SOME RESULTS ON SETTING THE CLOCK BACK TO ZERO PROPERTY

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CERTIFICATE

Certified that the thesis entitled "**Some Results on Setting the Clock Back to Zero Property**" is a bonafide record of work done by Smt.L.Mini under my guidance in the Department of Statistics, Cochin University of Science and Technology and no part of it has been included anywhere previously for the award of any degree or title.

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CHAPTER 1

INTRODUCTION

In early days of the development of Statistics, normal distribution played an important role in statistical analysis. The discovery of the central limit theorem was one of the reasons for this. Later it was observed that the use of normal distribution is not the appropriate model in many real life situation. The deviation from normality was noticed and need of using non-normal distributions was keenly felt. It was found that in many cases exponential distribution serves as an alternative to normal, when the form of variation in the population is known and is far from normal. The popularity of the exponential distribution is mainly due to the well known non-ageing or memory less property. If we see that the random life length of a device is as fresh as a new one, after an elapsed time, we say that it is not ageing, but its life is terminated by a sudden event. The probabilistic interpretation is that the remaining life time is the same as the initial life time. Due to the applications in real life situation, this property plays a crucial role in theoretical as well as in applied work. In engineering problems exponential distribution is more suitable than other distributions. Hence it is widely used in reliability and life testing. By reliability we mean the capability or the failure free operation of a device or component or organism to perform its functions adequately for a specified period without failure. The concepts and tool in reliability have found applications in disciplines like biology, medicine, engineering, economics, demography, etc. One of the most important problems in reliability analysis is to identify the underlying model which generate the observation when only the data on failure times is available. Some innovations in this regard are to identify a characteristic property of the distribution that is of interest. From this point of view, we change our direction to a new property which is called Setting the Clock Back to Zero (SCBZ) property. In reliability this property ensures that the conditional distribution of the additional length of survival of the device given that it has survived certain time units x_0 is the same as the unconditional distribution except for a slight change in the parameters. Evidently this property will be an extension of the lack of memory property . The main objective of the present study is to investigate the importance and uses of the SCBZ property in the context of reliability. Even though the concept was introduced in 1990 by Rao and Talwalker, they did not give a look into the scope of characterizations of the family of distributions having this innovative concept. Also the study of this property in discrete distributions are not yet considered. Under this background, through the present study we try to investigate the SCBZ property of the continuous univariate distributions in more detail and to seek the importance and uses in the context of reliability and life testing.

This thesis is divided into five chapters. First one is the present introductory one. In the second chapter we have mentioned the Lack of Memory Property (LMP) and various other concepts used in the present study. In chapter 2 itself we have introduced the concept of SCBZ property defined by Rao and Talwalker (1990). Chapter 3 starts with an idea of developing certain characterizations which brings the family of continuous univariate distributions possessing this property under a uniform framework. After deriving a partial differential equation (PDE) for the class of distributions with this property, some of its important members which admit the PDE are identified and are tabled. By envisaging the application of this property in reliability, the equivalence of this concept in terms of the quantities designed to measure the ageing phenomenon are also considered. Cox (1962) has shown that the asymptotic distribution of the residual life is the equilibrium distribution. The comparison of these distributions with the parent distribution brings an idea about the ageing aspects. In this viewpoint we have done some work on equilibrium distribution and is presented in chapter 3.

In the same way as the extension of the LMP to the bivariate as well as the multivariate cases, there is a scope for extending the concept of SCBZ property to higher dimensions. As there is no unique way of extension, the different ways, particularly in the bivariate case, are discussed in chapter 4. Most of the characteristics considered in chapter 3 are amenable for the extensions to the bivariate case.

In many practical situations, we face the cases in which the life time is measured in discrete time units. Hence it is desirable to have a consideration about this property in the discrete distributions. In chapter 5 we try to introduce the concept of SCBZ in discrete situations with emphasis on reliability. Both the univariate and multivariate cases are considered here.

CHAPTER 2

LACK OF MEMORY PROPERTY AND ITS VARIANTS

2.1 Introduction

This chapter covers a review work of the essential ideas to be utilised for this thesis. We start with the well known lack of memory property and then its multivariate extensions. While carrying out an extensive study, it is seen that, there are several situations where the LMP is not satisfied but properties which are very near to LMP is satisfied. This was one of the motivations for extending the LMP. In a sense the SCBZ property is an extension of the LMP.

2.2 Lack of memory property and exponential distribution

Let us consider a device which was functioning for sometime. If its future performance does not depend on the past, knowing its present condition, we say that the device is having LMP. In proper mathematical terms if X is a non-negative random variable (r.v.)

possessing absolutely continuous distribution with respect to the Lebesgue measure, we say that the random variable X or its distribution has LMP if for all x, $y \geq 0$,

$$P(X \geq x+y \mid X \geq y) = P(X \geq x) \tag{2.1}$$

with $P(X \geq y)$ strictly greater than zero or equivalently.

$$P(X \geq x+y) = P(X \geq x)\, P(X \geq y) \tag{2.2}$$

for all x, $y \geq 0$ and $P(X=0) \neq 1$.

In theoretical and applied work this property plays a crucial role due to its application to real life situation. If $R(x)=P(X \geq x)$ denote the survival function of the random variable X, then in terms of R(.), the LMP can be stated as

$$R(x+y) = R(x)\, R(y) \tag{2.3}$$

Galambos and Kotz (1978) have established the equivalence of LMP, constancy of failure rate and constancy of mean residual life. Cauchy (1821) and Darboux (1875) have established that the unique non-zero solution of (2.3) is $R(x) = e^{-\lambda x}$ for some constant λ, which is the survival function of the exponential distribution. So exponential distribution is the only distribution having this property.

The lack of memory property can be extended either to widen the domain of the random variable or to provide a class of distributions in which the exponential is included. We can observe different types of extensions in various directions suggested by different authors. Some of them are listed here.

If (2.3) is satisfied for almost all $x,y \geq 0$ with respect to a Lebesgue measure, then Fortet (1977) had shown that this amounts to a characterization of the exponential distribution. Sethuraman (1965) have considered another relation of the domain of X, by considering finite induction as given below. The equation (2.3) can be written as

$$R(x_1+x_2+\ldots+x_n) = R(x_1) \ldots R(x_n) \qquad (2.4)$$

If $x_1=x_2=\ldots=x_n=x\geq 0$, (2.4) becomes

$$R(nx)=[R(x)]^n \qquad (2.5)$$

If (2.4) holds for any two integers n_1 and n_2 such that $\frac{\log n_1}{\log n_2}$ is irrational, then it characterizes the exponential distribution. Marsaglia and Tubilla (1975) showed that (2.3) is valid for two values y_1 and y_2 of y such that $0< y_1< y_2$ and y_1/y_2 is irrational for all $x>0$, then X follows a negative exponential.

Another attempt towards the extension is due to the functional form of the conditional expectation of a function specifically,

$$\mathrm{E}(h(X)|X\geq x) = g(x), \qquad x\geq 0 \qquad (2.6)$$

or

$$E(h(X-x)|X\geq x) = g(x), \qquad x\geq 0 \tag{2.7}$$

Here $h(.)$ and $g(.)$ are known functions ending up with the solutions that are proper survival functions. More details are available in Kotlarski (1972), Laurent (1972), Shanbhag and Rao (1975), Dallas (1976) and Gupta (1976).

Mulier and Scarsini (1981) made an extension in the following manner. In place of (2.2) they used the equation

$$P(X> x* y)= P(X> x)\, P(X> y), \tag{2.8}$$

'*' being taken as an associative and reducible binary operator. In this case (2.3) reduce the form

$$R(x*y) = R(x)R(y). \tag{2.9}$$

The unique solution of (2.9) is

$$x*y = g^{-1}(g(x)+g(y)) \tag{2.10}$$

with $g(.)$ being continuous and strictly monotonic. Chukova and Dimitrov (1992) introduced the concept of almost lack of memory property (ALMP). X is said to have the ALMP if there exists a sequence of distinct constants $\{a_n\}_{n=1}^{\infty}$ such that

$$P(X\geq b+x|X\geq b) = P(X\geq x) \tag{2.11}$$

holds for any $b=a_n$, $n=1,2,\ldots$ and for all $x\geq 0$.

Galambos and Kotz (1978) bring out the equation

$$P(X \le uv | X \le v) = P(X \le u),\ 0 \le u, 0 \le v \le 1 \tag{2.12}$$

to characterize the uniform distribution in [0,1] of X. Also the equation

$$P(X \ge uv | X \ge v) = P(X \ge u), \quad u,\ v > 1 \tag{2.13}$$

characterizes the Pareto distribution with survival function

$$R(x) = x^{-r},\ x \ge 1,\ r > 0. \tag{2.14}$$

(2.12) and (2.13) are called the multiplicative lack of memory property. Dimitrov and Collani (1995) introduced the multiplicative almost lack of memory property.

A random variable X is said to have the multiplicative almost lack of memory property of type 1 (MALMI) if there exists a sequence of numbers $\{v_n\}_{n=1}^{\infty}$, $0 \le v_n \le 1$, $v_n \ne v_m$ for n≠m such that

$$P(X \le uv_n | X \le v_n) = P(X \le u) \tag{2.15}$$

for all $u \ge 0$. A random variable X is said to have multiplicative almost lack of memory property of type 2 (MALM2) if there exists a sequence $\{v_n\}_{n=1}^{\infty}$, $v_n \ge 1$, $v_n \ge v_m$ for all $n \ne m$ such that

$$P(X \ge uv_n | X \ge v_n) = P(X \ge u) \tag{2.16}$$

for all $u \ge 1$.

A somewhat different approach to the extension was considered by Huang (1981). If X and Y are two independent non-negative random variables, we say that X is ageless relative to Y if $P(X>Y)>0$ and

$$P(X>Y+x|X>Y) = P(X>x) \tag{2.17}$$

for all $x \geq 0$.

2.3 Lack of memory property and geometric distribution

Let us consider LMP in a discrete set up. If X is a non-negative integer valued random variable satisfying the condition

$$P(X \geq x+y | X \geq y) = P(X \geq x)$$

then X follows a geometric distribution with

$$P(X=x) = p(1-p)^x,\ x = 0,1,2,\ldots;\ 0<p<1. \tag{2.18}$$

The geometric random variable is the only discrete random variable having the LMP.

The lack of memory property is equivalent to the constancy of failure rate and constancy of mean residual life function. The characterization of geometric distribution and discrete IFR (DFR) using order statistics are established in Neweichi and Govindarajulu (1979). Another characterizations of geometric distribution in terms of order statistics are studied in Arnold (1980) and Srivastava (1974). Nair and Hitha (1989) characterizes the discrete models using distributions on partial sums. The motivation behind this was the work of Xekalaki (1983).

2.4 Multivariate Extensions.

The concepts of LMP explained in sections 2.2 and 2.3 can be extended to higher dimensions. Let us first present some important variations in the bivariate cases.

Let (X_1,X_2) represent a bivariate random vector with support $\{(x_1,x_2) : x_1,x_2 \geq 0\}$ and the survival function $R(x_1,x_2)$. A natural extension of the LMP in the bivariate case is defined by

$$R(x_1+y_1,\ x_2+y_2) = R(x_1,x_2)R(y_1,y_2) \tag{2.19}$$

where

$$R(x_1,x_2) = \mathrm{P}(X_1 \geq x_1, X_2 \geq x_2).$$

The unique solution of (2.19) turns out to be

$$R(x_1,x_2) = e^{-\lambda_1 x_1 - \lambda_2 x_2} \tag{2.20}$$

which is the product of the marginals. If (X_1,X_2) is taken to be the lifetimes of a two component system, (2.20) shows that the life times of the components are independent, which does not have any relevance in life testing. To introduce the dependency of the component life times we consider the equation

$$R(x_1+t, x_2+t) = R(t,t)\ R(x_1,x_2) \tag{2.21}$$

for all $x_1,x_2,\ t \geq 0$.

Marshall and Olkin (1967) obtained the unique solution of (2.21) as

$$R(x_1,x_2) = \exp\{-\lambda_1 x_1-\lambda_2 x_2-\lambda_{12}\max(x_1,x_2)\}. \qquad (2.22)$$

Here also the marginal distributions are again exponential but they are not independent. Also $P(X_1=X_2)>0$. That is the failure timesof the two components can be equal or the distribution is not absolutely continuous. It can be observed that LMP, absolute continuity and exponential marginals cannot occur simultaneously except for bivariate distribution with independent exponential marginals. Block and Basu (1974) derived a bivariate exponential distribution preserving LMP and absolute continuity in which marginals are not purely exponential but are mixtures of exponentials.

Another way of studying the equipment behaviour is to investigate the behaviour of one of the components, when lifetime of the other is pre-assigned. The first work in this direction is due to Johnson and Kotz (1975) who defined the vector valued failure rate$(h_1(x_1,x_2),\ h_2(x_1,x_2))$ where

$$h_i(x_1,x_2) = \frac{-\partial \log R(x_1,x_2)}{\partial x_i},\quad i = 1,2 \qquad (2.23)$$

and considered the situation of $h_i(x_1,x_2) = c_i$, a constant. It can be seen that this situation exists when the distribution is the product of the independent exponential marginals. Hence they considered the situation of the local constancy of the failure rate vector. That is

$$h_i(x_1,x_2) = A_i(x_j), \quad i,j = 1,2;\ i \neq j. \tag{2.24}$$

(2.24) characterizes the Gumbel's (1960) bivariate exponential distribution with survival function

$$R(x_1,x_2) = \exp\{-\lambda_1 x_1 - \lambda_2 x_2 - \delta\, x_1 x_2\}, \tag{2.25}$$

$\lambda_1,\ \lambda_2 > 0,\ 0 \leq \delta \leq \lambda_1\lambda_2$.

Johnson and Kotz (1975) defined the local lack of memory property of the random vector (X_1,X_2) by the relations

$$P(X_i > x_i + y_i \mid X_1 > x_1,\ X_2 > x_2) = P(X_i > y_i \mid X_j > x_j),\ i,j = 1,2;\ i \neq j \tag{2.26}$$

or

$$\left.\begin{aligned} G_1(x_1+y_1,\ x_2) &= G_1(x_1,x_2)G_1(y_1,x_2) \\ \text{and} \quad & \\ G_2(x_1,\ x_2+y_2) &= G_2(x_1,x_2)G_2(x_1,y_2), \end{aligned}\right\} \tag{2.27}$$

for all $x_1,x_2,\ y_1,y_2 > 0$ where $G_i(x_1,x_2) = P(X_i > x_i \mid X_j > x_j),\ i,j = 1,2;\ i \neq j$.

The equivalence of (2.24) and (2.27) is given in Nair and Nair (1988b). Nair and Nair (1991) defined the notion of conditional lack of memory for a random vector in the support of $R_2^+=\{(x_1,x_2),\ x_1,x_2>0\}$ by

$$P(X_i \geq x_i+y_i | X_i \geq x_i,\ X_j=x_j) = P(X_i \geq y_i |\ X_j=x_j),\ i,j=1,2;\ i \neq j \qquad (2.28)$$

for all $x_i,\ y_i>0$ and proved that (2.28) holds iff the distribution is the bivariate exponential by Arnold and Strauss (1988) with joint probability density function (p.d.f.)

$$f(x_1,x_2) = \lambda_1\lambda_2\theta \exp\{-\lambda_1 x_1 - \lambda_2 x_2 - \delta\lambda_1\lambda_2 x_1 x_2\},\ \lambda_1,\lambda_2>0,\ \delta\geq 0,\ x_1,x_2>0 \qquad (2.29)$$

where $\theta = \theta(\delta) = \delta e^{-1/\delta} / -E_i(1/\delta)$ with $E_i(u) = -\int_u^\infty e^{-w} w^{-1} dw$

[E_i is the exponential integral].

Analogous to the various extensions of the lack of memory property in the bivariate continuous case, a similar approach can be taken in the discrete case also. The bivariate discrete Lack of Memory Property in the usual case is

$$P(X_1 \geq x_1+t, X_2 \geq x_2+t \mid X_1 \geq t, X_2 \geq t) = P(X_1 \geq x_1,\ X_2 \geq x_2), \qquad (2.30)$$

$x_1,\ x_2, t = 0,1,2,\ldots$.

In Nair and Asha (1994), it is shown that the relation (2.30) holds if and only if

$$R(x_1,x_2) = \begin{cases} p^{x_2} p_1^{x_1-x_2} & x_1 \geq x_2 \\ p^{x_1} p_2^{x_2-x_1} & x_1 \leq x_2 \end{cases} \tag{2.31}$$

$1+p \geq p_1+p_2$; $0<p \leq p_j<1$, $p = p_1+p_2-c_3-1$ with $c_3 = \dfrac{P(X_1 = x_1, X_2 = x_2)}{P(X_1 \geq x_1, X_2 = x_2)}$.

The local LMP

$$P(X_i \geq x_i+t \mid X_1 \geq x_1, X_2 \geq x_2) = P(X_i \geq t \mid X_j \geq x_j) \tag{2.32}$$

for all x_1, x_2, $t = 0,1,2, \ldots$ is characteristic of the bivariate geometric distribution with survival function given in Nair and Nair (1988a). The conditional LMP in the discrete case is studied by Nair and Nair (1991). For multivariate results both in the continuous and discrete vectors we refer Puri and Rubin (1974), Puri (1973), Shaked et.al. (1995) and Zahedi (1975). Pointing out the inconsistency in the definition of failure rate defined by Nair and Hitha (1989), Kotz and Johnson (1991) extended the use of partial sums to the bivariate discrete case. The possible distributions are geometric, waring and negative hypergeometric.

