SOME PROBLEMS IN FLUID MECHANICS AND APPLICATIONS OF DIFFERENTIAL EQUATIONS

NUMERICAL AND ANALYTICAL STUDIES OF A KdVB TYPE EQUATION FOR WATER WAVES

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CERTIFICATE

This is to certify that the thesis bound herewith is an authentic record of the research work carried out by Miss. Shailaja R., Department of Mathematics and Statistics, Cochin University of Science and Technology, Cochin 22, under my supervision and guidance, for the degree of Doctor of Philosophy and further that no part thereof has been submitted for any other degree or diploma.

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C O N T E N T S

CHAPTER 1

INTRODUCTION

1.1 KORTEWEG-de VRIES EQUATION

It is well known that the celebrated Korteweg-de Vries (KdV) equation

$$U_t + 6UU_x + U_{xxx} = 0, \qquad (1.1)$$

is the simplest model equation describing a nonlinear dispersive non-dissipative phenomenon. The most important property of the KdV equation is that it admits steady progressive wave solutions called solitary waves, which are long waves of small amplitude travelling without change of form.

This equation represents physical phenomena arising out of a balance between weak nonlinearity and weak dispersion and describes the unidirectional propagation of small but finite amplitude nonlinear waves in a dispersive medium. The equation was first derived by Korteweg and de Vries (1895) as an approximation to the Navier-Stokes equation assuming that the waves being considered have small amplitude and large wave length compared to the undisturbed depth.

The first recorded observation of a solitary wave was made in 1834 by the navel architect Scott Russell (1844, 1845). Boussinesq (1872) and Rayleigh (1876) analysed mathematically the phenomenon of solitary waves and derived approximate results for the shape and velocity of such waves. Rayleigh considered a solitary wave in a Eulerian frame of reference moving at a velocity that brings the wave to rest. Boussinesq's equation have travelling wave solutions moving to both left and right. Going a step ahead from Boussinesq theory Korteweg and de Vries restricted attention to waves moving to the right only. They deduced Rayleigh's solitary wave as the limiting case of the cnoidal waves for long wavelength.

Though solitary waves continued to attract attention in the ensuing decades (Weinstein, 1926; Lamb,1932; Keulegan and Patterson, 1940; Ursell, 1953; Friedrichs and Hyers, 1954; Lavrentiff, 1954; Stoker, 1957 to mention a few), the significance of KdV equation as a basic equation in mathematical physics was brought out when Gardner and Morikawa (1960) obtained the equation as a model for waves in a cold collisionless plasma. The current interest in the KdV equation has its origin in a numerical experiment by Zabusky and Kruskal (1965) in which they observed that the solution of KdV equation might exhibit FPU recurrence (Fermi, Pasta, and Ulam, 1955, 1974). They found that a smooth initial profile evolving under the KdV developed into a train of solitary waves. The most remarkable property of solitary waves they observed was that after interaction two such waves emerge unaffected in shape and suffered only phase shifts. The interaction was clean in the sense that no residual disturbances were created. This particle like behaviour led to the nomenclature 'solitons'.

In the case of infinitesimal waves, the classical theory of linear waves gives a complete description of their evolution. If the waves have small but finite amplitude, the linear theory breaks down and nonlinear corrections are to be made to extend the range of validity of the theory to a longer time scale. The development of the soliton theories during the last two decades has enriched the theory of nonlinear waves. Typically, soliton theories provide the nonlinear corrections to render the linear theory valid on a longer time scale. There is a short time on which the linear theory applies followed by a longer time scale on which the soliton theory applies, and may be followed by even longer time scale on which this theory also breaks down.

KdV equation has been found to be an evolution equation for a large number of rather distinct physical systems as it represents a balance between some form of dispersion (or variation of dispersion in the case of wave-packet evolution) and weak nonlinearity in an appropriate reference frame. This includes water waves (Korteweg and de Vries,1895; Freeman and Johnson, 1970; Madsen and Mei, 1969; Madsen *et al.* 1970), plasma waves (Gardner and Morikawa, 1960; 1965) and anharmonic lattices and waves in elastic rods (Zabusky, 1967, 1968, 1969, 1973).

The study of solitons has richly contributed to different branches of mathematics. A major development in the theory of differential equations was the Inverse Scattering Transformation (IST) method by Gardner *et al.* (1967, 1974) by which an exact solution for KdV equation was obtained. It has also given birth to new mathematical methods having applications ranging from `practical' problems of wave propagation to rather `pure', topics in Algebraic Geometry (Dubrovin *et al.*, 1976).

It is to be noted that though the KdV equation was first derived in the context of water waves, the recent revival of interest in its studies owes much to development in other branches of physics. In this context it is worth mentioning that the KdV equation, inspite of its fame and popularity has not remained unchallenged as a model equation for long waves in a channel (Peregrine, 1966).

It is well known that the waves reaching a shelf are well separated and hence can be considered as solitary waves. Thus the development of a solitary wave over a region of varying depth is of great practical importance. Notable contributions in this direction are due to Peregrine (1967) and Ippen and Kulin (1970). The results available concerning KdV equation was first employed in the study of water waves by Madsen and Mei (1969). Grimshaw (1970) considered the problem of waves on water of slowly varying depth and investigated the condition for solution to be a solitary wave. Ott and Sudan (1970) modified the KdV equation to include energy dissipation. The dissipative term they treated was the Fourier transform of the linear damping, and obtained a time evolution of a solitary wave. Kakutani (1971) has shown that a modification of a KdV equation can describe shallow water waves propagation over gently slopping beaches. Johnson (1972, 1973a,b) has independently derived a KdV type equation for the same problem. These equations have later led to the study of a perturbed KdV equation. Zabusky and Galvin (1971) in their experimental studies on shallow water waves have shown that the KdV like evolution equation

$$U_t + UU_x + \delta^2 U_{xxx} = 0, \qquad (1.2)$$

of shallow water waves is very accurate even for large nonlinearities and found nontrival amounts of energy in wave number $k > 0.5$. While Zabusky and Galvin (1971) considered an initial value problem having spatial periodicity, Hammack (1973) and Hammack and Segur (1974) considered an initial value problem posed on the real line.

Benjamin, Bona, and Mahony (1972) have proposed an alternative model

$$U_t + U_x + UU_x - U_{xxt} = 0, \qquad (1.3)$$

which they call a regularised KdV equation (BBM equation). Bona and Bryant (1973) have studied the system as a model for long water waves of small but finite amplitude generated in a uniform open channel by a wave maker at one end. But the solitary wave solutions of the BBM equation may not be solitons (Jeffrey, 1979). Specific examples of other types of model equations for long waves are given by Bona and Smith (1976) and Bona and Dougalis (1980). It has been shown by Bona and Smith (1975) that an exact relation exists between equation (1.3) and the KdV equation

$$U_t + U_x + UU_x + U_{xxx} = 0, \qquad (1.4)$$

in the sense that for the same initial data both equations have unique smooth solutions.

Longuet-Higgins (1974) has established some exact relations between the momentum and potential energy in the case of solitary waves. Extensive numerical studies carried out by Longuet-Higgins and Fenton (1974) have shown that speed, mass, momentum, and kinetic and potential energies for waves of amplitude less than the maximum do not increase monotonically with wave amplitude. This result has been confirmed by Byatt-Smith and Longuet-Higgins (1976), and

Longuet-Higgins and Fox (1977, 1978).

Miura (1974, 1976) has given extensive review and detailed analysis of the works in this field. Some other review articles are due to Jeffrey and Kakutani (1972), Scott *et al.* (1973), Benjamin (1974), Whitham (1974), Lax (1976), Cercignani (1977), Makhankov (1978), Miles (1980, 1981a), Newell (1983) and Sander and Hutter (1991).

Stability of solitary waves has been investigated by Benjamin (1972) and Bona (1975). Berryman (1976) have shown that a KdV soliton is stable whereas the Boussinesq solitary wave is unstable to infinitesimal perturbations. A general method for the study of stability of a solitary wave solution to KdV or BBM type equation has been given by Souganidis and Strauss (1990).

Stoke's (1847) investigations in water waves are the starting point for the nonlinear theory of dispersive waves. Russell's solitary waves are regarded as the limiting cases of Stoke's oscillatory waves of permanent types, the wavelength being considerably large compared to the depth of the channel. So the widely separated elevations are independent of one another. But Stoke's theory fails when the wave length much exceeds the depth and hence it cannot unravel the physical causes leading to the formation of solitary waves. Amick and Toland (1981) have investigated the validity of Stoke's conjecture.

Bampi and Morro (1979) and Bona (1983) have investigated the physical and mathematical approximations that are at the basis of the KdV equation as a model when nonlinear and dispersive effects are of comparable small order. Justification of KdV approximation in the case of N-soliton water waves has been investigated by Sachs (1984).

In the derivation of the equation, Korteweg and de Vries (1895) had taken into account the effect of surface tension also. They showed that the solitary waves of depression exist for sufficiently large value of surface tension. Shinbrot (1981/1982) has attempted to study solitary waves with surface tension. By a formal perturbation expansion Vanden-Broeck *et al.* (1983) have considered the effect of surface tension on cnoidal waves by a systematic perturbation method. Solitary waves at the interface have been studied by Miles (1980), Koop and Butler (1981), Segur and Hammack (1982), Dai (1982, 1983), Gear and Grimshaw (1983), Mirie and Su (1984, 1986), Dai and Jeffrey (1989a) and Bona and Sachs (1989). Huang *et al.* (1989) have obtained exact and explicit solitary wave solution for higher order KdV equation for water waves with surface tension.

Grimshaw (1983) has investigated solitary wave propagation in density stratified fluids and described applications to waves in atmosphere and ocean. Fission of a solitary wave in a stratified fluid with a free surface has been investigated by Zhou (1988). Gabov (1989) has shown that nonlinear waves on the surface of shallow floating fluid can be described by the KdV equation.

Naumkin and Shishmarev (1990) have studied the local and global existence of the solitons and asymptotic properties of the equations of surface waves which include KdV equation.

1.2 KORTEWEG-de VRIES-BURGERS' EQUATION

Burgers' equation is the simplest model of diffusive waves. The equation

$$U_t + UU_x = \nu U_{xx} , \qquad \nu > 0 \tag{1.5}$$

was first introduced by Bateman (1915). Burgers (1948) showed that it is the simplest equation to combine nonlinearity with diffusion. The equation gained its significance when Hopf (1950) and Cole (1951) showed that general solution could be obtained explicitly. It has found application in different fields like turbulence, sound waves in viscous media, waves in fluid filled visco-elastic tubes and magnetohydrodynamic waves. In the context of fluid dynamics, the nonlinear term represents convection while the second order term represents the viscous force. The effect of nonlinearity produces progressively more and more deformation in the wave profile with time. It achieves the smooth joining of two asymptotic uniform states through continuously varying states, while the only bounded solution of linearised Burgers' equation is a constant state. The second order term counteract the effect of nonlinearity and check the development of steep slopes in the wave profile.

Burgers' equation has unidirectional travelling wave solutions known as Burgers' shock waves. The speed of the travelling wave is determined only by solutions at infinity.

It has been shown by Jeffrey and Kakutani (1972) that these shocks waves are either asymptotically stable or stable to infinitesimal disturbances. Like KdV equations, Burgers' equation also is a model for a wide class of nonlinear Galilean invariant systems under weak nonlinearity with long wave length approximations (Su and Gardner, 1969). While KdV equation is a limiting form for nonlinear dispersive systems, Burgers' equation is a limiting form for nonlinear dissipative systems.

There are many physical systems, the modelling of which requires incorporating the effects of nonlinearity, dispersion and dissipation, represented by an equation of the form

$$U_t + a\, UU_x + b\, U_{xx} + c\, U_{xxx} = 0, \qquad (1.6)$$

where a, b and c are constants, which is called Korteweg-de Vries-Burgers' (KdVB) equation. Grad and Hu (1967) have used a steady state version of these equations to describe a weak-shock profile in plasmas. Johnson (1970) has obtained this equation for waves propagating in a liquid-filled elastic tube. A comprehensive account of the travelling wave solution to the KdVB equation can be found in Jeffrey and Kakutani (1972). Jeffrey (1979) has also obtained an asymptotic shock wave solution of the KdVB equation that applies when dissipative effects predominate over dispersive effects. He has proposed a new Time Regularised Long Wave (TRLW) equation which is in the form of a conservation law and is capable of characterising bidirectional wave propagation. Numerical investigations of the solution have been carried out by Canosa and Gazdag (1977). Korebeinikov (1983) has studied solutions of KdVB equations for plane, cylindrical and spherical waves, and established all invariant solutions using group theoretic methods. Gibbon *et al.* (1985) have showed that KdVB equation does not have Painleve property. Shu (1987) has studied asymptotic behaviour of solutions of KdVB equation. Jeffrey and Xu (1989) have shown that the equation can be reduced to a quadratic form involving new dependent variable in its partial derivatives. Melkonian (1989) has used the equation for nonlinear waves in thin films. Uniqueness of solution has been investigated by Vlieg and Halford (1991).

1.3 SOLITONS AND INVERSE SCATTERING TRANSFORMATION

The most important development in the theory of partial differential equations during the 1960's was the invention of Inverse Scattering Transform (IST) method by Gardner, Greene, Kruskal and Miura (GGKM) (1967, 1974) for finding exact solution of the initial value problem for KdV equation. Till then the only known exact solutions were the solitary waves and cnoidal waves. IST method provides a procedure for explicitly obtaining the pure soliton solutions and qualitative information about the general solutions. This method can be viewed as a generalization of Fourier analysis in the sense that it provides the exact solution to certain nonlinear evolution equations just as the Fourier transform does for linear evolution equations.

A remarkable property of KdV equation is that it is in the form of a conservation law. Further conservation laws can be derived from the equation and Miura *et al.* (1968) have found that KdV equation has an infinite number of conserved densities. The existence of infinitely many conservation laws of multiple soliton solutions and solvability by IST are closely related. For any dynamical system there exist true connections between solvability and integrability conditions. Nonlinear evolution equations which are exactly solvable by IST are said to satisfy the integrability condition. The existence of infinite number of conserved quantities can also be termed as a necessary and sufficient condition for integrability and hence solvability by IST.

The IST method can be briefly described as follows. Let us consider, following Miura (1976), the KdV equation

$$U_t - 6UU_x + U_{xxx} = 0. \tag{1.7}$$

If V is a solution of the modified KdV equation

$$V_t - 6V^2Vx + V_{xxx} = 0, \tag{1.8}$$

then

$$U = V^2 + V_x , \tag{1.9}$$

is a solution of equation (1.7). Equation (1.9) defines a transformation analogous to the Hopf-Cole transformation of Burgers' equation. If U is known, we shall write equation (1.9) in the form

$$V_x + V^2 = U,$$

and is a Riccati equation for V.

Then the transformation $V = \psi_x / \psi$ yields the linear equation

$$\psi_{xx} - U\psi = 0. \tag{1.10}$$

Equation (1.10) is the time-independent Schrodinger equation; however missing the energy level term. Now U is a solution of the KdV equation which is invariant under Galilean transformation. Thus U can be shifted by a constant and x differentiations remain unchanged. Without loss of generality we can replace the equation (1.10) by

$$\psi_{xx} - (u-\lambda)\psi = 0. \tag{1.11}$$

This is time-independent Schrodinger equation of quantum mechanics where ψ is the wave function, U is the potential and λ represents energy levels. The variable t only plays the role of a parameter. The usual problem in quantum mechanics is, given the potential U, to find the bound state energy levels and the wave functions, ie the eigenvalues and the proper and improper eigenfunctions. This is called the direct scattering problem. But here the problem is to find U from apriori knowledge of certain information called scattering data, which includes the discrete eigenvalues, the normalizing coefficients for the corresponding eigenfunctions, and the reflection coefficient and this is called the inverse scattering problem. First it is solved in the quantum mechanics context, ie t is fixed a constant. Then the dependence on t can be taken into account and thereby effects a solution of the initial-value problem for the KdV equation. This method reduces the problem of solving a KdV equation to that of solving a linear integral equation, the Gel'fand-Levitan or the Marchenko equation.

This method was soon expressed in an elegant form by Lax (1968) and later by Ablowitz, Kaup, Newell and Segur (AKNS)

(1974). One of the most important feature of IST is that it maps an integrable field into a set of action-angle variable (the scattering data) in scattering space where the equations of motion for the scattering data become very simple, the spectrum becomes invariant and the phases execute a very simple rotation about the spectrum. Lax obtained exact expressions for how this scattering data evolved in time for any system integrable or not. The evolution of the scattering data from its initial spectrum follows from the appropriate AKNS equations.

The special class of initial data for which exact solutions can be obtained corresponds to the N-soliton solutions and are characterised by zero reflection coefficient. There are contributions only from discrete spectrum. The solitons which propagate with positive velocity are the physical manifestation of the discrete spectrum for each eigenvalue. The continuous spectrum gives rise to a component of the solution which although nonlinear bears a close resemblance to the solution of the linearised KdV equation. The discrete eigenvalues are independent of the parameter t and are constants of motion for the KdV equation. The two striking features of the method are that for the associated eigenvalue problem, the discrete eigenvalues are constants in time and the time-dependence of the other scattering data (continuous eigenvalue) can be determined apriori. Soliton solutions arise when the reflection coefficient in the scattering data is zero (Miura, 1976). With zero reflection coefficient the kernal and inhomogeneous term in the Gel'fand-Levitan integral equation are reduced to finite sums over the discrete spectrum and the equation becomes degenerate.

Zakharov and Faddeev (1972) have shown that KdV equation is a completely integrable Hamiltonian system, that the inverse scattering data may be viewed as another set of canonical co-ordinates for this system, and that an infinite number of integrals of the motion arise rather naturally from this interpretation.

Zakharov and Shabat (1972) and Ablowitz *et al.* (1973) have shown that the inverse scattering method is applicable to other nonlinear evolution equations also. Zakharov-Shabat IST provides a basis for a subsequent synthesis of the various equations that were known to exhibit soliton behaviour (Ablowitz *et al.*, 1974; Zakharov and Shabat, 1974; Flaschka and Newell, 1975). A detailed description of the IST associated with the generalized Zakharov-Shabat and Schrodinger eigenvalue problem is available in Newell (1980). Wadati (1980) and Calagero and Degasperis (1980) have given a review on the matrix generalization of the IST method.

Ablowitz and Newell (1973) have used the IST method to study the asymptotic behaviour of the KdV solution. IST method has been extended to study nonlinear equations with periodic boundary conditions (McKean and Van Moerbeke, 1975; Novikov, 1974). The particle like behaviour of solitons has given rise to renewed interest in Backlund transformation to construct N-soliton solutions (Hirota and Satsuma, 1976).

Jeffrey and Dai (1988) have obtained one soliton solution for a variable coefficient KdV equation by IST method. However, the particular equation considered by them can be transformed into the general KdV equation, so that it does not represent the most general case. Later Dai and Jeffrey (1989b) have extended the Zakharov-Shabat IST for the class of variable coefficient KdV equations which are directly integrable. Hirota (1980) has developed an important technique to find N-soliton solutions. This is a direct method which does not require a solution of the IST problem.

1.4 KdVB EQUATION FOR WAVES ON WATER WHERE THE DEPTH CHANGES FORMING A SHELF

Pramod and Vedan (1992) have derived a KdVB type equation for long wave propagation in water when there is a sudden change in depth forming a shelf. Here we give briefly the derivation of the equation.

Denoting the dimensional variables by primes, the depth of the water below the equilibrium level is h'. This depth is defined by

$$h' = h_0' - \epsilon' H(x'-x_0'), \tag{1.12}$$

where $H(x'-x_0')$ is the Heaviside step-function. The flow is irrotational. The velocity potential ϕ' is given by

$$\phi' = f'(x') - \frac{(y'+h')^2}{2!} \frac{\partial^2 f'(x')}{\partial_x{'}^2} + \frac{(y'+h')^4}{4!} \frac{\partial^4 f'(x')}{\partial_x{'}^4} \tag{1.13}$$

Dimensionless variables are defined by

$$x' = lx, \quad x_0' = lx_0 , \quad \epsilon' = h_0\epsilon , \quad y' = h_0 y, \tag{1.14a}$$

$$\phi' = \frac{gla}{c_0} \phi \quad \text{and} \quad f' = \frac{gla}{c_0} f, \tag{1.14b}$$

where

$$c_0 = \sqrt{gh_0} , \quad \alpha = \frac{a}{h_0} , \quad \beta = \frac{h_0^{\ 2}}{l^2} ,$$

and l and a are characteristic wavelength and amplitude respectively. As in the derivation of KdV equation terms of $O(\alpha^2)$, $O(\beta^2)$ and $O(\alpha\beta)$ are neglected. Retaining terms in $\epsilon\alpha$ and $\epsilon\beta$ it is shown that the long wave propagation is governed by the equation

$$\eta_t + \left(1 - \frac{1}{2}\epsilon H(x-x_0) + \frac{5}{12}\epsilon\beta\delta'(x-x_0)\right)\eta_x$$

$$+ \left(\frac{3}{2}\alpha + \frac{5}{4}\epsilon\alpha H(x-x_0)\right)\eta\eta_x + \frac{1}{3}\epsilon\beta\delta(x-x_0)\eta_{xx}$$

$$+ \left(\frac{1}{6}\beta - \frac{1}{3}\epsilon\beta H(x-x_0)\right)\eta_{xxx} = 0, \qquad (1.15)$$

where

$$\delta(x-x_0) = \frac{d}{dx} H(x-x_0),$$

is the Dirac delta function and the primes denotes differentiation with respect to x.

From equation (1.15), we find that

$$\eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x + \frac{1}{6}\beta\eta_{xxx} = 0, \qquad x<x_0 \qquad (1.16)$$

and

$$\eta_t + \left(1 - \frac{1}{2}\epsilon\right)\eta_x + \left(\frac{3}{2}\alpha + \frac{5}{4}\epsilon\alpha\right)\eta\eta_x$$

$$+ \left(\frac{1}{6}\beta - \frac{1}{3}\epsilon\beta\right)\eta_{xxx} = 0, \qquad x>x_0 \qquad (1.17)$$

Equation (1.16) is identical with the classical KdV equation for constant depth and equation (1.17) also reduces to it as $\epsilon \longrightarrow 0$.

