STUDIES IN SURVIVAL ANALYSIS

SURVIVAL ANALYSIS OF CHICKEN AND ASSOCIATED INVENTORY PROBLEMS

THESIS SUBMITTED FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

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CERTIFICATE

Certified that the work reported in the present thesis is based on the bonafide work done by Sri.Ravindranathan N. under my guidance in the School of Mathematical Sciences, Cochin University of Science and Technology, and has not been included in any other thesis submitted previously for the award of any degree.

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CHAPTER ONE

A BRIEF SURVEY

1.1 Introduction

Survival Analysis is mainly concerned with statistical models and methods for analysing data representing life times, waiting times or more generally times to the occurence of some specific events. Such data denoted as survival data can arise in various scientific fields. The statistical analysis on life time data has developed into an important topic especially in the Bio-medical Sciences and in the field of Engineering. Basically situations are considered in which the time to the occurence of some event are measured from some particular point. Mathematically one can think of life time as a non-negative valued variable and survival time is used not in literal sense but in figurative sense.

Numerous parametric and non-parametric models are used in the analysis of life time data and in the problems related to failure times. Among univariate models, distribution like exponential, Weibull, gamma and log normal occupy central role in survival analysis. Similarly life table techniques have been used widely to describe survival pattern using non-parametric models. A useful reference in this context is Johnson and Kotz (1970) which extensively covers all probability distributions. Cox (1959), Watson and Leadbetter (1964) Chiang (1968) have important research findings in this direction. The work of Kaplan and Meier (1958), Barclay (1958), Cox (1972) provide analytical methods for survival analysis using life table techniques. The stochastic study of the life table and its applications by Chiang (1961) is of relevence to this context. The work of Kalbfleisch and Prentice (1980) and Lawless (1982) on statistical models and methods for life time data describe numerous applications in different fields. Nelson (1982), Namboodiri and Suchindran (1987) review life time analysis and comparison of survival models. Empirical Bayesian estimates of age standardised relative risks for use in disease mapping is explained by Clayton (1987). Jones and Crowley (1989) presents a general class of non-parametric tests for survival analysis. Edmund and Siddique (1973) propose least square estimates for the parameters of survival distributions and a method is given for selecting distribution, based on the likelihood, under four survival models. Computerised simple regression methods for survival time studies are proposed by Kennedy and Gehan (1971).

Cox regression model (1972) has been analysed by White-Head (1980) using GLIM. The method of estimating survival functions here are based on the work of Baker and Nelder (1978) and Aitkin and Clayton (1980).

Comparative Bayesian and traditional inference on gamma modelled survival data is made by Alfred (1977) wherein two distinct methodologies are developed and compared for inference on gamma scale parameters in one and two population problems. Both approach permit concommitant variables and censored observations in the exponential case. Cornfield and Katherine (1977) have done life table analysis by taking the moments of the posterior probability density functions of the probability of surviving upto time t, p(t), are obtained assuming a time dependent poisson process for failures.

1.2 Survival Analysis of chicken

In most of Bio-medical studies, the basic observation is the time elapsed from one well defined event (say day of birth) to another well defined event (last day of productive life). Two difficulties arise in the statistical analysis of survival time. First, survival time distributions are positive valued and most of them are highly skewed in the positive direction. This positive

Skewness suggests the use of transformations or non-parametric procedures to reduce the influence of the infrequent extra ordinarily long survival time to provide better approximations by assymptotic theory. The second difficulty is the presence of censoring. In many studies of non-human population, it is necessary or at least desirable to analyse the data before all individuals in the population experience their terminating event. This phenomena is true for the survival analysis of chicken also.

Singh (1981), while studying poultry production, states that the most costly age at which mortality occurs is of sexual maturity. Nesheim et al (1979) conclude their study on chicken stating that the mortality rates among laying pullets is found high and less in older ages and most commercial farms experience a death loss of not more than ten percent. If this is distributed uniformly through out the year the effect on the cost of producing eggs is small especially in white leghorns and in other similar breeds. The difference between the inevitable depreciation and total loss by death is not significant. Portsmouth (1978) states that certain economic survey show that approximately ten percent of the birds die in their first laying year while a large portion succumbed when the birds reach maturity.

1.3 Poultry development in India and Kerala

Poultry farming is emerging as an important activity for enhancing nutrition and providing employment. The decade of 1980 has seen poultry emerge as the fastest growing sector. In the next five years, the annual production is expected to cross 450 million broilers. The scenario represents a challenge which safely predicts poultry to spread to its wings far and wide. Today India ranks as the world's fifth largest egg producing country but in terms of per capita availability it would rank among the lowest. A network of 500 commercial hatcheries and breeding farms, 100 commercial feed mills, large number of veterinary, pharmaceuticals and equipment manufacturers, units of Indian Council of Agricultural research including Agricultural Universities have made poultry farming a dynamic agro-business, duly supported by research and development. Of late there is a growing realisation about the importance of good quality, balanced and nutritive feeds and higher production. There is also an alert for minimising the incidence of disease outbreaks, through disease control projects. Inspite of all precautions, outbreaks of diseases continue to impede the progress of poultry production. The infra-structure for providing health cover to the birds in the country therefore needs to be strengthened.

It is in this context that importance of monitoring of disease outbreaks as well as alerting farmers about emerging diseases are of great significance to survival analysis. Disease surveillance and disease control methods at the required time periods of productive life of birds will help to devise disease control projects. Indian Council of Agricultural Research on Poultry has fixed the productive life of chicken as seventy two weeks from the day of hatch and breeding programmes are planned in this direction.

1.4 Study of Mortality pattern in chicken

There are many published reports on mortality patterns in chicken but the criteria adopted seem to be different. The analysis of mortality pattern by Duncliff (1913), Card and Kirkpatrik (1919), Alder (1934), Brunson and Godfrey (1952), Blakstone (1954), Barger et al. (1958), Tudor (1963) North et al (1972), Nesheim Maldem et al (1979) are all instances of mortality studies conducted abroad. In India, reports of Sundaram et al (1962), Prakash and Rajya (1970), Sivadas et al (1970), Jagadeesh Babu et al (1974) Srivastava (1984), Thyagarajan (1984), Khan et al. (1985), Chakraborthy et al (1985), Amritha Viswanathan et al. (1985), Pannerselvam (1987), Kalita et al (1988), Panda (1989), Rai et al (1989), Ravindranathan et al (1990), Ravindranathan (1994) show the large volume of research work carried out in chicken. From the studies it is reported that "Lymphold leucosis" disease occured in twenty percent of mortality cases. Similarly "Marek's disease","Coccidiosis", and other miscellaneous groups of diseases occured in twenty three, twenty six and twenty one percent respectively. In all studies is seen that the mortality occurs at a high level, around seven percent before fifth week and a peak of twenty percent in the age group of ten to fifteen weeks. A fall in mortality is observed since then and declines to almost Zero in the end. Another important finding in most of the research work especially of Jagadeesh Babu et al (1974) is that seventy five percent of the total mortality occurs before fifteen weeks. In the study of mortality pattern, Ravindranathan et al (1990) observes an exponential hazard function when the interval is grouped into class width of eight weeks age. Most of the research studies reveal that there are no differences between strains while studying mortality pattern when the extraneous factors are removed from the data.

Based on the above research findings, an attempt is made in the present study to develop a statistical model for mortality pattern of White leghorn chicken. The objective of the study is to predict the probability of survival at any instant of life time. A life table technique is also attempted to work out survival probability using non-parametric method to validate the statistical model. The study also aims to understand the death rate of chicken in their productive life to formulate different disease control projects. Another aspect of study in this thesis is about inventory management of Poultry birds. It appears that very few studies have been made in this direction. This study has been necessiated by the fact that the stock at hand should be known at least probabilistically in order to meet (possibly all) demands that take place for the chicken and at the time minimize the loss to the farm due to death of birds. The information that is gained from the survival analysis is of great advantage in the determination of the stock on hand.

Inventory of perishable items have been studied by several authors, Kaspi and Perry (1983), Kalpakam and Arivarignam (1985), Manoharan and Krishnamoorthy (1989), Krishnamoorthy, Narasimhalu and Basha (1992) describe models in this context. A review of perishable inventory upto 1982 can be found in Nahmias (1982).

Analysis of perishable inventories become more and more complex with weaker assumptions on the life times of items and the inter-arrival times between demands. A trade off between holding cost, loss due to perishability of items on one hand and the loss due to not meeting the demands on the other hand is what is needed. An attempt to investigate this is also made in this thesis.

Multicommodity inventory systems are analysed mostly in very simple situations, like deterministic arrival of demands, lead-time and so on. A departure from this is done by Sivazlian (1975). However, the method adopted in it is so complex that its practical utility is over shadowed. Recently Krishnamoorthy, Lakshmi and Basha (1994a) have considered two commodity inventory problems with demands arising for commodity at each demand epoch with specified probabilities. They (1994b) also examined a two commodity inventory problem with Markov shift in demand for the type of commodity demanded. In both these works the authors have investigated the system state probabilities in finite time and in the long run and also obtained the optimal policy. They also establish characterisation theoroms for the limiting probability distributions. An attempt is made in this thesis to introduce bulk demand of commodities thereby generalising their results.

1.5 Outline of the work done in this thesis

The thesis is devided into five chapters. Chapter one reviews the research work being carried out in the veterinary and allied fields on mortality of chicken and the objectives of the study. Chapter two describes important survival distributions and their role in survival analysis. These functional forms are made as a basis to develop a parabolic cum exponential hazard function for the productive life of chicken. Based on the mathematical form of hazard function, corresponding survival function is worked out with five unknown parameters. These parameters are estimated using method of least squares and conditional likelihood techniques. The survival probabilities obtained from the model and those from the observed data are found significantly correlated and maintain a good fit.

In Chapter three, a demographic approach is made to work out survival probabilities. The theory applicable to Cohort life tables is applied and seperate life tables are made for each strain of cohort of twenty thousand numbers each and also for whole data of one lakh numbers. The life history of chicken is presented through life tables and survival probability is worked out for each group at different ages by different methods. The survival probabilities obtained by these methods are then compared with those obtained from model values using appropriate test criteria. Detail discussion together with graphical presentations are made in this chapter.

In Chapter four an age dependent replacement inventory model for chicken is worked out. Here the assumption made is that the birds are replaced on attaining the age T (72 weeks) or death, whichever occurs first. In general, an exponential life time is assumed and demand pattern also is assumed to follow a compound Poisson process. The expression for the system state probability both for finite time and in the long run are obtained. Models using the data are also worked out for optimum ordering quantity.

In Chapter five, a two species inventory model is discussed. The joint distribution of the demand quantity is assumed to be general. Inter-arrival timings are assumed to follow a renewal process. An optimisation problem associated with this model is also worked out. Numerical illustration is also provided.

CHAPTER TWO

STATISTICAL MODEL OF MORTALITY IN CHICKEN*

2.1 Introduction

The high rate of mortality prevailing among chicken is an important factor besetting Poultry development in India. Even though techniques for controlling diseases have been identified and practiced to check onset of diseases, a substantial reduction in the mortality figures has not yet been achieved. There are number of research studies on chicken mortality but most of the studies seem to be concentrated on the causes of death as well as on differentials among various breeds. For instance, Mohan et. al.(1978) report the disease-wise survival pattern of chicken during the period from day of hatch to eight weeks of age while Chakraborthy et.al. (1985) discuss the incidence of mortality among four white leghorn strains and conclude that all strains have more or less the same pattern of mortality. Similar inferences have also been made by Jalaludin et.al.(1989).

It is widely acknowledged that the events, survival and death of an organism, are heavily dependent on the age and accordingly most analysis proceed along this line. However, in the case of chicken, other than some empirical studies like that of Suneja et.al. (1986) who observe that the

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percentage of deaths were high in the first week of life and thereafter it exhibited a decreasing trend, no worthwhile theoretical basis has been provided towards mortality analysis. A traditional way of explaining mortality pattern is by expressing the proportion (probability of) surviving as a function of age and this could be accomplished by rationalising observed facts through certain models of mortality behaviour. Such an approach would provide a more general theory valid over space, time and different species than empirical studies that give results specific to data it represents. Further, it can often result in deeper insight into the phenomena under investigation. With this objective in mind, a statistical model that depicts the mortality behaviour in chicken is worked out to draw certain inferences on mortality differentials with respect to age. For an associated work in this context, reference is made to Ravindranathan and Nair (1990) and also Ravindranathan(1994) which present survival analysis of chicken.

2.2 Basic concepts of survival distributions

Let X be the random variable representing life time. The survival function which gives the probability that a person chosen at random survives beyond age x is

$$S(x) = P\left(X > x\right) \tag{2.1}$$

This function provides the tool by which various characteristics that govern and influence the events, survival and death, are derived. The mathematical form of S(x) is obtained from the formula

$$S(x) = \exp\left[-\int_0^x h(t)\,dt\right] \tag{2.2}$$

Where h(x) stands for the probability that death occurs between the ages x and x+dx, conditioned on its survival to age x. The function h(x) is called the instantaneous death rate or force of mortality at age x, and its form is often postulated on the basis of knowledge about the process that governs the incidence of mortality. Two other quantities of interest are:

q_i = P (an individual dies between ages x_i and x_{i+1})

and $e_x = E\left(X-x \mid X > x\right)$

= average life time remaining of a unit which has survived age x

and as calculated as

$$e_x = \left[S(x)\right]^{-1} \int_x^{\infty} S(t)\,dt \tag{2.3}$$

The details regarding the above concepts and formulas are available in Lawless (1984).

2.3 Some important survival models

Numerous parametric models are used in the analysis of survival data and problems related to the modelling of failure process. In this context some important univariate distributions are to be mentioned because of their demonstrated usefulness in a wide range of situations. As a matter of fact, the motivation for using a particular model in a given situation is often mainly empirical which does not imply any absolute correctness of the model. The following are some of the important probability distributions used for survival analysis as stated by Lawless (1984) and Namboodiri (1987).

(a) Exponential distribution

The general form of probability density function of exponential type was considered by Sukhatme (1937), Epstein and Sobel (1953), Johnson and Kotz (1970) and by Galambos and Kotz (1978) for development of life time models. The Pdf of exponential distribution is

$$f(x) = \alpha \exp(-\alpha x) \tag{2.4}$$

with survival function

$$S(x) = \exp(-\alpha x) \tag{2.5}$$

and hazard function

$$h(x) = f(x)/S(x) = \alpha \tag{2.6}$$

(b) Weibull distribution

This is considered as an important distribution in survival analysis. This distribution was used by

Lieblein and Zelen (1956) for the study of life of deep groove ball bearings. This is perhaps the most widely used life time distribution and its applications in connection with life time of manufactured items have been widely advocated. It has been used as a model with diverse types of items such as in vaccum tubes by Kao(1959), in electrical insulation by Nelson (1972), in Bio-medical applications by Whittemore et al. (1976) and in many other situations. This distribution has a hazard function of the form $h(x) = \lambda\beta(\lambda x)^{\beta-1}$ (2.7)

where $\lambda > 0$, $\beta > 0$ are parameters. It includes the exponential distribution when β takes the value one. The survival function of this distribution is

$$S(x) = \exp\left[-(\lambda x)^{\beta}\right] \quad x > 0 \qquad (2.8)$$

and the p.d.f. is

$$f(x) = \lambda\beta(\lambda x)^{\beta-1} \quad \exp\left[-(\lambda x)^{\beta}\right], x > 0 \qquad (2.9)$$

(c) Extreme value distribution

This is a very closely related distribution to Weibull distribution and usually is referred to as Gumbel distribution (1958). In the situation where modelling is to be done for the data on natural calamity, the extreme value distribution plays an important role. The p.d.f. and survival function of this distribution are, respectively,

$$f(x) = \beta^{-1} \exp\left[\frac{x-u}{\beta} - \exp\left(\frac{x-u}{\beta}\right)\right] \quad -\infty < x < \infty, \; u > x \qquad (2.10)$$

$$S(x) = \exp\left[-\exp\left(\frac{x-u}{\beta}\right)\right] \quad -\infty < x < \infty, \; u > x \qquad (2.11)$$

(d) Gamma distribution

The Gamma distribution has a p.d.f. of the form

$$f(x) = \frac{\lambda(\lambda x)^{k-1} e^{-\lambda x}}{\Gamma K}, \quad x > 0 \qquad (2.12)$$

Where $k > 0$, $\lambda > 0$ are parameters: λ is a scale parameter and K is some times called the shape parameter. This distribution like the Weibull includes exponential as a special case (K=1). Integrating (2.12) we find the survival function as

$$S(x) = 1 - I(K, \lambda x) \qquad (2.13)$$

Where $I(K,x) = \frac{1}{\Gamma K} \int_0^x u^{K-1} e^{-u} \, du$

This distribution was used as a life time model by Gupta (1961) and Buckland (1964). Since the survival and hazard functions of this distribution are not expressible in a simple closed form, applications are found very much limited.

(e) Log normal distribution

This distribution has been widely used as a life time distribution model. It has been used in the analysis of survival time of electrical insulation by Nelson (1972) and for the study of bio-medical applications by Whittemore et al (1976). This distribution is described by saying that the life time X is log normally distributed if the logarithm Y=log X is normally distributed, say with mean μ and variance σ^2. The p.d.f of Y is

$$\frac{1}{(2\pi)^{\frac{1}{2}}\ \sigma}\ \exp\left[-\tfrac{1}{2}\left(\frac{Y-\mu}{\sigma}\right)^2\right] \qquad -\infty < Y < \infty$$

and from this p.d.f. of X = exp Y is found out as

$$f(x) = \frac{1}{(2\pi)^{\frac{1}{2}}\sigma . x}\ \exp\left[-\tfrac{1}{2}\left(\frac{\log x-\mu}{\sigma}\right)^2\right], \quad x > 0 \qquad (2.14)$$

The survival and hazard functions for the log normal distribution involve the standard normal distribution function

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{(2\pi)^{1/2}}\, e^{-u^2/2}\, du$$

The log normal survival function is easily seen to be

$S(x)=1-\Phi\left(\frac{\log x-\mu}{\sigma}\right)$ and the hazard function is given as $h(x) = f(x)/S(x)$

In addition, many other models are available and are in use. However, depending on the nature of applications and the form of hazard functions, a decision is to be taken on the selection of model.

2.4 Choice of a life time distribution

In the survival analysis, selection of the appropriate model is to be made by considering the context of study to select a particular family of models which may fit data on hand well. In some cases, past experience may have shown the model to give a good description of life time distributions from similar populations and so on. However, in situations where no model is singled out as being particularly appropriate, choice of a model is made as suggested by Lawless (1984) on the basis of (1) convenience of mathematically handling the model (2) statistical methods available in connection with the model and (3) the degree of complication of the calculations involved in using the model. A point to be noted here is that most of the commonly used models can handle situations that call for a monotone hazard function but are not capable to handle non-monotone functions. Hence three additional points are to be considered while developing survival models in the situations where a non-linear hazard function such as parabolic model is assumed in a given situation. First, the test of any model is to understand that it fits the available data. Second, even though model fits the data well, consequences of departures from the assumed model is to be studied. Finally, it is desirable to avoid strong assunptions about the model and non-parametric methods may be used to validate the model.