2.5 Reliability characteristics

Let X be a non-negative continuous random variable denoting the lifetime of a device or a component or an organism. If the survival function $P(X \geq x)$ is denoted by $R(x)$ and the probability density function by $f(x)$, then the failure rate $h(x)$ is given by

$$h(x) = \frac{f(x)}{R(x)}$$

$$= -\frac{d \log R(x)}{dx}. \tag{2.33}$$

The mean residual life

$$r(x) = E(X-x \mid X>x)$$

$$= \frac{1}{R(x)} \int_x^{\infty} R(t)dt \tag{2.34}$$

is another important concept in reliability. It can be seen that the $h(x)$ and $r(x)$ uniquely determines the distribution through the relations

$$R(x) = \exp\left\{-\int_0^x h(t)dt\right\} \tag{2.35}$$

and

$$R(x) = \frac{r(0)}{r(x)} \exp\left\{-\int_0^x \frac{1}{r(t)}dt\right\}. \tag{2.36}$$

An interesting feature of the extension of the univariate concepts of the failure mechanism into higher dimensions is that there is no unique way of representation. In the bivariate setup (X_1,X_2) is a non-negative random vector admitting the continuous distribution function $F(x_1,x_2)$. The survival function is denoted by

$$R(x_1,x_2) = P(X_1 \geq x_1, X_2 \geq x_2) \tag{2.37}$$

and the density of (X_1,X_2) by

$$f(x_1,x_2) = \frac{\partial^2 R}{\partial x_1 \partial x_2}. \tag{2.38}$$

If (X_1,X_2) is treated as the lives of the components in a two-component system, the bivariate scalar failure rate defined by Basu (1971) is

$$a(x_1,x_2) = \frac{f(x_1,x_2)}{R(x_1,x_2)} \tag{2.39}$$

$a(x_1,x_2)$ is a constant independent of X_1 and X_2 if and only if X_1 and X_2 are independent and exponentially distributed. Galambos and Kotz (1978) derived a differential equation connecting the failure rate and the survival function as

$$a(x_1,x_2) = \frac{\partial H}{\partial x_1}\frac{\partial H}{\partial x_2} - \frac{\partial^2 H}{\partial x_1 \partial x_2} \tag{2.40}$$

where

$$H = -\log R(x_1,x_2).$$

Johnson and Kotz (1975) gives another approach to define the bivariate failure rate as a vector $(h_1(x_1,x_2), h_2(x_1,x_2))$ where

$$h_i(x_1,x_2) = \frac{-\partial \log R(x_1,x_2)}{\partial x_i}, \quad i=1,2. \tag{2.41}$$

This vector valued failure rate determines the distribution uniquely through the equation

$$R(x_1,x_2) = \exp\left\{-\int_0^{x_1} h_1(t_1,0)dt_1 - \int_0^{x_2} h_2(x_1,t_2)dt_2\right\} \tag{2.42}$$

or

$$R(x_1,x_2) = \exp\left\{-\int_0^{x_2} h_2(0,t_2)dt_2 - \int_0^{x_1} h_1(t_1,x_2)dt_1\right\} \tag{2.43}$$

The third approach is to define the failure rate vector as $(c_1(x_1,x_2),\ c_2(x_1,x_2))$ where

$$c_i(x_1,x_2) = \frac{f(x_1,x_2)}{\frac{-\partial R(x_1,x_2)}{\partial x_j}},\ i,j=1,2;\ i\neq j. \tag{2.44}$$

Buchanan and Singpurwalla(1977) define the bivariate mean residual life (m.r.l.) function $r(x_1,x_2)$ by

$$r(x_1,x_2) = E((X_1-x_1)\ (X_2-x_2) \mid X_1>x_1,\ X_2>x_2) \tag{2.45}$$

The second definition is provided by Shanbhag and Kotz(1987) and Arnold and Zahedi (1988). The vector valued Borel measurable function on $R_2^+ = \{(x_1,x_2),\ 0<x_1,x_2<\infty\}$ is defined by $(r_1(x_1,x_2),\ r_2(x_1,x_2))$ for all $(x_1,x_2)\in R_2^+$ where

$$r_i(x_1,x_2) = E(X_i-x_i \mid X_1>x_1,\ X_2>x_2) \tag{2.46}$$

$$= \frac{1}{R(x_1,x_2)}\int_{x_i}^{\infty} R(t,x_j)dt,\ i,\ j=1,2;\ i\neq j \tag{2.47}$$

Nair and Nair (1989) provides the unique representation of $R(x_1,x_2)$ in terms of $r_1(x_1,x_2)$ and $r_2(x_1,x_2)$ as

$$R(x_1,x_2) = \frac{r_1(0,0)r_2(x_1,0)}{r_1(x_1,0)r_2(x_1,x_2)} \exp\left\{-\int_0^{x_1}\frac{dt}{r_1(t,0)}-\int_0^{x_2}\frac{dt}{r_2(x_1,t)}\right\} \tag{2.48}$$

or

$$R(x_1,x_2) = \frac{r_1(0,x_2)r_2(0,0)}{r_1(x_1,x_2)r_2(0,x_2)} \exp\left\{-\int_0^{x_1}\frac{dt}{r_1(t,x_2)}-\int_0^{x_2}\frac{dt}{r_2(0,t)}\right\} \tag{2.49}$$

The basic formulation to the study of discrete life distributions are provided by Cox(1972), Kalbfleish and Prentice (1980) and Lawless(1982). Let X be a non-negative integer valued random variable having the survival function $R(x)$= P($X \geq x$) and probability mass function $f(x)$. The failure rate $h(x)$ is defined to be

$$h(x) = \frac{f(x)}{R(x)} \tag{2.50}$$

and the mean residual life function by

$$r(x) = \mathrm{E}(X-x \mid X>x)$$

$$= \frac{1}{R(x+1)}\sum_{t=x+1}^{\infty}R(t). \tag{2.51}$$

$R(x)$ is uniquely determined by $h(x)$ and $r(x)$ through the relations (Salvia and Bollinger, 1982)

$$R(x) = \prod_{y=0}^{x-1}[1-h(y)] \tag{2.52}$$

and

$$R(x) = \prod_{u=1}^{x-1}\left[\frac{r(u-1)-1}{r(u)}\right][1-f(0)] \tag{2.53}$$

where $f(0)$ is determined such that $\sum_x f(x)=1$. Hitha and Nair (1989) have established the relationship between $h(x)$ and $r(x)$ as

$$1-h(x+1) = \frac{r(x)-1}{r(x+1)},\ x = 0,1,\ \ldots \tag{2.54}$$

Life distributions with virtual hazard rate, mean residual life etc. were studied in Abouammoh(1990) and Roy and Gupta (1992).

Coming to the case of bivariate distributions it is desirable to have a single quantity for failure rate as provided by Puri and Rubin (1974) and Puri (1973). They define the multivariate failure rate as

$$a(x_1,\ldots,x_n) = \frac{P(X_1 = x_1,\ldots,X_n = x_n)}{P(X_1 \geq x_1,\ldots,X_n \geq x_n)}. \tag{2.55}$$

In particular the bivariate failure rate is

$$a(x_1,x_2) = \frac{P(X_1 = x_1, X_2 = x_2)}{P(X_1 \geq x_1, X_2 \geq x_2)}. \tag{2.56}$$

The second alternative definition is introduced by Nair and Nair (1990) as a vector failure rate $(h_1(x_1,x_2),\ h_2(x_1,x_2))$ with

$$h_i(x_1,x_2) = \frac{P(X_i = x_i, X_j \geq x_j)}{P(X_1 \geq x_1, X_2 \geq x_2)},\ i,\ j=1,2,\ i\neq j \tag{2.57}$$

and this vector determines the survival function uniquely through the formula

$$R(x_1,x_2) = \prod_{r=1}^{x_1}[1-h_1(x_1-r,x_2)]\prod_{r=1}^{x_2}[1-h_2(0,x_2-r)]. \qquad (2.58)$$

The third alternative definition of the failure rate is suggested in Kotz and Johnson (1991) who view it as the vector $(c_1(x_1,x_2),\ c_2(x_1,x_2))$ where

$$c_i(x_1,x_2) = \frac{P(X_1 = x_1, X_2 = x_2)}{P(X_i \geq x_i, X_j = x_j)},\ i,\ j=1,2;\ i\neq j \qquad (2.59)$$

The failure rate in the multivariate set up is attempted by Shaked et. al.(1995).

In the case of m.r.l. it can be observed that the first work in higher dimentions is due to Nair and Nair (1988a) in which they define the m.r.l. in the support of $I_2^+ = \{(x_1,x_2):\ x_1,x_2=0,1,\ldots\}$ as $(r_1(x_1,x_2),$ $r_2(x_1,x_2))$

where

$$r_i(x_1,x_2) = E(X_i-x_i \mid X_1\geq x_1,\ X_2\geq x_2). \qquad (2.60)$$

It is also proved that $r_i(x_1,x_2)= c_i$ for $i=1,2$ if and only if X_i's are independent geometric random variables and $r_i(x_1,x_2) = A_i(x_j)$, $i,\ j=1,2;$

$i \neq j$ iff (X_1, X_2) is distributed as bivariate geometric with survival function

$$R(x_1,x_2)=p_1^{x_1}\, p_2^{x_2}\, \theta^{x_1x_2},\ 0\le p_1, p_2\le 1;\ 0\le\theta\le 1;\ 1-\theta\le(1-p_1\theta)(1-p_2\,\theta). \quad (2.61)$$

2.6 Equilibrium Distribution

Let X be a random variable admitting an absolutely continuous distribution function $F(x)$ with respect to the Lebesgue measure in the support of the set of non-negative reals and having a finite mean μ. Associated with X, a new random variable Y is defined, whose p.d.f is

$$g(x) = \frac{R(x)}{\mu},\ x>0 \quad (2.62)$$

where $R(x) = P(X \geq x)$ is the survival function of X. This distribution has a special significance in the context of renewal theory.

Consider a system of components whose times to failure are of interest. Let us start the experiment with a single new component at time zero and replace it upon failure by a new second component and so on. These failure times X_i, $i=1,2,..$ are independent and let they are identically distributed with distribution function $F(x)$. Then $S_n=X_1+X_2+\ldots+X_n$ will constitute renewal process. Take a sampling point t at random over a very long time interval. Define the $r.v.$ U_t as the length of time measured backwards from t to the last renewal at or before t. That is U_t denotes the age of the component in use at t. Let V_t be the time measured from t to the next renewal to occur after it. That is V_t is the residual life time of the component in use at time t. Cox (1962) proved that the limiting distribution of U_t and V_t is common, called the equilibrium distribution and has a density of the form specified by (2.62). In this physical situation Y represents the residual life of the component whose life length is X. For the applications of equilibrium distribution in reliability studies, we refer Scheaffer (1972), Rao(1985), Deshpande et. al.(1986) and Blumenthal (1967).

The probabilistic comparison of Y with parent population of X is utilised to explain the phenomenon of ageing. Gupta (1984) obtained the equilibrium distribution as a weighted distribution with weight $[h(x)]^{-1}$ where $h(.)$ is the failure rate. Let $G(.)$ denote the survival function of Y.

The relationship of the characteristics of equilibrium distribution with that of the parent distribution in the context of reliability are studied by Gupta (1984), Gupta and Kirmani (1990) and Hitha and Nair (1989). Some of the important identites among them are

(i) $G(x) = \frac{1}{\mu}\int_x^{\infty} R(t)dt$

(ii) $h_y(x) = \frac{1}{r(x)}$, where $h_y(x)$ is the failure rate of Y.

In the point of view of Deshpande et. al. (1986) the life distribution of a unit which ages more rapidly will come off worse in a comparison of $R(x)$ and $G(x)$. The wide spread applicability of weighted distribution in univariate case has prompted many researchers to extend the concept to the higher dimensions. However the applications to real problems in such cases have rarely been pointed out.

Let (X_1,X_2) be a random vector in the support of $\{(x_1,x_2)$: $0<x_1,x_2<\infty\}$ with an absolutely continuous distribution function $F(x_1,x_2)$ or the survival function $R(x_1,x_2)$. Defining $w(x_1,x_2)$ be a non-negative weighted function with $E[w(X_1,X_2)] <\infty$, Mahfoud and Patil (1982) defined a bivariate weighted distribution as the distribution of the vector (Y_1,Y_2) with p.d.f

$$g(x_1,x_2) = \frac{w(x_1,x_2)f(x_1,x_2)}{E[w(X_1,X_2)]}. \qquad (2.63)$$

When

$$w(x_1,x_2) = 1/h(x_1,x_2)$$

$$= \frac{R(x_1,x_2)}{f(x_1,x_2)},$$

$$E[w(X_1,X_2)] = \int_0^\infty\int_0^\infty \frac{R(x_1,x_2)}{f(x_1,x_2)} f(x_1,x_2)dx_1dx_2.$$

That is

$$E[w(X_1,X_2)] = E(X_1X_2) = \mu.$$

Hence

$$g(x_1,x_2) = \frac{R(x_1,x_2)}{E(X_1X_2)} = \frac{R(x_1,x_2)}{\mu}. \quad (2.64)$$

Let $G(.,.)$ denote the survival function of (Y_1,Y_2). Then

$$G(x_1,x_2) = \int_{x_1}^\infty\int_{x_2}^\infty \frac{R(t_1,t_2)}{\mu} dt_1dt_2. \quad (2.65)$$

We can see that

$$h_y(x_1,x_2) = 1/r(x_1,x_2) \quad (2.66)$$

where $h_y(.,.)$ is the scalar failure rate of (Y_1,Y_2) and $r(x_1,x_2)$ is the scalar m.r.l function of (X_1,X_2).

2.7 Setting the Clock Back to Zero Property

Having the discussion in the previous section let us now introduce the concept of setting the clock back to zero property. As mentioned earlier, this setting the clock back to zero property can be defined as an extension of the lack of memory property. Rao and

Talwalker (1990) introduced this concept of setting the clock back to zero(SCBZ) property.

A family of life distributions $\{f(x,\theta),\ x \geq 0,\ \theta \in \Theta\}$ is said to have the SCBZ property if the form of $f(\mathbf{x},\ \theta)$ remains unchanged under the following three operations, except for
the value of the parameters, that is

$$f(\mathbf{x},\ \theta) \rightarrow f(\mathbf{x},\ \theta^*) \tag{2.67}$$

where $\theta^* \in \Theta$,

1. Truncating the orginal distribution at some point $x_0 \geq 0$.
2. Considering the observable distribution for life time $X \geq x_0$ and
3. Changing the origin by means of the transformation given by $X_1 = X - x_0$, so that $X_1 \geq 0$.

In terms of the survival function $R(x,\theta)$, the definition can be restated as the following

A family of life distributions $\{R(x,\theta),\ x \geq 0,\ \theta \in \Theta\}$ is said to have the SCBZ property if for each $x_0 \geq 0$ and $\theta \in \Theta$, the survival function satisfies the equation

$$R(x+x_0,\theta) = R(x_0,\ \theta)\, R(x,\theta^*) \tag{2.68}$$

with $\theta^* = \theta^*(x_0) \in \Theta$.

Here θ may be a single or vector of parameters.

The random variable X is said to have SCBZ property if

$$P(X \geq x+x_0 | X \geq x_0) = P(X^* \geq x), \qquad (2.69)$$

where X^* has the same distribution as that of X, except for the parameters. It is not necessary that all the parameters are to be changed. The parameters which does not undergo any change under the SCBZ transformation has been called normalizing constants.

In this operation of SCBZ, truncating the distribution at time x_0 and then setting the origin at x_0 leaves the form of the distribution invariant expect for the parameters.

In reliability, this property ensures that the conditional distribution of the additional length or time of survival of a living organism or a device, given that it has survived x_0 time units is the same as the unconditional distribution except for a slight change in the parameters. That is the residual life distribution (RLD)has the same form as that of the original distribution except for a change in the parameters.

Rao (1990) proved that the life expectancy has a simple form if the family has SCBZ property and used this property of the time to tumor distribution with survival function

$$R(x_0,\theta) = \exp\left\{\frac{-\beta}{\gamma}\left(x_0 - \frac{\alpha}{\gamma}(1-e^{\gamma x_0})\right)\right\},\ \alpha,\ \beta,\ \gamma>0$$

which is the generalization of the simple time to tumor model by Chiang and Conforti (1989). If $r(x_0,\theta)$ denote the mean residual life, that is, it shows how long an organism or a device of age x_0 would survive, on the average

$$r(x_0,\theta) = E_\theta(X-x_0 \mid X>x_0)$$

$$= \int_{x_0}^{\infty}\frac{R(x,\theta)}{R(x_0,\theta)}dx$$

$$= \int_{0}^{\infty}\frac{R(x+x_0,\theta)}{R(x_0,\theta)}dx.$$

That is

$$r(x_0,\theta) = \int_{0}^{\infty} R(x,\theta^*)dx \qquad (2.70)$$

if $R(x,\theta)$ has SCBZ property.

Exponential, Pareto type II, finite range, Gompertz, linear hazard model, the model for time to tumor given by Rao (1990) possess this property. The growth model, Gompertz distribution has a biological significance that the same growth curve can be described from any point on it taken as the origin. Rao (1992) have proved that the tampered random variable model is equivalent to tampered failure rate model. Another application of this SCBZ property is given in Rao et. al. (1993a). They have shown that the family of survival distributions under the proportional hazard model and accelerated life models have SCBZ property if the baseline survival distributions have. The effect of the covariate vector X on the tumor free life expectancy are also considered there. Rao and Damaraju (1992) have shown that the inequalities in the definitions of the measures New Better than Used (NBU) and New Better than Used in Expectation (NBUE) of the maintance policies become equalities iff the family of distribution has SCBZ property.

Rao et. al. (1993b) extended the notion of SCBZ in the bivariate case which he called the extended SCBZ property.

Consider an individual exposed simultaneously to two risks R_1 and R_2 with hypothetical life times X_1 and X_2 respectively. The joint survival function of X_1 and X_2 is defined by $R(x_1,x_2,\theta)$, $0\leq x_1,x_2<\infty$, where θ is the parameter or a vector of parameters. The survival function of the individual upto age x_0 can have the idea that the individual's hypothetical life times satisfy $X_1 \geq x_0$ and $X_2 \geq x_0$. The conditional distribution of the additional survival time of an induvidual due to risk R_1 given that the individual has survived for a time of x_0 units is

$$P(X_1 \geq x_1 + x_0 \mid X_1 \geq x_0,\ X_2 \geq x_0) = \frac{R(x_1 + x_0, x_0, \theta)}{R(x_0, x_0, \theta)}. \tag{2.71}$$

In a similar way,

$$P(X_2 \geq x_2 + x_0 \mid X_1 \geq x_0,\ X_2 \geq x_0) = \frac{R(x_1, x_2 + x_0, \theta)}{R(x_0, x_0, \theta)}. \tag{2.72}$$

Using this notations Rao et. al.(1993b) defined SCBZ property in the bivariate case as follows.

A class of bivariate life distributions$\{ R(x_1,x_2,\theta)$, $x_1,x_3 \geq 0, \theta \in \Theta\}$ is said to have the SCBZ property if for each $\theta \in \Theta$ and $x_0 \geq 0$, the survival function satisfies the pair of equations

$$\frac{R(x_1 + x_0, x_0, \theta)}{R(x_0, x_0, \theta)} = R(x_1, x_0, \theta^*)$$

and (2.73)

$$\frac{R(x_0, x_2 + x_0, \theta)}{R(x_0, x_0, \theta)} = R(x_0, x_2, \theta^{**})$$

where $\theta^* = \theta^*(x_0)$ and $\theta^{**} = \theta^{**}(x_0) \in \Theta_0$ where Θ_0 denote the boundary of Θ.

They have showed that the life expectancy vector($r_1(x_0,x_0,\theta)$, $r_2(x_0,x_0,\theta)$) has a closed form, since

$$r_1(x_0,x_0,\theta) = E_\theta(X_1 - x_0 \mid X_1 \geq x_0,\ X_2 \geq x_0)$$

$$= \frac{1}{R(x_0,x_0,\theta)} \int_0^\infty \int_0^\infty (x_1 - x_0) f(x_1,x_2,\theta) dx_1 dx_2 ,$$

where $f(x_1,x_2,\theta)$ is the joint p.d.f of (X_1,X_2). That is

$$r_1(x_0,x_0,\theta) = \int_0^\infty \frac{R(x_1 + x_0, x_0, \theta)}{R(x_0,x_0,\theta)} dx_1$$

$$= \int_0^\infty R(x_1,x_0,\theta^*) dx_1 .$$

Similiraliy

$$r_2(x_0,x_0,\theta) = \int_0^\infty R(x_0,x_2,\theta^{**}) dx_2 .$$

The examples cited in Rao et. al. (1993b) include the bivariate exponential distributions proposed by Marshall-Olkin (1967) and Gumbel (1960), Bivariate Gompertz and Bivariate Pareto models.

CHAPTER 3

SETTING THE CLOCK BACK TO ZERO PROPERTY IN CONTINUOUS UNIVARIATE SET UP

3.1 Introduction

In the previous two chapters we have introduced the concept of SCBZ property and reviewed the important results in that connection. As identified in section 2.7, the problems that required further investigation will be considered in this and subsequent chapters. In particular the concern in the present chapter is to develop some new results in the univariate SCBZ property, especially characterizations of probability models.

3.2 Characterization of the probability distributions with SCBZ property (Mini and Nair (1994))

In the discussions that follow in this section we continue the notations in the previous chapters and assume that the two limit

$$a(\theta) = \lim_{x \to 0} \frac{\theta^* - \theta}{x}$$

and

$$b(\theta) = \lim_{y \to 0} \frac{\log R(y,\theta)}{y}$$

exist and are finite. A characterization of the family of distributions possessing SCBZ property and satisfying the above mentioned regularity conditions is presented in the theorem below. We call a family of distributions possessing the regularity conditions as a regular family.

Theorem 3.1

Let the family of survival functions $\{R(y,\theta),\ y>0,\ \theta \in \Theta\}$ be regular. Then a necessary condition that the SCBZ property holds for the family is that the partial differential equation

$$\frac{\partial Z}{\partial y} - a(\theta)\frac{\partial Z}{\partial \theta} = b(\theta) \qquad (3.1)$$

where

$$Z = \log R(y,\theta)$$

is satisfied by $R(y,\theta)$.