It has been shown that a local generalized solution can be obtained in the form

$$\eta = \eta_1 + H(x-x_0)\ (\eta_2 - \eta_1), \qquad (1.18)$$

where η_1 and η_2 are solutions of equations (1.16) and (1.17) respectively.

Assuming that η has compact support we find that at $x = 0$

$$\frac{\partial}{\partial t} \int_{-\infty}^{\infty} \eta dt = \epsilon \left[-\frac{1}{2} \eta + \frac{5}{8} \alpha\eta^2 - \frac{5}{12} \beta\eta_{xx} \right]_{x=0} . \qquad (1.19)$$

If $\epsilon = 0$ we have

$$\frac{\partial}{\partial t} \int_{-\infty}^{\infty} \eta dt = 0.$$

For $\epsilon \neq 0$ the mass flux is changed by the right hand side of equation (1.19) unless η satisfies

$$\frac{1}{2} \eta - \frac{5}{8} \alpha\eta^2 + \frac{5}{12} \beta\eta_{xx} = 0 \quad \text{at } x = 0. \qquad (1.20)$$

1.5 SCOPE OF THE THESIS

Equation (1.15) is a KdVB type equation. Nonlinearity and dispersion are measured by the same parameters as in the classical KdV equation. The additional effect of change in depth is taken into account by a parameter ϵ and by retaining terms linear in this parameter. As $\epsilon \longrightarrow 0$, we find that the equation reduces to the classical KdV equation. An additional feature is a diffusion term as in Burgers' equation. The thesis contains the results of various studies of equation (1.15).

Because of the complexity of equation (1.15) we start from a study of equations (1.16) and (1.17). If we consider the two equations as defined for $-\infty < x < \infty$, equation (1.17) can be considered as a perturbation of equation (1.16) due to the parameter ϵ.

In chapter 2 equation (1.17) is studied. Method of derivative expansion is used to study wave interaction.

In chapter 3 we consider IST analysis of the equations (1.16) and (1.17) separately. We further consider numerical study of the equations (1.16) and (1.17) using a soliton as initial condition. The two equations are used for the two domains, upstream and downstream of the shelf respectively. A finite difference scheme is used.

Unlike equations (1.16) and (1.17), equation (1.15) contains a diffusion term as in Burgers' equation. To study the effect of diffusion we consider the following equation

$$\eta_t + \left(1 - \frac{1}{2}\epsilon\right)\eta_x + \left(\frac{3}{2}\alpha + \frac{5}{4}\epsilon\alpha\right)\eta\eta_x + \frac{1}{3}\epsilon\beta\eta_{xx}$$

$$+ \left(\frac{1}{6}\beta - \frac{1}{3}\epsilon\beta\right)\eta_{xxx} = 0 \qquad (1.21)$$

This is a KdVB type equation.

In chapter 4 we study equation (1.21). We first obtain an asymptotic form for travelling wave solution of the equation (1.21). Then exact solutions are also obtained using two different methods. The first method is a direct one based on a combination of solutions to the KdV and Burgers' equations. The second one is an extension of Hirota's method.

Kaup and Newell (1978) have used IST theory for exact KdV equation as a basis for a perturbation scheme. The study involves all perturbations of equations which are integrable by using the IST associated with the Zakharov-Shabat eigenvalue problem (Zakharov and Shabat, 1972) or the Schrodinger equations. An exact expression for the solution in terms of the scattering data and squared eigenfunctions was used to avoid inverse procedure given by the Marchenko equations. In chapter 5 we use this method to study the equation (1.15).

The final chapter we summarise the results of chapters 2 to 5 and point out the direction for further research work.

CHAPTER 2

WAVE INTERACTION

2.1 INTRODUCTION

The study of wave-wave interactions has its origin in the fundamental paper by Peierls (1929) on heat conduction in solids. Since then it has been applied to other branches of physics, particularly in quantum field theory. Litvak (1960) has applied the theory to study plasma wave interactions. The same has found applications in scattering geophysical fields through the works of Phillips (1960), Hasselmann (1960, 1962), Benney (1962) and Longuet-Higgins (1962). The theory has also been applied to study exchange of energy in internal and surface waves (Ball, 1964), capillary waves (McGoldrick, 1965), waves in stratified fluids (Thorpe, 1966) and nonlinear interaction between gravity waves and turbulent atmospheric boundary layer (Hasselmann, 1967).

Nishikawa *et al.* (1974), Kawahara *et al.* (1975) and Benney (1976, 1977) have studied interactions between short and long waves by means of the coupled equations for a single monochromatic wave and a long wave. The interaction and the statistics of many localised waves have been investigated by

Zakharov (1972) in connection with Langmuir turbulence. Miles (1977a, b) has studied the general interaction of two oblique solitary waves and interaction associated with the parametric end points of the singular regime.

In the case of two solitary waves propagating in opposite directions, there are significant differences between experimental results (Maxworthy, 1976) and theory based on Boussinesq equation (Oikawa and Yajima, 1973). Su and Mirie (1980) recasted nonlinear surface boundary conditions into a pair of equations involving the free surface elevation and the velocity along the horizontal bottom boundary and determined a third-order perturbation solution to the head-on collision of two solitary waves. They have shown that although the waves emerged from the collision without any change in height, were symmetric and changed slowly in time.

Fenton and Rienecker (1982) have investigated the interaction of one solitary wave overtaking another, and the results supported experimental evidence for the applicability of the KdV equation. The phase-shift due to the interaction of large and small solitary waves has been studied by Johnson (1983).

Various authors have investigated solitary wave propagation at the interface of an inviscid two-fluid system (Miles, 1980; Koop and Butler, 1981; Segur and Hammack, 1982; Gear and Grimshaw, 1983). Mirie and Su (1984) have studied internal solitary waves and their head-on collision by a perturbation method.

Strong interactions between solitary waves belonging to different wave modes have been studied by Gear (1985). Mirie and Su (1986) have investigated the head-on collision between two modified KdV solitary waves where cubic and quadratic nonlinearities balance dispersion. It is shown that the collision is elastic because of a dispersive wave train generated behind each emerging solitary wave.

Byatt-Smith (1988, 1989) has studied the reflection of a solitary wave by a vertical wall by considering the head-on collision of two equal solitary waves. He has found analytically that the amplitude of the solitary wave after reflection is reduced.

The resonant interaction between two internal gravity waves in a shallow stratified liquid can be modelled by a system of two KdV equations coupled by small linear and nonlinear terms. Kivshar and Boris (1989) have used this system. It is shown that two solitons belonging to different wave modes form an oscillatory bound state (bi-soliton). They have calculated the frequency of internal oscillations of a bi-soliton and the intensity of the radiation emitted by a weakly excited bi-soliton.

It has been pointed out by Kawahara (1973) that the derivative expansion method can be applied in a systematic way to the analysis of weak nonlinear dispersive waves in uniform media. He (1975a) has studied the weak nonlinear self-interactions of capillary gravity waves using this method. Derivative expansion method that avoids secularity incorporates partial sums in the sense that the solution thus obtained by a perturbation is not a simple power series solution (Jeffrey and Kawahara, 1981). Kakutani and Michihiro (1976) have applied this method to study the far-field modulation of stationary water waves and the same has been applied by Kawahara (1975b) to problems of wave propagation in nonhomogeneous medium. Using this method Kawahara and Jeffrey (1979) have derived several asymptotic kinematic equations for a wave system composed of an ensemble of many monochromatic waves having a continuous spectrum together with a long wave. Since the introduction of multiple scale concepts simplifies the order estimation necessary in a perturbation analysis, this method systematizes the wave-packet formalism.

Nirmala and Vedan (1990) have used derivative expansion method to study wave interaction on water of variable depth based on Johnson's (1973a) equation.

Here we use derivative expansion method considering equation (1.17) as a perturbation of system (1.16) due to the parameter ϵ.

2.2 STUDY OF WAVE INTERACTION USING DERIVATIVE-EXPANSION METHOD

We consider the asymptotic series expansion

$$\eta = \eta_1 + \epsilon\, \eta_2 + \epsilon^2\, \eta_3 + \ldots\ldots \qquad (2.1)$$

regarding η as a function of multiple scales of the parameter ϵ. The partial derivatives with respect to t and x are also expanded as

$$\frac{\partial}{\partial t} \equiv \frac{\partial}{\partial t_0} + \epsilon \frac{\partial}{\partial t_1} + \epsilon^2 \frac{\partial}{\partial t_2} + \ldots\ldots \qquad (2.2)$$

$$\frac{\partial}{\partial x} \equiv \frac{\partial}{\partial x_0} + \epsilon \frac{\partial}{\partial x_1} + \epsilon^2 \frac{\partial}{\partial x_2} + \ldots\ldots \qquad (2.3)$$

Then substituting for η_t , η_x , $\eta\eta_x$, η_{xxx} in equation (1.17) we get,

$$\frac{\partial\eta_1}{\partial t_0} + \epsilon \frac{\partial\eta_2}{\partial t_0} + \epsilon \frac{\partial\eta_1}{\partial t_1} + \left[1 - \frac{1}{2}\epsilon\right]\left[\frac{\partial\eta_1}{\partial x_0} + \epsilon \frac{\partial\eta_2}{\partial x_0} + \epsilon \frac{\partial\eta_1}{\partial x_1}\right]$$
$$+ \left[\frac{3}{2}\alpha + \frac{5}{4}\epsilon\alpha\right]\left[\eta_1 \frac{\partial\eta_1}{\partial x_0} + \epsilon\eta_1 \frac{\partial\eta_2}{\partial x_1} + \epsilon\eta_1 \frac{\partial\eta_1}{\partial x_1}\right.$$
$$\left. + \epsilon\eta_2 \frac{\partial\eta_1}{\partial x_0}\right] + \left[\frac{1}{6}\beta - \frac{1}{3}\epsilon\beta\right]\left[\frac{\partial^3\eta_1}{\partial x_0^3} + \epsilon \frac{\partial^3\eta_2}{\partial x_0^3}\right.$$
$$\left. + 3\epsilon \frac{\partial^3\eta_1}{\partial x_0^2 \partial x_1}\right] = 0 \ . \qquad (2.4)$$

Collecting $O(\epsilon^0)$,$O(\epsilon^1)$ terms in equation (2.4) we get,

$$L_0 \eta_1 = 0, \tag{2.5}$$

$$L_0 \eta_2 + L_1 \eta_1 + L_2 \eta_2 = N_0\left[\eta_1^2\right]. \tag{2.6}$$

where

$$L_0 = \frac{\partial}{\partial t_0} + \frac{\partial}{\partial x_0} + \frac{3}{2}\alpha\eta_1 \frac{\partial}{\partial x_0} + \frac{1}{6}\beta \frac{\partial^3}{\partial x_0^3}, \tag{2.7}$$

$$L_1 = \frac{\partial}{\partial t_1} + \frac{\partial}{\partial x_1} + \frac{1}{2}\left[3\alpha\eta_1 \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_0}\right] + \frac{1}{6}\beta\left[3\frac{\partial^3}{\partial x_0^2 \partial x_1} - 2\frac{\partial^3}{\partial x_0^3}\right], \tag{2.8}$$

$$L_2 = \frac{3}{2}\alpha \frac{\partial \eta_1}{\partial x_0}, \tag{2.9}$$

and

$$N_0 = -\frac{5}{8}\alpha \frac{\partial}{\partial x_0}. \tag{2.10}$$

We note that equation (2.5) is a nonlinear homogeneous equation in η_1. Solving this and substituting in equation (2.6) we get a nonlinear nonhomogeneous equation in η_2.

Now we study the nonlinear interaction between a long wave and an ensemble of short waves (Jeffrey and Kawahara, 1982), ie. a superposition of a number of monochromatic waves with different wave numbers, or with a continuous spectrum. For this purpose, to the lowest order of approximation, we consider a solution of equation (2.5) in the form

$$\eta_1 = \int_{-\infty}^{\infty} A_1\left(k;\ x_1, t_1 \ \ldots\ldots\right) \exp\left[i\left(kx_0 - \omega t_0\right)\right] dk$$

$$+ B_1\left(x_1,\ t_1, \ldots\ldots\right), \tag{2.11}$$

where $A_1(k)$ is a slowly varying complex amplitude with the wave number k and B_1 is a slowly varying real function representing the long-wave component. The reality of η_1 requires that $A_1^*\ (k) = A_1(k)$ where the asterisk denote complex conjugate . The dispersion relation of the linear equation (2.5) is

$$D(k,\omega) = -\ i\omega + ik - \frac{i}{6}\beta k^3 = 0, \tag{2.12a}$$

Then we have

$$\omega(k) = k - \frac{1}{6}\beta k^3\ , \tag{2.12b}$$

and the group velocity is

$$V(g) = \frac{\partial\omega}{\partial k} = 1 - \frac{1}{2}\beta k^2. \tag{2.13}$$

Substituting equation (2.11) in equation (2.6) we get,

$$\left[L_0 + L_2\right]\eta_2 = \int_{-\infty}^{\infty}\left[-\left(\frac{\partial}{\partial t_1} + Vg\frac{\partial}{\partial x_1}\right) + \frac{ik}{2} + \frac{i\beta k^3}{3}\right].$$

$$A_1(k)\exp\left[i\left(kx_0 - \omega t_0\right)\right]dk - \frac{3}{2}\alpha\left\{\int_{-\infty}^{\infty}\ \int_{-\infty}^{\infty} A_1(k').\right.$$

$$\frac{\partial A_1(k'')}{\partial x_1} \exp i\left[\left(k'+k''\right)x_0-\left(\omega'+\omega''\right)t_0\right]dk'dk''$$

$$+ B_1\left(x_1,t_1\right) \int_{-\infty}^{\infty} \frac{\partial A_1(k'')}{\partial x_1} \exp\left[i\left(k''x_0-\omega''t_0\right)\right]dk''$$

$$+ \frac{\partial B_1}{\partial x_1} \int_{-\infty}^{\infty} A_1\left(k',x_1,t_1\right)\exp i\left[\left(k'x_0-\omega't_0\right)\right]dk'$$

$$+ B_1\left(x_1,t_1\right) \left.\frac{\partial B_1}{\partial x_1}\right\}$$

$$- \frac{5}{8}\alpha \left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} i\left(k' + k''\right)A_1(k')A_1(k'').\right.$$

$$\exp i \left[\left(k'+k''\right)x_0-\left(\omega'+\omega''\right)t_0\right]dk'dk''$$

$$+ B_1\left(x_1,t_1\right)\int_{-\infty}^{\infty} A_1(k')ik'\exp\left[i\left(k'x_0-\omega't_0\right)\right]dk'$$

$$+ B_1\left(x_1,t_1\right)\int_{-\infty}^{\infty} A_1(k'')ik''\exp\left[i\left(k''x_0-\omega''t_0\right)\right]dK''$$

$$- \frac{\partial B_1}{\partial t_1} + \frac{\partial B_1}{\partial x_1} + \frac{1}{2}\beta\frac{\partial^3 B_1}{\partial x_0^2\,\partial x_1}, \qquad (2.14)$$

where V_g denotes the group velocity and ω' and ω'' denote $\omega(k')$ and $\omega(k'')$ respectively.

2.3 THREE-WAVE INTERACTION

We now consider the resonant wave interaction between different wave modes. Two primary components of wave numbers

k_1 and k_2 and frequencies ω_1 and ω_2 give rise to an interaction term with the magnitudes of the wave number k_3 and corresponding frequency ω_3 lying within the limits $|k_1+k_2|$ and $|k_1-k_2|$. Phillips (1960, 1977) has pointed out that a resonance is possible if the interaction frequencies $\omega_1+\omega_2$ and $\omega_1-\omega_2$ corresponds to wave numbers lying within that range and exchange of energy among wave modes is analogous to resonance in a forced linear oscillator. He has further shown that for three-wave interactions, the energy exchange is significant only when the conditions

$$k_1 + k_2 \pm k_3 = 0,$$

and

$$\omega_1 \pm \omega_2 \pm \omega_3 = 0.$$

are satisfied or nearly satisfied simultaneously.

Linear dispersion relation (2.12) admits the three-wave interaction process if $\omega'+\omega'' = \omega$ for $k'+k'' = k$. Here the three-wave interaction process does not occur since this condition is not satisfied.

In this case we obtain from equation (2.14) the following condition for the nonsecularity of the $O(\epsilon^1)$ solution,

$$\left[\frac{\partial}{\partial t_1} + Vg \frac{\partial}{\partial x_1} - \frac{ik}{2} - \frac{i\beta k^3}{3} \right] A_1(k)$$

$$+ \frac{3}{2}\alpha\left[B_1\left(x_1,t_1\right)\frac{\partial A_1(k)}{\partial x_1} + A_1(k)\frac{\partial B_1}{\partial x_1}\right]$$

$$+ \frac{5}{4}\alpha\left[A_1(k)ik\ B_1\left(x_1,t_1\right)\right] = 0, \qquad (2.15)$$

with

$$\frac{\partial B_1}{\partial t_1} = 0, \qquad \frac{\partial B_1}{\partial x_1} = 0.$$

There is a corresponding equation for the complex conjugate $A_1^*(k)$ also.

Equation (2.15) can be written as

$$\left[\frac{\partial}{\partial t_1} + Vg\frac{\partial}{\partial x_1}\right]A_1(k) + \frac{3}{2}\alpha\frac{\partial}{\partial x_1}\left[A_1(k)B_1\right]$$
$$= \text{Im}\left\{\left[\frac{k}{2} + \frac{\beta k^3}{3}\right]A_1(k) - \frac{5}{4}\alpha k\ A_1(k)B_1\left(x_1,t_1\right)\right\}.$$

(2.16)

Multiplying by $A_1^*(k)$ we get,

$$\left[\frac{\partial}{\partial t_1} + Vg\frac{\partial}{\partial x_1}\right]|A_1(k)|^2 + \frac{3}{2}\alpha\frac{\partial}{\partial x_1}\left[A_1(k)B_1\right]A_1^*(k)$$
$$= \text{Im}\ A_1^*(k)\left\{\left[\frac{k}{2} + \frac{\beta k^3}{3}\right]A_1(k) - \frac{5}{4}\alpha k\ A_1(k)B_1\left(x_1,t_1\right)\right\}.$$

(2.17)

Equating the real parts in equation (2.17) we get,

$$\left[\frac{\partial}{\partial t_1} + Vg\,\frac{\partial}{\partial x_1}\right] |A_1(k)|^2 + \frac{3}{2}\,\alpha\,\frac{\partial}{\partial x_1} |A_1(k)|^2\, B_1 = 0\ .$$

(2.18)

Equation (2.18) can be written as

$$\frac{\partial N}{\partial t_1} + \frac{\partial}{\partial x_1}\left[N\left(V+R\right)\right] = 0\ , \tag{2.19}$$

where

$$n(k) = |A_1(k)|^2\ , \tag{2.20a}$$

$$N = \int_{-\infty}^{\infty} n(k)dk, \tag{2.20b}$$

$$V = \frac{1}{N} \int_{-\infty}^{\infty} Vg\ n(k)dk\ , \tag{2.20c}$$

and

$$R = \frac{1}{N} \int_{-\infty}^{\infty} \frac{3}{2}\,\alpha B_1\ n(k)dk\ . \tag{2.20d}$$

Here $n(k)$ and N represent, energy density and total energy density of the short waves respectively.

2.4 DISCUSSION

Perturbation method can be applied in the study of a wide range of physical phenomena. The guiding principle for obtaining asymptotic equations is merely the nonsecularity of the perturbation.

Equation (2.19) is a conservation law. It is found that the dispersion relation (2.12) does not admit three-wave interaction process. Thus there is no transfer of energy between different wave numbers. But the total energy of the short wave components is conserved by transfer of energy between the short wave components and the interacting long wave.

CHAPTER 3

IST ANALYSIS AND NUMERICAL STUDY OF A PERTURBED KdV EQUATION

3.1 INTRODUCTION

In this chapter we consider IST analysis and numerical study of equations (1.16) and (1.17).

As has been pointed out earlier IST method provides a procedure for obtaining pure soliton solutions and quantitative information about the general solutions of the KdV equation. Johnson (1973a) has briefly discussed development of solitary wave moving over an uneven bottom using IST method. He has obtained what he calls the eigendepths relating number of solitons formed to the depth of the shelves. Soliton solutions for various depths have also been examined by numerically integrating the relevant KdV equation.

The IST method for one-dimensional Schrodinger operator on a straight line has been used by Mel'nikov (1990) to derive solutions for the KdV equation with self consistant source which describe creation and annihilation of solitons. Meinhold (1991) has used IST method to find solutions for KdV equation and in particular soliton solutions that do not vanish at infinity.

Apart from the analytical studies, numerical methods have also contributed equally to the qualitative study of KdV equation. Johnson (1972) has numerically studied the development of solitary waves moving over an uneven bottom. The reflection of the solitary wave in shallow water has been studied numerically using an improved MAC method by Funakoshi and Oikawa (1982) and a fourth order accurate difference scheme has been proposed by Wu and Guo (1983) for KdV, Burgers' and regularised long wave equations.

Johnson's equation (1973a) belongs to a class of perturbed KdV equation. From this point of view Knickerbocker and Newell (1980) have numerically studied the equation and pointed out the possibility of a shelf formation.

Iskandar (1989) has used a combined approach of linearization and finite difference method to solve the KdV equation and discussed the accuracy and efficiency of the scheme. The study involves interaction of solitary waves with different amplitudes. Pramod *et al.* (1989) have discussed solitary wave propagation and interactions using a KdV equation with variable coefficient. This study includes both the cases when the system is integrable and non-integrable.

In this section, we briefly point out the possibility of soliton creations due to the discontinuity at $x=x_0$.

3.2 IST ANALYSIS

Equations (1.16) and (1.17) can be written as

$$\eta_t + p\eta_x + q\eta\eta_x + r\eta_{xxx} = 0. \tag{3.1}$$

Let

$$\eta = \gamma V_x + V^2, \tag{3.2}$$

where

$$\gamma = i\sqrt{\frac{6r}{q}} .$$

If V is a solution of

$$V_t + pV_x + qV^2V_x + rV_{xxx} = 0,$$

then

$$\eta = \gamma V_x + V^2 ,$$

is a solution of (3.1).

