frac{1}{h_i(x_1,x_2)}\, dx_i\right]^{-1}. \qquad (3.17)$$

In the next section we discuss some properties of AFR, GFR and HFR.

3.5 PROPERTIES OF THE MEAN FAILURE RATES.

Theorem 3.4

The three means of failure rate, viz AFR,GFR and

HFR each uniquely determines the distribution function of the bivariate random vector.

Proof:

From (3.12),

$$x_i\ A_i(x_1,x_2) = \int_0^{x_i} h_i(x_1,x_2)\ dx_i$$

$$= \int_0^{x_i} \frac{-\partial \log R(x_1,x_2)}{\partial x_i}\ dx_i$$

$$= \log\left[\frac{R(x_1,x_2)}{R_j(x_j)}\right],\ i,j=1,2;i\neq j \qquad (3.18)$$

or

$$R(x_1,x_2) = R_j(x_j)\ \exp[-x_i A_i(x_1,x_2)]. \qquad (3.19)$$

That is,

$$R(x_1,x_2) = R_2(x_2)\ \exp[-x_1 A_1(x_1,x_2)] \qquad (3.20)$$

and

$$R(x_1,x_2) = R_1(x_1)\ \exp[-x_2 A_2(x_1,x_2)]. \qquad (3.21)$$

As $x_1 \rightarrow 0^+$ in (3.21), we have

$$R_2(x_2) = \exp[-x_2 A_2(0^+,x_2)].$$

Substituting for $R_2(x_2)$ in (3.20) it reduces to

$$R(x_1,x_2) = \exp\,[-x_1A_1(x_1,x_2)-x_2A_2(0^+,x_2)]. \qquad (3.22)$$

By analogy we obtain,

$$R(x_1,x_2) = \exp\,[-x_1A_1(x_1,0^+)-x_2A_2(x_1,x_2)]. \qquad (3.23)$$

Using one of these equations (3.22) and (3.23) we obtain the survival function of (X_1,X_2) uniquely.

Now we shall consider equation (3.16) which can be written as

$$x_1 \log G_1(x_1,x_2) = \int_0^{x_1} \log h_1(x_1,x_2)\, dx_1$$

for i=1. Differentiating with respect to x_1,

$$x_1 \frac{G_1'(x_1,x_2)}{G_1(x_1,x_2)} + \log G_1(x_1,x_2) = \log h_1(x_1,x_2). \qquad (3.24)$$

From (3.4),

$$\frac{-\partial \log R(x_1,x_2)}{\partial x_1} = G_1(x_1,x_2) \exp\left[x_1 \frac{G_1'(x_1,x_2)}{G_1(x_1,x_2)} \right] \qquad (3.25)$$

Integrating (3.25) in the range $(0,x_1)$

$$R(x_1,x_2) = R_2(x_2) \exp\left\{-\int_0^{x_1} G_1(x_1,x_2) \exp\left[x_1 \frac{G_1'(x_1,x_2)}{G_1(x_1,x_2)} \right] dx_1\right\}$$

(3.26)

Similarly,

$$R(x_1,x_2) = R_1(x_1)\ \exp\left\{-\int_0^{x_2} G_2(x_1,x_2)\ \exp\left[x_2\ \frac{G_2'(x_1,x_2)}{G_2(x_1,x_2)}\right]dx_2\right\}.$$

(3.27)

As $x_1 \rightarrow 0^+$ in (3.27) yields,

$$R_2(x_2) = \exp\left\{-\int_0^{x_2} G_2(0^+,x_2)\ \exp\left[\ x_2\ \frac{G_2'(0^+,x_2)}{G_2(0^+,x_2)}\right]dx_2\right\}$$

and substitution of this value in (3.26) gives

$$R(x_1,x_2) = \exp\left\{-\int_0^{x_1} G_1(x_1,x_2)\ \exp\left[\ x_1\ \frac{G_1'(x_1,x_2)}{G_1(x_1,x_2)}\right]dx_1 - \int_0^{x_2} G_2(0^+,x_2)\ \exp\left[\ x_2\ \frac{G_2'(0^+,x_2)}{G_2(0^+,x_2)}\right]dx_2\right\}. \quad (3.28)$$

Similarly,

$$R(x_1,x_2) = \exp\left\{-\int_0^{x_1} G_1(x_1,0^+)\ \exp\left[\ x_1\ \frac{G_1'(x_1,0^+)}{G_1(x_1,0^+)}\right]dx_1 - \int_0^{x_2} G_2(x_1,x_2)\ \exp\left[\ x_2\ \frac{G_2'(x_1,x_2)}{G_2(x_1,x_2)}\right]dx_2\right\}. \quad (3.29)$$

Equations (3.28) and (3.29) provide the necessary expressions for $R(x_1,x_2)$. Proceeding along similar lines, using equation (3.17) we can arrive at the survival function in terms of HFR as

$$R(x_1,x_2) = \exp\left\{-\int_0^{x_1} \left[\frac{\partial}{\partial x_1} \frac{x_1}{H_1(x_1,x_2)}\right]^{-1} dx_1 - \int_0^{x_2} \left[\frac{\partial}{\partial x_2} \frac{x_2}{H_2(0^+,x_2)}\right]^{-1} dx_2\right\} \qquad (3.30)$$

or

$$R(x_1,x_2) = \exp\left\{-\int_0^{x_1} \left[\frac{\partial}{\partial x_1} \frac{x_1}{H_1(x_1,0^+)}\right]^{-1} dx_1 - \int_0^{x_2} \left[\frac{\partial}{\partial x_2} \frac{x_2}{H_2(x_1,x_2)}\right]^{-1} dx_2\right\}. \qquad (3.31)$$

Examples.

1. When $A_i(x_1,x_2) = (\alpha_i+\theta x_j^{\beta})x_i^{\beta-1}$, $x_i>0$, $i,j=1,2;i\neq j$.

Then

$$A_1(x_1,0^+) = \alpha_1 x_1^{\beta-1}$$

and

$$A_2(0^+,x_2) = \alpha_2 x_2^{\beta-1}.$$

Substituting the corresponding values in equations (3.22) or (3.23) we get

$$R(x_1,x_2) = \exp[-x_1\alpha_1 x_1^{\beta-1} - x_2(\alpha_2+\theta x_1^{\beta})x_2^{\beta-1}]$$

$$= \exp[-\alpha_1 x_1^{\beta} - \alpha_2 x_2^{\beta} - \theta x_1^{\beta}x_2^{\beta}]$$

which is bivariate Weibull distribution.

