STOCHASTIC MODELLING, ANALYSIS
AND APPLICATIONS

ANALYSIS OF SOME STOCHASTIC INVENTORY SYSTEMS SUBJECT TO DECAY AND DISASTER

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CERTIFICATE

Certified that the thesis entitled **"ANALYSIS OF SOME STOCHASTIC INVENTORY SYSTEMS SUBJECT TO DECAY AND DISASTER"** is a bonafide record of work done by Sri. Varghese T. V. under my guidance in the Department of Mathematics, Cochin University of Science and Technology, and that no part of it has been included any where previously for the award of any degree.

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Chapter I

Introduction

In this thesis we study some problems in stochastic inventories with a special reference to the factors of *decay* and *disaster* affecting the stock. The problems are analyzed by identifying certain stochastic processes underlying these systems. Our main objectives are to find transient and steady state probabilities of the inventory states and the optimum values of the decision variables that minimize the cost functions. Most of the results are illustrated with numerical examples.

This introductory chapter contains some preliminary concepts in inventory and stochastic process, a brief review of the literature relevant to our topic and an outline of the work done in the present thesis.

1.1 INVENTORY SYSTEMS

Inventory is the stock kept for future use to synchronize the inflow and outflow of goods in a transaction. Examples of inventory are physical goods stored for sale, raw materials to be processed in a production plant, a group of personnel undergoing training for a firm, space available for books in a library, power stored in a storage battery, water kept in a dam, etc. Thus inventory models have a wide range of applications in the decision making of governments, military organizations, industries, hospitals, banks, educational institutions, etc. Study and research in this fast growing field of Applied Mathematics taking models from practical situations will contribute significantly to the progress and development of human society.

There are several factors affecting the inventory. They are demand, life times of items stored, damage due to external disaster, production rate, the time lag between order and supply, availability of space in the store, etc. If all these parameters are known beforehand, then the inventory is called deterministic. If some or all of these parameters are not known with certainty, then it is justifiable to consider them as random variables with some probability distributions and the resulting inventory is then called stochastic or probabilistic. Systems in which one commodity is held independent of other commodities are analyzed as single commodity inventory problems. Multi-commodity inventory problems deal with two or more commodities held together with some form of dependence. Inventory systems may again be classified as continuous review or periodic review. A continuous review policy is to check the inventory level continuously in time and a periodic review policy is to monitor the system at discrete, equally spaced instants of time.

Efficient management of inventory systems is done by finding out optimal values of the decision variables. The important decision variables in an inventory system are order level or maximum capacity of the inventory, re-ordering point, scheduling period and lot size or order quantity. They are usually represented by the letters, S, s, t and q respectively. Different policies are obtained when different combinations of decision variables are selected. Existing prominent inventory policies are: i) (s, S) - policy in which an order is placed for a quantity up to S whenever the inventory level falls to or below s,

ii) (s, q) - policy where the order is given for q quantity when the inventory level is s or below it, iii) (t, S) - policy which places an order at scheduling periods of t lengths so as to bring back the inventory level up to S and iv) (t, q)- policy that gives an order for q quantity at epochs of t interval length.

In multi-commodity inventory systems there are different replenishment policies. A single ordering policy is to order separately for each commodity whenever its inventory level falls to or below its re-ordering point. A joint ordering policy is to order for all the commodities whenever the inventory levels are equal to or below a pre-fixed state. The pre-fixed state may be the re-ordering point of at least one of the commodities, of at least some of the commodities, or of all the commodities. In the latter two cases there is a possibility of shortages of inventory.

The period between an order and a replenishment is termed as lead time. If the replenishment is instantaneous, then lead time is zero and the system is then called an inventory system without lead time. Inventory models with positive lead time are complex to analyze; still more complex are the models where the lead times are taken to be random variables.

Shortages of inventory occur in systems with positive lead time, in systems with negative re-ordering points, or in multi-commodity inventory systems in which an order is placed only when the inventory levels of at least two commodities fall to or below their re-ordering points. There are different methods to face the stock out periods of the inventory. One of the methods is to consider the demands during the dry periods as lost sales. The other is partial or full backlogging of the demands during these periods. Partial backlogging policy is an interesting field for recent researchers, with the adaptation of N-policy, T-policy and D-policy from queueing theory, in which local purchase is made when either the number of backlogs or the lead time exceeds a pre-fixed number.

In most of the analysis of inventory systems the decay and disaster factors are ignored. But in several practical situations these factors play an important role in decision making. Examples are electronic equipment stored and exhibited on a sales counter, perishable goods like food stuffs, chemicals, pharmaceuticals preserved in storage, crops vulnerable to insects and natural calamity, etc.

Large stores usually stock more than one commodity at a time that are also inter-related. For example, computer and its peripherals, electric equipment and voltage stabilizers, sanitary wares and their fittings, automobile spare parts, clothes for shirts and other suits, etc.

In this thesis we study single and multi-commodity stochastic inventory problems with continuous review (s, S) policy. Among the eight models discussed, four models are about single commodity inventory systems with a special focus on natural decay and external disaster. The next two are their extensions to multi-commodity. The last two models are two commodity problems with Markov shift in demand. These problems are analyzed with the help of the theories of stochastic processes, namely, Markov processes, renewal process, Markov renewal processes and semi-regenerative processes.

1.2 SOME BASIC CONCEPTS IN STOCHASTIC PROCESSES

Many a phenomenon, occurring in physical and life sciences, engineering and management studies are widely studied now not only as a random phenomenon but also as one changing with time or space. The study of random phenomena which are also functions of time or space leads to stochastic processes.

1.2.1 Stochastic Process

A stochastic process is a family of random variables $\{X(t),\ t\in I\}$ taking values from a set E. The parameter t is generally interpreted as time though it may represent a counting number, distance, length, thickness and so on. The sets I and E are called the index set and the state space of the process respectively. There are four types of stochastic processes depending on whether I and E are discrete or not. A discrete parameter stochastic process is usually written as $\{X_n\ ,\ n\in I\}$. If the members of the family of random variables $\{X(t),\ t\in I\}$ are mutually independent, it is an independent process. In the simplest form of dependency the random variables depend only on their immediate predecessors or only on their immediate successors, not on any other. A stochastic process possessing this type of dependency is known as a Markov process.

1.2.2 Markov Process

A stochastic process $\{X(t),\ t\in I\}$ with index set I and state space E is said to be a Markov process if it satisfies the following conditional probability statement:

$$\begin{aligned}\Pr\{X(t_n)\le x_n \mid X(t_0)=x_0, X(t_1)=x_1,\ldots\ldots, X(t_{n-1})=x_{n-1}\} = \\ \Pr\{X(t_n)\le x_n \mid X(t_{n-1})=x_{n-1}\} \quad \text{for all } t_0 < t_1 < \ldots\ldots < t_n.\end{aligned} \tag{1.1}$$

Discrete valued Markov processes are often called Markov chains. A Markov process can be completely specified with i) the marginal probability

$\Pr\{X(t_0) = x_0\}$, called the initial condition and ii) a set of conditional density functions $\Pr\{X(t_r) = x_r \mid X(t_s) = x_s \,;\, t_r < t_s\}$, called the transition probability densities. The Markov process is said to be stationary or time homogeneous if

$$\Pr\{X(t_r+\alpha)=j \mid X(t_r)=i\} = \Pr\{X(t_s+\alpha)=j \mid X(t_s)=i\};\ \text{for all r and s};\ \alpha > 0 \tag{1.2}$$

In that case (1.2) is denoted as p_{ij}^{α} or $p_{ij}(\alpha)$. A discrete parameter stationary Markov chain can be completely specified by the initial condition and the one step transition probability matrix $\mathbf{P} = (p_{ij});\ i, j \in E$, where $p_{ij} = \Pr\{X_{r+1} = j \mid X_r = i\}$. For a stationary continuous parameter Markov chain the role of the one step transition probabilities is played by the infinitesimal generator or the transition intensity matrix, $\mathbf{Q} = (q_{ij});\ i, j \in E$ where

$$q_{ij} = \begin{cases} -\dfrac{d}{dt} p_{jj}(0); & \text{for} \quad i = j \\ \dfrac{d}{dt} p_{ij}(0); & \text{for} \quad i \neq j \end{cases} \tag{1.3}$$

The following results on limiting probabilities of stationary Markov chains have wide range applications in many practical situations. Proofs of the results quoted in this chapter can be found in standard books on stochastic process.

Theorem 1.1

Let $\{X_n\,,\ n \in I\}$ be an irreducible and aperiodic Markov chain with discrete index set I and state space E. Then all states are recurrent non-null if and only if the system of linear equations

$$\sum_{i \in E} \pi_i p_{ij} = \pi_j;\quad j \in E \quad \text{and} \quad \sum_{i \in E} \pi_i = 1 \tag{1.4}$$

has a solution $\Pi = (\pi_1, \pi_2, \ldots\ldots)$. If there is a solution Π, then it is strictly positive, unique and $\pi_j = \lim_{n\to\infty} p_{ij}^n$ for all $i,j \in E$. □

When E is finite all the states are recurrent non-null, therefore, a unique solution Π exits always. Π is called the invariant measure of the Markov chain $\{X_n, n\in I\}$.

Theorem 1.2

Suppose $\{X(t),\ t \in R_+\}$ be an irreducible recurrent continuous time Markov chain with discrete state space E. Then

$$\pi(j) = \lim_{t\to\infty} \Pr\{X(t) = j\}; \quad j \in E \tag{1.5}$$

exists and is independent of X(0). If E is finite, then $\pi(j)$'s are given by the unique solution of $\sum_{i\in E} \pi(i) q_{ij} = 0; \quad j \in E$ and $\sum_{i\in E} \pi(i) = 1.$ (1.6)

1.2.3 Renewal Process

Suppose a certain event occurs repeatedly in time with the property that the interarrival times $\{X_n, n = 1, 2, \ldots\}$ form a sequence of non-negative independent identically distributed random variables with a common distribution F(.) and $\Pr\{X_n = 0\}<1$. Let us call each occurrence of the event a renewal. Since X_n 's are non-negative, $E(X_n)$ exists. Let $S_0 = 0$, $S_n = X_1 + X_2 + \ldots\ldots + X_n$ for $n > 0$. Then S_n denotes the time of the n^{th} renewal. If $F_n(t) = \Pr\{S_n \le t\}$ is the distribution of S_n, then $F_n(t) = F^{*n}(t)$ (n-fold convolution of F(.) with itself). Define $N(t) = \mathrm{Sup}\{ n \mid S_n \leq t\}$. Then N(t) represents the number of renewals in (0, t). The three interrelated processes, $\{X_n, n = 1, 2,....\}$, $\{S_n, n = 0, 1,....\}$ and $\{N(t), t \geq 0\}$ constitute a renewal process. Since one can be derived from the other, customarily one of the processes is called a renewal process.

The function M(t)= E[N(t)] is called the renewal function, and it can easily be seen that $M(t) = \sum_{n=1}^{\infty} F^{*n}(t)$. The derivative of M(t) is called the renewal density, which is the expected number of renewals per unit time. The integral equation satisfied by the renewal function,

$$M(t) = F(t) + \int_0^t M(t-u)\, dF(u) \tag{1.7}$$

is called the renewal equation. Suppose X_1 has a distribution different from the common distribution of $\{X_n, n > 1\}$, then the process is called delayed or modified renewal process.

The following two asymptotic results are used in the sequel.

Theorem 1.3 (Elementary Renewal Theorem)

Let $\mu = E(X_n)$ with the convention, $1/\mu = 0$ when $\mu = \infty$. Then,

$$\lim_{t \to \infty} \frac{M(t)}{t} = \frac{1}{\mu} \tag{1.8}$$

Theorem 1.4 (Key Renewal Theorem)

If $H(t)$ is a non-negative function of t such that $\int_0^\infty H(t)dt < \infty$, then

$$\lim_{t\to\infty} \int_0^t H(t-u)\, dM(u) = \frac{1}{\mu}\int_0^\infty H(t)dt \tag{1.9}$$

1.2.4 Markov Renewal Process

Markov renewal process is a generalization of both Markov process and renewal process. Consider a two dimensional stochastic process $\{(X_n, T_n), n \in N^0\}$ in which transitions from X_n to X_{n+1} constitute a Markov chain with state space E, and the sojourn times $T_{n+1} - T_n$ constitute another stochastic process with state space R_+ which depends only on X_n and X_{n+1}. Then $\{(X_n, T_n), n \in N^0\}$ is called a Markov renewal process on the state space E. We restrict our discussion to the case where E is finite. Formally Markov renewal process can be defined as follows: Let E, a finite set, be the state space of the Markov chain $\{X_n, n \in N^0\}$ and R_+, the set of non-negative real numbers, be the state space of T_n ($T_0 = 0$, $T_n < T_{n+1}$, $n = 0, 1, 2, \ldots$). If

$$\begin{aligned}\Pr\{X_{n+1} = k, T_{n+1} - T_n \le t \mid X_0, X_1, \ldots X_n, T_0, T_1, \ldots T_n\}& \\ = \Pr\{X_{n+1} = k, T_{n+1} - T_n \le t \mid X_n\}&\end{aligned} \tag{1.10}$$

for all n, k $\in$E, and t$\in R_+$, then $\{(X_n, T_n), n \in N^0\}$ is called a Markov renewal process on the state space E.

We assume that the process, $\{(X_n, T_n), n \in N^0\}$ is stationary and denote

$$Q(i,j,t) = \Pr\{X_{n+1} = j, T_{n+1} - T_n \le t \mid X_n = i\} \quad \text{for all } i,j \in \text{E},\ t \in R_+ \tag{1.11}$$

$\{Q(i,j,t);\ i,j \in \mathrm{E},\ t \in \mathrm{R}_+\}$ is called semi-Markov kernel. The functions

$$R(i,j,t) = E[\text{the number of transitions into state j in } (0,t) \mid X_0 = i,\quad i,j \in E]$$

are called Markov renewal functions and are given by

$$R(i,j,t) = \sum_{m=0}^{\infty} Q^{*m}(i,j,t);\ \text{where } Q^{*} \text{ denotes convolution of } Q \text{ with itself.} \quad (1.12)$$

$\{R(i,j,t);\ i,j \in \mathrm{E},\ t \in \mathrm{R}_+\}$ is known as Markov renewal kernel.

The stochastic process $\{X(t),\ t \in R_+\}$ defined by $X(t) = X_n$ for $T_n \le t < T_{n+1}$ is called the semi-Markov process in which the Markov renewal process $\{(X_n, T_n),\ n \in N^0\}$ is embedded. Let $p(i, j, t) = \Pr\{X(t) = j \mid X(0) = i\}$. Then $p(i, j, t)$ satisfies the Markov renewal equations

$$p(i,j,t) = \delta(i,j)\, h(i,t) + \sum_{k \in E} \int_0^t Q(i,k,du)\, p(k,j,t-u); \quad \text{for} \quad i,j \in E \quad (1.13)$$

where,
$$h(i,t) = 1 - \sum_{k \in E} Q(i,k,t)$$

and
$$\delta(i,j) = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1.5

The solution of the Markov renewal equation (1.13) is

$$p(i,j,t) = \int_0^t R(i,j,du)\, h(j,t-u); \quad \text{for } i,j \in E. \quad (1.14)$$

Theorem 1.6

If the Markov renewal process is aperiodic, recurrent and non-null, the limiting probabilities are given by

$$\lim_{t\to\infty} p(i,j,t) = \frac{\pi_j m_j}{\sum\limits_{k \in E} \pi_k m_k} ; j \in E \tag{1.15}$$

and are independent of the initial state, where $\Pi = (\pi_j)$, $j \in E$ is the invariant measure of the Markov chain $\{X_n, n \in N^0\}$ and m_j is the sojourn time in the state j.

1.2.5 Semi-Regenerative Process

Let $Z = \{Z(t), t \geq 0\}$ be a stochastic Process with topological space F, and suppose that the function $t \to Z(t, \omega)$ is right continuous and has left-hand limits for almost all ω. A random variable T taking values in $[0, \infty]$ is called a stopping time for Z provided that for any $t < \infty$, the occurrence or non-occurrence of the event $\{T \leq t\}$ can be determined once the history $\{Z(u), u \leq t\}$ of Z before t is known.

The Process Z is said to be semi-regenerative if there exists a Markov renewal process $\{ (X_n, T_n), n \in N^0 \}$ satisfying the following:

i) For each $n \in N^0$, T_n is a stopping time for Z.

ii) For each $n \in N^0$, X_n is determined by $\{Z(u), u \leq T_n \}$

iii) For each $n \in N^0$, $m \geq 1$, $0 \leq t_1 < t_2 < \ldots\ldots < t_m$, the function f defined on F^m and positive,

$$E[f\{Z(T_n+t_1),\ldots,Z(T_n+t_m)\} \mid \{Z(u),u\le T_n\},\{X_0=i\}]$$
$$=E[f\{Z(t_1),\ldots,Z(t_m)\} \mid \{X_0=j\}] \qquad \text{on } \{X_n=j\}.$$

Theorem 1.7

Let Z be a semi-regenerative process with state space E_1 and let $\{(X_n, T_n), n \in N^0\}$ be the Markov renewal process imbedded in Z. Let the semi-Markov kernel and Markov renewal kernel of $\{(X_n, T_n), n \in N^0\}$ be as defined in (1.11) and (1.12) respectively. Then

$$p(i,j,t)=\Pr\{Z(t)=j \mid Z(0)=X_0=i\}$$
$$=\sum_{k\in E}\int_0^t R(i,k,ds)\ K(k,j,t-s);\quad for\quad i\in E,\quad j\in E_1 \tag{1.16}$$

where

$$K(i,j,t)=\Pr\{Z(t)=j,\ T_1>t \mid Z(0)=X_0=i\}$$

The limiting probabilities are given by the following

Theorem 1.8

In addition to the hypotheses and notations of Theorem 1.7 assume further that $\{(X_n, T_n), n \in N^0\}$ is irreducible, recurrent and aperiodic and the sojourn time m_j in the state j is finite. Then

$$\lim_{t\to\infty} p(i,j,t)=\frac{\sum_{k\in E}\pi_k\int_0^\infty K(k,j,t)dt}{\sum_{k\in E}\pi_k m_k};j\in E_1,\quad i\in E. \tag{1.17}$$

where π_k's are as in (1.15).

1.3 REVIEW OF THE LITERATURE

1.3.1 Earlier Works

The mathematical analysis of inventory problem was started by Harris (1915). He proposed the famous EOQ formula that was popularized by Wilson. The first paper closely related to (s, S) policy is by Arrow, Harris and Marchak (1951). Dvorestzky, Kiefer and Wolfowitz (1952) have given some sufficient conditions to establish that the optimal policy is an (s, S) policy for the single-stage inventory problem. Whitin (1953) and Gani (1957) have summarized several results in storage systems.

A systematic account of the (s, S) inventory type is provided by Arrow, Karlin and Scarf (1958) based on renewal theory. Hadley and Whitin (1963) give several applications of different inventory models. In the review article Veinott (1966) provides a detailed account of the work carried out in inventory theory. Naddor (1966) compares different inventory policies by discussing their cost analysis. Gross and Harris (1971) consider the inventory systems with state dependent lead times. In a later work (1973) they deal with the idea of dependence between replenishment times and the number of outstanding orders. Tijms(1972) gives a detailed analysis of the inventory system under (s, S) policy.

1.3.2 Works on (s, S) Continuous Review Policy

Sivazlian (1974) analyzes the continuous review (s, S) inventory system with general interarrival times and unit demands. He shows that the limiting distribution of the position inventory is uniform and independent of the interarrival time distribution. Richards (1975) proves the same result for compound renewal demands. Later (1978) he deals with a continuous review (s, S) inventory system in which the demand for items in inventory is dependent on an external environment. Archibald and Silver (1978) discuss exact and approximate procedures for continuous review (s, S) inventory policy with constant lead time and compound Poisson demand.

Sahin (1979) discusses continuous review (s, S) inventory with continuous state space and constant lead times. Srinivasan (1979) extends Sivazlian's result to the case of random lead times. He derives explicit expression for probability mass function of the stock level and extracts steady state results from the general formulae. This is further extended by Manoharan, Krishnamoorthy and Madhusoodanan (1987) to the case of non-identically distributed interarrival times.

Ramaswami (1981) obtains algorithms for an (s, S) model where demand is a Markovian point process. Sahin (1983) derives the binomial moments of the transient and stationary distributions of the number of backlogs in a continuous review (s, S) model with arbitrary lead time and compound renewal demand. Kalpakam and Arivarignan (1984) discuss a single item (s, S) inventory model in which demands from a finite number of different types of sources form a Markov chain. Thangaraj and Ramanarayanan (1983) deal with an inventory system with random lead time and having two ordering levels. Jacob (1988) considers the same problem with varying re-order levels. Ramanarayanan and Jacob (1987) obtain time dependent system state probability using matrix convolution method for an inventory system with random lead time and bulk demands. Srinivasan (1988) examines (s, S) inventory systems with adjustable reorder sizes. Chikan (1990) and Sahin (1990) discuss extensively a number of continuous review inventory systems in their books.

An inventory system with varying re-order levels and random lead time is discussed by Krishnamoorthy and Manoharan (1991). Krishnamoorthy and Lakshmy (1991) investigate an (s, S) inventory system in which the successive demand quantities form a Markov chain. They (1990) further discuss problems with Markov dependent re-ordering levels and Markov dependent replenishment quantities. Zheng (1991) develops an algorithm for computing optimal (s, S) policies that applies to both periodic review and continuous review inventory systems. Sinha (1991) presents a computational algorithm by a search routine using numerical methods for an (s, S) inventory system having arbitrary demands and exponential interarrival times.

Ishigaki and Sawaki (1991) show that (s, S) policy is optimal among other policies even in the case of fixed inventory costs. Dohi et al. (1992) compare well-known continuous and deterministic inventory models and propose optimal inventory policies. Azoury and Brill (1992) derive the steady state distribution of net inventory in which demand process is Poisson, ordering decisions are based on net inventory and lead times are random. The analysis of the model applies level crossing theory. Sulem and Tapiero (1993) emphasize the mutual effect of lead time and shortage cost in an (s, S) inventory policy.

Kalpakam and Sapna (1993a) analyze an (s, S) ordering policy in which items are procured on an emergency basis during stock out period. Again they (1993b) deal with the problem of controlling the replenishment rates in a lost sales inventory system with compound Poisson demands and two types of re-orders with varying order quantities. Prasad (1994) develops a new classification system that compares different inventory systems. Zheng (1994) studies a continuous review inventory system with Poisson demand allowing special opportunities for placing orders at a discounted setup cost. He proves that the (s, c, S) policy is optimal and developed an efficient algorithm for computing optimal control parameters of the policy. Hill (1994) analyzes a continuous review lost sales inventory model in which more than one order may be outstanding. In an earlier work (1992) he describes a numerical procedure for computing the steady state characteristics where two orders may be outstanding.

Moon and Gallego (1994) discuss inventory models with unknown distribution of lead time but with the knowledge of only the first two moments of it. Mak and Lai (1995) present an (s, S) inventory model with cut-off point for lumpy demand quantities where the excess demands are refused. Hollier, Mak and Lam (1995, 1996) deal with similar problems in which the excess demands are filtered out and treated as special orders. Dhandra and Prasad (1995a) study a continuous review inventory policy in which the demand rate changes at a random point of time. Perry et al. (1995) analyze continuous review inventory systems with exponential random yields by the techniques of level crossing theory. Sapna (1996) deals with (s, S) inventory system with priority customers and arbitrary lead time distribution. Kalpakam and Sapna (1997) discuss an environment dependent (s, S) inventory system with renewal demands and lost sales where the environment changes between available and unavailable periods according to a Markov chain.

1.3.3 Works on Perishable Inventory

Ghare and Schrader (1963) introduce the concept of exponential decay in inventory problems. Nahmias and Wang (1979) derive a heuristic lot size re-order policy for an inventory problem subject to exponential decay. Weiss (1980) discusses an optimal policy for a continuous review inventory system with fixed life time. Graves (1982) apply the theory of impatient servers to some continuous review perishable inventory models. An exhaustive review of the work done in perishable inventory until 1982 can be seen in Nahmias(1982). Kaspi and Perri (1983, 1984) deal with inventory systems with constant life times applicable to blood banks. Pandit and Rao (1984) study an inventory system in which only good items are sold. These are selected from the stock including defective items with known probabilities until a good item is picked up.

Kalpakam and Arivarignan (1985a, 1985b) study a continuous review inventory system having an exhibiting item subject to random failure. They (1989) extend the result to exhibiting items having Erlangian life times under renewal demands. Again they (1988) deal with a perishable inventory model having exponential life times for all the items. Ravichandran (1988) analyzes a system with Poisson demand and Erlangian life time where lead time is assumed to be positive. Manoharan and Krishnamoorthy (1989) consider an inventory problem with all items subject to decay having arbitrary interarrival times and derive the limiting probabilities.

Srinivasan (1989) investigates an inventory model of decaying items with positive lead time under (s, S) operating policy. Incorporating adjustable re-order size he discusses a solution procedure for inventory model for decaying items. Liu (1990) considers an inventory system with random life times allowing backlogs, but having zero lead time. He gives a closed form of the long run cost function and discusses its analytic properties. Raafat (1991) presents an up-to-date survey of decaying inventory models.

Goh et al.(1993) consider a perishable inventory system with finite life times in which arrival and quantities of demands are batch Poisson process with geometrically sized batches. Kalpakam and Sapna (1994) analyze a perishable inventory system with Poisson demand and exponentially distributed lead times and derive steady state probabilities of the inventory level. Later they (1996) extend it to the case of arbitrary lead time distribution. Su et al. (1996) propose an inventory model under inflation for stock dependent consumption rate and exponential decay with no shortages. Bulinskaya (1996) discusses the stability of inventory problems taking into account deterioration and production.

1.3.4 Works on Multi-commodity Inventory

Balintfy (1964) analyses a continuous review multi-item inventory problem. Silver (1965) derives some characteristics of a special joint ordering inventory model. Ignall (1969) deals with two product continuous review inventory systems with joint setup costs. Some models of multi-item continuous review inventory problems can be seen in Schrady et al. (1971). Sivazlian (1975) discusses the stationary characteristics of a multi-commodity inventory system. Sivazlian and Stanfel (1975) study a single period two commodity inventory problem. Multi-item (s, S) inventory systems with a service objective are discussed in Mitchell (1988). Cohen et al. (1992) study multi-item service constrained (s, S) inventory systems. Golany and Lev-Er (1992) compare several multi-item joint replenish-ment inventory models by simulation study. Kalpakam and Arivarignan (1993) analyze a multi-item inventory model with unit renewal demands under joint ordering policy.

Krishnamoorthy, Iqbal and Lakshmy (1994) discuss a continuous review two commodity inventory problem in which the type of commodity demanded is governed by a discrete probability distribution. Krishnamoorthy and Varghese (1995a) consider a two commodity inventory problem with Markov shift in demand for the type of the commodity. The quantity demanded at each epoch is arbitrary but limited. Dhandra and Prasad (1995b) analyze two commodity inventory problems for substitutable items. Krishnamoorthy and Merlymole (1997) investigate a two commodity inventory problem with correlated demands. Krishnamoorthy, Lakshmy and Iqbal (1997) study a two commodity inventory problem with Markov shift in demand and characterize the limiting distributions of the inventory states.

1.4 AN OUTLINE OF THE PRESENT WORK

The thesis is divided into eight chapters including this introductory chapter. Chapter II deals with a single commodity continuous review (s, S) inventory system in which items are damaged due to decay and disaster. We assume that demands for items follow Poisson process. The lifetime of items and the times between the disasters are independently exponentially distributed. Due to disaster a unit in the inventory is either destroyed completely, independent of others, or survives without any damage. Shortages are not permitted and lead time is assumed to be zero. By identifying a suitable Markov Process transient and steady state probabilities of the inventory levels are derived . The probability distribution of the replenishment periods are found to be phase type and explicit expression for the expectation is obtained. Some special cases are deduced. Optimization problem is discussed and optimum value of the re-ordering level, s, is proved to be zero. Some numerical examples are provided to find out optimum values of S.

Chapter III is an extension of the model discussed in chapter II to positive lead time case. Shortages are allowed and demands during dry periods are considered as lost. We derive transient and steady state probabilities of the inventory levels by assuming arbitrary lead time distribution. A special case in which the stock is brought back to the maximum capacity at each instant of replenishment by an immediate second order is also discussed. The case in which the lead time distribution is exponentially distributed is discussed in detail. Expected replenishment cycle time is shown as minimum when $s = 0$. The cost analysis is illustrated with numerical examples.

In chapter IV we study a single commodity inventory problem with general interarrival times and exponential disaster periods. Here we assume that the damage is due to disaster only. The quantity demanded at each epoch follows an arbitrary distribution depending only on the time elapsed from the previous demand point. Other assumptions are same as in chapter II. Transient and steady state probabilities of the inventory level are derived with the help of the theory of semi-regenerative processes. A special case in which the disaster affecting only the exhibiting items and arriving customers demanding unit item is discussed and steady state distribution is obtained as uniform. Illustrations are provided by replacing the general distribution by gamma distribution.

Chapter V considers a single commodity inventory problem with general disaster periods and Poisson demand process. Here also the damage of item is restricted to disaster. Concentrating on the disaster epochs which form a renewal process, the transient and steady state probabilities of the inventory level are derived. Special cases are discussed and numerical illustrations are provided. In the special case where the disaster affects only an exhibiting item the steady state probabilities of the inventory levels are proved to be uniform.

Chapter VI generalizes the results of chapter II to multi-commodity inventory. There are *n* commodities and an arriving customer can demand only one type of commodity. Demands for an item follow Poisson process and life times of items are independent exponential distributions. Disaster periods are also exponential distribution and the disaster affects each unit in the inventory independently of others. Fresh orders are placed and instantaneously replenished whenever the inventory level of at least one of the commodities falls to or below its re-ordering point. The inventory level process is an n-dimensional continuous time Markov chain. Hence the time dependent and long run system state solutions are arrived at. Cost function for the steady state inventory is formulated and re-ordering levels are found out to be zeroes at optimum value. Numerical examples help to choose optimum values for maximum inventory levels.