Proof

The SCBZ property of the family $\{R(y,\theta),\ y>0,\ \theta\in\Theta\}$ implies

$$R(x+y,\theta) = R(x,\theta)\, R(y,\theta^*) \tag{3.2}$$

where θ^* is a function of θ and x.

That is,

$$\log R(x+y,\theta) = \log R(x,\theta) + \log R(y,\theta^*). \tag{3.3}$$

Now (3.3) can be written as

$$\frac{\log R(x+y,\theta)-\log R(y,\theta)}{x} - \frac{\log R(y,\theta^*)-\log R(y,\theta)}{x} = \frac{\log R(x,\theta)}{x}. \tag{3.4}$$

Taking the limit as $x \to 0$ in (3.4), we find that

$$\frac{\partial \log R(y,\theta)}{\partial y} - \lim_{x\to 0} \frac{\log R(y,\theta^*)-\log R(y,\theta)}{\theta^*-\theta}\left(\frac{\theta^*-\theta}{x}\right) = \lim_{x\to 0}\frac{\log R(x,\theta)}{x}. \tag{3.5}$$

Since $x \to 0$, $\theta^* \to \theta$, we have from (3.5)

$$\frac{\partial \log R(y,\theta)}{\partial y} - \frac{\partial \log R(y,\theta)}{\partial \theta}\lim_{x\to 0}\left(\frac{\theta^*-\theta}{x}\right) = \lim_{x\to 0}\frac{\log R(x,\theta)}{x}$$

or

$$\frac{\partial Z}{\partial y} - a(\theta)\frac{\partial Z}{\partial \theta} = b(\theta)$$

as stated in Theorem 3.1.

Some well known distributions possessing this partial differential equation (3.1) with the corresponding $a(\theta)$, $b(\theta)$ and θ^* are given in Table 1. Since a limiting process is involved in Theorem 3.1, converse of the Theorem 3.1 does not hold in general. However a partial converse exists under the following conditions.

Theorem 3.2

If there exists functions $a(\theta)$ and $b(\theta)$ in Θ such that

$$\frac{\partial \log R(x+y,\theta)}{\partial \theta} = \left[\frac{\partial \log R(y,\theta^*)}{\partial y} - b(\theta)\right] / a(\theta), \tag{3.6}$$

then (3.1) implies that the SCBZ property is satisfied for the family of survival functions $\{R(y,\theta),\ y>0,\ \theta\in\Theta\}$.

Proof

Assuming (3.1), we have by direct integration

$$R(y,\theta) = \exp\left\{\int_0^y a(\theta)\frac{\partial \log R(t,\theta)}{\partial \theta}dt + b(\theta)y\right\}$$

Thus

$$R(x+y,\theta) = \exp\left\{\int_0^{x+y} a(\theta)\frac{\partial \log R(t,\theta)}{\partial\theta}dt + b(\theta)(x+y)\right\}$$

$$= \exp\left\{\int_0^{x} a(\theta)\frac{\partial \log R(t,\theta)}{\partial\theta}dt + b(\theta)x\right\}$$

$$\exp\left\{\int_x^{x+y} a(\theta)\frac{\partial \log R(t,\theta)}{\partial\theta}dt + b(\theta)y\right\}$$

$$= R(x,\theta)\ R(y,\theta^*)$$

where

$$R(y,\theta^*) = \exp\left\{\int_0^{y} a(\theta)\frac{\partial \log R(t+x,\theta)}{\partial\theta}dt + b(\theta)y\right\}$$

which implies that $\{R(y,\theta), y>0, \theta\in\Theta\}$ has the desired SCBZ property.

It is of some interest to look at a general solution of the partial differential equation (PDE) (3.1). This is explained in the following theorem.

Theorem 3.3

The general solution of (3.1) is of the form

$$F(U(y,\theta,Z), V(y,\theta,Z) = 0 \qquad (3.7)$$

where $F(.\ ,\ .)$ is an arbitrary function of U and V with $U(y,\theta,Z)=Z-b(\theta)y$ and $V(y,\theta,Z)= y + g(\theta)$.

Proof

From Sneddon (1957), the general solution of (3.1) can be identified as $F(U,V)=0$ where U and V are the solutions of the ordinary differential equation

$$\frac{dy}{1} = \frac{-d\theta}{a(\theta)} = \frac{dZ}{b(\theta)}, \; a(\theta)\neq 0, \; b(\theta)\neq 0. \tag{3.8}$$

From the first pair of equations in (3.8), we get

$$y = \int \frac{-d\theta}{a(\theta)} + c_1.$$

That is

$$y + g(\theta) = c_1$$

where

$$g(\theta) = \int \frac{d\theta}{a(\theta)}.$$

From the first and third terms of (3.8), we get

$$Z - b(\theta)y = c_2.$$

Hence the general solution of the system is $F(U,V)=0$ where

$$U = Z - b(\theta)y \tag{3.9}$$

and

$$V = y + g(\theta). \tag{3.10}$$

By imposing suitable boundary conditions on θ and y, the particular solutions can be obtained, but all such solutions need not be satisfying SCBZ property conforming our earlier remark. In view of (3.6) and (3.8) through (3.10) we can offer the following results of the probability distributions. Let V_0 be the value of V at $y = 0$.*

Theorem 3.4

The general solution of (3.1) is of the form

$$U = - c \log\left(\frac{V}{V_0}\right) + c\left(\frac{V}{V_0} - 1\right) \quad (3.11)$$

with $b(\theta) = \frac{-c}{\theta}$ and $g(\theta)=\theta$ for any $c>0$ iff the distribution is Pareto type II.

Proof

(3.11) with $b(\theta) = \frac{-c}{\theta}$ and $g(\theta)=\theta$ implies

$$\log R(y,\theta) + \frac{c}{\theta} y = - c \log\left(\frac{\theta + y}{\theta}\right) + c\left(\frac{\theta + y}{\theta} - 1\right)$$

$$= -c \log\left(1 + \frac{y}{\theta}\right) + \frac{c}{\theta} y$$

That is

$$\log R(y,\theta) = -c \log\left(1+\frac{y}{\theta}\right)$$

or

$$R(y,\theta) = \left(1+\frac{y}{\theta}\right)^{-c},$$

which is the survival function of Pareto II distribution. The converse part can be easily verified from the given survival function.

Theorem 3.5

The general solution of (3.1) is of the form

$$U = c \log\left(\frac{V}{V_0}\right) + c\left(\frac{V}{V_0}-1\right) \tag{3.12}$$

with $b(\theta) = \frac{c}{\theta}$ and $g(\theta) = -\theta$ for any $c>0$ if and only if the distribution is finite range with survival function

$$R(y,\theta) = \left(1-\frac{y}{\theta}\right)^{c}.$$

Proof

By giving the values of U and V in the given form (3.12) it can be obtained that $R(y,\theta) = \left(1-\frac{y}{\theta}\right)^{c}$, which is the survival function of finite range distribution. The converse is obtained from the functional form of $\left(1-\frac{y}{\theta}\right)^c$.

Theorem 3.6

The general solution of (3.1) is of the form

$$U = -b(V-V_0)^2 \tag{3.13}$$

with $b(\theta) = -\theta$ and $a(\theta) = 2b$ iff

$$R(y, \theta) = \exp\{-(\theta y+by^2)\}.$$

Proof

By giving the values of U and V with $b(\theta) = -\theta$ and $a(\theta) = 2b$ in (3.13), we get

$$\log R(y, \theta)+\theta y = -b \left(\frac{\theta}{2b}+y-\frac{\theta}{2b}\right)^2$$

$$= - by^2$$

or

$$R(y, \theta) = \exp\{-(\theta y+by^2)\},$$

which is the linear hazard exponential. The converse also can be verified.

As in the earlier theorems the Gompertz distribution holds the following result.

Theorem 3.7

The general solution of (3.1) is of the form

$$U = -(e^{\alpha V}/\alpha) + e^{\alpha V_0}[(1/\alpha)+V-V_0]$$

with $b(\theta) = -\theta$ and $a(\theta) = \theta\alpha$ if and only if

$$R(y,\theta) = \exp\left\{\frac{-\theta}{\alpha}(e^{\alpha y}-1)\right\}. \tag{3.14}$$

The proof is similar to the earlier cases.

3.3 Characterization by functional form of θ^* (Mini and Nair (1994))

Now we establish characterizations of probability distributions in which θ^* has certain simple functional forms.

Theorem 3.8

Let X be a continuous random variable in the supp-ort of $T \subset R^+$ with survival function $R(x,\theta)$. Then $R(x,\theta)$ has SCBZ property with $\theta^*=\theta+x$, iff

$$R(x,\theta) = \frac{k(x+\theta)}{k(\theta)} \tag{3.15}$$

where $k(.)$ is non-negative, non-increasing continuous function satisfying $k(\infty)=0$

Proof ·

Suppose $R(x,\theta)$ has SCBZ property with $\theta^{*}=\theta+x$. Then from (2.68)

$$R(x+y,\theta) = R(x,\theta)\ R(y,\ x+\theta)$$

for all $x,\ y \in T$ and for all $\theta\in\Theta$.

For a fixed θ, the last equation implies

$$R(x+y) = R(x)\ R(y,x)$$

or

$$R(y,x) = \frac{R(x+y)}{R(x)}.$$

Thus $R(x,\theta)$ has the desired form with $k(x) = R(x)$, so that, conditions on $k(x)$ are satisfied. When $R(x,\theta)$ has the given form, we have

$$\frac{R(x+y,\theta)}{R(x,\theta)} = \frac{k(x+y+\theta)/k(\theta)}{k(x+\theta)/k(\theta)}$$

$$= \frac{k(x+y+\theta)}{k(x+\theta)}$$

$$= R(y,\ x+\theta),$$

which implies the converse.

Examples

(i) Pareto distribution with survival function

$$R(x,\theta) = \left(1+\frac{x}{\theta}\right)^{-a}, \ x,\theta>0,\ a>1$$

so that $k(\theta) = \theta^{a}$.

(ii) Linear hazard model with survival function

$$R(x,\theta) = \exp\left\{-\left(\theta x+\frac{x^2}{2}\right)\right\}, \ x,\theta>0$$

so that $k(\theta) = \exp\left\{-\frac{\theta^2}{2}\right\}$.

Theorem 3.9

Let X be a continuous random variable in the support of $(0,\theta)$ with survival function $R(x,\theta)$. Then $R(x,\theta)$ has SCBZ property with $\theta^{*}=\theta-x$ if and only if

$$R(x,\theta) = \frac{k(\theta)}{k(\theta-x)}$$

where $k(.)$ is a non-negative, non-increasing continuous function satisfying $k(0) = \infty$.

The proof follows in a similar line as that of Theorem 3.8.

Example

(i) Finite range distribution with survival function

$$R(x,\theta) = \left(1-\frac{x}{\theta}\right)^{a}, \; 0<x< \theta, \theta>0, \; a>1$$

so that $k(\theta) = \theta^{a}$.

3.4 Reliability Measures

In this section we establishes the equivalent condition of SCBZ property in terms of the failure rate and some properties of the distributions. These relationship enable the interpretation of SCBZ in terms of reliability concepts and there by enabling the use of the former in reliability and life testing.

Rao and Talwalker (1990) have proved that the SCBZ property is equivalent to

$$h(x_1+x_2,\theta) = h(x_2,\theta^{*}) \tag{3.16}$$

where $\theta^{*} = \theta^{*}(x_i)\in\Theta$.

Theorem 3.10

A continuous distribution has reciprocal linear hazard function only if it has the SCBZ property.

Proof

We have, from (2.35)

$$R(x,\theta) = \exp\left\{-\int_0^x h(t,\theta)dt\right\}.$$

Let $h(x,\theta) = 1/(ax+b)$. Then we have

$$R(x,\theta) = \exp\left\{-\int_0^x \frac{1}{at+b}dt\right\}$$

$$= \exp\left\{\frac{-1}{a}\log(ax+b)+\frac{1}{a}\log b\right\}$$

$$= b^{1/a}(ax+b)^{-1/a}.$$

$$\frac{R(x_1+x_2,\theta)}{R(x_1,\theta)} = \frac{b^{1/a}(a(x_1+x_2)+b)^{-1/a}}{b^{1/a}(ax_1+b)^{-1/a}}$$

$$= (ax_1+b)^{1/a}(ax_2+ax_1+b)^{-1/a}$$

$$= R(x_2,\theta^*)$$

where $\theta^* = ax_1+b$.

That is, the family $\{R(x,\theta),\ x\geq 0\}$ has SCBZ property.

Note 1.

If $\{R(x,\theta),\ x\geq 0,\ \theta\in\Theta\}$has SCBZ property, then in terms of the failure rate we have the condition (3.16)

Taking $x_2 = 0$ in (3.16), we get

$$h(x_1,\theta) = h(0,\theta^*)$$

$$= g(\theta^*).$$

That is, the failure rate is a function of θ^* only.

Theorem 3.11

If $h(x,\theta)$ is a one to one function of θ^*, then θ^* uniquely determines the distribution.

Proof

Let the one to one function from θ^* to $h(x,\theta)$ be $g(\theta^*)$. We know that the failure rate $h(x,\theta)$ uniquely determines the distribution through the formula (2.35). That is

$$R(x,\theta) = \exp\left\{-\int_0^x h(x,\theta)dx\right\}.$$

Since $h(x,\theta) = g(\theta^*)$, we have

$$R(x,\theta) = \exp\left\{-\int_0^x g(\theta^*)dx\right\}$$

which implies that $g(\theta^*)$ uniquely determines the distribution.

The mean residual life is a superior concept than the failure rate. For the reasons we refer Muth (1977). The following theorem gives the equivalent condition of the SCBZ property in terms of the mean residual life function.

Theorem 3.12

The SCBZ property is equivalent to

$$r(x_1+x_2, \theta) = r(x_2, \theta^*) \tag{3.17}$$

where $\theta^* = \theta^*(x_1) \in \Theta$ and $r(., \theta)$ is the mean residual life function of the random variable X having survival function $R(x, \theta)$.

Proof

SCBZ property implies, for x_1, x_2, $t \geq 0$

$$R(x_1+x_2, \theta) = R(x_1, \theta)\, R(x_2, \theta^*) \tag{3.18}$$

and

$$R(x_1+t, \theta) = R(x_1, \theta)\, R(t, \theta^*) \tag{3.19}$$

where $\theta^* = \theta^*(x_1) \in \Theta$.

Then on dividing (3.19) by (3.18), we get

$$\frac{R(x_1+t,\theta)}{R(x_1+x_2,\theta)} = \frac{R(t,\theta^*)}{R(x_2,\theta^*)} \tag{3.20}$$

Integrating with respect to t within the limit (x_2,∞), we get

$$\frac{1}{R(x_1+x_2,\theta)}\int_{x_2}^{\infty} R(x_1+t,\theta)dt = \frac{1}{R(x_2,\theta^*)}\int_{x_2}^{\infty} R(t,\theta^*)dt\,.$$

That is

$$r(x_1+x_2,\theta) = r(x_2,\theta^*).$$

Retracing the steps backward, we can arrive at (3.20). Then

$$\frac{R(x_1+x_2,\theta)}{R(x_2,\theta^*)} = \frac{R(x_1+t,\theta)}{R(t,\theta^*)} = c(x_1,\theta)$$

or

$$R(x_1+x_2,\theta) = c(x_1,\theta)\,R(x_2,\theta^*) \tag{3.21}$$

Taking $x_2 = 0$, we get

$$R(x_1,\theta) = c(x_1,\theta). \tag{3.22}$$

On substituting (3.22) in (3.21), we get (3.18), which shows the SCBZ property of X.

In connection with Theorem 3.10 we can offer the following theorem.

Theorem 3.13

A continuous distribution has linear mean residual life function only if it has the SCBZ property.

Proof

Hitha (1991) has proved that reciprocal linear hazard rate implies and is implied by linear mean residual life function. Hence by Theorem 3.10 the present assertion holds.

Note 2.

If $\{R(x,\theta),\ x\geq 0,\ \theta\in\Theta\}$has SCBZ property, then in terms of the mean residual life we have the condition (3.17).

Taking $x_2 = 0$, we get

$$r(x_1,\theta) = r(0,\theta^*)$$

$$= f(\theta^*). \tag{3.23}$$

Hence mean residual life is a function of θ^* alone.

Observation: It can be observed that if $r(x,\theta)$ is a one to one function of θ^*, then θ^* uniquely determines the distribution.

3.5 SCBZ property in Equilibrium Distributions

Let X be a random variable admitting an absolutely continuous survival function $R(x,\theta)$ with respect to the Lebesgue measure in the support of the set of non-negative reals with a finite mean μ. Then a new random variable Y with p.d.f

$$g(x, \theta) = \frac{R(x,\theta)}{\mu}, \; x>0$$

is said to be the random variable corresponding to the equilibrium distribution. As explained in section 2.6, Y has a p.d.f which is the limiting distribution of the forward and backward recurrence times. The probabilistic comparison of Y with X can be used to explain the phenomenon of ageing. In this regard here we established that the SCBZ property of X preserves in Y and vice versa.

Theorem 3.14

X has SCBZ property if and only if Y has SCBZ property.

Proof

Let $R(x, \theta)$ denotes the survival function of X and $G(x, \theta)$ that of Y. Then $G(., \theta)$ and $R(., \theta)$ are related through

$$G(x, \theta) = \mu^{-1} \int_x^\infty R(t,\theta)dt$$

Hence

$$G(x_2, \theta^*) = \mu^{*-1} \int_{x_2}^\infty R(t,\theta^*)dt$$

where $\mu^* = E_{\theta^*}(X) = \int_0^\infty R(t,\theta^*)dt$, and

$$G(x_1+x_2,\theta) = \mu^{-1} \int_{x_1+x_2}^\infty R(t,\theta)dt$$

$$= \mu^{-1} \int_0^\infty R(t+x_1+x_2,\theta)dt.$$

Then by the SCBZ property of X, we have

$$R(t+x_1+x_2,\theta) = R(x_1,\theta)\, R(t+x_2,\theta^*)$$

where $\theta^* = \theta^*(x_1) \in \Theta$, the parametric space.

Hence

$$G(x_1+x_2,\theta) = \mu^{-1} R(x_1,\theta) \int_0^\infty R(t+x_2,\theta^*)dt$$

But

$$\mu^* R(x_1,\theta) = R(x_1,\theta) \int_0^\infty R(t,\theta^*)dt$$

$$= \int_0^\infty R(t+x_1,\theta)dt,$$

by the SCBZ property of X.

Then

$$R(x_1,\theta) = \mu^{*-1} \int_0^\infty R(t+x_1,\theta)dt$$

and hence

$$G(x_1+x_2,\theta) = \mu^{-1}\mu^{*-1}\int_0^\infty R(t+x_1,\theta)dt\int_0^\infty R(t+x_2,\theta^*)dt$$

$$= G(x_1,\theta)\,G(x_2,\theta^*),$$

which implies the SCBZ property of Y.

Let $h_y(.,\theta)$ denote that failure rate of Y and $r(x,\theta)$, the mean residual life of X. For the equilibrium distribution from Gupta and Kirmani (1990) and Hitha and Nair (1989) we have

$$h_y(x,\theta) = 1/r(x,\theta).$$

Hence if Y has the SCBZ property we have the condition that

$$1/r(x_1+x_2,\theta) = 1/r(x_2,\theta^*)$$

where $\theta^* = \theta^*(x_1,\theta) \in \Theta$.

or

$$r(x_1+x_2,\theta) = r(x_2,\theta^*).$$

Then by Theorem 3.12, we can have the result that X also holds this SCBZ property.

By connecting the above theorem with Theorem 3.4 and 3.5, we can have the following corollaries.

Corollary 1

Y has SCBZ property with $\theta^* = \theta+x_1$ if and only if the survival function $R(.,\theta)$ of X is of the form

$$R(x,\theta) = \frac{S_1(x+\theta)}{S_1(\theta)}$$

where $S_1(.)$ is a non-negative, non-increasing function satisfying $S_1(\infty)=0$.