2. By taking $G_i(x_1,x_2) = (\alpha_i+\theta x_j)$, $x_i>0, i,j=1,2;\ i\neq j$.

$$G_1(x_1,0^+) = \alpha_1 \quad \text{and} \quad G_2(0^+,x_2)=\alpha_2.$$

Substituting in (3.28) or in (3.29)

$$R(x_1,x_2) = \exp[-(\alpha_1+\theta x_2)x_1-\alpha_2 x_2]$$

$$= \exp[-\alpha_1 x_1 - \alpha_2 x_2 - \theta x_1 x_2],$$

which is Gumbel's bivariate exponential distribution

3. when $H_i(x_1,x_2) = (\alpha_i+\theta\log x_j)\dfrac{2x_i}{x_i^2-1}$, $x_i>1, i,j=1,2;\ i\neq j$.

$$H_1(x_1,1^+) = \frac{2\alpha_1 x_1}{x_1^2-1} \quad \text{and} \quad H_2(1^+,x_2) = \frac{2\alpha_2 x_2}{x_2^2-1} .$$

Substituting in (3.30)

$$R(x_1,x_2) = \exp\left[-\int_1^{x_1} \left(\frac{x_1}{\alpha_1+\theta \log x_2}\right)^{-1} dx_1 - \int_1^{x_2} \left(\frac{x_2}{\alpha_2}\right)^{-1} dx_2 \right]$$

$$= \exp[-(\alpha_1+\theta \log x_2)\log x_1 - \alpha_2 \log x_2]$$

$$= \exp[-\alpha_1 \log x_1 - \alpha_2 \log x_2 - \theta \log x_1 \log x_2].$$

which is bivariate Pareto Type I distribution.

Theorem 3.5

A set of necessary conditions that an AFR function $A(x_1,x_2) = (A_1(x_1,x_2),A_2(x_1,x_2))$ should satisfy are,

(i) $A_i(x_1,x_2) \geq 0,\ i=1,2$

(ii)
$$\frac{A_1(x_1,x_2)}{A_2(0^+,x_2)} \geq -\frac{x_2}{x_1} \qquad (3.32)$$

$$\frac{A_2(x_1,x_2)}{A_2(x_1,0^+)} \geq -\frac{x_1}{x_2} \qquad (3.33)$$

(iii) $\lim_{x_1\to\infty} x_1 A_1(x_1,x_2)=\infty,\quad \lim_{x_2\to\infty} x_2 A_2(x_1,x_2)=\infty.$

Proof:

Condition (i) is obvious. To prove condition (ii) (3.22) gives

$$R(x_1,x_2) = \exp\left[-x_1 A_1(x_1,x_2) - x_2 A_2(0^+,x_2)\right].$$

For $R(x_1,x_2)$ to be a survival function one must have $R(0^+,0^+) = 1$ and $0 \le R(x_1,x_2) \le 1$. Hence

$$x_1 A_1(x_1,x_2) + x_2 A_2(0^+,x_2) \ge 0$$

$$\frac{A_1(x_1,x_2)}{A_2(0^+,x_2)} \ge -\frac{x_2}{x_1}$$

also

$$\frac{A_2(x_1,x_2)}{A_1(x_1,0^+)} \ge -\frac{x_1}{x_2}.$$

Condition (iii) follows from

$$R(\infty,x_2) = \lim_{x_1 \to \infty} R(x_1,x_2) = 0.$$

which implies

$$\lim_{x_1 \to \infty} x_1 A_1(x_1,x_2) + x_2 A_2(0^+,x_2) = \infty$$

or

$$\lim_{x_1 \to \infty} x_1 A_1(x_1,x_2) = \infty.$$

Similarly from $R(x_1,\infty)=0$

$$\lim_{x_2 \to \infty} x_2 A_2(x_1,x_2) = \infty.$$

3.6 CHARACTERIZATIONS OF BIVARIATE WEIBULL DISTRIBUTION AND GUMBEL'S BIVARIATE EXPONENTIAL DISTRIBUTION

We shall obtain certain characterizations of bivariate Weibull distribution and Gumbel's bivariate exponential distribution using these measures.

Theorem 3.6

If $h_i(x_1,x_2)$ is differentiable in both the arguments then X follows bivariate Weibull distribution specified by (2.27) if and only if any of the following statements hold.

(1) $A_i(x_1,x_2) = a\ h_i(x_1,x_2)$, $i=1,2$, $a>0$.

(2) $G_i(x_1,x_2) = b\ h_i(x_1,x_2)$, $i=1,2$, $0< \log b<1$

(3) $H_i(x_1,x_2) = c\ h_i(x_1,x_2)$, $i=1,2$, $c<2$.

Proof:

Assume that (1) holds.

Taking $i=1$,

$$ah_1(x_1,x_2)x_1 = \int_0^{x_1} h_1(x_1,x_2)dx_1. \qquad (3.34)$$

Taking i=1 and differentiating (3.34) with respect to x_1, we get

$$a\,\frac{h_1'(x_1,x_2)}{h_1(x_1,x_2)}\,x_1 + a = 1$$

$$\frac{h_1'(x_1,x_2)}{h_1(x_1,x_2)} = \left(\frac{1-a}{a}\right)\frac{1}{x_1}. \qquad (3.35)$$

Integrating (3.35) with respect to x_1

$$\log h_1(x_1,x_2) = \log k_1(x_2) + \left(\frac{1}{a} - 1\right)\log x_1$$

or

$$h_1(x_1,x_2) = k_1(x_2)x_1^{\frac{1}{a}-1}.$$

Similarly for i=2,

$$h_2(x_1,x_2) = k_2(x_1)x_2^{\frac{1}{a}-1}.$$

Using the relation (1.26) we can reach at the functional equation,

$$\frac{k_1(x_2)-k_1(0^+)}{x_2^{1/a}} = \frac{k_2(x_1)-k_2(0^+)}{x_1^{1/a}}$$

which gives solution as

$$k_i(x_j) = k_i(0^+) + \delta x_j^{1/a}.$$

On substitution, the survival function reduces to that of Weibull distribution, with $\alpha_i = ak_i(0^+)$ and $\theta=\delta a$.

Conversely, if it is assumed that $X=(X_1,X_2)$ has bivariate Weibull distribution specified by (2.27), then from Table (2.3)

$$h_1(x_1,x_2) = (\alpha_1+\theta x_2^{\beta})x_1^{\beta-1}$$

which gives

$$A_1(x_1,x_2) = \beta^{-1}h_1(x_1,x_2),\ \beta^{-1}>0$$

and similarly from the form of $h_2(x_1,x_2)$

$$A_2(x_1,x_2) = \beta^{-1}h_2(x_1,x_2),$$

So that the specified form holds.