For equation (1.16),

$$\gamma \equiv \gamma_1 = i\sqrt{\frac{2\beta}{3\alpha}} . \tag{3.3a}$$

For equation (1.17),

$$\gamma \equiv \gamma_1 = i\sqrt{\frac{4\beta(1-2\epsilon)}{\alpha(6+5\epsilon)}} . \tag{3.3b}$$

In equation (3.2) we take η to be known, then this corresponds to a Riccati equation for V and can be linearised by the transformation,

$$V = \gamma \frac{\psi_x}{\psi} , \tag{3.4}$$

yielding

$$\gamma^2 \psi_{xx} - \eta\psi = 0.$$

This is the time-independent Schrodinger equation; however it is missing the energy term. Now we use the fact that η is to be a solution of the KdV equation which is invariant under a Galilean transformation. Under such a transformation η can be shifted by a constant and x-differentiations remain unchanged. Therefore, without loss of generality we can replace the above equation by

$$\gamma^2 \psi_{xx} - \left(\eta - \lambda\gamma^2\right)\psi = 0,$$

which is the time-independent Schrodinger equation, where η is the potential, λ's are the energy levels and ψ is the wave function.

Thus the Schrodinger equation for the two equations can be written as

$$\gamma^2 \psi_{xx} - \left(\eta - \lambda\gamma^2\right)\psi = 0. \tag{3.5}$$

The solution of the KdV equation (1.16) is the potential of the Schrodinger equation. Consider a wave moving to right in the region $x < x_0$, it is the potential of the Schrodinger equation

$$\gamma_1^2\psi_{xx} - \left(\eta - \lambda\gamma_1^2\right)\psi = 0 \qquad (3.6)$$

Let it reach $x=x_0$ at $t=t_0$. Then it becomes the initial condition for the KdV equation (1.17). Thus the potential in the Schrodinger equation

$$\gamma_2^2\psi_{xx} - \left(\eta - \lambda\gamma_2^2\right)\psi = 0 \qquad (3.7)$$

which is the solution for the KdV equation (1.17), is the solution of equation (1.16) at $t=t_0$ and we expect the spectrum also different in this case. This will lead to emergence of new solitons by splitting of the wave reaching $x=x_0$.

3.3 NUMERICAL SCHEME

Now we discuss numerical solution of equations (1.16) and (1.17). The numerical method is based on a finite difference scheme and the criterion for stability as proposed by Vliegenthart (1971) is used.

Consider the transformations T=t and X=x-t then equations (1.16) and (1.17) can be written as

$$\eta_T + \frac{3}{2}\alpha\eta\eta_X + \frac{1}{6}\beta\eta_{XXX} = 0, \qquad (3.8)$$

and

$$\eta_T - \frac{1}{2}\epsilon\eta_X + \left(\frac{3}{2}\alpha + \frac{5}{4}\epsilon\alpha\right)\eta\eta_X$$

$$+\left(\frac{1}{6}\beta - \frac{1}{3}\epsilon\beta\right)\eta_{XXX} = 0. \qquad (3.9)$$

Equations (3.8) and (3.9) can be written as

$$\eta_T + \left[\frac{3}{4}\alpha\eta^2 + \frac{1}{6}\beta\eta_{XX}\right]_X = 0, \qquad (3.10)$$

and

$$\eta_T + \left[-\frac{1}{2}\epsilon\eta + \left(\frac{3}{4}\alpha + \frac{5}{8}\epsilon\alpha\right)\eta^2\right.$$

$$\left. + \left(\frac{1}{6}\beta - \frac{1}{3}\epsilon\beta\right)\eta_{XX}\right]_X = 0. \qquad (3.11)$$

Multiplying (3.10) and (3.11) by η, then the equations can be written as

$$\left[\frac{1}{2}\eta^2\right]_T + \left[\frac{1}{2}\alpha\eta^3 + \frac{1}{6}\beta\eta\eta_{XX} - \frac{1}{2}\beta{\eta_X}^2\right]_X = 0, \qquad (3.12)$$

and

$$\left[\frac{1}{2}\eta^2\right]_T + \left[-\frac{1}{4}\epsilon\eta^2 + \left(\frac{3}{2}\alpha + \frac{5}{4}\epsilon\alpha\right)\frac{\eta^3}{3} + \left(\frac{1}{6}\beta - \frac{1}{3}\epsilon\beta\right)\eta\eta_{XX}\right.$$

$$\left. - \left(\frac{1}{6}\beta - \frac{1}{3}\epsilon\beta\right)\frac{{\eta_X}^2}{2}\right]_X = 0. \qquad (3.13)$$

Equations (3.10), (3.11) and (3.12), (3.13) can be interpreted as the conservation laws of momentum and energy respectively of the equations (3.8) and (3.9).

Difference schemes that approximate the equations (3.8) and (3.9) are

$$\eta_j^{n+1} = \eta_j^{n-1} - \frac{\alpha}{2} \frac{\Delta T}{\Delta X} \left(\eta_{j+1}^n + \eta_j^n + \eta_{j-1}^n\right)\left(\eta_{j+1}^n - \eta_{j-1}^n\right)$$

$$- \frac{1}{6} \beta \left(\frac{\Delta T}{(\Delta X)^3}\right)\left(\eta_{j+2}^n - 2\eta_{j+1}^n + 2\eta_{j+1}^n - \eta_{j-1}^n\right), X<X_0 \tag{3.14}$$

and

$$\eta_{j+1}^n = \eta_j^{n-1} + \frac{\epsilon}{2} \frac{\Delta T}{\Delta X}\left(\eta_{j+1}^n - \eta_{j+1}^n\right) - \left(\frac{3}{2}\alpha + \frac{5}{4} \epsilon\alpha\right).$$

$$\frac{\Delta T}{3\Delta X}\left(\eta_{j+1}^n + \eta_j^n + \eta_{j-1}^n\right)\left(\eta_{j-1}^n - \eta_{j-1}^n\right)$$

$$- \left(\frac{1}{6}\beta - \frac{1}{3}\epsilon\beta\right)\left(\frac{\Delta T}{(\Delta X)^3}\right).$$

$$\left(\eta_{j+2}^n - 2\eta_{j+1}^n + 2\eta_{j+1}^n - \eta_{j-2}^n\right), \quad X>X_0 \tag{3.15}$$

where

$$\eta_j^n = \eta\left(j\Delta X, n\Delta T\right),$$

and $T_n = n\Delta T$, ΔX and ΔT being the step lengths in X and T respectively. Following Vliegenthart (1971) we see that the above difference scheme is stable if

$$\frac{\Delta T}{\Delta X}\left[\frac{3}{2}\alpha|\eta| + \frac{2}{3}\frac{\beta}{(\Delta X)^2}\right] \le 1, \tag{3.16}$$

$$\frac{\Delta T}{\Delta X}\left[\left(\frac{3}{2}\,\alpha + \frac{5}{4}\,\epsilon\alpha\right)|\eta| + \frac{1}{2}\,\epsilon + \frac{4}{(\Delta X)^2}\left(\frac{1}{6}\,\beta - \frac{1}{3}\,\epsilon\beta\right)\right] \le 1. \tag{3.17}$$

The computations are carried out for the following values of parameters,

(1) $\alpha = 0.00111111$, $\beta = 0.01$,

(2) $\alpha = 0.01$, $\beta = 0.02$,

(3) $\alpha = 0.01$, $\beta = 0.04$,

In all the above cases, the parameter ϵ takes values $\epsilon=0$ and $\epsilon=0.1$.

The initial condition is taken as

$$\eta_j^0 = 12\ \mathrm{sech}^2(j.\Delta X - C), \tag{3.18}$$

where C is a constant which prescribes the position of the peak of the solitary wave. The integration is performed on a CYBER 180/830 machine with 500 steps in X. Taking $\Delta X=0.1$ and assuming a maximum value $|\eta|=24$, the step length ΔT is chosen to satisfy the conditions (3.16) and (3.17). We take the following step lengths for ΔT,

(1) $\Delta T = 0.14$,

(2) $\Delta T = 0.05$,

(3) $\Delta T = 0.03$,

we take $X_0 = 0$ and 3000 steps in T and the initial profile centered at $X = -5$ throughout the computation.

3.4 RESULTS OF NUMERICAL INTEGRATION

Figures 1-3 show the results of the computations. Figure 1(*A*) shows the wave propagation in the case when the nonlinearity and dispersion counterbalance each other and $\epsilon=0$. In this case the integrability condition is satisfied and the system is equivalent to the plane KdV equation. Here the wave is propagated without change of shape. The momentum and energy are conserved as shown by Table 1(a). Figure 1(*B*) shows the wave propagation in the case $\epsilon=0.1$. In this case momentum and energy are not conserved as shown by Table 1(b). Thus figure shows solitons propagating off to the right together with the oscillatory waves. Figures 2 and 3 and Tables 2 and 3 are results of computations in which case the nonlinearity and dispersion are not balancing for $\epsilon = 0$ also. In all the cases we see that the effect of the parameter ϵ is to give rise to oscillatory waves. The nonlinearity also become more prominent, resulting in the narrowing down of the peaks and increasing of the amplitude.

3.5 DISCUSSION

We have studied the equation (1.15) representing waves on water of variable depth due to a sudden change in depth. Equations (1.16) and (1.17) represents the wave into the domains $x<x_0$ and $x>x_0$ respectively. These are the two equations considered in our analysis. For $\epsilon=0$, equation (1.17) reduces to equation (1.16).

In section 3.2, we have considered IST analysis taking the two equations separately.

In section 3.3, we have considered numerical study of the equations (1.16) and (1.17). The equations are transformed to equations (3.8) and (3.9) respectively using a scaling transformation. Shelf corresponds to $X=0$. Wave travelling from left to right is governed by (3.8) in the domain $X<0$ and (3.9) in $X>0$. A finite difference scheme is used. From computations we find that the nonlinearity and dispersion are not balanced and harmonic waves are excited.

Using IST technique Kaup and Newell (1978) have pointed out that in the case of a perturbed KdV equation, continuous spectrum is excited due to interaction between soliton and the perturbation and the reflection coefficient will have a Dirac delta function behaviour and conservation of momentum is possible only by the formation of a shelf. Equation (1.20) also points out the excitation of solitons due to the shelf in our case. The IST analysis in our case also points out the enlargement of spectrum as the wave crosses the shelf. Our numerical study shows that momentum and energy are not conserved and this is due to the neglecting of singularity at $X=0$.

Table 1(a) Computational values of momentum and energy

$\epsilon=0.0$, $\alpha=0.00111111$, $\beta=0.01$

Time	Total Momentum	Total Energy
0	239.9999999999	959.9999999998
140	239.9812488449	960.0005975577
280	240.0007749207	960.0006134471
420	239.9975195932	960.0000169623

Table 1(b) Computational values of momentum and energy

$\epsilon=0.1$, $\alpha=0.00111111$, $\beta=0.01$

Time	Total Momentum	Total Energy
0	239.9999999999	959.9999999998
140	446.6326680072	1517.617343429
280	44.88500865291	266.8922448879
420	274.4761013643	898.015530322

Table 2 Computational values of momentum and energy

$\alpha=0.01$, $\beta=0.02$

Value of ϵ	Time	Total Momentum	Total Energy
0.0	0	239.9999999999	959.9999999998
	50	239.9305707873	960.0342806722
	100	240.0009459306	960.0467549429
	150	239.97573907	960.0470581366
0.1	0	239.9999999999	959.9999999998
	50	211.4214591654	941.79225341
	100	239.3741884093	959.9956914168
	150	226.3308302433	954.7334964146

Table 3 Computational values of momentum and energy

$\alpha=0.01$ $\beta=0.04$

Value of ϵ	Time	Total Momentum	Total Energy
0.0	0	239.9999999999	959.9999999998
	30	240.1065617646	960.0002702658
	60	239.9956813727	960.0032560182
	90	240.0445399648	960.0057730142
0.1	0	239.9999999999	959.9999999998
	30	237.5590856265	959.8403146036
	60	239.9584327767	960.0025670368
	90	239.271377768	959.9818525025