Corollary 2

X has SCBZ property with $\theta^* = \theta + x_1$ if and only if the survival function $G(.,\theta)$ of Y is of the form

$$G(x,\theta) = \frac{S_2(x+\theta)}{S_2(\theta)}$$

where $S_2(.)$ is a non-negative, non-increasing function satisfying $S_2(\infty)=0$.

Corollary 3

$R(x,\theta) = \dfrac{S_1(x+\theta)}{S_1(\theta)}$ where $S_1(.)$ is as defined in Corollary 1 iff

$G(x,\theta) = \dfrac{S_2(x+\theta)}{S_2(\theta)}$ where $S_2(.)$ is as mentioned in Corollary. 2.

Corollary 4

$$R(x,\theta) = \frac{S_1(\theta)}{S_1(\theta - x)}$$

where $S_1(.)$ is a non-negative, non-increasing continuous function satisfying $S_1(0)=\infty$ iff

$$G(x,\theta) = \frac{S_2(\theta)}{S_2(\theta - x)}$$

where $S_2(.)$ is a non-negative, non-increasing continuous function satisfying $S_2(0)=\infty$.

Theorem 3.15

If X has SCBZ property then

$$h_y(x,\theta) + a(\theta)\frac{\partial \log G(x,\theta)}{\partial \theta} = \mu^{-1} \qquad (3.24)$$

where $a(\theta) = \lim_{x\to 0}\left(\frac{\theta^* - \theta}{x}\right)$.

Proof

From Theorem 3.14, the SCBZ property of X imples the SCBZ property of Y. When Y has SCBZ property its survival function should satisfy the partial differential equation

$$\frac{\partial \log G(x,\theta)}{\partial x} - a(\theta)\frac{\partial \log G(x,\theta)}{\partial \theta} = b(\theta) \qquad (3.25)$$

where $a(\theta) = \lim_{x\to 0}\left(\frac{\theta^* - \theta}{x}\right)$ and $b(\theta) = \lim_{x\to 0}\left(\frac{\log G(x,\theta)}{x}\right)$.

We have

$$\frac{\partial \log G(x,\theta)}{\partial x} = \frac{G'(x,\theta)}{G(x,\theta)}$$

$$= - h_y(x, \theta) \tag{3.26}$$

$$b(\theta) = \lim_{x\to 0}\left(\frac{\log G(x,\theta)}{x}\right)$$

$$= \lim_{x\to 0}\left(\frac{\partial \log G(x,\theta)}{\partial x}\right)$$

$$= \lim_{x\to 0}\frac{\partial}{\partial x}\left(\log \mu^{-1} + \log \int_x^{\infty} R(t,\theta)dt\right)$$

$$= \lim_{x\to 0}\frac{\frac{\partial}{\partial x}\left(\int_x^{\infty} R(t,\theta)dt\right)}{\int_x^{\infty} R(t,\theta)dt}$$

$$= \lim_{x\to 0}\frac{-R(x,\theta)}{\int_x^{\infty} R(t,\theta)dt}$$

$$= \frac{-R(0,\theta)}{\int_0^{\infty} R(t,\theta)dt} = -\mu^{-1}. \tag{3.27}$$

On substituting (3.26) and (3.27) in (3.25), we get

$$- h_y(x, \theta) - a(\theta)\frac{\partial \log G(x,\theta)}{\partial \theta} = -\mu^{-1}$$

or it is same as (3.24).

Table 1

Name of Distribution	Survival Function	Residual Life Distribution	θ^*	$a(\theta)$	$b(\theta)$
Exponential	$e^{-\theta x}$ $(x\geq 0; \theta>0)$	$e^{-\theta y}$	θ	0	$-\theta$
Pareto	$\left(1+\frac{x}{\theta}\right)^{-a}$ $(x, \theta>0; a>1)$	$\left(1+\frac{y}{\theta+x}\right)^{-a}$	$\theta+x$	1	$-a/\theta$
Finite Range	$\left(1-\frac{x}{\theta}\right)^{a}$ $(0<x<\theta, \theta>0; a>1)$	$\left(1-\frac{y}{\theta-x}\right)^{a}$	$\theta-x$	-1	$-a/\theta$
Linear hazard model	$\exp\{-(\theta x+bx^2)\}$ $(x\geq 0, \theta, b>0)$	$\exp\{-(\theta x+2bx)y-by^2\}$	$\theta+2bx$	$2b$	$-\theta$
Gompertz	$\exp\left\{-\frac{\theta}{\alpha}\left(e^{\alpha x}-1\right)\right\}$ $(x\geq 0, \theta>0; -\infty<\alpha<\infty)$	$\exp\left\{-\frac{\theta}{\alpha}e^{\alpha x}\left(e^{y}-1\right)\right\}$	$\theta e^{\alpha x}$	$\alpha\theta$	$-\theta$
Chiang-Conforti	$\exp\left\{-\frac{\beta}{\nu}\left(x-\frac{\theta}{\nu}\left(1-e^{\nu x}\right)\right)\right\}$ $(x\geq 0, \theta>0, \beta>0, \eta>0)$	$\exp\left\{-\frac{\beta}{\nu}\left(y-\frac{\theta e^{\nu x}}{\nu}(1-e^{\nu y})\right)\right\}$	$\theta e^{\nu x}$	$\theta\nu$	$\frac{-\beta}{\nu}\left(1-\frac{\theta}{\nu}\right)$

CHAPTER 4

CONTINUOUS MULTIVARIATE SCBZ PROPERTY

4.1 Introduction

Having the discussion of univariate SCBZ property in chapter 3, let us now draw our attention to the concept in multivariate case with special relevance to bivariate cases. Since there is no unique way of extension of univariate concepts in higher dimensions, we can define the bivariate SCBZ property in four different ways. These are discussed in the succeeding five sections. The various equivalent forms of bivariate SCBZ properties in terms of reliability measures are explained in section 4.7. Just as the main result of chapter 3, we can bring the distributions having bivariate SCBZ(1) property into a class, which is shown in section 4.3. In section 4.9 it is shown that the bivariate SCBZ(2) property is preserved in the equilibrium distributions. The various SCBZ properties of n variables are discussed in the concluding section of this chapter.

4.2 Bivariate Setting the Clock Back to Zero (1) Property

As in the case of the natural extension of lack of memory property due to Marshall and Olkin (1967), the SCBZ property in the univariate case also can be extended to the bivariate case.

Definition

A class of survival distribution $\{R(x_1,x_2,\theta),\ x_1,x_2\geq 0,\ \theta\in\Theta\}$ is said to have bivariate SCBZ(1) property if

$$R(x_1+t,x_2+t,\theta) = R(t,t,\theta)\ R(x_1,x_2,\theta^*) \qquad (4.1)$$

with $\theta^* = \theta^*(t) \in\Theta$.

Consider a two component system. By this property we mean that if the system has an age t, then the conditional distribution of the remaining life time of the system is again in that family of orginal distributions. In the relaibility context this property ensures that the residual life-time of a sytem belongs to the same family of distributions.

Examples

1. Bivariate Pareto distribution with survival function

$$R(x_1,x_2,\theta) = (1+\sigma_1 x_1+\ \sigma_2 x_2)^{-\alpha},\ x_1,x_2\geq 0 \qquad (4.2)$$

where $\theta = (\sigma_1, \sigma_2, \alpha)$ and the parametric space is

$$\Theta = \{ (\sigma_1, \sigma_2, \alpha): \sigma_1, \sigma_2 > 0, \alpha > 1 \}.$$

In this case the new parameter vector $\theta^* = (\sigma_1^*, \sigma_2^*, \alpha)$ where

$\sigma_i^* = \sigma_i (1+\sigma_1 t + \sigma_2 t)^{-1}$ for $i=1,2$.

2. Marshall-Olkin class of bivariate exponential distribution with survival function

$$R(x_1,x_2,\theta) = \exp\{-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_{12} \max(x_1, x_2)\}, \quad x_1,x_2 \geq 0 \qquad (4.3)$$

with $\theta = (\lambda_1, \lambda_2, \lambda_{12})$ and the parametric space is

$$\Theta = \{ (\lambda_1, \lambda_2, \lambda_{12}): \lambda_1, \lambda_2, \lambda_{12} \geq 0, \lambda_i + \lambda_{12} > 0, i=1,2 \}.$$

Here $\theta^* = \theta$ itself.

3. Gumbel's bivariate exponential distribution with survival function

$$R(x_1,x_2,\theta) = \exp\{-\lambda_1 x_1 - \lambda_2 x_2 - \delta x_1 x_2\}, \quad x_1,x_2 \geq 0, \qquad (4.4)$$

with $\theta = (\lambda_1, \lambda_2, \delta)$ and $\Theta = \{ (\lambda_1, \lambda_2, \delta): \lambda_1, \lambda_2 > 0, \delta \geq 0 \}$.

Here $\theta^* = (\lambda_1^*, \lambda_2^*, \delta)$ with $\lambda_i^* = \lambda_i + \delta t$, $i = 1,2$.

4. Bivariate Gompertz distribution with survival function

$$R(x_1,x_2,\theta) = \exp\left\{\gamma\left(1 - e^{ax_1 + bx_2}\right)\right\}, \quad x_1,x_2 \geq 0 \qquad (4.5)$$

where $\theta = (a, b, \gamma)$ and $\Theta = \{ (a, b, \gamma) : a, b > 0, \gamma \geq 1 \}$.

Here $\theta^* = (a, b, \gamma^*)$ with $\gamma^* = \gamma e^{at+bt}$.

5. Bivariate finite range distribution with survival function

$$R(x_1,x_2,\theta) = (1-p_1x_1-p_2x_2)^d,\ 0<x_1<\frac{1}{p_1},\ 0<x_2<\frac{1-p_1x_1}{p_2},\ d\geq 1 \qquad (4.6)$$

with $\theta = (p_1, p_2, d)$ and the parametric space is

$$\Theta = \{\ (p_1, p_2, d):\ p_1>0,\ p_2>0,\ d\geq 1\}$$

In this case $\theta^* = (\,p_1^*,\ p_2^*,\ d)$ where $p_1^* = p_1(1-p_1t-p_2t)^{-1}$ and $p_2^* = p_2(1-p_1t-p_2t)^{-1}$.

4.3 Characterization of the Probability Distributions with Bivariate SCBZ(1) Property.

In the following discussions we assume that the two limits

$$a(\theta) = \lim_{t\to 0}\left(\frac{\theta^* - \theta}{t}\right)$$

and

$$b(\theta) = \lim_{t\to 0}\left(\frac{\log R(t,t,\theta)}{t}\right)$$

exist and are finite. A characterization theorem concerning the bivariate SCBZ(1) property is stated below.

Theorem 4.1

Let the family of survival functions $\{R(x_1,x_2,\theta),\ x_1,x_2>0,\ \theta\in\Theta\}$ be regular. Then a necessary condition that bivariate SCBZ(1) property holds for the family is that, the auxiliary system of partial differential equation

$$\frac{\partial Z}{\partial x_1} + \frac{\partial Z}{\partial x_2} - a(\theta)\,\frac{\partial Z}{\partial \theta} = b(\theta)$$

where Z = log $R(x_1,x_2,\theta)$, is satisfied by $R(x_1,x_2,\theta)$.

Proof

Bivariate SCBZ(1) property implies (4.1).

On taking logarithm on both sides of (4.1), we get

$$\log R(x_1+t,x_2+t,\theta) - \log R(x_1,x_2,\theta^*) = \log R(t,t,\theta). \tag{4.7}$$

Now (4.7) can be written as

$$\begin{aligned}&[\log R(x_1+t,x_2+t,\theta) - \log R(x_1+t,x_2,\theta)]/t\\ &+ [\log R(x_1+t,x_2,\theta) - \log R(x_1,x_2,\theta)]/t\\ &- [\log R(x_1,x_2,\theta^*) - \log R(x_1,x_2,\theta)]/t = [\log R(t,t,\theta)]/t.\end{aligned} \tag{4.8}$$

Taking the limit as $t \to 0$ in (4.8), we get

$$\frac{\partial \log R(x_1,x_2,\theta)}{\partial x_2} + \frac{\partial \log R(x_1,x_2,\theta)}{\partial x_1} - \frac{\partial \log R(x_1,x_2,\theta)}{\partial \theta}\lim_{t\to 0}\left(\frac{\theta^* - \theta}{t}\right) = \lim_{t\to 0}\left(\frac{\log R(t,t,\theta)}{t}\right)$$

That is,

$$\frac{\partial Z}{\partial x_1} + \frac{\partial Z}{\partial x_2} - a(\theta)\,\frac{\partial Z}{\partial \theta} = b(\theta) \tag{4.9}$$

as stated in the Theorem 4.1.

It is of some interest to have a general solution of the partial differential equation (4.9). It is explained in the following theorem.

Theorem 4.2

The general solution of (4.9) is of the form

$$F(u(x_1,x_2, Z,\theta), v(x_1,x_2, Z,\theta), w(x_1,x_2, Z,\theta)) = 0 \qquad (4.10)$$

where $F(.,.,.,\theta)$ is an arbitrary function of x_1,x_2,Z,θ with $u = x_1 - g(\theta)$; $v=x_2 - x_1$ and $w= Z-b(\theta)x_2$.

Proof

Sneddon (1957) has shown that a general solution of the linear partial differential equation

$$P_1\frac{\partial Z}{\partial x_1} + P_2 \frac{\partial Z}{\partial x_2} + \ldots + P_n\frac{\partial Z}{\partial x_n} = R$$

is of the form $F(u_1, u_2, \ldots, u_n)=0$, where $u_i(x_1, x_2, \ldots, x_n,Z)=c_i$ $(i=1,2,\ldots,n)$ are the independent solutions of the equations

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \ldots = \frac{dx_n}{P_n} = \frac{dZ}{R}.$$

Using this result, it can be seen that the general solution of (4.9) is of the form

$$F(u(x_1,x_2,Z,\theta), v(x_1,x_2,Z,\theta), w(x_1,x_2,Z,\theta)) = 0$$

where u, v and w which are the functions of x_1,x_2,Z and θ are the solutions of the ordinary differential equation

$$\frac{dx_1}{1} = \frac{dx_2}{1} = \frac{-d\theta}{a(\theta)} = \frac{dZ}{b(\theta)}.$$

From the first pair, we get

$$x_2 = x_1 + c_1$$

or

$$c_1 = x_2 - x_1.$$

From the first and third, we get

$$x_1 = \int \frac{-1}{a(\theta)} d\theta + c_2$$

or

$$c_2 = x_1 + g(\theta), \text{ where } g(\theta) = \int \frac{d\theta}{a(\theta)}.$$

From the second and fourth, we get

$$z = b(\theta)\, x_2 + c_3$$

or

$$c_3 = z - b(\theta)\, x_2$$

Therefore we have $u = x_1 + g(\theta)$; $v = x_2 - x_1$ and $w = z - b(\theta)x_2$ in (4.10).

4.4 Bivariate Setting the Clock Back to Zero (2) Property

In terms of the bivariate LMP with independent marginals SCBZ property in bivariate case can be defined which we call as bivariate SCBZ(2) property.

Definition

A class of bivariate survival functions $\{R(x_1,x_2,\theta),\ x_1,x_2\geq 0,\ \theta\in\Theta\}$ is said to have bivariate SCBZ(2) property if the survival function satisfies the condition

$$R(x_1+t,x_2+s,\theta) = R(t,s,\theta)\ R(x_1,x_2,\theta^*) \qquad (4.11)$$

for every $x_1,x_2,t,s\geq 0$, $\theta^* = \theta^*(t,s) \in\Theta$.

By this property we mean that the conditional distribution of the additional time of survival of the components of the system given that the two components have survived t and s units respectively, remains in the original family of distributions itself.

Examples

1. Bivariate Pareto distribution with survival function (4.2)

In this case the new parameter vector $\theta^* = \left(\sigma_1^*,\sigma_2^*,\alpha\right)$ with

$\sigma_i^* = \sigma_i (1+\sigma_1 t+ \sigma_2 s)^{-1}$ for $i=1,2$.

2. Gumbel's bivariate exponential distribution with survival function (4.4)

Here the changed parameter is $\theta^*=(\lambda_1^*,\lambda_2^*,\delta)$ with $\lambda_1^*= \lambda_1+\delta s$, $\lambda_2^*= \lambda_2+\delta t$.

3. Bivariate Gompertz distribution with survival function (4.5)

where the new parameter vector is $\theta^* = (a,b,\gamma^*)$ with $\gamma^* = \gamma e^{at+bs}$.

4. Bivariate finite range distribution with survival function (4.6). Here $\theta^* = (p_1^*,\ p_2^*,\ d)$ where $p_1^* = p_1(1- p_1 t- p_2 s)^{-1}$ and $p_2^*= p_2(1- p_1 t- p_2 s)^{-1}$.

4.5 Conditional Setting the Clock Back to Zero (1) Property

In the direction of the local lack of memory property due to Nair and Nair (1988a), we can define the SCBZ property in a more precise manner, which we call, the conditional setting the clock back to zero (1) property. Let (X_1,X_2) be a random vector defined on R_2^+ with an absolutely continuous survival distribution $R(.,.,\theta)$. Let us denote the conditional survival function of X_i given $X_j>x_j$ by $G_i(.,x_j)$ for all $i,j= 1,2$, $i\neq j$.

That is

$$G_i(x_i,x_j)= P(X_i>x_i \mid X_j>x_j).$$

Definition

A family of survival distributions $\{R(x_1,x_2,\theta): x_1,x_2>0, \theta\in\Theta\}$ is said to have conditional SCBZ(1) property if it satisfies the equations

$$\left.\begin{aligned} \frac{G_1(t_1+s_1,t_2,\theta)}{G_1(t_1,t_2,\theta)} &= G_1(s_1,t_2,\theta^*) \\ \text{and} \qquad & \\ \frac{G_2(t_1,t_2+s_2,\theta)}{G_2(t_1,t_2,\theta)} &= G_2(t_1,s_2,\theta^{**}) \end{aligned}\right\} \qquad (4.12)$$

for all t_1,t_2, $s_1,s_2>0$ where θ^* and θ^{**} belong to Θ.

$$G_1(t_1,t_2,\theta) = P(X_1>t_1 \mid X_2>t_2,\theta)$$

$$= \frac{P(X_1>t_1,X_2>t_2,\theta)}{P(X_2>t_2,\theta)}$$

$$= \frac{R(t_1,t_2,\theta)}{R(0,t_2,\theta)} \qquad (4.13)$$

and

$$G_2(t_1,t_2,\theta) = \frac{R(t_1,t_2,\theta)}{R(t_1,0,\theta)}. \qquad (4.14)$$

This property ensures that the conditional distribution of the additional length of survival of a component given that it has already survived t_i units and the other has survived t_j units belongs to the same family of conditional distribution of the former component given the other.

Examples

1. Gumbel's bivariate exponential distribution with survival function (4.4). In this case $\theta^* = \theta^{**} = \theta$.
2. Bivariate Gompertz distribution with survival function (4.5). Here $\theta^* = (a, b, \gamma^*)$ with $\gamma^* = \gamma e^{at_1}$ and $\theta^{**} = (a, b, \gamma^{**})$ with $\gamma^{**} = \gamma e^{bt_2}$.

4.6 Conditional Setting the Clock Back to Zero (2) Property

In accordance with the conditional lack of memory property defined by Nair and Nair (1991), a new approach can be taken to define the SCBZ property in the bivariate case. We call that property as conditional SCBZ(2) property.

Let (X_1,X_2) be a non-negative random vector defined on R_2^+ with an absolutely continuous survival distribution $R(.,.,\theta)$. Denote the conditional survival function of X_i given $X_j=t_j$ by $S_i(t_i,t_j,\theta)$ for all $i,j=1,2$; $i\neq j$.