2.5 Statistical model of Mortality in chicken

As mentioned already, the statistical model for survival among chicken dictated by formula (2.2) requires that an appropriate functional form for hazard function h(x) has to be arrived at. To achieve this objective, for various Cohorts under observation, the data on deaths at successive ages (in weeks) show that there are two distinct phases in their mode of depletion. The first phase running from the day of hatch (reckoned as age Zero) to the end of the Fifteenth week exhibit a more or less uniform pattern of mortality that decreases from the initial stages of life for a few weeks and then gradually increases till the fifteenth week. A second degree curve of the form $h(x)=ax^2+bx+c$ is adopted to accommodate this behaviour. In the remaining period of life (taken as 16-72 week in the present study since the birds are having productive life only upto 72 weeks as per the norms of Indian Council of

Agricultural Research) there is a steady decline in the mortality rate suggesting an exponential form. Accordingly it is assumed that $h(x) = \alpha \exp(\beta x)$ for this period. These considerations lead to the following expressions for the survival function.

$$S(x) = \begin{cases} \exp\left[-\left(\frac{ax^3}{3} + \frac{bx^2}{2} + cx\right)\right], & 0 < x \leq 15 \\ \exp\left[-\frac{(15)^3}{3}a - \frac{(15)^2}{2}b - (15)c - \frac{\alpha}{\beta}\left(\exp(\beta x) - \exp(15\beta)\right)\right], & x > 15 \end{cases} \quad (2.15)$$

The corresponding probability density function of x is derived from (2.15) as $f(x) = -\frac{ds(x)}{dx}$. Equation (2.15) will be taken as the mathematical model of survival time of chicken. In this connection, it is observed that, except for reasons other than biological, the cut off point of 15 weeks between the two phases has remained stable in the follow up studies. It is also evident that any change in the boundary point can easily be accomodated as it is required to replace 15 with the new value.

2.6 Estimation of Parameters

In the interval (0,15), it can be seen that

$$Y_x = Ax^2+Bx+C \quad (2.16)$$

Where $Y_x = -\frac{1}{x}\log S(x)$, $A = \frac{a}{3}$, $B = \frac{b}{2}$. The method of least squares is applied to evaluate A,B and C after replacing Y_x by $\left[\log\left(lx/lo\right)\right]^{-1/x}$ where lo is the number of birds at age 0 (day of hatch) and lx is the number that has survived x weeks. For the second phase

$$S(x) = e^{P} \exp\left[-\frac{\alpha}{\beta}\exp(\beta x) - \exp(15\beta)\right] \qquad (2.17)$$

with $P = -\left[\frac{(15)^3}{3}a + \frac{(15)^2}{2}b + 15c\right]$

Equation (2.17) is equivalant to

$$Z_x = \log S_x = P - \frac{\alpha}{\beta}\left[\exp(\beta x) - \exp(15\beta)\right] \qquad (2.18)$$

From (2.18)

$$Z_{x+h} = p - \frac{\alpha}{\beta}\left[\exp(\beta x + \beta h) - \exp(15\beta)\right] \qquad (2.19)$$

$$Z_{x+2h} = P - \frac{\alpha}{\beta}\left[\exp(\beta x + 2\beta h) - \exp(15\beta)\right] \qquad (2.20)$$

so that

$$\exp(\beta h) = \frac{Z_{x+2h} - Z_{x+h}}{Z_{x+h} - Z_x} \qquad (2.21)$$

With Z_x equated to log (lx/lo), the value of β is estimated. Once β is evaluated, the least square estimates resulting from (2.16) are used in (2.18) to provide the estimate of α. Newton-Raphson method is used to get refined estimates of α

through successive iterations and let these estimates be α_0 and β_0.

The estimates obtained for α and β have been further refined by using conditional likelihood technique. Thus the following derivations are made to get refined estimates for α and β.

The likelihood function $L = \prod_{x_i > 15} f(x_i)$

Therefore

$$L = K^n \exp\left(-\frac{\alpha}{\beta}\right)\left[e^{\Sigma \beta x_i} - ne^{\beta(15)}\right] \frac{n}{\alpha} e^{\Sigma \beta x_i} \tag{2.22}$$

$$\log L = n \log K - \frac{\alpha}{\beta}\left[e^{\Sigma\beta x_i} - ne^{\beta(15)}\right] + n \log\alpha + \Sigma\beta x_i$$

$$\frac{\partial \log L}{\partial \alpha} = \left[-\frac{1}{\beta}\left[\Sigma f_i\, e^{\beta x_i} - ne^{\beta(15)}\right] + \frac{n}{\alpha}\right]$$

$$\frac{\partial \log L}{\partial \beta} = -\frac{\alpha}{\beta^2}\left[\beta(\Sigma f_i x_i\, e^{\beta x_i} - n(15)e^{15\beta}) - \left[\Sigma f_i\, e^{\beta x_i} - n.e^{15\beta}\right]\right] + \left[\Sigma x_i f_i\right]$$

$$\frac{\partial^2 \log L}{\partial \alpha^2} = -\frac{n}{\alpha^2}$$

$$\frac{\partial^2 \log L}{\partial \beta^2} = -\frac{\alpha}{\beta^4}\Big[\beta^3(\Sigma f_i x_i{}^2 e^{\beta x_i} - ne^{15\beta}(15)^2)$$

$$-2\beta^2(\Sigma f_i x_i e^{\beta x_i} - (15)e^{15\beta}(n))$$

$$+2\beta(\Sigma f_i e^{\beta x_i} - ne^{15\beta})\Big]$$

$$\frac{\partial^2 \log L}{\partial \alpha \partial \beta} = -\frac{1}{\beta^2}\Big[\beta(\Sigma f_i x_i e^{\beta x_i} - n(15)e^{15\beta}) - (\Sigma e^{\beta x_i} - ne^{15\beta})\Big]$$

If α_o, β_o are the initial solutions of α, β, the next approximation is given by

$$\begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix} = \begin{pmatrix}\alpha_0\\ \beta_0\end{pmatrix} - D^{-1}(X)$$ where D is the

information matrix = $\begin{pmatrix} \frac{\partial^2 \log L}{\partial \alpha^2} & \frac{\partial^2 \log L}{\partial \alpha \partial \beta} \\ \frac{\partial^2 \log L}{\partial \alpha \partial \beta} & \frac{\partial^2 \log L}{\partial \beta^2} \end{pmatrix}_{\text{at}(\alpha_o, \beta_o)}$

and $X = \begin{pmatrix} \frac{\partial \log L}{\partial \alpha} & \frac{\partial \log L}{\partial \beta} \end{pmatrix}_{\text{at}(\alpha_o, \beta_o)}$

Thus all the parameters in the model (2.15) have been estimated. The methods used for the estimation of parameters are the method of least squares and the conditional likelihood in which the model is expressed as a linear function of the parameters. In this context a reference is made to the study of Edmund et al. (1973) in which least square method is used for the estimation of parameters of survival distributions such as exponential, Gompertz and Weibull and Monte-Carlo technique is applied to validate the estimation procedure based on least square method and maximum likelihood method.

A software developed in the above lines for the estimation of the parameters a,b,c, α and β is given in Appendix 2

2.7 Data Analysis

The model proposed in (2.15) is applied to the data recorded in the All India Co-ordinated Research projects on Poultry breeding for the period 1987-1990 (Appendix 1) and the current data is collected from these units situated in Kerala, Madras and Hyderabad. The productive life considered in all these cases is from the day of hatch to seventy two weeks as prescribed by Government of India norms. Altogether, data on one lakh birds batched during the months of January to April (months fixed for hatching) and reared under homogeneous management practices are subjected to analysis. Care has been taken to exclude those data sets that are affected by extraneous factors like epidemics, heat stroke etc. Twenty thousand numbers each (ten sets of homogeneous two thousand numbers each)of IWN,IWP, IWK,IWD and IWF White leghorn strains are followed up from day of hatch to seventy two weeks to record the number of deaths at the various ages of each cohort. Since all the strains are homogeneous and of the same breed, the data is pooled and analysis has been carried out for a batch of one lakh birds also for the same period. The earlier studies referred in the veterinary field justify the homogenity of strains of white leghorn birds(Chakraborty et al (1985), Khan et al (1985) Yadav (1991) and Ravindranathan and Nair (1990)).

2.8 Statistical Inference on the model values

Five typical data sets (with each cohort size 20,000) one each from IWN, IWP, IWK, IWD and INF and a pooled data set of one lakh birds, including all strains along with the estimated survival probabilities using the model are presented in Table 1 and their corresponding graphs in GRAPH-1. . It is seen that the model explains quite well all data sets and this encourages to conclude that the assumptions made about the mortality pattern is realistic enough to be chosen as a basis to draw further conclusions. In this sense, the interpretation for the parameters and their general behaviour is attempted. It is noticed here that correlation analysis carried out between the observed and the model survival probabilities also justifies that there exists significant correlation between them (r = .98814) as detailed in the studies of Edmund et al (1973).

2.9 Statistical Interpretation of Parameters

For all the strains the parameter C describes the mortality rate in the neighbourhood of the time of hatch. It can be seen that the value of 'C' ranges from .004434$\pm$.000264 (IWK Strain) to .007854$\pm$.000115 (IWN Strain) and has got a value .006863$\pm$.000125 while considering all the strains together. This parameter depends upon mortality rate and is minimum for IWK Strain, even though there is very little to choose between the Strains in this respect. The parameter 'b' measuring the rate at which the mortality change is found to be negative in the order -.001248$\pm$.000061 (IWK), -.001981$\pm$.000095 (IWD), -.002056$\pm$.000112 (IWP), -.002074$\pm$.000379 (IWF), -.002212$\pm$.000049 (IWN). For all the strains 'b' is a decreasing function of 'a' and whenever the initial mortality is high, it is off set by a corresponding decrease in b. The mortality for (0,15) age group is minimum at $x = -\frac{b}{2a}$ which according to the parameter value happens for all the cases between the age six to seven weeks from the day of hatch. Taking all the Strains, it is seen that the parameter 'a' takes the value between .000099±.000012(IWK Strain) and .000156±.000015(IWN Strain) and increase or decrease according to the mortality rate (b). On the other hand no functional relation is established between parameters α and β from the sixteenth week. The value of α ranges from .000196±.000027(IWD Strain) to .000337±.000051(IWN) and the parameter β from .011052±.005030(IWF) to .012580±.00059(IWP). The mortality rates are found almost stable and take smaller values and no apparant increase or decrease is observed from 24th week onwards The parameters α and β take positive values for all the strains. Contrary to the earlier period (0-15 week) the latter period (16 to 72 week) shows IWF strains shows the lowest mortality and IWP highest in the numerical values. These interpretations are based on Table 1.

3.0 Conclusion

The major contribution to the total number of deaths in the productive life time hails from the first week to fifteen weeks and hence efforts to achieve over all mortality reduction have to be applied here through various controls. In terms of model parameters this would mean that a and c have to be decreased so that b will get increased and the mortality will remain at a uniformly low level. Similarly measures can be taken to reduce the value of α and β .

An important application of the model is its capability for prediction or interpolation using the functional form obtained in (2.15) to realise the probabilities of survival, number dying etc at any point of productive life. It is seen that the prediction of values using the model proved to be quite useful and in confirmity with the observed. The model can be used for formulating disease-control projects enabling to reduce the over all mortality rates and to develop better genetic variety of chicken in the organised sectors.

TABLE 1

SURVIVAL PATTERN IN FIVE STRAINS OF CHICKEN

Strain	Age x	No. died in the interval	Probability of surviving age x		Parameter values
			Observed	From the model	(Estimated)
(1)	(2)	(3)	(4)	(5)	(6)
IWN	1	157	.992150	.993223	
	3	117	.986300	.985100	
	5	94	.981600	.982043	$a = .000156 \pm$
	7	51	.979050	.981556	$.000015$
	9	74	.975350	.981172	$b = -.002212 \pm$
	11	103	.970200	.978455	$.000049$
	13	124	.964000	.970997	$c = .007854 \pm$
	15	139	.957028	.956514	$.000115$
	16	115	.951278	.956125	$\alpha = .000337 \pm$
	20	90	.946778	.954521	$.000051$
	24	73	.943128	.952837	$\beta = .012191 \pm$
	28	59	.940178	.951077	$.000808$
	32	44	.937978	.949231	
	36	38	.936078	.947296	
	40	30	.934578	.945269	
	44	26	.933278	.943144	
	48	21	.932228	.940919	
	52	21	.931178	.938589	
	56	21	.930378	.936148	
	60	16	.929578	.933592	
	64	16	.928828	.930916	
	68	15	.928078	.928114	

(1)	(2)	(3)	(4)	(5)	(6)
IWP	1	142	.992900	.993585	
	3	92	.988300	.985786	
	5	76	.984500	.982652	a=.000147 ± .000003
	7	54	.981800	.981827	
	9	58	.978900	.980915	b=-.002056± .000112
	11	86	.974600	.977853	
	13	111	.969050	.970139	c=.007415 ± .000391
	15	138	.962150	.955711	
	16	112	.956550	.955412	α =.000257 ± .000041
	20	85	.952300	.954181	
	24	61	.949250	.952888	β =.01258 ± .000591
	28	58	.946350	.951531	
	32	48	.943950	.950105	
	36	41	.941900	.948608	
	40	34	.940200	.947036	
	44	27	.938850	.945386	
	48	22	.937750	.943654	
	52	22	.936650	.941836	
	56	18	.935750	.939928	
	60	18	.934950	.937925	
	64	16	.934200	.935824	
	68	15	.933600	.933620	

(1)	(2)	(3)	(4)	(5)	(6)
IWK	1	85	.995750	.996164	
	3	57	.992900	.991460	
	5	49	.990450	.989362	
	7	33	.988800	.988288	
	9	73	.985150	.986671	
	11	101	.980100	.982954	
	13	122	.974000	.975615	a=.000099 ± .000012
	15	142	.966900	.963209	
	16	94	.962200	.962922	b=-.001248± .000061
	20	76	.958400	.961742	
	24	62	.955300	.960508	c=.004434 ± .000264
	28	47	.952950	.959218	
	32	41	.950900	.957868	α=.000244 ± .000053
	36	33	.949250	.956456	
	40	27	.947900	.954980	β=.01154 ± .0001213
	44	22	.946800	.953437	
	48	17	.945950	.951824	
	52	16	.945150	.950135	
	56	16	.944350	.948374	
	60	14	.943650	.946531	
	64	11	.943100	.944606	
	68	10	.942600	.942591	

(1)	(2)	(3)	(4)	(5)	(6)
IWD	1	130	.993500	.994146	
	3	81	.989450	.987213	
	5	58	.986550	.984578	
	7	35	.984800	.983842	
	9	86	.980500	.982641	
	11	114	.974800	.978624	a=.000150 ±
	13	130	.968300	.969497	.000043
	15	148	.960900	.953112	b=.001981 ±
	16	95	.956150	.952879	.000949
	20	74	.952450	.951913	c=.006812 ±
	24	63	.949300	.950892	.000332
	28	52	.946700	.949611	α=.000196 ±
	32	44	.944500	.948668	.000027
	36	32	.942900	.947460	β =.014293 ±
	40	28	.941500	.946182	.000379
	44	23	.940350	.944831	
	48	22	.939250	.943402	
	52	22	.938250	.941891	
	56	20	.937350	.940295	
	60	20	.936450	.938607	
	64	16	.935650	.936824	
	68	16	.934900	.934939	

(1)	(2)	(3)	(4)	(5)	(6)
IWF	1	141	.992950	.993617	
	3	96	.988150	.985913	
	5	68	.984750	.982873	
	7	47	.982400	.982095	
	9	64	.979198	.981216	
	11	102	.974089	.977885	a=.00015 ±
	13	120	.968072	.969798	.000016
	15	142	.960952	.954780	b=-.00206 ±
	16	104	.955747	.954521	.000216
	20	77	.951893	.953459	c=.007390 ±
	24	61	.948840	.952351	.000379
	28	50	.946338	.951194	α=.000228 ±
	32	40	.944336	.949986	.000036
	36	33	.942684	.948725	β=.011052 ±
	40	25	.941433	.947409	.000503
	44	19	.940482	.946036	
	48	19	.939531	.944692	
	52	15	.938780	.943106	
	56	15	.938029	.941545	
	60	15	.937429	.939916	
	64	10	.936928	.938217	
	68	10	.936428	.936444	

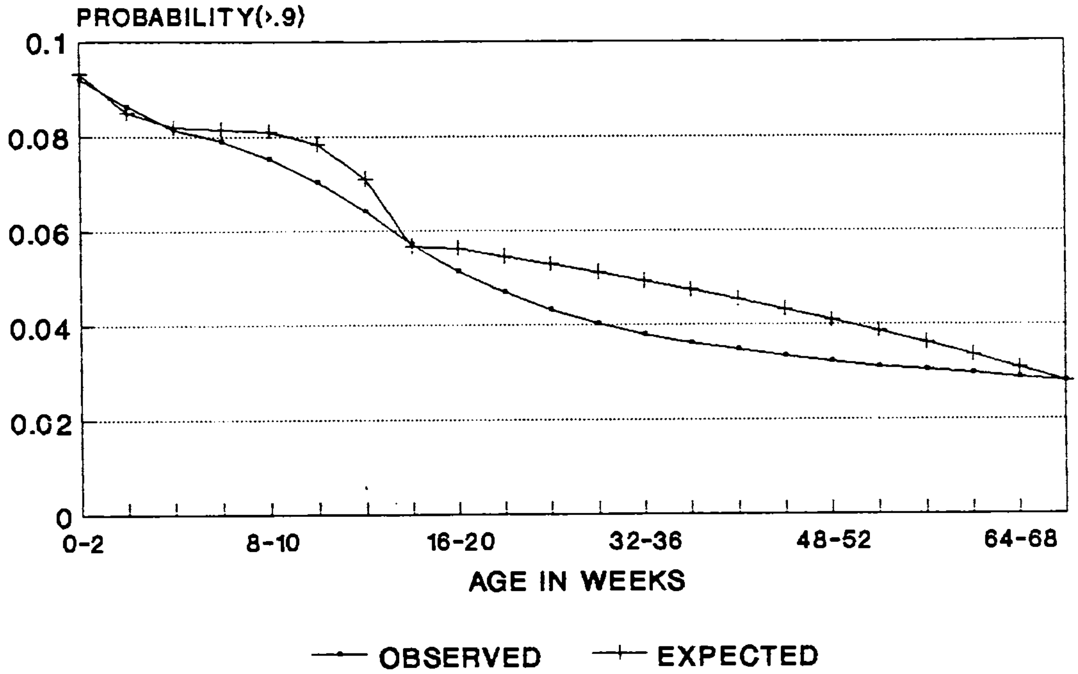
TABLE 1 (CONTD.)