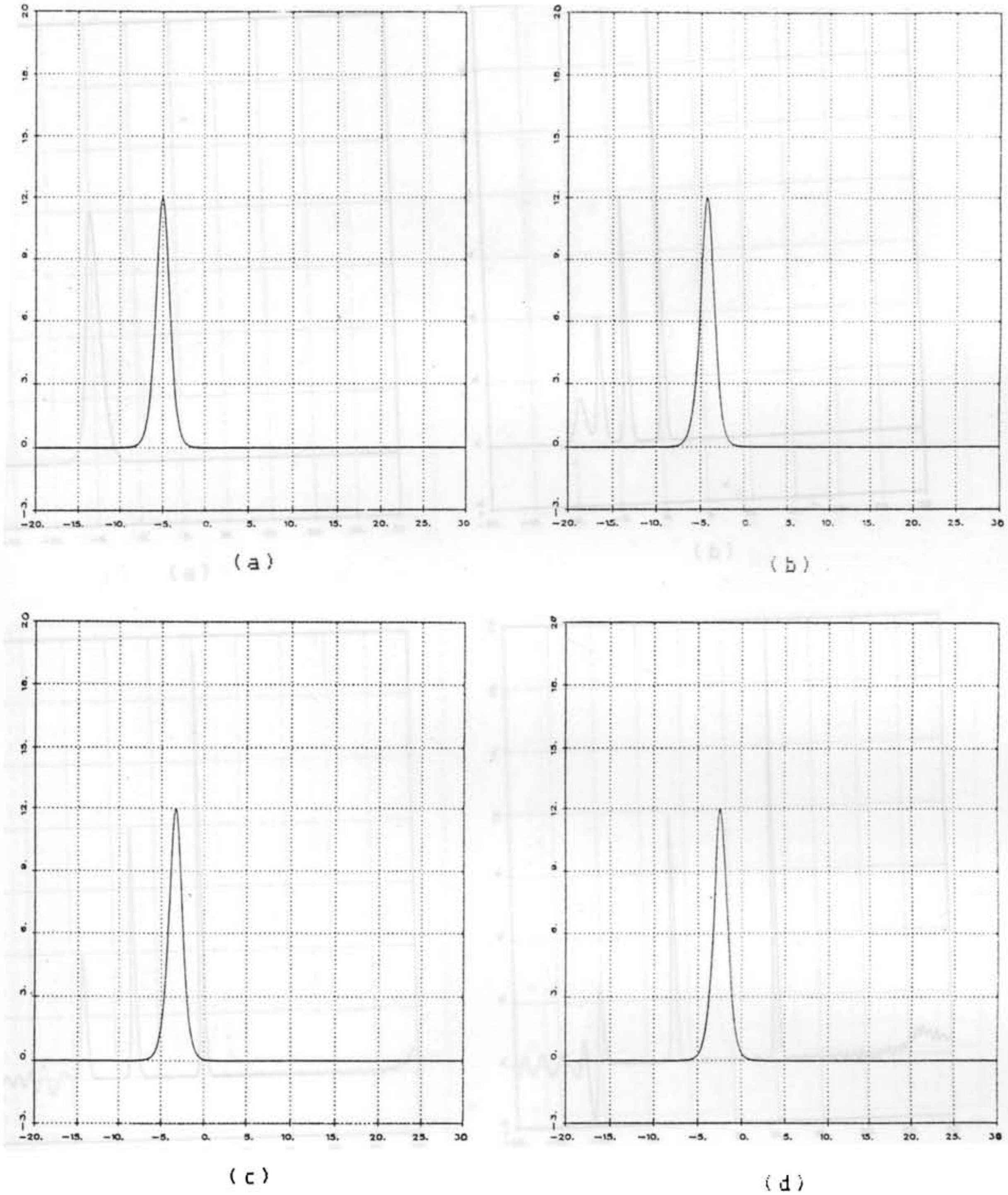

Fig. 1A Wave propagation corresponding to ϵ=0.0,

(a) T=0 (b) T=140 (c) T=280 (d) T=420

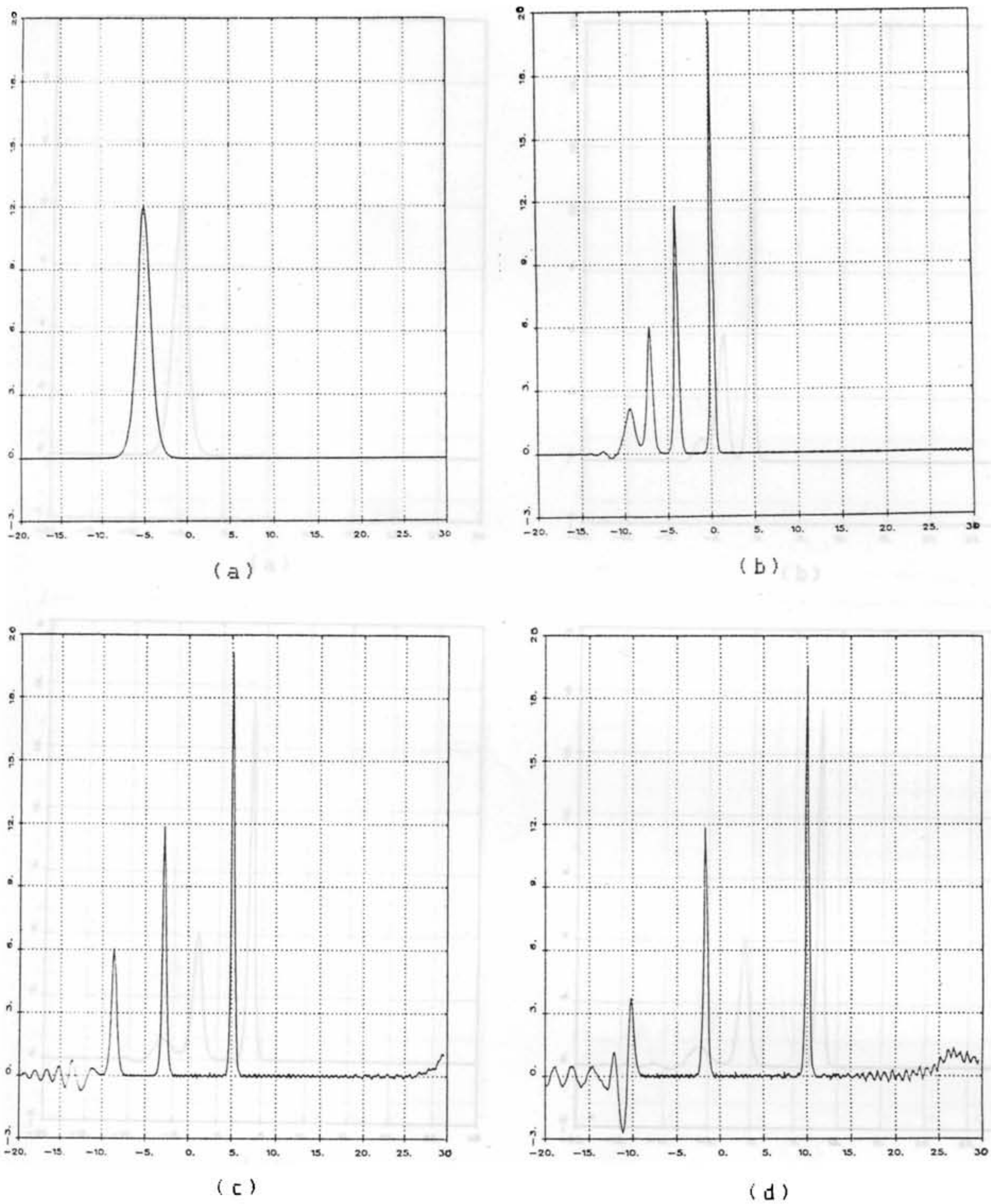

Fig. 1B Wave propagation corresponding to $\epsilon=0.1$,

(a) T=0 (b) T=140 (c) T=280 (d) T=420

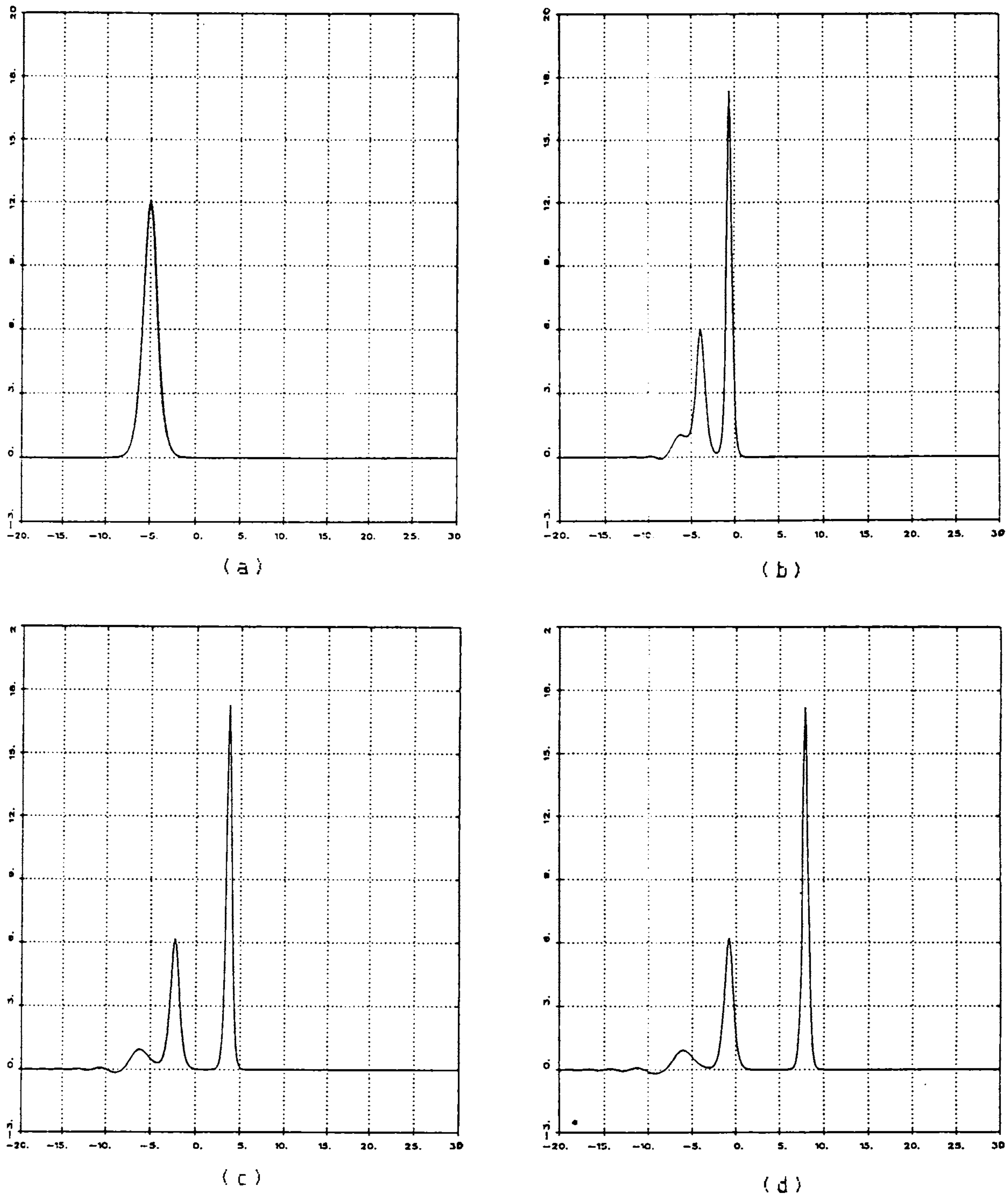

Fig. 2A Wave propagation corresponding to ϵ=0.0,

(a) T=0 (b) T=50 (c) T=100 (d) T=150

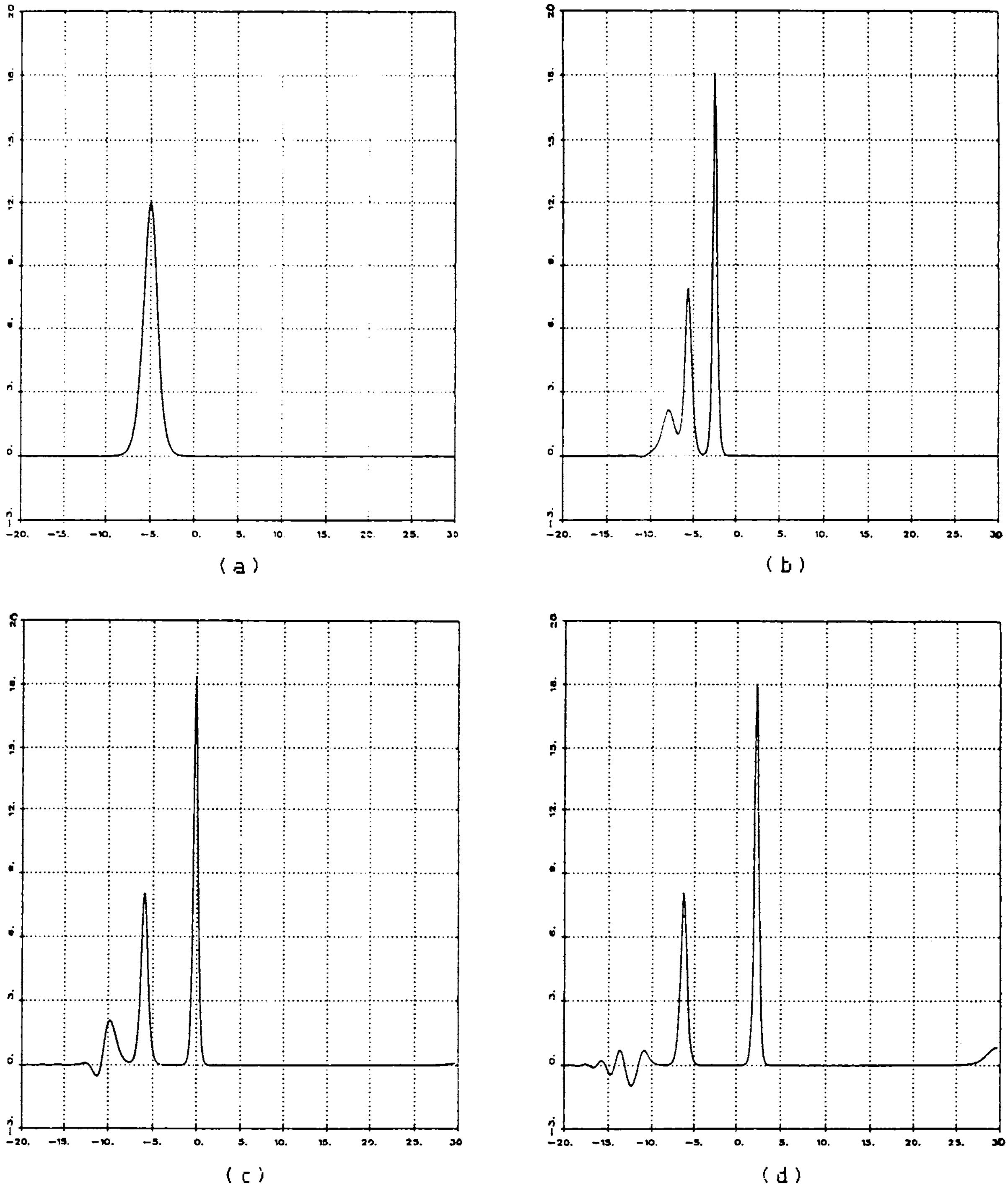

Fig. 2B Wave propagation corresponding to $\epsilon=0.1$,

(a) T=0 (b) T=50 (c) T=100 (d) T=150

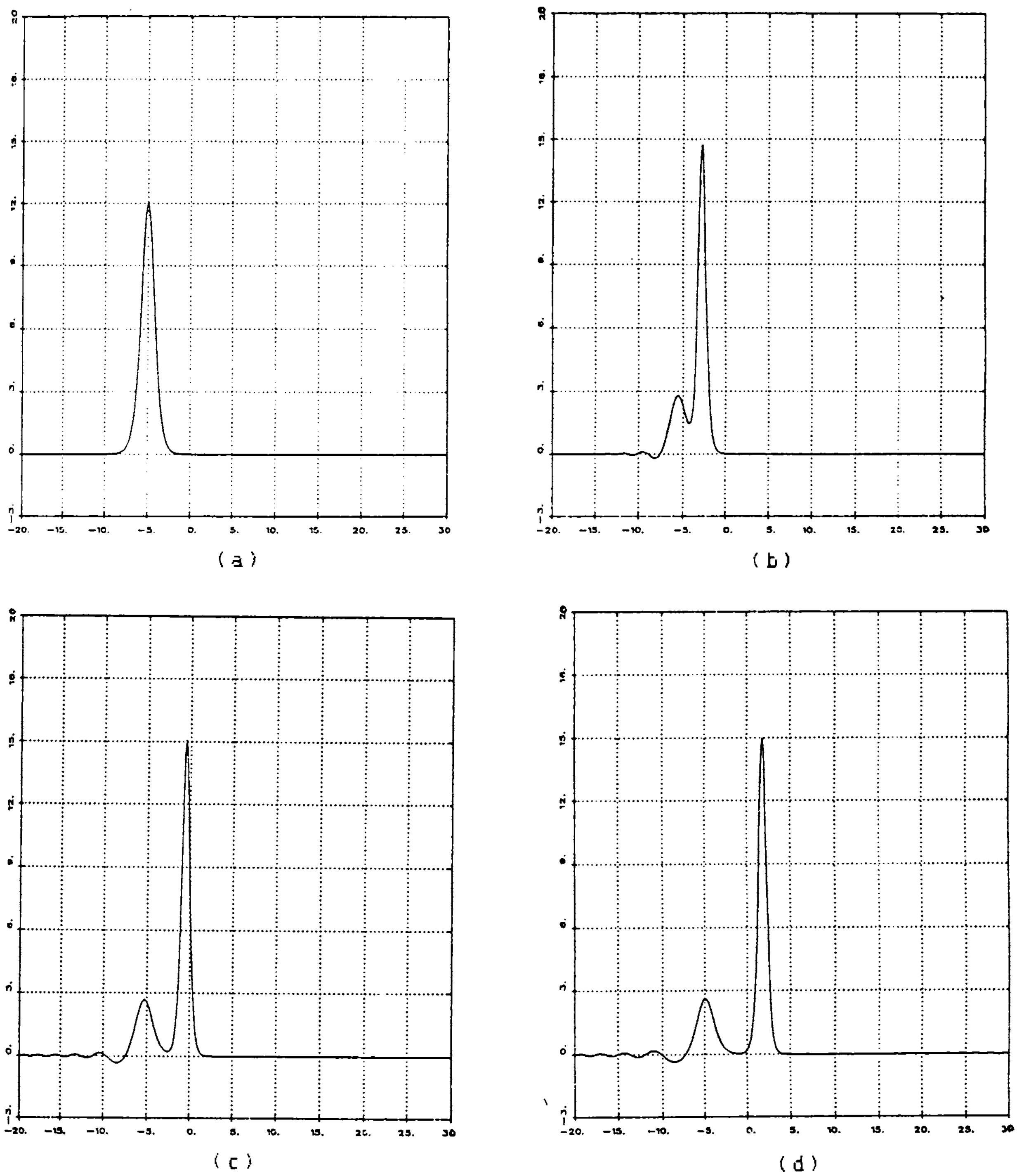

Fig. 3A Wave propagation corresponding to ϵ=0.0,

(a) T=0 (b) T=30 (c) T=60 (d) T=90

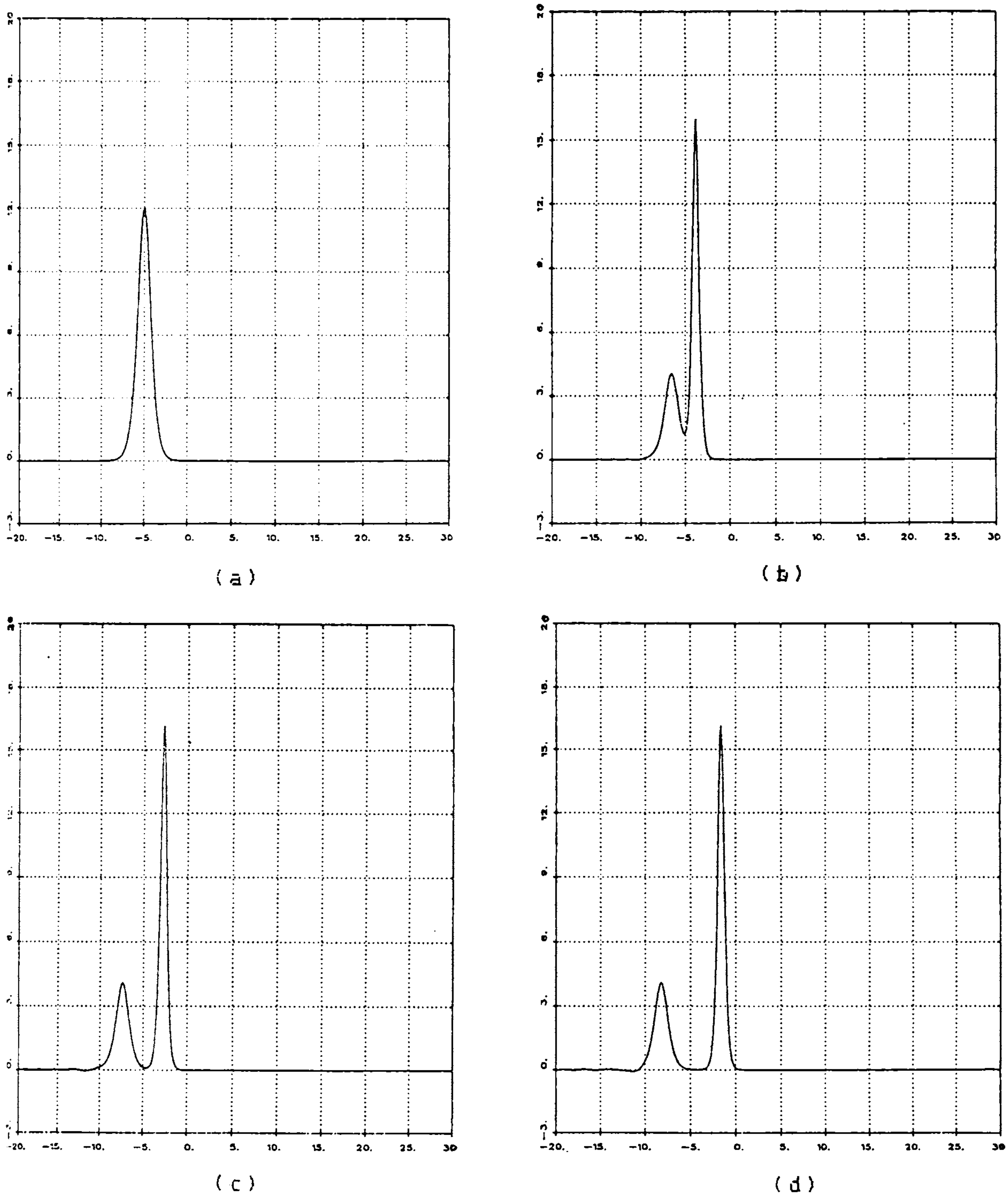

Fig. 3B Wave propagation corresponding to ϵ=0.1,

(a) T=0 (b) T=30 (c) T=60 (d) T=90

CHAPTER 4

ANALYTICAL SOLUTIONS OF A KdVB EQUATION

4.1 INTRODUCTION

In chapter 2, we have considered the effect of perturbations on a KdV equation. There we have the case of nonlinearity and dispersion. In this chapter we consider the effect of dissipation on the KdV equation. Thus in addition to nonlinearity and dispersion, we consider the dissipation term $\frac{1}{3}\epsilon\beta\eta_{xx}$. The resulting equation is a KdVB type equation.

It is to be noted that we are interested in the effect of perturbation on the perturbed KdV type equation which incorporates the dissipation.

While analytical solution exists for the travelling wave solutions of Burgers' and KdV equations, no comparable analytical solution exists for the KdVB equation. Numerical solutions by Grad and Hu (1967) and Johnson (1970) show that when dispersion dominates on dissipation the solution represents an oscillatory shock wave. Studies by Grad and Hu (1967) and Jeffrey and Kakutani (1972) show that when dissipative effect predominates the solution behaves like a

Burgers' shock wave. In this case Jeffrey (1979) has obtained an analytical solution for a KdVB travelling wave.

In our analysis we follow Jeffrey (1979) and Jeffrey and Mohamad (1991). In the case of KdV equation solitary wave solutions exists due to a balancing of nonlinearity and dispersion. Unlike Jeffrey (1979), we are interested in a perturbation of the solitary wave solution due to dissipation.

4.2 ASYMPTOTIC SOLUTION

In this chapter we study the effect of dissipation using three methods. First we consider the asymptotic solution for KdVB travelling wave solution interms of the parameter ϵ.

We consider the equation (1.21) as

$$\eta_t + \left(1 - \frac{1}{2}\epsilon\right)\eta_x + \left(\frac{3}{2}\alpha + \frac{5}{4}\epsilon\alpha\right)\eta\eta_x + \frac{1}{3}\epsilon\beta\eta_{xx} + \left(\frac{1}{6}\beta - \frac{1}{3}\epsilon\beta\right)\eta_{xxx} = 0. \tag{4.1}$$

Equation (4.1) can be written as

$$\eta_t + \left(1 - \frac{1}{2}\epsilon\right)\eta_x + \left(\frac{3}{2}\alpha + \frac{5}{4}\epsilon\alpha\right)\eta\eta_x + \gamma\eta_{xx} + \mu\eta_{xxx} = 0, \tag{4.2}$$

where

$$\gamma = \frac{1}{3} \epsilon\beta, \quad \text{and} \quad \mu = \frac{1}{6} \beta - \frac{1}{3} \epsilon\beta \ . \tag{4.3}$$

As pointed above by Jeffrey (1979), equation (4.2) has travelling wave solutions. Such solutions have the form

$$\eta(x,t) = \tilde{\eta}(\zeta), \tag{4.4}$$

where $\zeta = x - \lambda t$.

The boundary conditions at infinity determine the permissible range of values of λ. We consider equation (4.2) with the boundary conditions

$$\tilde{\eta}(-\infty) = \eta_{\infty}^{-} \quad \text{and} \quad \tilde{\eta}(\infty) = \eta_{\infty}^{+}$$

Then $\tilde{\eta}$ must satisfy the equation

$$- \lambda \frac{d\tilde{\eta}}{d\zeta} + \left(1 - \frac{1}{2} \epsilon\right)\frac{d\tilde{\eta}}{d\zeta} + \left(\frac{3}{2} \alpha + \frac{5}{4} \epsilon\alpha\right)\tilde{\eta} \frac{d\tilde{\eta}}{d\zeta}$$

$$+ \gamma \frac{d^2\tilde{\eta}}{d\zeta^2} + \mu \frac{d^3\tilde{\eta}}{d\zeta^3} = 0. \tag{4.5}$$

Integrating equation (4.5) with respect to ζ we obtain

$$\mu \frac{d^2\tilde{\eta}}{d\zeta^2} = - \frac{1}{2}\left(\frac{3}{2} \alpha + \frac{5}{4} \epsilon\alpha\right)\tilde{\eta}^{\,2} + \lambda\tilde{\eta} - \left(1 - \frac{1}{2} \epsilon\right)\tilde{\eta}$$

$$+ A - \gamma \frac{d\tilde{\eta}}{d\zeta} , \tag{4.6}$$

where A is the constant of integration.

This equation may be interpreted as an equation of motion for a particle or of an anharmonic oscillator under the action of a nonlinear force together with a friction proportional to velocity provided that we regarded ζ and η as the time and space co-ordinates respectively.

Using the boundary conditions and the vanishing of derivatives at infinity we get

$$\lambda = \left(1 - \frac{1}{2}\epsilon\right) + \frac{1}{2}\left(\frac{3}{2}\alpha + \frac{5}{4}\epsilon\alpha\right)\left(\eta_\infty^+ + \eta_\infty^-\right), \tag{4.7}$$

$$A = -\frac{1}{2}\left(\frac{3}{2}\alpha + \frac{5}{4}\epsilon\alpha\right)\eta_\infty^+\,\eta_\infty^- . \tag{4.8}$$

Therefore equation (4.6) becomes

$$\mu\frac{d^2\tilde{\eta}}{d\zeta^2} + \surd\frac{d\tilde{\eta}}{d\zeta} + \frac{1}{2}\left(\frac{3}{2}\alpha + \frac{5}{4}\epsilon\alpha\right)\tilde{\eta}^2$$
$$- \left[\frac{1}{2}\left(\frac{3}{2}\alpha + \frac{5}{4}\epsilon\alpha\right)\left(\eta_\infty^+ + \eta_\infty^-\right)\right]\tilde{\eta}$$
$$+ \frac{1}{2}\left(\frac{3}{2}\alpha + \frac{5}{4}\epsilon\alpha\right)\eta_\infty^+\,\eta_\infty^- = 0. \tag{4.9}$$

Making the variable changes

$$\phi = \frac{\tilde{\eta} - \eta_\infty^-}{\eta_\infty^+ - \eta_\infty^-}, \quad \xi = \frac{\left(\eta_\infty^+ - \eta_\infty^-\right)\zeta}{2\surd}, \quad \tau = \frac{\mu\left(\eta_\infty^+ - \eta_\infty^-\right)}{2\surd^2}. \tag{4.10}$$

reduces equation (4.9) to

$$\tau \frac{d^2\phi}{d\xi^2} + \frac{d\phi}{d\zeta} + \left(\frac{3}{2}\alpha + \frac{5}{4}\epsilon\alpha\right)\phi^2 - \left(\frac{3}{2}\alpha + \frac{5}{4}\epsilon\alpha\right)\phi = 0, \tag{4.11}$$

with the boundary conditions

$$\phi(-\infty) = 0 \quad , \quad \phi(\infty) = 1. \tag{4.12}$$

We look for an asymptotic solution of equation (4.11) in the form

$$\phi(\xi) = \epsilon\,\phi_1(\xi) + \epsilon^2\phi_2(\xi) + \epsilon^3\phi_3(\xi) + \ldots\ldots \tag{4.13}$$

we match an asymptotic solution of $0(\epsilon)$ to the value of ϕ at the point where the curvature of the travelling wave changes sign. Because of the invariant of the equation under an arbitrary fixed translation we can take the origin at this point. Now to determine $\phi(0)$ we consider the (ϕ,s)-phase plane with $s=d\phi/d\xi$. Then equation (4.11) becomes

$$\tau \frac{ds}{d\xi} = -s - \left(\frac{3}{2}\alpha + \frac{5}{4}\epsilon\alpha\right)\phi^2 + \left(\frac{3}{2}\alpha + \frac{5}{4}\epsilon\alpha\right)\phi\,, \tag{4.14}$$

$$\frac{d\phi}{d\xi} = s. \tag{4.15}$$

This system has critical points at the origin (0,0) and at point (1,0), with the origin representing a saddle point and (1,0) a node or focus. These two points corresponds to the boundary conditions to be satisfied by the solution to equation (4.11). From this we conclude that the solution corresponding to the trajectory joining these two critical points must be unique. Thus the point at which $ds/d\phi=0$ will correspond to the point where the curvature of the wave changes sign.