That is

$$S_i(t_i,t_j,\theta) = P(X_i \geq t_i \mid t_j \leq X_j \leq t_j+dt_j),\ i,j = 1,2;\ i \neq j.$$

where dt_j is a small increment in t_j.

Definition

A family of survival distributions $\{R(t_1,t_2,\theta),\ t_1,t_2>0,\ \theta\in\Theta\}$ is said to have conditional SCBZ(2) property if it satisfies the equations

$$\left.\begin{aligned} \frac{S_1(t_1+s_1,t_2,\theta)}{S_1(t_1,t_2,\theta)} &= S_1(s_1,t_2,\theta^*) \\ \text{and} \\ \frac{S_2(t_1,s_2+t_2,\theta)}{S_2(t_1,t_2,\theta)} &= S_2(t_1,s_2,\theta^{**}) \end{aligned}\right\} \tag{4.15}$$

for all $t_1,t_2,\ s_1,s_2>0$ where $\theta^*,\ \theta^{**}\in\Theta$.

$$S_1(t_1,t_2,\theta) = P(X_1\geq t_1 \mid t_2\leq X_2\leq t_2+dt_2,\ \theta)$$

$$= \frac{P(X_1\geq t_1, t_2\leq X_2\leq t_2+dt_2,\theta)}{P(t_2\leq X_2\leq t_2+dt_2,\theta)}$$

$$= \frac{\dfrac{-\partial R(t_1,t_2,\theta)}{\partial t_2}}{\dfrac{-\partial R(0,t_2,\theta)}{\partial t_2}} \tag{4.16}$$

and

$$S_2(t_1,t_2,\theta) = \frac{\dfrac{-\partial R(t_1,t_2,\theta)}{\partial t_1}}{\dfrac{-\partial R(t_1,0,\theta)}{\partial t_1}}. \tag{4.17}$$

Examples

1. Gumbel's bivariate exponential distribution with survival function (4.4). In this case $\theta^* = (\lambda_1, \lambda_2^*, \delta)$ with $\lambda_2^* = \lambda_2 + \delta t_1$ and $\theta^{**} = (\lambda_1^*, \lambda_2, \delta)$ with $\lambda_1^* = \lambda_1 + \delta t_2$.
2. Bivariate Gompertz distribution with survival function (4.5). Here $\theta^* = (a, b, \gamma^*)$ with $\gamma^* = \gamma e^{at_1}$ and $\theta^{**} = (a, b, \gamma^{**})$ with $\gamma^{**} = \gamma e^{bt_2}$.
3. Consider the bivariate distribution with exponential conditionals due to Arnold and Strauss (1988) with joint probability density function

$$f(t_1, t_2, \theta) = \exp\{m t_1 t_2 - a t_1 - b t_2 + c\}, \quad t_1, t_2 > 0$$

where $\theta = (m, a, b)$ with $\Theta = \{\theta\colon a, b > 0,\ m \leq 0\}$. Here $\theta^* = \theta^{**} = \theta$.

4.7 Reliability Measures

In view of the importance of SCBZ property in reliability theory, this section establishes the various relationships of SCBZ property with reliability characteristics.

The following theorem gives the condition of bivariate SCBZ(2) property in terms of the scalar failure rate.

Theorem 4.3

Bivariate SCBZ(2) property implies

$$a(x_1+t,\ x_2+s,\ \theta) = a(x_1,\ x_2, \theta^*)$$

where $a(.,.,\theta)$ is the scalar failure rate defined by Basu (1971).

Proof

For the proof of the theorem let us assume the condition (4.11). Then on taking the logarithms on both sides of (4.11), we get

$$\log R(x_1+t, x_2+s, \theta) = \log R(t, s, \theta) + \log R(x_1, x_2, \theta^*)$$

Differentiating w.r.t. x_1 and x_2, we get

$$\frac{\partial^2 \log R(x_1+t, x_2+s, \theta)}{\partial x_1 \partial x_2} = \frac{\partial^2 \log R(x_1, x_2, \theta^*)}{\partial x_1 \partial x_2}.$$

That is

$$a(x_1+t,\ x_2+s,\ \theta) = a(x_1,\ x_2, \theta^*) \qquad (4.18)$$

The bivariate scalar mean residual life $r(x_1, x_2, \theta)$ has a closed form when the family has SCBZ(2) property.

$$r(x_1, x_2, \theta) = E_\theta[(X_1 - x_1)(X_2 - x_2) \mid X_1 > x_1,\ X_2 > x_2]$$

$$= \frac{1}{R(x_1, x_2, \theta)} \int_{x_1}^{\infty}\int_{x_2}^{\infty} R(t_1, t_2, \theta) dt_1 dt_2$$

$$= \int_0^\infty \int_0^\infty \frac{R(x_1 + t_1, x_2 + t_2, \theta)}{R(x_1, x_2, \theta)} dt_1 dt_2$$

$$= \int_0^\infty \int_0^\infty R(t_1, t_2, \theta^*) dt_1 dt_2$$

$$= E_{\theta^*}(X_1 X_2), \tag{4.19}$$

which is the expectation of X_1 X_2 with parameter θ^*.

Theorem 4.4

Bivariate SCBZ(2) property implies

$$r(x_1+t,\ x_2+s,\ \theta) = r(x_1,\ x_2, \theta^*).$$

Proof

The bivariate SCBZ(2) property implies (4.11), which is

$$R(x_1+t, x_2+s, \theta) = R(t,s,\theta)\ R(x_1,x_2,\theta^*)$$

with $\theta^*(t,s) \in \Theta$ and $x_1,x_2,t,s \geq 0$.

Then for any y_1, y_2 we have

$$R(y_1+t, y_2+s, \theta) = R(t,s,\theta)\ R(y_1,y_2,\theta^*) \tag{4.20}$$

On dividing (4.20) by (4.11) we get

$$\frac{R(y_1 + t, y_2 + s, \theta)}{R(x_1 + t, x_2 + s, \theta)} = \frac{R(y_1, y_2, \theta^*)}{R(x_1, x_2, \theta^*)}$$

or

$$\int_{x_1}^{\infty}\int_{x_2}^{\infty}\frac{R(y_1+t,y_2+s,\theta)}{R(x_1+t,x_2+s,\theta)}dy_1dy_2 = \int_{x_1}^{\infty}\int_{x_2}^{\infty}\frac{R(y_1,y_2,\theta^*)}{R(x_1,x_2,\theta^*)}dy_1dy_2 .$$

That is

$$r(x_1+t,\ x_2+s,\ \theta) = r(x_1,\ x_2,\theta^*). \qquad (4.21)$$

As the mean residual life is practically more useful than the failure rates, Theorem 4.4 is any way necessitate our study.

The equivalent condition of conditional SCBZ(1) property in terms of the vector valued failure rate defined in (2.41) is established in the following theorem.

Theorem 4.5

The conditional SCBZ(1) property is equivalent to

$$h_1(t_1+s_1,\ t_2,\ \theta) = h_1(s_1,\ t_2,\theta^*) \qquad (4.22)$$

and

$$h_2(t_1,\ t_2+s_2,\ \theta) = h_2(t_1,\ s_2,\theta^{**}) \qquad (4.23)$$

with $\theta^* = \theta^*(t_1)\in\Theta$ and $\theta^{**} = \theta^{**}(t_2)\in\Theta$ where $h_i(.,.,\theta)$ is the i^{th} component of the vector failure rate

Proof

Conditional SCBZ(1) property means (4.12).

That is

$$G_1(t_1+s_1,t_2,\theta) = G_1(t_1,t_2,\theta)\ G_1(s_1,t_2,\theta^*) \tag{4.24}$$

and

$$G_2(t_1,\ t_2+s_2,\theta) = G_2(t_1,t_2,\theta)\ G_2(t_1,s_2,\theta^{**}). \tag{4.25}$$

Taking the logarithm on both sides of (4.24), we get

$$\log G_1(t_1+s_1,t_2,\theta) = \log G_1(t_1,t_2,\theta) + \log G_1(s_1,t_2,\theta^*)$$

Differentiating with respect to s_1, we get

$$\frac{\dfrac{\partial G_1(t_1+s_1,t_2,\theta)}{\partial s_1}}{G_1(t_1+s_1,t_2,\theta)} = \frac{\dfrac{\partial G_1(s_1,t_2,\theta^*)}{\partial s_1}}{G_1(s_1,t_2,\theta^*)}$$

That is

$$\frac{\dfrac{-\partial\dfrac{R(t_1+s_1,t_2,\theta)}{R(0,t_2,\theta)}}{\partial s_1}}{\dfrac{R(t_1+s_1,t_2,\theta)}{R(0,t_2,\theta)}} = \frac{\dfrac{-\partial\dfrac{R(s_1,t_2,\theta^*)}{R(0,t_2,\theta^*)}}{\partial s_1}}{\dfrac{R(s_1,t_2,\theta^*)}{R(0,t_2,\theta^*)}} \tag{4.26}$$

That is

$$\frac{\dfrac{-\partial R(t_1+s_1,t_2,\theta)}{\partial s_1}}{R(t_1+s_1,t_2,\theta)} = \frac{\dfrac{-\partial R(s_1,t_2,\theta^*)}{\partial s_1}}{R(s_1,t_2,\theta^*)}$$

or we have (4.22).In a similar line we can obtain (4.23).

Retracing the steps backward we get (4.26) and then on integrating within the range (s_1,∞), we have

$$\log G_1(t_1+s_1,t_2,\theta) = \log G_1(s_1,t_2,\theta^*) + c_1(t_1,t_2,\theta)$$

Taking $s_1 \to 0$, we get

$$\log G_1(t_1,t_2,\theta) = c_1(t_1,t_2,\theta)$$

Therefore

$$\log G_1(t_1+s_1,t_2,\theta) = \log G_1(t_1,t_2,\theta) + \log G_1(s_1,t_2,\theta^*)$$

or

$$G_1(t_1+s_1,t_2,\theta) = G_1(t_1,t_2,\theta)\, G_1(s_1,t_2,\theta^*),$$

which is (4.24).

Similarly we can obtain (4.25) also and they together show the conditional SCBZ(1) property.

As in the case of failure rate, we can obtain the necessary and sufficient condition of conditional SCBZ(1) property in terms of the vector valued mean residual life, which is presented through the following theorem.

Theorem 4.6

The conditional SCBZ(1) property is equivalent to

$$r_1(t_1+s_1,\ t_2,\ \theta) = r_1(s_1,\ t_2,\theta^*) \tag{4.27}$$

and

$$r_2(t_1,\ t_2 + s_2,\ \theta) = r_2(t_1,\ s_2,\theta^{**}) \tag{4.28}$$

where $r_i(.,.,\theta)$ is the i^{th} component of the vector valued mean residual life.

Proof

Conditional SCBZ(1) property implies (4.24) with $\theta^* = \theta^*(t_1)\in\Theta$.

Then for $y_1>0$, we have

$$G_1(t_1+y_1,t_2,\theta) = G_1(t_1,t_2,\theta)\ G_1(y_1,t_2,\theta^*) \tag{4.29}$$

Dividing (4.29) by (4.22), we get

$$\frac{G_1(t_1+y_1,t_2,\theta)}{G_1(t_1+s_1,t_2,\theta)} = \frac{G_1(y_1,t_2,\theta^*)}{G_1(s_1,t_2,\theta^*)}$$

That is

$$\frac{R(t_1+y_1,t_2,\theta)}{R(t_1+s_1,t_2,\theta)} = \frac{R(y_1,t_2,\theta^*)}{R(s_1,t_2,\theta^*)}$$

or

$$\frac{1}{R(t_1+s_1,t_2,\theta)}\int_{s_1}^{\infty} R(t_1+y_1,t_2,\theta)dy_1 = \frac{1}{R(s_1,t_2,\theta^*)}\int_{s_1}^{\infty} R(y_1,t_2,\theta^*)dy_1$$

That is, we have (4.27).

In a similar way we can obtain

$$r_2(t_1,\ t_2+s_2,\ \theta) = r_2(t_1,\ s_2,\theta^{**}),$$

which is (4.28). For the converse part, retrace the steps backward to get

$$\frac{G_1(t_1+y_1,t_2,\theta)}{G_1(y_1,t_2,\theta^*)} = \frac{G_1(t_1+s_1,t_2,\theta^*)}{G_1(s_1,t_2,\theta^*)} = c_1(t_1,\ t_2,\theta) \tag{4.30}$$

Letting $s_1\to 0$ in (4.30), we get

$$c_1(t_1,\ t_2,\theta) = G_1(t_1,\ t_2,\theta)$$

Hence, we have (4.24). Similarly we can obtain (4.25) also.

Thus the desired conditional SCBZ(1) property is provred.

Remark

When the random vector (X_1, X_2) has conditional SCBZ(1) property, then $r_i(x_1, x_2, \theta)$ has a closed form

$$
\begin{aligned}
r_1(x_1, x_2, \theta) &= \frac{1}{R(x_1, x_2, \theta)} \int_{x_1}^{\infty} R(t_1, x_2, \theta)\, dt_1 \\
&= \frac{\int_0^{\infty} R(t_1 + x_1, x_2, \theta)\, dt_1}{R(x_1, x_2, \theta)} \\
&= \int_0^{\infty} R(t_1, x_2, \theta^*)\, dt_1 ,\ \theta^*(x_1) \in \Theta \\
&= E_{\theta^*}(X_1 | X_2 > x_2). \qquad (4.31)
\end{aligned}
$$

Similarly

$$
r_2(x_1, x_2, \theta) = E_{\theta^{**}}(X_2 | X_1 > x_1), \qquad (4.32)
$$

where $\theta^{**} = \theta^{**}(x_2) \in \Theta$.

When $s_1 \to 0$ in the condition

$$
h_1(t_1 + s_1, t_2, \theta) = h_1(s_1, t_2, \theta^*)
$$

we get

$$
\begin{aligned}
h_1(t_1, t_2, \theta) &= h_1(0, t_2, \theta^*) \\
&= k_1(t_2, \theta^*),
\end{aligned}
$$

which is independent of t_1. It can also be shown that $h_2(t_1, t_2, \theta)$ is independent of t_2.

Just as in the case of conditional SCBZ(1) property the equivalence of conditional SCBZ(2) property in terms of reliability characteristics are studied in the following theorems.

Theorem 4.7

The conditional SCBZ(2) property is equivalent to

$$c_1(t_1+s_1,\ t_2,\ \theta) = c_1(s_1,\ t_2, \theta^*) \tag{4.33}$$

and

$$c_2(t_1,\ t_2 + s_2,\ \theta) = c_2(t_1,\ s_2, \theta^{**}) \tag{4.34}$$

where

$$c_i(t_1,\ t_2,\theta) = \frac{\dfrac{\partial^2 R(x_1,x_2,\theta)}{\partial t_1 \partial t_2}}{\dfrac{-\partial R(x_1,x_2,\theta)}{\partial t_j}} \quad \text{for } i,\ j = 1,2;\ i \neq j.$$

Proof

Conditional SCBZ(2) property implies (4.15). That is

$$S_1(t_1+s_1,t_2, \theta) = S_1(t_1,t_2,\theta)\ S_1(s_1,t_2,\theta^*) \tag{4.35}$$

and

$$S_2(t_1,\ t_2+s_2, \theta) = S_2(t_1,t_2,\theta)\ S_2(t_1,s_2,\theta^{**}). \tag{4.36}$$

Differentiating the first equation (4.35) with respect to s_1, we get

$$\frac{\partial \log S_1(t_1+s_1,t_2,\theta\ )}{\partial s_1} = \frac{\partial \log S_1(s_1,t_2,\theta^*)}{\partial s_1}$$

or

$$\frac{\dfrac{\partial S_1(t_1+s_1,t_2,\theta\ )}{\partial s_1}}{S_1(t_1+s_1,t_2,\theta\ )} = \frac{\dfrac{\partial S_1(s_1,t_2,\theta^*)}{\partial s_1}}{S_1(s_1,t_2,\theta^*)}$$

That is

$$\frac{-\dfrac{\partial^2 R(t_1+s_1,t_2,\theta)}{\partial t_1 \partial t_2}}{-\dfrac{\partial^2 R(t_1+s_1,t_2,\theta)}{\partial t_2}} = \frac{-\dfrac{\partial^2 R(t_1+s_1,t_2,\theta^*)}{\partial t_1 \partial t_2}}{-\dfrac{\partial^2 R(t_1+s_1,t_2,\theta^*)}{\partial t_2}}$$

and is same as (4.33). Similarly, we can obtain (4.34) also.

Retracing the steps backward and on integration, we get

$$\log S_1(t_1+s_1,t_2,\theta) = \log S_1(s_1,t_2,\theta^*) + k_1(t_1,t_2,\theta)$$

Taking $s_1 \to 0$, we get

$$\log S_1(t_1,t_2,\theta) = k_1(t_1,t_2,\theta).$$

Hence

$$\log S_1(t_1+s_1,t_2,\theta) = \log S_1(t_1,t_2,\theta) + \log S_1(s_1,t_2,\theta^*)$$

and it implies (4.35). Similarly we will get (4.36) also and shows the conditional SCBZ(2) property.

According as the failure rate defined in (2.44), the mean residual life vector can be defined by $(r_1^*(x_1,x_2,\theta),\ r_2^*(x_1,x_2,\ \theta))$ where

$$r_i^*(x_1,x_2,\theta) = \mathrm{E}_\theta(X_i - x_i \mid X_i \ge x_i, X_j = x_j)\ i, j = 1,2;\ i \ne j.$$

$$= \frac{1}{\dfrac{\partial R(x_1,x_2,\theta)}{\partial x_j}} \int_{x_i}^{\infty} \frac{\partial R(t_i,x_j,\theta)}{\partial x_j} dt_i. \tag{4.37}$$

The following theorem gives the necessary and sufficient condition of conditional SCBZ(2) property.

Theorem 4.8

The conditional SCBZ(2) property is equivalent to

$$r_1^*(t_1+s_1, t_2, \theta) = r_1^*(s_1, t_2, \theta^*) \tag{4.38}$$

and

$$r_2^*(t_1, t_2+s_2, \theta) = r_2^*(t_1, s_2, \theta^{**}) \tag{4.39}$$

with $\theta^* = \theta^*(t_1) \in \Theta$ and $\theta^{**} = \theta^{**}(t_2) \in \Theta$.

Proof

Conditional SCBZ(2) property implies (4.35) and

$$S_1(t_1+y_1,t_2,\theta) = S_1(t_1,t_2,\theta)\, S_1(y_1,t_2,\theta^*) \tag{4.40}$$

Hence on dividing (4.40) by (4.35), we get

$$\frac{S_1(t_1+y_1,t_2,\theta)}{S_1(t_1+s_1,t_2,\theta)} = \frac{S_1(y_1,t_2,\theta^*)}{S_1(s_1,t_2,\theta^*)}$$

That is

$$\frac{\dfrac{\partial R(t_1+y_1,t_2,\theta)}{\partial t_2}}{\dfrac{\partial R(t_1+s_1,t_2,\theta)}{\partial t_2}} = \frac{\dfrac{\partial R(y_1,t_2,\theta^*)}{\partial t_2}}{\dfrac{\partial R(s_1,t_2,\theta^*)}{\partial t_2}}$$

or

$$\frac{\displaystyle\int_{s_1}^{\infty}\frac{\partial R(t_1+y_1,t_2,\theta)}{\partial t_2}dy_1}{\dfrac{\partial R(t_1+s_1,t_2,\theta)}{\partial t_2}} = \frac{\displaystyle\int_{s_1}^{\infty}\frac{\partial R(y_1,t_2,\theta^*)}{\partial t_2}dy_1}{\dfrac{\partial R(s_1,t_2,\theta^*)}{\partial t_2}}$$

which implies (4.38). In a similar line we can obtain (4.39) also Retracing the steps backward and by taking $s_1 \to 0$, we get (4.35) and (4.36), which shows the conditional SCBZ(2) property.