SURVIVAL PATTERN OF ALL STRAINS TOGETHER CONSIDERED

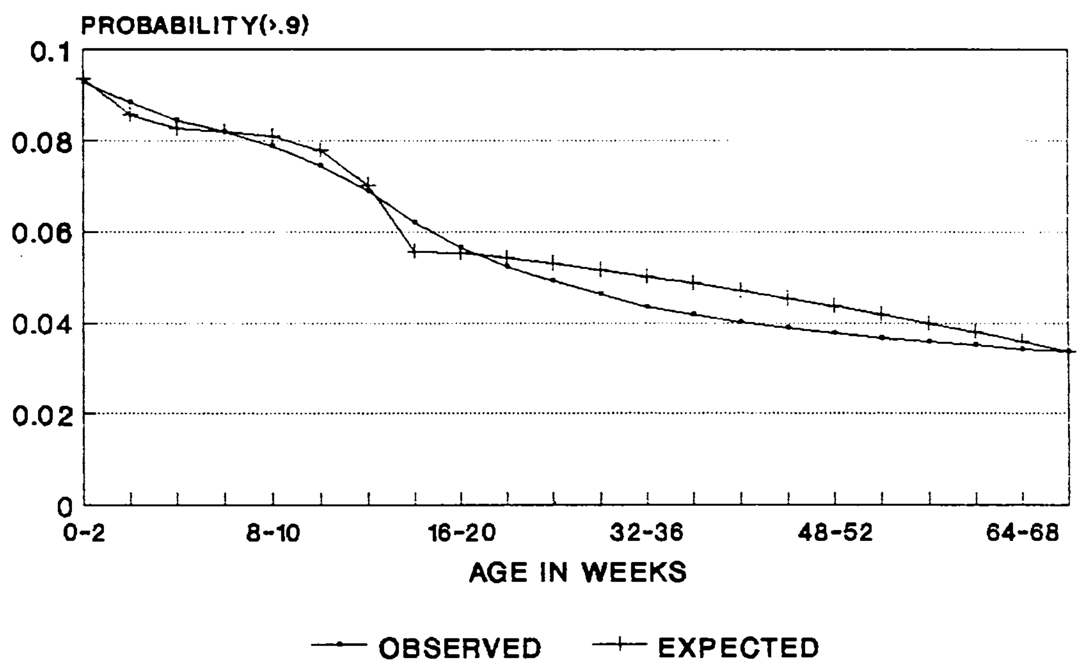
Age	No.died	Probability of surviving age x		
		Observed	From the model	
1	655	.993450	.994065	
3	443	.989020	.986842	a=.000141 ± .000001
5	345	.985570	.983867	
7	220	.983370	.982879	b=-.00191 ± .000014
9	355	.979820	.981657	
11	506	.974760	.977993	c=.006863 ± .000125
13	607	.968690	.969723	
15	709	.961600	.954808	α =.000249 ± .000031
16	520	.956400	.954520	
20	402	.952380	.953330	β =.012481 ± .000220
24	320	.949180	.952081	
28	266	.946520	.950770	
32	217	.944350	.949394	
36	177	.942580	.947949	
40	144	.941140	.946433	
44	117	.939970	.944842	
48	102	.938950	.943173	
52	98	.937970	.941421	
56	76	.937210	.939583	
60	75	.936460	.937654	
64	70	.935760	.935632	
68	68	.935080	.935510	

SURVIVAL PROBABILITY

WHITE LEGHORN IWN

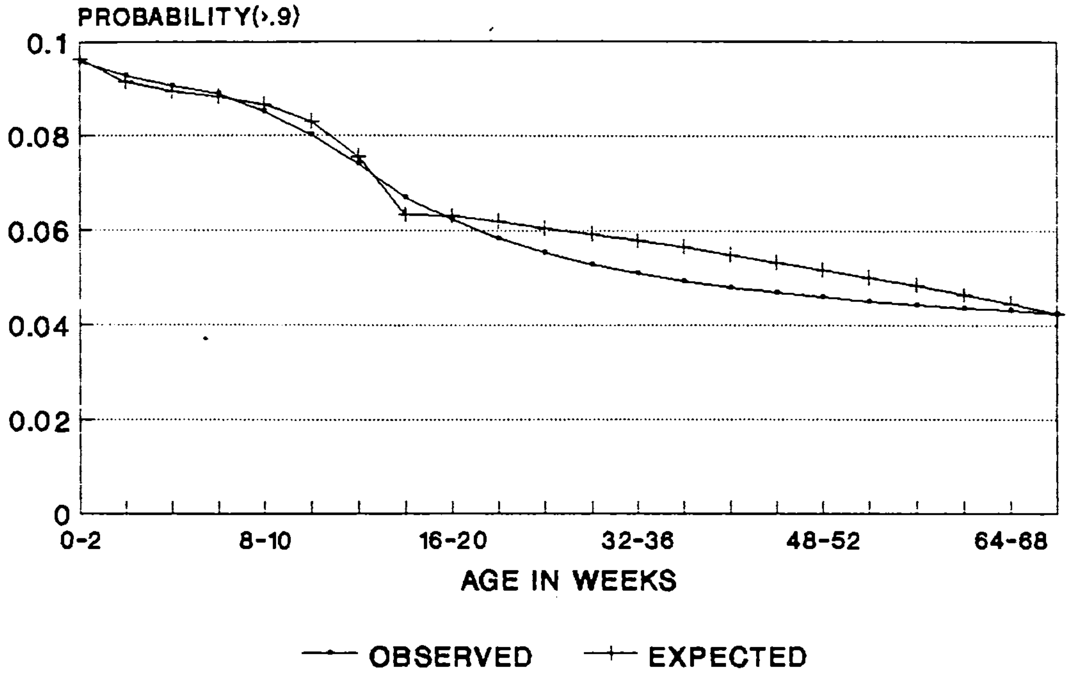


WHITE LEGHORN IWP

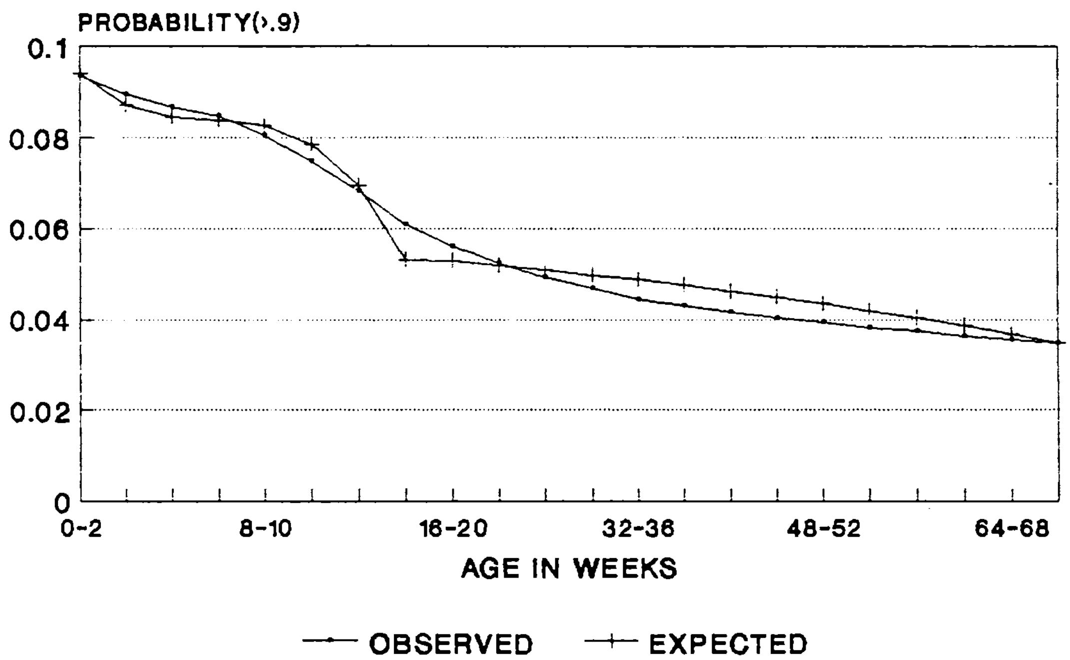


SURVIVAL PROBABILITY

WHITE LEGHORN IWK



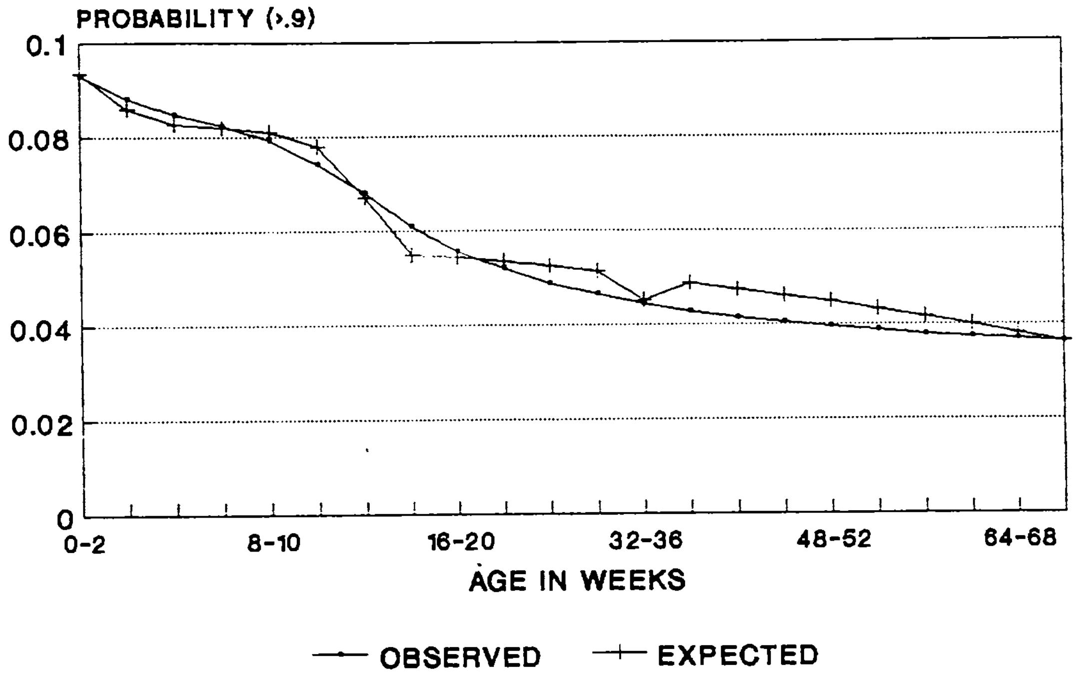
WHITE LEGHORN IWD



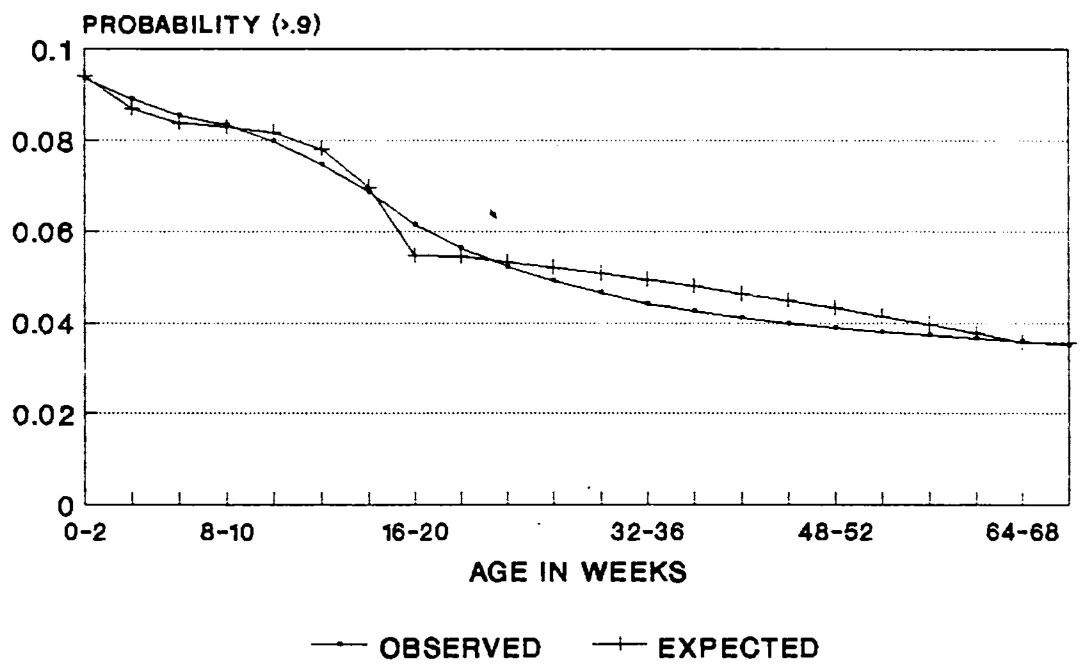
GRAPH 1

SURVIVAL PROBABILITY

WHITE LEGHORN IWF



WHITE LEGHORN (All Strains)



CHAPTER THREE

LIFE TABLE MODEL OF MORTALITY IN CHICKEN*

3.1 Introduction

A life table is a statistical technique for presenting the survival experience of a population at any instant of time during its life period. This technique is also used for analysing data for different quantitative measurements. However, it is found that this technique is widely used in the survival analysis of human population. It can be seen from the work of Calvin W.Schwabe(1977) that the life tables are used for the studies of cattle, chicken, horses etc. by defining "productive life" as the life period. Similar studies have been reported by scientists from Indian Council of Agricultural Research by defining productive life as the lactation period of cows. The main advantage in all these studies is found to be that the method helps to give a clear picture of life history of a population for easy interpretation. The work of Ravindranathan (1994) on demographic analysis on chicken is very much relevant in this context.

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3.2 Significance of Productive life in non-human population

In most of the studies on life time of animals and birds, available information on failure time may be incomplete. This is mainly because that animals are being observed for their productive period only. This type of situation thus provide censored information only. As in the case of most of the life testing experiments, starting from zero, n items are placed on the test and the experiment is terminated at time t_c, then the failure time will be known exactly for items that fail before t_c. When individuals 1,2,3,...n are kept under observation for periods of length $c_1, c_2, \ldots c_n$ respectively so that the i^{th} person's failure time T_i is observed only if $T_i \leq c_i$, the resulting sample is said to be Type one censored. The data of this type can be represented by n pairs of random variables (t_i, δ_i) where t_i=min (T_i, δ_i) and

$$\delta_i = 1 \text{ if } T_i \leq c_i \qquad (a)$$

$$\delta_i = 0 \text{ if } T_i > c_i \qquad (b)$$

It can be seen that case (a) comes under uncensored and (b) under censored. If T_i are assumed to be independently and identically distributed random variables possessing continuous distribution with pdf f(t) and survival function

S(t), then the likelihood on the data may be written as

$$\left.\begin{aligned} L &= \prod_{i=1}^{n} \left[f(t_i)\right]^{\delta_i} \left[S(c_i)\right]^{1-\delta_i} \\ &= \prod_{u} f(t_i) \prod_{c} S(c_i) \end{aligned}\right\} \quad (3.1$$

Where the first product is over uncensored cases and the second over censored cases.

In the study of chicken mortality censored data only can be used because the chicken are studied only on their productive life which is from the day of hatch to seventy two weeks.

3.3 Construction of Life Tables

The life tables are constructed mainly on two ways - Complete Life Table and Abridged Life Tables. A Complete Life Table gives information for each single period age interval starting from an integer value. In the case of latter type, the mortality experience of the 'cohort' will be observed from their birth till the end of productive life and thus a "follow up study" is attempted. The important columns considered for abridged life table are the following.

(1) The period of life between two exact ages (age interval) (x to x+n)

(2) The probability that a person who is alive at the beginning of age interval will die before the end of the interval ($^{n}q_{x}$)

(3) The number alive at the beginning of the indicated age interval say ℓ_x. In all cases, a cohort size is taken and assumes that they experience attrition due to mortality according to the pattern exhibited by $^{n}q_{x}$ (Column 2)

(4) The number of death in the indicated age interval (d_x)

(5) Average fraction of time lived by those in the age interval who died in the interval, say $^{n}a_{x}$. This is a very vital information which helps to work out mortality pattern. The value is calculated by using the formula

$$^{n}a_{x} = \frac{\int_0^n t\ell(x+t)\,\mu(x+t)\,dt}{\int_0^n \ell(x+t)\,\mu(x+t)\,dt} \qquad (3.2)$$

(6) The period of life lived by the cohort within the indicated age interval $^{n}L_{x} = {}^{n}\ell_{x+n} + {}^{n}a_{x}dx$

(3.3)

(7) Total number of years of life remaining for the cohort after surviving till the beginning of the indicated age interval. This is obtained by adding $^{n}L_{x}$ for the considered age interval and those for the subsequent age intervals and denoted by $^{n}T_{x}$.

(8) The average period of life remaining for the body of lives in question after attaining the age and denoted by e_x and calculated by dividing T_x by ℓ_x.

3.4 Mathematical Interpretation of Life Tables

In life table techniques, age is treated as continuous variable and use the notation $\ell(x)$ for the number living whose age is x. The force of mortality (q_x) is calculated as

$$q_x = \lim_{\Delta x \to 0} \left[\frac{\ell(x) - \ell(x+\Delta x)}{\ell x \, \Delta x} \right] \quad (3.4)$$

$$= - \frac{\ell'(x)}{\ell(x)} \quad (3.5)$$

Taking (3.5) as a differential equation, a solution is obtained as

$$\ell(x) = \ell(0) \exp \left[- \int_0^x \mathcal{l}(u) du \right] \quad (3.6)$$

From (3.4) it can be seen that the force of mortality $(q_x) = \frac{\Delta x\, q_x}{\Delta x}$ when $\Delta x \to 0$ where $\Delta x\, q_x$ is the conditional probability of dying in the age interval $(x, x+\Delta x)$ given survival till age x which helps to write and expression $\Delta_x q_x = \Delta_x q_x + 0(\Delta x)$ where $0(\Delta x)$ is a function of Δx such that $\frac{0(\Delta x)}{\Delta x}$ tends to zero as Δx tends to zero. This means that for very small values of Δx, the conditional probability of dying in the age interval x to $x+\Delta x$ given survival until x, is closely approximated as $\Delta x\, q(x)$.