Now to find $\phi(0)$ we seek an expansion of s in the form

$$s(\phi) = \epsilon f_1(\phi) + \epsilon^2 f_2(\phi) + \ldots\ldots \tag{4.16}$$

From equations (4.14) and (4.15)

$$\tau s \frac{ds}{d\xi} = -s - \left(\frac{3}{2}\alpha + \frac{5}{4}\epsilon\alpha\right)\phi^2 + \left(\frac{3}{2}\alpha + \frac{5}{4}\epsilon\alpha\right)\phi. \tag{4.17}$$

Substituting for s and equating terms of $0(\epsilon)$ we get

$$s(\phi) = \epsilon \frac{5}{4}\alpha \left(\phi - \phi^2\right) + \ldots\ldots\ldots \tag{4.18}$$

Then to the same order it follows that $ds/d\phi=0$ when $\phi_1(0)=0$ and $\phi_1'(0)=\frac{5\alpha}{16}$. It is to be noted that the equation (4.13) implies that the $0(1)$ solution corresponds to $\phi=0$ or $\phi=1$. Thus we are considering the solution corresponds to the perturbation only. The same thing follows, if we consider the corresponding terms in equation (4.16) also. From equation (4.11) we thus obtain the boundary value problem

$$\tau \phi_1''(\xi) + \phi_1'(\xi) - \frac{3}{2}\alpha \phi_1(\xi) = 0, \tag{4.19}$$

$$\text{with} \quad \phi_1(0) = \frac{1}{2} \quad \text{and} \quad \phi_1'(0) = \frac{5}{16}\alpha . \tag{4.20}$$

Solving equation (4.19) we get

$$\phi_1(\xi) = \left(4 + \frac{(5\alpha + 8)\tau}{\sqrt{6\alpha\tau+1}}\right).\exp\left(\frac{\sqrt{6\alpha\tau+1} - 1}{2\tau}\right)\xi$$

$$- \left(\frac{7}{2} + \frac{(5\alpha + 8)\tau}{\sqrt{6\alpha\tau+1}}\right).\exp -\left(\frac{\sqrt{6\alpha\tau+1} + 1}{2\tau}\right)\xi \ . \tag{4.21}$$

By using equation (4.10) we get

$$\bar{\eta} = \eta_\infty^- + \epsilon\left(\eta_\infty^+ - \eta_\infty^-\right)\left\{\left(4 + \frac{(5\alpha + 8)\tau}{\sqrt{6\alpha\tau+1}}\right).\right.$$

$$\exp\left(\frac{\sqrt{6\alpha\tau+1} - 1}{2\tau}\right)\xi - \left(\frac{7}{2} + \frac{(5\alpha + 8)\tau}{\sqrt{6\alpha\tau+1}}\right).$$

$$\left.\exp -\left(\frac{\sqrt{6\alpha\tau+1} + 1}{2\tau}\right)\xi\right\} + 0(\epsilon^2). \tag{4.22}$$

Unlike in Jeffrey (1979) τ and hence μ enters as a nonlinear factor in the solution.

It is to be noted that to the order of ϵ we are not directly making use of the boundary condition at $\xi=\pm\infty$. This corresponds to the fact that we cannot prescribe such a boundary condition in our physical problem.

In the next two sections we give exact solutions of equation (4.1). The study involves the applications of the methods proposed by Jeffrey and Mohamad (1991) for the general KdVB equation.

4.3 DIRECT METHOD

Equation (4.1) can be written as

$$\eta_t + a\eta_x + b\eta\eta_x + c\eta_{xx} + d\eta_{xxx} = 0. \qquad (4.23)$$

where a, b, c and d are the constant coefficients of η_x, $\eta\eta_x$ η_{xx}, and η_{xxx} respectively.

We look for a solution of the KdVB equation (4.23) is of the form

$$\eta = \eta(\xi), \quad \text{where} \quad \xi = kx - \omega t. \qquad (4.24)$$

Here k and ω are constants to be determined. Substituting in equation (4.23) we get,

$$- \omega\eta' + ak\eta' + bk\eta\eta' + ck^2\eta'' + dk^3\eta''' = 0. \qquad (4.25)$$

Equation (4.25) can be integrated to get

$$- \omega\eta + ak\eta + \frac{1}{2} bk\eta^2 + ck^2\eta' + dk^3\eta'' = C, \qquad (4.26)$$

where C is the constant of integration.

The basis of the method is to assume a travelling wave solution of the form

$$\eta = A \operatorname{sech}^n\xi + B \tanh^m\xi + D, \qquad (4.27)$$

which is a superposition of solutions of Burgers' equation and KdV equation. In equation (4.27) A, B and D are constants to be determined.

There are five undetermined constants and inorder to obtain a unique solution, it is necessary to find and then solve five independent algebraic equations.

By setting the integration constant C equal to zero and substituting equation (4.27) into equation (4.26) we get,

$$\left(-\omega + ak\right)\left(A\,\mathrm{sech}^{n}\xi + B\,\tanh^{m}\xi + D\right)$$
$$+ \frac{1}{2}\,bk\left(A\,\mathrm{sech}^{n}\xi + B\,\tanh^{m}\xi + D\right)^{2}$$
$$+ ck^{2}\left(-An\,\mathrm{sech}^{n}\xi\,\tanh\xi + Bm\,\mathrm{sech}^{2}\xi\,\tanh^{m-1}\xi\right)$$
$$+ dk^{3}\Big(An^{2}\,\mathrm{sech}^{n}\xi\,\tanh^{2}\xi - An\,\mathrm{sech}^{n+2}\xi$$
$$+ Bm(m-1)\,\mathrm{sech}^{4}\xi\,\tanh^{m-2}\xi$$
$$- 2Bm\,\mathrm{sech}^{2}\xi\,\tanh^{m}\xi\Big) = 0. \qquad (4.28)$$

Then only when n = 2 and m = 1 in equation (4.28) we can arrive at five simultaneous equations from which to determine the five remaining unknowns. These equations are

$$\frac{1}{2}\,bk\left(B^{2} + D^{2}\right) - D\left(\omega - ak\right) = 0, \qquad (4.29)$$

$$4dk^{3}A + ck^{2}B + \frac{1}{2}\,bk\left(2AD - B^{2}\right) - A\left(\omega - ak\right) = 0, \qquad (4.30)$$

$$bkD - \left(\omega - ak\right) = 0, \qquad (4.31)$$

$$-6dk^{2} + \frac{1}{2}\,bA = 0, \qquad (4.32)$$

$$\frac{1}{2}\, bAB - ckA - dk^2B = 0. \tag{4.33}$$

It follows from equations (4.29) to (4.33) that

$$k = \pm \frac{c}{10d}\,, \tag{4.34}$$

$$\omega = \frac{6c^3}{250d^2} \pm \frac{ac}{10d}\,, \tag{4.35}$$

$$A = \frac{3c^2}{25bd}\,, \qquad B = \pm \frac{6c^2}{25bd} \qquad \text{and} \qquad D = \pm \frac{6c^2}{25bd}\,. \tag{4.36}$$

Substituting these results into equation (4.24) and equation (4.27) with $k = \frac{c}{10d}$ we get

$$\eta(x,t) = \frac{3c^2}{25bd}\left\{\operatorname{sech}^2(\xi/2) + 2\tanh(\xi/2) + 2\right\}, \tag{4.37a}$$

where

$$\xi = \frac{c}{5d}\left[x - \left(\frac{6c^2}{25bd} + a\right)t\right].$$

Similarly when $k = -\frac{c}{10d}$,

$$\eta(x,t) = \frac{3c^2}{25bd}\left\{\operatorname{sech}^2(\xi/2) - 2\tanh(\xi/2) - 2\right\}, \tag{4.37b}$$

where

$$\xi = -\frac{c}{5d}\left[x + \left(\frac{6c^2}{25bd} - a\right)t\right].$$

These results are in agreement with those obtained by

Jeffrey and Xu (1989).

Equation (4.37) represent a travelling wave solution to the KdVB equation (4.23).

We also find the travelling wave solution η given by equation (4.37a) has the limit 0 as $\xi \longrightarrow -\infty$ and the limit $12c^2/25bd$ as $\xi \longrightarrow \infty$, while the travelling wave solution η given by equation (4.37b) has the limit 0 as $\xi \longrightarrow -\infty$ and the limit $-\ 12c^2/25bd$ as $\xi \longrightarrow \infty$.

It is also to be noted that the travelling wave solutions of either Burgers' equation or KdV equation cannot be obtained as limiting cases from the solution (4.37).

4.4 SERIES METHOD

We seek a solution of the KdVB equation (4.23) in the form

$$\eta(x,t) = \sum_{j=0}^{2} \eta_j \ F^{j-2} \ , \tag{4.38}$$

where η_j and F are functions of x and t respectively. Substituting equation (4.38) into equation (4.23) we find that

$$\eta_0 = -\ 12(d/b)\ F_x^{\ 2} \ , \tag{4.39a}$$

$$\eta_1 = \frac{12}{5}\ (c/b)\ F_x + 12(d/b)\ F_{xx} \ , \tag{4.39b}$$

where η_0 and η_1 are the coefficients of the powers of F^{-5} and F^{-4} respectively. Using equation (4.39), the coefficient of

F^{-3} can be written in the form

$$F_t + \left(a - \frac{c^2}{25d}\right)F_x + \frac{6}{5}cF_{xx} + 4dF_{xxx}$$

$$- 3dF_x^{-1} F_{xx}^{\;2} + 6F_x \eta_2 = 0. \qquad (4.40)$$

A solution $\eta(x,t)$ of the KdVB equation (4.23) may then be written as

$$\eta(x,t) = \frac{\eta_0}{F^2} + \frac{\eta_1}{F} + \eta_2(x,t). \qquad (4.41)$$

Substituting from equations (4.39), equation (4.41) can be written as

$$\eta(x.t) = 12(d/b)\frac{\partial^2}{\partial x^2}\log F$$

$$+ \frac{12c}{5b}\frac{\partial}{\partial x}\log F + \eta_2(x,t). \qquad (4.42)$$

Equation (4.42) includes as particular cases, the transformation used by Jeffrey and Xu (1989), Weiss *et al* (1983) and the well known Hopf-Cole transformation (Cole, 1951; Hopf, 1950).

We set

$$F(x,t) = 1 + \exp(kx - \omega t), \qquad (4.43)$$

where k and ω are constants to be determined.

Substituting equation (4.43) into equations (4.40) and (4.42) and setting $\eta_2 = 0$ we find that a function F of the form of equation (4.43) will be a solution of the KdVB equation provided

$$6c^2 k^2 + 5ack - 5c\omega = 0, \tag{4.44a}$$

and

$$c^2 k^2 - 25d^2 k^4 = 0. \tag{4.44b}$$

From equation (4.44) we have,

$$\omega = \frac{6c^3}{125d^2} \pm \frac{ac}{5d}, \tag{4.45a}$$

and

$$k = \pm \frac{c}{5d}. \tag{4.45b}$$

Thus corresponding to $k = \frac{c}{5d}$ *and*

$$\xi = \frac{c}{5d}\left[x - \left(\frac{6c^2}{25bd} + a\right)t\right],$$

we have

$$\eta(x,t) = 12(d/b)\frac{\partial^2}{\partial x^2}\left\{\log\left(1+\exp(kx - \omega t)\right)\right\}$$

$$+ \frac{12c}{5b}\frac{\partial}{\partial x}\left\{\log\left(1+\exp(kx - \omega t)\right)\right\}$$

$$= \frac{3c^2}{25bd}\left\{\operatorname{sech}^2(\xi/2) + 2\tanh(\xi/2) + 2\right\}. \tag{4.46}$$

Similarly corresponding to $k = -\frac{c}{5d}$ and

$$\xi = -\frac{c}{5d}\left[x + \left(\frac{6c^2}{25bd} + a\right)t\right],$$

we have

$$\eta(x,t) = \frac{3c^2}{25bd}\left\{\text{sech}^2(\xi/2) - 2\tanh(\xi/2) - 2\right\} . \tag{4.47}$$

These results are identical with those we have obtained using the direct method.

4.5 DISCUSSION

As a further step in the study of equation (1.15) we have considered here equation (1.21) which includes a term $\frac{1}{3}\epsilon\beta\eta_{xx}$ in addition to nonlinearity and dispersion. Unlike the usual cases considered else where, $\frac{1}{3}\epsilon\beta>0$ in our case.

First, we have obtained an asymptotic solution. Here we see that unlike in Jeffrey (1979), the dispersion coefficient enters nonlinearly in the solution. It has already been pointed out by Jeffrey that KdVB shock solution is sensitive to a perturbation at the origin.

We have also obtained analytic solutions using two different methods. It is found that the travelling wave solutions of either Burgers' equation or KdV equation cannot be obtained as limitting cases from the solution of KdVB equation. The results obtained by series method are identical with those obtained using direct method.

CHAPTER 5

IST ANALYSIS OF KdVB EQUATION

5.1 INTRODUCTION

The celebrated KdV equation derived by Korteweg and de Vries (1895) is a model for unidirectional long waves of small amplitude travelling over a constant depth. In order to compute the deformation of a solitary wave climbing a beach, Peregrine (1967) used a finite-difference scheme and has given the quantitative results. He has derived a long wave equation in water of variable depth, which corresponds to the Boussinesq equation for water of constant depth.

Madsen and Mei (1969) have investigated the problem of solitary wave propagating over a mild slope on to a shelf of constant depth. They have observed that while propagating over a shelf, the wave is disintegrating into a train of solitary waves of decreasing amplitude.

If the depth of water is changing slowly the resulting equation is a perturbed KdV equation. Several successful attempts were made to find an asymptotic solution to this problem (Grimshaw, 1970, 1971; Leibovich and Randall, 1973). A remarkable development in this direction is the derivation of such an equation by Kakutani (1971), and Johnson (1972). Johnson (1973a) proved that if the depth decreases to form a shelf, then for particular new depths a solitary wave breaks up into a finite number of solitons asymptotically far along the shelf.

KdV equation with variable coefficients appears as a model for wave propagation in an inhomogeneous medium. In the case of water waves the inhomogenity may be due to change in depth or cross-section. The equations due to Kakutani (1971) and Johnson (1972) belong to this class. Nirmala *et al.* (1986a, b) have considered the integrability of a particular class of KdV equations with variable coefficients. Some other works are due to Shen and Zhong (1981) Zhou (1981, 1983) and Sobezyk (1992).

As pointed out in 3.1, Johnson's (1973a) equation belongs to a class of perturbed KdV equation. Kaup and Newell (1978) used the IST theory for the exact KdV equation as a basis for a perturbation scheme, and found that the zero wave number mode of the continuous spectrum was excited by the solitary wave interacting with the perturbation. This excitation is manifested by the creation of a shelf in the wake of a solitary wave extending between the rear of the wave and the point to which the largest linear disturbances would have travelled. The role of the shelf is to provide a balance in the mass flux relation. Kaup and Newell also showed that the shelf acts as a precursor to the formation of new solitons. Their study involves all perturbations of equations which are exactly integrable by using the inverse scattering transforms associated with the Zakharov-Shabat eigenvalue problem (Zakharov and Shabat, 1972) or the Schrodinger equations. An exact expression for the solution interms of the scattering data and squared eigenfunctions was used to avoid the inverse procedure given by the Marchenko equations. Zakharov-Shabat method has another advantage that the unperturbed system is integrable and this system can be used as the basis for analysing the perturbed system. The fixed parameters (the action variables) of the unperturbed system will take on a slowly varying behaviour in the perturbed system.

Knickerbocker and Newell (1980) conducted numerical studies of the analytical results of Kaup and Newell (1978) concerning the effect of a perturbation on a solitary wave of the KdV equation.Kalyakin (1991), Byatt-Smith (1992) and Grimshaw (1992) have also studied solutions of perturbed KdV equations.

5.2 PERTURBATION METHOD

We consider the KdVB equation (Pramod and Vedan, 1992),

$$\eta_t + \left(1 - \frac{1}{2}\,\epsilon H(x-x_0) + \frac{5}{12}\,\epsilon\beta\delta'(x-x_0)\right)\eta_x$$

$$+ \left(\frac{3}{2} \alpha + \frac{5}{4} \epsilon\alpha H(x-x_0) \right) \eta\eta_x + \frac{1}{3} \epsilon\beta\delta(x-x_0)\eta_{xx}$$

$$+ \left(\frac{1}{6} \beta - \frac{1}{3} \epsilon\beta H(x-x_0) \right) \eta_{xxx} = 0. \tag{5.1}$$

Equation (5.1) can be written as

$$\eta_t + \eta_x + \frac{3}{2} \alpha\eta\eta_x + \frac{1}{6} \beta\eta_{xxx}$$

$$= \epsilon \left\{ \left(\frac{1}{2} H(x-x_0) - \frac{5}{12} \beta\delta'(x-x_0) \right) \eta_x \right.$$

$$- \frac{5}{4} \alpha H(x-x_0)\eta\eta_x - \frac{1}{3} \beta\delta(x-x_0)\eta_{xx} +$$

$$\left. + \frac{1}{3} \beta H(x-x_0) \eta_{xxx} \right\} . \tag{5.2}$$

By the definition of a Dirac delta function we have,

$$\delta(x-x_0) = \frac{1}{2\Pi} \int_{-\infty}^{\infty} \exp ik(x-x_0) dk,$$

and

$$\frac{d}{dx} H(x-x_0) = \delta(x-x_0).$$

Then equation (5.2) can be written as,

$$\eta_t + \eta_x + \frac{3}{2} \alpha\eta\eta_x + \frac{1}{6} \beta\eta_{xxx}$$

$$= \epsilon \left\{ \left(\frac{1}{2ik} - \frac{5}{12} \beta ik \right) \eta_x \frac{1}{2\Pi} \int_{-\infty}^{\infty} \exp\ ik(x-x_0) dk \right.$$

$$- \frac{5}{4} \alpha \frac{1}{ik} \eta\eta_x \frac{1}{2\Pi} \int_{-\infty}^{\infty} \exp\ ik(x-x_0) dk$$

$$- \frac{1}{3} \beta\eta_{xx} \frac{1}{2\Pi} \int_{-\infty}^{\infty} \exp\ ik(x-x_0)\ dk$$

$$\left. + \frac{1}{3} \beta \frac{1}{ik} \eta_{xxx} \frac{1}{2\Pi} \int_{-\infty}^{\infty} \exp\ ik(x-x_0) dk \right\}$$

$$= \epsilon \left\{ \left(\frac{1}{2ik} - \frac{5}{12} \beta ik \right) \eta_x\ \delta(x-x_0) - \frac{5}{4} \alpha \frac{1}{ik} \eta\eta_x\ \delta(x-x_0) \right.$$

$$\left. - \frac{1}{3} \beta\eta_{xx}\ \delta(x-x_0) + \frac{1}{3} \beta \frac{1}{ik} \eta_{xxx}\ \delta(x-x_0) \right\}. \qquad (5.3)$$

Equation (5.3) can be again written as,

$$\eta_t + \eta_x + \frac{3}{2} \alpha\eta\eta_x + \frac{1}{6} \beta\eta_{xxx} = \epsilon F, \qquad (5.4)$$

where

$$F = \left(\frac{1}{2ik} - \frac{5}{12} \beta ik \right) \eta_x\ \delta(x-x_0) - \frac{5}{4} \alpha \frac{1}{ik}\ \eta\eta_x\ \ \delta(x-x_0)$$

$$- \frac{1}{3} \beta\eta_{xx}\ \delta(x-x_0) + \frac{1}{3} \beta \frac{1}{ik} \eta_{xxx}\ \delta(x-x_0). \qquad (5.5)$$

Equation (5.4) is a perturbed KdV equation. We consider the integrable system (in equation (5.4))

$$\eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x + \frac{1}{6}\beta\eta_{xxx} = 0. \qquad (5.6)$$

Let

$$\eta = \gamma U_x + U^2, \qquad (5.7)$$

where

$$\gamma = i\sqrt{\frac{2\beta}{3\alpha}} .$$

If U is a solution of

$$U_t + U_x + \frac{3}{2}\alpha U^2 U_x + \frac{1}{6}\beta U_{xxx} = 0,$$

then $\eta = \gamma U_x + U^2$ is a solution of equation (5.6).

In equation (5.7) we take η to be known; then this corresponds to a Riccati equation for V and can be linearised by the transformation,

$$U = \gamma \frac{V_x}{V} , \qquad (5.8)$$

yielding

$$V_{xx} - \bar{\eta}V = 0, \qquad (5.9a)$$

where

$$\bar{\eta} = \frac{\eta}{\gamma^2} . \qquad (5.9b)$$

This is the time-independent Schrodinger equation; however it is missing the energy level term. In IST method we consider the time-independent Schrodinger equation,

$$V_{xx} - \left(\bar{\eta} - \zeta^2\right)V = 0, \tag{5.10}$$

where $\bar{\eta}$ is the potential, ζ^2 's are the energy levels, where $\zeta = \xi + i\bar{\eta}$ and V is the wave function.

5.3 ZAKHAROV-SHABAT EIGENVALUE PROBLEM

Equation (5.10) can be written as Zakharov-Shabat (Z.S.) eigenvalue problem (Zakharov and Shabat, 1972),

$$V_{1x} - i\zeta V_1 = -\bar{\eta}\, V_2 , \tag{5.11a}$$

and

$$V_{2x} + i\zeta V_2 = -V_1 . \tag{5.11b}$$

Following Kaup and Newell (1978) and Newell (1980), we define for real ζ, two pairs of linearly independent solutions. The solutions are $\phi(x,t,\zeta)$ and $\bar{\phi}(x,t,\zeta)$ where $\bar{\phi} = (\phi_2^*, -\phi_1^*)$ and $\psi(x,t,\zeta)$ and $\bar{\psi}(x,t,\zeta)$ where $\bar{\psi} = (\psi_2^*, -\psi_1^*)$ and * denotes the complex conjugate, which have the following asymptotic properties,

$$\phi \longrightarrow e^{-i\zeta x}, \qquad x \longrightarrow -\infty \tag{5.12a}$$

$$\phi \longrightarrow a(\zeta,t)\, e^{-i\zeta x} + b(\zeta,t)\, e^{i\zeta x}, \qquad x \longrightarrow \infty \tag{5.12b}$$

$$\psi \longrightarrow e^{i\zeta x}, \qquad x \longrightarrow \infty \tag{5.12c}$$

$$\bar{\psi} \longrightarrow a(\zeta,t)\, e^{i\zeta x} - b(-\zeta,t)\, e^{-i\zeta x}, \qquad x \longrightarrow -\infty$$

(5.12d)

Here $1/a$ is the transmission coefficient and b/a is the reflection coefficient. The solutions are interrelated by

$$\phi(x,t,\zeta) = a(\zeta,t)\, \psi(x,t,-\zeta) + b(\zeta,t)\, \psi(x,t,\zeta), \qquad (5.13a)$$

$$\phi(x,t,-\zeta) = a(-\zeta,t)\, \psi(x,t,\zeta) + b(-\zeta,t)\, \psi(x,t,-\zeta). \qquad (5.13b)$$

and the inverse relations can be found by using the fact that

$$a(\zeta,t)\, a(-\zeta,t) - b(\zeta,t)\, b(-\zeta,t) = 1.$$

The reality of $\bar{\eta}(x,t)$ implies that,

$$a(-\zeta) = a^{*}(\zeta^{*}),$$

$$b(-\zeta) = b^{*}(\zeta^{*}).$$

The zeros of $a(\zeta,t)$, when $\mathrm{Im}(\zeta)>0$, are the discrete eigenvalues of equation (5.10). Assume that these zeros are simple. The linear integral equation which allows one to reconstruct the potential $\bar{\eta}(x,t)$ from the scattering data uses the following combinations,

$$S_{+} \; ; \left\{ b(\zeta,t)/a(\zeta,t), \quad \zeta \text{ real} \; ; \; (\zeta_k, \gamma_k), \quad k=1,\ldots\ldots,N \right\}$$

(5.14a)

$$S_{-} \; ; \left\{ b^{*}(\zeta,t)/a(\zeta,t), \quad \zeta \text{ real} \; ; \; (\zeta_k, \beta_k), \quad k=1,\ldots\ldots,N \right\}$$

(5.14b)

Here $\gamma_k = b_k / a_k$, $\beta_k = -1/b_k a_k{}'$, and $b_k(t)$ is defined by the relation, $\phi(\zeta_k ,t) = b_k(t)\, \psi(\zeta_k ,t)$, $b_k(t)$ is the analytic extension of $b(\zeta,t)$ to ζ_k if there is one. We will work with S_-. We make the substitution $\bar{\eta} = \frac{\eta}{\gamma^2}$ in equations (5.4) - (5.6).