Remark

When the bivariate distribution has conditional SCBZ(2) property, r_i^* (t_1, t_2) has the closed form

$$r_1^*(x_1,x_2,) = E_\theta(X_1-x_1 \mid X_1 \geq x_1,\ X_2=x_2)$$

$$= \frac{\displaystyle\int_{x_1}^{\infty}\frac{\partial R(t_1,x_2,\theta)}{\partial x_2}dt_1}{\dfrac{\partial R(x_1,x_2,\theta)}{\partial x_2}}$$

$$= \frac{\displaystyle\int_{0}^{\infty}\frac{\partial R(x_1+t_1,x_2,\theta)}{\partial x_2}dt_1}{\dfrac{\partial R(x_1,x_2,\theta)}{\partial x_2}}$$

$$= \int_0^\infty \frac{\partial R(t_1, x_2, \theta^*)}{\partial x_2} dt_1$$

$$= E_{\theta^*}(X_1 | X_2 = x_2). \tag{4.41}$$

Similarly, we can obtain

$$r_2^*(x_1, x_2, \theta) = E_{\theta^{**}}(X_2 | X_1 = x_1). \tag{4.42}$$

4.8 Properties of Distributions with SCBZ Properties

In this section we establish the implications between the definitions of bivariate SCBZ(1) and bivariate SCBZ(2) properties and certain identities and characterizations of the property.

Theorem 4.9

Bivariate SCBZ(2) property implies the bivariate SCBZ(1) property.

Proof

Bivariate SCBZ(2) property implies (4.11). Puting $s=t$ in (4.11), we get

$$R(x_1+t, x_2+t, \theta) = R(t,t,\theta)\, R(x_1, x_2, \theta^*),$$

which is the SCBZ(1) property.

Theorem 4.10

The converse of the Theorem 4.9 need not be true

Proof

The theorem can be established by considering a counter example.

Consider the bivariate distribution with survival function

$$R(x_1,x_2,\theta) = \alpha e^{-(x_1+x_2)} + \beta e^{-\max(x_1,x_2)}\ ,\ x_1,x_2 \geq 0 \qquad (4.43)$$

with $\theta = (\alpha,\beta)$ where $0<\alpha<1$, $\alpha+\beta=1$.

$$\frac{R(x_1+t,x_2+t,\theta)}{R(t,t,\theta)} = \frac{\alpha e^{-(x_1+t+x_2+t)} + \beta e^{-\max(x_1+t,x_2+t)}}{\alpha e^{-(t+t)} + \beta e^{-\max(t,t)}}$$

$$= \frac{\alpha e^{-2t-(x_1+x_2)} + \beta e^{-t-\max(x_1,x_2)}}{\alpha e^{-2t} + \beta e^{-t}}$$

$$= \alpha^* \ e^{-(x_1+x_2)} + \beta^* \ e^{-\max(x_1,x_2)}$$

$$=' R(x_1,x_2,\theta^*)$$

where $\alpha^* = \dfrac{\alpha}{\alpha e^{-2t} + \beta e^{-t}} e^{-2t}$ and $\beta^* = \dfrac{\beta}{\alpha e^{-2t} + \beta e^{-t}}\ e^{-t}$ and $\theta^*=(\alpha^*,\beta^*)\in\Theta$ and

$0<\alpha^*<1$, $\alpha^*+\beta^*=1$.

That is, the given

$R(x_1,x_2,\theta)$ satisfies SCBZ(1) property. But

$$\frac{R(x_1+t,x_2+s,\theta)}{R(t,s,\theta)} = \frac{\alpha e^{-(x_1+t+x_2+s)} + \beta e^{-\max(x_1+t,x_2+s)}}{\alpha e^{-(t+s)} + \beta e^{-\max(t,s)}}$$

$$\neq R(x_1,x_2,\ \alpha^*,\beta^*),$$

for (α^*,β^*), element of the same parametric space. Hence $R(x_1,x_2,\theta)$ does not satisfies SCBZ(2) property.

Characterization of the family of survival functions having SCBZ(2) property in terms of the unique representation of the new parameters θ^* is given through the following theorem.

Theorem 4.11

The family of survival distributions $\{R(x_1,x_2,\theta)\colon \theta\in\Theta,\ x_1,x_2>0\}$ has bivariate SCBZ(1) property with $\theta^* = \theta+\delta t$ if and only if $R(x_1,x_2,\theta)$ is of the form

$$R(x_1,x_2,\theta) = \frac{s\left(x_1+\frac{\theta}{\delta},x_2+\frac{\theta}{\delta}\right)}{s\left(\frac{\theta}{\delta},\frac{\theta}{\delta}\right)}$$

Where $s(.,.)$ is a non-negative, non-increasing continuous function satisfying $s(0,\infty)= s(\infty,0)= s(\infty,\infty)=0$.

Proof

Bivariate SCBZ(1) property with $\theta^* = \theta+\delta t$ implies (4.1) with $\theta+\delta t$ in the place of θ^*. Then on taking limit as $\theta \to 0^+$ in (4.1), we get

$$R(x_1+t,x_2+t,0^+) = R(t,t,0^+)\ R(x_1,x_2,\delta t)$$

and hence

$$R(x_1,x_2,\delta t) = \frac{R(x_1+t,x_2+t,0^+)}{R(t,t,0^+)}$$

$$= \frac{s(x_1+t, x_2+t, 0^+)}{s(t,t)}$$

or

$$R(x_1,x_2,\theta) = \frac{s\left(x_1+\frac{\theta}{\delta}, x_2+\frac{\theta}{\delta}\right)}{s\left(\frac{\theta}{\delta}, \frac{\theta}{\delta}\right)}. \qquad (4.44)$$

For the converse part, let us assume the form (4.44) for $R(x_1,x_2,\theta)$. Then

$$\frac{R(x_1+t, x_2+t, \theta)}{R(t,t,\theta)} = \frac{s\left(x_1+t+\frac{\theta}{\delta}, x_2+t+\frac{\theta}{\delta}\right)}{s\left(t+\frac{\theta}{\delta}, t+\frac{\theta}{\delta}\right)}$$

$$= \frac{s\left(x_1+\frac{\theta+\delta t}{\delta}, x_2+\frac{\theta+\delta t}{\delta}\right)}{s\left(\frac{\theta+\delta t}{\delta}, \frac{\theta+\delta t}{\delta}\right)}$$

$$= R(x_1,x_2,\theta^*),$$

where $\theta^* = \frac{\theta+\delta t}{\delta} \in \Theta$, which implies the bivariate SCBZ(1) property.

The following theorems show that the bivariate SCBZ(2) property preserves also in the marginals and the family of the distributions of the minimum.

Theorem 4.12

The bivariate SCBZ(2) property of the vector (X_1,X_2) implies the univariate SCBZ property of the marginals.

Proof

Bivariate SCBZ(2) property implies (4.11) for every $x_1,x_2,t,s\geq 0$, $\theta^* = \theta^*(t,s) \in \Theta$. Letting t=0 and x_1=0, we get

$$R(0,x_2+s,\theta) = R(0,s,\theta)\ R(0,x_2,\theta^*)$$

That is

$$R_2(x_2+s,\theta) = R_2(s,\theta)\ R_2(x_2,\theta^*) \qquad (4.45)$$

where $R_2(.,\theta)$ denote the survival function of X_2. The relation (4.45) shows the SCBZ property of X_2.

In a similar way we can prove the SCBZ property of the random variable X_1.

Theorem 4.13

If (X_1,X_2) is a random vector having bivariate SCBZ(2) property, then Z=min(X_1,X_2) has univariate SCBZ property

Proof

Let $R(.,\theta)$ denote the survival function of Z. Then

$$R(x,\theta) = \mathrm{P}(Z\geq x,\theta)$$

$$= P(X_1 \geq x, X_2 \geq x, \theta)$$

$$= R(x,x,\theta)$$

and hence

$$R(x+s,\theta) = R(x+s,x+s,\theta)$$

$$= R(x,x,\theta)\ R(s,s,\theta^*),$$

$$= R(x,\theta)\ R(s,\theta^*),$$

where $\theta^* = \theta^*(x) \in \Theta$, which implies the SCBZ propertyof Z.

A similar property of the bivariate SCBZ(2) property explained in Theorem 4.12 also holds in the case of distributions having conditional SCBZ(1) property. And this is explained in the following theorem.

Theorem 4.14

The conditional SCBZ(1) property implies the SCBZ property of the marginals

Proof

Conditional SCBZ(1) property implies (4.24) and (4.25) for every $t_1,t_2,s_1,s_2>0$, θ^* and $\theta^{**} \in \Theta$. When $t_2 \to 0$ in the equation (4.24) we have

$$G_1(t_1+s_1,0,\theta) = G_1(t_1,0,\theta)\ G_1(s_1,0,\theta^*).$$

That is

$$R(t_1+s_1,0,\theta) = R(t_1,0,\theta)\ R(s_1,0,\theta^*)$$

or

$$R_1(t_1+s_1,\theta) = R_1(t_1,\theta)\ R_1(s_1,\theta^*),$$

which implies the SCBZ property of X_1. In a similar line, by setting $t_1 \to 0$ we can show that X_2 has univariate SCBZ property.

4.9 Preservation of SCBZ properties in bivariate equilibrium distribution

As mentioned in section 2.6, the bivariate equilibrium distribution has a density of the form

$$g(x_1,x_2,\theta) = R(x_1,x_2,\theta)/\mu, \quad x_1,x_2>0$$

where $R(x_1,x_2,\theta)$ is the survival function of the parent distribution. Let the original random vector be (X_1,X_2) and the random vector corresponding to the equilibrium distribution be (Y_1,Y_2). Let $G(y_1,y_2,\theta)$ be the survival function of (Y_1,Y_2). As in the case of univariate situation if $h_y(x_1,x_2,\theta)$ denote the scalar failure rate of (Y_1,Y_2) and $r(x_1,x_2,\theta)$ denote the scalar mean residual life of (X_1, X_2), we will get the identity (2.66), which is

$$h_y(x_1,x_2,\theta) = 1/r(x_1,x_2,\theta).$$

Theorem 4.15

The random vector (X_1,X_2) has the bivariate SCBZ(2) property if and only if (Y_1,Y_2) has the SCBZ(2) property.

Proof

Assume that (X_1,X_2) has the bivariate SCBZ(2) property. Then

$$G(x_1+t_1,x_2+t_2,\theta) = \mu^{-1} \int_{x_1+t_1}^{\infty} \int_{x_2+t_2}^{\infty} R(u,v,\theta)dudv$$

$$= \mu^{-1} \int_0^{\infty}\int_0^{\infty} R(u+x_1+t_1, v+x_2+t_2,\theta)dudv$$

$$= \mu^{-1} R(x_1,x_2,\theta) \int_0^{\infty}\int_0^{\infty} R(u+t_1, v+t_2,\theta^*)dudv$$

where $\theta^* = \theta^*(x_1,x_2)\in\Theta$. If μ^* denote $E_{\theta^*}(X_1X_2)$,

$$\mu^* R(x_1,x_2,\theta) = R(x_1,x_2,\theta) \int_0^{\infty}\int_0^{\infty} R(u,v,\theta^*)dudv$$

$$= \int_0^{\infty}\int_0^{\infty} R(u+x_1, v+x_2,\theta)dudv .$$

Hence

$$R(x_1,x_2,\theta) = \mu^{*-1} \int_0^{\infty}\int_0^{\infty} R(u+x_1, v+x_2,\theta)dudv .$$

Therefore

$$G(x_1+t_1,x_2+t_2,\theta)=\mu^{-1}\mu^{*-1}\int_0^\infty\int_0^\infty R(u+x_1,v+x_2,\theta)dudv\int_0^\infty\int_0^\infty R(u+t_1,v+t_2,\theta^*)dudv.$$

$$= G(x_1,x_2,\theta)\ G(t_1,t_2,\theta^*), \qquad (4.46)$$

which shows the bivariate SCBZ(2) property of the vector (Y_1,Y_2).

Bivariate SCBZ(2) property of the vector (Y_1,Y_2) implies (4.46) with $\theta^* = \theta^*(x_1,x_2)\in\Theta$.

Taking the derivative with respect to t_1 and t_2, we get

$$\frac{\partial^2 G(x_1+t_1,x_2+t_2,\theta)}{\partial t_1 \partial t_2} = G_1(x_1,x_2,\theta)\ \frac{\partial^2 G(t_1,t_2,\theta^*)}{\partial t_1 \partial t_2}. \qquad (4.47)$$

On dividing (4.47) by (4.46) we get

$$h_y(x_1+t_1,x_2+t_2,\theta) = h_y(t_1,t_2,\theta^*). \qquad (4.48)$$

That is

$$\int_{t_1}^\infty\int_{t_2}^\infty \left[\frac{R(x_1+s_1,x_2+s_2,\theta)}{R(x_1+t_1,x_2+t_2,\theta)} - \frac{R(s_1,s_2,\theta^*)}{R(t_1,t_2,\theta^*)}\right] ds_1 ds_2 = 0.$$

Since $R(.,.,\theta)$ is continuous, we have

$$\frac{R(x_1+s_1,x_2+s_2,\theta)}{R(x_1+t_1,x_2+t_2,\theta)} = \frac{R(s_1,s_2,\theta^*)}{R(t_1,t_2,\theta^*)}$$

or

$$\frac{R(x_1+t_1,x_2+t_2,\theta)}{R(t_1,t_2,\theta^*)} = \frac{R(x_1+s_1,x_2+s_2,\theta)}{R(s_1,s_2,\theta^*)} = c_1(x_1,x_2,\theta).$$

That is

$$R(x_1+t_1, x_2+t_2, \theta) = R(t_1, t_2, \theta^*)\ c_1(x_1, x_2, \theta). \tag{4.49}$$

Taking $t_1 \to 0$ and $t_2 \to 0$, we get

$$c_1(x_1, x_2, \theta) = R(x_1, x_2, \theta). \tag{4.50}$$

Hence, on substituting (4.50) in (4.49), we get

$$R(x_1+t_1, x_2+t_2, \theta) = R(t_1, t_2, \theta^*)\ R(x_1, x_2, \theta),$$

which shows the bivariate SCBZ(2) property of (X_1, X_2).

4.10 Multivariate Setting the Clock Back to Zero Properties

The SCBZ properties in the bivariate case can also be extended to more than two variables cases. In this section, we present the corresponding definitions, examples and some important results.

Let $(X_1, X_2, \ldots, X_n)$ be a non-negative random vector in the support of $R_n^+ = \{(x_1, x_2, \ldots, x_n, \theta): x_i, \geq 0, i=1,2,\ldots,n\}$ with the survival distribution $R(x_1, x_2, \ldots, x_n, \theta)$, $\theta \in \Theta$.

Definition 1

A class of multivariate life distributions $\{R(x_1, x_2, \ldots, x_n, \theta),\ x_i \geq 0, \theta \in \Theta\}$ is said to have multivariate SCBZ(1) property if for each $\theta \in \Theta$ and $x_0 \geq 0$, the survival function satisfies the condition

$$\frac{R(x_1+x_0,x_2+x_0,\ldots,x_n+x_0,\theta)}{R(x_0,x_0,\ldots,x_0,\theta)} = R(x_1,x_2,\ldots,x_n,\theta^*) \qquad (4.51)$$

where $\theta^* = \theta^*(x_0) \in \Theta$.

Examples

1. Multivariate Exponential distributions with survival function

$$R(x_1,x_2,\ldots,x_n,\theta) = \exp\left\{-\sum_{i=1}^{n} a_i x_i - \sum_{i<j} a_{ij} x_i x_j - \ldots - a_{12\ldots n} x_1 x_2 \ldots x_n\right\} \qquad (4.52)$$

where $\theta = (a_1, \ldots, a_n, a_{11}, \ldots, a_{12\ldots n})$ with $\Theta = \{\theta$: a_i's are non-negative$\}$.

Here $\theta^* = (a_1^*, \ldots, a_n^*, a_{11}^*, \ldots, a_{12\ldots n}^*)$ where

$$a_i^* = a_i + \sum_{i<j} a_{ij}\, x_0 + \ldots + a_{12\ldots n}\, x_0^{n-1}$$

and

$$a_{ij}^* = a_{ij} + \sum_{i<j<k} a_{ijk}\, x_0 + \ldots + a_{12\ldots n}\, x_0^{n-2}. \text{ etc.}$$

2. Multivariate analogue of the Marshall-Olkin bivariate exponential.

$$R(x_1,\ldots,x_n,\theta) = \exp\left\{-\sum_{j=1}^{n} \lambda_j x_j - \sum_{j_1<j_2} \lambda_{j_1 j_2} \max(x_{j_1}, x_{j_2}) - \ldots - \lambda_{12\ldots n} \max(x_1, x_2, \ldots, x_n)\right\}$$

(4.53)

where $\theta = (\lambda_1, \ldots, \lambda_n, \lambda_{12}, \ldots, \lambda_{12\ldots n})$

with $\Theta = \{\theta$: $\lambda_1, \ldots, \lambda_n, \lambda_{12}, \ldots, \lambda_{12\ldots n} \geq 0\}$. Here $\theta^* = \theta$.

3. Multivariate Lomax distribution with survival function

$$R(x_1,x_2,\ldots,x_n,\theta) = \left(1+\sum_{i=1}^{n} a_i x_i\right)^{-k}, \; x_i>0 \tag{4.54}$$

where $\theta = (a_1, \ldots, a_n, k)$ with $\Theta=\{\theta: a_i \geq 0, \forall\ i=1,2,\ldots,n, k>1\}$. In this case $\theta^* = (a_1^*, \ldots, a_n^*, k)$ where $a_i^* = \dfrac{a_i}{\left(1+\sum_{i=1}^{n} a_i x_0\right)}$.

4. Multivariate Gompertz distribution with survival function

$$R(x_1,x_2,\ldots,x_n,\theta) = \exp\left\{\gamma(1-e^{\alpha_1 x_1+\alpha_2 x_2+\ldots+\alpha_n x_n})\right\}, \; x_i\geq 0, \; i=1,2\ldots,n \tag{4.55}$$

$\theta = (\alpha_1, \ldots, \alpha_n, \gamma)$ with $\Theta=\{\theta: \alpha_i >0, \gamma\geq 1\}$. Here $\theta^* = (\alpha_1, \ldots, \alpha_n, \gamma^*)$ with $\gamma^* = \gamma\, e^{(\alpha_1+\alpha_2+\ldots+\alpha_n)x_0}$.

Definition 2

A class of multivariate life distributions $\{R(x_1,x_2,\ldots,x_n,\theta),\ x_i \geq 0, \theta\in\Theta\}$ is said to have multivariate SCBZ(2) property if for each $\theta\in\Theta$ and $t_1, t_2, \ldots, t_n \geq 0$, the survival function satisfies the condition

$$\frac{R(x_1+t_1, x_2+t_2, \ldots, x_n+t_n, \theta)}{R(t_1, t_2, \ldots, t_n, \theta)} = R(x_1,x_2,\ldots,x_n,\theta^*) \tag{4.56}$$

where $\theta^* = \theta^*(t_1, t_2, \ldots, t_n) \in\Theta$.

Examples

1. Multivariate Exponential distribution specified in (4.52). In this case $\theta^* = (a_1^*, \ldots, a_n^*, a_{11}^*, \ldots, a_{12\ldots n}^*)$ where

$$a_i^* = a_i + \sum_{i<j} a_{ij}\, t_j + \ldots + a_{12\ldots n}\, t_1 t_2 \ldots t_n$$

and

$$a_{ij}^* = a_{ij} + \sum_{i<j<k} a_{ijk}\, t_{ijk} + \ldots + a_{12\ldots n}\, t_1 t_2 \ldots t_n. \text{ etc.}$$

2. Multivariate Lomax distribution specified by (4.54)

Here $\theta^* = (a_1^*, \ldots, a_n^*, k)$ where $a_i^* = \dfrac{a_i}{\left(1+\sum_{i=1}^{n} a_i t_i\right)}$.