From (3.4), $\ell(x) = \ell o \exp\left[-\int_0^x q(u)\,du\right]$ (3.7)

With $\ell(0)=1$, $f(x)=q(x)\,\ell(x)$ is the probability density function of the age at death.

$$\therefore \int_0^\infty f(a)\,da = \int_0^\infty q(a)\ \ell(a)\,da \qquad (3.8)$$

$$= \Big[-\ell(a)\Big]_0^\infty = 1 \qquad (3.9)$$

$$\text{and} \int_x^{x+n} f(a)\,da = -\int_0^{x+n} \ell^1(a)\,da$$

$$= \Big[-\ell(a)\Big]_x^{x+n} = \ell(x) - \ell(x+n)$$

$$= {}^n dx \qquad (3.10)$$

The conditional probability of dying in the age interval $(x,x+n)$ given ${}^{n}q_x$ is the same as the probability that the age at death and so ${}^{n}q_x = \frac{\ell(x)-\ell(x+n)}{\ell(x)}$ (3.11)

It follows that the conditional probability of surviving till age $(x+n)$ given survival until age x is

$$
\begin{aligned}
{}^{n}p_x &= 1-{}^{n}q_x \\
&= \frac{\ell(x+n)}{\ell(x)} \\
&\frac{\exp\left[-\int_0^{x+n} q(a)da\right]}{\exp\left[-\int_0^{x} q(a)da\right]}
\end{aligned}
\tag{3.12}
$$

for $0 < s < t < u < v < w < x$, if $1(0)=1$

$$
\begin{aligned}
\text{Thus } \ell(x) &= \exp\left[-\int_0^x q(a)da\right] \\
&= \exp\left[\int_0^s q(a)da\right] \exp\left[-\int_s^t q(a)\ da\right] \ldots\ldots \\
&\qquad \ldots\ldots \exp\left[-\int_\omega^s q(a)da\right]
\end{aligned}
\tag{3.13}
$$

Now denoting the age at death by the random variable X, its expected value, conditional on dying after attaining age x is

$$
E\left(X \middle| X \geq x\right) = \frac{\int_x^\infty a\ f(a)da}{\int_x^\infty f(a)da} = \frac{\int_x^\infty a\left[-\ell'(a)\right]da}{\int_x^\infty\left[-\ell^1(a)\right]da} = x + \frac{T(x)}{\ell(x)} \tag{3.14}
$$

Where $T_x = \int_x^{\infty} \ell(a)da$

The expected value of the age at death, conditional on dying in the age interval (x,x+n) is

$$E(X \mid x \leq X < x+n) = \frac{\int_x^{x+n} a\, f(a)da}{\int_x^{x+n} f(a)da}$$

$$= \frac{x+{}^{n}L_x - n\ell(x+n)}{{}^{n}d_x}$$

$$= n+{}^{n}a_x \qquad (3.15)$$

Where ${}^{n}L_x = \int_x^{x+n} \ell(a)da$

$${}^{n}d_x = \ell(x)-\ell(x+n)$$

$${}^{n}a_x = \frac{{}^{n}L_x - n\ell(x+n)}{{}^{n}d_x}$$

${}^{n}a_x$ is the expected length of life of the life time lived in the age interval (x,x+n), conditional on dying in that age interval. Thus ${}^{n}L_x = {}^{n}a_x dx+n\ell(x+n)$ (3.16)

The above concepts can be seen from Barclay (1958), Lawless (1984) and Namboodiri (1987).

3.5 Estimation of mortality rate

Method 1: Kaplan-Meier Method (1958)

This method known as Product-limit estimation is one of the most common method used for calculation of q_x where

$$q_x = \frac{dx}{l(x)} \qquad (3.17)$$

From the estimated values of q_x, the probability of survival is worked out by using the formula

$$p_x = 1-q_x \qquad (3.18)$$

When the estimates of q_x and p_x are worked out, the product limit estimate for $p_j=p_1p_2\ldots\ldots p_j$ (3.19)

Where j=1,2,3.... K+1 wherein the probability of surviving is given as the product of conditional probabilities of surviving past intervals, given survival to the start of the interval.

Method 2: Chiang Method (1968)

Chiang presented a method to estimate the mortality rate considering censoring time and using a relationship between age-specification death rate and the estimate of probability of death. The death rate for age interval (x_i,x_{i+1}) is defined as

$$M_i = \frac{\text{Number of individuals dying in interval } (x_i, x_{i+1})}{\text{Number of years lived in interval } (x_i, x_{i+1}) \text{ by those alive at } x_i} \quad (3.20)$$

and the estimate of the probability as the ratio of the number of deaths in (x_i, x_{i+1}) to the number of individuals living at x_i and defined as

$$\hat{q}_i = \frac{n_i M_i}{1+(1-a_i) n_i M_i} \quad (3.21)$$

Consider an individual alive at age x_i and in the interval (x_i, x_{i+1}). Let $\mu(x)$ be the force of mortality at age x_i. Then the probability that the individual die in (x_i, x_{i+1}) is

$$q_i = 1-\exp\left[-\int_0^n \mu(x_i+\xi)\,d\xi\right] \quad (3.22)$$

From this the number of years lived in the interval (x_1, x_{1+n}) by ℓ_1 survivors at age x_1 is

$$L_1 = n(\ell_1-d_1)+a_1 n d_1 \quad (3.23)$$

Chiang method suggests to use the value of q_i derived from (3.21) as the probability of death on a basis to calculate survival probability by using Product limit formula(3.19).

<u>Method 3</u>: Parametric Method (1994)

The survival probabilities are calculated by using the parametric distribution values derived from the following expressions for survival function (2.15)

$$S(x)=\exp\left[-\left(\frac{ax^3}{3}+\frac{bx^2}{2}+cx\right)\right] \qquad 0<x\leq 15$$

$$=\exp\left[-\frac{(15)^3}{3}a-\frac{(15)^2}{2}b-(15)c-\frac{\alpha}{\beta}(\exp(\ x)-\exp(15\ ))\right],\quad x>15$$

and $q_x=1-p(x)$ is taken as the values for different age intervals.

3.6 Calculation of the fraction of last age interval $^n a_x$.

In the calculation of probability of dying in an interval, the value of $^n a_x$ is an important information to be used. This value gives the average number of period lived by those individuals in the age interval(x,x+n) who died in that interval and is defined as

$$^n a_x = \frac{\int_0^n t\,l(x+t)\mu(x+t)\,dt}{\int_0^n l(x+t)\mu(x+t)\,dt} \qquad (3.24)$$

For calculation of $^n a_x$, many methods are available like those suggested by Reed and Merrell (1939), Greville(1943) Keyfitz and Fraventhal (1975) Nair (1984) make use of techniques in the form of iteration, Taylor Series expansion etc. in arriving at a solution for ${}^{n}a_x$. In this context Chiang has given the formula (3.24) which is used as the basis for calculation. Different assumptions can be made for working out ${}^{n}a_x$. Nair (1984) worked out ${}^{n}a_x$, considering both $\ell(x)$ and $\mu(x)$ assuming linear forms. In this study, age specific death rate is calculated by using the formula

$$ {}^{n}m_x = \frac{\int_0^n \ell(x+t)\, \mu(x+t)\, dt}{\int_0^n \ell(x+t)\, dt} $$

Assuming $\ell(x+t)$ and $\mu(x+t)$ to be linear functions of t it is shown that ${}^{n}m_x = \mu(x+\frac{n}{2}) \left[1 - \frac{n^2}{12} \mu^{\prime}(x+\frac{n}{2})\right]$

and by using linearity of $\mu(x+t)$,

$$ \mu(x+\tfrac{n}{2}) = \frac{1}{n} \int_0^n \mu(x+t)\, dt \text{ so that} $$

$$ \int_x^{x+n} \mu(t)\, dt = \frac{{}^{n}m_x}{1 - \frac{n^2}{12} m_x^{\prime}} $$

where $m_x^{\prime} = \mu^{\prime}(x+\frac{n}{2})$

The equation (3.24) is re-written as

$$ {}^{n}a_x = \frac{n}{2} + \frac{I_1}{I_2} \qquad (3.25) $$

Where $I_1 = \int_{-n/2}^{n/2} t\,\ell(t+x+\frac{n}{2})\,\mu\,(t+x+\frac{n}{2})\,dt$

and $I_2 = \int_{-n/2}^{n/2} \ell(t+x+\frac{n}{2})\,\mu(t+x+\frac{n}{2})\,dt$

Case 1

Assuming both $\ell(x+t)$ and $\mu(x+t)$ to be linear functions of t, the above form of ${}^{n}a_x$ is written. Using linearity conditions for both $\ell(x)$ and $\mu(x)$

$$I_1 = \frac{n^2}{12}\left[\mu'(x+\frac{n}{2}) - \mu^2(x+\frac{n}{2})\right]$$

$$I_2 = \left[\mu(x+\frac{n}{2}) - \frac{n^2}{12}\,\mu(x+\frac{n}{2})\,\mu'(x+\frac{n}{2})\right]$$

With usual estimates

$$\mu(x+\frac{n}{2}) = M_x$$

$$\mu'(x+\frac{n}{2}) = \frac{M_{x+n}-M_x}{n}$$ (3.25) reduces to

$${}^{n}a_x = \frac{n}{2} + \frac{n(M_{x+n}-M_x-nM_x^2)}{M_x(12-nM_{x+n}+nM_x)} \qquad (3.26)$$

Case 2

Assuming $\ell(x)$ linear and $\mu(x)$ non-linear function of t, the above form of ${}^{n}a_x$ is written as $\frac{n}{2} + \frac{I_1}{I_2}$ where

$$\frac{I_1}{I_2} = \frac{(\mu' - \mu^2)\frac{n^2}{12} - \mu\mu''\frac{n^4}{160}}{\mu + (\mu'' - 2\mu\mu')\frac{n^2}{24}} \qquad (3.27)$$

Case 3

Assuming both $\ell(x)$ and $\mu(x)$ non-linear function of t, the value of I_1 and I_2 are

$$I_1 = (\mu' - \mu^2)\frac{n^2}{12} - \left[\mu\mu'' + (\mu')^2 - \mu^2\mu''\right]\frac{n^4}{160}$$

$$\text{and } I_2 = \mu + \left[\mu'' - 3\mu\mu' + \mu^3\right]\frac{n^4}{24} - (\mu' - \mu^2)\mu''\frac{n^4}{320} \qquad (3.28)$$

For both (3.27) and (3.28), $\mu;\ \mu',\ \mu''$ are estimated with special formulas because of changes in the age intervals in abridged life tables.

Formula 1

In all cases $\mu = \mu(x+\frac{n}{2}) = M_x$ = Death rate

Formula 2

$$\left.\begin{aligned} \mu' &= \frac{M_{x+n} - M_{x-n}}{2n} \\ \mu'' &= \frac{M_{x+n} - 2M_x + M_{x-n}}{n^2} \end{aligned}\right\} \quad \text{When } x = 3, 5, 7, 9, 11 \text{ and } 13$$

Formula 3

$$\left.\begin{aligned} M^{I} &= \frac{1}{2n}\left[-3M_x + 4M_{x+n} - M_{x+2n}\right] \\ M^{II} &= \frac{1}{n^2}\left[M_{x+2n} - 2M_{x+n} + M_x\right] \end{aligned}\right\} \text{When } x=1,\ 20 \text{ to } 50$$

Formula 4

$$\left.\begin{aligned} M^{I} &= \frac{1}{2n}\left[3M_x - 4M_{x-n} + M_{x-2n}\right] \\ M^{II} &= \frac{1}{n^2}\left[M_x - 2M_{x-n} + M_{x-2n}\right] \end{aligned}\right\} \text{When } x=52 \text{ to } 72$$

Formula 5

$$\left.\begin{aligned} M^{I} &= -\frac{3}{10} M_{x-2} - \frac{1}{6} M_x + \frac{2}{15} M_{x+3} \\ M^{II} &= \frac{1}{5} M_{x-2} - \frac{1}{3} M_x + \frac{2}{15} M_{x+3} \end{aligned}\right\} x=15$$

Formula 6

$$\left.\begin{aligned} M^{I} &= -\frac{4}{21} M_{x-3} + \frac{1}{12} M_x + \frac{3}{28} M_{x+4} \\ M^{II} &= \frac{2}{21} M_{x-3} - \frac{1}{6} M_x + \frac{1}{14} M_{x+4} \end{aligned}\right\} \text{When } x=16$$

3.7 Data Analysis

Abridged life tables are constructed for the data recorded in All India Co-ordinated Research Project on Poultry Breeding for the period 1987-1990 as detailed in (2.7) in Chapter 2. Life Tables are prepared for the productive life, ie. from day of batch to seventy weeks for all the strains together and also for each strain of white leghorn breeds and given in Table 3.1. The unit of measurement is taken as one week. The values of average expected life of birds died in an interval, ${}_{n}a_{x}$ also have been calculated using the special formulas developed as stated in the paragraph (3.6). These values are tabulated and given in Table 3.2

TABLE 3.1

COHORT LIFE TABLE CONSIDERING IWN STRAIN OF WHITE LEGHORN CHICKEN

Age (Weeks) (x)	Prob. of dying q_x	Number Living l_x	Number dying d_x	Fraction of last age a_x	Number of weeks lived L_x	Total Number of weeks beyond T_x	Expectation of life e_x
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
0-2	.00785	20000	157	.47	39834	1362177	68.1
2-4	.00590	19843	117	.41	39548	1322343	66.6
4-6	.00477	19726	94	.16	39294	1282795	65.0
6-8	.00260	19632	51	.30	39193	1243501	63.3
8-10	.00378	19581	74	.47	39084	1204309	61.5
10-12	.00528	19507	103	.48	38907	1165285	59.7
12-14	.00639	19404	124	.50	38684	1126318	58.0
14-16	.00723	19220	139	.47	38293	1087634	56.6
16-20	.00601	19141	115	.45	76279	1049342	54.8
20-24	.00473	19026	90	.38	75906	973063	51.1
24-28	.00386	18936	73	.45	75569	897157	47.4
28-32	.00313	18863	59	.40	75329	821588	43.6
32-36	.00234	18804	44	.48	75119	746259	39.7
36-40	.00203	18760	38	.45	74963	671140	35.8
40-44	.00160	18722	30	.49	74828	596177	31.8
44-48	.00139	18692	26	.50	74716	521349	27.9
48-52	.00113	18666	21	.50	74621	446633	23.9
52-56	.00113	18645	21	.49	74534	372012	20.0
56-60	.00086	18624	21	.45	74463	297478	16.0
60-64	.00086	18608	16	.48	74399	223015	12.0
64-68	.00080	18592	16	.49	74338	148616	8.0
68-72	.00080	18577	15	.50	74278	74728	4.0

Table 3.1 (Contd.)

IWP STRAIN

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
0-2	.00710	20000	142	.47	39849	1369599	68.5
2-4	.00463	19858	92	.41	39607	1329750	67.0
4-6	.00385	19766	76	.16	39404	1290142	65.3
6-8	.00274	19690	54	.30	39304	1270738	63.5
8-10	.00295	19636	58	.47	39211	1211433	61.7
10-12	.00439	19578	86	.48	39067	1172223	59.9
12-14	.00570	19492	111	.50	38873	1133156	58.1
14-16	.00712	19381	138	.47	38616	1094283	56.5
16-20	.00582	19243	112	.45	76694	1055668	54.9
20-24	.00444	19131	85	.38	76337	978973	51.2
24-28	.00320	19046	61	.45	76038	902636	47.4
28-32	.00306	18985	58	.40	75819	826599	43.5
32-36	.00254	18927	48	.48	75602	750779	39.7
36-40	.00217	18879	41	.45	75432	675177	35.7
40-44	.00181	18838	34	.49	75284	599745	31.8
44-48	.00144	18804	27	.50	75162	524461	27.9
48-52	.00117	18777	22	.50	75063	449299	23.9
52-56	.00117	18755	22	.49	74972	374236	20.0
56-60	.00096	18737	18	.45	74911	299264	16.0
60-64	.00086	18719	18	.48	74843	224353	12.0
64-68	.00080	18703	16	.49	74782	149510	8.0
68-72	.00064	18688	15	.50	74728	74728	4.0

Table 3.1 (Contd.)

IWK STRAIN

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
0-2	.00425	20000	85	.47	39910	1379546	69.0
2-4	.00286	19915	57	.41	39763	1339636	67.3
4-6	.00247	19858	49	.16	39634	1299874	65.5
6-8	.00167	19809	33	.30	39572	1260240	63.6
8-10	.00369	19776	73	.47	39475	1220668	61.7
10-12	.00513	19703	101	.48	39301	1181194	59.9
12-14	.00622	19602	122	.50	39082	1141893	58.3
14-16	.00729	19480	142	.47	38809	1102811	56.6
16-20	.00486	19338	94	.45	77145	1064001	55.0
20-24	.00395	19244	76	.38	76788	986856	51.3
24-28	.00324	19168	62	.45	76536	910068	47.5
28-32	.00246	19106	47	.40	76311	833533	43.6
32-36	.00215	19059	41	.48	76151	757222	39.7
36-40	.00174	19018	33	.45	75999	681071	35.8
40-44	.00142	18985	27	.49	75884	605072	31.9
44-48	.00116	18958	22	.50	75788	529187	27.9
48-52	.00090	18936	17	.50	75710	453399	23.9
52-56	.00085	18919	16	.49	75643	377689	20.0
56-60	.00085	18903	16	.45	75577	302045	16.0
60-64	.00074	18889	14	.48	75527	226468	12.0
64-68	.00058	18878	11	.49	75490	150942	8.0
68-72	.00053	18868	10	.50	75452	75452	4.0

Table 3.1 (Contd.)

IWD STRAIN

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
0-2	.00650	20000	130	.47	39862	1370832	68.5
2-4	.00408	19870	81	.41	39644	1330969	67.0
4-6	.00293	19789	58	.16	39481	1291325	65.3
6-8	.00177	19731	35	.30	39413	1251844	63.4
8-10	.00437	19696	86	.47	39301	1212431	61.6
10-12	.00581	19610	114	.48	39101	1173130	59.8
12-14	.00669	19496	130	.50	38862	1134029	58.2
14-16	.00764	19366	148	.47	38575	1095167	56.6
16-20	.00494	19218	95	.45	76663	1056592	55.0
20-24	.00387	19123	74	.38	76308	979929	51.2
24-28	.00331	19049	63	.45	76057	903621	47.4
28-32	.00274	18986	52	.40	75819	827563	43.6
32-36	.00232	18934	44	.48	75644	751744	39.7
36-40	.00169	18890	32	.45	75490	676099	35.8
40-44	.00148	18858	28	.49	75375	600610	31.8
44-48	.00122	18830	23	.50	75274	525235	27.9
48-52	.00117	18807	22	.50	75184	449961	23.9
52-56	.00106	18785	22	.49	75099	374777	20.0
56-60	.00096	18765	20	.45	75021	299678	16.0
60-64	.00096	18745	20	.48	74950	224656	12.0
64-68	.00085	18729	16	.49	74883	149705	8.0
68-72	.00080	18713	16	.50	74822	74822	4.0

Table 3.1 (Contd.)