Equation (5.6) is the integrable system,

$$\bar{\eta}_t + \frac{\partial}{\partial x}\left[P_0(M_s)\bar{\eta}\right] = 0. \tag{5.15}$$

In equation (5.15) above, $P_0(\zeta)$ is an entire function (real for ζ real) and M_s is the operator,

$$M_s(\psi^2) = \zeta^2 \psi^2 ,$$

where

$$P_0(\zeta^2) = -4\zeta^2 , \tag{5.16a}$$

and

$$M_s = -\frac{1}{4} - \frac{1}{24}\beta \frac{\partial^2}{\partial x^2} - \alpha\gamma^2 \bar{\eta} - \frac{13}{8}\alpha\gamma^2 \int_x^\infty dy \bar{\eta}_y . \tag{5.16b}$$

The scattering data for the above integrable system is,

$$\zeta_{kt} = 0, \tag{5.17a}$$

$$\beta_{kt} = 2i\zeta_k P_0(\zeta_k^2)\beta_k , \tag{5.17b}$$

$$(b^*/a)_t = 2i\zeta P_0(\zeta^2) b^*/a . \tag{5.17c}$$

Now we can solve

$$\bar{\eta}_t + \frac{\partial}{\partial x}\left[P_0(M_s)\bar{\eta}\right] = \epsilon F, \tag{5.18}$$

where

$$F = \left(\frac{1}{2ik} - \frac{5}{12}\beta ik\right)\bar{\eta}_x\,\delta(x-x_0) - \frac{5}{4}\alpha\gamma^2\frac{1}{ik}\bar{\eta}\bar{\eta}_x\,\delta(x-x_0)$$

$$-\frac{1}{3}\beta\bar{\eta}_{xx}\,\delta(x-x_0) + \frac{1}{3}\beta\frac{1}{ik}\bar{\eta}_{xxx}\,\delta(x-x_0). \tag{5.18a}$$

By mapping $\bar{\eta}(x,t)$ in to the scattering functions associated with equation (5.10). Then,

$$\beta_k\,\zeta_{kt} = \frac{\epsilon}{2i\zeta_k\,a_k{}'^2}\int_{-\infty}^{\infty} F\,\psi_k{}^2\,dx, \tag{5.19a}$$

$$\beta_{kt} + \left(\frac{a_k''}{a_k'} + \frac{1}{\zeta_k}\right)\beta_k\zeta_{kt} = 2i\zeta_k\,P_0(\zeta_k^2)\beta_k +$$

$$+\frac{\epsilon}{2i\zeta_k\,a_k{}'^2}\int_{-\infty}^{\infty} F\left(\frac{\partial}{\partial\zeta}\psi^2\right)_k dx, \tag{5.19b}$$

for $k = 1, 2, \ldots\ldots, N$ and for real ζ

and

$$(b^*/a)_t = 2i\zeta\,P_0(\zeta^2)b^*/a + \frac{\epsilon}{2i\zeta\,a^2}\int_{-\infty}^{\infty} F\,\psi^2\,dx. \tag{5.19c}$$

For the unperturbed system the multi-soliton state, following methods similar to Zakharov and Shabat (1972) is given by

$$\psi(\zeta) = \exp i(\zeta x)\left[1 - \sum_{k=1}^{N} \frac{\gamma_k \psi_k \exp i(\zeta_k x)}{(\zeta_k+\zeta)}\right], \qquad (5.20a)$$

$$\left(\frac{\partial}{\partial\zeta}\psi\right)_{\zeta j} = ix(\psi)_j + \sum_{k=1}^{N} \frac{\gamma_k \psi_k \exp i(\zeta_k+\zeta_j)x}{(\zeta_k+\zeta_j)^2}, \qquad (5.20b)$$

where ψ_j , j = 1,2,....,N are found from the equations

$$\psi_j + \sum_{k=1}^{N} \frac{\gamma_k \exp i(\zeta_k+\zeta_1)x}{(\zeta_k+\zeta_j)} \psi_k = \exp i(\zeta_j x), \qquad (5.20c)$$

where j = 1,....,N.

Using the property of delta function, equations (5.19) becomes

$$\beta_k \zeta_{kt} = \frac{\epsilon}{2i\zeta_k a_k'^2} \left\{\left[\left(\frac{1}{2ik} - \frac{5}{12}\beta ik\right)\bar{\eta}_x\right.\right.$$

$$\left. - \frac{5}{4}\alpha\frac{1}{ik}\bar{\eta}\bar{\eta}_x - \frac{1}{3}\beta\bar{\eta}_{xx} + \frac{1}{3}\beta\frac{1}{ik}\bar{\eta}_{xxx}\right].$$

$$\exp(2i\zeta_k x)\left[1 - 2\sum_{k=1}^{N} \frac{\gamma_k \psi_k \exp(i\zeta_k x)}{2\zeta_k}\right.$$

$$+ \sum_{k=1}^{N} \sum_{l=1}^{N} \frac{\gamma_k \gamma_1 \psi_k \psi_1 \exp i(\zeta_k+\zeta_1)x}{2\zeta_k(\zeta_k+\zeta_1)} \Bigg] \Bigg\}_{x=x_0} ,$$

(5.21a)

$$\beta_{kt} + \left(\frac{a_k''}{a_k'} + \frac{1}{\zeta_k} \right) \beta_k \zeta_{kt} = 2i\zeta_k P_0(\zeta_k^2) \beta_k$$

$$+ \frac{\epsilon}{2i\zeta_k a_k'^2} \left\{ \left[\left(\frac{1}{2ik} - \frac{5}{12} \beta i k \right) \bar{\eta}_x - \frac{5}{4} \alpha \frac{1}{ik} \bar{\eta}\bar{\eta}_x \right. \right.$$

$$\left. - \frac{1}{3} \beta \bar{\eta}_{xx} + \frac{1}{3} \beta \frac{1}{ik} \bar{\eta}_{xxx} \right] 2ix \exp (2i\zeta_k)x .$$

$$\left[1 - 2 \sum_{k=1}^{N} \frac{\gamma_k \psi_k \exp (i\zeta_k)x}{2\zeta_k} \right.$$

$$\left. + \sum_{k=1}^{N} \sum_{l=1}^{N} \frac{\gamma_k \gamma_1 \psi_k \psi_1 \exp i(\zeta_k+\zeta_1)x}{2\zeta_k(\zeta_k+\zeta_1)} \right]$$

$$+ 2 \sum_{k=1}^{N} \frac{\gamma_k \psi_k \exp (3i\zeta_k)x}{(2\zeta_k)^2}$$

$$+ \sum_{k=1}^{N} \sum_{l=1}^{N} \frac{\gamma_k \gamma_1 \psi_k \psi_1 \exp i(3\zeta_k+\zeta_1)x}{(\zeta_1-\zeta_k)} .$$

$$\left[\frac{1}{(\zeta_k+\zeta_1)^2} - \frac{1}{(2\zeta_k)^2}\right]\Bigg\}_{x=x_0} , \qquad (5.21b)$$

$$(b^*/a)_t = 2i\zeta\, P_0(\zeta^2)\, b^*/a + \frac{\epsilon}{2i\zeta\, a^2} \cdot$$

$$\Bigg\{\Bigg[\left(\frac{1}{2ik} - \frac{5}{12}\beta ik\right)\bar{\eta}_x - \frac{5}{4}\alpha\frac{1}{ik}\bar{\eta}\bar{\eta}_x - \frac{1}{3}\beta\bar{\eta}_{xx}$$

$$+ \frac{1}{3}\beta\frac{1}{ik}\bar{\eta}_{xxx}\Bigg] . \exp(i2\zeta)x \; .$$

$$\Bigg[1 - 2\sum_{k=1}^{N}\frac{\gamma_k\psi_k\exp(i\zeta_k x)}{(\zeta_k+\zeta)}$$

$$+\sum_{k=1}^{N}\sum_{l=1}^{N}\frac{\gamma_k\,\gamma_l\,\psi_k\,\psi_l\,\exp i(\zeta_k+\zeta_l)x}{(\zeta_k+\zeta)(\zeta_l+\zeta)}\Bigg]\Bigg\}_{x=x_0} . \qquad (5.21c)$$

It has been pointed out by Kaup and Newell (1978) that perturbed KdV equation cannot conserve mass flux. While conservation of energy is consistent with the fact that the eigenvalue of the Schrodinger operator for real potentials adjust its value in an adiabatic way to change in depth, the growing or decaying of soliton cannot correspond to supply of water across the shelf to keep a constant flux. We have already noted (chapter 3, discussion) the excitation of solitons due to the shelf. Now to study the propagation of soliton past the shelf, we consider the contribution of continuous spectrum. For this we calculated the reflection coefficient b/a from equation (5.21c) with $P_0 = -4\zeta^2$ and $(b/a)_{t=0} = 0$.

From equation (5.21c),

$$(b^*/a)_t + 8i\zeta^3(b^*/a) = \frac{\epsilon}{2i\zeta\, a^2}\left\{\left[\left(\frac{1}{2ik} - \frac{5}{12}\beta ik\right)\bar{\eta}_x - \frac{5}{4}\alpha\frac{1}{ik}\bar{\eta}\bar{\eta}_x - \frac{1}{3}\beta\bar{\eta}_{xx} + \frac{1}{3}\beta\frac{1}{ik}\bar{\eta}_{xxx}\right]\cdot\right.$$

$$\exp(2i\zeta x)\left[1 - 2\sum_{k=1}^{N}\frac{\gamma_k\,\psi_k\exp(i\zeta_k x)}{(\zeta_k+\zeta)} + \left.\sum_{k=1}^{N}\sum_{l=1}^{N}\frac{\gamma_k\,\gamma_l\,\psi_k\,\psi_l\exp i(\zeta_k+\zeta_l)x}{(\zeta_k+\zeta)(\zeta_l+\zeta)}\right]\right\}_{x=x_0} \cdot \tag{5.22}$$

Integrating we get (with $\bar{\eta}_t = 0$ to first order) equation (5.22) becomes

$$b(\xi,t) = \frac{-\epsilon}{16\xi^4 a}\left\{\left[\left(\frac{1}{2ik} - \frac{5}{12}\beta i k\right)\bar{\eta}_x - \frac{5}{4}\alpha\frac{1}{ik}\bar{\eta}\bar{\eta}_x + \right.\right.$$

$$\left. + \frac{1}{3}\beta\bar{\eta}_{xx} + \frac{1}{3}\beta\frac{1}{ik}\bar{\eta}_{xxx}\right] \exp(-2i\xi x).$$

$$\left[1 - 2\sum_{k=1}^{N}\frac{\gamma_k\psi_k \exp(-i\xi_k x)}{(\xi_k+\xi)}\right.$$

$$\left.\left. + \sum_{k=1}^{N}\sum_{l=1}^{N}\frac{\gamma_k\,\gamma_l\,\psi_k\,\psi_l \exp -i(\xi_k+\xi_l)x}{(\xi_k+\xi)\,(\xi_l+\xi)}\right]\right\}_{x=x_0}$$

$$\left(\exp(8i\xi^3 t) - 1\right). \tag{5.23}$$

From equation (5.23) we find that $b(\xi,t)$ is not defined for $\xi = 0$.

In fact we have to solve the perturbed equation in an iterative manner treating ϵ as a small parameter and using well known ideas from singular perturbation theory. Our interest is to obtain an asymptotic expansion for $\bar{\eta}(x,t)$ which is uniformly valid for times $t = O(\epsilon^{-\tau})$ for some $\tau > 0$.

Uniformly valid asymptotic expansions in scattering space result in a uniform expansion in the physical space. But when the asymptotic expansions in scattering space is nonuniform we resolve the question by demanding that the resulting asymptotic expansion (Kaup and Newell, 1978; Newell (1980) for $\bar{\eta}(x,t)$ in physical space interms of the scattering data and squared eigenfunctions

$$\bar{\eta}(x,t) = \frac{2}{i\Pi} \int_{-\infty}^{\infty} \zeta \, \frac{b}{a} \, \psi^2 d\zeta - 4 \sum_{k=1}^{N} \gamma_k \, \zeta_k \, \psi_k^2 \, , \qquad (5.24)$$

is uniformly valid. Here the summation corresponds to the contribution of the discrete spectrum and the integration corresponds to the contribution of the continuous spectrum.

In equation (5.20),

$$\text{If} \quad N = 1, \qquad \zeta_1 = i\bar{\eta}, \qquad \text{and} \qquad \gamma_1 = 2i\bar{\eta}e^{2\bar{\theta}} \, ,$$

where $\theta = -\bar{\eta}x + \bar{\theta}$, then

$$\psi_1 = \frac{e^{-\bar{\eta}x}}{1+\exp\,(-2\bar{\eta}x+2\bar{\theta})} = \frac{1}{2}\exp\,(-\bar{\theta})\,\text{Sech}\,\theta.$$

Therefore equation (5.20a) becomes

$$\psi(\xi) = e^{i\xi x}\left(1 - \frac{2\bar{\eta}}{\bar{\eta}-i\xi}\,\frac{e^{2\theta}}{1+e^{2\theta}}\right). \qquad (5.25)$$

Now we find out the contribution of continuous spectrum $\bar{\eta}_c(x,t)$ of this term to $\bar{\eta}(x,t)$ which can be computed from equation (5.24). We get,

$$\bar{\eta}_c(x,t) = \frac{\epsilon i}{8\pi} \int_{-\infty}^{\infty} \frac{d\xi}{\xi^3 a^2(\xi)} \exp(2i\xi x) \left[1 - \frac{2\bar{\eta}}{\bar{\eta}-i\xi} \frac{e^{2\theta}}{1+e^{2\theta}}\right]^2$$

$$\left(\exp(8i\xi^3 t) - 1\right) \left\{\left[\left(\frac{1}{2ik} - \frac{5}{12}\beta ik\right)\bar{\eta}_x\right.\right.$$

$$\left. - \frac{5}{4}\alpha \frac{1}{ik}\bar{\eta}\bar{\eta}_x + \frac{1}{3}\beta\bar{\eta}_{xx} + \frac{1}{3}\beta\frac{1}{ik}\bar{\eta}_{xxx}\right]$$

$$\exp(-2i\xi x)\left[1 - 2\sum_{k=1}^{N} \frac{\gamma_k \psi_k \exp(-i\xi_k x)}{(\xi_k+\xi)}\right.$$

$$\left.\left. + \sum_{k=1}^{N}\sum_{l=1}^{N} \frac{\gamma_k\,\gamma_l\,\psi_k\,\psi_l\,\exp -i(\xi_k+\xi_l)x}{(\xi_k+\xi)\,(\xi_l+\xi)}\right]\right\}_{x=x_0} . \qquad (5.26)$$

Clearly this has non-vanishing contribution in the neighbourhood of $\xi = 0$.

5.4 ONE-SOLITON CASE

As a particular example we consider the motion of a soliton

$$\bar{\eta}(x,t) = 2\mu^2 \mathrm{sech}^2 \mu(x-\bar{x}), \tag{5.27}$$

after it has passed the position $x=x_0$ at which the depth changes. We assume that the soliton reaches the position at $t=0$. We obtain the equations of motion for the amplitude $\mu(t)$ and position $\bar{x}(t)$ of the soliton. Then equation (5.18a) becomes

$$F = \left(\frac{1}{2ik} - \frac{5}{12}\beta ik\right) . - 4\mu^3 \mathrm{sech}^2 \mu(x-\bar{x}) \tanh\mu(x-\bar{x})\delta(x-x_0)$$

$$+ \frac{10\alpha}{ik}\mu^5 \mathrm{sech}^4 \mu(x-\bar{x}) \tanh\mu(x-\bar{x})\ \delta(x-x_0)$$

$$- \frac{\beta}{3}\left(8\mu^4 \mathrm{sech}^2 \mu(x-\bar{x}) - 12\mu^4 \mathrm{sech}^4 \mu(x-\bar{x})\right)\delta(x-x_0)$$

$$+ \frac{\beta}{3ik}\left(- 16\mu^5 \mathrm{sech}^2 \mu(x-\bar{x}) \tanh\mu(x-\bar{x})\right.$$

$$\left. + 48\mu^5 \mathrm{sech}^4 \mu(x-\bar{x}) \tanh\mu(x-\bar{x})\right)\delta(x-x_0) \tag{5.28}$$

For the scattering functions associated with equation (5.10) we get,

$$\beta_k \zeta_{kt} = \frac{\epsilon}{2i\zeta_k a_k'^2}\left\{\left[\left(\frac{1}{2ik} - \frac{5}{12}\beta ik\right) .\right.\right.$$

$$- 4\mu^3 \mathrm{sech}^2 \mu(x-\bar{x}) . \tanh\mu(x-\bar{x})$$

$$+ \frac{10\alpha}{ik}\mu^5 \mathrm{sech}^4 \mu(x-\bar{x}) \tanh\mu(x-\bar{x})$$

$$- \frac{\beta}{3} \left(8\mu^4 \operatorname{sech}^2 \mu(x-\bar{x}) - 12\mu^4 \operatorname{sech}^4 \mu(x-\bar{x})\right)$$

$$+ \frac{\beta}{3ik} \left(- 16\mu^5 \operatorname{sech}^2 \mu(x-\bar{x}) \tanh \mu(x-\bar{x})\right.$$

$$\left.\left. + 48\mu^5 \operatorname{sech}^4 \mu(x-\bar{x}) \tanh \mu(x-\bar{x})\right)\right] \cdot$$

$$\exp(2i\zeta_k x)\left[1 - 2\sum_{k=1}^{N} \frac{\gamma_k \psi_k \exp i\zeta_k x}{2\zeta_k}\right.$$

$$\left.\left. + \sum_{k=1}^{N} \sum_{l=1}^{N} \frac{\gamma_k \gamma_l \psi_k \psi_l \exp i(\zeta_k+\zeta_l)x}{2\zeta_k(\zeta_k+\zeta_l)}\right]\right\}_{x=x_0} ,$$

(5.29a)

$$\beta_{kt} + \left(\frac{a_k''}{a_k'} + \frac{1}{\zeta_k}\right)\beta_k \zeta_{kt} = 2i\zeta_k \quad P_0(\zeta_k^2)\beta_k$$

$$+ \frac{\epsilon}{2i\zeta_k\, a_k'^2} \left\{\left[\left(\frac{1}{2ik} - \frac{5}{12}\beta ik\right) \cdot\right.\right.$$

$$- 4\mu^3 \operatorname{sech}^2 \mu(x-\bar{x}) \tanh \mu(x-\bar{x})$$

$$+ \frac{10\alpha}{ik} \mu^5 \operatorname{sech}^4 \mu(x-\bar{x}) \tanh \mu(x-\bar{x})$$

$$- \frac{\beta}{3} \left(8\mu^4 \operatorname{sech}^2 \mu(x-\bar{x}) - 12\mu^4 \operatorname{sech}^4 \mu(x-\bar{x})\right)$$

$$+ \frac{\beta}{3ik} \left(- 16\mu^5 \operatorname{sech}^2 \mu(x-\bar{x}) \tanh \mu(x-\bar{x})\right.$$

$$+ 48\mu^5 \text{ sech}^4 \mu(x-\bar{x}) \tanh \mu(x-\bar{x})\Big)\Big].$$

$$2ix \exp(2i\zeta_k x)\Big[1 - 2\sum_{k=1}^{N} \frac{\gamma_k \psi_k \exp(i\zeta_k x)}{2\zeta_k}$$

$$+\sum_{k=1}^{N}\sum_{l=1}^{N} \frac{\gamma_k \gamma_l \psi_k \psi_l \exp i(\zeta_k+\zeta_l)x}{2\zeta_k(\zeta_k+\zeta_l)}\Big]$$

$$+ 2\sum_{k=1}^{N} \frac{\gamma_k \psi_k \exp(3i\zeta_k x)}{(2\zeta_k)^2}$$

$$+\sum_{k=1}^{N}\sum_{l=1}^{N} \frac{\gamma_k \gamma_l \psi_k \psi_l \exp i(3\zeta_k+\zeta_l)x}{(\zeta_l-\zeta_k)} \cdot$$

$$\left[\frac{1}{(\zeta_k+\zeta_l)^2} - \frac{1}{(2\zeta_k)^2}\right]\Bigg\}_{x=x_0}, \qquad (5.29b)$$

and

$$(b^*/a)_t = 2i\zeta\, P_0(\zeta^2) b^*/a + \frac{\epsilon}{2i\zeta\, a^2}\Bigg\{\Bigg[\left(\frac{1}{2ik} - \frac{5}{12}\beta ik\right).$$

$$- 4\mu^3 \text{ sech}^2 \mu(x-\bar{x}) \tanh \mu(x-\bar{x})$$

$$+ \frac{10\alpha}{ik}\mu^5 \text{ sech}^4 \mu(x-\bar{x}) \tanh \mu(x-\bar{x})$$

$$- \frac{\beta}{3}\left(8\mu^4 \operatorname{sech}^2 \mu(x-\bar{x}) - 12\mu^4 \operatorname{sech}^4 \mu(x-\bar{x})\right)$$

$$+ \frac{\beta}{3ik}\left(- 16\mu^5 \operatorname{sech}^2 \mu(x-\bar{x}) \tanh \mu(x-\bar{x})\right.$$

$$\left.\left. + 48\mu^5 \operatorname{sech}^4 \mu(x-\bar{x}) \tanh \mu(x-\bar{x})\right)\right] .$$

$$\exp(2i\zeta x)\left[1 - 2\sum_{k=1}^{N} \frac{\gamma_k \psi_k \exp(i\zeta_k x)}{(\zeta_k+\zeta)}\right.$$

$$\left.\left. + \sum_{k=1}^{N}\sum_{1=1}^{N} \frac{\gamma_k \gamma_1 \psi_k \psi_1 \exp i(\zeta_k+\zeta_1)x}{(\zeta_k+\zeta)(\zeta_1+\zeta)}\right]\right\}_{x=x_0} . \tag{5.29c}$$