3 Multivariate Gompertz distribution with survival function (4.55).

In this case $\theta^* = (\alpha_1, \ldots, \alpha_n, \gamma^*)$ with $\gamma^* = \gamma\, e^{\alpha_1 t_1 + \alpha_2 t_2 + \ldots + \alpha_n t_n}$.

Theorem 4.16

Multivariate SCBZ(2) property implies Multivariate SCBZ(1) property, but the converse does not holds good.

Proof

Multivariate SCBZ(2) property implies (4.56), when $t_1 = t_2 = \ldots = t_n = x_0$, we get

$$\frac{R(x_1 + x_0, x_2 + x_0, \ldots, x_n + x_0, \theta)}{R(x_0, x_0, \ldots, x_0, \theta)} = R(x_1, x_2, \ldots, x_n, \theta^*)$$

which shows the multivariate SCBZ(1) property.

Now consider the multivariate analogue of Marshall-Olkin bivariate exponential distribution with survival function (4.53). We can observe that the condition for multivariate SCBZ(1) is satisfied by it while it does not satisfy the condition for multivariate SCBZ(2) property.

Theorem 4.17

Let $(X_1,X_2,\ldots,X_n)$ be a non-negative random vector in the support of R_n^+ having multivariate SCBZ(2) property. Then $Y=\min(X_1,X_2,\ldots,X_n)$ has univariate SCBZ property.

Proof

Let $R_y(.,\theta)$ denote the survival function of Y. Then

$$\frac{R_y(y+x_0,\theta)}{R_y(x_0,\theta)}=\frac{R(y+x_0,y+x_0,\ldots,y+x_0,\theta)}{R(x_0,x_0,\ldots,x_0,\theta)}$$

$$=R(y,y,\ldots,y,\theta^*)$$

where $\theta^* = \theta^*(x_0) \in \Theta$.

That is

$$\frac{R_y(y+x_0,\theta)}{R_y(x_0,\theta)}=R_y(y,\theta^*),$$

which shows the SCBZ property of Y.

Theorem 4.18

Multivariate SCBZ(2) property implies the SCBZ property of the marginals of less orders.

Proof

We have the marginal survival of k random variable as

$R_{j_1, j_2, \ldots, j_k}(x_{j_1}, x_{j_2}, \ldots, x_{j_k}, \theta) = R(0,0,\ldots,0,x_{j_1},x_{j_2},\ldots,x_{j_k},0,\ldots,0,\theta)$,

$j_1 < j_2 < \ldots < j_k$. Hence

$$\frac{R_{j_1,j_2,\ldots,j_k}(x_{j_1}+t_{j_1},\ldots,x_{j_k}+t_{j_k},\theta)}{R_{j_1,j_2,\ldots,j_k}(t_{j_1},\ldots,t_{j_k},\theta)} = \frac{R_{j_1,j_2,\ldots,j_k}(0,\ldots,0,x_{j_1}+t_{j_1},0,\ldots,0,x_{j_k}+t_{j_k},0,\ldots,0,\theta)}{R_{j_1,j_2,\ldots,j_k}(0,\ldots,0,t_{j_1},0,\ldots,0,t_{j_k},0,\ldots 0,\theta)}$$

$$= R(0,0,\ldots,0,x_{j_1},x_{j_2},\ldots,x_{j_k},0,\ldots,0,\theta^*)$$

$$= R_{j_1, j_2, \ldots, j_k}(x_{j_1},x_{j_2},\ldots,x_{j_k},\theta^*)$$

where $\theta^* = \theta^*(t_{j_1},t_{j_2},\ldots,t_{j_k})$, which implies the SCBZ property of the marginal survival of k random variables out of n random variables.

Definition 3

A class of multivariate life distributions $\{R(x_1,x_2,\ldots,x_n,\theta),\ x_i > 0,\ \theta\in\Theta\}$ is said to have multivariate conditional SCBZ property if for each $\theta\in\Theta$ it satisfies the equation

$$\frac{G_i(x_1,\ldots,x_i+t_i,x_{i+1},\ldots,x_n)}{G_i(x_1,\ldots,x_i,x_{i+1},\ldots,x_n)} = G_i(x_1,x_2,\ldots,x_{i-1},t_i,x_{i+1},\ldots,x_n,\theta^*) \quad (4.57)$$

for each $i = 1,2,\ldots,n$ and $\theta^* = \theta^*(x_i) \in\Theta$, where

$$G_i(x_1,x_2,\ldots,x_n,\theta) = \mathrm{P}(X_i>x_i|\ X_1>x_1,\ldots,\ X_{i-1}>x_{i-1},\ X_{i+1}>x_{i+1},\ldots,\ X_n>x_n,\theta)$$

$$= \frac{R(x_1,\ldots,x_n)}{R(x_1,\ldots,x_{i-1},0,x_{i=1},\ldots,x_n)}.$$

Example

Multivariate Lomax distribution with survival function (4.54). If

$b_i = \dfrac{1}{1+\sum_{j\neq i}^{n} a_j x_j}$, then $b_i^* = \dfrac{1}{1+b_i x_i}$.

CHAPTER 5

SETTING THE CLOCK BACK TO ZERO PROPERTY OF DISCRETE DISTRIBUTIONS

5.1 Introduction

In the last two chapters we were discussing the concept of SCBZ property in the continuous univariate as well as multivariate situations. The study of this property in the discrete set up is of more interest, since in actual practice the life of the components are measured in discrete time units, that is the time is measured discretely as the completed years of life or as number of cycles. The difficulties, in measuring the time continuously are discussed by authors like Xekalaki (1983), Cox (1972) and Kalbfleish and Prentice (1980). From this view point we try to define SCBZ property in the discrete cases- both for univariate and multivariate distributions. It can be noticed that, we will get almost all the results parallel to those we have obtained for continuous distributions with necessary modifications.

5.2 Univariate discrete SCBZ property. (Nair and Mini, 1999)

Let X be a discrete random variable defined on the support of $I^+=\{0, 1, 2, \ldots\}$ with a survival function $R(x, \theta)=P(X\geq x)$ and probability mass function $f(x, \theta)$. As in the case of continuous random variable the SCBZ property can be defined in the discrete case as follows.

Definition

A non-negative discrete random variable X defined on I^+ or its family of survival functions $\{R(x, \theta)\colon x\in I^+, \theta\in\Theta\}$ is said to have the SCBZ property if for each $\theta\in\Theta$ and x, $t \in I^+$, the survival function $R(x, \theta)$ satisfies the condition

$$R(x+t, \theta) = R(t, \theta)\, R(x, \theta^*) \tag{5.1}$$

where $\theta^* = \theta^*(t)\in\Theta$, the parametric space.

By this property we mean that the conditional distribution of additional time of survival of a device or an organism given that it has already survived t units of time remains in the same family. In the reliability context, this property ensures that the residual life distribution remains in the original family of distributions.

Examples

1. Consider the geometric distribution with survival function

$$R(x, \theta) = q^x,\ x = 0, 1, 2, \ldots;\ 0 \le q \le 1. \tag{5.2}$$

where $\theta = q$ and $\Theta = \{q: 0 \le q \le 1\}$. It can be noticed that

$$\frac{R(x+t,\theta)}{R(t,\theta)} = \theta^x = R(x, \theta^*),$$

where $\theta^* = \theta$.

2. Consider the Waring distribution specified in Irwin (1975) with survival function

$$R(x, \theta) = \frac{(b)_x}{(a)_x},\ x = 0, 1, 2, \ldots;\ a, b > 0 \tag{5.3}$$

where $(a)_x = a(a+1)\ldots(a+x-1)$; with $\theta=(a,b)$ and $\Theta=\{(a,b): a,b>0; a>b+1\}$. In this case

$$\frac{R(x+t,\theta)}{R(t,\theta)} = \frac{(b+t)_x}{(a+t)_x}$$

$$= R(x, \theta^*),$$

where $\theta^* = (a+t,\ b+t)$.

3. Consider the case of negative hypergeometric distribution with survival function

$$R(x,\theta) = \frac{\binom{k+n-x}{n-x}}{\binom{k+n}{n}}, \; x = 0,\, 1,\, 2,\, \ldots, n; \tag{5.4}$$

where $\theta=(k,n)$ and $\Theta = \{(k,n)\colon k,n>0\}$.

Here

$$\frac{R(x+t,\theta)}{R(t,\theta)} = \frac{\binom{k+n-t-x}{n-t-x}}{\binom{k+n-t}{n-t}}$$

$$= R(x,\theta^*),$$

where $\theta^* = (k,n\text{-}t)$.

5.2.1 Reliability Characteristics (Nair and Mini, 1999)

In this section we establish the equivalent conditions of SCBZ property in terms of reliability characteristics especially the failure rate and mean residual life.

Theorem 5.1

The SCBZ property is equivalent to

$$h(x+t,\theta) = h(x,\theta^*)$$

where $h(.,\theta)$ is the failure rate defined in (2.50).

Proof

The SCBZ property of the random variable X implies (5.1). Hence

$$R(x+t+1,\theta) = R(t,\theta)\, R(x+1,\theta^*). \tag{5.5}$$

On taking the difference between (5.1) and (5.5) and dividing by (5.1) we get

$$\frac{R(x+t,\theta)-R(x+t+1,\theta)}{R(x+t,\theta)} = \frac{R(x,\theta^*)-R(x+1,\theta^*)}{R(x,\theta^*)}$$

That is in terms of the failure rate $h(.,\theta)$, it can be written as

$$h(x+t,\theta) = h(x,\theta^*). \tag{5.6}$$

For the converse part use the relation (2.52) connecting $h(x,\theta)$ and $R(x,\theta)$

$$\frac{R(x+t,\theta)}{R(t,\theta)} = \frac{\prod_{y=0}^{x+t-1}[1-h(y,\theta)]}{\prod_{y=0}^{t-1}[1-h(y,\theta)]}$$

$$= \prod_{y=t}^{x+t-1}[1-h(y,\theta)]$$

$$= \prod_{y=0}^{x-1}[1-h(y+t,\theta)]$$

$$= \prod_{y=0}^{x-1}[1-h(y,\theta^*)].$$

Assuming (5.6), we can write

$$\frac{R(x+t,\theta)}{R(t,\theta)} = R(x,\theta^*),$$

which establishes the desired result.

From the above theorem, it is noticed that the failure rate is a function of the transformed parameter. That is

$$\begin{aligned} h(x,\theta) &= h(0,\theta^*) \\ &= g(\theta^*) \quad \text{(say)}. \end{aligned}$$

The following theorems give the two important results concerning the functional form of the failure rate and SCBZ property.

Theorem 5.2

If the failure rate is linear, then the family of survival function possesses the SCBZ propery.

Proof

The survival function of a discrete random variable defined on I^+ is uniquely determined by the equation (2.52). If $h(x,\theta) = a+bx$, we have

$$\frac{R(x+t,\theta)}{R(t,\theta)} = \frac{\prod_{y=0}^{x+t-1}(1-a-by)}{\prod_{y=0}^{t-1}(1-a-by)}$$

$$= \prod_{y=0}^{x-1}(1-a-bt-by)$$

$$= R(x,\theta^*),$$

where $\theta^* = (a+bt,\ b)$.

Theorem 5.3

If $h(x,\theta)$ = is a one to one function $g(.)$ of θ^*, then $g(\theta^*)$ uniquely determines the distribution.

Proof

Let the one to one function of $h(x,\theta)$ is $g(\theta^*)$. Since the failure rate uniquely related to the survival function by (2.52)

$$R(x,\theta) = \prod_{t=0}^{x-1}[1-h(t,\theta)]$$

$$= \prod_{t=0}^{x-1}[1-g(\theta^*)]$$

which implies that $g(\theta^*)$ uniquely determines the distribution.

Now we can think about the mean residual life. The following result gives a characterization of SCBZ property in terms of the mean residual life.

Theorem 5.4

SCBZ property of a discrete random variable with $f(0)=0$ is equivalent to

$$r(x+t,\theta) = r(x,\theta^*) \tag{5.7}$$

where $r(.,\theta)$ is the mean residual life defined in (2.51).

Proof

SCBZ property of the random variable implies (5.1) and (5.5). Also

$$R(u+t+1,\theta) = R(t,\theta)\, R(u+1,\theta^*).$$

Therefore

$$\frac{R(u+t+1,\theta)}{R(x+t+1,\theta)} = \frac{R(u+1,\theta^*)}{R(x+1,\theta^*)}.$$

That is

$$\frac{1}{R(x+t+1,\theta)} \sum_{u=x}^{\infty} R(u+t+1,\theta) = \frac{1}{R(x+1,\theta^*)} \sum_{u=x}^{\infty} R(u+1,\theta^*).$$

That is

$$r(x+t,\theta) = r(x,\theta^*)$$

For the converse part, let us assume the relation (5.7) with $f(0)=0$. Using (2.53),

$$\frac{R(x+t,\theta)}{R(t,\theta)} = \frac{\prod_{u=1}^{x+t-1}\left[\frac{r(u-1,\theta)-1}{r(u,\theta)}\right]}{\prod_{u=1}^{t-1}\left[\frac{r(u-1,\theta)-1}{r(u,\theta)}\right]}$$

$$= \prod_{u=t}^{x+t-1}\left[\frac{r(u-1,\theta)-1}{r(u,\theta)}\right]$$

$$= \prod_{u=0}^{x-1}\left[\frac{r(u+t-1,\theta)-1}{r(u+t,\theta)}\right]$$

$$= \prod_{u=0}^{x-1}\left[\frac{r(u-1,\theta^*)-1}{r(u,\theta^*)}\right]$$

$$= \prod_{u=1}^{x-1}\left[\frac{r(u-1,\theta^*)-1}{r(u,\theta^*)}\right]$$

$$= R(x,\theta^*),$$

which shows the SCBZ property.

5.2.2 Distribution of Partial Sums (Nair and Mini, 1999)

In this section we investigate certain characteristics of the distributions based on the partial sums. Let X be a discrete random variable defined on the set of non-negative integers with probability mass function $f(x, \theta)$, survival function $R(x, \theta) = P(X \geq x, \theta)$ and a finite mean μ. Then the random variable Y specified by

$$\begin{aligned} g(x, \theta) &= P(Y{=}x) \\ &= \mu^{-1} P(X{>}x) \\ &= \mu^{-1} R(x+1, \theta) \qquad (5.8) \end{aligned}$$

is said to have the distribution based on partial sums corresponding to X. Gupta (1979) has shown that Y is the residual lifetime of a component in a system where a component of life length X is replaced upon failure by another having the same distribution, so that it forms a renewal process. He also showed that the failure rate of Y is the reciprocal of the mean residual life of X. Some other properties are studied by Johnson and Kotz (1969). Nair and Hitha (1989) obtained certain relations between the failure rate and mean residual life function to characterize certain discrete distributions by considering the relevance of these models in reliability analysis.

Let $h_y(x, \theta)$ denote the failure rate of Y and $r(x, \theta)$ denote the mean residual life of X. Then we have

$$h_y(x, \theta) = [r(x, \theta)]^{-1}. \qquad (5.9)$$

The following theorem shows that the SCBZ property of the parent distribution preserves in the distribution based on partial sums.

Theorem 5.5

The SCBZ property of X implies the SCBZ property of Y.

Proof

Let $G(.,\theta)$ denote the survival function of Y.

$$G(x+t,\theta) = \sum_{u=x+t}^{\infty} g(u,\theta)$$

$$= \mu^{-1} \sum_{u=x+t}^{\infty} R(u+1,\theta)$$

$$= \mu^{-1} \sum_{u=0}^{\infty} R(x+t+u+1,\theta)$$

$$= \mu^{-1} \sum_{u=0}^{\infty} R(t,\theta)R(x+u+1,\theta^{*})$$

$$\mu^{*}R(t,\theta) = R(t,\theta) \sum_{u=1}^{\infty} R(u,\theta^{*})$$

$$= R(t,\theta) \sum_{u=0}^{\infty} R(u+1,\theta^{*})$$

$$= \sum_{u=0}^{\infty} R(t+u+1,\theta)$$

Therefore

$$R(t,\theta) = (\mu^*)^{-1}\sum_{u=0}^{\infty} R(t+u+1,\theta)$$

and

$$G(x+t,\theta) = (\mu\mu^*)^{-1}\sum_{u=0}^{\infty} R(t+u+1,\theta)\sum_{u=0}^{\infty} R(x+u+1,\theta^*)$$

$$= G(t,\theta)\, G(x,\theta^*), \tag{5.10}$$

which shows the SCBZ property of Y.

The converse of the above theorem need not holds always. If $f(0)=0$, then it holds. It is established in the following theorem.

Theorem 5.6

The SCBZ property of Y implies SCBZ property of X if $f(0)=0$.

Proof

SCBZ property of Y implies (5.10). Then as in the line of proof of Theorem 5.1 we can have

$$h_y(x+t,\theta) = h_y(x,\theta^*).$$

By the relation (5.9), we have

$$r(x+t,\theta) = r(x,\theta^*).$$

Then by Theorem 5.4, we have the desired result that X has SCBZ property.

5.3 SCBZ Property of Bivariate Distributions

The analogous discrete situations of the Chapter IV is discussed in this section. In the previous section we define the SCBZ property of univariate case. The extension of this property to the bivariate case is not unique and hence we can define it in various ways. Let (X_1,X_2) be a vector of random variables with support $I_2^+ = \{(x_1,x_2): x_1,x_2=0,1,2,\ldots\}$ and family of survival functions $\{R(x_1,x_2,\theta):(x_1,x_2)\in I_2^+, \theta \in\Theta\}$, where $R(x_1,x_2,\theta) = P(X_1\geq x_1,X_2 \geq x_2,\theta)$.

5.3.1 Bivariate SCBZ(1) Property

The natural extension of the definition of SCBZ property in univariate case leads to the following definition.

Definition

A family of survival distributions with support I_2^+ or the random vector (X_1,X_2) is said to have bivariate SCBZ(1) property if for each $\theta\in\Theta$ and x_1, x_2, $t=$ 0, 1, ..., the following condition

$$R(x_1+t,x_2+t,\theta) = R(t,t,\theta)\, R(x_1,x_2,\theta^*) \tag{5.11}$$

where $\theta^* = \theta^*(t) \in \Theta$, the parametric space holds.

Examples

1. Bivariate Waring distribution with survival function

$$R(x_1,x_2,\theta) = \frac{B(\alpha,\beta+x_1+x_2)}{B(\alpha,\beta)}, \ \alpha,\beta>0, \ x_1, x_2 = 0, 1, \ldots, \qquad (5.12)$$

where $B(\alpha,\beta) = \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ with $\theta = (\alpha,\beta)$.

(This distribution can be regarded as the discrete analogue of bivariate Pareto distribution and it belongs to the Pearson's system of discrete distributions(Ord, 1972)). Here

$$\frac{R(x_1+t,x_2+t,\theta)}{R(t,t,\theta)} = \frac{B(\alpha,\beta+x_1+x_2+2t)}{B(\alpha,\beta+2t)}$$

$$= R(x_1,x_2,\theta^*)$$

where $\theta^* = (\alpha,\beta+2t)$.

2. Bivariate negative hypergeometric distribution with survival function

$$R(x_1,x_2,\theta) = \frac{\binom{k+n-x_1-x_2}{n-x_1-x_2}}{\binom{k+n}{n}}, \qquad (5.13)$$

$x_1, x_2 = 0, 1, \ldots, n;\ n,k>0;\ x_1+x_2 \le n$ with $\theta = (k,n)$.

(Xekalaki (1983), by the slope to mean ordinate ratio method has shown that the continuous analogue of negative hypergeometric distribution is finite range distribution).

$$\frac{R(x_1+t, x_2+t, \theta)}{R(t,t,\theta)} = \frac{\binom{k+n-x_1-x_2-2t}{n-x_1-x_2-2t}}{\binom{k+n-2t}{n-2t}}$$

$$= R(x_1, x_2, \theta^*)$$

where $\theta^* = (k, n\text{-}2t)$.