IWF STRAIN

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
0-2	.00705	20000	141	.47	39851	1369450	68.5
2-4	.00483	19859	96	.41	39605	1329600	67.0
4-6	.00344	19763	68	.16	39412	1289995	65.3
6-8	.00239	19695	47	.30	39324	1250583	63.5
8-10	.00326	19640	64	.47	39212	1211259	61.7
10-12	.00522	19546	102	.48	38986	1172047	60.0
12-14	.00618	19426	120	.50	38732	1133061	58.3
14-16	.00736	19306	142	.47	38461	1094329	56.7
16-20	.00542	19202	104	.45	76579	1055867	55.0
20-24	.00403	19098	77	.38	76201	979288	51.3
24-28	.00321	19021	61	.45	75950	903087	47.5
28-32	.00264	18960	50	.40	75720	827137	43.6
32-36	.00212	18910	40	.48	75557	751417	39.7
36-40	.00175	18870	33	.45	75407	675861	35.8
40-44	.00133	18837	25	.49	75297	600453	31.9
44-48	.00101	18812	19	.50	75210	525156	27.9
48-52	.00101	18793	19	.50	75134	449946	23.9
52-56	.00080	18774	15	.49	75065	374812	20.0
56-60	.00080	18759	15	.45	75004	299747	16.0
60-64	.00064	18744	15	.48	74951	224743	12.0
64-68	.00053	18734	10	.49	74916	149792	8.0
68-72	.00053	18724	10	.50	74876	74876	4.0

Table 3.1 (Contd.)

ALL STRAINS TOGETHER

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
0-2	.00655	100000	655	.47	199306	6852900	68.5
2-4	.00446	99345	443	.41	198167	6653595	67.0
4-6	.00349	98902	345	.16	197224	6455428	65.3
6-8	.00223	98557	220	.30	196806	6258203	63.5
8-10	.00361	98337	355	.47	196298	6061397	61.6
10-12	.00516	97982	506	.48	195438	5865099	59.9
12-14	.00623	97476	607	.50	194345	5669662	58.2
14-16	.00732	96889	709	.47	192986	5475317	56.5
16-20	.00541	96160	520	.45	383496	5282330	54.9
20-24	.00420	95640	402	.38	381563	4898834	51.2
24-28	.00336	95238	320	.45	380248	4517271	47.4
28-32	.00280	94918	266	.40	379034	4137023	43.6
32-36	.00229	94652	217	.48	378157	3757990	39.7
36-40	.00187	94435	177	.45	377351	3379833	35.8
40-44	.00153	94258	144	.49	376738	3002482	31.9
44-48	.00124	94114	117	.50	376222	2625744	27.9
48-52	.00109	93997	102	.50	375784	2249522	23.9
52-56	.00104	93895	98	.49	375380	1873738	20.0
56-60	.00081	93797	76	.45	375020	1498358	16.0
60-64	.00080	93721	75	.48	374728	1123337	12.0
64-68	.00075	93646	70	.49	374441	748609	8.0
68-72	.00072	93576	68	.50	374168	374168	4.0

TABLE 3.2

EXPECTED LENGTH OF LIFE TIME IN THE AGE INTERVAL, CONDITIONAL ON DYING IN THAT INTERVAL (^na_x) OF ALL STRAINS

Age interval (Weeks)	Strain	Value of: Linear fn of t	Value of: Non-linear fn of t	Death rate (Mortality Pattern) Approximate value taken for life table
(1)	(2)	(3)	(4)	(5) (Observed from Mortality Registers)
0-2	IWN	.4693	.4701	
	IWP	.4685	.4693	
	IWK	.4651	.4722	.47
	IWD	.4689	.4713	
	IWF	.4691	.4703	
2-4	IWN	.4092	.4113	
	IWP	.4102	.4109	
	IWK	.3998	.4092	.41
	IWD	.4104	.4181	
	IWF	.4096	.4093	
4-6	IWN	.1572	.1610	
	IWP	.1602	.1609	
	IWK	.1592	.1598	.16
	IWD	.1611	.1609	
	IWF	.1672	.1682	
6-8	IWN	.2982	.3013	
	IWP	.2991	.3112	
	IWK	.3012	.3083	.30
	IWD	.2918	.3012	
	IWF	.2885	.3101	

Table 3.2 (Contd.)

(1)	(2)	(3)	(4)	(5)
8-10	IWN	.4713	.4722	
	IWP	.4802	.4762	
	IWK	.4599	.4681	.47
	IWD	.4601	.4611	
	IWF	.4613	.4631	
10-12	IWN	.4822	.4831	
	IWP	.4891	.4822	
	IWK	.4801	.4823	.48
	IWD	.4798	.4817	
	IWF	.4813	.4822	
12-14	IWN	.4923	.5013	
	IWP	.4981	.5102	
	IWK	.4963	.5001	.50
	IWD	.4812	.4961	
	IWF	.4867	.4983	
14-16	IWN	.4822	.4831	
	IWP	.4791	.4812	
	IWK	.4652	.4703	.47
	IWD	.4712	.4722	
	IWF	.4709	.4801	

Table 3.2 (Contd.)

(1)	(2)	(3)	(4)	(5)
16-20	IWN	.4613	.4581	
	IWP	.4582	.4584	
	IWK	.4503	.4511	.45
	IWD	.4612	.4622	
	IWF	.4519	.4582	
20-24	IWN	.3672	.3689	
	IWP	.3771	.3812	
	IWK	.3813	.3802	.38
	IWD	.3694	.3714	
	IWF	.3712	.3722	
24-28	IWN	.4513	.4602	
	IWP	.4522	.4519	
	IWK	.4514	.4501	.45
	IWD	.4545	.4601	
	IWF	.4589	.4522	
28-32	IWN	.3919	.4102	
	IWP	.3989	.4004	
	IWK	.3892	.4013	.40
	IWD	.4101	.4109	
	IWF	.3859	.4001	

Table 3.2 (Contd.)

(1)	(2)	(3)	(4)	(5)
32-36	IWN	.4822	.4831	
	IWP	.4802	.4811	
	IWK	.4821	.4801	.48
	IWD	.4834	.4851	
	IWF	.4862	.4833	
36-40	IWN	.4503	.4522	
	IWP	.4524	.4563	
	IWK	.4531	.4503	.45
	IWD	.4511	.4582	
	IWF	.4462	.4491	
40-44	IWN	.5010	.5014	
	IWP	.4985	.5102	
	IWK	.4992	.5004	.49
	IWD	.4891	.5010	
	IWF	.4983	.5014	
44-48	IWN	.4986	.5011	
	IWP	.5013	.5046	
	IWK	.4991	.5024	.50
	IWD	.4985	.5013	
	IWF	.4896	.5001	

Table 3.2 (Contd.)

(1)	(2)	(3)	(4)	(5)
48-52	IWN	.4902	.5014	
	IWP	.4893	.5122	
	IWK	.4995	.5013	.50
	IWD	.4996	.5019	
	IWF	.4985	.5062	
52-56	IWN	.5016	.5024	
	IWP	.5082	.4981	
	IWK	.5001	.4965	.49
	IWD	.4988	.5013	
	IWF	.5011	.5102	
56-60	IWN	.4682	.4701	
	IWP	.4539	.4613	
	IWK	.4512	.4532	.45
	IWD	.4503	.4521	
	INF	.4524	.4533	
60-64	IWN	.4811	.4824	
	IWP	.4822	.4841	
	IWK	.4801	.4834	.48
	IWD	.4794	.4812	
	IWF	.4810	.4821	

Table 3.2 (Contd.)

(1)	(2)	(3)	(4)	(5)
64-68	IWN	.4981	.5001	
	IWP	.4892	.4903	
	IWK	.4835	.4864	.49
	IWD	.4869	.4892	
	IWF	.4897	.4901	
68-72	IWN	.5001	.5011	
	IWP	.5112	.5124	
	IWK	.5013	.5028	.50
	IWD	.5049	.5054	
	IWF	.5022	.5019	

3.8 Comparison of survival probabilities obtained through three methods

Comparison of survival probabilities worked out by Kaplan Meier, Chiang and parametric methods has been made. The survival probabilities are given in Table 3.3 and plotted in graph 2.

To choose among the three methods the idea introduced and developed in several papers by Cox (1961) has been considered. Since the three methods involve the equal number of parameters, it is sensible to calculate log likelihood of the observed data under various methods. The method yielding the largest log L and consistent with what is known about the data has to be chosen as the acceptable one. Confirmation of the choice has been made by examining with additional sets of data.

To decide if the best fitting method yields a good fit to the data, twice the difference between the log likelihoods under the parametric method and the other two methods is approximated as a chi-square with twenty degrees of freedom. Graphical analysis also has been made and it is seen that there is no difference between the methods and the survival probabilities are found not significantly different. The chi-square values obtained by this test procedure are 21.47, 18.78, 14.43, 17.29 and 17.20 for IWN, IWP, IWK, IWD and IWF strains respectively. It is noted that the procedures as outlined have been tried out with different sets of data and found consistent as described in a similar work done by Edmund and Siddiqui (1973).

TABLE 3.3

COMPARISON OF SURVIVAL PROBABILITIES

Strain	Age x	Probability of surviving age x: Kaplan-Meier Method	Chiang Method	Parametric Method
(1)	(2)	(3)	(4)	(5)
IWN	1	.99215	.98824	.993223
	3	.98630	.97960	.985100
	5	.98160	.97265	.982043
	7	.97905	.96888	.981556
	9	.97535	.96342	.981172
	11	.97030	.95590	.978455
	13	.96400	.94677	.970997
	15	.95703	.93662	.956514
	16	.95128	.92290	.956125
	20	.94678	.91227	.954521
	24	.94312	.90419	.952837
	28	.94018	.89689	.951077
	32	.93798	.89182	.949231
	36	.93608	.88746	.947296
	40	.93458	.88453	.945269
	44	.93228	.88092	.943144
	48	.93223	.87868	.940919
	52	.93118	.87630	.938589
	56	.93038	.87449	.936148
	60	.92958	.87269	.933592
	64	.92883	.87100	.930916
	68	.92808	.86391	.928114

Table 3.3 (Contd.)

(1)	(2)	(3)	(4)	(5)
IWP	1	.99290	.98919	.993585
	3	.98830	.98256	.985786
	5	.98450	.97688	.982652
	7	.98180	.97288	.981827
	9	.97890	.96964	.980915
	11	.97460	.96230	.977853
	13	.96905	.95415	.970139
	15	.96215	.94400	.955711
	16	.95655	.93063	.955412
	20	.95230	.92055	.954181
	24	.94925	.91337	.952888
	28	.94635	.90660	.951531
	32	.94395	.90102	.950105
	36	.94190	.89629	.948608
	40	.94020	.89238	.947036
	44	.93885	.88928	.945386
	48	.93775	.88677	.943654
	52	.93665	.88296	.941836
	56	.93575	.88215	.939928
	60	.93495	.88040	.937925
	64	.93420	.87871	.935824
	68	.93360	.87735	.933629

Table 3.3 (Contd.)

(1)	(2)	(3)	(4)	(5)
IWK	1	.99575	.99364	.996164
	3	.99290	.98939	.991460
	5	.99045	.98574	.989362
	7	.98880	.98328	.988288
	9	.98515	.97931	.986671
	11	.98010	.97039	.982954
	13	.97400	.96090	.975615
	15	.96690	.95108	.963209
	16	.96220	.93959	.962922
	20	.95840	.93057	.961742
	24	.95530	.92322	.960508
	28	.95295	.91794	.959218
	32	.95090	.91288	.957868
	36	.94925	.90903	.956456
	40	.94790	.90590	.954980
	44	.94680	.90335	.953437
	48	.94595	.90138	.951824
	52	.94515	.89953	.950135
	56	.94435	.89768	.948374
	60	.94365	.89708	.946531
	64	.94310	.89482	.944606
	68	.94260	.89367	.942591

Table 3.3 (Contd.)

(1)	(2)	(3)	(4)	(5)
IWD	1	.99350	.99030	.994146
	3	.98945	.98426	.987213
	5	.98655	.97991	.954578
	7	.98480	.97736	.983842
	9	.98050	.97100	.982641
	11	.97480	.96259	.978624
	13	.96830	.95349	.969497
	15	.96090	.94218	.953112
	16	.95615	.93083	.952879
	20	.95245	.92204	.951913
	24	.94390	.91461	.950892
	28	.94670	.90852	.949611
	32	.94450	.90340	.948668
	36	.94290	.89969	.947460
	40	.94150	.89645	.946182
	44	.94035	.89380	.944831
	48	.93925	.89127	.943402
	52	.93825	.88898	.941891
	56	.93735	.88692	.940295
	60	.93645	.88487	.938607
	64	.93565	.88305	.936824
	68	.93490	.88135	.934939

Table 3.3 (Contd.)

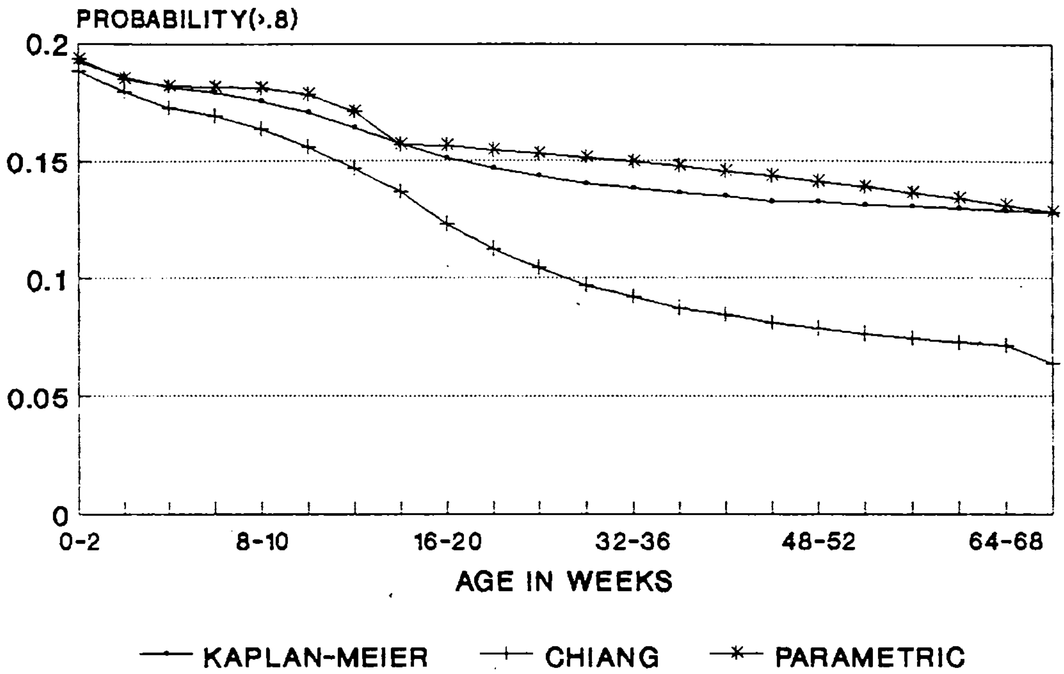
(1)	(2)	(3)	(4)	(5)
IWF	1	.99295	.98947	.993617
	3	.98815	.98234	.985913
	5	.98475	.97730	.982873
	7	.98240	.97377	.982095
	9	.97920	.96908	.981216
	11	.97409	.95755	.977885
	13	.96807	.95271	.969798
	15	.96095	.94231	.954780
	16	.95575	.92983	.954521
	20	.95189	.92068	.953459
	24	.94884	.91349	.952351
	28	.94634	.90764	.951194
	32	.94434	.90298	.949986
	36	.94268	.89915	.948725
	40	.94143	.89623	.947409
	44	.94048	.89407	.946036
	48	.93953	.89188	.944692
	52	.93878	.89016	.943106
	56	.93803	.88844	.941545
	60	.93743	.88722	.939916
	64	.93693	.88592	.938217
	68	.93643	.88478	.936444

3.9 Conclusion

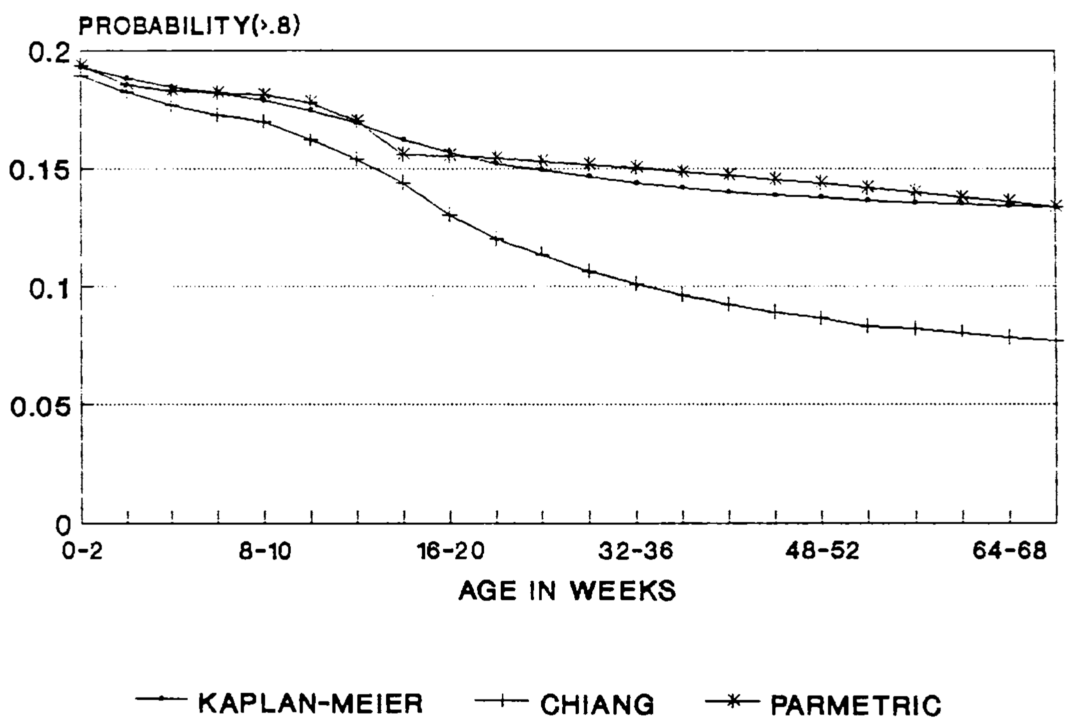
The survival probabilities derived for different age groups of IWN, IWP, IWK, IWD and IWF strains of chicken using the life table technique confirm the validity of the parametric model and survival probabilities. Besides, the method gives values of death rate among chicken in the same lines as prepared by Chiang (1972) for life table preparation of California Human Population (1970). It is noted that all strains possess almost equal survival probability through out the productive life which justifies the earlier studies in this regard. The vital information of death rate of chicken is very useful for formulating insurance policies of birds in a scientific manner. The life table technique gives deeper insight to take measures for rearing chicken of superior genetic type with a higher productivity. The results can be used to formulate plans for organising "health clinics" in the field of veterinary and Animal Sciences as envisaged in the annual plans of our country.