Assume that condition (2) holds . Proceeding as in the previous case we get

$$h_i(x_1,x_2) = m_i(x_j)x_i^{-\log b}$$

where $m_i(x_j)$ is the constant of integration, and this gives the required survival function. Converse also is similar to that in case (1).

In case (3), we have

$$\left[\frac{1}{x_1}\int_0^{x_1}\frac{1}{h_1(x_1,x_2)}\,dx_1\right]^{-1} = ch_1(x_1,x_2)$$

$$\int_0^{x_i}\frac{1}{h_1(x_1,x_2)}\,dx_1 = \frac{x_1}{ch_1(x_1,x_2)}$$

Differentiating with respect to x_1,

$$\frac{1}{ch_1(x_1,x_2)} = \frac{h_1(x_1,x_2)-x_1h_1'(x_1,x_2)}{c^2h_1^2(x_1,x_2)}$$

$$\frac{c}{h_1(x_1,x_2)} = \frac{h_1(x_1,x_2)-x_1h_1'(x_1,x_2)}{h_1^2(x_1,x_2)}$$

$$(1-c)h_1(x_1,x_2) = x_1h_1'(x_1,x_2)$$

$$\frac{h_1'(x_1,x_2)}{h_1(x_1,x_2)} = \frac{1-c}{x_1}\ .$$

Integration yields,

$$\log h_1(x_1,x_2) = \log P_1(x_2) + (1-c)\log x_1$$

$$h_1(x_1,x_2) = P_1(x_2)\ x_1^{1-c}$$

and similarly

$$h_2(x_1,x_2) = P_2(x_1)\ x_2^{1-c}.$$

and therefore the proof is analogous.

Corollary 3.1

Let $X = (X_1,X_2)$ be a bivariate random vector admitting absolutely continuous distribution with respect to Lebesgue measure. Then

$$A_i(x_1,x_2) = G_i(x_1,x_2) = H_i(x_1,x_2) = M_i(x_j),\ i,j=1,2,\ i\neq j$$

of function X_j alone, if and only if X has Gumbel's bivariate exponential distribution.

Proof:

Proof follows from equations (3.22), (3.28) and (3.30) and Theorem 3.6.

Corollary 3.2

$A_i(x_1,x_2) = G_i(x_1,x_2) = H_i(x_1,x_2) = k$, a constant independent of X_1 and X_2 if and only if X_1 and X_2 are independent exponential variates. A restatement of Theorem 3.6 is as follows

Theorem 3.7

If $h_i(x_1,x_2)$ is differentiable, the following four statements are equivalent.

(i) $A_i(x_1,x_2)$ is proportional to $G_i(x_1,x_2)$ for all x_i, $i=1,2$.

(ii) $G_i(x_1,x_2)$ is proportional to $H_i(x_1,x_2)$ for all x_i, $i=1,2$.

(iii) $H_i(x_1,x_2)$ is proportional to $A_i(x_1,x_2)$ for all x_i, $i=1,2$.

and

(iv) $X=(X_1,X_2)$ follows bivariate Weibull distribution, specified by (2.27) wherever these measures are defined.

3.7 CHARACTERIZATION OF PARETO TYPE I DISTRIBUTION.

In this section we establish a characterization of

Pareto type I distribution which is an extension to the bivariate case of the result in Dallas (1976) mentioned in Section 1.2.

Theorem 3.8

Let $X=(X_1,X_2)$ be a random vector admitting absolutely continuous distribution with respect to Lebesgue measure in the support of $(1,\infty)x(1,\infty)$ with $E(X_i^r)<\infty$ for any positive integer r. Then X follows bivariate Pareto I distribution specified by (2.23) if and only if

$$E(X_i^r|X_1>c_1,X_2>c_2)=E(c_iX_i^r|X_j>c_j),\ c_i>1,i,j=1,2;\ i\neq j, \qquad (3.36)$$

Proof:

Assume that the survival function of X is of the form (2.23).

$$E(X_1^r|X_1>c_1,X_2>c_2) = c_1^r + r\ [R(c_1,c_2)]^{-1} \int_{c_1}^{\infty} x_1^{r-1}R(x_1,c_2)dx_1 \qquad (3.37)$$

Substituting for $R(c_1,c_2)$ from (2.23) we get

$$E(X_1^r | X_1 > c_1, X_2 > c_2) = c_1^r + \frac{rc_1^r}{(\alpha_1 + \theta \log c_2 - r)}$$

$$= c_1^r \left(\frac{\alpha_1 + \theta \log c_2}{\alpha_1 + \theta \log c_2 - r}\right). \qquad (3.38)$$

Now consider

$$E((c_1 X_1)^r | X_2 > c_2) = c_1^r + rc_1^r [R_2(c_2)]^{-1} \int_1^\infty x_1^{r-1} R(x_1, c_2) dx_1$$

Substituting for $R_2(c_2)$ and $R(x_1,c_2)$ yields,

$$E((c_1 X_1)^r | X_2 > c_2) = c_1^r + \frac{rc_1^r}{(\alpha_1 + \theta \log c_2 - r)}$$

$$= c_1^r \left(\frac{\alpha_1 + \theta \log c_2}{\alpha_1 + \theta \log c_2 - r}\right). \qquad (3.39)$$

Thus (3.36) is true for i=1. The case for i=2 holds similarly.

Conversely, assume that condition (3.36) holds. Denote by

$$\rho_1(c_2) = \left[\int_1^\infty \int_{c_2}^\infty x_1^r \, f(x_1, x_2) dx_1 dx_2 \right] [R_2(c_2)]^{-1}$$

also we have,

$$E(X_1^r | X_1>c_1, X_2>c_2) = c_1^r\ R(c_1,c_2) + r\int_{c_1}^{\infty} x_1^{r-1} R(x_1,c_2)dx_1 .$$

Then (3.36) can be written as,

$$c_1^r\ R(c_1,c_2)+r\int_{c_1}^{\infty} x_1^{r-1}R(x_1,c_2)dx_1 = c_1^r\ \rho_1(c_2)R(c_1,c_2). \quad (3.40)$$

Since R(.,.) is a monotone, continuous function, it is differentiable. Differentiating (3.40) with respect to c_1, we have

$$r\ \rho_1(c_2)R(c_1,c_2) + c_1\ \rho_1(c_2)R'(c_1,c_2) = c_1\ R'(c_1,c_2)$$

$$c_1\ R'(c_1,c_2) = \frac{r\rho_1(c_2)}{1-\rho_1(c_2)}\ R(c_1,c_2)$$

$$= -\theta_1(c_2)R(c_1,c_2)$$

where $\theta_1(c_2) = \dfrac{r\rho_1(c_2)}{\rho_1(c_2)-1}$, $c_1>1$ and $\rho_1>1$.