The assumptions of chapter VII are similar to those in the previous chapter except those concerning the replenishment policy and shortages. A new order is placed only when the inventory levels of all the commodities fall to or below their re-ordering levels. Hence there are shortages and the sales are considered as lost during stock out period. Results are illustrated with numerical examples.

In the last chapter there are two models of two commodity inventory problems. Each arrival can demand one unit of commodity I, one unit of commodity II or one unit each of both. The type of commodity demanded at successive demand epochs constitutes a Makov chain. Shortages are not allowed and lead time is assumed to be zero. Neither decay nor disaster affects the inventory. The interarrival times of demands are i.i.d. random variables following a general distribution. In the first model fresh orders are placed for each commodity separately whenever its inventory level falls to its re-ordering level for the first time after the previous replenishment. In the second model an order is placed for both commodities whenever the inventory level of at least one of the commodities falls to its re-ordering level for the first time after the previous replenishment. Transient and steady state probabilities of the system states are computed with the help of the theory of semi-Markov processes. Distributions of the replenishment periods and that of replenishment quantities are formulated to discuss optimization problem. Numerical examples are given to illustrate each model and to compare the two.

The notations used in this thesis are explained in each chapter. Numerical examples provided at the end of each chapter are solved with the help of a computer; for brevity, the respective computer programs are not presented. The thesis ends with a list of references.

Chapter II

Single Commodity Inventory Problem Perishable due to Decay and Disaster*

2.1 INTRODUCTION

In this chapter we discuss a continuous review inventory system in which commodities are damaged due to decay and disaster. The maximum capacity of the warehouse is S and the sock is brought to S whenever the inventory level falls to or below the re-ordering point, s . Shortages are not permitted and lead time is zero . Demands are assumed to follow Poisson process with rate λ . The times between disasters and life times of an item have exponential distributions with parameters μ and ω respectively. Each unit in the inventory , independent of others, survives a disaster with probability p and succumbs to it with probability 1-p.

Our objectives are to find transient and steady state probabilities of the inventory level and long run optimum value of the pair, (s, S). Numerical examples provided in the last section illustrate the results.

The review by Nahmias (1982) discusses most of the earlier perishable inventory models. Kalpakam and Arivarignan (1988) deal with a perishable

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* The results of this chapter are published in *Optimization* , 35, 85 - 93, (1995).

inventory model in which the life time of an item is exponentially distributed and the demands form a Poisson process. This chapter is an attempt to generalize their model by adding the possibility of a disaster.

2. 2 NOTATIONS

S	: Maximum inventory level
s	: reordering point
M	: $S - s$
q	: $1 - p$
R_+	: The set of non-negative real numbers
N^0	: The set of non-negative integers
E	: $\{s+1, s+2, \ldots\ldots\ldots, S\}$
E_1	: $\{s+1, s+2, \ldots\ldots, S-1\}$
E_s	: $\{s, s+1, \ldots\ldots\ldots\ldots\ldots, S\}$
E_σ	: $\{\sigma, s+1, s+2, \ldots\ldots\ldots, S\}$
E_M	: $\{1, 2, \ldots\ldots\ldots\ldots\ldots\ldots, M\}$
Π	: $(\pi_{s+1}, \pi_{s+2}, \ldots\ldots\ldots, \pi_S\}$
e	: $(1, 1, \ldots\ldots\ldots\ldots\ \ldots\ldots)^T$; $e^T \in R^M$
α	: $(0, 0, \ldots\ldots\ldots\ldots\ldots, 0, 1) \in R^M$
α_1	: $(0, 0, \ldots\ldots\ldots\ldots\ldots, 0, 1) \in R^{M+1}$
A	: $(a_{i,j})_{M\times M}$; $a_{i,j}$'s are defined by (2.5).
D_i	: the determinant of the sub matrix obtained from A by deleting the first i-s rows, the last and first i-s-1 columns; $i \in E_1$.
D_S	: 1
$\Delta C(0,S)$	: $C(0, S) - C(0, S-1)$.
$\Delta^2 C(0,S)$	: $\Delta C(0, S) - \Delta C(0, S-1)$

2.3 ANALYSIS OF THE INVENTORY LEVEL

Let X(t) denote the inventory level at any time $t \geq 0$. Then $\{X(t),\ t \in R_+\}$ is a continuous time Markov chain with state space E. We assume that the initial probability vector of this chain is α.

Let

$$P_{ij}(t) = \Pr\{X(t) = i \mid X(0) = i\}; \qquad i, j \in E. \tag{2.1}$$

Then the transition probability matrix,

$$\mathbf{P}(t) = (P_{ij}(t))_{M \times M}; \quad i, j \in E \tag{2.2}$$

together with α will uniquely determine the Markov chain $\{X(t)\}$.

Theorem 2.1

The transition probability matrix **P**(t) is uniquely determined by

$$\mathbf{P}(t) = \exp(\mathbf{B}t) = \mathbf{I} + \sum_{n=1}^{\infty} \frac{\mathbf{B}^n t^n}{n!} \tag{2.3}$$

where matrix **B** = **A**+**C**, in which **A** and **C** are defined as follows :

$$\mathbf{A} = \begin{bmatrix} a_{s+1,s+1} & a_{s+1,s+2} & a_{s+1,S} \\ a_{s+2,s+1} & a_{s+2,s+2} & a_{s+2,S} \\ & & \cdots\cdots \\ \cdots\cdots & & \cdots\cdots \\ a_{S,s+1} & a_{S,s+2} & a_{S,S} \end{bmatrix} \tag{2.4}$$

with

$$a_{ij} = \begin{cases} -(\lambda + \mu + i\omega) + p^i \mu & if \ \ i = j \\ \lambda + i\omega + \binom{i}{j} p^j q^{i-j} \mu & if \ \ i = j+1 \\ \binom{i}{j} p^j q^{i-j} \mu & if \ \ i > j+1 \\ 0 & otherwise \end{cases} \tag{2.5}$$

and $\mathbf{C} = (c_{ij})_{M \times M}$; i,j$\in$E, with

$$c_{ij}=\begin{cases} \lambda+i\omega+\sum\limits_{k=i-s}^{i}\binom{i}{k}p^{i-k}q^{k}\mu & \text{if } j=S;\ i=s+1 \\ \sum\limits_{k=i-s}^{i}\binom{i}{k}p^{i-k}q^{k}\mu & \text{if } j=S;\ i>s+1 \\ 0 & \text{otherwise} \end{cases} \qquad (2.6)$$

Proof:

For a fixed i, we have the following:

$$P_{ij}(t+\delta\ t)=P_{ij}(t)\{1-(\lambda+\mu+j\omega)\delta\ t\}+P_{i\,j+1}(t)[\lambda+(j+1)\omega]\delta\ t+ \\ \sum_{k=0}^{S-j}P_{i\,j+k}(t)\binom{j+k}{j}p^{j}q^{k}\mu\,\delta\ t+o(\delta\ t);\qquad j\in E_1 \qquad (2.7)$$

$$P_{iS}(t+\delta\ t)=P_{iS}(t)\{1-(\lambda+\mu+S\omega)\delta\ t+P^{S}\mu\delta\ t\} \\ +P_{iS+1}(t)[\lambda+(S+1)\omega]\delta\ t \qquad (2.8) \\ +\sum_{r=s+1}^{S}\ \sum_{\mathrm{k=r\text{-}s}}^{\mathrm{r}}P_{i\,r}(t)\binom{r}{k}p^{r-k}q^{k}\mu\,\delta\ t+o(\delta\ t)$$

Hence the difference differential equations are

$$P'_{ij}(t)=P_{ij}(t)[-(\lambda+\mu+j\omega)]+P_{i\,j+1}(t)[\lambda+(j+1)\omega]+ \\ \sum_{k=0}^{S-j}P_{i\,j+k}(t)\binom{j+k}{j}p^{j}q^{k}\mu\,;\qquad j\in E_1 \qquad (2.9)$$

$$P'_{iS}(t)=P_{iS}(t)[-(\lambda+\mu+S\omega)+P^{S}\mu]+P_{iS+1}(t)[\lambda+(S+1)\omega] \\ +\sum_{r=s+1}^{S}\ \sum_{\mathrm{k=r\text{-}s}}^{\mathrm{r}}P_{i\,r}(t)\binom{r}{k}p^{r-k}q^{k}\mu \qquad (2.10)$$

From (2.4) - (2.6), (2.9) and (2.10) we can easily see that the Kolmogorov equations,

$$\mathbf{P'}(\mathrm{t})=\mathbf{P}(\mathrm{t})\mathbf{B} \text{ and } \mathbf{P'}(\mathrm{t})=\mathbf{B}\mathbf{P}(\mathrm{t}) \qquad (2.11)$$

with the condition,

$$\mathbf{P}(0)=\mathbf{I} \qquad (2.12)$$

are satisfied by **P**(t). The solution of (2.11) with (2.12) is (2.3). Since **B** is a finite matrix the series in (2.3) is convergent and the solution is unique. Hence the theorem.

2.3.1 Steady State Probabilities

Since in the Markov chain $\{X(t), t \geq 0\}$ transition from any state i(i $\in$ E) to any state j (j $\in$ E) is possible with positive probability, it is irreducible. Hence the limiting probabilities, $\lim_{t\to\infty} P_{ij}(t) = \pi_j;\quad j \in E$ exist and are given by the unique solution of

$$\Pi\mathbf{B} = \mathbf{0} \tag{2.13}$$

and

$$\Pi\mathbf{e} = 1 \tag{2.14}$$

Theorem 2.2

The steady state probabilities π_i (i $\in$ E) are given by

$$\pi_i = \frac{D_i}{F(s,S)\prod_{k=i}^{S}(-a_{kk})};\quad i \in E \tag{2.15}$$

where

$$F(s,S) = \sum_{i=s+1}^{S} \frac{D_i}{\prod_{k=i}^{S}(-a_{kk})} \tag{2.16}$$

Proof:

Because of (2.14), the last column of **B** is not needed for solving the equations. Hence it is enough to take **A** instead of **B**. Construct a series of determinants from **A** as follows: Let D_i be the determinant of the sub matrix obtained from **A** by deleting the first i–s rows the last and first i–s–1 columns,

$i \in E_1$, and $D_S = 1$. Then we can easily see that the solution of (2.13) and (2.14) is

$$\pi_i = \frac{D_i \pi_S}{\prod_{k=i}^{S-1} (-a_{kk})} ; i \in E_1 \qquad (2.17)$$

$$\text{and} \quad \pi_S = \frac{1}{(-a_{ss})F(s,S)} \qquad (2.18)$$

Substitution of (2.18) in (2.17) yields (2.15). Hence the theorem.

Corollary 2.2.1

When there is no disaster and the items are non-perishable, then the stationary probabilities are uniformly distributed.

Proof:

When there is no disaster and the items are non-perishable, $\mu = 0$ and $\omega = 0$. Then $a_{k,k} = -\lambda$ for every k, $D_i = \lambda^{S-i}$, $i \in E$ and $F(s,S) = M/\lambda$. Therefore from (2.15), $\pi_i = 1/M$, hence uniform distribution. This agrees with the result of Sivazlian (1974).

Corollary 2.2.2

If there is only decay and no disaster, then

$$\pi_i = \frac{1}{(\lambda + i\omega) \sum_{j=s+1}^{S} \frac{1}{(\lambda + j\omega)}} ; i \in E \qquad (2.19)$$

Proof:

In case of perishable items with no disaster, $\mu = 0$. Then $a_{k,k} = -(\lambda + k\omega)$ for every k and $D_i = \Pi_{j=i}^{S-1} (-a_{j+1,j+1})$; $i \in E_1$. Therefore from (2.16),

$$F(s,S)=\sum_{i=s+1}^{S}\frac{1}{(\lambda+i\omega)} \tag{2.20}$$

and from (2.15) the corollary follows. (See Kalpakam and Arivarignan (1988)).

Corollary 2.2.3

If the goods are non-perishable and only an exhibited item affects the disaster, then the stationary probabilities are uniform.

Proof:

In this case,

$$a_{ij}=\begin{cases}-(\lambda+q\mu) & if\ i=j\\ \lambda+q\mu & if\ i=j+1\\ 0 & otherwise\end{cases} \tag{2.21}$$

Then $a_{kk} = -(\lambda + \mu q)$ for every k, $D_i = (\lambda + \mu q)^{S-i}$, $i \in E$. Hence from (2.15) and (2.16),

$$\pi_i = 1/M. \tag{2.22}$$

2.4 PROBABILITY DISTRIBUTION OF THE REPLENISHMENT CYCLES

Let $0 = T_0 < T_1 < T_2 < \ldots\ldots\ldots\ldots$ be the epochs when orders are placed. The inventory level at T_n is S, $n \in N^0$. Therefore $\{T_n, n \in N^0\}$ is a renewal process.

Theorem 2. 3

The probability distribution of the replenishment cycles is phase type on $[0, \infty)$ and is given by

$$G(t) = 1-\alpha \exp(At)e \quad \text{for } t \geq 0 \tag{2.23}$$

Proof:

Since lead time is zero, the inventory is replenished whenever stock level is reduced to s or below it for the first time after each replenishment. Let σ denote the instantaneous state representing the states s, s−1,.....1, 0. Assume that the stock level is σ for an infinitesimally small interval before making it S. Define the Markov chain $\{Y(t),\ t \in R_+\}$ with state space E_σ and initial probability vector $\boldsymbol{\alpha}_1$ and transition probability matrix

$$\overline{\mathbf{B}} = \begin{bmatrix} 0 & 0 \\ \overline{\mathbf{C}} & \mathbf{A} \end{bmatrix} \quad where \quad \overline{\mathbf{C}} = \mathbf{Ce}. \tag{2.24}$$

Since matrix **A** is non-singular, state σ is absorbing and all other states are transient (see Neuts (1978)) for the Markov chain $\{Y(t), t \in R_+\}$. If G(.) is the probability distribution of the time until absorption into the instantaneous state σ with initial probability vector $\boldsymbol{\alpha}_1$, then G(.) is the distribution of the phase type on $[0, \infty)$ and is given by (2.23). When the time spent in σ tends to zero G(.) becomes the probability distribution of the replenishment cycles of the Markov chain $\{X(t), t \in R_+\}$. Hence the theorem.

Theorem 2.4

The expected time between two successive re-orders,

$$E(T) = F(s,S) = \frac{1}{-a_{SS}\pi_S} \tag{2.25}$$

Proof:

The characteristic values of the lower triangular matrix **A** are a_{ii} 's, hence distinct. Therefore **A** can be represented as

$$\mathbf{Q}\begin{bmatrix} a_{s+1,s+1} & & & \\ & a_{s+2,s+2} & & 0 \\ & & \ddots & \\ 0 & & & a_{S,S} \end{bmatrix}\mathbf{Q}^{-1} \tag{2.26}$$

where $\mathbf{Q} = (R_{s+1}, R_{s+2}, \ldots R_S)$ and R_i is the right eigen vector corresponding to the eigen value a_{ii} ($i \in E$).

Thus,

$$\exp(\mathbf{A}t) = \mathbf{Q}\begin{bmatrix} \exp(a_{s+1,s+1}\, t) & & 0 \\ & \exp(a_{s+2,s+2}\, t) & \\ & & \ddots \\ 0 & & \exp(a_{S,S}\, t) \end{bmatrix}\mathbf{Q}^{-1} \tag{2.27}$$

and

$$\int_0^{\infty} \exp(\mathbf{A}t)dt = -\,\mathbf{Q}\begin{bmatrix} \frac{1}{a_{s+1,s+1}} & & 0 \\ & \frac{1}{a_{s+2,s+2}} & \\ & & \ddots \\ 0 & & \frac{1}{a_{S,S}} \end{bmatrix}\mathbf{Q}^{-1} \tag{2.28}$$

$$= -\,\mathbf{A}^{-1}$$

Therefore from (2.23)

$$E(T) = \int_0^{\infty} \alpha \exp(\mathbf{A}t)\mathbf{e}\, dt \tag{2.29}$$

$$= -\,\alpha\, \mathbf{A}^{-1}\,\mathbf{e} \tag{2.30}$$

Let $\mathbf{A}^{-1} = (a'_{ij});\quad i, j \in E$. Then

$$a'_{S\,i} = \frac{D_i}{\prod_{k=i}^{S}(-a_{kk})}; i \in E \tag{2.31}$$

Hence (2.30) becomes

$$E(T) = \sum_{i=s+1}^{S} a'_{Si} = \sum_{i=s+1}^{S} \frac{D_i}{\prod_{k=i}^{S} (-a_{kk})} \tag{2.32}$$

and from (2.18) and (2.16) the theorem follows.

Corollary 2.4.1

When there is no disaster and the items are non-perishable, then

$$E\,(T) = M/\lambda.$$

Corollary 2.4.2

In case of perishable inventory, with no disaster, we have from (2.20),

$$E(T) = \sum_{i=s+1}^{S} \frac{1}{(\lambda + i\omega)} \tag{2.33}$$

and the result reduces to Kalpakam and Arivarignan (1988).

Corollary 2.4.3

When the disaster affects only an exhibiting item,

$$E(T) = \frac{M}{\lambda + q\mu} \tag{2.34}$$

2.5 OPTIMIZATION PROBLEM

Due to disaster, the stock level may go below s, at any instant. Hence the re-ordering quantity is not always M = S – s. If M* represents the expected re-ordering quantity at steady state, then

$$M^* = E(T)\left[\lambda + \sum_{i=s+1}^{S} \pi_i \left(i\omega + \mu \sum_{j=0}^{i} j \binom{i}{j} p^{i-j} q^j \right)\right] \tag{2.35}$$

$$= E(T)\left[\lambda + (\omega + q\mu)\, H(s,S)\right], \quad where\ H(s,S) = \sum_{i=s+1}^{S} i\pi_i \tag{2.36}$$

Let h be the unit holding cost per unit time, c the unit procurement cost of the item, K the fixed ordering cost and d the unit cost for the damaged item. Then the cost function to be minimized is

$$C(s,S)=\frac{K+cM^{*}}{E(T)}+h\,H(s,S)+d(\omega+\mu\, q)H(s,S) \tag{2.37}$$

$$=\frac{K}{F(s,S)}c\lambda+\left[(c+d)\,(\omega+\mu\, q)+h\,\right]H(s,S) \tag{2.38}$$

Theorem 2.5

The cost function C (s,S) is minimum for s = 0.

Proof:

Consider the matrix $\tilde{\mathbf{A}}=(\tilde{a}_{ij})$, i, j ∈ E_s, where

$$\tilde{a}_{ij}=\begin{cases}-(\lambda+\mu+i\omega)+p^{i}\mu & \text{if } i=j\\ \lambda+i\omega+\binom{i}{j}p^{j}q^{i-j}\mu & \text{if } i=j+1\\ \binom{i}{j}p^{j}q^{i-j}\mu & \text{if } i>j+1\\ 0 & \text{otherwise}\end{cases} \tag{2.39}$$

Let $\tilde{D}_i$ = be the determinant of the sub matrix obtained from $\tilde{\mathbf{A}}$ by deleting the first i–s+1 rows, the last and first i–s columns (i ≠ S), and $\tilde{D}_S=1$. Then $\tilde{D}_i$ = D_i for i ∈ E . Also observe that $\tilde{D}_i$ is positive for every i since it has negative entries on the super diagonal, non-negative entries in the lower triangular portion and zeros elsewhere. Then

$$F(s-1,S)=\sum_{i=s}^{S}\frac{\tilde{D}_i}{\prod_{k=i}^{S}\left(\lambda+k\omega+\mu-p^{k}\mu\right)}$$

$$=\sum_{i=s+1}^{S}\frac{D_i}{\prod_{k=i}^{S}\left(\lambda+\mu+k\omega-p^{k}\mu\right)}+\frac{\tilde{D}_s}{\prod_{k=s}^{S}\left(\lambda+\mu+k\omega-p^{k}\mu\right)}$$

$$>F\,(s,\,S) \tag{2.40}$$

Also

$$H(s-1,S) = s + \frac{1}{F(s-1,S)} \sum_{i=0}^{S-s} \frac{i\widetilde{D}_{i+s}}{\prod\limits_{k=i+s}^{S} \left(\lambda + \mu + k\omega - p^k \mu\right)}$$

$$< s + \frac{1}{F(s,S)} \sum_{i=1}^{S-s} \frac{iD_{i+s}}{\prod\limits_{k=i+s}^{S} \left(\lambda + \mu + k\omega - p^k \mu\right)} \qquad by \quad (2.40)$$

$$= H(s,\ S) \tag{2.41}$$

Thus from (2.38) , (2.40) and (2.41),

C(s–1, S) < C(s, S).

Hence the proof.

Let Φ(S) =F(0,S) and Ψ (S) =H(0,S). Then (2.38) becomes

$$C(0,S) = \frac{K}{\Phi(S)} + c\lambda + \left[(c+d)(\omega + \mu\, q) + h\right]\Psi(S)$$

$$= \frac{K + cS}{\Phi(S)} + \left[d(\omega + \mu\, q) + h\right]\Psi(S) \tag{2.42}$$

2.6 NUMERICAL ILLUSTRATIONS

In general C(0,S) is not a convex function as evidenced by table 2.1. However, numerical examples indicate that when S is large ΔC(0,S) tends to a constant . This can be seen in figure 2.1. In practice the maximum capacity of the warehouse is also delimited by other constraints and hence given an upper limit we can easily find out the optimum value of S for a minimum value of C(0,S) . Tables 2.2, 2.3 and 2.4 show variation of the optimal values of S for different values of p, λ,ω, and μ. The effect of decay and disaster on the cost function is illustrated in figure 2.2.

Table 2.1

(Showing that the function C(0, S) in not convex)

$\lambda = 2$, $\omega = 1$, $\mu = 10$, $p = 0.1$, $K = 50$, $c = 10$, $h = 2$, $d = 0.4$.

S	C(0,S)	ΔC(0,S)	Δ^2C(0,S)
3	693.566		
		42.351	
4	735.917		2.663
		45.014	
5	780.931		-0.149
		44.865	
6	825.796		-0.792
		44.073	
7	869.869		

Figure 2.1

(The graph of ΔC(0,S))

$\lambda = 2$, $\omega = 1$, $\mu = 10$, $p = 0.1$, $K = 50$, $c = 10$, $h = 2$, $d = 0.4$.

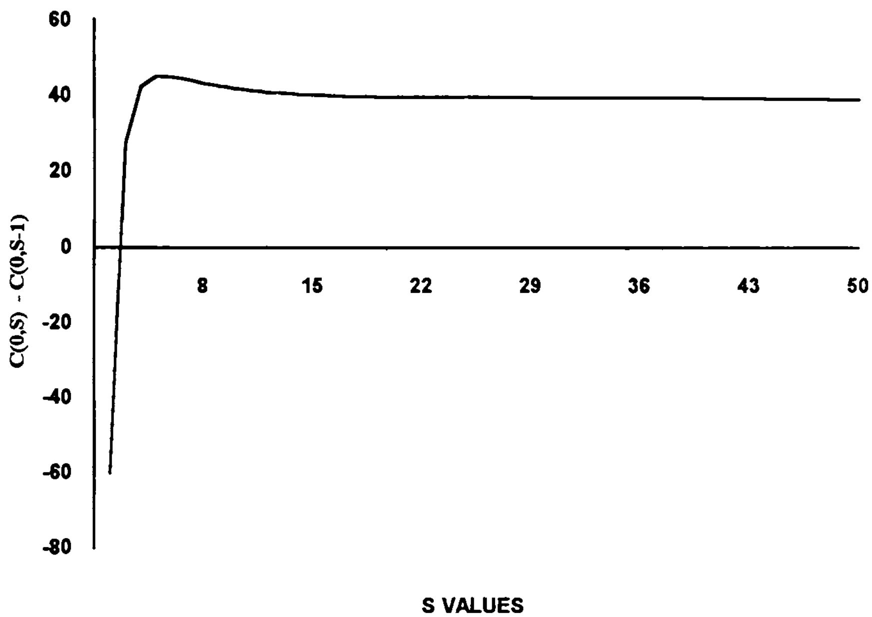


Figure 2.2

(The effect of decay and disaster on the cost function)

$\lambda = 4$, $p = 0.5$, $K = 200$, $c = 10$, $h = 2$, $d = 0.4$.

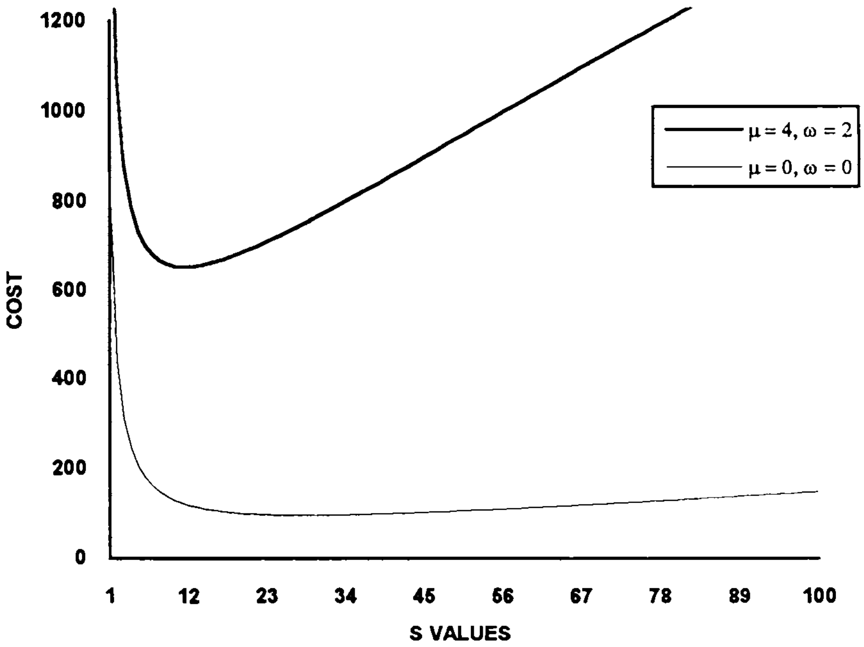


Table 2.2

p= .1, K=200, c=10, h=2, d=.4

ω	$\lambda\downarrow$ \ $\mu\rightarrow$	4	3.5	3	2.5	2	1.5	1	.5	0
	1	6	6	6	7	7	7	8	9	11
2	2	6	7	7	7	8	8	9	10	12
	3	7	7	8	8	8	9	10	11	13
	4	7	8	8	9	9	10	11	12	14
	1	6	6	6	6	6	7	8	9	11
1	2	6	6	6	7	7	8	9	10	14
	3	6	7	7	7	8	9	10	12	15
	4	7	7	8	8	9	10	11	13	17
	1	6	6	6	6	6	6	6	7	14
0	2	6	6	6	6	6	7	7	9	20
	3	6	6	6	7	7	8	9	12	25
	4	6	7	7	7	8	9	10	14	28

Table 2.3

p= .5, K=200, c=10, h=2, d=.4

ω	μ→ / λ↓	4	3.5	3	2.5	2	1.5	1	.5	0
2	1	9	9	10	10	10	10	10	10	11
	2	10	10	10	11	11	11	11	12	12
	3	11	11	11	11	12	12	12	13	13
	4	11	12	12	12	12	13	13	14	14
1	1	9	9	10	10	10	10	10	11	11
	2	10	10	11	11	11	11	12	13	14
	3	11	11	12	12	12	13	13	14	15
	4	12	12	12	13	13	14	14	15	17
0	1	9	9	9	10	10	10	11	11	14
	2	10	11	11	11	12	12	13	15	20
	3	11	12	12	13	13	14	15	17	25
	4	12	13	13	14	14	15	17	20	28

Table 2.4

p= .9, K=200, c=10, h=2, d=.4

ω	μ→ / λ↓	4	3.5	3	2.5	2	1.5	1	.5	0
2	1	10	10	11	11	11	11	11	11	11
	2	12	12	12	12	12	12	12	12	12
	3	13	13	13	13	13	13	13	13	13
	4	14	14	14	14	14	14	14	14	14
1	1	11	11	11	11	11	11	11	11	11
	2	13	13	13	13	13	13	13	13	14
	3	14	14	14	14	15	15	15	15	15
	4	15	16	16	16	16	16	16	17	17
0	1	12	12	13	13	13	13	14	14	14
	2	15	16	16	17	17	18	18	19	20
	3	18	18	19	19	20	21	22	23	25
	4	20	21	21	22	23	24	25	26	28

Chapter III

Single Commodity Perishable Inventory Problem with Lead Time

3.1 INTRODUCTION

A continuous review inventory system with arbitrary lead time distribution in which the commodities are damaged due to decay and disaster is discussed in this chapter. The re-ordering level is s and the maximum capacity of the ware house is S. Assume that $S > 2s$. The demands during stock out period are assumed to be lost. Demands follow Poisson process with rate λ and the life times of an item follow exponential distribution with parameter ω. The interarrival times of disasters is also exponential distribution, but with parameter μ. The lead times are i.i.d. random variables with absolutely continuous distribution function $G(.)$ having finite mean m. A unit in the inventory, independent of others, survives or not with respective probabilities p and $1-p$. All the distributions mentioned are independent of each other.

Kalpakam and Sapna (1994) deal with a continuous review (s, S) perishable inventory system with exponential lead times. Later they (1996) have extended it to the case of arbitrary lead time distribution. We further extend the problem to disaster case.

Section 3.6 deals with a special case of the problem where the inventory level is brought back to S at each replenishment epoch. We come accross this situation in the common market where the suppliers bring some items additional to the prior order so as to get instantaneous order from the stockists to fill the inventory. In section 3.7 exponentially distributed lead time case is discussed in detail and useful results are obtained which are illustrated with numerical examples.