SURVIVAL PROBABILITY

WHITE LEGHORN IWN

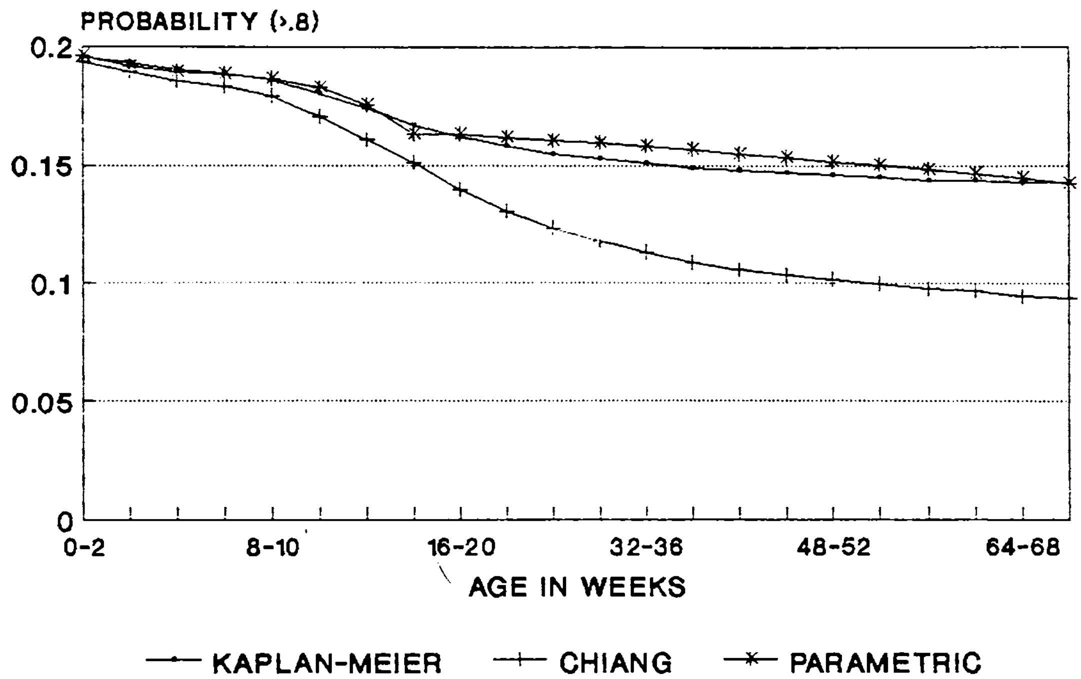


WHITE LEGHORN IWP

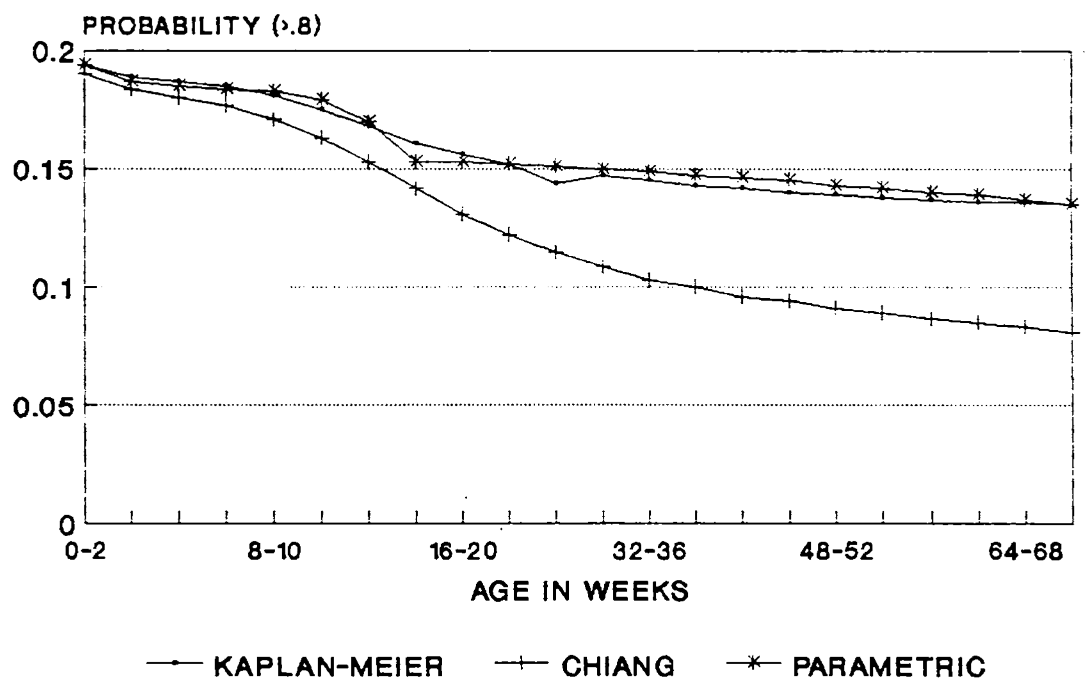


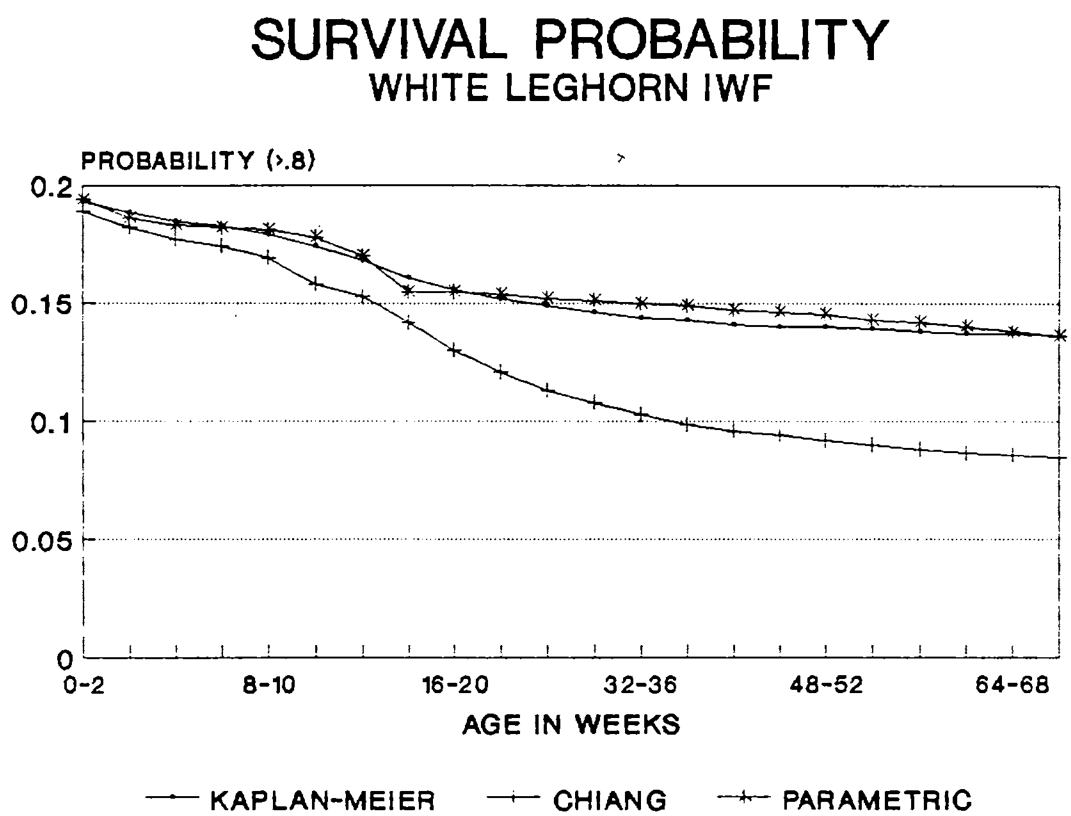
SURVIVAL PROBABILITY

WHITE LEGHORN IWK



WHITE LEGHORN IWD




SURVIVAL PROBABILITY
WHITE LEGHORN IWF
PROBABILITY (>.8)
0.2
0.15
0.1
0.05
0
0-2
8-10
16-20
32-36
48-52
64-68
AGE IN WEEKS
KAPLAN-MEIER
CHIANG
PARAMETRIC

CHAPTER FOUR

PERISHABLE INVENTORY PROBLEM WITH AGE-DEPENDENT REPLACEMENT POLICY

4.1 Introduction

In this chapter an inventory model of a single breed of chicken (white leghorn) is considered. The policy adopted is (S,s) and lead time is assumed to be zero. Further shortage cost is infinity. Chicken are disposed of on attaining age T (here 72 weeks). The life time of chicken are assumed to be independent and identically distributed random variables following exponential distribution with parameter μ. The demand process form a compound poisson process. The rate of arrival of demand is λ per unit time. The quantity demanded at an epoch is independent of the quantity demanded at any other epoch and q_i is probability that i units (i=1,2,...) are demanded at a demand epoch. Since lead time is zero we may assume that the optimal 's' value is zero. The replenishment rate is assumed to be infinite. The time-dependent and also long run system state probabilities are calculated. The optimal 'S' value also is computed.

Single Commodity Inventory Problem has been analysed by several researchers. An account of the work in its initial stage can be had from Hadley and Whittin (1963) and Naddor (1966). Stochastic Inventory system is studied in depth by Arrow, Karlin and Scarf (1958). Sivazlian (1975) considers a single commodity inventory system with a demand forming a renewal process. Lead time is taken to be zero and no shortage is permitted. He obtains the limiting inventory level distribution as a discrete uniform and derives the optimal values of the ordering quantity. This is extended by Srinivasan (1979) to include lead time having arbitrary distribution function. Sahin (1979) considers an inventory problem with continuous state space and constant lead time. The binomial moments are computed in the case of an inventory problem with random lead time and demand taking place according to a compound renewal processes by Sahin (1983). An excellent review of perishable single commodity inventory problem is contained in Nahmias (1982). Kalpakam and Arivarignam (1985) deal with an inventory model with one exhibiting item having exponential life time distribution. They establish the limiting inventory level distribution. Krishnamoorthy and Lakshmi (1991) deal with an inventory problem with Markov dependent demand quantities. This is especially useful in production inventory. Perishable Inventory problems are also considered, among others by Manoharan and Krishnamoorthy (1989), and Krishnamoorthy, Narasimhalu and Iqbal Basha (1992).

4.2 Mathematical Modelling and analysis of the problem

Let $0 < T_1 < T_2 \ldots < T_n < \ldots$ be the successive demand epochs. The successive replenishment epochs are identified as $T_0'(=0)$, $T_1', T_2' \ldots T_n' \ldots$ Note that the replenishment epochs need not coincide with a demand epoch since inventory level may fall to zero due to death of chicken. Further the successive replenishment epochs $T_0', T_1', \ldots T_n' \ldots$ constitute a renewal process since at these epochs the inventory levels are brought back to S.

The distribution of the time between two consecutive S to S transition is computed. This is then made use of compute the system state probabilities at any time (both finite and long run). The following notations are used:

$I(t)$ = Number of birds alive at time t; $t \geq 0$

$P_n(t)$ = $P\{I(t)=n/I(0)=S\}$, $n = s+1 \ldots\ldots S$

$$\phi_{\ell,m}(u) = P\left\{\begin{array}{l} I(u)=m/I(0)=\ell \text{ without a demand epoch and} \\ \text{no replenishment in the time interval} \\ (0,u) \end{array}\right\}$$

$\gamma_{\lambda,k}(u)$ denotes the gamma density with scale parameter λ and shape parameter k.

Thus $\phi_{\ell,m}(u) \geq 0$ for $\ell \geq m$
$= 0$ otherwise

$$\lim_{t\to\infty} P_n(t) = p_n, \; n=1,2,\ldots S.$$

Obviously $O_{,m}(u)$ stands for the probability that during an interval of duration u, the number of deaths is m

Thus

$$\phi_{\ell,m}(u) = \binom{\ell}{m} e^{-\mu um} (1-e^{-\mu u})^{\ell-m}$$

While proceeding to compute $P_n(t)$, for $t>0$, note that upto time t there might have been none, one or more replenishments. These may happen with or without any demands in between. So the distribution of the time between two consecutive replenishments is computed first. There are three cases.

(i) No demand in between consecutive replenishment epochs and the inventory level falls from S to ℓ (due to deaths) at the end of T time units from the

× previous replenishments epoch. The remaining $S-\ell$ birds are disposed off as their productive life has been completed on attaining age T. The probability of this event is

$$e^{-\lambda T} \phi_{S,\ell}(T) \qquad (1)$$

(ii) There are one or more demands between two replenishments. All the birds are either sold off and/or some of them died between these two epochs. Thus replenishment time (time between two replenishment epochs) is less than T in this case. The probability of this event is

$$\sum_{k=1}^{S} \int_{u_1=0}^{T} \cdots \int_{u=u_k}^{T} \phi_{S,\ell_1}(u_1)\, q_{m_1}\, \phi_{\ell_1-m_1,\ell_2}(u_2)\, q_{m_2} \cdots$$

$$\phi_{\ell_k-m_k}(u_k)\, q_{m_k}\, \lambda e^{-\lambda u_1}\, \lambda e^{-\lambda(u_2-u_1)} \cdots \lambda e^{-\lambda(u_k-u_{k-1})}$$

$$\phi_{S-\sum_{i=1}^{k} m_i-\ell_k,0}(u-u_k)\, du\, du_k \cdots du_1 \qquad (2)$$

Here the factor $\phi_{S-\sum_{i=1}^{k} m_i-\ell_k,0}(u-u_k)$ includes probability of left over, if any, dying before attaining age T.

(iii) There are one or more demands between two consecutive replenishments (time duration of this is T). Some birds are sold off and some die between these two epochs. The remaining are disposed off on attaining age T at which the next replenishment takes place. The probability for this denoted by H(x) equal to

$$\sum_{k=1}^{S} \int_{u_1=0}^{T} \cdots \int_{u_k=u_{k-1}}^{T} \sum_{\substack{l_1, l_2, \ldots l_k \geq 0 \\ m_1, m_2, \ldots m_k \geq 1 \\ l_1 + \cdots + l_k}} \sum_{j=1}^{S-\sum_{i=1}^{k} m_i - l_k} \phi_{S,l_1}^{(u_1)} q_{m_1} \phi_{l_1-m_1, l_2}^{(u_2)} q_{m_2} \cdots \phi_{l_{k-1}-m_{k-1}, l_k}^{(u_k)} q_{m_k}$$
$$\times \lambda e^{-\lambda u_1} \lambda e^{-\lambda(u_2-u_1)} \cdots \lambda e^{-\lambda(u_k-u_{k-1})}$$
$$\times \phi_{S-\sum_{i=1}^{k} m_i - l_k, j}^{(T-u_k)} e^{-\lambda(T-u_k)} \qquad (3)$$

Thus the distribution of any $Y_n = T_n' - T_{n-1}'$ is given by

$P[Y_n \leq x]$ = expression (2) for $x < T$

and $P[Y_n = T]$ is expression (i) + expression (iii)

Let the n-fold convolution (n=1,2,...) of H(x) be denoted by $H^{*n}(x)$ and its density by $h^{*n}(x)$ ($H^{*0}(u)$ is defined to be identically equal to one)

Now the inventory level probabilities can be computed at arbitrary (finite) time. For $t < T$,

$$P_S(t) = e^{-\lambda t} \phi_{S,S}(t) + \int_0^t \sum_{n=1}^{\infty} h^{*n}(u)\, e^{-\lambda(t-u)} \phi_{S,S}(t-u)\, du$$

and for $n = s+1, \ldots, S-1$,

$$P_n(t) = \sum_{\substack{l_1,\ldots,l_k \geq 0 \\ m_1,\ldots,m_k \geq 1 \\ \Sigma l_i + \Sigma m_i \leq S-1}} \int_0^t q_{m_1} \cdots q_{m_k}\, \phi_{S,l_1}(u_1) \cdots \phi_{l_k-m_k,\, m_k+n}(t-u) \times \lambda e^{-\lambda u_1}\, \lambda e^{-\lambda(u_2-u_1)} \cdots \lambda e^{-\lambda(u_k-u_{k-1})} \times e^{-\lambda(t-u)}\, du_1 \cdots du_k.$$

$$+ \int_{u=0}^{t}\int_{u_1=u}^{t} \cdots \int_{u_k=u_{k-1}}^{t} \sum_{\substack{l_1,\ldots,l_k \geq 0 \\ m_1,\ldots,m_k \geq 1 \\ \Sigma l_i + \Sigma m_i \leq S-1}} \sum_{n=1}^{\infty} h^{*n}(u)\, q_{m_1} q_{m_2} \cdots q_{m_k}\, \phi_{S,l_1}(u_1) \cdots \phi_{l_k-m_k,\, n}(t-u_k)\, \lambda e^{-\lambda(u_1-u)} \cdots \lambda e^{-\lambda(u_k-u_{k-1})}\, e^{-\lambda(t-u_k)}\, du_1\, du_2 \cdots du_k\, du.$$

For $t \geq T$

$$P_S(t) = \int_{u=t-T}^{t} \sum_{m=1}^{\infty} h^{*m}(u)\, e^{-\lambda(t-u)}\, \phi_{S,S}(t-u)\, du \qquad (4)$$

and for n satisfying $s+1 \leq n \leq S-1$

$$P_n(t) = \sum_{\substack{l_1,\ldots,l_k \geq 0 \\ m_1,\ldots,m_k \geq 1 \\ l_1+\cdots+l_k+m_1+\cdots+m_k \leq S-n}} \int_{u=t-T}^{t} \sum_{m=1}^{\infty} h^{*m}(u)\, q_{m_1} \cdots q_{m_k} \int_{u_1=u}^{t} \cdots \int_{u_k=u_{k-1}}^{t} \phi_{S,l_1}(u_1-u) \cdots \phi_{l_k-m_k,\, n}(t-u_k)\, \lambda e^{-\lambda(u_1-u)}\, \lambda e^{-\lambda(u_2-u_1)} \cdots \lambda e^{-\lambda(u_k-u_{k-1})}\, e^{-\lambda(t-u_k)}\, du_1 \cdots du_k$$

(5

4.3 Limiting distribution

Now the limiting distribution of the system state can be computed. To this end $P_S(t)$ and $P_n(t)$ (given above by (4) and (5)) for $t > T$ are made use of. The Laplace transform of a function is defined by

$$\hat{f}(z) = \int_0^\infty e^{-zt} f(t)\,dt$$

Taking the Laplace transform on both sides of (4) and (5) we get

$$P_S(z) = \sum_{m=1}^{\infty} (\hat{h}(z))^m \phi_{S,S}(\lambda+z) \tag{6}$$

and for n such that $1 \leq n \leq S-1$

$$\hat{P}_n(z) = \sum_{\substack{l_1, l_2, \dots l_k \geq 0 \\ m_1, m_2, \dots, m_k \geq 1 \\ l_1+\dots+l_k+m_1+m_2+\dots+m_k \leq S-m}} q_{m_1} q_{m_2} \cdots q_{m_k} \sum_{m=1}^{\infty} (\hat{h}(z))^m \lambda^k \frac{1}{\lambda+z} \hat{\phi}_{S,l_1}(z) \cdots \hat{\phi}_{l_k-m_k,n}(z) \quad (\text{:}$$

These can be inverted to obtain the required probabilities.