$$\frac{R'(c_1,c_2)}{R(c_1,c_2)} = \frac{-\theta_1(c_2)}{c_1} . \quad (3.41)$$

Integrating this we get

$$R(c_1,c_2) = A_1(c_2)c_1^{-\theta_1(c_2)} \tag{3.42}$$

where $A_1(c_2)$ is the constant of integration. Letting $c_1 \to 1$, we get $R_2(c_2)=A_1(c_2)$. Thus (3.42) becomes

$$R(c_1,c_2) = R_2(c_2)c_1^{-\theta_1(c_2)}$$

Similarly by assuming,

$$E(X_2^r|X_1>c_1,X_2>c_2) = E((c_2X_2)^r|X_1>c_1)$$

we get

$$R(c_1,c_2) = R_1(c_1)c_2^{-\theta_2(c_1)}. \tag{3.43}$$

Setting $c_2=1$ in (3.42), it reduces to

$$R_1(c_1) = c_1^{-\theta_1}, \text{ where } \theta_1(1) = \theta_1.$$

Substituting this value of $R_1(c_1)$ in (3.43) we get

$$R(c_1,c_2) = c_1^{-\theta_1} c_2^{-\theta_2(c_1)} \tag{3.44}$$

and similarly

$$R(c_1,c_2) = c_1^{-\theta_1(c_2)} c_2^{-\theta_2}. \tag{3.45}$$

Equating (3.44) and (3.45)

$$c_1^{\theta_1(c_2)-\theta_1} = c_2^{\theta_2(c_1)-\theta_2}.$$

Taking logarithms on both sides and re-arranging

$$\frac{\theta_2(c_1)-\theta_2}{\log c_1} = \frac{\theta_1(c_2)-\theta_1}{\log c_2} = \phi \text{ a constant.}$$

This gives

$$\theta_i(c_j) = \theta_i+\phi \log c_j, \quad i,j=1,2;i\neq j.$$

Then substituting $\theta_i(c_j)$ in (3.44) or in (3.45) we get

$$R(c_1,c_2) = c_1^{-\theta_1} c_2^{-\theta_2} c_2^{-\phi\log c_1}$$

which is of the form (2.23). This completes the proof.

Chapter IV

SOME DERIVED MODELS

4.1. INTRODUCTION

There are several attempts in literature to construct bivariate distributions which has specified forms for its marginal and conditional distributions of which the systems with specified marginals are reviewed in Johnson and Kotz (1972). Seshadri and Patil (1964) studied the problem of determining the joint distribution of X_1 and X_2 given the marginal distribution of X_i and conditional distribution of X_j given $X_i = x_i$, $i,j = 1,2$, $i \neq j$. They showed that a sufficient condition for the uniqueness of the joint density function of X_1 and X_2 is that the conditional distribution of X_i given X_j is of the exponential form. The question of determining the joint distribution using the conditional distributions has received considerable attention in the recent times, on the ground that information about the conditional densities are available in many real life phenomenon. Some recent papers in this area are of Castillo and Galambos (1987) Arnold (1987), Arnold and Strauss (1988), etc. Arnold and Press (1989) determined a necessary and sufficient condition for the existence of joint density given the conditional densities. Gourieroux and Monfort (1979). Most of the attempts in these papers were to obtain joint distributions which have a specified form for their conditionals, such as bivariate distributions whose conditionals are normal, Weibull, Pareto etc. In the following section, we provide a uniform framework in which a class of bivariate distributions can be generated. This class contains models whose conditionals are exponential, Weibull, Pareto I, Pareto II and finite range distributions.

4.2. DERIVATION OF THE FAMILY.

The lack of memory property defined by Nair and Nair (1991) given in equation (1.36) is generalised here as follows.

$$P(X_i > G(t_i, s_i) \mid X_i \geq s_i, X_j = x_j) = P(X_i \geq t_i \mid X_j = x_j) \qquad (4.1)$$

for all s_i, t_i, x_j in (u,c) holds, i,j = 1,2 $i \neq j$ where G(.,.), u, c etc; are all as explained in the beginning of Chapter II.

Writing the conditional survival function of X_i given $X_j = x_j$ as,

$$P(X_i \geq x_i | X_j = x_j) = S(x_i, x_j) \qquad (4.2)$$

equation (4.1) becomes,

$$\frac{P(X_i \geq G(t_i, s_i) | X_j = x_j)}{P(X_i \geq s_i | X_j = x_j)} = P(X_i \geq t_i | X_j = x_j). \qquad (4.3)$$

Using (4.2) and (4.3) we have

$$S(G(t_i, s_i), x_j) = S(t_i, x_j).S(s_i, x_j). \qquad (4.4)$$

For a fixed, but otherwise arbitrary x_j (4.4) has the solution, following the arguments in Muliere and Scarsini (1987)

$$S(x_i, x_j) = \exp[-\lambda_i(x_j) g(x_i)], \ \lambda_i(x_j) > 0. \qquad (4.5)$$

Thus the problem of finding the bivariate distributions characterized by (4.1) reduces to find the joint distribution of (X_1, X_2), where the conditional distribution of X_i given $X_j = x_j$ forms,

$$P(X_1 \geq x_1 | X_2 = x_2) = \exp[-\lambda_1(x_2) g(x_1)] \quad (4.6)$$

and

$$P(X_2 \geq x_2 | X_1 = x_1) = \exp[-\lambda_2(x_1) g(x_2)] \quad (4.7)$$

The probability density function corresponding to (4.6) and (4.7) are then

$$f(x_1 | x_2) = \lambda_1(x_2)\, e^{-\lambda_1(x_2) g(x_1)} g'(x_1) \quad (4.8)$$

and

$$f(x_2 | x_1) = \lambda_2(x_1)\, e^{-\lambda_2(x_1) g(x_2)} g'(x_2) \quad (4.9)$$

respectively.