Notations

M	:S – s
E	:{0, 1,......., S}
E_M	:{M, M+1,....., S}
N^0	:{0, 1, 2,........}
$\boldsymbol{e}$	: $(1, 1, \ldots\ldots)^T$; $\boldsymbol{e}^T \in R^{S+1}$
α	: $(0, 0, \ldots\ldots 0, 1) \in R^{S+1}$
$Q^{*n}(i, j, t)$	: n-fold convolution of Q with itself where

$$Q^{*0}(i,j,t) = \begin{cases} 1 \text{ if } & i = j \\ 0 \text{ if } & i \neq j \end{cases}$$

3.2 MODEL FORMULATION AND ANALYSIS

Let X(t) be the inventory level at time $t \geq 0$. Then X(t) takes values from E. Let $0 = T_0 < T_1 < T_2 < \ldots\ldots$ be the epochs at which the replenishments take place. Concentrating on the pure death process X(t), in between two replenishment epochs, and disregarding the order placement, let

$$\phi_{ij}(t) = \Pr\{X(\rho+t) = j \mid X(\rho+) = i,\ T_n < \rho < \rho+t < T_{n+1}\text{, no order placed even if } i \text{ and } j \leq s\};\quad \rho > 0;\ i, j \in E; \quad (3.1)$$

and $$\Phi(t) = (\phi_{ij}(t)); \quad i, j \in E \tag{3.2}$$

be the square matrix of order S+1. Assume that

$$\Phi(0) = \mathbf{I}, \tag{3.3}$$

then the difference differential equations satisfied by the components $\phi_{ij}(t)$ are

$$\phi_{ij}'(t) = \begin{cases} -(\lambda+\mu+j\omega)\phi_{ij}(t) + [\lambda+(j+1)\omega]\phi_{i\,j+1}(t) + \mu\sum_{k=0}^{i-j}\binom{j+k}{j}p^j q^k \phi_{i\,j+k} & 0 < j \le i-1 \\ [-(\lambda+\mu+j\omega)+\mu p^j]\phi_{ij}(t) & i = j \ne 0 \\ (\lambda+\omega)\phi_{i1}(t) + \mu\sum_{k=0}^{i} q^k \phi_{i\,k}(t) & i > 0;\ j = 0 \\ 0 & \text{otherwise} \end{cases} \quad i, j \in E \tag{3.4}$$

Let $\mathbf{A} = (a_{ij})_{(S+1)\times(S+1)}$; i, j $\in$ E, be the infinitesimal generator of the pure death process. Then,

$$a_{ij} = \begin{cases} -(\lambda+\mu+i\omega) + p^i\mu & i = j \ne 0 \\ \lambda + i\omega + \binom{i}{j} p^j q^{i-j}\mu & i = j+1 \\ \binom{i}{j} p^j q^{i-j}\mu & i > j+1 \\ 0 & \text{otherwise} \end{cases} \tag{3.5}$$

We have

Theorem 3.1

The matrix $\Phi(t)$ is given by

$$\Phi(t) = \mathbf{H}\exp(\mathbf{B}t)\,\mathbf{H}^{-1} \tag{3.6}$$

where **H** is a non-singular matrix formed with the right eigen vectors of **A** and

$$\exp(\mathbf{B}t) = \begin{bmatrix} 1 & 0 & & 0 \\ 0 & e^{a_{11}t} & & 0 \\ \cdots & & & \\ \cdots & \cdots & & \cdots \\ 0 & 0 & & e^{a_{SS}t} \end{bmatrix} \tag{3.7}$$

Proof:

From (3.4) and (3.5) we can see that Φ(t) satisfies the Kolmogorov differential equations,

$$\Phi'(t) = \Phi(t)\mathbf{A} \qquad \text{and} \qquad \Phi'(t) = \mathbf{A}\Phi(t) \tag{3.8}$$

The solution of (3.8) with (3.3) is

$$\Phi(t) = \exp(\mathbf{A}t) = \mathbf{I} + \sum_{n=1}^{\infty} \frac{\mathbf{A}^n\, t^n}{n!} \tag{3.9}$$

The eigen values of the lower triangular matrix **A** are a_{ii} ($i \in E$), hence distinct. Let

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & & 0 \\ 0 & a_{11} & & 0 \\ \cdots & & & \cdots \\ \cdots & \cdots & & \cdots \\ 0 & 0 & & a_{SS} \end{bmatrix} \tag{3.10}$$

Then

$$\mathbf{A}^n = \mathbf{H}\,\mathbf{B}^n\,\mathbf{H}^{-1} \tag{3.11}$$

Substituting in (3.9), we get (3.6). Hence the theorem.

Now define $X_n = X(T_n+)$; $n \in N^0$. Then we have

Theorem 3.2

$\{(X_n, T_n); n \in N^0\}$ is a Markov renewal process with state space E_M and semi Markov kernel, $\mathbf{Q} = \{Q(i, j, t);\ i, j \in E_M,\ t \geq 0\}$ where

$$Q(i,j,t) = \Pr\{X_{n+1} = j,\ T_{n+1} - T_n \le t\ \mid X_n = i\} \\ = \Theta_1(i,j,t) + \Theta_2(i,j,t) \tag{3.12}$$

where

$$\Theta_1(i,j,t) = [\lambda + (s+1)\omega] \int_0^t \int_u^t \phi_{i\ s+1}(u)\ \ \phi_{s\ j-M}(v)\ dG(v)\,du \qquad \text{and}$$

$$\Theta_2(i,j,t) = \mu \sum_{k=s+1}^{i} \ \sum_{r=k-s}^{k-(S-j)} \binom{k}{r} p^{k-r} q^r \int_0^t \int_u^t \phi_{ik}(u)\ \ \phi_{k-r\ j-S+k-r}(v)\ dG(v)\,du \tag{3.13}$$

with ϕ_{ij} defined in (3.1).

Proof:

From the assumptions it is clear that $\{X_n, T_n\}$ is a Markov renewal process. To derive the expression for Q(i, j, t) note that the transition from i to j ($i, j \in E_M$) occurs in the following two mutually exhaustive and exclusive ways: 1) The inventory level reduces to (s+1) and by a demand or by natural decay it becomes s in between ($u, u+\delta u$) causing placement of an order which materializes at time v ($v < t$). 2) The inventory level reduces to k ($k = s+1, \ldots, i$) and by a disaster it again falls to or below s resulting in placement of an order in between ($u, u+\delta u$) which materializes at time v ($v < t$). The Probability for the first event is $\Theta_1(i,j,t)$ and for the second event is $\Theta_2(i,j,t)$.

3.3 TIME DEPENDENT PROBABILITIES

Let $p(i, j, t) = \Pr\{X(t) = j \mid X(0+) = i\}$, $i \in E_M$, $j \in E$. Once the inventory level at $T_n = \sup_i\{T_i < t\}$ is known, the history of X(t) prior to T_n loses its predictive value. Hence $\{T_n ; n \in N^0\}$ are stopping times and $\{X(t); t \ge 0\}$ is a semi-regenerative process with embedded Markov renewal process, (X_n, T_n).

The funçtions p(i, j, t) satisfy the following Markov renewal equations,

$$p(i,j,t) = k(i,j,t) + \sum_{r=M}^{S} \int_0^t Q(i,r,du) p(r,j,t-u); \quad i \in E_M; j \in E \qquad (3.14)$$

where

$$k(i,j,t) = \Pr\{X(t) = j, \quad T_1 > t \mid X(0+) = i\}; \quad i \in E_M; \quad j \in E \qquad (3.15)$$

$$= \begin{cases} \phi_{ij}(t) & s+1 \le j \le i \\ \beta_1(i,j,t) + \beta_2(i,j,t) & 0 \le j \le s \\ 0 & i < j \le S \end{cases} \qquad (3.16)$$

in which

$$\beta_1(i,j,t) = [\lambda + (s+1)\omega] \int_0^t \phi_{i\, s+1}(u) \ \phi_{s\, j}(t-u) \ [1 - G(t-u)]\, du \qquad \text{and}$$

$$\beta_2(i,j,t) = \mu \sum_{k=s+1}^{i} \sum_{r=k-s}^{k-j} \binom{k}{r} p^{k-r} q^r \int_0^t \phi_{ik}(u) \ \phi_{k-r\, j}(t-u) \ [1 - G(t-u)]\, du \qquad (3.17)$$

The solution of (3.14) can be formulated as the following

Theorem 3.3

$$p(i,j,t) = \sum_{r=M}^{S} \int_o^t R(i,r,du)\, k(r,j,t-u); \quad i \in E_M, \quad j \in E; \qquad (3.18)$$

where $$R(i,j,t) = \sum_{n=0}^{\infty} Q^{*n}(i,j,t); \quad i,j \in E_M$$

3.4 STEADY STATE SOLUTION

Let $\mathbf{Q}_1 = (q_{ij})$, i, j $\in$ E_M , be the transition probability matrix of the underlying Markov chain $\{X_n , n \in N^0\}$ associated with the Markov renewal process (X_n , T_n). Then

$$q_{ij} = \lim_{t \to \infty} Q(i,j,t)$$

$$= [\lambda + (s+1)\omega] \int_0^\infty \int_u^\infty \phi_{i\,s+1}(u)\ \phi_{s\,j-M}(v)\ dG(v)\,du \tag{3.19}$$

$$+ \mu \sum_{k=s+1}^{i} \sum_{r=k-s}^{k-(S-j)} \binom{k}{r} p^{k-r} q^{r} \int_0^\infty \int_u^\infty \phi_{ik}(u)\ \phi_{k-r\,j-S+k-r}(v)\ dG(v)\,du$$

Since $q_{ij} > 0$ for every i, j $\in E_M$, the finite Markov chain $\{X_n, n \in N^0\}$ is irreducible and hence it is recurrent. Therefore it possesses a unique stationary distribution,

$$\bar{\pi} = (\pi'_M, \pi'_{M+1}, \ldots\ldots \pi'_S) \text{ which satisfies } \bar{\pi}\mathbf{Q}_1 = \bar{\pi} \text{ and } \Sigma\pi'_j = \tag{3.20}$$

Let $\bar{v} = (v_0, v_1, \ldots\ldots\ldots, v_S)$ denote the steady state probability vector of the inventory level. Since G(t) is absolutely continuous with finite expectation, we get from (3.14) and (3.18) the following

Theorem 3.4

The limiting probabilities of the inventory levels are given by

$$v_j = \frac{\sum_{i \in E_M} \pi'_i \int_o^\infty k(i,j,t)dt}{\sum_{i \in E_M} \pi'_i m_i}; \qquad j \in E \tag{3.21}$$

where

$$m_i = E[T_{n+1} - T_n | X_n = i] = E[T_1 | X_0 = i] = \sum_{j \in E} \int_0^\infty k(i,j,t)dt \tag{3.22}$$

3.5 COST FUNCTION

Let the various costs associated with the inventory be: K, fixed ordering cost per order; c_1, unit procurement cost of an item; h, unit holding cost per unit time; d, cost for a damaged unit; c_3 , unit shortage cost. Let the different expected rates at steady state be: R_1 rate of depletion of inventory due to decay and disaster; R_2 ,rate of shortage; R_3 , rate of re-order. Then

$$R_1 = (\omega + \mu\, q)\sum_{j=1}^{S} j v_j ; \qquad R_2 = \lambda v_0 \tag{3.23}$$

$$R_3 = [\lambda + (s+1)\omega]\, v_{s+1} + \mu \sum_{j=s+1}^{S} v_j \sum_{k=j-s}^{j} \binom{j}{k} p^{j-k} q^k \tag{3.24}$$

Let M* be the expected re-ordering quantity at steady state, then

$$M^* = \frac{1}{R_3}[\lambda \sum_{j=s+1}^{S} v_j + (\omega + \mu\, q) \sum_{j=s+1}^{S} j v_j] \tag{3.25}$$

Therefore, the steady state expected total cost,

$$C(s,S) = (K + c_1 M^*)R_3 + h\sum_{j=1}^{S} j\nu_j + dR_1 + c_3 R_2$$

$$= KR_3 + c_1[\lambda \sum_{j=s+1}^{S} \nu_j + (\omega + \mu\, q) \sum_{j=s+1}^{S} j\nu_j] + [d(\omega + \mu\, q) + h]\sum_{j=1}^{S} j\nu_j + c_3\lambda\nu_0 \tag{3.26}$$

3.6 SPECIAL CASE

Suppose the inventory level at T_0 is S and at each replenishment epoch it is brought back to S by a fresh order, if necessary, which is met instantaneously. Then $\{X(t), t \geq 0\}$, is a regenerative process and $0 = T_0 < T_1 < T_2 < \ldots\ldots$ are regenerative epochs.

Theorem 3.5

The transient probabilities are given by

$$p(S,j,t) = \sum_{r=M}^{S} \int_0^t R(S,r,du)\; k(S,j,t-u) \qquad j \in E \tag{3.27}$$

where $p(S, j, t)$, $k(S, j, t)$, and $R(S, r, t)$ are as defined in section 3.3

Proof:

Conditioning on the first replenishment epoch T_1 we have,

$$p(S,j,t) = k(S,j,t) + \int_0^t p(S,j,t-u)\, dF(u) \qquad j \in E$$

$$\text{where} \qquad F(t) = \sum_{r=M}^{S} Q(S,r,t) \tag{3.28}$$

The solution of the above renewal equations is

$$P(S,j,t) = k(S,j,t) + \int_0^t k(S,j,t-u)\, dM_1(u)\,; \qquad j \in E,$$

$$\text{where} \qquad M_1(t) = \sum_{n=1}^{\infty} F^{*n}(t) = \sum_{r=M}^{S} \sum_{n=1}^{\infty} Q^{*n}(S,r,t) \tag{3.29}$$

Therefore we get

$$p(S,j,t) = \sum_{r=M}^{S} \int_0^t R(S,r,du)\; k(S,j,t-u)\,; \qquad j \in E$$

In the limiting case, since $k(S, j, t)$ is non-negative, non-increasing and tends to zero as t tends to infinity, such that $\int_0^\infty k(S,j,t)dt < \infty$, the application of Key Renewal Theorem yields.

Theorem 3.6

$$\lim_{t \to \infty} P(S,j,t) = \eta_j = \frac{\int_0^\infty k(S,j,t)dt}{\int_0^\infty [1-F(t)]dt} = \frac{\int_0^\infty k(S,j,t)dt}{\sum_{j \in E} \int_0^\infty k(S,j,t)dt}\,; \qquad j \in E. \tag{3.30}$$

3.6.1 Cost Analysis

As in section 3.5 the expected quantity ordered at time of usual order is

$$M_1^* = \frac{1}{R_3}[\lambda \sum_{j=s+1}^{S} \eta_j + (\omega + \mu q) \sum_{j=s+1}^{S} j\eta_j] \tag{3.31}$$

where $$R_3 = [\lambda + (s+1)\omega]\,\eta_{s+1} + \mu \sum_{j=s+1}^{S} \eta_j \sum_{k=j-s}^{j} \binom{j}{k} p^{j-k} q^k$$

The expected long run quantity to be ordered at the time of replenishment is

$$M_2^* = \frac{m}{\sum_{j=0}^{s} \eta_j}[\lambda(-\eta_0 + \sum_{j=0}^{s} \eta_j) + (\omega + \mu q) \sum_{j=0}^{s} j\eta_j] \tag{3.31}$$

where m is the mean of the lead time distribution. If c_2 is the unit procurement cost of this order, and all other costs same as in Section 3.5, then the total cost function in this case is

$$C_1(s,S) = [K + c_1 M_1^* + c_2 M_2^*]R_3 + [d(\omega + \mu q) + h]\sum_{j=1}^{S} j\eta_j + c_3 \lambda \eta_0 \tag{3.32}$$

3.7 EXPONENTIAL LEAD TIMES

When the lead time distribution is exponential with parameter γ, the inventory level process $\{X(t), t \geq 0\}$ is a continuous time Markov chain with state space E. As in Section 3.6 assume that the inventory level is brought to S at the time of replenishment by a new order, if necessary. The analysis is done as in chapter 2.

Let $$P_{ij}(t) = \Pr\{X(t) = j \mid X(0) = i\} \quad i, j \in E$$

and $$\mathbf{P}(t) = [P_{ij}(t)]_{(S+1)\times(S+1)} \quad i, j \in E \tag{3.32}$$

If we assume that the initial probability vector is α then (3.32) uniquely determine the Markov chain {X(t)}.

Theorem 3.7

The matrix **P**(t) is uniquely given by

$$\mathbf{P}(t) = \exp(\mathbf{A}_1 t) = \mathbf{I} + \sum_{m=1}^{\infty} \frac{\mathbf{A}_1{}^m t^m}{m!} \tag{3.33}$$

where $\mathbf{A}_1 = (\bar{a}_{ij})_{(S+1)\times(S+1)}$ and

$$\bar{a}_{ij} = \begin{cases} -\gamma & \text{if } i=j=0 \\ -(\lambda+\mu+i\omega+\gamma)+p^i\mu & \text{if } 1\le i=j\le s \\ -(\lambda+\mu+i\omega)+p^i\mu & \text{if } s<i=j\le S \\ \lambda+i\omega+\binom{i}{j}p^j q^{i-j}\mu & \text{if } i=j+1 \\ \binom{i}{j}p^j q^{i-j}\mu & \text{if } i>j+1 \\ \gamma & 0\le i\le s;\quad j=S \\ 0 & \text{otherwise} \end{cases} \tag{3.34}$$

Proof:

For a fixed i∈E, we have the following difference differential equations.

$$P'_{ij}(t) = -(\lambda+\mu+j\omega+\gamma)P_{ij}(t)+[\lambda+(j+1)\omega]P_{i\,j+1}(t) + \sum_{k=0}^{S-j}\binom{j+k}{j}p^j q^k \mu P_{i\,j+k}(t) \qquad 0<j\le s \tag{3.35}$$

$$P'_{ij}(t) = -(\lambda+\mu+j\omega)P_{ij}(t)+[\lambda+(j+1)\omega]P_{i\,j+1}(t) + \sum_{k=0}^{S-j}\binom{j+k}{j}p^j q^k \mu P_{i\,j+k}(t) \qquad s<j\le S-1 \tag{3.36}$$

$$P'_{i0}(t) = -\gamma\, P_{i0}(t) + (\lambda+\omega)P_{i1}(t) + \sum_{k=0}^{S} q^k \mu P_{ik}(t) \tag{3.37}$$

$$P'_{iS}(t) = -(\lambda+\mu+S\omega)P_{iS}(t) + p^S\mu\, P_{iS}(t) + \sum_{k=0}^{s}\gamma\, P_{ik}(t) \tag{3.38}$$

Therefore, $$\pi_i = \frac{D_i}{F(s,S)\prod_{k=i}^{S}(-\bar{a}_{k,k})}; \qquad i \in E. \tag{3.44}$$

3.7.2 First Passage Times

Let $T_0 = 0 < T_1 < T_2 < \ldots\ldots$ be the epochs when the stock is replenished. Then $\{T_m, m \in N^0\}$ is a renewal process.

Theorem 3.8

If E(T) represents the expected time between two successive replenishments,

$$E(T) = F(s,S) = \frac{1}{-\bar{a}_{S,S}\pi_{S,S}} \tag{3.45}$$

Proof:

By a similar argument as in Section 4 of Chapter 2 we can derive the expression

$$E(T) = \int_0^\infty \alpha \exp(\mathbf{A}_1 t)\mathbf{e}dt \tag{3.46}$$

$$= -\alpha A^{-1} e$$

$$= \sum_{i=0}^{S} \frac{D_i}{\prod_{k=i}^{S}(-\bar{a}_{k,k})} \tag{3.47}$$

From (3.42) and (3.43) the theorem follows.

Theorem 3.9

Let E(T*) represent the expected time between a replenishment and the successive order. Then

$$E(T^*) = \sum_{i=s+1}^{S} \frac{D_i}{\prod_{k=i}^{S}(-\bar{a}_{k,k})} \tag{3.48}$$

Proof:

When the lead time is zero, the replenishment epochs and order epochs coincide. Therefore, E(T*) is the expected time between two successive re-orders in the zero lead time case. From (2.5) and (3.34) and from the definition of D_i we observe that a_{ii} and D_i are equal in both cases for $i \in \{s+1, s+2, \ldots\ldots, S\}$. Hence from 2.32,

$$E(T^*) = \sum_{i=s+1}^{S} \frac{D_i}{\prod_{k=i}^{S}(-\bar{a}_{k,k})}$$

Corollary 3.9.1

$$\frac{1}{\gamma} = \sum_{i=0}^{s} \frac{D_i}{\prod_{k=i}^{S}(-\bar{a}_{k,k})} \tag{3.49}$$

Proof:

Since the random variables involved are independent of each other and the mean of the lead time distribution is $1/\gamma$,

$$E(T) = E(T^*) + 1/\gamma. \tag{3.50}$$

Therefore from (3.48) and (3.47) the corollary follows.

3.7.3 Optimization of the Cost Function

Let Γ_1 and Γ_2 be the expected rates of depletion of inventory due to decay and disaster during the period between a replenishment and the successive order, and during the lead time period respectively, Γ_3 be the shortage rate. Then

$$\Gamma_1 = \frac{(\omega+\mu q)\sum_{i=s+1}^{S} i\pi_i}{\sum_{i=s+1}^{S}\pi_i}; \qquad \Gamma_2 = \frac{(\omega+\mu q)\sum_{i=0}^{s} i\pi_i}{\sum_{i=0}^{S}\pi_i}; \qquad \Gamma_3 = \lambda\pi_0 \tag{3.51}$$

Theorem 3.10

If N_1* and N_2* represent the expected quantities ordered at the time of usual order and at the instant of replenishment respectively, then

$$N_1^* = E(T)\sum_{i=s+1}^{S}\pi_i[\lambda + i(\omega+\mu q)] \tag{3.52}$$

$$N_2^* = E(T)\{-\pi_0\lambda + \sum_{i=0}^{s}\pi_i[\lambda + i(\omega+\mu q)]\} \tag{3.53}$$

Proof:

Since the expected quantity ordered is the product of expected time and rate of depletion of inventory during the period,

$$N_1^* = E(T^*)\,[\lambda + \Gamma_1]. \tag{3.54}$$

$$= \frac{E(T^*)[\lambda\sum_{i=s+1}^{S}\pi_i + (\omega+\mu q)\sum_{i=s+1}^{S} i\pi_i]}{\sum_{i=s+1}^{S}\pi_i} \tag{3.55}$$

Therefore from (3.43) and (3.48)

$$N_1^* = \frac{E(T^*)[\lambda\sum_{i=s+1}^{S}\pi_i + (\omega+\mu q)\sum_{i=s+1}^{S} i\pi_i]}{E(T^*)/F(s,S)} \tag{3.56}$$

and substitution of (3.45) gives (3.52).

$$N_2^* = \frac{1}{\gamma}[\lambda(1-\frac{\pi_0}{\sum_{i=0}^{s}\pi_i}) + \Gamma_2]. \tag{3.57}$$

$$= \frac{[\lambda(-\pi_0 + \sum_{i=0}^{s}\pi_i) \quad + (\omega + \mu\, q)\sum_{i=0}^{s} i\pi_i]}{\gamma \sum_{i=0}^{s}\pi_i} \tag{3.58}$$

From (3.43) and (3.49) we get,

$$N_2^* = \frac{[\lambda(-\pi_0 + \sum_{i=0}^{s}\pi_i) \quad + (\omega + \mu\, q)\sum_{i=0}^{s} i\pi_i]}{1/F(s,S)} \tag{3.59}$$

which is same as (3.53). Hence the theorem.

Let the various costs be as in section 3.6. Then the total cost function is

$$C_2(s,S) = \frac{K + c_1 N_1^* + c_2 N_2^*}{E(T)} + h\sum_{i=1}^{S} i\pi_i + d[\Gamma_1 + \Gamma_2] + c_3\Gamma_3$$

$$= \frac{K}{E(T)} + c_1 \sum_{i=s+1}^{S}\pi_i[\lambda + i(\omega + \mu q)] + c_2\{-\lambda\pi_0 + \sum_{i=0}^{s}\pi_i[\lambda + i(\omega + \mu q)]\} \tag{3.60}$$

$$+ h\sum_{i=1}^{S} i\pi_i + d[\Gamma_1 + \Gamma_2] + c_3\lambda\pi_0$$

3.7.4 Numerical Illustrations

From (3.50) and (2.40) we get E(T) is maximum when s = 0. Naturally $c_1 \le c_2$. When $c_3 \le c_1$ numerical examples indicate that C(s, S) is minimum when s = 0. However, when c_3 is large the optimum value of s need not be zero. These are illustrated by Table 3.1. Table 3.2 gives optimum values of the pair (s, S) for different values of μ and γ which explains the effect of disaster and lead time on the inventory system. Figures 3.1 and 3.2 show that s and S ecrease with the increase of μ. Comparing these two figures we can conclude ıat higher the lead time greater the values of s and S when μ is small.

Table 3.1

(Optimum values of (s, S) for different values of c_1 and c_3)

$\lambda = 10$, $\mu = 1$, $p = .7$ $\omega = 1$, $\gamma = 3$, $K = 300$, $h = 10$, $d = 5$, $c_2 = c_1 \times 1.5$

$c_1 \rightarrow$ / $c_3 \downarrow$	10	20	30	40	50	60	70	80	90	100
20	0,15	0,12	0,9	0,8	0,6	0,5	0,4	0,4	0,4	0,4
40	0,17	0,13	0,10	0,9	0,7	0,6	0,6	0,5	0,4	0,4
60	0,18	0,14	0,11	0,10	0,8	0,7	0,6	0,6	0,5	0,5
80	0,20	0,15	0,12	0,10	0,9	0,8	0,7	0,6	0,6	0,5
100	0,21	0,16	0,13	0,11	0,10	0,9	0,8	0,7	0,6	0,6
120	1,23	0,17	0,14	0,12	0,11	0,9	0,8	0,8	0,7	0,6
140	1,24	0,18	0,15	0,13	0,11	0,10	0,9	0,8	0,7	0,7
160	2,25	0,19	0,16	0,14	0,12	0,11	0,10	0,9	0,8	0,7
180	2,26	1,21	0,17	0,14	0,13	0,11	0,10	0,9	0,8	0,8
200	3,28	1,22	0,18	-0,15	0,13	0,12	0,11	0,10	0,9	0,8
220	3,28	2,23	0,19	0,16	0,14	0,13	0,12	0,11	0,10	0,9
240	4,29	2,23	1,19	0,16	0,14	0,13	0,12	0,11	0,10	0,9
260	4,30	2,24	1,20	0,17	0,15	0,13	0,12	0,11	0,10	0,9
280	4,31	3,25	2,21	1,18	0,16	0,14	0,13	0,12	0,11	0,10
300	5,32	3,25	2,22	1,19	0,16	0,15	0,13	0,12	0,11	0,10

Figure 3.1
(Optimum (s, S) values when $\gamma = 0.5$)
$\lambda = 10$, $p = .7$ $\omega = 1$, $K = 300$, $h = 10$, $d = 5$, $c_1 = 10$, $c_3 = 200$, $c_2 = 15$

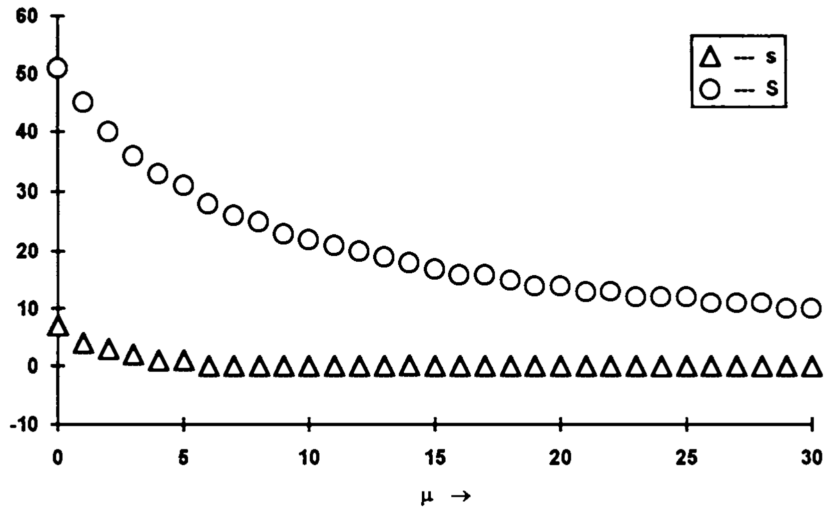


Figure 3.2
(Optimum (s, S) Values when $\gamma = 20$)
$\lambda = 10$, $p = .7$ $\omega = 1$, $K = 300$, $h = 10$, $d = 5$, $c_1 = 10$, $c_3 = 200$, $c_2 = 15$

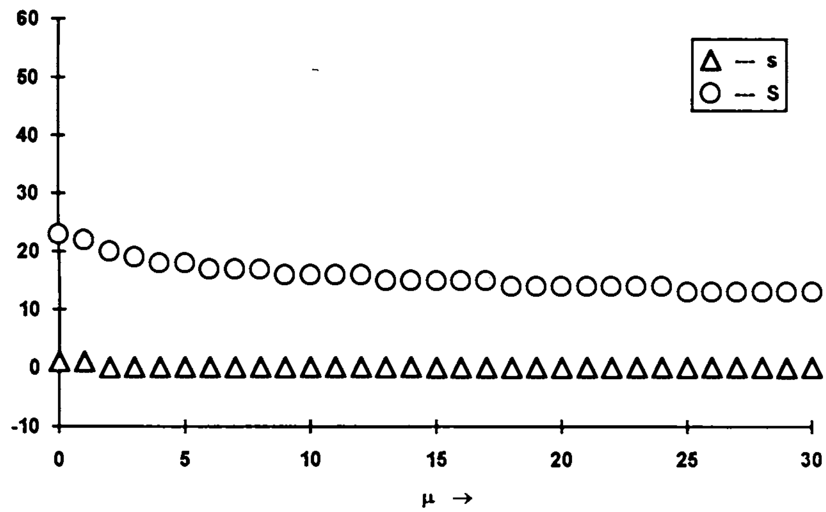


Table 3.2

(Optimum values of (s, S) for change of μ and γ)

$\lambda = 10$, $p = .7$ $\omega = 1$, $K = 300$, $h = 10$, $d = 5$, $c_1 = 10$, $c_3 = 200$, $c_2 = 15$

$\mu\rightarrow$ / $\gamma\downarrow$	0	1	2	3	4	5	6	7	8	9
20	1,23	1,22	0,20	0,19	0,18	0,18	0,17	0,17	0,17	0,16
16	1,23	1,22	1,21	0,19	0,19	0,18	0,18	0,17	0,17	0,16
12	1,23	1,22	1,21	1,20	0,19	0,18	0,18	0,17	0,17	0,17
8	2,25	1,23	1,22	1,21	1,20	1,20	0,18	0,18	0,17	0,17
6	2,25	2,24	1,23	1,22	1,21	1,20	1,20	0,19	0,18	0,17
4	3,27	2,26	2,24	2,23	1,22	1,21	1,21	1,20	1,19	0,18
2	5,33	4,31	3,29	2,27	2,26	2,24	1,23	1,22	1,21	0,20
1.5	5,36	4,33	3,31	3,29	2,27	2,26	1,24	1,23	1,22	0,21
1	6,41	5,38	4,35	3,32	2,30	1,28	1,26	1,24	0,23	0,22
.5	7,51	4,45	3,40	2,36	1,33	1,31	0,28	0,26	0,25	0,23

Chapter IV

Single Commodity Inventory System Subject to Disaster with General Interarrival Times

4.1. INTRODUCTION

In this chapter we discuss a single commodity continuous review (s,S) inventory system in which commodities are damaged due to disaster only. Shortages are not permitted and lead time is assumed to be zero so that inventory is replenished instantaneously whenever the inventory level falls to or below the re-ordering point s. The times between disasters follow independent exponential distribution with parameter μ. Each unit in the stock, independent of others, survives a disaster with probability p, and perishes with probability (1-p) = q. The failed items are disposed off immediately. The interarrival times of demands constitute a family of i.i.d. random variables with common distribution function F(.). The quantity demanded at a demand epoch is r and has an arbitrary distribution $b_r(t)$ (r = 1, 2,....) depending only on the time t elapsed from the previous demand point.