We calculate the reflection coefficient b/a from equation (5.29c) with $p_0 = -4\zeta^2$ and $(b/a)_{t=0} = 0$. Then the equation becomes

$$(b^*/a)_t + 8i\zeta^3(b^*/a) = \frac{\epsilon}{2i\zeta\, a^2}\left\{\left[\left(\frac{1}{2ik} - \frac{5}{12}\beta ik\right).\right.\right.$$

$$- 4\mu^3 \operatorname{sech}^2 \mu(x-\bar{x}) \tanh \mu(x-\bar{x})$$

$$+ \frac{10\alpha}{ik}\mu^5 \operatorname{sech}^4 \mu(x-\bar{x}) \tanh \mu(x-\bar{x})$$

$$- \frac{\beta}{3} \left(8\mu^4 \operatorname{sech}^2 \mu(x-\bar{x}) - 12\mu^4 \operatorname{sech}^4 \mu(x-\bar{x})\right)$$

$$+ \frac{\beta}{3ik} \left[- 16\mu^5 \operatorname{sech}^2 \mu(x-\bar{x}) \tanh \mu(x-\bar{x})\right.$$

$$\left.+ 48\mu^5 \operatorname{sech}^4 \mu(x-\bar{x}) \tanh \mu(x-\bar{x})\right)\Bigg].$$

$$\exp(2i\zeta x)\left[1 - 2 \sum_{k=1}^{N} \frac{\gamma_k \psi_k \exp(i\zeta_k)x}{(\zeta_k+\zeta)}\right.$$

$$\left.\left. + \sum_{k=1}^{N} \sum_{l=1}^{N} \frac{\gamma_k \gamma_l \psi_k \psi_l \exp i(\zeta_k+\zeta_l)x}{(\zeta_k+\zeta)(\zeta_l+\zeta)}\right]\right\}_{x=x_0} . \tag{5.30}$$

Integrating (with $\mu_t = 0$ to first order) equation (5.30) we get,

$$\frac{b^*(\xi,t)}{a} = \exp(-8i\xi^3 t) \int dt. \frac{\epsilon}{2i\xi a^2} \left\{\left[\left(\frac{1}{2ik} - \frac{5}{12} \beta i k\right).\right.\right.$$

$$- 4\mu^3 \operatorname{sech}^2 \mu(x-\bar{x}) \tanh\mu(x-\bar{x})$$

$$+ \frac{10\alpha}{ik} \mu^5 \operatorname{sech}^4 \mu(x-\bar{x}) \tanh \mu(x-\bar{x})$$

$$- \frac{\beta}{3} \left(8\mu^4 \operatorname{sech}^2 \mu(x-\bar{x}) - 12\mu^4 \operatorname{sech}^4 \mu(x-\bar{x})\right)$$

$$+ \frac{\beta}{3ik}\left(- 16\mu^5 \mathrm{sech}^2\, \mu(x-\bar{x})\, \tanh\, \mu(x-\bar{x})\right.$$

$$\left.\left. + 48\mu^5\, \mathrm{sech}^4\, \mu(x-\bar{x})\, \tanh\, \mu(x-\bar{x})\right)\right].$$

$$\exp\,(2i\xi x)\left[1 - 2\sum_{k=1}^{N} \frac{\gamma_k\, \psi_k\, \exp\,(i\xi_k x)}{(\xi_k+\xi)}\right.$$

$$\left.\left. + \sum_{k=1}^{N}\sum_{l=1}^{N} \frac{\gamma_k\, \gamma_l\, \psi_k\, \psi_l\, \exp\, i(\xi_k+\xi_l)x}{(\xi_k+\xi)(\xi_l+\xi)}\right]\right\}_{x=x_0}.$$

$$\left(\exp\,(8i\xi^3 t) - 1\right).$$

The reflection coefficient $b(\xi,t)$ becomes

$$b(\xi,t) = \frac{\epsilon}{2i}\, \exp\,(8i\xi^3 t)\int dt.\frac{1}{\xi a}\left(\exp\,(-8i\xi^3 t) - 1\right)$$

$$\left\{\left[\left(\frac{1}{2ik} - \frac{5}{12}\,\beta ik\right).-4\mu^3\, \mathrm{sech}^2\, \mu(x-\bar{x})\, \tanh\, \mu(x-\bar{x})\right.\right.$$

$$+ \frac{10\alpha}{ik}\, \mu^5\, \mathrm{sech}^4\, \mu(x-\bar{x})\, \tanh\, \mu(x-\bar{x})$$

$$+ \frac{\beta}{3}\left(8\mu^4\, \mathrm{sech}^2\, \mu(x-\bar{x}) - 12\mu^4\, \mathrm{sech}^4\, \mu(x-\bar{x})\right)$$

$$+ \frac{\beta}{3ik}\left(- 16\mu^5 \mathrm{sech}^2\, \mu(x-\bar{x})\, \tanh \mu(x-\bar{x})\right.$$

$$\left.\left. + 48\mu^5\, \mathrm{sech}^4\, \mu(x-\bar{x})\, \tanh \mu(x-\bar{x})\right)\right] .$$

$$\exp(-2i\xi x)\left[1 - 2 \sum_{k=1}^{N} \frac{\gamma_k\, \psi_k \exp(-i\xi_k x)}{(\xi_k+\xi)}\right.$$

$$\left.\left. + \sum_{k=1}^{N} \sum_{l=1}^{N} \frac{\gamma_k\, \gamma_l\, \psi_k\, \psi_l \exp -i(\xi_k+\xi_l)x}{(\xi_k+\xi)(\xi_l+\xi)}\right]\right\}_{x=x_0} . \tag{5.31}$$

From equation (5.31) we find that $b(\xi,t)$ is not defined for $\xi=0$. Now we find out the contribution of continuous spectrum $\bar{\eta}_c(x,t)$ of this term to $\bar{\eta}(x,t)$ which can be computed from equation (5.24). We get,

$$\bar{\eta}_c(x,t) = - \frac{\epsilon}{\Pi} \int_{-\infty}^{\infty} d\xi .\ \frac{\exp i(8\xi^3 t+2\xi x)}{a^2(\xi)} \left[1 - \frac{2\bar{\eta}}{\bar{\eta}-i\xi}\ \frac{e^{2\theta}}{1+e^{2\theta}}\right]^2 .$$

$$\int dt . \left(\exp(-8i\xi^3 t) - 1\right) \left\{\left[\left(\frac{1}{2ik} - \frac{5}{12}\ \beta ik\right) .\right.\right.$$

$$- 4\mu^3\ \mathrm{sech}^2\ \mu(x-\bar{x}) .\ \tanh \mu(x-\bar{x}) .$$

$$+ \frac{10\alpha}{ik}\ \mu^5\ \mathrm{sech}^4\ \mu(x-\bar{x})\ \tanh \mu(x-\bar{x})$$

$$+ \frac{\beta}{3}\left(8\mu^4 \operatorname{sech}^2 \mu(x-\bar{x}) - 12\mu^4 \operatorname{sech}^4 \mu(x-\bar{x})\right)$$

$$+ \frac{\beta}{3ik}\left(- 16\mu^5 \operatorname{sech}^2 \mu(x-\bar{x}) \tanh \mu(x-\bar{x})\right.$$

$$\left.\left. + 48\mu^5 \operatorname{sech}^4 \mu(x-\bar{x}) \tanh \mu(x-\bar{x})\right)\right] .$$

$$\exp(-2i\xi x)\left[1 - 2 \sum_{k=1}^{N} \frac{\gamma_k \psi_k \exp -(i\xi_k x)}{(\xi_k+\xi)}\right.$$

$$\left.\left. + \sum_{k=1}^{N} \sum_{l=1}^{N} \frac{\gamma_k \gamma_l \psi_k \psi_l \exp -i(\xi_k+\xi_l)x}{(\xi_k+\xi)(\xi_l+\xi)}\right]\right\}_{x=x_0} . \tag{5.32}$$

Clearly this has non-vanishing contribution in the neighbourhood of $\xi=0$.

5.5 DISCUSSION

The equations of Kakutani (1971) and Johnson (1973b) has led to the study of a perturbed KdV equation as a model for diverse physical systems. Some of the major contributions in this field are due to Kaup and Newell (1978), Knickerbocker and Newell (1980) and Newell (1980), especially in the context of water waves.

The classical KdV equation is known to have an infinite number of conserved quantities. In the context of water waves this includes the conservation laws of mass and energy. The asymptotic solutions of perturbed KdV equation show that mass is not conserved which points out that the solution is nonuniform.

To circumvent this difficulty Kaup and Newell (1978) studied this problem using a different method. They exploited the fact that the classical KdV equation is exactly integrable, ie an infinite dimensional Hamiltonian system. The IST transform is a canonical transformation which carries the old co-ordinates (wavefunction) to the scattering data of the corresponding Schrodinger equation. Here the bound state eigenvalues are the action variables which prescribe the constant amplitude, shape and speed of the soliton; the normalization constants corresponds to angle variable and defines its position. The reflection coefficient measures the degree to which the continuous spectrum is excited.

In the case of the unperturbed system, reflection coefficient is identically zero. When the system is perturbed it is no more exactly separable. The normal modes become mixed so that an initial state consisting only of solitons can stimulate radiation and conversely the radiation can result in creation of new solitons. Thus the reflection coefficient is no more identically zero and the system doesn't have an infinite number of conservation laws.

The physical argument given by Kaup and Newell is as follows: KdV soliton has only one parameter, its amplitude, to adjust as it faces a perturbation. So it cannot simultaneously satisfy conservation of mass and energy flux. It chooses to satisfy the latter. The failure to satisfy conservation of mass flux means that another solution component must be excited to preserve the mass balance. The physical manifestation of this is the creation of a shelf and mathematically it means that the reflection coefficient develops a Dirac delta function behaviour.

In this chapter we have studied equation (1.15) as a perturbed KdV equation. Following Kaup and Newell (1978), we have formulated the problem as a Zakharov-Shabat system (Zakharov and Shabat, 1972). Our study here shows the excitation of continuous spectrum and the evolution of new solitons. We have also considered an example in which the excitation of continuous spectrum by a one-soliton as it passes the shelf is pointed out.

CHAPTER 6

CONCLUSIONS

The general approach throughout the thesis is treating the KdVB equation (1.15) as a perturbed KdV equation. Though a rich literature exists in the case of KdV equation, not much works have appeared about KdVB equation.

As pointed out in chapter 1, equation (1.15) gives rise to two equations of KdV type for the two domains to the left and right of the position of shelf. In chapter 2, we have considered the two equations to be defined in the whole domain $-\infty < x < \infty$ and treated the second equation as a perturbation of the first one. In the next chapter we have considered the two equations defined in two domains and studied the solution numerically.

The studies in chapters 2 and 3 involves only nonlinearity and dispersion. To take into account the effect of dissipation in chapter 4, an additional term is considered and studied the case when dissipation dominates. In chapter 5, we have studied equation (1.15) again treating it as a perturbed KdV equation.

It is well known that energy exchange among different wave modes is significant only when three-wave interaction is possible. When we consider only an ensemble of short waves, the total energy is conserved by exchange of energy between different wave modes in the case of three-wave interaction. In our case the total energy of short wave components is conserved as a result of transfer of energy between short wave components and the interacting long wave. Mathematically the study of interaction between short waves and long wave is justifiable as KdV equation is known to be dispersive and the dispersion relation is that of the corresponding linearised equation. The study is also significant in the case of a perturbed KdV equation as the equation is known to give rise to oscillatory tails.

The results of chapter 3 points towards the need for considering the equation (1.15) in detail. The time-independent Schrodinger equations corresponding to unperturbed and perturbed KdV equation show the enlargement of spectrum as the wave crosses the shelf. Numerical computations show that mass and energy are not conserved. It is possible only if we take into account the physical phenomena occurring as the wave crosses the shelf.

To consider energy dissipation we have used a KdVB equation with a term $\frac{1}{3}\epsilon\beta\eta_{xx}$. Here the coefficient $\frac{1}{3}\epsilon\beta>0$ so that this corresponds to addition of energy and not dissipation. This is justifiable physically also since the phenomena occurring at the shelf is not fully known. The dispersion coefficient is seen to enter nonlinearly in the asymptotic solution.

The perturbation method due to Kaup and Newell (1978) is used in chapter 5, to study the equation (1.15). We see that the continuous spectrum is excited. This can lead to new solitons arising due to the shelf. This result has been obtained by Kaup and Newell also and they have drawn the conclusion that a perturbed KdV equation can conserve mass only if a shelf of elevation is created. In this context it is to be noted that a perturbed KdV equation is no more integrable and need not have conserved quantities like mass and energy. The creation of shelf is a result of failure to conserve mass.

KdV equation is completely integrable and belongs to a class of nonlinear partial differential equations exactly solvable by IST method. Long wave propagation on uneven bottom leads to perturbed KdV equations which are not completely integrable and cannot be solved by IST method. Solution of such a system is an open problem.

REFERENCES

ABLOWITZ, M. J. AND NEWELL, A. C. 1973. The decay of the continuous spectrum for solutions of the Korteweg-de Vries equation. *J. Math. Phys.*, 14, 9, 1277-1284.

ABLOWITZ, M. J., KAUP, D. J., NEWELL, A. C. AND SEGUR, H. 1973. Method for solving Sine-Gordon equation. *Phys. Rev. Lett.*, 30, 25, 1262-1264.

ABLOWITZ, M. J., KAUP, D. J., NEWELL, A. C. AND SEGUR, H. 1974. The inverse scattering transform-Fourier analysis for nonlinear problems. *Stud. Appl. Math.*, 53, 249-315.

AMICK, C. J. AND TOLAND, J. F. 1981. On solitary water-waves of finite amplitude. *Arch. Rational Mech. Anal.*, 76, 1, 9-95.

BALL, K. 1964. Energy transfer between external and internal gravity waves. *J. Fluid Mech.*, 19, 465-478.

BAMPI, F. AND MORRO, A. 1979. Korteweg-de Vries equation and nonlinear waves. *Lett. Nuovo Cim.*, 26, 2, 61-63.

BATEMAN, H. 1915. Some recent researches on the motion of fluids. *Monthly Weather Review*, 43, 163-170.

BENJAMIN, T. B. 1972. The stability of solitary waves. *Proc. Roy. Soc. Lond.*, A, 328, 153-183.

BENJAMIN, T. B. 1974. Lectures on nonlinear waves. *Nonlinear Wave Motion Proc. Summer Sem.* Potsdam (New York) 1972.

Lectures in Applied Math., 15, American Math. Soc. Providence, R. I. 3-47.

BENJAMIN, T. B., BONA, J. L. AND MAHONY, J. J. 1972. Model equations for long waves in nonlinear dispersive systems. *Phil. Trans. Roy. Soc. Lond.*, A. 272, 47-78.

BENNEY, D. J. 1962. Non-linear gravity wave interactions. *J. Fluid Mech.*, 14, 577-584.

BENNEY, D. J. 1976. Significant interaction between small and large scale surface waves. *Stud. Appl. Math.*, 55, 93-106.

BENNEY, D. J. 1977. A general theory for interactions between short and long waves. *Stud. Appl. Math.*, 56, 81-94.

BERRYMAN, J. G. 1976. Stability of solitary waves in shallow water. *Phys. Fluids*. 19, 6, 771-777.

BONA, J. 1975. On the stability theory of solitary waves. *Proc. Roy. Soc. Lond.*, A, 344, 1638, 363-374.

BONA, J. L. 1983. The Korteweg-de Vries equation posed in a quarter plane. *SIAM. J. Math. Anal.*, 14, 6, 1050-1106.

BONA, J. L. AND BRYANT, P. J. 1973. A mathematical model for long waves generated by wave makers in nonlinear dispersive systems. *Proc. Camb. Phil. Soc.*, 73, 391-405.

BONA, J. L. AND DOUGALIS, V. A. 1980. An initial and boundary value problem for a model equation for propagation of long waves. *J. Math. Anal. Appl.*, 75, 503-522.

BONA, J. L. AND SACHS, R. L. 1989. The existence of internal solitary waves in a two-fluid system near the KdV limit. *Geophys. Astrophys. Fluid Dynamics*, 48, 1-3, 25-51.

BONA, J. L. AND SMITH, R. 1975. The initial value problem for the Korteweg-de Vries equation. *Phil. Trans. Roy. Soc. Lond.*, A. 278, 555-604.

BONA, J. L. AND SMITH, R. 1976. A model for the two-way propagation of water waves in a channel. *Math. Proc. Camp. Phil. Soc.*, 79, 167-182.

BOUSSINESQ, J. 1872. The'orie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide contenu dans ce canal des vitesses sensiblement paveilles de la surface au fond. *J. Math. Pures Appl.*, 2, 17, 55-108.

BURGERS, J. M. 1948. A Mathematical model illustrating the theory of turbulence. *Adv. Appl. Mech.*, 1, 171-199.

BYATT-SMITH, J. G. B. 1988. The reflection of a solitary wave by a vertical wall. *J. Fluid Mech.*, 197, 503-512.

BYATT-SMITH, J. G. B. 1989. The head-on interaction of two solitary waves of unequal amplitude. *J. Fluid Mech.* 205, 573-579.

BYATT-SMITH, J. G. B. 1992. Solutions of the perturbed Korteweg-de Vries equation. *Nonlinear dispersive wave systems*. ed. Lokenath Debnath, World Scientific, Singapore, 157-179.

BYATT-SMITH, J. G. B. AND LONGUET-HIGGINS, M. S. 1976. On the speed and profile of steep solitary waves. *Proc. Roy. Soc. Lond.*, A. 350, 175-189.

CALOGERO, F. AND DEGASPERIS, A. 1980. Nonlinear evolution equations solvable by the inverse spectral transform associated with the matrix Schrodinger equation. In *Topics in current physics: Solitons.* eds. Bullough, R. K. and Caudrey, P. J., Springer-Verlag, Berlin.

CANOSA, J. AND GAZDAG, J. 1977. The Korteweg-de Vries-Burgers' equation. *J. Comput. Phys.*, 23, 393-403.

CERCIGANANI, C. 1977. Solitons: Theory and application. *Riv. Nuovo. Cim.*, 2, 7(4), 429-469.

COLE, J. D. 1951. On a quasi-linear parabolic equation occurring in aerodynamics. *Quart. Appl. Math.*, 9, 225-236.

DAI, S. Q. 1982. Solitary waves at the interface of a two-layer fluid. *Appl. Math. Mech.*, 3, 6, 777-788.

DAI, S. Q. 1983. Head-on collisions between two interfacial solitary waves. *Acta. Mech. Sinica*, 6, 623-632.

DAI, H. H. AND JEFFREY, A. 1989a. Reflection of interface solitary waves at a slope. *Wave Motion*, 11, 463-479.

DAI, H. H. AND JEFFREY, A. 1989b. The inverse scattering transforms for certain types of variable coefficient Korteweg-de Vries equations. *Phys. Lett.*, A, 139, 8, 369-372.

DUBROVIN, V. A., MATVEEV, V. B. AND NOVIKOV, S. P. 1976. Nonlinear equations of Korteweg-de Vries type, finite-zone linear operators and varieties. *Russ. Math. Surveys*, 31, 59-146.

FENTON, J. D. AND RIENECKER, M. M. 1982. A Fourier method for solving nonlinear water-wave problems: application to solitary-wave interactions. *J. Fluid Mech.*, 118, 411-443.

FERMI, E., PASTA, J. AND ULAM, S. 1955. *Studies of nonlinear problems.* Vol. 1 Technical Report LA-1940, Los Alamos Sci. Lab., also in Collected works of E. Fermi, Vol. 2., Chicago: Univ. Chicago Press, 1965, 978-988.

FERMI, E., PASTA, J. AND ULAM, S. 1974. Studies of nonlinear problems I. *Nonlinear Wave Motion, Lectures Applied Mathematics*. Vol. 15 ed. A. C. Newell 143-156. *Amer. Math. Soc.*, Providence R. I.

FLASCHKA, H. AND NEWELL, A. C. 1975. Integrable systems of nonlinear evolution equations. In *Dynamical systems, Theory and Application,* Lecture notes in physics. 38, ed. J. Moser, Springer-Verlag, Berlin.

FREEMAN, N. C. AND JOHNSON, R. S. 1970. Shallow water waves on shear flows. *J. Fluid Mech.*, 44, 195-208.

FRIEDRICHS, K. O. AND HYERS, D. H. 1954. The existence of solitary waves. *Commu. Pure Appl. Math.*, 7, 517-550.

FUNAKOSHI, M. AND OIKAWA, M. 1982. A numerical study on the reflection of a solitary wave in shallow water. *J. Phys. Soc. Japan*, 51, 3, 1018-1023.

GABOV, S. A. 1989. Shallow floating water and the KdV equation. (Russian). *Vestnik Moskow. Univ. Ser. III Fiz. Astronom*, 30, 2, 9-13.

GARDNER, C. S. AND MORIKAWA, G. K. 1960. Similarity in the asymptotic behaviour of collision-free hydromagnetic waves and water waves. *New York Univ. Courant Inst. Math. Sci. Res. Rep., NYO-* 9082, 1-30.

GARDNER, C. S. AND MORIKAWA, G. K. 1965. *Commu. pure and Appl. Math.* 18, 35.

GARDNER, C. S., GREENE, J. M., KRUSKAL, M. D. AND MIURA, R. M. 1967. Method for solving the Korteweg-de Vries equation. *Phys. Rev. Lett.*, 19, 1095-1097.

GARDNER, C. S., GREENE, J. M., KRUSKAL, M. D. AND MIURA, R. M. 1974. Korteweg-de Vries equation and generalizations. V1. *Comm. Pure Appl. Math.,* 27, 97-133.

GEAR, J. A. 1985. Strong interactions between solitary waves belonging to different wave modes. *Stud. Appl. Math.*, 72, 2, 95-124.