Representing the marginal densities of X_1 and X_2 by $f_1(x_1)$ and $f_2(x_2)$ we arrive at the identity,

$$\lambda_1(x_2)\, e^{-\lambda_1(x_2) g(x_1)} g'(x_1)\, f_2(x_2)$$

$$= \lambda_2(x_1)\, e^{-\lambda_2(x_1) g(x_2)} g'(x_2)\, f_1(x_1) \quad (4.10)$$

or equivalently for all x_1, x_2 in (u,c).

$$\log \lambda_1(x_2) - \lambda_1(x_2)g(x_1) + \log g'(x_1) + \log f_2(x_2)$$

$$= \log \lambda_2(x_1) - \lambda_2(x_1)g(x_2) + \log g'(x_2) + \log f_1(x_1) \tag{4.11}$$

with primes indicating differentiation. Differentiating (4.11) with respect to x_2,

$$\frac{\partial \log \lambda_1(x_2)}{\partial x_2} - \frac{\partial \lambda_1(x_2)}{\partial x_2} g(x_1) + \frac{\frac{\partial f_2(x_2)}{\partial x_2}}{f_2(x_2)}$$

$$= -\lambda_2(x_1) \frac{\partial g(x_2)}{\partial x_2} + \frac{\frac{\partial g'(x_2)}{\partial x_2}}{g'(x_2)} . \tag{4.12}$$

Now differentiating (4.12) with respect to x_1, we have

$$\frac{\partial \lambda_1(x_2)}{\partial x_2} \cdot \frac{\partial g(x_1)}{\partial x_1} = \frac{\partial \lambda_2(x_1)}{\partial x_1} \cdot \frac{\partial g(x_2)}{\partial x_2}$$

or

$$\frac{\frac{\partial \lambda_1(x_2)}{\partial x_2}}{\frac{\partial g(x_2)}{\partial x_2}} = \frac{\frac{\partial \lambda_2(x_1)}{\partial x_1}}{\frac{\partial g(x_1)}{\partial x_1}} . \tag{4.13}$$

For equation (4.11) to be true for all x_1, x_2 it must be true that

$$\frac{\dfrac{\partial\lambda_i(x_j)}{\partial x_j}}{\dfrac{\partial g(x_j)}{\partial x_j}} = \theta, \quad i,j = 1,2 \quad i \neq j \tag{4.14}$$

where θ is a constant independent of both X_1 and X_2. Since this solution is unique, the value of $\lambda_i(x_j)$ that satisfy (4.10) is

$$\lambda_i(x_j) = (\alpha_i + \theta\, g(x_j)). \tag{4.15}$$

Introducing this value of $\lambda_i(x_j)$ in (4.10) and simplifying

$$[g'(x_2)]^{-1} f_2(x_2) \; (\alpha_1+\theta\, g(x_2)) \exp(\alpha_2 g(x_2))$$

$$= [g'(x_1)]^{-1} f_1(x_1) \; (\alpha_2+\theta\, g(x_1)) \exp(\alpha_1 g(x_1))$$

for all x_1, x_2. This however means that for some constant $c>0$,

$$f_i(x_i) = c\, g'(x_i) \; (\alpha_j+\theta\, g(x_i))^{-1} \exp(\alpha_i g(x_i)). \tag{4.16}$$

From (4.10),(4.14) and (4.16) the joint density of (x_1, x_2) is

$$f(x_1,x_2) = C\, g'(x_1)g'(x_2)\, \exp(-\alpha_1 g(x_1)-\alpha_2 g(x_2)-\theta g(x_1)g(x_2)) \qquad (4.17)$$

x_1, x_2 belonging to (u,c), $\alpha_i>0$, $\theta>0$, $i=1,2$.

In particular when $\theta=0$, we have the case of independence of x_1, x_2. The constant C can be obtained as follows.

We have

$$\int_u^\infty \int_u^\infty f(x_1,x_2)dx_1 dx_2 = 1$$

$$C\int_u^\infty e^{-\alpha_1 g(x_1)} g'(x_1)\ (\alpha_2+\theta g(x_1))^{-1} dx_1 = 1.$$

That is,

$$C\theta^{-1}\exp(\alpha_1\alpha_2\theta^{-1})\ E_1(\alpha_1\alpha_2\theta^{-1}) = 1$$

or

$$C = \theta\ \exp(-\alpha_1\alpha_2\theta^{-1})\ \left[E_1(\alpha_1\alpha_2\theta^{-1})\right]^{-1}. \qquad (4.18)$$

Corresponding survival function is obtained as

$$R(x_1,x_2) = \int_{x_1}^\infty \int_{x_2}^\infty f(x_1,x_2)\ dx_1\ dx_2.$$

$$= C \int_{x_1}^\infty \int_{x_2}^\infty g'(x_1)\ g'(x_2)\ e^{-\alpha_1 g(x_1)-\alpha_2 g(x_2)-\theta g(x_1)g(x_2)} dx_2 dx_1$$

$$= C\int_{x_1}^{\infty} e^{-\alpha_2 g(x_2)}(\alpha_2+\theta g(x_1))^{-1}\, e^{-\alpha_1 g(x_1)-\theta g(x_1)g(x_2)}\, g'(x_1)dx_1$$

$$= C\, e^{-\alpha_2 g(x_2)}\int_{x_1}^{\infty} (\alpha_2+\theta g(x_1))^{-1}\, e^{-(\alpha_1+\theta g(x_2))g(x_1)}\, g'(x_1)\, dx_1$$

$$= E_1(\alpha_1\alpha_2\theta^{-1})^{-1}\, E_1(\alpha_1\alpha_2\theta^{-1}+\alpha_1 g(x_1)+\alpha_2 g(x_2)+\theta g(x_1)g(x_2)). \tag{4.19}$$

4.3. PARTICULAR CASES.

1. Taking $G(x_1,x_2) = x_1+x_2$

$$g(x_1+x_2) = g(x_1) + g(x_2)$$

and this reduces to the density function of the form

$$f(x_1,x_2) = C\exp(-\alpha_1 x_1-\alpha_2 x_2-\theta x_1 x_2)$$

$x_1,x_2 > 0$, which is the bivariate exponential distribution obtained in Arnold and Strauss (1988) and Abrahams and Thomas (1984).

2. $G(x_1,x_2) = x_1 x_2$

implies

$$g(x_i) = a \log x_i \quad i = 1,2$$

and gives

$$f(x_1,x_2) = C\, x_1^{-(\alpha_1+1)} x_2^{-(\alpha_2+1)} x_1^{-\theta \log x_2},$$

$$\alpha_1,\alpha_2 > 0,\ \theta \geq 0,\ x_1,x_2 > 1.$$

This is the bivariate distribution with Pareto I model as conditionals.