The main objective in this chapter is to derive the transient and steady state probabilities of the inventory level. We have done this with the help of the theory of semi-regenerative processes. A special case in which the disaster affects only the exhibiting item and arriving customers demand unit item is discussed in Section 4.4. In this case the steady state distribution of the inventory level is obtained as uniform. Optimization analysis is done and results are illustrated with examples in Section 4.4.1.

Notations

E	: $\{s+1, s+2, \ldots\ldots\ldots, S\}$
M	: $S-s$
q	: $1-p$
N^0	: $\{0, 1, 2, \ldots\ldots\ldots\ldots\ldots\}$
$\overline{F}(t)$	: $1-F(t)$
$\overline{b}_r(t)$	: $\sum\limits_{r=k}^{\infty} b_r(t)$; $(k = 1, 2, \ldots\ldots)$
(j)	: $\begin{cases} 1 & if \quad j \geq 0 \\ 0 & if \quad j < 0 \end{cases}$
$f(t) * g(t)$	: Convolution of the functions f(t) and g(t)
$f^{*n}(t)$	: n - fold convolution of $f(t)$; where $f^{*0}(t) \equiv 1$
$Q^{*0}(i,j,t)$	: $\begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

4.2. FORMULATION AND ANALYSIS

Let X(t) be the inventory level at time t. Then X(t) assumes values from E = {s+1, s+2, ...S}. Assume that the times at which the demand occurs are $0 = T_0 < T_1 < T_2 < T_3 < \ldots\ldots$ Define $X_n = X(T_n+)$, $n \in N^0$.

Let $\theta_1 < \theta_2 < \theta_3 < \ldots\ldots$be the times at which the inventory is replenished. Define

$${}_i\Psi_j(t) = \Pr\{X(\rho+t) = j \mid X(\rho+) = i, \text{ no demands in } (\rho, \rho+t],$$
$$\theta_r < \rho < \rho+t < \theta_{r+1}\}; \text{ for some r } (r = 1,2,\ldots\ldots).$$

Let $\tau_1^r < \tau_2^r < \tau_3^r \ldots\ldots$ $(\theta_r < \rho < \tau_1^r < \tau_2^r < \tau_3^r \ldots\ldots\ldots \leq \rho+t < \theta_{r+1})$ be the successive disaster epochs in (ρ, ρ+t]. Let $i_1, i_2, i_3, \ldots\ldots$ be the inventory levels just after the disaster at these epochs and $j_1, j_2, j_3, \ldots\ldots$ be the units destroyed at these epochs. Then

$$ {}_i\Psi_j(t) = \begin{cases} \sum\limits_{n=0}^{\infty} \dfrac{e^{-\mu t}(\mu t)^n}{n!} p^{ni} & \text{if } i=j \\ \sum\limits_{n=1}^{\infty} \sum\limits_{j_1+j_2+\ldots\ldots+j_n=i-j} \dfrac{e^{-\mu t}(\mu t)^n}{n!} z_1(j_1)\, z_2(j_2) \ldots\ldots\ldots z_n(j_n); & \text{if } i>j \\ 0 & \text{otherwise} \end{cases} \tag{4.1} $$

where

$$ z_k(j_k) = \binom{i_{k-1}-j_k}{j_k} p^{i_{k-1}-j_k}\, q^{j_k}; \qquad i_0 = i; \quad i_k = i_{k-1} - j_k $$

which reduces to

$$ {}_i\Psi_j(t) = \begin{cases} e^{-\mu t(1-p^i)} & \text{if } i=j \\ \sum\limits_{n=1}^{\infty} \sum\limits_{j_1+j_2+\ldots\ldots+j_n=i-j} \dfrac{e^{-\mu t}(\mu t)^n}{n!} \dfrac{i!}{j!\, j_1!\, j_2! \ldots j_n!} p^{i_1+i_2+\ldots+i_n}\, q^{j_1+j_2+\ldots+j_n}; & \text{if } i>j \\ 0 & \text{otherwise} \end{cases} \tag{4.2} $$

Define ${}_i\Phi_S(t)$ = the conditional probability that the inventory level reaches for the first time at S by a replenishment in (t, t+δt) given that the inventory level was initially at i and only disasters (at least one) in between. Then

$$ {}_i\Phi_S(t) = \sum_{n=1}^{\infty} \sum_{\substack{j_1+j_2+\ldots\ldots+j_{n-1}<i-s \\ j_1+j_2+\ldots\ldots+j_n \geq i-s}} \frac{e^{-\mu t}\mu(\mu t)^{n-1}}{(n-1)!} \frac{i!}{[i-(j_1+j_2+\ldots+j_n)]!\, j_1!\, j_2! \ldots j_n!} p^{i+i_1+i_2+\ldots+i_{n-1}-(j_1+j_2+\ldots+j_n)}\, q^{j_1+j_2+\ldots+j_n} \tag{4.3} $$

Define $g_n(i,t) = \lim\limits_{\delta t \to 0} \Pr\{t < \theta_n \leq t+\delta t \mid X(0+) = i, \quad T_1 > t\} / \delta t$ (4.4)

Then $g_n(i,t)$ denotes the conditional probability that the n^{th} replenishment takes place in (t, t+δt) given that the inventory level is initially at i and $T_1 > t$.

Since $\theta_n = \theta_1 + \theta_2 - \theta_1 + \ldots\ldots + \theta_n - \theta_{n-1}$ and θ_1 , $\theta_2 - \theta_1$,......$\theta_n - \theta_{n-1}$ are independent random variables, we have

$$\begin{aligned} g_1(i,t) &= {}_i\Phi_S(t) \\ g_n(i,t) &= {}_i\Phi_S(t) * {}_S\Phi_S^{*(n-1)}(t) \end{aligned} \tag{4.5}$$

Theorem 4.1

The stochastic process $\{ (X_n, T_n), n \in N^0 \}$ is a Markov renewal process with state space E and semi-Markov kernel $\{Q(i, j, t);\ i, j \in E, t \geq 0 \}$ where

$$Q(i, j, t) = \Pr\{X_{n+1} = j,\ T_{n+1} - T_n \leq t \mid X_n = i\} \tag{4.6}$$

$$= \begin{cases} \delta(i-j-1)\displaystyle\sum_{r=1}^{i-j} \int_0^t b_r(u)\ {}_i\Psi_{j+r}(u)\, dF(u) \\ \quad + \displaystyle\sum_{n=1}^{\infty}\sum_{r=1}^{S-j} \int_0^t b_r(u)\, [g_n(i,u) * {}_S\Psi_{j+r}(u)]\, dF(u) & \text{for} \quad j \neq S \\ \displaystyle\sum_{r=1}^{i-s} \int_0^t \bar{b}_r(u)\ {}_i\Psi_{s+r}(u)\, dF(u) \\ \quad + \displaystyle\sum_{n=1}^{\infty}\sum_{r=1}^{S-s} \int_0^t \bar{b}_r(u)\, [g_n(i,u) * {}_S\Psi_{s+r}(u)]\, dF(u) & \text{for} \quad j = S \end{cases} \tag{4.7}$$

Proof:

The fact that $\{(X_n, T_n), n \in N^0\}$ is a Markov Renewal Process with state space E is clear from the assumptions. If we denote the number of replenishments in (0, t) by N(t) and define

$$\Omega_n(i,j,t) = \Pr\{X_1 = j,\quad N(T_1) = n,\quad T_1 \leq t \mid X(0+) = i\};\quad n = 0,1,2,\ldots \tag{4.8}$$

then the semi-Markov kernel is given by

$$Q(i,j,t) = \sum_{n=0}^{\infty} \Omega_n(i,j,t) \tag{4.9}$$

To derive the expression for Ω_n (i, j, t), (n = 0, 1, 2, ...), assume that the next demand after the initial one occurs in (u, u+δu) where u < t. There are four cases.

(1) n = 0 and j ≠ S

In this case there is no replenishment in (0, u]. Assume that the demand that occurred at time u is for r items (r = 1, 2,......., i–j ; if i > j). In order that the inventory level is j at time u, the inventory level must have reduced to j+r (j+r < i) due to disasters in (0, u) from the initial level i. Therefore

$$\Omega_0(i,j,t) = \delta(i-j-1)\sum_{r=1}^{i-j} \int_0^t b_r(u) \quad {}_i\Psi_{j+r}(u)\, dF(u) \tag{4.10}$$

(2) n ≠ 0, j ≠ S

Here the n^{th} replenishment occurs at some time v (< u) and the inventory level is instantaneously brought to S. If the demand at u is for r items (r = 1, 2, ..., S–j), in order to have the inventory level j, the stock must have reduced from S to j+r due to disaster in (v, u). Hence

$$\Omega_n(i,j,t) = \sum_{r=1}^{S-j} \int_0^t b_r(u)\, [g_n(i,u) * {}_S\Psi_{j+r}(u)] dF(u) \tag{4.11}$$

(3) n = 1, j = S

Assume that there is no replenishment in (0, u) and a replenishment is triggered by the demand at u. This will happen when the inventory level is s+r and there is a demand for at least r units (r = 1, 2, ..., i–s) at time u. The disasters in (0, u) must have destroyed (i–s–r) items of the stock. So we have

$$\Omega_1(i,S,t) = \sum_{r=1}^{i-s} \int_0^t \bar{b}_r(u) \;\; {}_i\Psi_{s+r}(u)\, dF(u) \tag{4.12}$$

(4) n > 1, j = S

Suppose that the $(n-1)^{th}$ replacement is at some time v (< u) and another replenishment is triggered by the demand at u. For this the inventory level is brought down from S to s+r (r = 1, 2, ..., S–s) by the disasters in (v, u) and there must have been a demand for at least r items of inventory at u. Then

$$\Omega_n(i,S,t) = \sum_{r=1}^{S-s} \int_0^t \bar{b}_r(u)\, [g_{n-1}(i,u) * {}_S\Psi_{s+r}(u)] dF(u) \tag{4.13}$$

Substituting (4.10) - (4.13) in (4.9) we get (4.7). Hence the theorem.

4.3 TRANSIENT AND STEADY STATE SOLUTIONS

Let p(i, j , t) = Pr {X(t) = j | X(0+) = i }, i, j ∈ E. Then we have

Theorem 4.2

The transient solution of the inventory levels is given by

$$p(i,j,t) = \sum_{r=s+1}^{S} \int_0^t R(i,r,du)\, k(r,j,t-u); \quad i,j \in E \tag{4.14}$$

where $$R(i,j,t) = \sum_{n=0}^{\infty} Q^{*n}(i,j,t); \quad i,j \in E$$

and
$$k(i,j,t) = \begin{cases} \bar{F}(t)[\;{}_i\Psi_j(t) + \sum_{n=1}^{\infty} g_n(i,t) *_S \Psi_{S-j}(t)]; & j \neq S \\ \bar{F}(t)[\;{}_i\Psi_S(t) + \sum_{n=1}^{\infty} g_n(i,t) *_S \Psi_S(t)]; & j = S \end{cases} \tag{4.15}$$

Proof:

The stochastic process $\{X(t), t \geq 0\}$ is a semi-regenerative process with the embedded MRP $\{(X_n, T_n), n \in N^0\}$. Conditioning on the first demand epoch T_1 we can find that p(i, j, t) satisfy the following Markov renewal equations,

$$p(i,j,t) = k(i,j,t) + \sum_{r=s+1}^{S} \int_0^t Q(i,r,du)\, p(r,j,t-u); \quad i,j \in E \quad (4.16)$$

where $\qquad k(i,j,t) = \Pr\{X(t) = j, \ \ T_1 > t \mid X(0+) = i\}; \quad i,j \in E.$

To derive the expressions of k(i, j, t) in (4.15) note that, since $T_1 > t$, the depletion of inventory is only due to disaster and there may be n (n = 0, 1, 2,.......) replenishments in (0, t).The solution of (4.16) is given by (4.14). Hence the theorem.

Consider the underlying Markov chain $\{ X_n, n \in N^0 \}$ associated with the MRP $\{(X_n, T_n), n \in N^0\}$. Its transition probability matrix $\mathbf{Q} = (q_{ij})$; $i, j \in E$, is given by

$$q_{ij} = \lim_{t \to \infty} Q(i,j,t)$$

$$= \begin{cases} \delta(i-j-1)\sum_{r=1}^{i-j} \int_0^\infty b_r(u) \ \ {}_i\Psi_{j+r}(u)\, dF(u) \\ \quad + \sum_{n=1}^{\infty} \sum_{r=1}^{S-j} \int_0^\infty b_r(u)\,[g_n(i,u) * {}_S\Psi_{j+r}(u)]\, dF(u) \quad \text{for} \quad j \neq S \\ \sum_{r=1}^{i-s} \int_0^\infty \bar{b}_r(u) \ \ {}_i\Psi_{s+r}(u)\, dF(u) \\ \quad + \sum_{n=1}^{\infty} \sum_{r=1}^{S-s} \int_0^\infty \bar{b}_r(u)\,[g_n(i,u) * {}_S\Psi_{s+r}(u)]\, dF(u) \quad \text{for} \quad j = S \end{cases} \quad (4.17)$$

If $b_1(t) \neq 0$ for some interval in $[0, \infty)$ it can easily be seen that the finite Markov chain $\{X_n, n \in N^0\}$ is irreducible and hence it is recurrent. Since the chain is irreducible, it possesses a unique stationary distribution,

$$\Pi = (\pi_{s+1}, \pi_{s+2}, \ldots\ldots\pi_S) \text{ which satisfies } \Pi\mathbf{Q} = \Pi \text{ and } \Sigma\pi_j = 1. \qquad (4.18)$$

Let $\mathbf{P} = (p_{s+1}, p_{s+2}, \ldots\ldots, p_S)$ denote the steady state probability vector of the inventory level where $p_j = \lim_{t\to\infty} p(i,j,t)$.

Theorem 4.3

If F(t) is absolutely continuous with finite expectation, m, then the steady state probabilities of the inventory level are given by

$$p_j = \frac{\sum_{i\in E} \pi_i \int_0^\infty k(i,j,t)dt}{m}; \quad j \in E. \qquad (4.19)$$

Proof:

Since F(t) is absolutely continuous with finite expectation, it follows from (4.14) that

$$p_j = \frac{\sum_{i\in E} \pi_i \int_0^\infty k(i,j,t)dt}{\sum_{i\in E} \pi_i m_i}; \quad j \in E \qquad (4.20)$$

where m_i = mean sojourn time in state i = $\int_0^\infty t dF(t) = m$. Substitution yields (4.19). Hence the theorem.

4.4 A PARTICULAR CASE

Suppose the disaster affects only an exhibiting item which is replaced instantaneously by another one upon failure, and the arriving customers demand only unit item, then the rate of disaster is μq and that of survival is μp whence

$$b_r(t) = \begin{cases} 1 & \text{if } \; r = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{4.21}$$

This results in

Theorem 4.4

If the disaster affects only an exhibiting item and each arrival demands exactly one unit of the item, then the steady state probabilities of the inventory level are uniformly distributed.

Proof:

Because of the special assumptions in this section,

$$_i\Psi_j(t) = \begin{cases} \dfrac{(\mu\, qt)^{(i-j)}\, e^{-\mu\, qt}}{(i-j)!} & \text{if } \; i \ge j \\ 0 & \text{otherwise} \end{cases} \tag{4.22}$$

and $$g_n(i,t) = \frac{\mu\, q(\mu\, qt)^{[(n-1)M+i-s-1]}\, e^{-\mu\, qt}}{[(n-1)M+i-s-1]!} \qquad \text{for } \; n = 1,2,\ldots\ldots \tag{4.23}$$

Therefore from (4.21)

$$Q(i,j,t) = \sum_{\substack{n=\delta\,(j-i) \\ k=(i-j+nM-1)}}^{\infty} \int_0^t \frac{e^{-\mu\, qu}(\mu\, qu)^k}{k!} dF(u) \qquad \text{for } i,j \in E \tag{4.24}$$

and $$q_{ij} = Q(i,j,\infty) = \sum_{\substack{n=\delta\,(j-i) \\ k=(i-j+nM-1)}}^{\infty} \int_0^\infty \frac{e^{-\mu\, qu}(\mu\, qu)^k}{k!} dF(u) \quad \text{for } i,j \in E \tag{4.25}$$

Since q_{ij} is a function of (i-j),

$$\begin{aligned} \sum_{i=s+1}^{S} q_{ij} &= \sum_{k=0}^{\infty} \int_0^\infty \frac{e^{-\mu qu}(\mu qu)^k}{k!} dF(u) \\ &= \int_0^\infty dF(u) = 1 \qquad \text{for } j \in E \end{aligned} \tag{4.26}$$

Therefore the transition probability matrix (q_{ij}) is doubly stochastic and from the uniqueness it follows that the invariant measure $\pi_j = 1/M$ for $j \in E$. Also note that

$$\sum_{i=s+1}^{S} \int_0^\infty k(i,j,du) = \int_0^\infty (1 - F(u))du = m \tag{4.27}$$

Therefore from (4.19) we get $p_j = 1/M$ for $j \in E$. Hence the theorem.

4.4.1 Illustrations

Now suppose that the interarrival times follow a gamma distribution with parameters (ν, λ), then

$$Q(i,j,t) = \sum_{\substack{n=\delta\,(j-i) \\ k=(i-j+nM-1)}}^{\infty} \int_0^t \frac{e^{-(\mu q+\lambda)u}(\mu q)^k \lambda^\nu u^{k+\nu-1}}{k!(\nu-1)!} du \quad \text{for } i,j \in E \tag{4.28}$$

and

$$\begin{aligned} q_{ij} = Q(i,j,\infty) &= \sum_{\substack{n=\delta(j-i) \\ k=(i-j+nM-1)}}^{\infty} \int_0^\infty \frac{e^{-(\mu q+\lambda)u}(\mu q)^k \lambda^\nu u^{k+\nu-1}}{k!(\nu-1)!} du \quad \text{for } i,j \in E \\ &= \sum_{\substack{n=\delta(j-i) \\ k=(i-j+nM-1)}}^{\infty} \binom{k+\nu-1}{k} \left(\frac{\mu q}{\mu q+\lambda}\right)^k \left(\frac{\lambda}{\mu q+\lambda}\right)^\nu \quad \text{for } i,j \in E. \end{aligned} \tag{4.29}$$

Since $\qquad \overline{F}(t) = e^{-\lambda t} \sum_{a=0}^{\nu-1} \frac{(\lambda t)^a}{(a-1)!}$

$$\int_0^\infty k(i,j,t) = \sum_{\substack{n=\delta\,(j-i-1) \\ b=(i-j+nM)}}^{\infty} \frac{1}{\mu q+\lambda} \sum_{a=0}^{\nu-1} \binom{a+b}{b} \left(\frac{\mu q}{\mu q+\lambda}\right)^b \left(\frac{\lambda}{\mu q+\lambda}\right)^a \quad \text{for } i,j \in E. \tag{4.30}$$

Therefore the steady state probabilities of the inventory level are given by

$$
\begin{aligned}
p_j &= \frac{\lambda}{\nu}\sum_{i=s+1}^{S}\pi_i \sum_{\substack{n=\delta(j-i-1)\\ b=(i-j+nM)}}^{\infty} \frac{1}{\mu q+\lambda}\sum_{a=0}^{\nu-1}\binom{a+b}{b}\left(\frac{\mu q}{\mu q+\lambda}\right)^b\left(\frac{\lambda}{\mu q+\lambda}\right)^a; \ \mathrm{j} \in \mathrm{E} \\
&= \frac{\lambda}{\nu}\sum_{i=s+1}^{S}\pi_i \sum_{\substack{n=\delta(j-i-1)\\ b=(i-j+nM)}}^{\infty} \frac{1}{\mu q+\lambda}\sum_{a=0}^{\nu-1}\binom{a+b}{b}\left(\frac{\mu q}{\mu q+\lambda}\right)^b\left(\frac{\lambda}{\mu q+\lambda}\right)^a; \ \mathrm{j} \in \mathrm{E} \\
&= \frac{\lambda}{\nu\mathrm{M}}\sum_{i=s+1}^{S} \sum_{\substack{n=\delta(j-i-1)\\ b=(i-j+nM)}}^{\infty} \frac{1}{\mu q+\lambda}\sum_{a=0}^{\nu-1}\binom{a+b}{b}\left(\frac{\mu q}{\mu q+\lambda}\right)^b\left(\frac{\lambda}{\mu q+\lambda}\right)^a; \qquad (4.31) \\
&= \frac{1}{\nu\mathrm{M}}\sum_{a=0}^{\nu-1}\left(\frac{\lambda}{\mu q+\lambda}\right)^{a+1}\sum_{b=0}^{\infty}\binom{a+b}{b}\left(\frac{\mu q}{\mu q+\lambda}\right)^b \\
&= \frac{1}{\nu\mathrm{M}}\sum_{a=0}^{\nu-1}\left(\frac{\lambda}{\mu q+\lambda}\right)^{a+1}\left(1-\frac{\mu q}{\mu q+\lambda}\right)^{-(a+1)} = \frac{1}{\mathrm{M}}
\end{aligned}
$$

In this case, the expected replenishment cycle time is $M/(\mu q+\lambda/\nu)$. Therefore, if the fixed ordering cost is K , unit purchase cost of the item is c, and the holding cost per unit time is h, the unit cost for a damaged item is d, the cost function to be minimized is

$$
\begin{aligned}
C(s,S) &= \frac{(K+cM)}{M/(\mu q+\lambda/\nu)} + \frac{h}{M}\sum_{i=s+1}^{S} i + d\mu q \qquad (4.32) \\
&= [(K/M)+c][\mu q+\lambda/\nu] + hs + (h/2)(M+1) + d\mu q
\end{aligned}
$$

which is clearly minimum for s = 0. Therefore the cost function reduces to

$$
\mathrm{C(0,S)} = [(K/S)+c][\mu q+\lambda/\nu] + (h/2)(S+1) + d\mu q. \qquad (4.33)
$$

Clearly C(0,S) is a convex function since,

$$
\Delta^2 C(0,S) = \frac{2K[\mu q+(\lambda/\nu)]}{S(S+1)(S+2)} > 0. \qquad (4.34)
$$

If S* denotes the value of S minimizing C(0, S), then it is given by

$$
S^*(S^*-1) < \frac{2K[\mu q+(\lambda/\nu)]}{h} < S^*(S^*+1). \qquad (4.35)
$$

The following three tables give optimum values of S for different values of K, h, μ, ν, λ and q. Figure 4.1 Illustrates the effect of disaster on the cost function.

Table 4.1

(Optimum value of S when K = 100 and h = 2.5)

λ/ν μq	.2	.6	1	1.4	1.8	2.2	2.6	3	3.4	3.8
.1	5	8	9	11	12	14	15	16	17	18
.4	7	9	11	12	13	14	16	17	17	18
.7	9	10	12	13	14	15	16	17	18	19
1	10	11	13	14	15	16	17	18	19	20
1.3	11	12	14	15	16	17	18	19	19	20
1.6	12	13	14	15	17	17	18	19	20	21
1.9	13	14	15	16	17	18	19	20	21	21
2.2	14	15	16	17	18	19	20	20	21	22
2.5	15	16	17	18	19	19	20	21	22	22
2.8	15	16	17	18	19	20	21	22	22	23

Table 4.2

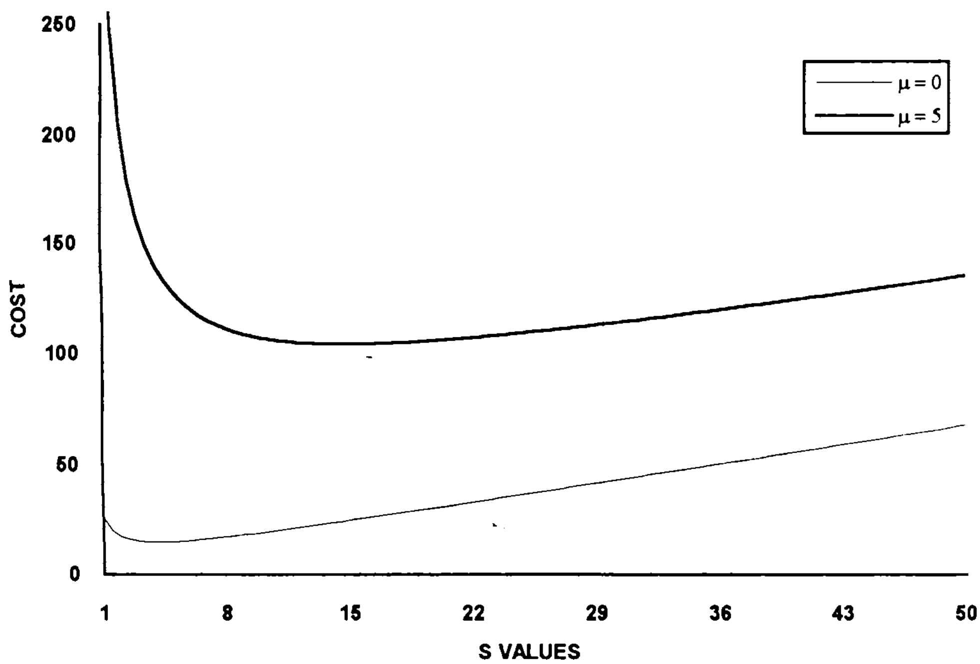
(Optimum value of S when K = 200 and h = 2.5)

λ/ν μq	.2	.6	1	1.4	1.8	2.2	2.6	3	3.4	3.8
.1	7	11	13	16	17	19	21	22	24	25
.4	10	13	15	17	19	20	22	23	25	26
.7	12	14	17	18	20	22	23	24	26	27
1	14	16	18	20	21	23	24	25	27	28
1.3	16	17	19	21	22	24	25	26	27	29
1.6	17	19	20	22	23	25	26	27	28	29
1.9	18	20	22	23	24	26	27	28	29	30
2.2	20	21	23	24	25	27	28	29	30	31
2.5	21	22	24	25	26	27	29	30	31	32
2.8	22	23	25	26	27	28	29	30	32	33

Table 4.3
(Optimum value of S when K = 300 and h = 2.5)

λ/ν \ μq	.2	.6	1	1.4	1.8	2.2	2.6	3	3.4	3.8
.1	9	13	16	19	21	24	25	27	29	31
.4	12	16	18	21	23	25	27	29	30	32
.7	15	18	20	22	25	26	28	30	31	33
1	17	20	22	24	26	28	29	31	33	34
1.3	19	21	24	25	27	29	31	32	34	35
1.6	21	23	25	27	29	30	32	33	35	36
1.9	22	25	26	28	30	31	33	34	36	37
2.2	24	26	28	29	31	33	34	35	37	38
2.5	25	27	29	31	32	34	35	36	38	39
2.8	27	29	30	32	33	35	36	37	39	40

Figure 4.1
(The effect of disaster on the cost function)
$K = 100$, $c = 20$, $h = 2.5$, $d = 5$, $\lambda = 2$, $\nu = 10$, $q = 0.5$.



Chapter V

Single Commodity Inventory System with General Disaster Periods

5.1. INTRODUCTION

This chapter deals with a single commodity continuous review (s,S) inventory system in which commodities are damaged due to disaster only. Shortages are not permitted and lead time is assumed to be zero. The demands constitute Poisson process with parameter λ. The times between disasters follow general distribution G(.) which is absolutely continuous with finite mean m. Each unit in the stock, independent of others, either survives a disaster with probability p, or damages completely with probability $(1 - p) = q$. The failed items are disposed off immediately.

The structure of this chapter is similar to chapter 4. As in the previous chapter, the principal aim of the present chapter is to derive the transient and steady state probabilities of the inventory level. A special case in which the disaster affects only the exhibiting item is discussed in Section 5.4. For this special case, the steady state distribution of the inventory levels is shown to be uniform. Illustrations are provided in Section 5.4.1.

Notations

E : $\{s+1, s+2, \ldots\ldots\ldots S\}$

M : $S-s$

q $1-p$

N^0 : $\{0, 1, 2, \ldots\ldots\ldots\ldots\ldots\ldots\}$

$\overline{G}(t)$: $1-G(t)$

$\delta(j)$: $\begin{cases} 1 & if \quad j \geq 0 \\ 0 & if \quad j < 0 \end{cases}$

$Q^{*n}(i,j,t)$: n - fold convolution of $Q(i,j,t)$ with itself.