4.4 Optimisation problem

In this section the minimisation of total cost of running the system is discussed.

Let C_1 = fixed cost of ordering

C_2 = procurement cost per unit

C_3 = holding cost per unit per unit time

C_4 = loss due to death of a bird

C_5 = loss due to disposal of the bird on attaining age T if before that time it could not be sold off

The expected inventory(undecayed) held per unit time can be obtained from the inventory level distribution as given by (4) and (5). This provides the average holding cost per unit time. The average number of deaths is also obtained. Further the expected number of birds disposed off on attaining age T can be calculated. These taken together provide the expression for the expected total cost incurred per unit time. The S value that minimises the total cost is easily obtained from this. It easily follows that the optimal re-ordering level is zero since lead time is zero and shortage cost is infinity.

CHAPTER FIVE

TWO STRAIN INVENTORY PROBLEM

5.1. Introduction

Sivazlian (1971) considers the stationary characteristics of a multi commodity inventory system. Sivazlian and Stanfel (1975) deal with a two commodity single period inventory problem. Recently Krishnamoorthy, Lakshmi and Basha (1993,1994) have dealt with two strain inventory system with demand quantities exactly one unit of either type at each demand epoch. Here we generalize their result (contained in 1993). Specifically we consider a bulk demand two strain inventory problem with the strains represented by W_1 and W_2 respectively. We follow (s_i, S_i) policies for the strain $W_i (i=1,2)$. The probability that a demand occurs for strain W_i alone is p_i $(i=1,2)$, $p_1+p_2=1$. Conditioned on a demand taking place for $W_1 (W_2)$, the probability for $i(j)$ units of $W_1 (W_2)$ demanded is $g_i (h_j)$, $i=1,2,\ldots a (j=1,2,\ldots,b)$. A demand for both W_1 and W_2 together never occurs since $p_1+p_2=1$. The interarrival times of demands are i.i.d. random variables following distribution function $G(.)$, with mean   . The demand quantities are independent of the type of the commodity demanded. No shortage is permitted. Replenishment is such that whenever the inventory level of

W_i falls to s_i (i=1,2) or below that due to a demand, after the previous replenishment, an order is placed and instantaneous replenishment of that occurs so as to bring the inventory level back to S_i.

In section 2 we deal with the analysis of the model. In section 3 stationary distribution of the inventory level is computed. Section 4 deals with an optimisation problem. An example is also provided in section 4. Numerical illustrations are given in section 5.

Notations:

$X(t)$ = Inventory level of W_1 at time t

$Y(t)$ = Inventory level of W_2 at time t

$I(t)$ = $X(t), Y(t)$

M_i = $S_i - s_i$ for i=1,2

$*$ denotes convolution

E_i = $s_i+1, \ldots, S_i$, i=1,2

E = $E_1 \times E_2$

g_i = probability that i units of W_1 are demanded at a demand epoch given that the type of the commodity demanded is W_1, i=1,2,...a

h_j = probability that j units of W_2 are demanded at a demand epoch given that the type of the commodity demanded is W_2, j=1,2,...b

$$\phi_1(z) = \sum_{i=1}^{a} g_i z^i \quad , \qquad \phi_2(z) = \sum_{j=1}^{b} h_j z^j$$

$$[\phi_i(z)]^{*\ell} = [\phi_i(z)]^{*\ell-1} [\phi_i(z)]_1 = 2,3,\ldots\ldots * \quad i=1,2$$

with $O_i(z)^{*0} = 1$

$g_i(\ell)$ = probability of ℓ demands for W_1 alone consuming i units of W_1. This is the coefficient of z^i in $[\phi_1(z)]^{*\ell}$

$h_j(\ell)$ = probability of ℓ demands for W_2 alone consuming j units of W_2. This is the coefficient of z^j in $[\phi_2(z)]^{*\ell}$

$g_{i\ell}$ = probability that i units of W_1 demanded at 1^{th} demand epoch of W_1 after the previous replenishment,

$i=1,2,\ldots.a;\ \ell = \left[\frac{M_1}{a}\right] + \delta_{\left[\frac{M_1}{a}\right]} \ldots\ldots\ M_1$

where $\left[\frac{M_1}{a}\right] = \begin{cases} 1 & \text{if } M_1/a \text{ is not an integer} \\ 0 & \text{otherwise.} \end{cases}$

$h_{j\ell}$ = probability that j units of W_2 demanded at ℓ th demand epoch of W_2 after the previous replenishment, $j=1,2\ldots.b;$

$\ell = \left[\frac{M_2}{b}\right] + \delta_{\left[\frac{M_2}{b}\right]}, \ldots\ldots\ldots, M_2.$

$g_{i_{u,w}}$ = probability that $i_{u,w}$ (=1,2,...a) units of W_1 is demanded at the w^{th} demand epoch of W_1 (w=1,2,... r_{u+1}) in the interval containing u^{th} and $(u+1)^{th}$ demand epochs of W_2 where u=0,1,2,..., $\ell-1$

$h_{j_{v,x}}$ = probability that $j_{v,x}$(=1,2,...b) units of W_2 is demanded at the x^{th} demand epoch of W_2 (x=1,2,...r_{v+1}) in the interval containing v^{th} and $(v+1)^{th}$ demand epochs of W_1, where v=0,1,2,...,ℓ-1

5.2. Analysis

Suppose a total of exactly ℓ demands for W_1 alone results in its replenishment. Thus ℓ-1 demands take away atmost (S_1-s_1-1) units of W_1. In between there can be a number of demands for W_2. We compute the distribution of time between two consecutive replenishments of W_1 alone (W_2 alone). Let $0=T_0$ $T_1 < \ldots < T_n < \ldots$ be the successive demand epochs and $X_0, X_1, \ldots, X_n, \ldots$ and $Y_0, Y_1, \ldots, Y_n, \ldots$ be the inventory levels of W_1 and W_2, respectively, immediately after the demands at these epochs. Let $F_1 \left[(S_1,j),(S_1,k),t\right]$ be the probability distribution of the time between two consecutive S_1 to S_1 transition of W_1, with none, one, or

Then,

$$F_1\left[(S_1,j),(S_1,k),t\right] = \sum_{= \frac{M_1}{a} + \delta_{\left[\frac{M_1}{a}\right]}}^{M_1} \quad \sum_{r_1,r_2,\ldots r_\ell \geq 0} \quad \sum_{\left\{\begin{array}{l} i_1+\ldots+i_{\ell-1} < M_1 \\ i_1+\ldots+i_\ell \geq M_1 \end{array}\right\}}$$

$$\sum_{k=0}^{\ell-1} \sum_{r_{k+1}\geq 0} \sum_{j_{k,r_{k+1}}=1;\, j_{0,1}+\ldots+j_{0,r_1}+\ldots+j_{\ell-1,r_\ell}\geq 0}^{b}$$

$$p_2^{\,r_1} p_1 p_2^{\,r_2} p_1 \ldots p_2^{\,r_\ell} p_1 q_{ij}^{(r_1+\ldots+r_\ell)}$$

$$(g_{i_1}\ldots g_{i_{\ell-1}} g_{i_\ell})(h_{j_{0,1}}\ldots h_{j_{0,r_1}})\ldots$$

$$(h_{j_{\ell-1,1}}\ldots h_{j_{\ell-1,r_\ell}}) G(t)^{*(r_1+\ldots+r_\ell+\ell)}$$

where, $\delta_{\left[\frac{M_1}{a}\right]} = \begin{cases} 1 & \text{if } M_1/a \text{ is not an integer} \\ 0 & \text{otherwise} \end{cases}$

and

$q_{ij}^{(r)}$ = probability of a transition from i to j of W_2 due to r demands, $r=1,2,\ldots;$ $i,j \in E_2$.

with

$$q_{ij}^{(1)} = \begin{cases} h_{i-j} & \text{if } i > j \\ \sum_{k=i-s_2}^{b} h_k & j = S_2 \end{cases}$$

Define $R_1\left[(S_1,S_2),(S_1,k),t\right] = \sum_{n=0}^{\infty} F_1^{*n}\left[(S_1,S_2),(S_1,k),t\right]$

$$(S_1,S_2),(S_1,k) \in E$$

Similarly $F_2\left[(i,S_2),(m,S_2),t\right]$, the probability distribution of the time between two consecutive S_2 to S_2 transition of W_2 with none, one or more demands for W_1 in between, is

$$F_2\ (i,S_2),(m,S_2),t = \sum_{\ell=\left[\frac{M_2}{b}\right]+\delta_{\left[\frac{M_2}{b}\right]}}^{M_2} \ \sum_{r_1,r_2,\ldots,r_\ell \geq 0}$$

$$\sum_{\left\{\begin{array}{l} j_1+\ldots+j_{\ell-1} < M_2 \\ j_1+\ldots+j_\ell \geq M_2 \end{array}\right\}} \ \sum_{k=0}^{\ell-1} \ \sum_{r_{k+1}\geq 0} \ \sum_{\left\{i_{k,r_{k+1}}=1;\, i_{0,1},\ldots i_0, r_1 + \ldots + i_{\ell-1,r_\ell} \geq 0\right\}}^{a}$$

$$p_1^{\,r_1}\, p_2 p_1^{\,r_2}\, p_2 p_1^{\,r_3}\, p_2, \ldots, p_1^{\,r_\ell}\, p_2\, Y_{ij}^{(r_1+\ldots+r_\ell)} \times$$

$$(h_{j_1}\cdots h_{j_{\ell-1}}\cdots h_{j_\ell})(g_{i_{0,1}}\cdots g_{i_{0,r_1}})\cdots$$

$$(g_{i_{\ell-1,1}}\cdots g_{i_{\ell-1,r}})G^{*(r_1+\cdots+r_\ell+\ell)}(t)$$

where,

$$\delta\left(\frac{M_2}{b}\right)=\begin{cases}1 & \text{if } M_2/b \text{ is not an integer}\\ 0 & \text{otherwise}\end{cases}$$

and

$Y_{ij}^{(r)}$ = probability of a transition from i to j of W_1 due to r demands, $r=1,2,\ldots;$ $i,j \in E_1$.

with

$$Y_{ij}^{(1)} = \begin{cases} g_{i-j} & \text{if } i>j \\ \sum_{k=i-s_1}^{a} g_k & \text{if } j=S_1 \end{cases}$$

Define $R_2\left[(i,S_2),(m,S_2),t\right] = \sum_{n=0}^{\infty} F_2^{*n}\left[(i,S_2),(m,S_2),t\right]$

$$(i,S_2),(m,S_2)\in E$$

Next we compute the time dependent system size probabilities.

Let $I(t) = (X(t),Y(t))$ be the inventory level at time t.

Then $I(t) = (X_n,Y_n)$, $T_n \leq t < T_{n+1}$ and $I(t)$, $t > 0)$ is a semi-Markov process on E. The system size probabilities at time t satisfies the equation

$$P\left[(S_1,S_2),(i,j),t\right] = H\left[(S_1,S_2),(i,j),t\right] + \int_0^t \sum_{k \in E_2} R_1\left[(S_1,S_2),(S_1,k),du\right] H\left[(S_1,k),(i,j),t-u\right]; \quad i,j \in E.$$

where,

$H\left[(S_1,j),(i,k),t\right]$ = probability of transition from (S_1,j) to (i,k) with $i \neq S_1$ and the state S_1 of W_1 never revisited in $[0,t]$ if atleast one demand for W_1 occurs.

Thus,

$$H\left[(S_1,j),(i,k),t\right] = \begin{cases} \sum_{n=1}^{\infty} \phi^{(n)}_{(S_1,j),(i,k)} \left[G^{*n}(t) - G^{*(n+1)}(t)\right] & \text{if } i \neq S_1 \\ \sum_{n=0}^{\infty} \phi^{(n)}_{(S_1,j),(i,k)} \left[G^{*n}(t) - G^{*(n+1)}(t)\right] & \text{if } i = S_1 \end{cases}$$

Hence the time dependent system size probabilities are given by

$$P\left[(S_1,S_2),(i,j),t\right] = \int_0^t \sum_{k \in E_2} R_1\left[(S_1,S_2),(S_1,k),du\right] H\left[(S_1,k)(i,j),t-u\right]$$

5.3. Limiting distributions

Let $\lim_{t \to \infty} P[(S_1,S_2),(i,j),t]=p(i,j);(i,j) \in E.$

From the transition probability matrix of the Markov chain $[(X_n,Y_n)]$, its stationary distribution $\pi=\{\pi(i,j)/(i,j) \in E\}$ can be computed using $\pi \mathbb{P} = \pi$ and $\pi \underline{e} = 1$ where $\underline{e} = (1,...1)^T$ and π is a row vector of $M_1 \times M_2$ elements.

Theorem 1

The limiting probabilities of the system size are given by $P(i,j) = \pi(i,j);\quad (i,j) \in E.$

Proof:

The mean sojourn time in any state (i,j) is $m(i,j) = \int_0^\infty [1-G(t)]dt = \mu$ assumed finite. Hence the expected sojourn time is same for every state $(i,j);(i,j) \in E.$

Thus

$$P(i,j)= \frac{(i,j) \times \int_0^\infty Pr[I(t)=(i,j),T_1>t \mid I(0)=(i,j)]\,dt}{\sum_{(i,j)\in E} \pi(i,j)\; m(i,j)}$$

$$= \pi(i,j)$$

From the above expression $\lim_{t \to \infty} P[(S_1,S_2),(i,j),t]=P(i,j)=\pi(i,j)$ and are independent of the initial state, as is expected from the theory of finite state irreducible Markov chains.

Theorem 2

If $P_1 = p_2 = p$ $(=\frac{1}{2})$ then the inventory level probabilities follow the discrete uniform distribution

$$\pi(i,j) = \frac{1}{M_1 \ M_2} \text{ for every } (i,j) \in E$$

Proof:

From $\pi \mathbb{P} = \pi$ and $\pi \underline{e} = 1$ we see that the equation $\pi(i,j+1)p + \pi(i+1,j)p = \ (i,j)$, for $i = s_1+1, \ldots S_1$ and $j = s_2+1, \ldots, S_2$, have a solution given by

$$\pi(i,j) = \frac{1}{M_1 M_2}$$ for $(i,j) \in E$. However, this solution is unique since the Markov chain has a finite state space.

If we assume $p_2 = 0$ so that $p_1 = 1$ or $p_1 = 0$ so that $p_2 = 1$, we have a single commodity inventory problem.

5.4. Optimisation Problem

The objective function corresponding to this model is the total expected cost per unit time under steady state. Here the decision variables are S_1, s_1, S_2, s_2. T be the time duration between two consecutive replenishments of W_1 alone. Then define this T as the length of a cycle. Then the expected length of a cycle is E(T).

Distribution of time for S_1 to S_1 transition

$$= \sum_{\ell = \frac{M_1}{a} + \delta\left[\frac{M_1}{a}\right]}^{M_1} \sum_{k=0}^{\infty} p_1^{\ell} p_2^{k} \sum_{r=1}^{a} \sum_{j=0}^{a-r} g_{M_1-r}^{(\ell-1)} g_{r+j}^{(1)} G^{*(\ell+k)}(t)$$

$$E(T) = \sum_{\ell = \frac{M_1}{a} + \delta\left[\frac{M_1}{a}\right]}^{M_1} \sum_{k=0}^{\infty} (\ell+k) E\,(\text{inter arrival time}) \times$$

$$p_1^{\ell} p_2^{k} \sum_{r=1}^{a} \sum_{j=0}^{a-r} g_{M_1-r}^{(\ell-1)} g_{r+j}^{(1)}$$

$$= \sum_{\ell = \frac{M_1}{a} + \delta\left[\frac{M_1}{a}\right]}^{M_1} \sum_{k=0}^{\infty} \mu(\ell+k)\, p_1^{\ell} p_2^{k} \sum_{r=1}^{a} \sum_{j=0}^{a-r} g_{M_1-r}^{(\ell-1)} g_{r+j}^{(1)} \qquad (*)$$

Hence the expected number of orders placed per unit time for W_1 is $\frac{1}{E(T)}$.

The expected number of demands for W_2 in time $E(T)$ is

$$\left[\frac{E(T)}{\mu} - M_1'\right]^{+} \text{ where } M_1' = \left[\frac{M_1}{\sum_{i=1}^{a} i\, g_i}\right] + \delta\left[\frac{M_1}{\sum_{i=1}^{a} i\, g_i}\right] \text{ and } x^{+} = \max[o, x]$$

Hence the expected number of orders placed per unit time for W_2 is

$$\frac{\left[\frac{E(T)}{\mu} - M_1'\right]^+}{M_2'\,E(T)} \quad \text{where } M_2' = \left[\frac{M_2}{\sum_{j=1}^{b} j\,h_j}\right] + \delta\left(\frac{M_2}{\sum_{j=1}^{b} j\,h_j}\right)$$

Let k_1 and k_2 be the fixed ordering costs for W_1 and W_2 respectively. Then the total expected cost of ordering for W_1 and W_2 per unit time is

$$\frac{k_1}{E(T)} + k_2\,\frac{\left[\frac{E(T)}{\mu} - M_1'\right]^+}{M_2'E(T)}$$

Let v_1 and v_2 be the holding cost of W_1 and W_2 per unit per unit time. Then the total average holding cost of W_1 and W_2 per unit time is

$$v_1\left[\sum_{i=s_1+1}^{S_1} i \sum_{j=s_2+1}^{S_2} \pi(i,j)\right] + v_2\left[\sum_{j=s_2+1}^{S_2} j \sum_{i=s_1+1}^{S_1} \pi(i,j)\right]$$

$$= V(S_1,S_2,s_1,s_2) \qquad (**)$$

Thus the total expected cost per unit time under steady state is $Z(S_1,s_1,S_2s_2)$ where

$$Z(S_1,s_1,S_2,s_2)=V(S_1,S_2,s_1,s_2)+\frac{k_1}{E(T)}+\frac{k_2\left[\frac{E(T)}{\mu}-M_1'\right]^+}{M_2'\,E(T)}$$

$$+\frac{r_1M_1'}{E(T)}+\frac{\left[\frac{E(T)}{\mu}-M_1'\right]^+}{E(T)}\,r_2,$$ where r_i is the unit procurement cost of item W_i $(i=1,2)$ and the values of $E(T)$ and $V(S_1,S_2,s_1,s_2)$ are given by (*) and (**). The optimal values of M_1 and M_2 can be calculated from the given values of $k_1,k_2,v_1,v_2,r_1,r_2,p_1,p_2,g_i$'s, h_j's and μ $(i=1,\dots a; j=1,2,..,b)$.