GEAR, J. A. AND GRIMSHAW, R. 1983. A second order theory for solitary waves in shallow fluids. *Phys. Fluids*, 26, 14-29.

GIBBON, J. D., RADMORE, P., TABOR, M. AND WOOD, D. 1985. The

Painleve and Hirota's method. *Stud. Appl. Math.*, 72, 39-63.

GRAD, H. AND HU, P. N. 1967. Unified shock profile in a plasma. *Phys. Fluids*, 10, 12, 2596-2602.

GRIMSHAW, R. 1970. The solitary wave in water of variable depth. *J. Fluid Mech.*, 42, 639-656.

GRIMSHAW, R. 1971. The solitary wave in water of variable depth. Part 2. *J. Fluid Mech.*, 46, 611-622.

GRIMSHAW, R. 1983. Solitary waves in density stratified fluids. *Nonlinear deformation waves.* (Tallinn 1982). 431-447. Springer-Verlag, Berlin.

GRIMSHAW, R. 1992. Nonlinear waves in fluids-the KdV paradigm. *Nonlinear dynamics and chaos.* (Canberra, 1991), 175-198. World Sci. Publishing, River Edge, New Jersy.

HAMMACK, J. L. 1973. A note on tsunamis: their generation and propagation in an ocean of uniform depth. *J. Fluid Mech.*, 60, 769-799.

HAMMACK, J. L. AND SEGUR, H. 1974. The Korteweg-de Vries equation and water waves. Part 2. Comparison with experiments. *J. Fluid. Mech.*, 65, 289-314.

HASSELMANN, K. 1960. *Schiffestechnik.* 7, pp 191.

HASSELMANN, K. 1962. On the nonlinear energy transfer in a gravity wave spectrum. *J. Fluid. Mech.*, 12, 1, 481-500.

HASSELMANN, K. 1967. Nonlinear interactions treated by the methods of theoretical physics (with application to the generation of waves by wind). *Proc. Roy. Soc. Lond.*, A. 299, 77-100.

HIROTA, R. 1980. Direct methods in soliton theory. In *Topics in current physics: Solitons.* eds. Bullough, R. K. and Caudrey, P. J. Springer-Verlag, Berlin.

HIROTA, R AND SATSUMA, J. 1976. *Prog. Theor. Phys. Suppl.*, 59.

HOPF, E. 1950. The partial differential equation $U_t+UU_x=\mu U_{xx}$. *Comm. Pure Appl. Math.*, 3, 201-230.

HUANG, G. X., LUO, S. Y. AND DAI, X. X. 1989. Exact and explicit solitary wave solutions to a model equation for water waves. *Phys. Lett.*, A, 139, 8, 373-374.

IPPEN, A. T. AND KULIN, G. 1970. Hydrodynamics. *Lamb. Tech. Rep.* 15.

ISKANDAR, L. 1989. New numerical solution of the Korteweg-de Vries equation. *Appl. Numer. Math.*, 5, 3, 215-221.

JEFFREY, A. 1979. Some aspects of the mathematical modeling of long nonlinear waves. *Arch. of Mech.*, 31, 4, 559-574.

JEFFREY, A. AND DAI, H. H. 1988. A variable coefficient version of Zakharov and Shabat's method: with applications to the integration of variable coefficient nonlinear equations. In *Proc. EUROCHEM Colloq.* 241: Nonlinear waves in active media. Tallinn, Estonia, Springer-Verlag.

JEFFREY, A. AND KAKUTANI, T. 1972. Weak nonlinear dispersive waves: A discussion centered around the Korteweg-de

Vries equation. *SIAM Review*, 14, 4, 582-643.

JEFFREY, A. AND KAWAHARA, T. 1981. A note on the multiple scale Fourier transform. *Nonlinear analysis, theory, methods and applications*, 5, 12, 1331-1340.

JEFFREY, A AND KAWAHARA, T. 1982. *Asymptotic methods in nonlinear wave theory.* Boston-London, Pitman advanced publishing programe, Melbourne.

JEFFREY, A. AND MOHAMAD, M. N. B. 1991. Exact solutions to the Korteweg-de Vries-Burgers' equation. *Wave Motion*, 14, 369-375.

JEFFREY, A. AND XU, S. 1989. Exact solutions to the Korteweg-de Vries-Burgers' equation. *Wave Motion*, 11, 559-564.

JOHNSON, R. S. 1970. A nonlinear equation incorporating damping and dispersion. *J. Fluid Mech.*, 42, 1, 49-60.

JOHNSON, R. S. 1972. Some numerical solutions of a variable coefficient Korteweg-de Vries equation (with applications to solitary wave development on a shelf). *J. Fluid Mech.*, 54, 1, 81-91.

JOHNSON, R. S. 1973a. On the development of solitary wave moving over an uneven bottom. *Proc. Camb. Phil. Soc.*, 73, 183-203.

JOHNSON, R. S. 1973b. Asymptotic solution of the Korteweg-de Vries equation with slowly varying coefficients. *J. Fluid Mech.*, 60, 4, 813-824.

JOHNSON, R. S. 1983. On the phase shift due to the interaction of a large and a small solitary wave. *Phys. Lett.*, A. 94, 1, 7-11.

KAKUTANI, T. 1971. Effect of an uneven bottom on gravity waves. *J. Phys. Soc. Japan*, 30, 272-276.

KAKUTANI, T. AND MICHIHIRO, K. 1976. Nonlinear modulation of stationary water waves. *J. Phys. Soc. Japan*, 41, 5, 1792-1799.

KALYAKIN, L. A. (1991). Asymptotics of an integral that arises in the perturbation theory of KdV solitons. *Math. Notes*, 50, 5-6, 1114-1122.

KAUP, D. J. AND NEWELL, A. C. 1978. Solitons as particles, oscillators, and in slowly changing media: a singular perturbation theory. *Proc. Roy. Soc. Lond.*, A. 361, 413-446.

KAWAHARA, T. 1973. The derivative expansion method and nonlinear dispersive waves. *J. Phys. Soc. Japan*, 35, 5, 1537-1544.

KAWAHARA, T. 1975a. Non-linear self-modulation of capillary-gravity waves on liquid layer. *J. Phys. Soc. Japan*, 38, 1, 265-270.

KAWAHARA, T. 1975b. Derivative expansion method for nonlinear waves on a liquid layer of slowly varying depth. *J. Phys. Soc. Japan*, 38, 4, 1200-1206.

KAWAHARA, T. AND JEFFREY, A. 1979. Asymptotic dynamical equations for a ensemble of nonlinear dispersive waves. *Wave Motion*, 1, 83-94.

KAWAHARA, T., SUGIMOTO, N. AND KAKUTANI, T. 1975. Nonlinear interaction between short and long capillary gravity waves. *J. Phys. Soc. Japan*, 39, 5, 1379-1386.

KEULEGAN, G. H. AND PATTERSON, G. W. 1940. Mathematical theory of irrotational translation waves. *U. S. Nat. Bureau of Standards. J. Res.*, 24, 47.

KEVER, H. AND MORIKAWA, G. K. 1969. Korteweg-de Vries equation for nonlinear hydromagnetic waves in a warm collision free plasma. *Phys. Fluids*, 12, 2090-2093.

KIVSHAR, Y. S. AND BORIS, M. A. 1989. Solitons in a system of coupled Korteweg-de Vries equations. *Wave Motion*, 11, 3, 261-269.

KNICKERBOCKER, C. J. AND NEWELL, A. C. 1980. Shelves and the Korteweg-de Vries equation. *J. Fluid Mech.*, 98, 804-818.

KOOP, C. G. AND BUTLER, G. 1981. An investigation of internal solitary waves in a two-fluid system. *J. Fluid Mech.*, 112, 225-251.

KOREBEINIKOV, V. P. 1983. Some exact solutions of Korteweg-de Vries-Burgers' equation for plane, cylindrical and spherical waves. *Nonlinear deformation waves* (Tallinn 1982), 149-154, Springer-Verlag, Berlin.

KORTEWEG, D. J. AND DE VRIES, G. 1895. On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. *Philos. Mag.*, 39, 422-443.

LAMB, H. 1932. *Hydromagnetics.* 6th ed. Cambridge University press.

LAVRENTIEF, M. A. 1954. On the theory of long waves and a contribution to the theory of long waves. *Amer. Math. Soc. Transl.*, 102, American Math. Soc. Prov. R. I.

LAX, P. D. 1968. Integrals of nonlinear equations of evolution and solitary waves. *Comm. Pure Appl. Math.*, 21, 467-490.

LAX, P. D. 1976. Almost periodic solutions of the Korteweg-de Vries equation. *SIAM Review,* 18, 351-375.

LEIBOVICH, S. AND RANDALL, J. D. 1973. Amplification and decay of long nonlinear waves. *J. Fluid Mech.*, 58, 481-493.

LITVAK, M. M. 1960. A transport equation for magnetohydrodynamic waves. *AVCO-ERERTT Res. Lab. Res. Rep.*, 92.

LONGUET-HIGGINS, M. S. 1962. Resonant interactions between two train of gravity waves. *J. Fluid Mech.*, 12, 321-332.

LONGUET-HIGGINS, M. S. 1974. On the mass, momentum, energy and circulation of a solitary wave. *Proc. Roy. Soc. Lond.*, A, 337, 1-13.

LONGUET-HIGGINS, M. S. AND FENTON, J. D. 1974. On the mass, momentum, energy and circulation of a solitary wave. II

Proc. Roy. Soc. Lond., A. 340, 471-493.

LONGUET-HIGGINS, M. S. AND FOX, M. J. H. 1977. Theory of the almost highest wave: the inner solution. *J. Fluid Mech.*, 80, 721-742.

LONGUET-HIGGINS, M. S. AND FOX, M. J. H. 1978. Theory of the almost highest wave. Part 2. Matching and analytic extension. *J. Fluid Mech.*, 85, 4, 769-786.

MADSEN, O. S. AND MEI, C. C. 1969. The transformation of a solitary wave over an uneven bottom. *J. Fluid Mech.*, 39, 781-791.

MADSEN, O. S., MEI, C. C. AND SAVAGE, R. P. 1970. The evolution of time-periodic long waves of finite amplitude. *J. Fluid Mech.*, **44**, 195-208.

MAKHANKOV, V. G. 1978. Dynamics of classical solitons (in non-integrable systems). *Phys. Rep.*, 35, 1-128.

MAXWORTHY, T. 1976. Experiments on collisions between solitary waves. *J. Fluid Mech.*, 76, 177-185.

McGOLDRICK, L. F. 1965. Resonant interactions among capillary-gravity waves. *J. Fluid Mech.*, 21, 305-332.

McKEAN, H. P. AND Van MOERBEKE. 1975. The spectrum of Hill's equation. *Invent Math.*, 30, p 217.

MEINHOLD, P. 1991. On soliton solutions of the Korteweg-de Vries equation. *Wiss. Z. Tech. Univ. Dresden*, **40**, 165-167.

MELKONIAN. 1989. Nonlinear waves on thin films. *Continuum* *mechanics and its applications*. (Burnably, BC, 1988). 411-423. Hemisphere, New York.

MEL'NIKOV, V. K. 1990. Creation and annihilation of solitons in the system described by the Korteweg-de Vries equation with a self consistant source. *Inverse Problems*, 6, 809-823.

MILES, J. W. 1977a. Obliquely interacting solitary wave. *J. Fluid Mech.*, 79, 157-169.

MILES, J. W. 1977b. Resonantly interacting solitary waves. *J. Fluid Mech.*, 79, 1, 171-179.

MILES, J. W. 1980. Solitary waves. *Ann. Rev. Fluid Mech.*, 12, 11-43.

MILES, J. W. 1981. The Korteweg-de Vries equation: a historical essay. *J. Fluid Mech.*, 106, 131-147.

MIRIE, R. M. AND SU, C. H. 1982. Collisions between two solitary waves, Part 2 A numerical study. *J. Fluid Mech.*, 115, 475-492.

MIRIE, R. M. AND SU, C. H. 1984. Internal solitary waves and their head-on collision, Part I. *J. Fluid Mech.*, 147, 213-231.

MIRIE, R. M. AND SU, C. H. 1986. Internal solitary waves and their head-on collision, Part II. *Phys.Fluids*, 29, 1, 31-37.

MIURA, R. M. 1974. The Korteweg-de Vries equation*: A model equation for nonlinear dispersive waves*. eds.

Leibovich, S. and Seebass, A. R. Cornell University press, Ithaca, New York. 212-234.

MIURA, R. M. 1976. The Korteweg-de Vries equation: A survey of results. *SIAM Review*, 18, 3, 412-458.

MIURA, R. M., GARDNER, C. S. AND KRUSKAL, M. D. 1968. Korteweg-de Vries equation and generalizations. II Existence of conservation laws and constants of motion. *J. Math. Phys.*, 9, 1204-1209.

NAUMKIN, P. I.AND SHISHMAREV, I. A. 1990. Nonlinear nonlocal equations in wave theory II. A system of equations of surface waves. Asymptotics of dissipative equations. *Vestnik Moskov Univ. Ser. II Fiz. Astronom*, 31, 6, 3-17.

NEWELL, A. C. 1980. The inverse scattering transform. In *Topics in modern physics. Solitons.* eds. R. Bullough and P. Caudrey, Springer-Verlag, Berlin.

NEWELL, A. C. 1983. The history of the soliton. *J. Appl. Mech.*, 105, 1127-1138.

NIRMALA, N. AND VEDAN, M. J. 1990. Wave interaction on water of variable depth. *J. Math. Phy. Sci.*, 24, 2, 107-114.

NIRMALA, N., VEDAN, M. J. AND BABY, B. V. 1986a. Auto-Backlund transformation, Lax pairs and Painleve property of a variable coefficient Korteweg-de Vries equation I. *J. Math. Phys.*, 27, 2640-2643.

NIRMALA, N., VEDAN, M. J. AND BABY, B. V. 1986b. A variable coefficient Korteweg-de Vries equation: Similarity analysis and exact solution. *J. Math. Phys.*, 27, 2644-2646.

NISHIKAWA, K., HOJO, H., MIMA, K. AND IKEZI, H. 1974. Coupled nonlinear electron-plasma and ion-acoustic waves. *Phys. Rev. Lett.*, 33, 148-151.

NOVIKOV, S. P. 1974. The periodic problem for the Korteweg-de Vries equation. *Funct. Anal. Appl.*, 8, 236-246.

OIKAWA, M. AND YAJIMA, N. 1973. Interactions of solitary waves - a perturbation approach to nonlinear systems. *J. Phys. Soc. Japan*, 34, 1093-1099.

OTT, E. AND SUDAN, R. N. 1970. Damping of solitary waves. *Phys. Fluids*, 13, 6, 1432-1434.

PEIERLS, R. 1929. *Ann. Phys.*, 3, 1055.

PEREGRINE, D. H. 1966. Calculations of the development of an undular bore. *J. Fluid Mech.*, 25, 321-330.

PEREGRINE, D. H. 1967. Long waves on a beach. *J. Fluid Mech.*, 27, 815-827.

PHILLIPS, O. M. 1960. On the dynamics of unsteady gravity waves of finite amplitude. *J. Fluid Mech.*, 9, 1, 193-217.

PHILLIPS, O. M. 1977. *The dynamics of the upper ocean.* second edition. Cambridge University press, Cambridge.

PRAMOD, K. V., NIRMALA, N., VEDAN, M. J. AND SAKOVICH, S. YU. 1989. Integrability study and numerical analysis of a

Korteweg-de Vries equation with variable coefficients. *Int. J. Nonlinear Mech.*, 24, 5, 431-439.

PRAMOD, K. V. AND VEDAN, M. J. 1992. Long-wave propagation in water with bottom discontinuity. *Int. J. Non-Linear Mech.*, 27, 197-201.

RAYLEIGH LORD, 1876. On waves. *Phil. Mag.*, 5, 1, 257-279.

SACHS, R. L. 1984. A justification of the Korteweg-de Vries approximation to first order in the case of N-soliton water waves in a canal. *SIAM. J. Math. Anal.*, 15, 3, 468-489.

SANDER, J. AND HUTTER, K. 1991. On the development of the theory of the solitary wave. A historical essay. *Acta. Mech.*, 86, 1-4, 111-152.

SCOTT RUSSEL, L. 1844. Report on waves. In *Rep. 14th meeting of the British Assoc. for the Advancement of Science,* London: John Murray, pp. 311-390 + 57 plates.

SCOTT RUSSEL, J. 1845. The wave of translation: London.

SCOTT, A. C., CHU, F. Y. F. AND McLAUGHLIN, D. W. 1973. The soliton: A new concept in applied science. *Proc. IEEE.*, 61, 10, 1443-1483.

SEGUR, H. AND HAMMACK, J. L. 1982. Soliton models for long internal waves. *J. Fluid Mech.*, 118, 285-304.

SHEN, M. C. AND ZHONG, X. C. 1981. Derivation of KdV equations for water waves in a channel with variable cross-section. *J. Mecanique,* 20, 4, 789-801.

SHINBROT, M. C. 1981/1982. The solitary wave with surface tension. *Quart. Appl. Math.*, 39, 2, 287-291.

SHU, J. J. 1987. The proper analytical solution of the Korteweg-de Vries-Burgers' equation. *J. Phys.*, A, 20, 2, 49-56.

SOBEZYK, K. 1992. KdV solitons in a randomly varying medium. *Internat. J. Non-Linear Mech.*, 27, 1, 1-8.

SOUGANIDIS, P. E. AND STRAUSS, W. A. 1990. Instability of a class of dispersive solitary waves. *Proc. Roy. Soc. Edinburgh.* A, 114, 3-4, 195-212.

STOKER, J. J. 1957. *Water waves.* The mathematical theory with applications. Interscience Publishers, New York.

STOKES, G. G. 1847. On the theory of oscillatory waves. *Trans. Camb. Phil. Soc.*, 8, 441-455.

SU, C. H. AND GARDNER, C. S. 1969. Korteweg-de Vries equation and generalizations. III Derivation of the Korteweg-de Vries equation and Burgers' equation. *J. Math. Phys.*, 10, 3, 536-539.

SU, C. H. AND MIRIE, R. M. 1980. On head-on collisions between two solitary waves. *J. Fluid Mech.*, 98, 509-525.

THORPE, S. A. 1966. On wave interactions on a stratified fluid. *J. Fluid Mech.*, 24, 737-752.

URSELL, F. 1953. The long wave paradox in the theory of gravity waves. *Proc. Camb. Phil. Soc.*, 49, 685-694.

VANDEN-BROECK, JEAN-MARC AND SHEN, M. C. 1983. A note on solitary and cnoidal waves with surface tension. *Z. Angew. Math. Phys.*, 34, 1, 112-117.

VLIEG, H. M. AND HALFORD, W. D. 1991. The Korteweg-de Vries-Burgers' equation: a reconstruction of exact solutions. *Wave Motion*, 14, 3, 267-271.

VLIEGENTHART, A. C. 1971. On finite-difference methods for the Korteweg-de Vries equation. *J. Engng. Math.*, 5, 137-155

WADATI, M. 1980. Generalized matrix form of the inverse scattering method: In *Topics in current physics: Solitons.* 17, eds. Bullough, R. K. and Caudrey, P. J. Springer-Verlag, Berlin.

WEINSTEIN, A. 1926. Sur la Vitesse de propagation de l'onde solitaire. *Accad. Naz. Lincei, Cl. Sci. Fis., Mat. Nat. Rendiconti*, 6, 3, 463-468

WEISS, J., TABOR, M. AND CARNEVALE, G. 1983. The Painleve property for partial differential equations. *J. Math. Phys.*, 24, 522-526.

WHITHAM, G. B. 1974. *Linear and Nonlinear Waves.* Wiley-Interscience, New York, 636 pp.

WU, H. MO. AND GUO, B.YU. 1983. High order accurate difference schemes for the Korteweg-de Vries, Burgers' and RLW equations. *Math. Numer. Sinica*, 5, 1, 90-98.

ZABUSKY, N. J. 1967. A synergetic approach to problems of nonlinear dispersive wave propagation and interaction. In *Nonlinear Partial Differential Equations.* ed. W.

Ames, New York, Academic Press. 223-258.

ZABUSKY, N. J. 1968. Solitons and bound states of the time-independent Schrodinger equation. *Phys. Rev.*, 168, 124-128.

ZABUSKY, N. J. 1969. Nonlinear lattice dynamics and energy sharing. In *Proc. Int. Conf. on Statistical Mechanics*. 1968. Also *J. Phys. Soc. Japan*, 26, 196-202.

ZABUSKY, N. J. 1973. Solitons and energy transport in nonlinear lattices. *Comput. Phys. Commu.*, 5, 1-10.

ZABUSKY, N. J. AND GALVIN, C. J. 1971. Shallow-water waves, the Korteweg-de Vries equation and solitons. *J. Fluid Mech.*, 47, 4, 811-824.

ZABUSKY, N. J. AND KRUSKAL, M. D. 1965. Interaction of solitons in a collisionless plasma and the recurrence of initial states. *Phys. Rev. Lett.*, 15, 240-243.

ZAKHAROV, V. E. 1972. Collapse of Langmuir waves. *Soviet Phys. JETP.*, 35, 908-914.

ZAKHAROV, V. E. AND FADDEEV, L. D. 1972. Korteweg-de Vries equation: A completely integrable Hamiltonian system. *Funct. Anal. Appl.*, 5, 280-287.

ZAKHAROV, V. E. AND SHABAT, A. B. 1972. Exact theory of two-dimensional self-focussing and one-dimensional self-modulation of waves in nonlinear media. *Soviet Phys. JETP.*, 34, 1, 62-69.

ZAKHAROV, V. E. AND SHABAT, A. B. 1974. A scheme for integrating the nonlinear equations of mathematical physics by the method of the inverse scattering problem I. *Funct. Anal. Appl.*, 8, 226-235.

ZHOU, X. C. 1981. The solitary waves in a gradually varying channel of arbitrary cross-section. *Appl. Math. Mech.*, 2, 4, 429-440.

ZHOU, X. C. 1983. Nonlinear periodic wave and fission of a solitary wave in a slowly varying channel with arbitrary cross-section. *Sci. Sinica*, A, 26, 6, 626-636.

ZHOU, X. C. 1988. Fission of solitary wave in stratified fluid. *Sci. Sinica*, A, 31, 5, 551-562.