$Q^{*0}(i,j,t)$: $\begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

5.2. ANALYSIS OF THE INVENTORY LEVEL

Let X(t) be the inventory level at time t ($t \geq 0$). Then X(t) takes values on $E = \{s+1, s+2, \ldots, S\}$. Assume that the disaster epochs are $0 = T_0 < T_1 < T_2 < \ldots$. Define $X_n = X(T_n+)$, $n \in N^0$.

Theorem 5.1

The stochastic process $\{ (X_n, T_n), n \in N^0 \}$ is a Markov renewal process with state space E and the semi-Markov kernel $\{Q(i, j, t), i, j \in E, t \geq 0\}$ where

$$Q(i,j,t) = \Pr\{X_{n+1} = j, \quad T_{n+1} - T_n \leq t \quad | \quad X_n = i\} \tag{5.1}$$

$$
= \begin{cases}
\delta(i-j)\displaystyle\sum_{r=0}^{i-j} \binom{j+r}{j} p^j q^r \int_0^t \frac{(\lambda u)^{i-j-r}}{(i-j-r)!} e^{-\lambda u}\, dG(u) \\
\quad + \displaystyle\sum_{k=1}^{\infty} \sum_{r=0}^{S-j} \binom{j+r}{j} p^j q^r \int_0^t \frac{(\lambda u)^{kM+i-j-r}}{(kM+i-j-r)!} e^{-\lambda u}\, dG(u) \qquad \text{for } j \neq S \\
\\
\delta(i-S)\, p^S \displaystyle\int_0^t e^{-\lambda u} dG(u) + \sum_{k=0}^{\infty} p^S \int_0^t \frac{(\lambda u)^{kM+i-s}}{(kM+i-s)!} e^{-\lambda u}\, dG(u) \\
\quad + \displaystyle\sum_{a=s+1}^{i} \sum_{r=a-s}^{a} \binom{a}{r} p^{a-r} q^r \int_0^t \frac{(\lambda u)^{i-a}}{(i-a)!} e^{-\lambda u}\, dG(u) \\
\quad + \displaystyle\sum_{k=1}^{\infty} \sum_{a=s+1}^{S} \sum_{r=a-s}^{a} \binom{a}{r} p^{a-r} q^r \int_0^t \frac{(\lambda u)^{kM+i-a}}{(kM+i-a)!} e^{-\lambda u}\, dG(u) \quad \text{for } j = S
\end{cases} \tag{5.2}
$$

Proof:

Since the demand process is Poisson, the interarrival times are exponentially distributed. Hence X_{n+1} depends only on X_n and $T_{n+1} - T_n$. Therefore $\{(X_n, T_n), n \in N^0\}$ is a Markov Renewal Process with state space E. Let the number of replenishments in (0, t) be N(t) and define

$$\Omega_k(i,j,t) = \Pr\{X_1 = j, N(T_1) = k, T_1 \le t \mid X(0+) = i\};\ k = 0, 1, 2, \ldots. \tag{5.3}$$

then semi-Markov kernel $Q(i, j, t)$ is given by

$$Q(i,j,t) = \sum_{k=0}^{\infty} \Omega_k(i,j,t) \tag{5.4}$$

To derive the expression for $\Omega_k(i, j, t)$ (k = 0, 1, 2, ...) assume that the next disaster after the initial one occurs in (u, u+δu) where u < t. There are five cases.

(1) $k = 0$ and $j \neq S$.

In this case there is no replenishment in (0, u]. Assume that the disaster that happened at time u destroys r items ($r = 0, 1, 2, \ldots\ldots, i - j$; if $i \ge j$). In order that the inventory level is j just after this disaster, the inventory level must have reduced to j+r (j+r ≤ i) due to demands in (0, u) from the initial inventory level i. Therefore

$$\Omega_0(i,j,t) = \delta(i-j)\sum_{r=0}^{i-j}\binom{j+r}{r}p^j q^r \int_0^t \frac{(\lambda u)^{i-j-r}e^{-\lambda u}}{(i-j-r)!}dG(u) \qquad (5.5)$$

(2) $k \neq 0, j \neq S$.

Here there are k replenishments in (0, u) due to depletion of inventory by demand and the stock level is instantaneously brought to S each time. If the disaster at u destroys r items (r = 0, 1, 2,......, S − j), in order to have the inventory level j just after the disaster at u, the arrivals in (0, u) must have demanded (kM + i - j - r) units in (0, u). Hence

$$\Omega_k(i,j,t) = \sum_{r=0}^{S-j}\binom{j+r}{r}p^j q^r \int_0^t \frac{(\lambda u)^{kM+i-j-r}e^{-\lambda u}}{(kM+i-j-r)!}dG(u) \qquad (5.6)$$

(3) $k = 1, j = S$.

There are two possibilities. (i) There is exactly one replenishment due to demand in (0, u) and the S units in the inventory survives the disaster at u. (ii) There is no replenishment in (0, u) and a replenishment is triggered by the disaster at u. The former case will happen when the demands in (0, u) are exactly for i - s units and the disaster at u affects none of the items in the stock. The latter case happens when the inventory level is a just before the disaster and at least (a - s) units (s+1 ≤ a ≤ i) are destroyed by the disaster at u. So we have

$$\Omega_1(i,S,t) = p^S \int_0^t \frac{(\lambda u)^{i-s}e^{-\lambda u}}{(i-s)!}dG(u) + \sum_{a=s+1}^{i}\sum_{r=a-s}^{a}\binom{a}{r}p^{a-r}q^r \int_0^t \frac{(\lambda u)^{i-a}e^{-\lambda u}}{(i-a)!}dG(u) \qquad (5.7)$$

(4).k > 1, j = S.

In this case also there are two possibilities. (i) There are exactly k replenishments due to demand in (0, u) and the S units in the inventory survive the disaster at u. (ii) There are (k - 1) replenishments due to demand in (0, u) and a replenishment is triggered by the disaster at u. In the former case exactly [(k - 1)M + i - s] units are demanded in (0, u) and the disaster at u affects none of the items in the stock. The latter case happens when the inventory level is brought to a by [(k - 1)M + i - a] demands in (0, u) and at least a - s units (s+1 ≤ a ≤ S) are destroyed by the disaster at u. So we have

$$\Omega_k(i,S,t) = p^S \int_0^t \frac{(\lambda u)^{(k-1)M+i-s} \; e^{-\lambda u}}{[(k-1)M+i-s]!} dG(u) + \sum_{a=s+1}^{S} \sum_{r=a-s}^{a} \binom{a}{r} p^{a-r} q^r \int_0^t \frac{(\lambda u)^{(k-1)M+i-a} \; e^{-\lambda u}}{[(k-1)M+i-a]!} dG(u) \tag{5.8}$$

(5) k = 0, j = S.

This happens only when i = S, when there is no demand in (0, u) and the disaster at time u affects none of the units in the stock. So we get

$$\Omega_0(S,S,t) = p^S \int_0^t e^{-\lambda u} dG(u) \tag{5.9}$$

Substituting (5.5) – (5.9) in (5.4) we get (5.2). Hence the theorem.

5.3 TIME DEPENDENT AND LIMITING DISTRIBUTIONS

Let $p(i, j, t) = \Pr\{X(t) = j \mid X(0+) = i\}$, $i, j \in E$. Then we have

Theorem 5.2

The time dependent probabilities of the inventory levels are given by

$$p(i,j,t) = \sum_{r=s+1}^{S} \int_0^t R(i,r,du)\, k(r,j,t-u); \quad i,j \in E \tag{5.10}$$

where $$R(i,j,t) = \sum_{n=0}^{\infty} Q^{*n}(i,j,t); \quad i,j \in E$$

and $$k(i,j,t) = \overline{G}(t) \sum_{n=\delta(j-i-1)}^{\infty} \frac{(\lambda t)^{nM+i-j}\, e^{-\lambda t}}{(nM+i-j)!}. \tag{5.11}$$

Proof:

The stochastic process $\{X(t), t \geq 0\}$ is a semi-regenerative process with the embedded MRP $\{(X_n, T_n), n \in N^0\}$. Conditioning on the first disaster epoch T_1 we see that $p(i, j, t)$'s satisfy the following Markov renewal equations,

$$p(i,j,t) = k(i,j,t) + \sum_{r=s+1}^{S} \int_0^t Q(i,r,du)\, p(r,j,t-u); \quad i,j \in E \tag{5.12}$$

where $k(i,j,t) = \Pr\{X(t) = j, \ T_1 > t \mid X(0+) = i\}; \ \ i,j \in E.$

To derive the expressions of $k(i, j, t)$ in (5.11) note that, since $T_1 > t$, the depletion of inventory is only due to demand and there may be n replenishments in (0, t). If $i < j$ then there should be at least one replenishment and n varies from 1 to ∞ in (5.11), otherwise n varies from zero to ∞. The solution of (5.12) is given by (5.10). Hence the theorem.

Consider the underlying Markov chain $\{ X_n, n \in N^0 \}$ associated with the MRP $\{(X_n, T_n), n \in N^0\}$. Its transition probability matrix $\mathbf{Q} = (q_{ij})$; $i, j \in E$, is given by

$$q_{ij} = \lim_{t\to\infty} Q(i,j,t)$$

$$= \begin{cases} \delta(i-j)\sum\limits_{r=0}^{i-j} \binom{j+r}{j} p^j q^r \int\limits_0^\infty \frac{(\lambda u)^{i-j-r}}{(i-j-r)!} e^{-\lambda u}\, dG(u) \\ \quad + \sum\limits_{k=1}^{\infty} \sum\limits_{r=0}^{S-j} \binom{j+r}{j} p^j q^r \int\limits_0^\infty \frac{(\lambda u)^{kM+i-j-r}}{(kM+i-j-r)!} e^{-\lambda u}\, dG(u) \qquad \text{for } j \neq S \\ \\ \delta(i-S)\, p^S \int\limits_0^\infty e^{-\lambda u} dG(u) + \sum\limits_{k=0}^{\infty} p^S \int\limits_0^\infty \frac{(\lambda u)^{kM+i-s}}{(kM+i-s)!} e^{-\lambda u}\, dG(u) \\ \quad + \sum\limits_{a=s+1}^{i} \sum\limits_{r=a-s}^{a} \binom{a}{r} p^{a-r} q^r \int\limits_0^\infty \frac{(\lambda u)^{i-a}}{(i-a)!} e^{-\lambda u}\, dG(u) \\ \quad + \sum\limits_{k=1}^{\infty} \sum\limits_{a=s+1}^{S} \sum\limits_{r=a-s}^{a} \binom{a}{r} p^{a-r} q^r \int\limits_0^\infty \frac{(\lambda u)^{kM+i-a}}{(kM+i-a)!} e^{-\lambda u}\, dG(u) \quad \text{for } j = S \end{cases} \tag{5.13}$$

Since the transition from any state i to any state j (i, j $\in$ E) is possible with positive probability the finite Markov chain $\{X_n, n \in N^0\}$ is irreducible and hence it is recurrent. Since the chain is irreducible, it possesses a unique stationary distribution,

$$\Pi = (\pi_{s+1}, \pi_{s+2}, \ldots\ldots\ldots \pi_S) \text{ which satisfies } \Pi\mathbf{Q} = \Pi \text{ and } \Sigma\pi_j = 1. \tag{5.14}$$

Let $\mathbf{P} = (p_{s+1}, p_{s+2}, \ldots, p_S)$ denote the steady state probability vector of the inventory level where $p_j = \lim\limits_{t\to\infty} p(i,j,t)$. Then we have

Theorem 5.3

If G(t) is absolutely continuous with finite expectation, m, then the steady state probabilities of the inventory levels are given by

$$p_j = \frac{\sum\limits_{i\in E} \pi_i \int\limits_0^\infty k(i,j,t)dt}{m}; \quad j \in E. \tag{5.15}$$

Proof:

Since G(t) is absolutely continuous with finite expectation, it follows from (5.10) - (5.12) that

$$p_j = \frac{\sum_{i \in E} \pi_i \int_0^\infty k(i,j,t)dt}{\sum_{i \in E} \pi_i m_i}; \quad j \in E \tag{5.16}$$

where $m_i =$ mean sojourn time in state i $= \int_0^\infty t dG(t) = m$. Substitution yields (5.15).

Hence the theorem.

5.4 A SPECIAL CASE

In this sub-section we discuss a special case in which the disaster affects only an exhibiting item.

Theorem 5.4

If the disaster affects only an exhibiting item and it is replaced instantaneously by another one upon failure, then the steady state probabilities of the inventory level are uniformly distributed.

Proof:

In this case the semi-Markov kernel $\{Q\ (i\ ,j,\ t),$ i, j $\in$ E, t ≥ 0 $\}$ is given by

$$\begin{aligned} Q(i,j,t) = p &\sum_{k=\delta\,(j-i-1)}^{\infty} \int_0^t \frac{e^{-\lambda u} (\lambda u)^{(i-j+kM)}}{(i-j+kM)!} dG(u) \\ &+ q \sum_{k=\delta\,(j-i)}^{\infty} \int_0^t \frac{e^{-\lambda u} (\lambda u)^{(i-j+kM-1)}}{(i-j+kM-1)!} dG(u) \qquad \text{for i, j} \in \text{E} \end{aligned} \tag{5.17}$$

and the transition probabilities,

$$q_{ij} = Q(i,j,\infty) = p \sum_{k=\delta\,(j-i-1)}^{\infty} \int_0^{\infty} \frac{e^{-\lambda u}(\lambda u)^{(i-j+kM)}}{(i-j+kM)!} dG(u) + q \sum_{k=\delta\,(j-i)}^{\infty} \int_0^{\infty} \frac{e^{-\lambda u}(\lambda u)^{(i-j+kM-1)}}{(i-j+kM-1)!} dG(u) \quad \text{for } i,j \in E \tag{5.18}$$

Since q_{ij} is a function of (i – j),

$$\sum_{i=s+1}^{S} q_{ij} = p \sum_{n=0}^{\infty} \int_0^{\infty} \frac{e^{-\lambda u}(\lambda u)^n}{n!} dG(u) + q \sum_{n=0}^{\infty} \int_0^{\infty} \frac{e^{-\lambda u}(\lambda u)^n}{n!} dG(u) = (p+q)\int_0^{\infty} dG(u) = 1 \quad \text{for } j \in E \tag{5.19}$$

Therefore the transition probability matrix **Q** is doubly stochastic and from the uniqueness of solution it follows that the invariant measure $\pi_j = 1/M$ for $j \in E$. Also note that

$$\sum_{i=s+1}^{S} \int_0^{\infty} k(i,j,du) = \int_0^{\infty} \sum_{n=0}^{\infty} \frac{(\lambda u)^n e^{-\lambda u}}{n!} [1-G(u)]du = \int_0^{\infty} [1-G(u)]du = m \tag{5.20}$$

Therefore from (5.15) we get $p_j = 1/M$ for $j \in E$. Hence the theorem.

5.4.1 Illustrations

As in chapter 4 we shall illustrate the above results by taking the general distribution of disaster periods as gamma distribution with parameters (ν, μ). Then for $i, j \in E$,

$$Q(i,j,t) = p \sum_{k=\delta\,(j-i-1)}^{\infty} \int_0^t \frac{e^{-(\lambda+\mu)u}\ \lambda^{(i-j+kM)}\ \mu^{\nu}\ u^{(i-j+kM+\nu-1)}}{(i-j+kM)!(\nu-1)!} du$$
$$+ q \sum_{k=\delta\,(j-i)}^{\infty} \int_0^t \frac{e^{-(\lambda+\mu)u}\ \lambda^{(i-j+kM)}\ \mu^{\nu}\ u^{(i-j+kM+\nu-2)}}{(i-j+kM-1)!(\nu-1)!} du. \tag{5.21}$$

Therefore

$$q_{ij} = Q(i,j,\infty) = p \sum_{\substack{k=\delta\,(j-i-1) \\ n=(i-j+kM)}}^{\infty} \binom{n+\nu-1}{n} \left(\frac{\mu}{\mu+\lambda}\right)^{\nu} \left(\frac{\lambda}{\mu+\lambda}\right)^{n}$$
$$+ q \sum_{\substack{k=\delta\,(j-i) \\ n=(i-j+kM-1)}}^{\infty} \binom{n+\nu-1}{n} \left(\frac{\mu}{\mu+\lambda}\right)^{\nu} \left(\frac{\lambda}{\mu+\lambda}\right)^{n} \quad \text{for i, j} \in \text{E} \tag{5.22}$$

and

$$\sum_{i=s+1}^{S} q_{ij} = (p+q)\left(\frac{\mu}{\mu+\lambda}\right)^{\nu} \sum_{n=0}^{\infty} \binom{n+\nu-1}{n} \left(\frac{\lambda}{\mu+\lambda}\right)^{n}$$
$$= \left(\frac{\mu}{\mu+\lambda}\right)^{\nu} \left(1-\frac{\lambda}{\mu+\lambda}\right)^{-\nu} = 1 \qquad \text{for j} \in \text{E} \tag{5.23}$$

Since $\overline{G}(t) = e^{-\mu\, t} \sum_{a=0}^{\nu-1} \frac{(\mu t)^a}{(a-1)!}$ (5.24)

$$\int_0^{\infty} k(i,j,t)dt = \sum_{\substack{n=\delta\,(j-i-1) \\ b=(i-j+nM)}}^{\infty} \frac{1}{\mu+\lambda} \sum_{a=0}^{\nu-1} \binom{a+b}{b} \left(\frac{\mu}{\mu+\lambda}\right)^{a} \left(\frac{\lambda}{\mu+\lambda}\right)^{b}; \text{ for i, j} \in \text{E} \tag{5.25}$$

Therefore

$$\sum_{i=s+1}^{S} \int_0^{\infty} k(i,j,t)dt = \frac{1}{\mu+\lambda} \sum_{a=0}^{\nu-1} \left(\frac{\mu}{\mu+\lambda}\right)^{a} \sum_{b=0}^{\infty} \binom{a+b}{b} \left(\frac{\lambda}{\mu+\lambda}\right)^{b}$$
$$= \frac{1}{\mu} \sum_{a=0}^{\nu-1} \left(\frac{\mu}{\mu+\lambda}\right)^{a+1} \left(1-\frac{\lambda}{\mu+\lambda}\right)^{-(a+1)} \tag{5.26}$$
$$= \frac{\nu}{\mu} \qquad \text{for j} \in \text{E}$$

In this case, the expected replenishment cycle time is $M/(\lambda + \mu q/v)$. Therefore, if the fixed ordering cost is K , unit purchase cost of the item is c, the holding cost per unit time is h, and the unit damage cost is d, then the cost function to be minimized is

$$\begin{aligned} C(s,S) &= \frac{(K+cM)}{M/(\lambda+\mu q/v)} + \frac{h}{M}\sum_{i=s+1}^{S} i + d\mu q/v \\ &= [(K/M)+c][\lambda+\mu q/v] + hs + (h/2)(M+1) + d\mu q/v \end{aligned} \tag{5.27}$$

which is minimum for s = 0. Therefore the optimum cost function reduces to

$$C(0,S) = [(K/S)+c][\lambda+\mu q/v] + (h/2)(S+1) + d\mu q/v \tag{5.28}$$

Since

$$\Delta^2 C(0,S) = \frac{2K[\lambda+\mu q/v]}{S(S+1)(S+2)} > 0, \tag{5.29}$$

the cost function in (5.28) is convex. If S* denotes the optimum value of S, then it is given by

$$S^*(S^*-1) < \frac{2K[\lambda+\mu q/v]}{h} < S^*(S^*+1) \tag{5.30}$$

The following three tables show that there is increase in the optimum values of S with the increase of the values of λ, μ and q. In all the tables K = 200, h= 2.5 and v = 3. Figure 5.1 depicts the effect of disaster on the cost function.

Table 5.1
(Optimum values of S for λ = 1)

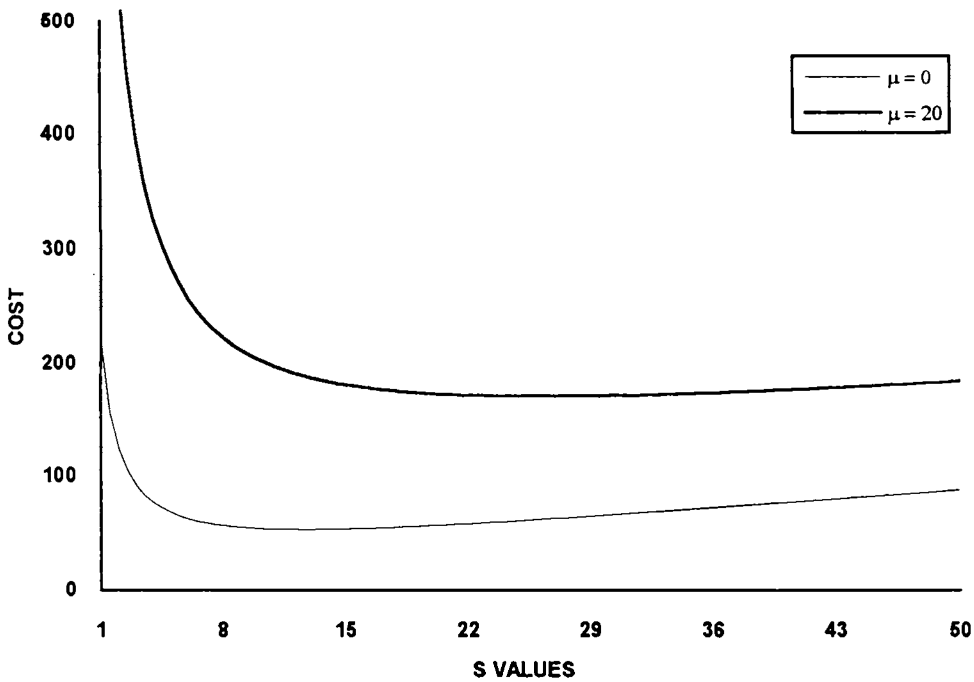
Table 5.2
(Optimum values of S for $\lambda = 6$)

$\mu\rightarrow$ / $q\downarrow$	0	5	10	15	20	25	30	35	40	45	50
.1	31	31	32	32	33	33	33	34	34	35	35
.2	31	32	33	33	34	35	36	37	37	38	39
.3	31	32	33	35	36	37	38	39	40	41	42
.4	31	33	34	36	37	39	40	41	43	44	45
.5	31	33	35	37	39	40	42	44	45	46	48
.6	31	33	36	38	40	42	44	46	47	49	51
.7	31	34	37	39	41	44	46	48	50	51	53

Table 5.3
(Optimum values of S for $\lambda = 11$)

$\mu\rightarrow$ / $q\downarrow$	0	5	10	15	20	25	30	35	40	45	50
.1	42	42	43	43	43	44	44	44	44	45	45
.2	42	43	43	44	44	45	46	46	47	47	48
.3	42	43	44	45	46	46	47	48	49	50	51
.4	42	43	44	46	47	48	49	50	51	52	53
.5	42	44	45	46	48	49	51	52	53	54	56
.6	42	44	46	47	49	51	52	54	55	57	58
.7	42	44	46	48	50	52	54	55	57	59	60

Figure 5.1
(The effect of disaster on the cost function)
$K = 200$, $c = 20$, $h = 2.5$, $d = 5$, $\lambda = 1$, $\nu = 3$, $q = 0.5$.



Chapter VI

Multi-Commodity Inventory Problem Perishable due to Decay and Disaster*

6.1 INTRODUCTION

In this chapter an attempt is made to study a continuous review multi-commodity perishable inventory system. The n commodities, $C_1, C_2, \ldots\ldots C_n$, are diminished from the inventory due to demands, decay and disaster. The maximum inventory level and the re-ordering point of commodity C_k are S_k and s_k respectively, ($k = 1, 2, \ldots\ldots n$). Shortages are not allowed and the lead time is assumed to be zero. Fresh orders are placed whenever the inventory level of at least one of the commodities falls to or below the re-ordering point for the first time after the previous replenishment. Demands for commodity C_k are assumed to follow Poisson process with rate λ_k. The life times of commodity C_k follow exponential distribution with parameter ω_k. The distribution of the times between the disasters is exponential with mean $1/\mu$. Each unit of commodity C_k survives a disaster with probability p_k and is destroyed completely with probability $1-p_k$ independently of others. The damaged items are removed from the inventory instantaneously.

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* The results of this chapter have been presented in the International Conference on Stochastic Processes held at Cochin (1996).

This chapter generalizes the results of chapter II to multi-commodity case. The objectives of this chapter are to find transient and stationary probabilities of the inventory states and the optimum value of the 2n-tuple, $(s_1,s_2,....s_n \quad S_1,S_2,....S_n)$ at steady state. The scheme of presentation of the chapter is as follows: In section 6.2 the notations used are explained while in section 6.3 the transient solution is arrived at. The stationary probabilities and the expected length of the replenishment periods are derived in section 6.4. Section 6.5 discusses optimization where as section 6.6 illustrates the model with numerical examples.

6.2 NOTATIONS

S_k	:Maximum inventory level of commodity $C_k(k = 1, 2,......, n)$
s_k	:Re-ordering level of commodity C_k $(k =1, 2,......, n)$
M_k	:S_k-s_k
M	:$M_1\times M_2\times....\times M_n$
q_k	:$1-p_k$
R_+	:The set of non-negative real numbers
N^0	:The set of non-negative integers
E_k	:$\{s_k+1, s_k+2,.......,S_k\}$
$E_{1\,0}$	:$\{s_1, s_1+1,.......,S_1\}$
E	:$E_1\times E_2\times.............\times E_n$
E_0	:$E_{1\,0}\times E_2\times.............\times E_n$
i*	: $i_n+(i_{n-1}-1)M_n+(i_{n-2}-1)M_{n-1}M_n+......+(i_1-1)M_2M_3....M_n$
s*	: $s_n+1+s_{n-1}M_n+s_{n-2}M_{n-1}M_n+....+s_1M_2...M_n$
s_1*	: $s_n+1+s_{n-1}M_n+s_{n-2}M_{n-1}M_n+....+(s_1-1)M_2...M_n$
S*	: $S_n+(S_{n-1}-1)M_n+(S_{n-2}-1)M_{n-1}M_n+....+(S_1-1)M_2...M_n$

E* :(s*, s*+1,............, S*)

α :(0,0,.........1); M components

e :$(1,1,.........1)^T$; M components

A :$(a_{i_1 i_2 ......i_n})_{M\times M}$ where $a_{i_1 i_2 ......i_n}$'s are given by (6.4)

A_{j_k} : $\sum_{r=0}^{j_k - s_k - 1} \binom{j_k}{r} p_k^{j_k - r} q_k^r$ $j_k = s_k+1, s_k+2, ...S_k$; k= 1, 2,......,n.

D_{i*} :The determinant of the submatrix obtained from A by deleting the first i*-s*+1 rows, the last and first i*-s* columns; i* ∈ E*−{S*}

D_{S*} :1

δ(i, j) :1 if i = j; 0 otherwise

6.3. ANALYSIS OF THE INVENTORY STATES

Let $X_k(t)$ denote the inventory level of commodity C_k (k = 1,2,.....n) at any time t ≥ 0. If $X(t) = \{X_1(t), X_2(t),......X_n(t)\}$, then $\{X(t),\ t\in R_+\}$ is a continuous time Markov chain with state space E. We assume that the initial probability vector of this chain is α.