3. $G(x_1,x_2) = (x_1^{\beta} + x_2^{\beta})^{1/\beta}$

gives

$$f(x_1,x_2) = C\beta^2 x_1^{\beta-1} x_2^{\beta-1} \exp(-\alpha_1 x_1^{\beta} - \alpha_2 x_2^{\beta} - \theta x_1^{\beta} x_2^{\beta})$$

$$\beta > 0,\ \alpha_i > 0,\ \theta \geq 0,\ i = 1,2,\ x_1,x_2 > 0.$$

A bivariate distribution with Weibull conditionals results.

4. $G(x_1,x_2) = x_1+x_2+ax_1x_2,$

implies

$$g(x_1+x_2+ax_1x_2) = g(x_1) + g(x_2)$$

and

$$g(x_i) = \log(1+ax_i)\ ,\ i = 1,2.$$

The joint density is

$$f(x_1,x_2) = C\,(1+ax_1)^{-\alpha_1}\,(1+ax_2)^{-\alpha_2}\,(1+ax_1)^{-\theta\log(1+ax_2)}$$

$$\alpha_1,\alpha_2,\ a > 0,\ \theta \geq 0.$$

5. $G(x_1,x_2) = x_1+x_2 \Big/ \left[1+ \dfrac{x_1x_2}{p^2}\right]$

implies

$$f(x_1,x_2) = \frac{4Cp^2}{(p^2-x_1^{\,2})(p^2-x_2^{\,2})}\left(\frac{p-x_1}{p+x_1}\right)^{\alpha_1}\left(\frac{p-x_2}{p+x_2}\right)^{\alpha_1}\left(\frac{p-x_1}{p+x_1}\right)^{\theta\log\left(\frac{p-x_2}{p+x_2}\right)}$$

$$0 < x_1 < p,\ 0 < x_2 < p,\ \alpha_1,\alpha_2 > 0,\ \theta \geq 0,\ p > 0.$$

In all these cases C is as in equation (4.18). The solution of the functional equations in the examples are available in Aczel (1966).

Remarks

The unique bivariate distribution with Pareto II conditionals obtained in example 4 of above differs from a similar model derived in Arnold (1987). Arnold chooses the scale parameters to depend on the conditioned variable and the shape parameter is fixed. In the model described here, the scale parameter remains unaltered, while the shape parameter changes with the values of the conditioned variable.

4.4. RANDOM ENVIRONMENTAL MODELS.

A working system is often affected by the changes in its surroundings. The environment in which the system is working need not be the same as the laboratory environment, under which the system was designed and the prospective reliability was determined. The working environment comprises of a number of observable and unobservable factors whose intensities change over time in a random manner. For example, the system might have been built on the premise that the components are structurally independent so that when they work in a common environment, the expectation is that they fail independently. However the common working condition may induce certain kind of relationships among the components that makes the assumption independent failure times untenable. Thus the reliability of the system is often affected sometimes adversely and sometimesfavourably, when the system operates in places different from the initial test site. It is important to assess the manner and extent by which the reliability is affected due to a change in environment and therefore extensive studies have been earned out by various researchers on models that can explain this fact.

Lindley and Singpurwalla (1986) have studied systems sharing common environment and Currit and Singpurwalla (1988) analysed the reliability function of Lindley and Singpurwalla model, in the parallel and series systems and have obtained a formula for making Bayesian inferences for the reliability function. Nayak (1987), Cinlar and Özeckici (1987), Roy (1989), Bandyopadhyay and Basu (1990), Gupta and Gupta (1990) Lee and Gross (1991), Sankaran and Nair (1993), Singpurwalla and Youngren (1993) etc; have considered environmental models in detail. It is customary in modelling problems to assume that the failure rate of the system working in the new environment is given by $\eta h(x_1,x_2)$ where $h(x_1,x_2) = (h_1(x_1,x_2),h_2(x_1,x_2))$ is the vector failure rate, when the system has worked in the test environments. In this representation, η stands for the effect on the failure rate due to the change in environment. Thus when the environment factor $\eta > 1$ ($\eta < 1$, $\eta = 1$) the new working conditions are assumed to be barsher (milder, same as) than the original work site. Since the influence of the changed environment is seldom known exactly, it is reasonable to take η as a random variable and to assume a suitable probability density function for it. One such choice for the distribution of `η' is the gamma density

$$f(\eta|m,p) = \frac{m^p}{\Gamma p} e^{-m\eta} \eta^{p-1}, \; m > 0, \; p > 0. \qquad (4.20)$$

Consider a two component system, with life lengths X_1 and X_2. Originally, the system is assumed to have a distribution function specified by (2.8). The vector valued failure rate of the system is given in equation (2.58). While working in an environment with environment factor η, its failure rate vector get changed to $\eta\, h(x_1,x_2)$, as

$$\eta\, h(x_1,x_2) = (\eta(\alpha_1+\theta g(x_2))g'(x_1),\eta(\alpha_2+\theta g(x_1))g'(x_2) \qquad (4.21)$$

which gives the new survival function of (X_1,X_2) as

$$\hat{R}(x_1,x_2) = \exp(-\eta\alpha_1 g(x_1)-\eta\alpha_2 g(x_2)-\eta\theta g(x_1)g(x_2)). \qquad (4.22)$$

Accounting the uncertainty of η, by averaging this over the distribution of η, given by (3.20)

$$R_\eta(x_1,x_2)=\int_0^\infty \frac{m^p}{\Gamma p} e^{-m\eta}\, \eta^{p-1}\, e^{-\eta(\alpha_1 g(x_1)+\alpha_2 g(x_2)+\theta g(x_1)g(x_2))} d\eta$$

$$= [1+a_1 g(x_1)+a_2 g(x_2)+bg(x_1)g(x_2)]^{-p} \qquad (4.23)$$

where $a_i = \frac{\alpha_i}{m}$, and $b = \theta/m$, $i = 1,2$.

The corresponding density function is given by

$$f(x_1,x_2) = p[p(a_1+bg(x_2)(a_2+bg(x_1))+a_1a_2-b]\ g'(x_1)g'(x_2)$$

$$[1+a_1 g(x_1)+a_2 g(x_2)+bg(x_1)g(x_2)]^{-(p+2)}$$

$$(4.24)$$

and the marginal density functions are

$$f_i(x_i) = a_i p(1+a_i g(x_i))^{-(p+1)} g'(x_i), \; i=1,2. \qquad (4.25)$$

The conditions on the parameters of the model derives from

$$R(x_1,u) \geq R(x_1,x_2)$$

or

$$(1+a_1 g(x_1))^{-p} \geq (1+a_1 g(x_1)+a_2 g(x_2)+bg(x_1)g(x_2))^{-p}$$

$$1+a_1 g(x_1) \leq (1+a_1 g(x_1)+a_2 g(x_2)+bg(x_1)g(x_2))$$

$$0 \leq (a_2+bg(x_1))g(x_2).$$

Since g(.) is monotonic increasing and g(u) = 0, the above inequality holds good for all x_1, x_2 if and only if $a_2 > 0$ and b > 0. Similarly we get

$$0 \leq (a_1+bg(x_2))g(x_1)$$

which gives $a_1 > 0$, $b > 0$. From the assumption of gamma density one gets $p > 0$.