3. A bivariate geometric distribution (discrete analogue of Gumbel's bivariate exponential) with survival function

$$R(x_1,x_2,\theta) = p_1^{x_1}\, p_2^{x_2}\, \alpha^{x_1 x_2},\ x_1,\ x_2 = 0,\ 1,\ \ldots \qquad (5.14)$$

$0<p_1,\ p_2<1;\ 0\le\alpha\le1;\ 1\text{-}\alpha \le(1\text{-}\ p_1\alpha)(1\text{-}\ p_2\alpha)$ where $\theta = (p_1,\ p_2,\alpha)$.

It can be noticed that $\theta^*=(p_1\alpha^t,\ p_2\alpha^t,\alpha)$.

5.3.2 Bivariate SCBZ(2) Property

The second definition to the SCBZ property of bivariate distributions is as follows

Definition

A random vector (X_1,X_2) defined on I_2^+ or its family of survival distributions $\{R(x_1,x_2,\theta),\ \theta\in\Theta,\ (x_1,x_2)\in I_2^+\}$ is said to have bivariate SCBZ(2) property if for each $\theta\in\Theta$ and all (x_1,x_2) and $(t_1,t_2)\in I_2^+$, it should satisfy the condition

$$R(x_1+t_1,x_2+t_2,\theta) = R(t_1,t_2,\theta)\ R(x_1,x_2,\theta^*) \qquad (5.15)$$

where $\theta^* = \theta^*(t_1,t_2)\in\Theta$.

Examples

1. Bivariate Waring distribution specified in (5.12). Here $\theta^*=(\alpha,\beta+t_1+t_2)$.
2. Bivariate negative hypergeometric distribution specified in (5.13) where $\theta^*=(k,n-t_1-t_2)$.
3. Bivariate geometric distribution with survival (5.14). It can be seen that $\theta^*=(p_1\alpha^{t_2}, p_2\alpha^{t_1},\alpha)$.

In terms of the local lack of memory property, the SCBZ property can be defined as in the subsequent section.

5.3.3 Conditional SCBZ (1) Property

Definition

A class of life distributions $\{R(x_1,x_2,\theta),\ \theta\in\Theta,\ (x_1,x_2)\in I_2^+\}$ or a vector (X_1,X_2) is said to have conditional SCBZ(1) property if for each $\theta\in\Theta$ and $s_1,\ s_2,\ t_1,\ t_2 = 0,\ 1,\ 2,\ \ldots$

$$\left.\begin{aligned} &G_1(t_1+s_1,\ t_2,\ \theta) = G_1(t_1,\ t_2,\ \theta)\ G_1(s_1,\ t_2,\ \theta^*) \\ &\text{and} \\ &G_2(t_1,\ t_2+s_2,\ \theta) = G_2(t_1,\ t_2,\ \theta)\ G_2(t_1,\ s_2,\ \theta^{**}) \end{aligned}\right\} \qquad (5.16)$$

where $\theta^* = \theta^*(t_1)$ and $\theta^{**} = \theta^{**}(t_2)$ belong to the same parametric space and $G_i(t_1,\ t_2,\ \theta) = \mathrm{P}(X_i\geq t_i|\ X_j\geq t_j,\ \theta),\ i,j = 1,2,\ i\neq j.$

Therefore

$$G_1(t_1,\ t_2,\ \theta) = \frac{R(t_1,t_2,\theta)}{R(0,t_2,\theta)}$$

and

$$G_2(t_1,\ t_2,\ \theta) = \frac{R(t_1,t_2,\theta)}{R(t_1,0,\theta)}.$$

Examples

1. In the case of the bivariate Waring distribution (5.12), $\theta^* = (\alpha,\beta+t_1)$ and $\theta^{**} = (\alpha,\beta+t_2)$.

2. Bivariate negative hypergeometric distribution specified in (5.13). Here $\theta = (k,n)$. In this case $\theta^{*}=(k,n-t_1)$ and $\theta^{**}=(k,n-t_2)$.

3. Bivariate geometric distribution with survival function (5.14). It can be seen that $\theta^{*}=(p_1, p_2\alpha^{t_1}, \alpha)$ and $\theta^{**}=(p_1\alpha^{t_2}, p_2, \alpha)$.

5.3.4 Conditional SCBZ(2) Property

In view of the conditional lack of memory property defined by Nair and Nair (1991), here we investigate to study the SCBZ property of one component when the value of other component is preassigned.

Definition

A class of bivariate survival functions $\{R(x_1,x_2,\theta),\ \theta\in\Theta,\ (x_1,x_2)\in I_2^{+}\}$ or a vector (X_1,X_2) is said to have conditional SCBZ(2) property if the following set of conditions is satisfied for each $\theta\in\Theta$ and $s_1, s_2, t_1, t_2 = 0, 1, 2, \ldots$

$$\left.\begin{aligned} &S_1(t_1+s_1, t_2, \theta) = S_1(t_1, t_2, \theta)\, S_1(s_1, t_2, \theta^{*}) \\ &\text{and} \\ &S_2(t_1, t_2+s_2, \theta) = S_2(t_1, t_2, \theta)\, S_2(t_1, s_2, \theta^{**}) \end{aligned}\right\} \tag{5.17}$$

where $\theta^{*}=\theta^{*}(t_1)$ and $\theta^{**}=\theta^{**}(t_2)\in\Theta$. $S_i(t_1, t_2, \theta) = P(X_i\geq t_i \mid X_j=t_j, \theta)$ for $i,j = 1,2;\ i\neq j$.

We have

$$S_1(t_1,\ t_2,\ \theta) = \mathrm{P}(X_1 \geq t_1 |\ X_2 = t_2,\ \theta)$$

$$= \frac{\mathrm{P}(X_1 \geq t_1, X_2 = t_2, \theta)}{\mathrm{P}(X_2 = t_2, \theta)}$$

$$= \frac{R(t_1, t_2, \theta) - R(t_1, t_2 + 1, \theta)}{R(0, t_2, \theta) - R(0, t_2 + 1, \theta)} \qquad (5.18)$$

and

$$S_2(t_1,\ t_2,\ \theta) = \frac{R(t_1, t_2, \theta) - R(t_1 + 1, t_2, \theta)}{R(t_1, 0, \theta) - R(t_1 + 1, 0, \theta)}. \qquad (5.19)$$

Examples

1. Consider the bivariate Waring distribution (5.12). Here

$$\frac{S_1(t_1 + s_1, t_2, \theta)}{S_1(t_1, t_2, \theta)} = \frac{B(\alpha, \beta + t_1 + t_2 + s_1) - B(\alpha, \beta + t_1 + t_2 + s_1 + 1)}{B(\alpha, \beta + t_1 + t_2) - B(\alpha, \beta + t_1 + t_2 + 1)}$$

$$= S_1(s_1,\ t_2,\ \theta^*)$$

where $\theta^* = (\alpha, \beta + t_1)$.

Similarly

$$\frac{S_2(t_1, t_2 + s_2, \theta)}{S_2(t_1, t_2, \theta)} = S_2(t_1,\ s_2,\ \theta^{**})$$

where $\theta^{**} = (\alpha, \beta + t_2)$.

2. For the bivariate negative hypergeometric distribution with survival function (5.13)

$$\frac{S_1(t_1+s_1,t_2,\theta)}{S_1(t_1,t_2,\theta)} = \frac{\binom{k+n-t_1-t_2-s_1}{n-t_1-t_2-s_1}-\binom{k+n-t_1-t_2-s_1-1}{n-t_1-t_2-s_1-1}}{\binom{k+n-t_1-t_2}{n-t_1-t_2}-\binom{k+n-t_1-t_2-1}{n-t_1-t_2-1}}$$

$$= S_1(s_1,\ t_2,\ \theta^*)$$

where $\theta^* =(k,n-t_1)$ and

$$\frac{S_2(t_1,t_2+s_2,\theta)}{S_2(t_1,t_2,\theta)} = S_2(t_1,\ s_2,\ \theta^{**})$$

where $\theta^{**}=(k,n-t_2)$.

3. In the case of bivariate geometric distribution with survival function (5.14),

$$\frac{S_1(t_1+s_1,t_2,\theta)}{S_1(t_1,t_2,\theta)} = \frac{p_1^{s_1}\alpha^{s_1t_2}(1-p_2\alpha^{t_1+s_1})}{(1-p_2\alpha^{t_1})}$$

$$= S_1(s_1,\ t_2,\ \theta^*)$$

where $\theta^*=(p_1,\ p_2\alpha^{t_1},\alpha)$ and

$$\frac{S_2(t_1,t_2+s_2,\theta)}{S_2(t_1,t_2,\theta)} = S_2(t_1,\ s_2,\ \theta^{**})$$

where $\theta^{**}=(p_1\alpha^{t_2},p_2,\alpha)$.

5.3.5 Extended Bivariate SCBZ Property

Rao et. al. (1993b) has extended the notion of SCBZ property to the bivariate continuous case as the one discussed in section 2.7. By applying a similar approach we can define the SCBZ property of the discrete case to that in bivariate case as follows.

Definition

A class of bivariate survival functions $\{R(x_1,x_2,\theta),\ \theta\in\Theta,\ (x_1,x_2)\in I_2^+\}$ or a vector (X_1,X_2) is said to have the extended bivaraite setting the clock back to zero property if for each $\theta\in\Theta$ and $x_1,x_2 = 0, 1, 2, \ldots$ the survival function satisfies the equations

$$\left.\begin{aligned} &R(x_1+t,\ t,\ \theta) = R(t,\ t,\ \theta)\, R(x_1,\ t,\ \theta^*) \\ \text{and} \\ &R(t,\ x_2+t,\ \theta) = R(t,\ t,\ \theta)\, R(t,\ x_2,\ \theta^{**}) \end{aligned}\right\} \tag{5.20}$$

with $\theta^*=\theta^*(t)\in\Theta_0$ and $\theta^{**}=\theta^{**}(t)\in\Theta_0$ where Θ_0 denotes the boundary of Θ.

Example

1. Consider the geometric distribution specified in (5.14)

$$\frac{R(x_1+t,t,\theta)}{R(t,t,\theta)} = \frac{p_1^{x_1+t}p_2^t\alpha^{x_1t+t^2}}{p_1^tp_2^t\alpha^{t^2}}$$

$$= p_1^{x_1}\alpha^{tx_1}$$

$$= R(x_1,\ t,\ \theta^*)$$

where $\theta^*=(p_1,\ p_2^*,\alpha)$ with $p_2^*=1$.

Also

$$\frac{R(t,x_2+t,\theta)}{R(t,t,\theta)} = p_2^{x_2}\alpha^{tx_2}$$

$$= R(x_1,\ t,\ \theta^{**})$$

where $\theta^{**}=(p_1^*, p_2,\ \alpha)$ with $p_1^*=1$. Here θ^* and θ^{**} belong to the boundary Θ_0, which includes $0\le p_1, p_2\le 1$.

5.4 Properties of SCBZ property

In this section we establish certain implications between various definitions and some of their properties.

Theorem 5.7

Bivariate SCBZ(2) property implies bivariate SCBZ(1) property, but the converse is not true.

Proof

When $t_1 = t_2 = t$ in the equation (5.15) of bivariate SCBZ(2) property, it reduces to (5.11), which is the condition for bivariate SCBZ(1) property. For proving the converse part, let us consider a bivariate geometric distribution (the discrete analogue of Marshall-Olkin exponential distribution) with survival function (2.32). (2.32) can also be written as

$$R(x_1,x_2,\theta) = p^{\min(x_1,x_2)}\, p_1^{\max(0,x_1-x_2)}\, p_2^{\max(0,x_2-x_1)}, \tag{5.21}$$

$1+p \geq p_1+p_2$, $0<p\leq p_j<1$, $p = p_1+p_2 - c_3 -1$, where $c_3 = \dfrac{P(X_1 = x_1, X_2 = x_2)}{P(X_1 \geq x_1, X_2 \geq x_2)}$.

Here

$$\frac{R(x_1+t, x_2+t, \theta)}{R(t,t,\theta)} = p^{\min(x_1,x_2)}\, p_1^{\max(0,x_1-x_2)}\, p_2^{\max(0,x_2-x_1)}$$

$$= R(x_1,x_2,\theta^*)$$

with $\theta^* = \theta$. But

$$\frac{R(x_1+t, x_2+t, \theta)}{R(t_1,t_2,\theta)} = \frac{p^{\min(x_1+t_1,x_2+t_2)}\, p_1^{\max(0,x_1-x_2+t_1-t_2)}\, p_2^{\max(0,x_2-x_1+t_2-t_1)}}{p^{\min(t_1,t_2)}\, p_1^{\max(0,+t_1-t_2)}\, p_2^{\max(0,+t_2-t_1)}}$$

$$\neq R(x_1,x_2,\theta^*)$$

with $\theta^* \in \Theta$. Then, it can be noticed that the bivariate geometric distribution with survival function (5.21) holds bivariate SCBZ(1) property but does not holds bivariate SCBZ(2) property. Hence we can conclude that bivariate SCBZ(1) need not imply bivariate SCBZ(2) property.

The next theorem shows that the bivaraiate SCBZ(2) property implies the SCBZ properties of the marginal distributions.

Theorem 5.8

Bivariate SCBZ(2) property implies the SCBZ property of marginal distributions

Proof

Bivariate SCBZ(2) property implies (5.15). On setting $x_2 = t_2 = 0$, (5.15) becomes

$$R(x_1+t_1,0,\theta) = R(t_1,0,\theta)\, R(x_1,0,\theta^*).$$

That is, if $R_i(x_i,\theta) = P(X_i \geq x_i,\theta)$, we have

$$R_1(x_1+t_1,\theta) = R_1(t_1,\theta)\, R_1(x_1,\theta^*)$$

with $\theta^* = \theta^*(t_1) \in \Theta$, which indicates the SCBZ property of the component X_1. Similarly we can show the SCBZ property of X_2 also.

As in the case of continuous variables a parallel result of Theorem 4.13 holds for the bivariate discrete case also.

Theorem 5.9

The bivariate SCBZ(2) property of (X_1,X_2) implies the univariate SCBZ property of $Z = \min(X_1,X_2)$.

Proof

Let $R(.,\theta)$ denote the survival function of Z. Then

$$\begin{aligned} R(x+t,\theta) &= P(Z \geq x+t,\theta) \\ &= P(X_1 \geq x+t,\ X_2 \geq x+t,\theta) \\ &= R(x+t,x+t,\theta) \\ &= R(t,t,\theta)\, R(x,x,\theta^*) \end{aligned}$$

with $\theta^* = \theta^*(t) \in \Theta$. That is

$$R(x+t,\theta) = R(t,\theta)\, R(x,\theta^*),$$

which shows the desired result.

The property of bivariate SCBZ(2) property described in Theorem 5.8 holds for the conditional SCBZ(1) property also and is established in the following theorem.

Theorem 5.10

Conditional SCBZ(1) property of (X_1,X_2) implies SCBZ properties of X_1 and X_2.

Proof

Conditional SCBZ(1) property implies (5.16). When $t_2 = 0$ in the first equation of (5.16), we get

$$G_1(t_1+ s_1,0,\ \theta) = G_1(t_1,0,\ \theta)\ G_1(s_1,0,\ \theta^*).$$

That is

$$R(t_1+ s_1,0,\ \theta) = R(t_1,0,\ \theta)\ R(s_1,0,\ \theta^*).$$

That is

$$R_1(t_1+ s_1,\theta) = R_1(t_1,\ \theta)\ R_1(s_1,\ \theta^*)$$

with $\theta^* = \theta^*(t_1) \in \Theta$, where $R_1(t_1,\theta)$ is the survival function of X_1 which implies the SCBZ property of the component X_1. In a similar way we can show that X_2 has SCBZ property.

5.5 Distribution based on Partial Sums in Bivariate case

An appropriate bivariate extension of partial sum distributions are firstly given by Kotz and Johnson (1991). Let X_1 and X_2 are the original random variables with survival function $R(x_1,x_2,\theta)$. Then the random vector (Y_1,Y_2) corresponding to the partial sums has a probability density function of the form

$$g(x_1,x_2,\theta) = \mathrm{P}(Y_1=x_1,\ Y_2=x_2,\theta)$$

$$= \frac{P(X_1 > x_1, X_2 > x_2, \theta)}{E(X_1 X_2)}$$

$$= \frac{R(x_1+1, x_2+1, \theta)}{\mu}, \qquad (5.22)$$

where $\mu = \mathrm{E}_\theta(X_1\ X_2)$.

Stipulated along the lines of the univariate case, we can obtain the analogous result of Theorem 5.5.

Theorem 5.11

The bivariate SCBZ(2) property of (X_1,X_2) implies the bivariate SCBZ(2) property of (Y_1,Y_2).

Proof

Let $\mathrm{G}(.,.,\theta)$ denote the survival function of (Y_1,Y_2). Then

$$G(x_1,x_2,\theta) = \sum_{u=x_1}^{\infty}\sum_{v=x_2}^{\infty} g(u,v,\theta).$$

That is

$$G(x_1,x_2,\theta) = \sum_{u=x_1}^{\infty}\sum_{v=x_2}^{\infty} \frac{R(u+1,v+1,\theta)}{\mu}.$$

Therefore

$$G(x_1+t_1,x_2+t_2,\theta) = \mu^{-1} \sum_{u=x_1+t_1}^{\infty}\ \sum_{v=x_2+t_2}^{\infty} R(u+1,v+1,\theta)$$

$$=\mu^{-1} \sum_{u=0}^{\infty}\sum_{v=0}^{\infty} R(u+x_1+t_1+1,v+x_2+t_2+1,\theta)$$

$$=\mu^{-1} \sum_{u=0}^{\infty}\sum_{v=0}^{\infty} R(t_1,t_2,\theta)R(u+x_1+1,v+x_2+1,\theta^*), \quad (5.23)$$

since (X_1,X_2) has bivariate SCBZ(2) property. We have

$$\mu^* R(t_1,t_2,\theta) = \sum_{u=0}^{\infty}\sum_{v=0}^{\infty} R(t_1,t_2,\theta)R(u+1,v+1,\theta^*)$$

$$= \sum_{u=0}^{\infty}\sum_{v=0}^{\infty} R(u+t_1+1,v+t_2+1,\theta)$$

Therefore

$$R(t_1,t_2,\theta) =(\mu^*)^{-1} \sum_{u=0}^{\infty}\sum_{v=0}^{\infty} R(u+t_1+1,v+t_2+1,\theta) \quad (5.24)$$

Hence on substituting (5.24) in (5.23), we get

$$G(x_1+t_1, x_2+t_2, \theta) = (\mu\mu^*)^{-1} \sum_{u=0}^{\infty}\sum_{v=0}^{\infty} R(u+t_1+1, v+t_2+1, \theta)$$

$$\sum_{u=0}^{\infty}\sum_{v=0}^{\infty} R(u+x_1+1, v+x_2+1, \theta^*).$$

That is

$$G(x_1+t_1, x_2+t_2, \theta) = G(t_1, t_2, \theta)\; G(x_1, x_2, \theta^*),$$

which shows the desired result.

5.6 Conclusion

In the present study we formed a class of univariate continuous distributions that admit a partial differential equation. The general solution of that equation also can be derived. But the formulation of the PDE in the case of all the bivariate models cannot be possible. Eventhough a PDE in the case of bivariate SCBZ (1) property is obtained, we are not able to show that all the distributions admitting that PDE should hold that property.

It can be proved that the SCBZ property preserves in the equilibrium distributions of univariate continuous and discrete cases. But the converse of that result can be proved only in the case of continuous distributions. In chapter 4 we have obtained that bivariate SCBZ (2) property preserves in the equilibrium distribution in continuous distributions and vice versa. But the converse of the parallel result in the bivariate discrete case cannot be proved. The study of the preservation of conditional SCBZ (1) property also requires some interest. These problems are set for future work. Also the study of the measures for maintanance policies in the distributions with SCBZ properties are meant for our future work.

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