Let the transition probability matrix of the Markov chain $\{X(t)\}$ be

$$\mathbf{P}(t) = [P_{i_1 i_2 ......i_n \quad j_1 j_2 ......j_n}(t)]_{M\times M}$$

where

$$P_{i_1 i_2 ....i_n \quad j_1 j_2 ....j_n}(t) = \Pr\{X_1(t) = j_1, \ldots, X_n(t) = j_n \ / \ X_1(0) = i_1, \ldots, X_n(0) = i_n\} \tag{6.1}$$

$$i_k, j_k \in E_k; \quad k = 1,2,.....n$$

Theorem 6.1

The transition probability matrix **P**(t) is uniquely determined by

$$\mathbf{P}(t) = \exp(\mathbf{B}t) = \mathbf{I} + \sum_{m=1}^{\infty} \frac{\mathbf{B}^m \ t^m}{m!} \tag{6.2}$$

where the matrix $\mathbf{B} = \mathbf{A} + \mathbf{G}$, in which $\mathbf{A}$ and $\mathbf{G}$ are defined as follows:

$$\mathbf{A} = \left[a_{i_1 i_2 \ldots\ldots i_n \quad j_1 j_2 \ldots\ldots j_n}\right]_{M\times M} \qquad \text{and} \quad \mathbf{G} = \left[g_{i_1 i_2 \ldots\ldots i_n \quad j_1 j_2 \ldots\ldots j_n}\right]_{M\times M} \tag{6.3}$$

with

$$a_{i_1 i_2 \ldots i_n \quad j_1 j_2 \ldots j_n} = \begin{cases} -\left[\mu + \sum_{k=1}^{n}(\lambda_k + i_k\omega_k)\right] + \mu\, p_1^{i_1} p_2^{i_2} \ldots\ldots p_k^{i_k} & \text{if} \quad i_k = j_k \quad k = 1,2,\ldots n \\ (\lambda_k + i_k\omega_k) + \mu \binom{i_k}{j_k} p_k^{j_k} q_k^{i_k - j_k} \prod_{\substack{l=1 \\ l\neq k}}^{n} p_l^{i_l} & \text{if } i_k = j_k + 1; i_l = j_l (l = 1,2,\ldots n; l \neq k) \\ \mu\left\{\prod_{k=1}^{n}\binom{i_k}{j_k} p_k^{j_k} q_k^{i_k - j_k}\right\} & \text{if} \quad \sum_{\substack{k=1 \\ (i_k - j_k)\geq 0}}^{n} (i_k - j_k) > 1 \\ 0 & \text{otherwise} \end{cases} \tag{6.4}$$

and

$$g_{i_1 i_2 \ldots i_n \quad j_1 j_2 \ldots j_n} = \begin{cases} \sum_{k=1}^{n} \delta(s_k + 1,\ i_k)[\lambda_k + (s_k + 1)\omega_k] + \mu\,(1 - A_{i_1} A_{i_2} \ldots A_{i_n}) & \text{if} \quad j_k = S_k \\ & \text{for every} \quad \mathrm{k} \\ 0 & \text{otherwise} \end{cases} \tag{6.5}$$

Proof:

For a fixed $i_0 = (i_1, i_2, \ldots\ldots, i_n)$ the difference-differential equations satisfied by the transition probabilities are:

$$P'_{i_0 \quad j_1 j_2 \ldots\ldots j_n}(t) = -[\mu + \sum_{k=1}^{n}(\lambda_k + j_k\omega_k)]P_{i_0 \quad j_1 j_2 \ldots\ldots j_n}(t)$$

$$+ \sum_{k=1}^{n}(\lambda_k + \overline{j_k + 1}\omega_k)[1-\delta(S_k, j_k)]P_{i_0 \quad j_1 j_2 \ldots \overline{j_k+1} \ldots j_n}(t) \tag{6.6}$$

$$+ \mu\left\{\sum_{l_1=0}^{S_1-j_1}\sum_{l_2=0}^{S_2-j_2}\cdots\cdots\sum_{l_n=0}^{S_n-j_n}\left[\prod_{k=1}^{n}\binom{j_k+l_k}{j_k}p_k^{j_k}q_k^{l_k}\right]P_{i_0 \quad j_1+l_1\, j_2+l_2 \ldots\ldots j_n+l_n}(t)\right\}$$

$$(j_1, j_2, \ldots\ldots j_n) \in E - \{(S_1, S_2, \ldots, S_n)\}$$

$$P'_{i_0 \quad S_1 S_2 \ldots\ldots S_n}(t) = -[\mu + \sum_{k=1}^{n}(\lambda_k + S_k\omega_k)]P_{i_0 \quad S_1 S_2 \ldots\ldots S_n}(t)$$

$$+ \sum_{\substack{k=1 \\ r_i = i \quad if \quad i<k \\ r_i = i+1 \quad if \quad i \geq k}}^{n} (\lambda_k + \overline{s_k + 1}\omega_k) \times$$

$$\sum_{j_{r_1}=s_{r_1}+1}^{S_{r_1}}\sum_{j_{r_2}=s_{r_2}+1}^{S_{r_2}}\cdots\cdots\sum_{j_{r_{n-1}}=s_{r_{n-1}}+1}^{S_{r_{n-1}}} P_{i_0 \quad j_1 j_2 \ldots\ldots j_n}(t)_{(j_k = s_k+1)} \tag{6.7}$$

$$+ \mu\left\{p_1^{S_1}p_2^{S_2}\cdots\cdots p_n^{S_n}P_{i_o \quad S_1 S_2 \ldots\ldots S_n}(t)\right.$$

$$\left. + \sum_{j_1=s_1+1}^{S_1}\sum_{j_2=s_2+1}^{S_2}\cdots\cdots\sum_{j_{r_{n-1}}=s_1+1}^{S_n} P_{i_o \quad j_1 j_2 \ldots\ldots j_n}(t)(1 - A_{j_1}A_{j_2}\cdots A_{j_n})\right\}$$

From equations (6.3) - (6.7) we can easily see that the Kolmogorov equations,

$$\mathbf{P}'(\mathrm{t}) = \mathbf{P}(\mathrm{t})\mathbf{B} \text{ and } \mathbf{P}'(\mathrm{t}) = \mathbf{B}\mathbf{P}(\mathrm{t}) \tag{6.8}$$

with the initial condition,

$$\mathbf{P}(0) = \mathbf{I} \tag{6.9}$$

are satisfied by **P**(t). The solution of (6.8) with (6.9) is (6.2). The finiteness of **B** guarantees the convergence of the series in (6.2) and the solution obtained is unique. Hence the thereom.

6.4. STEADY STATE PROBABILITIES AND REPLISHMENT PERIODS

Since the transition from any state (i_1, i_2,....., i_n) to any state (j_1, j_2,....., j_n) in E is possible with positive probability the Markov chain {X(t), t ≥ 0} is irreducible. Therefore

$$\lim_{t\to\infty} P_{i_1 i_2 \dots i_n \quad j_1 j_2 \dots j_n}(t) = \pi_{j_1 j_2 \dots j_n} \qquad (j_1, j_2, \dots\dots, j_n) \in E \qquad (6.10)$$

exist. $\pi_{j_1 j_2 \dots j_n}$'s are obtained by solving

$$\mathbf{\Pi B = 0} \qquad \text{and} \qquad \mathbf{\Pi e = 1} \qquad (6.11)$$

simultaneously. To solve (8) we define a function f from E to E^* as

$$f((i_1, i_2, \dots\dots, i_n)) = i^* = i_n + (i_{n-1} - 1)M_n + (i_{n-2} - 1)M_{n-1}M_n + \dots\dots + (i_1 - 1)M_2 M_3 \dots\dots M_n \;; \quad (i_1, i_2, \dots, i_n) \in E; \quad i^* \in E^* \qquad (6.12)$$

Since f is one-one and onto , henceforth $(i_1, i_2, \dots\dots, i_n)$ will be represented by i^*.

Theorem 6.2

The steady state probabilities of the inventory states are given by,

$$\pi_{i_1 i_2 \dots\dots i_n} = \frac{D_{i^*}}{F(s^*, S^*) \prod_{k^*=i^*}^{S^*} (-a_{k^*,k^*})}; \qquad i^* \in E^* \qquad (6.13)$$

where $$F(s^*, S^*) = \sum_{i=s^*}^{S^*} \frac{D_{i^*}}{\prod_{k^*=i^*}^{S^*} (-a_{k^*,k^*})} \qquad (6.14)$$

Proof:

Let D_{i^*} be the determinant of the submatrix obtained from **A** by deleting the first i∗ − s∗ + 1 rows , the last and first i∗ − s∗ columns, i∗ ∈ E∗ − {S∗}, and D_{S^*} = 1. With these notations we can see that the solution of (6.11) is

$$\pi_{i_1 i_2 \ldots\ldots i_n} = \pi_{i^*} = \frac{D_{i^*}\,\pi_{S^*}}{\prod_{k^*=i^*}^{S^*-1}(-a_{k^*,k^*})} \qquad i^* \in E^* - \{S^*\} \qquad (6.15)$$

and

$$\pi_{S_1 S_2 \ldots\ldots S_n} = \pi_{S^*} = \frac{1}{-a_{S^*,S^*}\,F(s^*, S^*)} \qquad (6.16)$$

Substituting (6.16) in (6.15) we get (6.13). Hence the theorem.

Let $T_0 = 0 < T_1 < T_2 < \ldots\ldots$ be the epochs when the orders are placed. This occurs whenever the inventory level of one of the commodities C_k falls to s_k or below it for the first time after the previous replenishment ($k = 1,2,\ldots\ldots n$). Since lead time is assumed to be zero, the stock level is immediately brought to $(S_1, S_2, \ldots\ldots, S_n)$. Thus clearly $\{T_m, m \in N^0\}$ is a renewal process.

Theorem 6.3

If $E(T)$ represents the expected time between two successive re-orders, then

$$E(T) = F(s^*, S^*) = \frac{1}{-a_{S^*S^*}\,\pi_{S^*}} \qquad (6.17)$$

Proof:

By a similar argument as in section 4 of chapter 2 the probability distribution of the replenishment cycles can be proved as phase type on $[0, \infty)$ and is given by

$$G(t) = 1 - \alpha \exp(\mathbf{A}t)\mathbf{e} \quad \text{for } t \geq 0 \qquad (6.18)$$

Therefore

$$E(T) = \int_0^\infty \alpha \exp(\mathbf{A}t)\mathbf{e}\, dt \qquad (6.19)$$

$$= -\alpha\, \mathbf{A}^{-1}\mathbf{e} \qquad (6.20)$$

$$= \sum_{i=s^*}^{S^*} \frac{D_{i^*}}{\prod_{k^*=i^*}^{S^*} (-a_{k^*,k^*})} \tag{6.21}$$

$$= F(s^*, S^*).$$

From (6.16), the theorem follows.

6.5 OPTIMIZATION PROBLEM

Let M_k^* represent the random variable of the re-ordering quantity of commodity C_k, then

$$E(M_k^*)$$

$$= E(T)\left[\lambda_k + \sum_{i_1=s_1+1}^{S_1} \sum_{i_2=s_2+1}^{S_2} \cdots\cdots \sum_{i_n=s_n+1}^{S_n} \pi_{i_1 i_2 \ldots i_n} \left\{ i_k \omega_k + \mu \sum_{j_k=0}^{i_k} j_k \binom{i_k}{j_k} p^{i_k - j_k} q^{j_k} \right\}\right]$$

$$= E(T)\left[\lambda_k + (\omega_k + q_k \mu) H_k(s^*, S^*)\right] \tag{6.22}$$

where $$H_k(s^*, S^*) = \sum_{i_1=s_1+1}^{S_1} \sum_{i_2=s_2+1}^{S_2} \cdots\cdots \sum_{i_n=s_n+1}^{S_n} i_k \pi_{i_1 i_2 \ldots i_n}$$

Let h_k be the unit holding cost per unit time, c_k the unit procurement cost and d_k the unit damage cost of commodity C_k (k = 1, 2,....., n). Assume that the fixed ordering cost for placing an order is K irrespective of the number of different items ordered for replenishment. Therefore the cost function is

$$C(s_1, s_2, \ldots, s_n \quad S_1, S_2, \ldots, S_n) = \frac{K + \sum_{k=1}^{n} c_k E(M_k^*)}{E(T)}$$

$$+ \sum_{k=1}^{n} \left[h_k + d_k (\omega_k + \mu q_k)\right] H_k(s^*, S^*) \tag{6.23}$$

$$= \frac{K}{F(s^*, S^*)} + \sum_{k=1}^{n} \left[c_k \lambda_k + \left\{(c_k + d_k)(\omega_k + \mu q_k) + h_k\right\} H_k(s^*, S^*)\right]$$

Since shortages are not allowed and lead time is assumed to be zero it is reasonable to expect that $s_k = 0$, $(k = 1, 2, \ldots, n)$ for the optimum cost function.

Theorem 6.4

The cost function $C(s_1, s_2, \ldots s_n \; S_1, S_2, \ldots S_n)$ is minimum for $s_1 = s_2 = \ldots = s_n = 0$

Proof:

Suppose $s_1 > 0$. Let $s_1^* = 1 + s_n + s_{n-1}M_n + s_{n-2}M_{n-1}M_n + \ldots + (s_1 - 1)M_2 \ldots M_n$

Consider the matrix $\widetilde{A} = (\widetilde{a}_{i_1 i_2 \ldots i_n \; j_1 j_2 \ldots j_n})$, $(i_1, i_2, \ldots i_n)$, $(j_1, j_2, \ldots j_n) \in E_0$ where

$$\widetilde{a}_{i_1 i_2 \ldots i_n \; j_1 j_2 \ldots j_n} = \begin{cases} -\left[\mu + \sum_{k=1}^{n}(\lambda_k + i_k\omega_k)\right] + \mu\, p_1^{i_1} p_2^{i_2} \ldots\ldots p_k^{i_k} & \text{if } \; i_k = j_k \quad k = 1, 2, \ldots n \\ (\lambda_k + i_k\omega_k) + \mu \binom{i_k}{j_k} p_k^{j_k} q_k^{i_k - j_k} \prod\limits_{\substack{l=1 \\ l \neq k}}^{n} p_l^{i_l} & \text{if } \; i_k = j_k + 1; i_l = j_l (l = 1, 2, \ldots n; l \neq k) \\ \mu \left\{ \prod\limits_{k=1}^{n} \binom{i_k}{j_k} p_k^{j_k} q_k^{i_k - j_k} \right\} & \text{if } \; \sum\limits_{\substack{k=1 \\ (i_k - j_k) \geq 0}}^{n} (i_k - j_k) > 1 \\ 0 & \text{otherwise} \end{cases} \tag{6.24}$$

Let $\widetilde{D}_{i^*}$ be the determinant of the submatrix obtained from $\widetilde{A}$ by deleting the first $i^* - s_1^* + 1$ rows, the last and first $i^* - s_1^*$ columns $(i^* \neq S^*)$, $\widetilde{D}_{S^*} = 1$. Then $\widetilde{D}_{i^*} = D_{i^*}$ for $i^* \in E^*$ and $\widetilde{D}_{i^*}$ is positive for every i^*.

From (6.15) we get

$$
\begin{aligned}
F(s_1-1,s_2,\dots,s_n \quad S_1,S_2,\dots\dots,S_n) = F(s_1^*,S^*) \\
&= \sum_{i^*=s_1^*}^{S^*} \frac{\widetilde{D}_{i^*}}{\prod_{k^*=i^*}^{S^*}\left(-a_{k^*,k^*}\right)} \\
&= \sum_{i^*=s^*}^{S^*} \frac{\widetilde{D}_{i^*}}{\prod_{k^*=i^*}^{S^*}\left(-a_{k^*,k^*}\right)} + \sum_{i^*=s_1^*}^{s^*-1} \frac{\widetilde{D}_{i^*}}{\prod_{k^*=i^*}^{S^*}\left(-a_{k^*,k^*}\right)} \\
&= \sum_{i^*=s^*}^{S^*} \frac{D_{i^*}}{\prod_{k^*=i^*}^{S^*}\left(-a_{k^*,k^*}\right)} + \sum_{i^*=s_1^*}^{s^*-1} \frac{\widetilde{D}_{i^*}}{\prod_{k^*=i^*}^{S^*}\left(-a_{k^*,k^*}\right)} \\
&> F(s^*,S^*) \qquad (6.25)
\end{aligned}
$$

Also

$$
\begin{aligned}
H_1(s_1-1,s_2,\dots s_n \quad S_1,S_2\dots S_n) &= \sum_{i_1=s_1}^{S_1} \sum_{i_2=s_2+1}^{S_2}\dots\dots \sum_{i_n=s_n+1}^{S_n} \frac{(i_1-s_1+s_1)\widetilde{D}_{i^*}}{F(s_1^*,S^*)\prod_{k^*=i^*}^{S^*}\left(-\tilde{a}_{k^*,k^*}\right)} \\
&= s_1 + \sum_{i_1=s_1+1}^{S_1} \sum_{i_2=s_2+1}^{S_2}\dots\dots \sum_{i_n=s_n+1}^{S_n} \frac{(i_1-s_1)D_{i^*}}{F(s_1^*,S^*)\prod_{k^*=i^*}^{S^*}\left(-a_{k^*,k^*}\right)} \\
&< s_1 + \sum_{i^*=s^*}^{S^*} \frac{(i_1-s_1)D_{i^*}}{F(s^*,S^*)\prod_{k^*=i^*}^{S^*}\left(-a_{k^*,k^*}\right)} \qquad \text{by (6.25)} \\
&= H_1(s_1,s_2,\dots s_n \quad S_1,S_2\dots S_n) \qquad (6.26)
\end{aligned}
$$

Thus, from (6.24) – (6.26) we have

$$C(s_1-1,s_2,\dots,s_n \quad S_1,S_2,\dots,S_n) < C(s_1,s_2,\dots,s_n \quad S_1,S_2,\dots\dots,S_n)$$

Therefore,

$$C(0,s_2,\dots,s_n \quad S_1,S_2,\dots,S_n) < C(s_1,s_2,\dots,s_n \quad S_1,S_2,\dots\dots,S_n).$$

If $s_k > 0$ for $k > 1$, then interchanging the position of s_1 and s_k we can similarly prove that the cost function is minimum for $s_k = 0$ for each k. Hence the theorem.

Let $F(0, S^*) = \Phi(S^*)$ and $H_k(0, S^*) = \psi_k(S^*)$

Then (6.24) will become

$$C(0,0,\ldots,0 \quad S_1,S_2,\ldots,S_n) = \frac{K}{\Phi(S^*)} + \sum_{k=1}^{n}\left[c_k\lambda_k + \{(c_k + d_k)(\omega_k + \mu q_k) + h_k\}\Psi_k(S^*)\right] \tag{6.27}$$

6.6 NUMERICAL ILLUSTRATIONS

In this section we provide some numerical examples. Table 6.1 gives the optimum (S_1, S_2, S_3) values of a three commodity problem when $\mu = 5$. Figure 6.1 depicts that optimum values of S_1 and S_2 decrease with the increase of value of μ. The last three tables compare the optimum (S_1, S_2) values of a two commodity inventory problem when the disaster rates are $\mu = 10, 5, 1$ respectively.

Table 6.1
(Optimum values of (S_1, S_2, S_3)

$\mu = 5$, $p_1 = .1$, $p_2 = .2$, $p_3 = .3$, $K = 100$, $c_1 = 25$, $c_2 = 10$, $c_3 = 30$,
$h_1 = 5$, $h_2 = 2$, $h_3 = 6$, $d_1 = 5/3$, $d_2 = 2/3$, $d_2 = 2$.

$(\lambda_1,\lambda_2,\lambda_3)\rightarrow$ $(\omega_1,\omega_2,\omega_3)\downarrow$	(1,1,1)	(1,1,4)	(1,2,4)	(3,1,1)	(3,1,4)	(3,2,4)
(0,0,0)	(1,2,1)	(1,2,2)	(1,2,2)	(2,2,1)	(2,2,2)	(2,2,2)
(0,3,5)	(1,3,2)	(1,3,3)	(1,2,4)	(2,3,2)	(2,3,3)	(2,3,3)
(2,0,5)	(2,2,2)	(2,2,3)	(2,2,3)	(2,2,2)	(2,2,3)	(2,2,2)
(2,3,0)	(2,3,1)	(2,3,2)	(2,3,2)	(2,3,1)	(2,3,2)	(2,3,2)
(2,3,5)	(2,3,2)	(2,3,3)	(2,3,3)	(2,3,2)	(2,3,2)	(2,3,2)

Figure 6.1
(Optimum values of (S_1, S_2)

$p_1 = .3$, $p_2 = .1$, $K = 100$, $c_1 = 20$, $c_2 = 10$, $h_1 = 4$, $h_2 = 2$,
$d_1 = 4/3$, $d_2 = 2/3$, $\lambda_1 = 2$, $\lambda_2 = 1$, $\omega_1 = 0$, $\omega_2 = 0$.

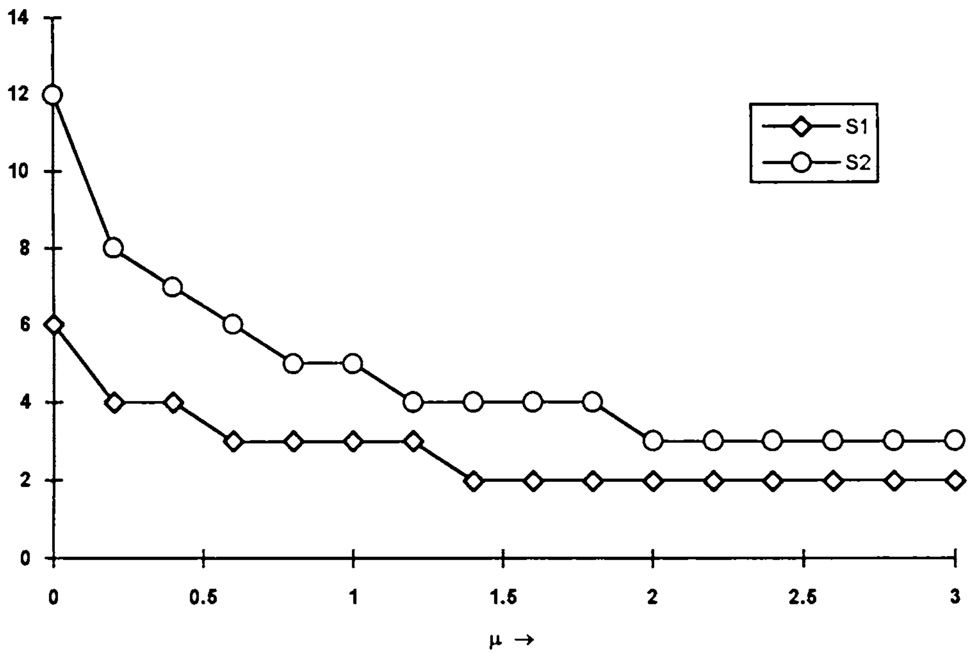


Table 6.2
(Optimum values of (S_1, S_2 when $\mu = 10$)

$p_1 = .3$, $p_2 = .1$, $K = 100$, $c_1 = 20$, $c_2 = 10$, $h_1 = 4$, $h_2 = 2$, $d_1 = 4/3$, $d_2 = 2/3$

$(\lambda_1,\lambda_2)\rightarrow$ $(\omega_1,\omega_2)\downarrow$	(1,1)	(1,2)	(1,3)	(2,1)	(2,2)	(2,3)	(3,1)	(3,2)	(3,3)
(0,0)	(1,2)	(1,2)	(1,2)	(2,2)	(2,2)	(2,2)	(2,2)	(2,2)	(2,2)
(0,1)	(1,2)	(1,2)	(1,2)	(2,2)	(2,2)	(2,3)	(2,2)	(2,2)	(2,3)
(1,0)	(2,2)	(2,2)	(2,2)	(2,2)	(2,2)	(2,2)	(2,2)	(2,2)	(2,2)
(1,1)	(2,2)	(2,2)	(2,3)	(2,2)	(2,2)	(2,3)	(2,2)	(2,2)	(2,2)

Table 6.3
(Optimum values of (S_1, S_2 when $\mu = 5$)

p_1=.3, p_2 =.1, K=100, c_1=20, c_2=10, $h_1 = 4$, $h_2 = 2$, $d_1 = 4/3$, $d_2 = 2/3$

$(\lambda_1,\lambda_2)\rightarrow$ $(\omega_1,\omega_2)\downarrow$	(1,1)	(1,2)	(1,3)	(2,1)	(2,2)	(2,3)	(3,1)	(3,2)	(3,3)
(0,0)	(2,2)	(2,3)	(2,3)	(2,2)	(2,2)	(2,3)	(2,2)	(2,2)	(2,3)
(0,1)	(2,3)	(2,3)	(2,3)	(2,2)	(2,3)	(2,3)	(2,2)	(2,3)	(2,3)
(1,0)	(2,2)	(2,2)	(2,3)	(2,2)	(2,2)	(2,3)	(3,2)	(3,2)	(3,3)
(1,1)	(2,2)	(2,3)	(2,3)	(2,2)	(2,3)	(2,3)	(3,2)	(3,3)	(3,3)

Table 6.4
(Optimum values of (S_1, S_2 when $\mu = 1$)

p_1=.3, p_2 =.1, K=100, c_1=20, c_2=10, $h_1 = 4$, $h_2 = 2$, $d_1 = 4/3$, $d_2 = 2/3$

$(\lambda_1,\lambda_2)\rightarrow$ $(\omega_1,\omega_2)\downarrow$	(1,1)	(1,2)	(1,3)	(2,1)	(2,2)	(2,3)	(3,1)	(3,2)	(3,3)
(0,0)	(3,3)	(3,5)	(3,6)	(4,3)	(4,5)	(4,6)	(5,3)	(5,4)	(4,5)
(0,1)	(3,4)	(3,5)	(2,6)	(4,4)	(3,5)	(3,6)	(4,4)	(4,5)	(4,6)
(1,0)	(3,3)	(3,4)	(3,5)	(4,3)	(4,4)	(4,5)	(5,3)	(5,4)	(4,5)
(1,1)	(3,4)	(3,5)	(3,6)	(4,4)	(4,5)	(4,6)	(4,4)	(4,4)	(4,5)

Chapter VII

Multi-Commodity Perishable Inventory Problem with Shortages

7.1 INTRODUCTION

A continuous review (s, S) multi-commodity inventory system perishable due to decay and disaster allowing shortages is studied in this chapter. The n commodities are denoted by $C_1, C_2, \ldots\ldots, C_n$. The maximum inventory level and the re-ordering point of commodity C_k are S_k and s_k respectively, $(k = 1,2,\ldots n)$. Lead time is assumed to be zero and the sales are considered as lost during stock out period. Fresh orders are placed whenever the inventory levels of all the commodities fall to or below their re-ordering points after the previous replenishment. Demands for commodity C_k are assumed to follow Poisson process with rate λ_k and the life times of commodity C_k follow exponential distribution with parameter ω_k. The distribution of the times between the disasters is exponential with mean $1/\mu$. Each unit of commodity C_k, independent of others, survives a disaster with probability p_k or is destroyed completely with probability $1-p_k$. The damaged items are disposed off from the inventory immediately.

The objectives of this chapter are to find transient and stationary probabilities of the inventory states and the optimal value of the $2n$-tuple,

($s_1,s_2,....,s_n$ $S_1,S_2,....,S_n$) at steady state. The scheme of presentation is similar to chapter VI. The time dependent solution is arrived at in section 7.3. Section 7.4 deals with the stationary probabilities and the replenishment periods where as section 7.5 discusses optimization problem. Some numerical examples are provided in the last section. The present chapter also generalizes the results of chapter II to multi-commodity case.

7.2 NOTATIONS

S_k	:Maximum inventory level of commodity C_k ($k = 1, 2,......, n$)
s_k	:Re-ordering level of commodity C_k ($k = 1, 2,......, n$)
M_k	:S_k-s_k
M	:$M_1 \times M_2 \times .... \times M_n$
q_k	:$1-p_k$
N^0	:$\{0, 1, 2, ...\}$
R_+	:The set of non-negative real numbers
E_k	:$\{0, 1, 2,......., S_k\}$
$E_{k\,s}$	:$\{0, 1, 2,......., s_k\}$
E	:$(E_1 \times E_2 \times ....\times E_n) - (E_{1\,s} \times E_{2\,s} \times ....\times E_{n\,s})$
Δ_k	:$\{(i_1,i_2,...i_n) \in E \mid i_k = 0\}$
s^*	:$f((s_1, s_2, ...s_n))$; f is defined in (7.12)
S^*	:$(S_1+1)\times(S_2+1)\times.........\times(S_n+1) - (s_1+1)\times(s_2+1)\times.........\times(s_n+1)$
E^*	:$\{1, 2, ..., S^*\}$
α	:$(0, 0,........., 1)$; S^* components
e	:$(1, 1,........., 1)^T$; S^* components
A	:$(a_{i_1 i_2 ......i_n})_{S^* \times S^*}$ $a_{i_1 i_2 ......i_n}$'s are given by (7.4)
$\delta(i, j)$	:1 if $i = j$; 0 otherwise

$$A_{j_k} \quad : \begin{cases} \sum_{r=j_k-s_k}^{j_k} \binom{j_k}{r} p_k^{j_k-r} q_k^r & \text{if} \quad j_k > s_k \\ 1 & \text{if} \quad j_k \le s_k \end{cases} \qquad k = 1, 2, \ldots\ldots, n.$$

D_{i*} :The determinant of the submatrix obtained from **A** by deleting the first i∗ rows , the last and first i∗ – 1 columns, $i* \in E^* - \{S^*\}$

D_{S*} :1

7.3. TRANSIENT PROBABILITIES

Let $X_k\ (t)$ denote the inventory level of commodity C_k (k=1,2,......,n) at any time t ≥0. If $X(t) = \{X_1(t), X_2(t), \ldots\ldots, X_n(t)\}$, then $\{X(t),\ t \in R\}$ is a continuous time Markov chain with state space E. We assume that the initial probability vector of this chain is α.