Also $f(u,u) \geq 0$ leaves the condition,

$$p(p(a_1 a_2)+a_1 a_2-b) > 0$$

or

$$(p+1)a_1 a_2 \geq b.$$

Thus $0 \le b \le (p+1)a_1a_2$.

Thus the conditions on the parameters are

$$a_i > 0,\ i = 1,2\ ,\ p > 0,\ 0 \le b \le (p+1)a_1a_2.$$

The family of distributions obtained under aforementioned framework includes a large class of distributions, like Pareto distributions of Hutchinson (1979), Lindley and Singpurwalla (1986), Burr distributions of Takahasi (1965), Durling et al (1970). These distributions are considered in the forthcoming section.

4.5. PARTICULAR CASES.

1. When $g(x_i) = x_i$, $i=1,2$, $u=0$, the form of original distribution is Gumbel's bivariate exponential distribution specified by (2.22) corresponding environmental model takes the form,

$$R_\eta(x_1,x_2) = [1+a_1x_1+a_2x_2+bx_1x_2]^{-p} \qquad (4.26)$$

which is bivariate Pareto of Hutchinson (1979). By taking b=0 in equation (3.26), bivariate Pareto distribution of Lindley and Singpurwalla (1986) is obtained.

2. $g(x_i) = \log x_i$ i=1,2 original distribution is bivariate Pareto Type I, specified by equation (2.23) and accordingly, equation (4.23) changes to

$$R_\eta(x_1,x_2) = [1+a_1\log x_1+a_2\log x_2+b\log x_1\log x_2]^{-p}, \quad x_1,x_2 > 1. \qquad (4.27)$$

3. $g(x_i) = x_i^\beta$, the parent distribution becomes bivariate Weibull given by (2.27), and the environmental model is

$$R_\eta(x_1,x_2) = [1+a_1x_1^\beta+a_2x_2^\beta+bx_1^\beta x_2^\beta]^{-p};\ x_1,x_2>0 \qquad (4.28)$$

which is bivariate Burr distribution of Durling et al (1970). When b = 0, bivariate Burr distribution of Takahasi (1965) results.

4. When $g(x_i) = \log(1+ax_i)$, the parent distribution becomes bivariate Pareto Type II, and the environmental model arising from (4.23) is

$$R_\eta(x_1,x_2) = [1+a_1\log(1+cx_1)+a_2\log(1+cx_2)$$

$$+b\log(1+cx_1)\log(1+cx_2)]^{-p}. \qquad (4.29)$$

5. When $g(x_i) = \log \frac{c+x_i}{c-x_i}$, with $0 < x_i < c$, $i=1,2$ the form of original distribution is bivariate finite range distribution specified by (2.32). Corresponding environmental model is

$$R_\eta(x_1,x_2) = \left[1 + a_1 \log \frac{c+x_1}{c-x_1} + a_2 \log \frac{c+x_2}{c-x_2} + b \log \frac{c+x_1}{c-x_1} \log \frac{c+x_2}{c-x_2} \right]^{-p}. \qquad (4.30)$$

To be able to analyse the reliability of this system in a changed environment, we note that, the vector valued failure rate of the system is

$$h(x_1,x_2) = (h_1(x_1,x_2),h_2(x_1,x_2))$$

with

$$h_i(x_1,x_2) = \frac{p(a_i+bg(x_j))\ g'(x_i)}{[1+a_1g(x_1)+a_2g(x_2)+bg(x_1)g(x_2)]}. \qquad (4.31)$$

Thus for a quantitative assessment of the effect of the new environment on the system the objects of comparison are the failure rates in (4.31) and (2.58). Using the superscripts `e' and `o' to differentiate the failure rates of changed and original environments the relative measures that facilitate comparison are

$$\frac{h_i^e(x_1,x_2)-h_i^o(x_1,x_2)}{h_i^o(x_1,x_2)} = \frac{h_i^e(x_1,x_2)}{h_i^o(x_1,x_2)} - 1. \tag{4.32}$$

Thus when actually the system is operated in a different set of conditions, the error that would be committed through the measurement of the failure rate would be positive or negative according as

$$\frac{h_i^e(x_1,x_2)}{h_i^o(x_1,x_2)} \gtrless 1.$$

With respect to our model, this happens when

$$\frac{p(a_i+bg(x_j))}{(1+a_ig(x_1)+a_2g(x_2)+bg(x_1)g(x_2))(\alpha_i+\theta g(x_j))} \gtrless 1.$$

For i = 1, the first condition reduces to

$$P(a_1+bg(x_2)) > (1+a_1g(x_1)+a_2g(x_2)+bg(x_1)g(x_2))(\alpha_1+\theta g(x_2))$$

$$= m(1+a_1g(x_1)+a_2g(x_2)+bg(x_1)g(x_2))(a_1+bg(x_2))$$

or when

$$p/m > (1+a_1 g(x_1)+a_2 g(x_2)+bg(x_1)g(x_2)).$$

Since $E(\eta) = p/m$, we conclude that whenever

$E(\eta) > (<) (1+a_1 g(x_1)+a_2 g(x_2)+bg(x_1)g(x_2))$ the changed environment would cause failures more (less) frequently than in the test condition, when

$E(\eta) = [1+ a_1 g(x_1) + a_2 g(x_2) + b g(x_1) g(x_2)]$, the two environments are identical.

4.6. CHARACTERIZATIONS

As discussed in Section 3.2, measures similar to AFR, GFR and HFR can be obtained if the concept of failure rate in them is replaced by mean residual life in the bivariate case. Accordingly, the arithmetic mean mean residual life (AM MRLF) is defined as the vector,

$$K(x_1,x_2) = (K_1(x_1,x_2),K_2(x_1,x_2))$$

where

$$K_i(x_1,x_2) = \frac{1}{x_i}\int_0^{x_i} r_i(x_1,x_2)\, dx_i \qquad (4.33)$$

and

$$r_i(x_1,x_2) = \frac{1}{R(x_1,x_2)} \int_{x_i}^{\infty} R(x_1,x_2)\, dx_i, \quad i=1,2$$

is the mean residual life of i^{th} component.