In the following illustration we compute the explicit expression for $E(T)$.

An Application

Suppose a system has S_1 identical components of type I and S_2 identical components of type II. The system is considered operating if at least s_1+1 type I and s_2+1 of type II of the components function. Otherwise the system is in the failed state. We assume that the life-time of all components of type I follow exponential distribution with mean μ_1 and that of type II follow exponential distribution with mean μ_2. At time origin all components are operating. Let T be the random variable denoting the time to failure of the system starting with S_1 type I and S_2 type II components at time zero. The system reliability in $[0,t]$ is given by

$$P[T>t]=\sum_{\ell=0}^{M_1-1}\sum_{k=0}^{M_2-1}\binom{S_1}{\ell}(1-e^{-\mu_1 t})^{\ell}(e^{-\mu_1 t})^{S_1-\ell}\binom{S_2}{k}(1-e^{-\mu_2 t})^{k}(e^{-\mu_2 t})^{S_2-k}$$

$P_0(t)$ denotes the probability that the system is in failed state at time t. Then

$$P_0(t) = 1-\sum_{\ell=0}^{M_1-1}\sum_{k=0}^{M_2-1}\binom{S_1}{\ell}(1-e^{-\mu_1 t})^{\ell}(e^{-\mu_1 t})^{S_1-\ell}\binom{S_2}{k}(1-e^{-\mu_2 t})^{k}(e^{-\mu_2 t})^{S_2-k}$$

Failed components are replaced by new identical components as soon as the system fails. Let Y be the random variable denoting the time elapsed between two successive replacements. We assume that $\mu_1=\mu_2$ and write $\mu_1 t = v$

Then,

$$E(Y) = \int_0^{\infty} P(Y>t)dt$$

$$=\int_0^{\infty}\sum_{\ell=0}^{M_1-1}\sum_{k=0}^{M_2-1}\binom{S_1}{\ell}(1-e^{-v})^{\ell}(e^{-v})^{S_1-\ell}\binom{S_2}{k}(1-e^{-v})^{k}(e^{-v})^{S_2-k}\frac{dv}{\mu_1}$$

$$= \int_0^\infty \sum_{\ell=0}^{M_1-1} \sum_{k=0}^{M_2-1} \frac{1}{\mu_1} \binom{S_1}{\ell} \binom{S_2}{k} B[S_1+S_2-(\ell+k), \ell+1]$$

$$F[-k, S_1+S_2-(\ell+k); S_1+S_2+1-k; 1]$$

$$= \frac{1}{\mu_1} \sum_{\ell=0}^{M_1-1} \sum_{k=0}^{M_2-1} \binom{S_1}{\ell} \binom{S_2}{k} B[S_1+S_2-(\ell+k), \ell+k+1]$$

see Abramowitz and Stegun (1970)

Particular case.

When there is only one type of components the above reduces to the problem of multiple satellite launch discussed by Sivazlian and Stanfel (1975).

5.5. Numerical Illustrations.

Consider a two strain inventory system with $k_1=10$, $k_2=12$, $r_1=5$, $r_2=7.5$, $v_1=1.00$, $v_2=1.50$, $a=5$, $b=4$ and mean of the distribution of the interarrival time of demands, $=4$. For four sets of fixed values of p_1, p_2, g_i's and h_j's, $i=1,2,\ldots,5$; $j=1,2,\ldots,4$ E(T) and the average cost are computed and tabulated. Then the optimal values of M_1 and M_2 are obtained.

Sl. No.	S_1	s_1	S_2	s_2	a	b	p_1	p_2	g_i	h_j	E(T)	Average cost
1	20	1	10	8	5	4	.4	.6	.2	.4	.02	2546.30
2							.5	.5	.2	.2	.04	1130.84
3							.6	.4	.3	.2	.10	523.74
4							.7	.3	.1	.2	.22	242.17
5							.8	.2	.2		.56	110.85
1	20	2	10	5	5	4	.4	.6	.2	.4	.08	606.71
2							.5	.5	.2	.2	.18	279.27
3							.6	.4	.3	.2	.40	138.89
4							.7	.3	.1	.2	.91	73.89
5							.8	.2	.2		2.28	43.66
1	20	3	10	6	5	4	.4	.6	.2	.4	.22	227.65
2							.5	.5	.2	.2	.50	113.74
3							.6	.4	.3	.2	1.10	64.93
4							.7	.3	.1	.2	2.51	42.38
5							.8	.2	.2		6.18	31.92
1	20	4	10	7	5	4	.4	.6	.2	.4	.54	105.03
2							.5	.5	.2	.2	1.23	60.67
3							.6	.4	.3	.2	2.72	41.67
4							.7	.3	.1	.2	6.15	32.92
5							.8	.2	.2	.2	14.85	28.87

From the table we see that for different values of M_1 and M_2, the optimal pair is M_1=15 and M_2=3. For different p_1,p_2,g_i's,h_j's values we can find out the optimal pair from a given set of values of (M_1,M_2).

Appendix-I

OBSERVED DATA OF WHITE LEGHORN CHICKENS
(Each group contains 2000 Nos.)

IWN STRAIN

Groups Age (Weeks)	Mortality observed										20,000 Total
	1	2	3	4	5	6	7	8	9	10	11
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
D-2	20	14	17	16	14	16	10	14	16	20	157
2-4	16	10	12	10	13	8	8	12	14	14	117
4-6	12	9	10	8	10	6	8	11	10	10	94
6-8	6	5	6	5	5	4	6	5	4	5	51
8-10	16	7	5	7	5	6	8	9	6	5	74
10-12	20	12	8	9	8	10	10	10	8	8	103
12-14	24	14	12	10	10	12	10	10	10	12	124
16-20	20	16	10	10	9	10	10	10	12	8	115
20-24	16	10	8	7	7	8	10	8	8	8	90
24-28	12	8	6	5	7	6	8	6	8	7	73
28-32	10	5	5	5	5	6	6	5	6	6	59
32-36	6	5	3	4	4	4	4	5	4	5	44
36-40	4	4	3	4	3	4	4	3	4	5	38
40-44	4	3	2	4	2	2	2	3	2	3	30
44-48	4	3	2	3	2	2	2	3	2	3	26
48-52	2	2	2	3	2	2	2	2	2	2	21
52-56	3	2	2	2	2	2	2	2	2	2	21
56-60	1	2	2	2	2	3	2	1	2	4	21
60-64	2	1	1	2	1	2	2	1	2	2	16
64-68	1	2	1	2	2	1	2	1	2	2	16
68-72	1	1	1	1	2	2	1	2	2	2	15

IWP STRAIN

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
0-2	18	16	16	12	10	14	10	18	16	12	142
2-4	8	10	6	6	8	12	8	12	14	8	92
4-6	10	8	4	4	6	8	6	8	12	10	76
6-8	6	7	4	4	5	4	4	4	8	8	54
8-10	8	9	6	6	7	2	2	4	10	8	58
10-12	10	9	10	8	9	6	4	6	14	10	86
12-14	12	10	12	12	10	8	7	12	16	12	111
14-16	16	14	16	10	10	16	10	12	20	14	138
16-20	12	13	10	10	9	14	10	10	12	12	112
20-24	8	10	8	8	7	10	8	8	10	8	85
24-28	6	8	6	8	5	12	6	6	8	8	61
28-32	6	8	4	6	4	8	4	6	6	6	58
32-36	4	6	4	4	4	6	4	6	4	6	48
36-40	4	6	4	4	3	6	2	4	4	4	41
40-44	4	5	2	3	4	4	2	2	4	4	34
44-48	2	4	2	2	4	4	2	2	2	3	27
48-52	2	3	2	2	2	2	2	2	2	3	22
52-56	2	3	2	2	2	2	2	2	2	3	22
56-60	2	1	2	2	1	3	2	1	2	2	18
60-64	3	2	1	2	1	2	1	2	2	2	18
64-68	1	1	2	2	2	1	2	3	1	1	16
68-72	2	1	1	1	1	2	1	2	2	2	15

IWK STRAIN

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
0-2	8	10	8	6	8	6	10	12	9	8	85
2-4	6	6	4	6	4	4	8	8	5	6	57
4-6	6	5	4	4	4	4	6	8	4	4	49
6-8	4	4	2	3	2	3	4	6	3	2	33
8-10	8	6	6	8	6	5	8	8	10	8	73
10-12	14	10	8	10	10	8	10	10	11	10	101
12-14	18	10	10	12	10	10	14	12	14	12	122
14-16	20	12	12	14	12	12	18	14	14	14	142
16-20	16	8	8	8	8	8	10	10	10	8	94
20-24	14	6	6	6	6	5	10	6	9	8	76
24-28	10	6	4	6	6	5	9	5	5	6	62
28-32	8	4	4	4	4	3	6	4	4	6	47
32-36	6	4	2	4	4	3	6	4	4	4	41
36-40	6	2	2	4	3	2	4	3	3	4	33
40-44	4	2	1	3	3	2	4	3	3	2	27
44-48	4	2	1	3	2	1	2	2	3	2	22
48-52	2	1	1	2	2	1	2	2	2	2	17
52-56	2	1	1	2	2	1	1	2	2	2	16
56-60	2	2	1	2	2	1	1	2	2	2	17
60-64	2	2	1	2	1	2	1	1	1	1	14
64-68	1	1	1	1	1	2	1	1	1	1	11
68-72	1	1	1	1	1	1	1	1	1	1	10

IWD STRAIN

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
0-2	10	14	16	12	14	16	10	12	14	12	130
2-4	8	6	12	8	10	10	6	8	7	6	81
4-6	6	6	10	6	6	6	4	4	6	4	58
6-8	4	2	6	4	2	4	2	4	3	4	35
8-10	10	10	8	6	8	12	8	10	6	8	86
10-12	14	12	10	8	12	12	12	14	10	10	114
12-14	14	12	12	10	14	16	14	14	12	12	130
14-16	16	14	12	14	14	18	16	16	14	14	148
16-20	10	8	10	9	10	10	12	12	8	6	95
20-24	8	6	8	6	8	8	10	8	6	6	74
24-28	6	6	8	5	8	6	8	6	6	4	63
28-32	4	4	6	4	6	6	8	6	4	4	52
32-36	4	4	6	4	6	4	6	4	4	2	44
36-40	3	2	4	3	4	4	6	2	2	2	32
40-44	3	2	4	2	4	3	4	2	2	2	28
44-48	2	2	2	2	2	3	4	2	2	2	23
48-52	2	2	2	2	2	3	3	2	2	2	22
52-56	1	2	1	2	3	2	3	3	2	3	22
56-60	2	2	3	2	2	2	1	2	3	1	20
60-64	3	2	2	1	2	2	2	1	3	2	20
64-68	1	1	2	1	2	2	2	2	2	1	16
68-72	1	1	1	1	3	2	2	1	2	2	16

IWF STRAIN

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
0-2	10	11	14	12	18	16	14	16	16	14	141
2-4	8	6	8	8	10	12	8	10	14	12	96
4-6	6	6	6	5	6	8	4	6	10	11	68
6-8	4	4	4	4	3	6	4	3	6	9	47
8-10	6	6	7	10	7	4	8	7	4	5	64
10-12	8	10	12	14	8	10	12	10	8	10	102
12-14	12	10	14	16	8	14	14	12	10	10	120
14-16	14	14	16	16	12	16	14	14	14	12	142
16-20	10	8	10	11	9	12	10	12	12	10	104
20-24	8	5	8	8	6	8	8	10	8	8	77
24-28	6	4	6	6	5	6	6	8	8	6	61
28-32	4	4	4	6	5	6	4	6	6	5	50
32-36	4	3	4	4	4	4	4	4	4	5	40
36-40	3	2	3	3	4	4	3	4	4	3	33
40-44	3	2	2	3	2	2	3	3	2	3	25
44-48	2	1	2	2	2	2	2	2	2	2	19
48-52	1	1	3	3	2	2	2	2	1	2	19
52-56	2	1	1	1	2	3	2	1	1	1	15
56-60	1	1	1	3	1	2	3	1	1	1	15
60-64	1	1	1	2	2	1	3	2	1	1	15
64-68	1	1	1	1	1	1	1	1	1	1	10
68-72	1	1	1	1	1	1	1	1	1	1	10

SOFTWARE DEVELOPED FOR ESTIMATION OF PARAMETERS

```
5  REM SURVIVAL PROBABILITY
10 DIM X(100),S(100),Y(100),F(100)
20 INPUT "Give the data file : ";F$
30 OPEN "I",#1,F$
40 INPUT "Give the no.of observations : ";N
50 FOR I=1 TO N
60 INPUT #1,X(I)
70 NEXT
80 FOR I=1 TO N
90 INPUT #1,S(I)
100 Y(I)=-LOG(S(I))/X(I)
110 NEXT
120 SX=0 : SX2=0 : SX3=0 : SXY=0 : SX4=0 : SX2Y=0 : N1=0 : SY=0
130 FOR I=1 TO N
135 IF X(I) > 15 THEN 160
140 N1=N1+1 : SX=SX+X(I):SX2=SX2+X(I)^2 : SX3=SX3+X(I)^3 : SY=SY+Y(I)
150 SXY=SXY+X(I)*Y(I) : SX2Y=SX2Y+X(I)^2*Y(I) : SX4=SX4+X(I)^4
160 NEXT
162 D1=SX2*(SX2^2-SX3*SX) : D2=SX*(SX3*SX2-SX4*SX) : D3=N1*(SX3^2-SX4*SX2)
170 DELTA=D1-D2+D3
180 D4=SY*(SX2^2-SX3*SX) : D5=SX*(SXY*SX2-SX2Y*SX) : D6=N1*(SXY*SX3-SX2Y*S
182 DL1=D4-D5+D6
190 D7=SX2*(SXY*SX2-SX2Y*SX) : D8=SY*(SX3*SX2-SX4*SX) :
    D9=N1*(SX3*SX2Y-SX4*SXY)
192 DL2=D7-D8+D9
200 D10=SX2*(SX2*SX2Y-SX3*SXY) : D11=SX*(SX3*SX2Y-SX4*SXY) :
    D12=SY*(SX3^2-SX4*SX2)
204 DL3=D10-D11+D12
210 A=DL1/DELTA : B=DL2/DELTA : C=DL3/DELTA
220 A1=3*A : B1=2*B
230 PRINT "The Equation is -1/x Log(Sx) =";A;" x^2 +";B;" x + ";C
240 PRINT
245 A$="#####.####"
250 PRINT TAB(20); "The Estimated Values"
260 PRINT SPC(10);"X";SPC(15);"Sx";SPC(10);"Sx.est."
270 FOR I=1 TO N
275 IF X(I) > 15 THEN 400
280 PRINT SPC(5) USING A$;X(I);
290 PRINT SPC(5) USING A$;S(I);
295 SEST=EXP(-X(I)*(A*X(I)^2+B*X(I)+C))
300 PRINT SPC(5) USING A$;SEST
310 NEXT
400 ALPHA=15 : H=28
410 K=EXP(-ALPHA*(A*ALPHA^2+B*ALPHA+C))
420 FOR I=1 TO N
430 Y(I)=LOG(S(I))
440 NEXT
```

```
450 Q=LOG((Y(23)-Y(16))/(Y(16)-Y(9)))/H : ALPQ=EXP(ALPHA*Q)
460 P=Q*(LOG(K)-Y(9))/(EXP(Q*X(9))-ALPQ)
470 SIGMAF=0 : SFEQX=0 : SFXEQX=0 : SFX=0 : SFX2E=0
480 FOR I=1 TO N
490 INPUT #1, F(I)
500 SIGMAF=SIGMAF+F(I) : SFEQX=SFEQX+F(I)*EXP(Q*X(I))


510 SFX=SFX+X(I)*F(I) : SFX2E=SFX2E+F(I)*X(I)^2*EXP(Q*X(I))
520 SFXEQX=SFXEQX+F(I)*X(I)*EXP(Q*X(I))
530 NEXT
540 DLDP=SIGMAF/P+(SIGMAF*ALPQ-SFEQX)/Q
550 DLDQ=-P/Q^2*(SFEQX-SIGMAF*ALPQ)-P/Q*(SFXEQX-SIGMAF*ALPHA*ALPQ)+SFX
560 D2LDP2=-SIGMAF/P^2
570 D2LDQ2=2*P/Q^3*(SFEQX-SIGMAF*ALPQ)-P/Q*(SFX2E-SIGMAF*ALPHA^2*ALPQ)
580 D2LDPQ=-1/Q^2*(SFXEQX-SIGMAF*ALPHA*ALPQ)
590 A1=D2LDP2 : B1 = D2LDPQ
600 A2=B1 : B2=D2LDQ2
610 DET=A1*B2-A2*B1
620 IA1=B2/DET : IB1=-B1/DET
630 IA2=-A2/DET : IB2=A1/DET
640 X1=DLDP
650 X2=DLDQ
660 IA1=IA1*X1+IB1*X2
670 IA2=IA2*X1+IB2*X2
680 PDASH=P-IA1 : QDASH=Q-IA2
700 FOR I=N1+1 TO N
710 PRINT SPC(5) USING A$;X(I);
720 PRINT SPC(5) USING A$;S(I);
725 ALPQ=EXP(QDASH*ALPHA)
730 SEST=K*EXP(-PDASH/QDASH*(EXP(QDASH*X(I))-ALPQ))
740 PRINT SPC(5) USING A$;SEST
750 NEXT
755 PRINT : PRINT "Initial Value of p = ";P;SPC(3);"q = ";Q
760 PRINT : PRINT "Final Value of p   = ";PDASH;SPC(3);"q = ";QDASH
770 PRINT
780 PRINT "The Model is ";K;" * Exp[";-PDASH/QDASH;" * {exp(";QDASH;"x)";
    -ALPQ;"}]"
```

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