Let the transition probability matrix of the Markov chain $\{X(t)\}$ be

$$\mathbf{P}(t) = [P_{i_1 i_2 \ldots\ldots i_n \quad j_1 j_2 \ldots\ldots j_n}(t)]_{S^* \times S^*}$$

where

$$P_{i_1 i_2 \ldots i_n \quad j_1 j_2 \ldots j_n}(t) = \Pr\{X_1(t) = j_1, \ldots, X_n(t) = j_n \mid X_1(0) = i_1, \ldots, X_n(0) = i_n\} \tag{7.1}$$

$$(i_1, i_2, \ldots\ldots i_n), (j_1, j_2, \ldots\ldots j_n) \in E$$

Theorem 7.1

The transition probability matrix **P**(t) is uniquely determined by

$$\mathbf{P}(t) = \exp(\mathbf{B}t) = \mathbf{I} + \sum_{m=1}^{\infty} \frac{\mathbf{B}^m\ t^m}{m!} \tag{7.2}$$

where the matrix **B** = **A** + **G** , in which **A** and **G** are defined as follows:

$$\mathbf{A} = \left[a_{i_1 i_2 \ldots i_n \quad j_1 j_2 \ldots j_n}\right]_{S^* \times S^*} \text{ and } \mathbf{G} = \left[g_{i_1 i_2 \ldots i_n \quad j_1 j_2 \ldots j_n}\right]_{S^* \times S^*} \tag{7.3}$$

$$(i_1, i_2, \ldots\ldots, i_n), \quad (j_1, j_2, \ldots\ldots, j_n) \in E$$

with

$$a_{i_1 i_2 \ldots i_n \quad j_1 j_2 \ldots j_n}$$

$$=\begin{cases} -\left[\mu + \sum_{k=1}^{n}[1-\delta(0,i_k)](\lambda_k + i_k\omega_k)\right] + \mu\, p_1^{i_1} p_2^{i_2} \ldots p_k^{i_k} & \text{if } i_k = j_k \ k = 1,2,\ldots,n \\ (\lambda_k + i_k\omega_k) + \mu \binom{i_k}{j_k} p_k^{j_k} q_k^{i_k - j_k} \prod_{\substack{l=1 \\ l\neq k}}^{n} p_l^{i_l} & \text{if } i_k = j_k + 1; i_l = j_l (l = 1,2,\ldots n; l \neq k) \\ \mu\left\{\prod_{k=1}^{n}\binom{i_k}{j_k} p_k^{j_k} q_k^{i_k - j_k}\right\} & \text{if } \sum_{\substack{k=1 \\ (i_k - j_k)\geq 0}}^{n}(i_k - j_k) > 1 \\ 0 & \text{otherwise} \end{cases} \tag{7.4}$$

and $g_{i_1 i_2 \ldots i_n \quad j_1 j_2 \ldots j_n}$

$$=\begin{cases} [\lambda_k + (s_k+1)\omega_k] + \mu\,(A_{i_1} A_{i_2} \ldots A_{i_n}) & \text{if } i_k = s_k + 1, \text{and } i_j \leq s_j \text{ for } j \neq k; \\ & \mathrm{k} = 1,2,\ldots\ldots\mathrm{n};\ (\mathrm{j}_1, \mathrm{j}_2, \ldots, \mathrm{j}_n) = (\mathrm{S}_1, \mathrm{S}_2, \ldots, \mathrm{S}_n) \\ \mu\,(A_{i_1} A_{i_2} \ldots A_{i_n}) \text{ if } & \sum_{\substack{k=1 \\ (i_k - j_k)\geq 0}}^{n}(i_k - j_k) > 1;(\mathrm{j}_1, \mathrm{j}_2, \ldots, \mathrm{j}_n) = (\mathrm{S}_1, \mathrm{S}_2, \ldots, \mathrm{S}_n) \\ 0 \quad \text{if} & (\mathrm{j}_1, \mathrm{j}_2, \ldots, \mathrm{j}_n) \neq (\mathrm{S}_1, \mathrm{S}_2, \ldots, \mathrm{S}_n) \end{cases} \tag{7.5}$$

Proof:

For a fixed $i_0 = (i_1, i_2, \ldots\ldots, i_n)$ the difference-differential equations satisfied by the transition probabilities are the following:

$$\begin{aligned} P'_{i_0 \quad j_1 j_2 \ldots j_n}(t) &= -[\mu + \sum_{k=1}^{n}\{1-\delta(0,j_k)\}(\lambda_k + j_k\omega_k)]P_{i_0 \quad j_1 j_2 \ldots j_n}(t) \\ &+ \sum_{k=1}^{n}[\lambda_k + (j_k+1)\omega_k][1-\delta(S_k, j_k)]P_{i_0 \quad j_1 j_2 \ldots (j_k+1) \ldots j_n}(t) \\ &+ \mu\left\{\sum_{l_1=0}^{S_1 - j_1}\sum_{l_2=0}^{S_2 - j_2} \ldots \sum_{l_n=0}^{S_n - j_n}\left[\prod_{k=1}^{n}\binom{j_k + l_k}{j_k} p_k^{j_k} q_k^{l_k}\right]P_{i_0 \quad j_1 + l_1\ j_2 + l_2 \ldots j_n + l_n}(t)\right\} \\ & \qquad (j_1, j_2, \ldots, j_n) \in E - \{(S_1, S_2, \ldots, S_n)\} \end{aligned} \tag{7.6}$$

$$P'_{i_0\ S_1S_2.....S_n}(t) = -[\mu + \sum_{k=1}^{n}(\lambda_k + S_k\omega_k)]P_{i_0\ S_1S_2.....S_n}(t)$$

$$+ \sum_{\substack{k=1 \\ r_i=i \ \ if \ \ i<k \\ r_i=i+1 \ \ if \ \ i\geq k}}^{n}[\lambda_k + (s_k+1)\omega_k]$$

$$\times \sum_{j_{r_1}=0}^{s_{r_1}} \sum_{j_{r_2}=0}^{s_{r_2}} \cdots\cdots \sum_{j_{r_{n-1}}=0}^{s_{r_{n-1}}} P_{i_0\ j_1j_2.....j_n}(t)_{(j_k=s_k+1)} \tag{7.7}$$

$$+ \mu \Big\{ p_1^{S_1} p_2^{S_2} \ldots\ldots p_n^{S_n} P_{i_o\ S_1S_2.....S_n}(t)$$

$$+ \sum_{(j_1,j_2,....j_n)\in E} P_{i_o\ j_1j_2.....j_n}(t)(A_{j_1}A_{j_2}\ldots A_{j_n}) \Big\}$$

From equations (7.3) - (7.7) we can easily see that the Kolmogorov equations,

$$\mathbf{P}'(t) = \mathbf{P}(t)\mathbf{B} \text{ and } \mathbf{P}'(t) = \mathbf{B}\mathbf{P}(t) \tag{7.8}$$

with the condition,

$$\mathbf{P}(0) = \mathbf{I} \tag{7.9}$$

are satisfied by **P**(t). The solution of (7.8) with (7.9) is (7.2). The finiteness of **B** guarantees the convergence of the series in (6.2) and the solution obtained is unique. Hence the theorem.

7.4. STEADY STATE PROBABILITIES AND REPLENISHMENT CYCLES

Since the transition from any state $(i_1, i_2, ...,i_n)$ to any state $(j_1, j_2, ...,j_n)$ in E is possible with positive probability the Markov chain $\{X(t), t \geq 0\}$ is irreducible. Therefore

$$\lim_{t\to\infty} P_{i_1i_2...i_n\ j_1j_2....j_n}(t) = \pi_{j_1j_2.....j_n} \qquad (j_1, j_2, \ldots\ldots, j_n) \in E \tag{7.10}$$

exist and are independent of the initial state. $\pi_{j_1 j_2 \ldots j_n}$'s are obtained by solving

$$\Pi\mathbf{B} = \mathbf{0} \qquad \text{and} \qquad \Pi\mathbf{e} = \mathbf{1} \tag{7.11}$$

simultaneously.

Define a relation $\leq$ in E as follows:

For $(i_1, i_2, \ldots, i_n), (j_1, j_2, \ldots, j_n) \in E$, $(i_1, i_2, \ldots, i_n) \leq (j_1, j_2, \ldots, j_n)$ if

(1) $i_1 < j_1$

or (2) $i_1 = j_1$; $i_2 < j_2$

or (3) $i_1 = j_1$; $i_2 = j_2$; $i_3 < j_3$

or ....

or (n) $i_1 = j_1$; $i_2 = j_2$; ....; $i_{n-1} = j_{n-1}$; $i_n \leq j_n$

Then clearly $\leq$ is a partial order relation in E. Arrange the elements of E in ascending order. In this arrangement $(0, 0, \ldots, s_n+1)$ will be the first element and $(S_1, S_2, \ldots, S_n)$ will be the S^*th element. Now define a function f from E to E^*. as

$f((i_1, i_2, \ldots, i_n)) = i^*$ if $(i_1, i_2, \ldots, i_n)$ is the i^*th element in the arrangement, $i^* \in E^*$. (7.12)

Since f is a bijection, henceforth $(i_1, i_2, \ldots, i_n)$ will be represented by i^*.

Let D_{i^*} be the determinant of the submatrix obtained from **A** by deleting the first i^* rows , the last and first $i^* - 1$ columns, $i^* \in E^* - \{S^*\}$, and $D_{S^*} = 1$. With these notations we have

Theorem 7.2

The steady state probabilities of the inventory states are given by

$$\pi_{i_1 i_2 \ldots i_n} = \frac{D_{i^*}}{F(s^*, S^*) \prod_{k^*=i^*}^{S^*} (-a_{k^*,k^*})}; \qquad i^* \in E^* \tag{7.13}$$

where $$F(s^*,S^*)=\sum_{i=1}^{S^*}\frac{D_{i^*}}{\prod_{k^*=i^*}^{S^*}(-a_{k^*,k^*})} \tag{7.14}$$

Proof:

As in the previous chapter, we can see that the solution of (7.11) is

$$\pi_{i_1 i_2 \ldots i_n}=\pi_{i^*}=\frac{D_{i^*}\pi_{S^*}}{\prod_{k^*=i^*}^{S^*-1}(-a_{k^*,k^*})} \qquad i^*\in E^*-\{S^*\} \tag{7.15}$$

and $$\pi_{S_1 S_2 \ldots S_n}=\pi_{S^*}=\frac{1}{-a_{S^*,S^*}F(s^*,S^*)} \tag{7.16}$$

Substituting (7.16) in (7.15) we get (7.13). Hence the theorem.

Let $T_0 = 0 < T_1 < T_2 < \ldots\ldots$ be the epochs when the orders are placed. This occurs whenever the inventory levels of all the commodities C_k fall to their reordering levels or below those for the first time after the previous replenishment (k = 1,2,....., n). Since lead time is assumed to be zero, the stock level is immediately brought to $(S_1,S_2,\ldots,S_n)$. Thus clearly $\{T_m, m\in N^0\}$ is a renewal process.

Arguing in the similar lines of Theorem 6.3, the expected replenishment cycle time is obtained by

Theorem 7.3

If $E(T)$ represents the expected time between two successive re-orders, then

$$E(T)=F(s^*,S^*)=\frac{1}{-a_{S^*S^*}\,\pi_{S^*}} \tag{7.17}$$

7.5 COST ANALYSIS

Let M_k* represent the random re-ordering quantity of commodity C_k. Then

$$E(M_k{}^*)$$
$$= E(T)\left[\sum_{(i_1,i_2,\ldots,i_n)\in E-\Delta_k} \pi_{i_1 i_2 \ldots i_n}\left\{\lambda_k + i_k\omega_k + \mu\sum_{j_k=0}^{i_k} j_k\binom{i_k}{j_k}p^{i_k-j_k}q^{j_k}\right\}\right]$$
$$= E(T)\left[\left(\sum_{(i_1,i_2,\ldots,i_n)\in E-\Delta_k}\pi_{i_1 i_2 \ldots i_n}\lambda_k\right) + (\omega_k + q_k\mu)H_k(s^*,S^*)\right] \tag{7.18}$$

where $$H_k(s^*,S^*) = \sum_{(i_1,i_2,\ldots,i_n)\in E-\Delta_k} i_k\pi_{i_1 i_2 \ldots i_n}$$

Let, h_k be the unit holding cost per unit time, c_k the unit procurement cost and d_k the unit damage cost, b_k be the unit shortage cost of commodity C_k (k=1,2,.....n), K be the fixed ordering cost per order. Then the cost function is

$$C(s_1,s_2,\ldots,s_n \quad S_1,S_2,\ldots,S_n) = \frac{K + \sum_{k=1}^{n} c_k E(M_k{}^*)}{E(T)}$$
$$+\sum_{k=1}^{n}\left[h_k + d_k(\omega_k + \mu q_k)\right]H_k(s^*,S^*) + \sum_{k=1}^{n}\sum_{(i_1,i_2,\ldots,i_n)\in\Delta_k}\lambda_k b_k \pi_{i_1 i_2 \ldots i_n}$$
$$= \frac{K}{E(T)} + \sum_{k=1}^{n}\left[c_k\lambda_k\left(1 - \sum_{(i_1,i_2,\ldots,i_n)\in\Delta_k}\pi_{i_1 i_2 \ldots i_n}\right)\right.$$
$$\left. + \left\{(c_k + d_k)(\omega_k + \mu q_k) + h_k\right\}H_k(s^*,S^*) + \sum_{(i_1,i_2,\ldots,i_n)\in\Delta_k}\lambda_k b_k\pi_{i_1 i_2 \ldots i_n}\right]$$
$$= \frac{K}{F(s^*,S^*)} + \sum_{k=1}^{n}\left[c_k\lambda_k + \left\{(c_k + d_k)(\omega_k + \mu q_k) + h_k\right\}H_k(s^*,S^*)\right.$$
$$\left. + \lambda_k(b_k - c_k)\sum_{(i_1,i_2,\ldots,i_n)\in\Delta_k}\pi_{i_1 i_2 \ldots i_n}\right] \tag{7.19}$$

7.6 NUMERICAL ILLUSTRATIONS

In this section we provide some numerical examples for two commodity inventory problems. Numerical examples show that optimal values of $s_k = 0$, ($k = 1, 2, ..., n$) when shortage cost is zero. This can be seen from tables 7.3 and 7.4. Figure 7.1 illusrates the effect of disaster on the optimum values of (s_1, S_1), (s_2, S_2). Tables 7.1 and 7.2 compare the optimum values (s_1, S_1), (s_2, S_2) of a two commodity problem for disaster rates $\mu = 1$ and 0 respectively when the shortage costs are $b_1 = 200$ and $b_2 = 100$. The third and fourth tables compare the same when shortage costs are set at zero. The effect of shortage on the optimum inventory level can be seen by comparing tables 7.1 and 7.3, and tables 7.2 and 7.4. The optimum values are found out with the aid of a computer giving upper bounds to $S_i = 9$ and assigning $s_i = 0, 1; i = 1,2$.

Table 7.1

Optimum values (s_1, S_1), (s_2, S_2) for $\mu = 1$ and $b_1=200$, $b_2=100$, $p_1=.3$, $p_2=.1$, $K=100$, $c_1=20$, $c_2=10$, $h_1 = 4$, $h_2 = 2$, $d_1= 4/3$, $d_2 = 2/3$.

$(\lambda_1,\lambda_2)\rightarrow$ $(\omega_1,\omega_2)\downarrow$	(1,1)	(1,2)	(1,3)	(2,1)	(2,2)	(2,3)
(0,0)	(0,3),(1,3)	(0,3),(0,5)	(1,3),(0,6)	(0,4),(1,2)	(0,4),(1,4)	(0,4),(1,6)
(0,1)	(0,3),(0,3)	(0,2),(0,5)	(1,2),(0,6)	(0,4),(1,3)	(0,4),(1,5)	(0,3),(1,6)
(1,0)	(0,3),(0,3)	(0,3),(0,4)	(1,3),(0,5)	(0,4),(1,2)	(0,4),(1,3)	(0,4),(1,4)
(1,1)	(0,3),(0,3)	(0,3),(0,5)	(0,2),(0,5)	(0,4),(1,2)	(0,4),(1,4)	(0,4),(1,5)

Table 7.2

Optimum values (s_1, S_1), (s_2, S_2) for $\mu = 0$ and $b_1=200$, $b_2=100$, $p_1=.3$, $p_2=.1$, $K=100$, $c_1=20$, $c_2=10$, $h_1 = 4$, $h_2 = 2$, $d_1= 4/3$, $d_2 = 2/3$.

$(\lambda_1,\lambda_2)\rightarrow$ $(\omega_1,\omega_2)\downarrow$	(1,1)	(1,2)	(1,3)	(2,1)	(2,2)	(2,3)
(0,0)	(0,8),(1,8)	(1,6),(1,9)	(1,5),(0,9)	(0,9),(1,5)	(1,9),(1,9)	(1,8),(1,9)
(0,1)	(0,4),(1,6)	(1,4),(0,9)	(1,3),(0,9)	(0,6),(1,6)	(0,6),(1,9)	(1,5),(0,9)
(1,0)	(0,5),(1,3)	(0,5),(1,6)	(1,5),(0,8)	(0,6),(1,2)	(0,7),(1,4)	(0,7),(1,6)
(1,1)	(0,4),(1,4)	(0,4),(0,7)	(1,4),(0,9)	(0,5),(1,3)	(0,5),(1,5)	(0,5),(1,7)

Table 7.3

Optimum values (s_1, S_1), (s_2, S_2) for $\mu = 1$ and $b_1=0$, $b_2=0$, $p_1=.3$, $p_2=.1$, $K=100$, $c_1=20$, $c_2=10$, $h_1 = 4$, $h_2 = 2$, $d_1= 4/3$, $d_2 = 2/3$.

$(\lambda_1,\lambda_2)\rightarrow$ $(\omega_1,\omega_2)\downarrow$	(1,1)	(1,2)	(1,3)	(2,1)	(2,2)	(2,3)
(0,0)	(0,2),(0,3)	(0,2),(0,4)	(0,2),(0,5)	(0,3),(0,3)	(0,3),(0,4)	(0,3),(0,5)
(0,1)	(0,3),(0,2)	(0,3),(0,3)	(0,2),(0,4)	(0,4),(0,2)	(0,3),(0,3)	(0,3),(0,4)
(1,0)	(0,2),(0,3)	(0,2),(0,4)	(0,2),(0,6)	(0,2),(0,3)	(0,2),(0,4)	(0,2),(0,5)
(1,1)	(0,2),(0,3)	(0,2),(0,4)	(0,2),(0,5)	(0,2),(0,3)	(0,2),(0,4)	(0,2),(0,5)

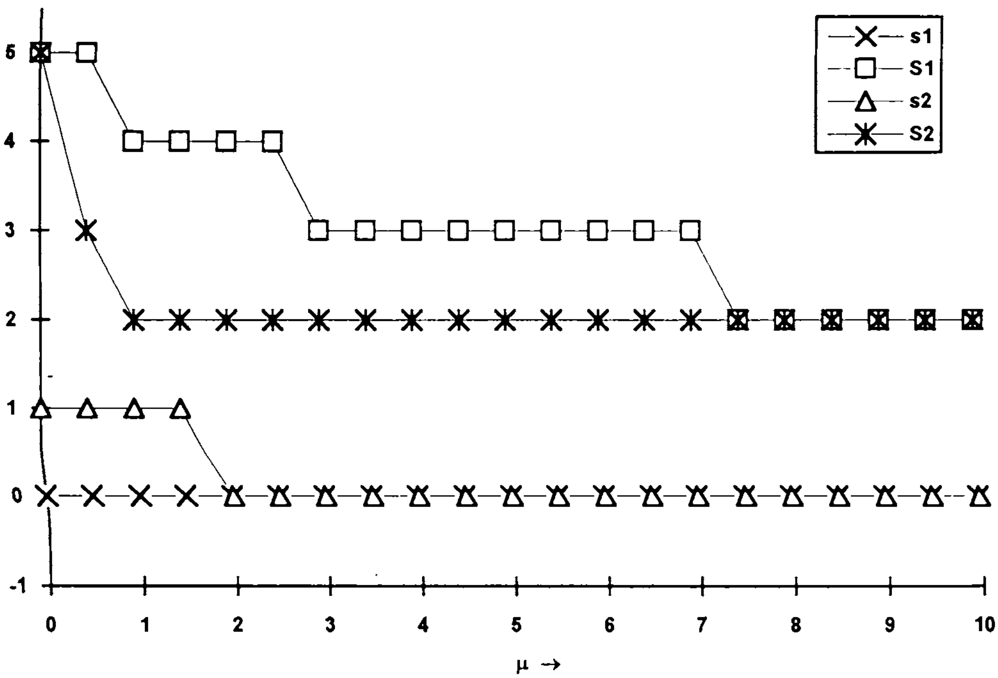
Table 7.4

Optimum values (s_1, S_1), (s_2, S_2) for $\mu = 0$ and $b_1=0$, $b_2=0$, $p_1=.3$, $p_2=.1$, $K=100$, $c_1=20$, $c_2=10$, $h_1 = 4$, $h_2 = 2$, $d_1= 4/3$, $d_2 = 2/3$.

$(\lambda_1,\lambda_2)\rightarrow$ $(\omega_1,\omega_2)\downarrow$	(1,1)	(1,2)	(1,3)	(2,1)	(2,2)	(2,3)
(0,0)	(0,5),(0,7)	(0,4),(0,9)	(0,4),(0,9)	(0,7),(0,6)	(0,7),(0,9)	(0,6),(0,9)
(0,1)	(0,4),(0,4)	(0,4),(0,6)	(0,4),(0,7)	(0,7),(0,4)	(0,6),(0,6)	(0,5),(0,7)
(1,0)	(0,2),(0,6)	(0,2),(0,8)	(0,2),(0,9)	(0,3),(0,5)	(0,3),(0,7)	(0,3),(0,8)

Figure 7.1
(Optimum values of $(s_1, S_1), (s_2, S_2)$)

$p_1 = .3$, $p_2 = .1$, $K = 100$, $c_1 = 20$, $c_2 = 10$, $h_1 = 4$, $h_2 = 2$, $d_1 = 4/3$, $d_2 = 2/3$,
$b_1 = 200$, $b_2 = 100$, $\lambda_1 = 2$, $\lambda_2 = 1$, $\omega_1 = 1$, $\omega_2 = 1$.



Chapter VIII

Two Commodity Inventory Problem with Markov Shift in Demand*

8.1 INTRODUCTION

Two models of a two commodity - C_1 and C_2 - inventory problem are discussed in this chapter. The type of commodity demanded at successive demand epochs constitutes a Markov chain. Each arrival can demand one unit of C_1, or one unit of C_2 or one unit each of C_1 and C_2. Shortages are not permitted and the lead time is assumed to be zero. The interarrival times of demands are i.i.d. random variables with absolutely continuous distribution function G(.) having finite mean μ.

In the first model, the replenishment policy is to order for C_i alone so as to bring the inventory level to S_i whenever the inventory level of C_i falls to the reordering point s_i after the previous replenishment (i = 1, 2). In the second model, the replenishment policy is to order for both C_1 and C_2 so as to make the inventory levels maximum (S_1 and S_2) whenever the inventory level of at least one of the commodities reaches its reordering point (s_1 or s_2) after the previous replenishment.

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The objectives are to find time dependent and steady state system state probabilities, to find optimum values of (s_i, S_i), i = 1, 2, at steady state and to compare the two replenishment policies in the two models. Section 8.2 explains the notations used in this chapter. Section 8.3 is devoted to Model I and section 8.4 deals with Model II. Numerical problems are discussed in section 8.5 in which the last two problems compare the two models.

Krishnamoorthy , Iqbal and Lakshmy (1997) deal with a two commodity inventory problem with Markov shift in demand in which each arrival can demand only unit item of either of the commodities but not both. They provide characterization for the limiting distribution of the system states. Krishnamoorthy and Varghese (1994a) generalize their model to arbitrary units of demands. In this chapter we generalize the problem to arrivals demanding unit item of either of the commodities or both. As an application of the models studied here consider computer and printer. Initially both are purchased and subsequently one or the other or both are replaced at various epochs.

8.2 NOTATIONS

C_i : Commodity of type i (i = 1, 2)

S_i : Maximum inventory level of C_i (i = 1, 2)

s_i : Reordering point of C_i (i = 1, 2)

M_i : $S_i - s_i$ (i = 1, 2)

$G(.)$: The distribution function of inter arrival times of demands.

R_+ : The set of non-negative real numbers.

N^0 : The set of non-negative integers.

$\delta[x, y]$: 1 if x = y ; 0 otherwise.

p_{ij} : Pr{demand is for C_j | the previous demand was for C_i}

P : (p_{ij}) i, j = 1, 2, 3.

$(\alpha_1, \alpha_2, \alpha_3)$: Invariant probability measure of **P**.

ρ_i : $\alpha_1 p_{1i} + \alpha_3 p_{3i}$; i = 1, 2, 3.

σ_i : $\alpha_2 p_{2i} + \alpha_3 p_{3i}$; i = 1, 2, 3.

$$b_{n_1}^{r_{1,i}} = \begin{cases} \alpha_1 & \text{if } n_1 = 0; r_{1,i} = 0; i = 0,1,\ldots\ldots n_1 \\ \alpha_2\, p_{22}^{r_{1,0}-1}\, p_{21} & \text{if } n_1 = 0; r_{1,i} \neq 0. \\ \bar{b}_{1,0}\, \bar{b}_{1,1} \ldots\ldots \bar{b}_{1,n_1} & \text{if } n_1 \neq 0. \end{cases}$$

$$\bar{b}_{1,0} = \begin{cases} \alpha_3 & \text{if } r_{1,0} = 0. \\ \alpha_2\, p_{22}^{r_{1,0}-1}\, p_{23} & \text{if } r_{1,0} \neq 0. \end{cases}$$

$$\bar{b}_{1,i} = \begin{cases} p_{33} & \text{if } r_{1,i} = 0. \\ p_{32}\, p_{22}^{r_{1,i}-1}\, p_{23} & \text{if } r_{1,i} \neq 0 \end{cases} \quad i = 1,2,\ldots. n_1 - 1$$

$$\bar{b}_{1,n_1} = \begin{cases} p_{31} & \text{if } r_{1,n_1} = 0. \\ p_{32}\, p_{22}^{r_{1,n_1}-1}\, p_{21} & \text{if } r_{1,n_1} \neq 0 \end{cases}$$

$$\bar{b}_{j,0} = \begin{cases} p_{33} & \text{if } r_{j,0} = 0. \\ p_{12}\, p_{22}^{r_{j,0}-1}\, p_{23} & \text{if } r_{j,0} \neq 0. \end{cases} \quad j = 2,3,\ldots\ldots, k$$

$$\bar{b}_{j,i} = \begin{cases} p_{33} & \text{if } r_{j,i} = 0. \\ p_{32}\, p_{22}^{r_{j,i}-1}\, p_{23} & \text{if } r_{j,i} \neq 0 \end{cases} \quad \begin{matrix} i = 1,2,\ldots. n_j - 1 \\ j = 2,3,\ldots\ldots\ldots, k \end{matrix}$$

$$\bar{b}_{j,n_j} = \begin{cases} p_{31} & \text{if } r_{j,n_j} = 0. \\ p_{32}\, p_{22}^{r_{j,n_j}-1}\, p_{21} & \text{if } r_{j,n_j} \neq 0 \end{cases} \quad j = 2,3,\ldots\ldots, k$$

$$b_{n_j}^{r_{j,i}} = \begin{cases} p_{11} & \text{if } n_j = 0; r_{j,i} = 0; i = 0,1,\ldots\ldots n_j. \\ p_{12}\, p_{22}^{r_{j,0}-1}\, p_{21} & \text{if } n_j = 0; r_{j,0} \neq 0. \quad j = 2,3,\ldots., k \\ \bar{b}_{j,0}\, \bar{b}_{j,1} \ldots\ldots \bar{b}_{j,n_1} & \text{if } n_j \neq 0. \end{cases}$$

$d_{n_j}^{r_{j,i}}$: are obtained from $b_{n_j}^{r_{j,i}}$ by interchanging the subscripts 1 and 2 of p_{ij} 's and α_i 's.

$h_{n_j}^{r_{j,i}}$: are obtained from $b_{n_j}^{r_{j,i}}$ by interchanging the subscripts 1 and 3 of p_{ij} 's and α_i 's.

$\tilde{b}_{n_1}^{r_{1,i}}$: are obtained from $b_{n_1}^{r_{1,i}}$ by replacing α_i by ρ_i

$\tilde{h}_{n_1}^{r_{1,i}}$: are obtained from $h_{n_1}^{r_{1,i}}$ by replacing α_i by ρ_i

$\approx{h}_{n_1}^{r_{1,i}}$: are obtained from $h_{n_1}^{r_{1,i}}$ by replacing α_i by σ_i

$\tilde{d}_{n_1}^{r_{1,i}}$: are obtained from $d_{n_1}^{r_{1,i}}$ by replacing α_i by σ_i

E_i : $\{s_i+1, s_i+2, \ldots\ldots, S_i\}$; i = 1, 2.

E_{ii} : $\{s_i+1, s_i+2, \ldots\ldots, S_i-1\}$; i = 1, 2.

E_3 : $\{1, 2, 3\}$

E^1 : $E_1 \times E_2 \times E_3$

E^2 : $(E_{11} \times E_{22} \times E_3) \cup (E_{11} \times \{S_2\} \times \{1\}) \cup (\{S_1\} \times E_{22} \times \{2\}) \cup (\{S_1\} \times \{S_2\} \times E_3)$

E^{*n} : $\{(i,j,k) \mid (i,j,k) = (i_1 - s_1, j_1 - s_2, k_1), (i_1, j_1, k_1) \in E^n\}$; n = 1, 2.

Π^n : $[\pi^n(i,j,k)]$; $(i, j, k) \in E^n$, n = 1, 2.

8.3 MODEL I

In this model, the replenishment policy is to order for C_i alone so as to bring the inventory level to S_i whenever the inventory level of C_i falls to the reordering point s_i after the previous replenishment (i = 1, 2).

8.3.1 Analysis of the Model

Let $0 = T_0 < T_1 < T_2 < \ldots\ldots < T_n < \ldots$ be the successive demand epochs. Denote by X_n^1, Y_n^1, $n \in N^0$, the inventory levels of C_1 and C_2 respectively, just after meeting the demand at T_n, and $X^1(t)$, $Y^1(t)$ the respective inventory levels at time t. We assume that $X^1{}_0 = X^1(0) = S_1$ and $Y^1{}_0 = Y^1(0) = S_2$ and the initial demand at T_0 is for C_1. Let

$$Z_n = \begin{cases} 1 & \text{if the demand at } T_n \text{ is for } C_1 \\ 2 & \text{if the demand at } T_n \text{ is for } C_2 \\ 3 & \text{if the demand at } T_n \text{ is for both } C_1 \text{ and } C_2 \end{cases}$$

and $\Pr\{Z_{n+1} = j|Z_n = i\} = p_{ij}$; (i,j = 1, 2, 3; n $\in N^0$). Then $\{Z_n, n\in N^0\}$ is a Markov chain on the state space E_3, with initial probability vector (1, 0, 0) and one step transition probability matrix $\mathbf{P} = (p_{ij})$; i, j = 1,2,3. Assuming that **P** is irreducible and aperiodic $\{Z_n, n\in N^0\}$ will be an irreducible ergodic Markov chain.