Likewise, the bivariate geometric mean mean residual life (GM MRLF) is defined as,

$$L(x_1,x_2) = (L_1(x_1,x_2), L_2(x_1,x_2))$$

where

$$L_i(x_1,x_2) = \exp\left\{\frac{1}{x_i} \int_0^{x_i} \log r_i(x_1,x_2)\, dx_i\right\} \qquad (4.34)$$

and the bivariate harmonic mean mean residual life (HM MRLF) is defined as

$$M(x_1,x_2) = (M_1(x_1,x_2), M_2(x_1,x_2))$$

with

$$M_i(x_1,x_2) = \left(\frac{1}{x_i} \int_0^{x_i} \frac{1}{r_i(x_1,x_2)}\, dx_i\right)^{-1}. \qquad (4.35)$$

Using the concepts of AM MRLF, GM MRLF and HM MRLF together with AFR, GFR and HFR we can characterize some of the models already considered in the sequel.

Theorem 4.1

A random vector $X = (x_1,x_2)$ in R_2^+ with absolutely continuous distribution satisfies the property

$$K(x_1,x_2) = r(x_1,x_2). \tag{4.36}$$

For every $x_1,x_2 > 0$ if and only if the distribution of (X_1,X_2) is bivariate exponential of Gumbel(1960).

Proof:

When (X_1,X_2) is of Gumbel's form,

$$r_i(x_1,x_2) = (\lambda_i+\theta x_j)^{-1}.$$

So that from (4.33) $r_i(x_1,x_2) = K_i(x_1,x_2)$, establishing (4.36). Conversely, if (4.36) holds differentiating the identity (4.33) with respect to x_i,

$$x_i \frac{\partial K_i(x_1,x_2)}{\partial x_i} + K_i(x_1,x_2) = r_i(x_1,x_2)$$

or

$$x_i \frac{\partial K_i(x_1,x_2)}{\partial x_i} = 0$$

giving $K_i(x_1,x_2) = r_i(x_1,x_2) = P_i(x_j)$.

Thus $r_i(x_1,x_2) = (p_1(x_2),p_2(x_1))$ and hence the result follows from Nair and Nair (1988).

Corollary :

$$K(x_1,x_2) = (p_1(x_2),p_2(x_1))$$

if and only if (X_1,X_2) has Gumbel's bivariate exponential distribution.

Adopting the same logic, but with a little different algebra, it can be seen that the following theorems hold.

Theorem 4.2

$L(x_1,x_2) = r(x_1,x_2)$ for every $x_1,x_2 > 0$ if and only if (X_1,X_2) has Gumbel's distribution.

Theorem 4.3

$M(x_1,x_2) = r(x_1,x_2)$ for every $x_1,x_2 > 0$ if and only if (X_1,X_2) has Gumbel's distribution.

Theorem 4.4

A necessary and sufficient condition for (X_1,X_2) to be an absolutely continuous random vector in the support of R_2^+ satisfies any one of the following conditions

(1) $K_i(x_1,x_2)\, H_i(x_1,x_2) = C$

(2) $L_i(x_1,x_2)\, G_i(x_1,x_2) = C$

(3) $M_i(x_1,x_2)\, A_i(x_1,x_2) = C$

for i = 1, 2, every x_1, $x_2 > 0$ and some positive real c is that (X_1,X_2) is distributed either as Gumbel's bivariate exponential distribution for c = 1 or a bivariate Pareto type in (4.26) for c>1, or as bivariate finite range with survival function

$$P(X_1>x_1,\ X_2>x_2) = (1-p_1x_1-p_2x_2+qx_1x_2)^d \tag{4.37}$$

p_1, p_2, d>0, $0<x_1<p_1^{-1}$, $0<x_2<\dfrac{(1-P_1x_1)}{(p_2-qx_1)}$, $1-d<qP_1^{-1}P_2^{-1}\le 1$

for 0 < C < 1.

Proof:

Suppose (1) holds. Then for i = 1

$$\left(\frac{1}{x_1}\int_0^{x_1} r_1(t,x_2)dt\right)\left(\frac{1}{x_1}\int_0^{x_1}\frac{dt}{h_1(t,x_2)}\right)^{-1} = C$$

or

$$\int_0^{x_1} r_1(t,x_2)dt = C \int_0^{x_1} \frac{dt}{h_1(t,x_2)}$$

Differentiating with respect to x_1,

$$r_1(x_1,x_2) \, . \, h_1(x_1,x_2) = C$$

Similarly for $i = 2$

$$r_2(x_1,x_2) \, . \, h_2(x_1,x_2) = C$$

This gives the form of

$$r_i(x_1,x_2) = Ax_i + B_i(x_j) \quad i,j = 1,2 \quad i \neq j$$

which characterizes the models in the Theorem for the specified values of C as given in Sankaran and Nair (1992).

When (2) holds. for $i = 1$,

$$\exp\left\{\frac{1}{x_1} \int_0^{x_1} \log\, r_1(t,x_2)dt\right\} \exp\left\{\frac{1}{x_1} \int_0^{x_1} \log\, h_1(t,x_2)dt\right\} = C$$

which gives the same expression for $r_1(x_1,x_2)$ as in the case of assumption (1). The proof for case (3) follows suit and this establishes the Theorem.

Theorem 4.5

An AM MRLF of the form,

$$K_i(x_1,x_2) = ax_i + b_i(x_j)$$

characterizes the Gumbel's bivariate law for a = 0 Pareto II distribution for a > 1/2 and the finite range distribution for 0 < a < 1/2.

Proof:

$$K_1(x_1,x_2) = ax_1 + b_1(x_2)$$

$$\leftrightarrow \frac{1}{x_1}\int_0^{x_1} \log r_1(t,x_2)dt = ax_1 + b_1(x_2)$$

$$\leftrightarrow r_1(x_1,x_2) = 2ax_1 + b_1(x_2).$$

4.7 CONCLUSION

The present study has considered three general families of distributions, each bringing a class of bivariate distributions under a uniform frame work. They provide new derivations for some well known distributions as well as certain new bivariate continuous distributions. Derivation of all the models are based on extensions of concepts that have found acceptance among a large audience. We have presented characterization theorems that will enable identification of the member which will suit the observations in a practical situation.

In view of the general functional form appearing in the survival function in each family, general characterization theorems were hard to establish, as in many cases the assumed properties lead to functional equations that are difficult to solve, by the existing methods. However, characterization theorems based on basic reliability concepts have been established, where the models are most apt to use. More characteristics of the families are being investigated and is hopefully expected to be presented in a future work.

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