We have the following

Lemma 8.1

$\{(X^1{}_n, Y^1{}_n, Z_n), n\in N^0\}$ is a Markov chain, whose states space is E^1 with initial probability,

$$\Pr\{(X_0^1, Y_0^1, Z_0) = (i,j,k)\} = \begin{cases} 1 & if\,(i,j,k) = (S_1, S_2, 1) \\ 0 & otherwise. \end{cases} \tag{8.1}$$

and one step transition probability matrix,

$$\mathbf{P}^1 = [q^1\{(i_1,j_1,k_1),(i_2,j_2,k_2)\}]; \quad (i_1,j_1,k_1),(i_2,j_2,k_2) \in E \tag{8.2}$$

$$where \quad q^1\{(i_1,j_1,k_1),(i_2,j_2,k_2)\}$$

$$= \begin{cases} p_{k_1k_2} & if\ k_2 = 1; j_1 = j_2; i_2 = i_1 - 1; s_1 + 1 < i_1 \\ p_{k_1k_2} & if\ k_2 = 1; j_1 = j_2; i_2 = S_1; i_1 = s_1 + 1 \\ p_{k_1k_2} & if\ k_2 = 2; j_1 - 1 = j_2; i_2 = i_1; s_2 + 1 < j_1 \\ p_{k_1k_2} & if\ k_2 = 2; i_1 = i_2; j_2 = S_2; j_1 = s_2 + 1 \\ p_{k_1k_2} & if\ k_2 = 3; i_2 = i_1 - 1; s_1 + 1 < i_1; j_2 = j_1 - 1; s_2 + 1 < j_1 \\ p_{k_1k_2} & if\ k_2 = 3; i_2 = S_1; j_2 = j_1 - 1; i_1 = s_1 + 1; j_1 > s_{2+1} \\ p_{k_1k_2} & if\ k_2 = 3; i_2 = j_1 - 1; j_2 = S_2; i_1 > s_1 + 1; j_1 = s_2 + 1 \\ p_{k_1k_2} & if\ k_2 = 3; i_2 = S_1; j_2 = S_2; i_1 = s_1 + 1; j_1 = s_2 + 1 \\ 0 & otherwise \end{cases} \tag{8.3}$$

Lemma 8.2

$\{(X_n^1, Y_n^1, Z_n), T_n; n \in N^0\}$ is a Markov renewal process on the state space E^1 with semi-Markov kernel,

$$\mathbf{Q}^1 = [Q^1\{(i_1, j_1, k_1), (i_2, j_2, k_2), t\}; \quad (i_1, j_1, k_1), (i_2, j_2, k_2) \in E^1; t \in R_+ \tag{8.4}$$

where

$$\begin{aligned} &Q^1\{(i_1, j_1, k_1),\ (i_2, j_2, k_2),\ t\} \\ &= \Pr\{(X_{n+1}^1 = i_2, Y_{n+1}^1 = j_2, Z_{n+1} = k_2), T_{n+1} - T_n \le t | (X_n^1 = i_1, Y_n^1 = j_1, Z_n = k_1)\} \quad (8.5) \\ &= q^1\{(i_1, j_1, k_1), (i_2, j_2, k_2)\}.G(t); \quad (i_1, j_1, k_1), (i_2, j_2, k_2) \in E^1 \end{aligned}$$

8.3.2 Time Dependent System State Probabilities

Since the depletion of inventory is only due to demand, we have,

$$\left.\begin{aligned} X^1(t) &= X_n^1 \\ Y^1(t) &= Y_n^1 \end{aligned}\right\} for \quad T_n \le t < T_{n+1}. \tag{8.6}$$

If we define $Z(t) = Z_n$ for $T_n \le t < T_{n+1}$, then $\{[X^1(t), Y^1(t), Z(t)], \quad t \in R_+\}$ is a semi-Markov process in which the Markov renewal process, $\{X_n^1, Y_n^1, Z_n), \quad n \in N^0\}$, is embedded with semi-Markov kernel $\mathbf{Q}^1$.

$$\begin{aligned} &If \quad p^1\{(i_1, j_1, k_1), (i_2, j_2, k_2), t\} \\ &\quad = \Pr\{X^1(t) = i_2, Y^1(t) = j_2, Z(t) = k_2)|(X^1(0) = i_1, Y^1(0) = j_1, Z(0) = k_1)\} \quad (8.7) \\ &\qquad \{(i_1, j_1, k_1), (i_2, j_2, k_2), \in E^1, t \in R_+ \end{aligned}$$

we have the following

Theorem 8.1

The transient probabilities of the inventory states are given by

$$p^1\{(S_1, S_2, 1), (i \ , j \ , k \ ), t\} = \int_0^t R^1\{(S_1, S_2, 1), (i \ , j \ , k \ ), du\}\{1 - G(t-u)\} \tag{8.8}$$

$$for \quad all \quad (i \ , j \ , k \ ) \in E^1, t \in R_+$$

where

$$R^1(S_1,S_2,1),(i\ ,j\ ,k\ ),t\} = \sum_{m=0}^{\infty} \quad Q^{1*^m}\{(S_1,S_2,1),(i\ ,j\ ,k\ ),t\}$$

with the convention, $\qquad Q^{1*^0}(x,y,t) = \begin{cases} 1 & if \quad x = y \\ 0 & \text{otherwise} \end{cases}$

Proof:

Define

$$\Theta^1\{(i,j,k),t\} = 1 - \sum_{(i_1,j_1,k_1)\in E^1} Q^1\{(i,j,k),(i_1,j_1,k_1),t\} \quad (i,j,k)\in E^1, t\in R_+$$

then, (8.9)

$\Theta^1\{(i,j,k),t\} = 1 - G(t)$ for all $(i,j,k)\in E^1, t\in R_+$.

Conditioning on the first demand epoch T_1, we have

$$\begin{aligned} & p^1\{(S_1,S_2,1),(i\ ,j\ ,k\ ),t\} \\ &= \Pr\{[X^1(t)=i, Y^1(t)=j, Z(t)=k];\ T_1 > t \mid [X^1(0)=S_1, Y^1(0)=S_2, Z(0)=1]\} \\ &+ \Pr\{[X^1(t)=i, Y^1(t)=j, Z(t)=k];\ T_1 \le t \mid [X^1(0)=S_1, Y^1(0)=S_2, Z(0)=1]\} \end{aligned} \tag{8.10}$$

$$\begin{aligned} &= \delta[(S_1,S_2,1),(i,j,k)]\ \Theta^1\{(S_1,S_2,1),t\} \\ &\quad + \int_0^t \sum_{(i_1,j_1,k_1)\in E^1} Q^1\{(S_1,S_2,1),(i_1,j_1,k_1),du\}\ p^1\{(i_1,j_1,k_1),(i,j,k),t-u\} \end{aligned} \tag{8.11}$$

The solution of the above Markov renewal equations are

$$\begin{aligned} & p^1\{(S_1,S_2,1),(i,j,k),t\} \\ &= \int_0^t \sum_{(i_1,j_1,k_1)\in E^1} R^1\{(S_1,S_2,1),(i_1,j_1,k_1),du\}\,\delta[(i_1,j_1,k_1),(i,j,k)]\Theta^1\{(i,j,k),t-u\} \end{aligned} \tag{8.12}$$

from which the theorem follows easily.

8.3.3 Limiting Probabilities

At steady state the one-step transition probability matrix ,

$$\lim_{t \to \infty} Q^1\{(i_1, j_1, k_1), (i_2, j_2, k_2), t\}]$$

is $\mathbf{P}^1$ (see (8.2) - (8.5)). Since the corresponding Markov chain, $\{X_n^1, Y_n^1, Z_n), n \in N^0\}$, is irreducible and ergodic, the invariant probability measure, Π^1, of this Markov chain obtained as the solutions of

$$\sum_{(i,j,k) \in E^1} \pi^1(i,j,k)\, q^1\{i,j,k), (i_1, j_1, k_1)\} = \pi^1(i_1, j_1, k_1) \quad \text{for all } (i_1, j_1, k_1) \in E^1 \tag{8.13}$$

with $\sum_{(i,j,k) \in E^1} \pi^1(i,j,k) = 1$, is unique. (8.14)

Theorem 8.2

The probabilities that the system state is at (i,j,k) at steady state,

$$\lim_{t \to \infty} p^1\{(S_1, S_2, 1), (i,j,k), t\} = \pi^1(i,j,k); \quad (i,j,k) \in E^1 \tag{8.15}$$

which are independent of the initial state.

Proof:

From theorem 8.1, we have

$$\lim_{t \to \infty} p^1\{(S_1, S_2, 1), (i,j,k), t\} = \frac{\pi^1(i,j,k)\ n(i,j,k)}{\sum_{(i_1, j_1, k_1) \in E^1} \pi^1(i_1, j_1, k_1)\ m^1(i_1, j_1, k_1)} \tag{8.16}$$

where $m^1(i_1, j_1, k_1)$ is the mean sojourn time in the state (i_1, j_1, k_1) of the Markov renewal process, $\{[(X_n^1, Y_n^1, Z_n), T_n], n \in N^0\}$ and

$$\begin{aligned} n(i,j,k) &= \int_0^\infty \Theta^1\{(i,j,k), t\} dt. \\ &= \int_0^\infty [1 - G(t)] dt. \end{aligned} \tag{8.17}$$

But $m^1(i_1, j_1, k_1) = E[T_{n+1} - T_n \mid (X_n^1 = i, Y_n^1 = j, Z_n = k)] \;= \int_0^\infty [1 - G(t)dt.$ (8.18)

Substituting these values in (8.16) we get (8.15). Hence the theorem.

Theorem 8.3.

The probabilities that the inventory system state is at (i, j, k) at steady state is independent of i and j and is given by

$$\lim_{t\to\infty} p^1\{(S_1, S_2, 1), (i, j, k), t\} = \frac{\alpha_k}{M_1 \times M_2}; \quad (i, j, k) \in E^1. \qquad (8.19)$$

They are uniformly distributed if and only if **P** is doubly stochastic.

Proof:

From (8.13) we get

$$\begin{aligned} \sum_{k=1}^{3} \pi^1(i, j, k) p_{k1} &= \pi^1(<i-1>, j, 1) \\ \sum_{k=1}^{3} \pi^1(i, j, k) p_{k2} &= \pi^1(i, <j-1>, 2) \\ \sum_{k=1}^{3} \pi^1(i, j, k) p_{k3} &= \pi^1(<i-1>, <j-1>, 3) \end{aligned} \qquad (8.20)$$

where

$$<i-1> = \begin{cases} i-1 & if \quad i \neq s_1 + 1 \\ S_1 & if \quad i = s_1 + 1 \end{cases} \quad and \quad <j-1> = \begin{cases} j-1 & if \quad j \neq s_2 + \\ S_2 & if \quad j = s_2 + 1 \end{cases}$$

It can be easily verified that

$$\pi^1(i, j, k) = \frac{\alpha_k}{M_1 \times M_2}; \quad (i, j, k) \in E^1 \qquad (8.21)$$

is a solution to the system (8.20) and (8.14) .Since the solution is unique, there is no other solution to the system. Substituting (8.21) in (8.15) we get (8.19). If **P** is doubly stochastic the uniqueness forces the invariant measures $(\alpha_1, \alpha_2, \alpha_3)$

to be uniform. Therefore from (8.21) $\pi^1(i,j,k);\quad (i,j,k)\in E^1$ will be uniform, hence from (8.19) the steady state probabilities are uniform. Conversely, if the probabilities at steady state are uniform, then from (8.19) and (8.21), $\alpha_1=\alpha_2=\alpha_3$ and substituting in $\sum_{i=1}^{3}\alpha_i\, p_{ij}=\alpha_j;\quad j=1,2,3$

we at once see that **P** is doubly stochastic. Hence the theorem.

8.3.4 Time between the replenishments

In the present model, the replenishment of one commodity is independent of the other. Hence considering the replenishment epochs of C_1 alone we have the following

Theorem 8.4

Let T^1 represent the time elapsed between two consecutive replenishment epochs of commodity C_1 and $F^1\{(S_1,j_1,k_1),(S_1,j_2,k_2),t\}$ be its distribution, then

$$F^1\{(S_1,j_1,k_1),(S_1,j_2,k_2),t\}=F_1^1\{(S_1,j_1,k_1),(S_1,j_2,1),t\}+F_2^1\{(S_1,j_1,k_1),(S_1,j_2,3),t\},\quad (k_1=1,3) \tag{8.22}$$

where

$F_1^1\{(S_1,j_1,k_1),(S_1,j_2,1),t\}$

$$=\sum_{k=1}^{M_1}\ \sum_{\substack{n_1,n_2,\dots,n_k=0\\ \sum n_j+k=M_1}}^{M_1-k}\ \sum_{z=z_0}^{\infty}\ \sum_{\substack{r_{1,0},r_{1,1},\dots,r_{1,n_1},r_{2,0},\dots,r_{k,n_k}=0\\ \sum r_{j,i}+\sum n_j=zM_2+j_1-j_2}}^{zM_2+j_1-j_2} \tilde{b}_{n_1}^{r_{1,i}}\, b_{n_2}^{r_{2,i}}\cdots\cdots b_{n_k}^{r_{k,i}} \times G^{*(k+\sum r_{j,i}+\sum n_j)}(t) \tag{8.23}$$

and $F_2^1\{(S_1,j_1,k_1),(S_1,j_2,3),t\}$

$$=\sum_{k=1}^{M_1}\ \sum_{\substack{n_1,n_2,\dots,n_k=0\\ \sum n_j+k=M_1}}^{M_1-k}\ \sum_{z=z_0}^{\infty}\ \sum_{\substack{r_{1,0},r_{1,1},\dots,r_{1,n_1},r_{2,0},\dots,r_{k,n_k}=0\\ \sum r_{j,i}+k=zM_2+j_1-j_2}}^{zM_2+j_1-j_2} \tilde{h}_{n_1}^{r_{1,i}}\, h_{n_2}^{r_{2,i}}\cdots\cdots h_{n_k}^{r_{k,i}} \times G^{*(k+\sum r_{j,i}+\sum n_j)}(t) \tag{8.24}$$

in which $$z_0 = \begin{cases} 0 & if\ j_1 \geq j_2 \\ 1 & if\ j_1 < j_2 \end{cases}$$

Proof:

Assume that there is a replenishment of commodity C_1 just after the n^{th} demand epoch ,then Z_n may be 1 or 3. Hence (8.22). Consider a replenishment period ending with the n^{th} arrival ,where $Z_n = 1$, in which there are exactly $k(k = 1,2,....M_1)$ demands for C_1 alone after the previous replenishment. In this replenishment period there may be z (varying from z_0 to ∞) replenishments of commodity C_2. Denote by $n_j(j = 1,2,.....k)$ the number of demands for both C_1 and C_2 in between the $(j-1)^{th}$ and j^{th} demands for C_1 alone and $r_{j,i}$ $(j = 1, 2,..., k,\ i = 1,2,....,n_j - 1)$ the number of demands for C_2 alone in between the i^{th} and $(i+1)^{th}$ demand for both C_1 and C_2 that happened after the $(j-1)^{th}$ demand for C_1 alone, and j^{th} demand for C_1 alone. Since there are exactly k demands for C_1 alone and the replenishment is due to a demand for C_1 alone, n_j can vary from zero to $M_1 - k$ with the condition that $k + \sum n_j = M_1$ and $r_{j,i}$'s can vary from zero to $zM_2 + j_1 - j_2$ with the condition that $\sum r_{j,i} + \sum n_j = zM_2 + j_1 - j_2$. With these notations (8.23) follows easily.

Now consider a replenishment epoch of C_1 ending with the n^{th} arrival, where $Z_n = 3$. Suppose that there are exactly k combined demands for C_1 and C_2 .Let n_j denote the number of demands of C_1 alone in between the $(j-1)^{th}$ and j^{th} combined demands, $r_{j,i}$ $(j = 1,2,...k;\ i = 1,2,....n_j - 1)$ denote the number of demands of C_2 alone in between the i^{th} and $(i+1)^{th}$ demand of C_1 that happened after the $(j-1)^{th}$ combined demand of C_1 and C_2 , r_{j,n_j} denote the number of demands of C_2 alone in between the n_j^{th} demand of C_1 that happened after the $(j-1)^{th}$ combined demand of C_1 and C_2 , and the j^{th} combined demand of C_1

and C_2. Since there are exactly k combined demands for C_1 and C_2, $\sum r_{j,i} + k = zM_2 + j_1 - j_2$ during this period. Hence (8.24).

In a similar way considering the replenishment epochs of commodity C_2 we can prove the following

Theorem 8.5

Let T^2 represent the time elapsed between two consecutive replenishment epochs of commodity C_2 and $F^2\{(i_1, S_2, k_1), (i_2, S_2, k_2), t\}$ be its distribution, then

$$F^2\{(i_1, S_2, k_1), (i_2, S_2, k_2), t\} = F_1^2\{(i_1, S_2, k_1), (i_2, S_2, 2), t\} + F_2^2\{(i_1, S_2, k_1), (i_2, S_2, 3), t\}; \ (k_1=2, 3) \tag{8.25}$$

where $F_1^2\{(i_1, S_2, k_1), (i_2, S_2, 2), t\}$

$$= \sum_{k=1}^{M_2} \sum_{\substack{n_1, n_2, \ldots, n_k = 0 \\ \sum n_j + k = M_2}}^{M_2 - k} \sum_{z=z_0}^{\infty} \sum_{\substack{r_{1,0}, r_{1,1}, \ldots, r_{1,n_1}, r_{2,0}, \ldots, r_{k,n_k} = 0 \\ \sum r_{j,i} + \sum n_j = zM_1 + i_1 - i_2}}^{zM_1 + i_1 - i_2} \tilde{d}_{n_1}^{\,r_{1,i}} d_{n_2}^{\,r_{2,i}} \ldots\ldots d_{n_k}^{\,r_{k,i}} \times G^{*(k + \sum r_{j,i} + \sum n_j)}(t) \tag{8.26}$$

and $F_2^2\{(i_1, s_2, k_1), (i_2, s_2, 3), t\}$,

$$= \sum_{k=1}^{M_2} \sum_{z=z_0}^{\infty} \sum_{\substack{n_1, n_2, \ldots, n_k = 0 \\ \sum n_j + k = zM_1 + i_1 - i_2}}^{zM_1 + i_1 - i_2} \sum_{\substack{r_{1,0}, r_{1,1}, \ldots, r_{1,n_1}, r_{2,0}, \ldots, r_{k,n_k} = 0 \\ \sum r_{j,i} + k = M_2}}^{M_2 - k} \tilde{\tilde{h}}_{n_1}^{\,r_{1,i}} h_{n_2}^{\,r_{2,i}} \ldots h_{n_k}^{\,r_{k,i}} \times G^{*(k + \sum r_{j,i} + \sum n_j)}(t) \tag{8.27}$$

in which $$z_0 = \begin{cases} 0 & if \ i_1 \geq i_2 \\ 1 & if \ i_1 < i_2 \end{cases}$$

8.3.5. Optimization Problem

Let w_j be the holding cost for one unit of C_j per unit time ($j = 1,2$). Then the expected holding cost per unit time,

$$E(H^1) = \sum_{(i,j,k)\in E^1} (iw_1 + jw_2)\pi^1(i,j,k)$$
$$= \sum_{(i_1,j_1,k)\in E^{*1}} (iw_1 + jw_2)\,\pi^1(i_1 + s_1, j_1 + s_2, k) + s_1 w_1 + s_2 w_2 \tag{8.28}$$

Hence $E(H^1)$ is minimum for $s_1 = s_2 = 0$. Also because of (8.19),

$$E(H^1) = \sum_{(i,j,k)\in E^{*1}} (iw_1 + jw_2)\frac{\alpha_k}{M_1 \times M_2} + s_1 w_1 + s_2 w_2$$
$$= \frac{1}{2}\left[w_1 (M_1 + 1) + w_2 (M_2 + 1)\right] + s_1 w_1 + s_2 w \tag{8.29}$$
$$= E(H_1) + E(H_2), \text{ where } E(H_i) = \frac{1}{2}\left[w_i (M_i + 1)\right] + s_i w_i, i = 1,2.$$

From (8.22) - (8.27) we can easily calculate the expected replenishment cycle times, $E(T^1)$ and $E(T^2)$. Also note that M_i is the quantity of C_i ordered at each replenishment epoch of C_i (i = 1, 2).

If K_i is the fixed ordering cost and c_i is the unit procurement cost for C_i (i = 1, 2), then the total expected cost (TEC^1) for the inventory system per unit time is

$$TEC^1 = \frac{K_1 + c_1 M_1}{E(T^1)} + \frac{K_2 + c_2 M_2}{E(T^2)} + E(H^1)$$
$$= \left[\frac{K_1 + c_1 M_1}{E(T^1)} + \frac{w_1}{2}(M_1 + 1) + w_1 s_1\right] + \left[\frac{K_2 + c_2 M_2}{E(T^2)} + \frac{w_2}{2}(M_2 + 1) + w_2 s_2\right] \tag{8.30}$$

= Total expected cost of C_1 + Total expected cost of C_2.

Since $E(T^i)$ is independent of s_i for a given value of M_i (i = 1,2), TEC^1 is minimum when $s_1 = s_2 = 0$.

8.4 MODEL II

In this model, the replenishment policy is to order for both C_1 and C_2 so as to make the inventory levels maximum (S_1 and S_2) whenever the inventory level of at least one of the commodities reaches to its reordering point (s_1 or s_2) after the previous replenishment.

8.4.1 Analysis of the Model

Let $0 = T_0 < T_1 < T_2 < \ldots\ldots < T_n < \ldots$ be the successive demand epochs. Denote by X_n^2, Y_n^2 , $n \in N^0$, the inventory levels of C_1 and C_2 respectively, just after meeting the demand at T_n, and $X^2(t)$, $Y^2(t)$ the respective inventory levels at time t. We assume that $X^2_0 = X^2(0) = S_1$ and $Y^2_0 = Y^2(0) = S_2$ and the initial demand at T_0 is for C_1. Let $\{Z_n, n \in N^0\}$ be the Markov chain as defined in section 8.3.1.

We observe that $\{X_n^2, Y_n^2, Z_n), n \in N^0\}$is also Markov chain, whose state space is E^{1}. Here, starting from the state (S_1, S_2, 1) some of the states are not visited by the process. Hence the Markov chain will not be irreducible. So excluding these states we have the following:

Lemma 8.3

$\{(X^2_n, Y^2_n, Z_n), n \in N^0\}$ with state space E^2 is an irreducible and ergodic Markov chain, having initial probability,

$$\Pr\{(X_0^2, Y_0^2, Z_0) = (i,j,k)\} = \begin{cases} 1 & if (i,j,k) = (S_1, S_2, 1) \\ 0 & otherwise. \end{cases} \tag{8.31}$$

and one step transition probability matrix,

$$\mathbf{P}^2 = [q^2\{(i_1, j_1, k_1), (i_2, j_2, k_2)\}]; \quad (i_1, j_1, k_1), (i_2, j_2, k_2) \in E^2 \tag{8.32}$$

where

$$q^2\{(i_1,j_1,k_1),(i_2,j_2,k_2)\}$$
$$=\begin{cases} p_{k_1k_2} & \text{if } k_2=1; j_1=j_2; i_2=i_1-1; s_1+1<i_1 \\ p_{k_1k_2} & \text{if } k_2=1; j_2=S_2; i_2=S_1; i_1=s_1+1 \\ p_{k_1k_2} & \text{if } k_2=2; i_1-i_2=j_2; j_2=j_1-1; s_2+1<j_1 \\ p_{k_1k_2} & \text{if } k_2=2; i_1=S_2; j_2=S_2; j_1=s_2+1 \\ p_{k_1k_2} & \text{if } k_2=3; i_2=i_1-1; s_1+1<i_1; j_2=j_1-1; s_2+1<j_1 \\ p_{k_1k_2} & \text{if } k_2=3; i_2=S_1; j_2=S_2; \quad i_1=s_1+1 \quad or \quad j_1=s_{2+1} \\ 0 & otherwise \end{cases} \tag{8.33}$$

Lemma 8.4

$\{(X_n^2,Y_n^2,Z_n),T_n;n\in N^0\}$ is a Markov renewal process on the state space E^2 with semi-Markov kernel,

$$\mathbf{Q}^2=[Q^2\{(i_1,j_1,k_1),(i_2,j_2,k_2),t\};\quad (i_1,j_1,k_1),(i_2,j_2,k_2)\in E^2;t\in R_+] \tag{8.34}$$

where

$$Q^2\{(i_1,j_1,k_1),(i_2,j_2,k_2),t\}$$
$$=\Pr\{(X_{n+1}^2=i_2,Y_{n+1}^2=j_2,Z_{n+1}=k_2),T_{n+1}-T_n\le t|(X_n^2=i_1,Y_n^2=j_1,Z_n=k_1)\}$$
$$=q^2\{(i_1,j_1,k_1),(i_2,j_2,k_2)\}.G(t);(i_1,j_1,k_1),(i_2,j_2,k_2)\in E^2 \tag{8.35}$$

8.4.2 Transient and Steady State Probabilities

Since

$$\left.\begin{array}{l} X^2(t)=X_n^2 \\ Y^2(t)=Y_n^2 \\ Z(t)=Z_n \end{array}\right\} for \quad T_n\le t<T_{n+1} \tag{8.36}$$

and if we denote $p^2\{(i_1,j_1,k_1),(i_2,j_2,k_2),t\}$

$$=\Pr\{X^2(t)=i_2,Y^2(t)=j_2,Z(t)=k_2)|(X^2(0)=i_1,Y^2(0)=j_1,Z(0)=k_1)\}$$
$$\{(i_1,j_1,k_1),(i_2,j_2,k_2),\in E^2,t\in R_+ \tag{8.37}$$

we have, similar to Theorem 8.1, the following

Theorem 8. 6

The transient probabilities of the inventory states are given by

$$p^2\{(S_1,S_2,1),(i\ ,j\ ,k\ ),t\} = \int_0^t R^2\{(S_1,S_2,1),(i\ ,j\ ,k\ ),du\}\{1-G(t-u)\}$$

$$for\quad all\quad (i\ ,j\ ,k\ )\in E^2, t\in R_+$$

where (8.38)

$$R^2(S_1,S_2,1),(i\ ,j\ ,k\ ),t\} = \sum_{m=0}^{\infty} \quad Q^{2*^m}\{(S_1,S_2,1),(i\ ,j\ ,k\ ),t\}$$

with the convention, $\quad Q^{2*^0}(x,y,t) = \begin{cases} 1 & if \quad x=y \\ 0 & \text{otherwise} \end{cases}$

As in section 8.3.3, the invariant probability measure, Π^2 of the Markov chain, $\{X_n^2, Y_n^2, Z_n), n\in N^0\}$, is obtained as the unique solution of

$$\sum_{(i,j,k)\in E^2} \pi^2(i,j,k)\, q^2\{i,j,k),(i_1,j_1,k_1)\} = \pi^2(i_1,j_1,k_1) \text{ for all } (i_1,j_1,k_1)\in E^2 \quad 8.39)$$

with

$$\sum_{(i,j,k)\in E^2} \pi^2(i,j,k) = 1. \qquad (8.40)$$

Also by the argument that led to theorem 8.2 we can prove that the probabilities that the system state is at *(i,j,k)* at steady state,

$$\lim_{t\to\infty} p^2\{(S_1,S_2,1),(i,j,k),t\} = \pi^2(i,j,k); \quad (i,j,k)\in E^2. \qquad (8.41)$$

8.4.3 Replenishment Cycles and Optimization

In this model, the stock levels of both C_1 and C_2 are brought to their maximum whenever the inventory level of at least one of the commodities has reached to its reordering point . The expression for the distribution of inter replenishment times is derived in the following

Theorem 8.7

Let T represent the time elapsed between two consecutive replenishment epochs and $F\{(S_1, S_2, k_1), (S_1, S_2, k_2), t\}$, its distribution. Then

$$F\{(S_1,S_2,k_1),(S_1,S_2,k_2),t\} = \sum_{k=1}^{3} F_k\{(S_1,S_2,k_1),(S_1,S_2,k),t\};\ k_1 = 1,2,3; \quad (8.42)$$

where $F_1\{(S_1, S_2, k_1), (S_1, S_2, 1), t\}$

$$= \sum_{k=1}^{M_1} \sum_{\substack{n_1,n_2,\ldots,n_k=0 \\ \sum n_j+k=M_1}}^{M_2-1} \sum_{\substack{r_{1,0},r_{1,1},\ldots,r_{1,n_1},r_{2,0},\ldots,r_{k,n_k}=0 \\ \sum r_{j,i}+\sum n_j<M_2}}^{M_2-1} b_{n_1}^{r_{1,i}} b_{n_2}^{r_{2,i}} \ldots b_{n_k}^{r_{k,i}} G^{*(k+\sum r_{j,i}+\sum n_j)}(t)$$

(8.43)

and $F_2\{(S_1, S_2, k_1), (S_1, S_2, 2), t\}$

$$= \sum_{k=1}^{M_2} \sum_{\substack{n_1,n_2,\ldots,n_k=0 \\ \sum n_j+k=M_2}}^{M_1-1} \sum_{\substack{r_{1,0},r_{1,1},\ldots,r_{1,n_1},r_{2,0},\ldots,r_{k,n_k}=0 \\ \sum r_{j,i}+\sum n_j<M_1}}^{M_1-1} d_{n_1}^{r_{1,i}} d_{n_2}^{r_{2,i}} \ldots d_{n_k}^{r_{k,i}} G^{*(k+\sum r_{j,i}+\sum n_j)}(t)$$

(8.44)

and $F_3\{(S_1, S_2, k_1), (S_1, S_2, 3), t\} \} = \Phi_1 + \Phi_2$ (8.45)

where

$$\Phi_1 = \sum_{\substack{k=1 \\ k\le M_2}}^{M_1} \sum_{\substack{n_1,n_2,\ldots,n_k=0 \\ \sum n_j+k=M_1}}^{M_1-k} \sum_{\substack{r_{1,0},r_{1,1},\ldots,r_{1,n_1},r_{2,0},\ldots,r_{k,n_k}=0 \\ \sum r_{j,i}+k\le M_2}}^{M_2-k} h_{n_1}^{r_{1,i}} h_{n_2}^{r_{2,i}} \ldots\ldots h_{n_k}^{r_{k,i}} \times G^{*(k+\sum r_{j,i}+\sum n_j)}(t) \quad (8.46)$$

and

$$\Phi_2 = \sum_{\substack{k=1 \\ k<M_1}}^{M_2} \sum_{\substack{n_1,n_2,\ldots,n_k=0 \\ \sum n_j+k<M_1}}^{M_1-k-1} \sum_{\substack{r_{1,0},r_{1,1},\ldots,r_{1,n_1},r_{2,0},\ldots,r_{k,n_k}=0 \\ \sum r_{j,i}+k=M_2}}^{M_2-k} h_{n_1}^{r_{1,i}} h_{n_2}^{r_{2,i}} \ldots\ldots h_{n_k}^{r_{k,i}} \times G^{*(k+\sum r_{j,i}+\sum n_j)}(t) \quad (8.47)$$

Proof:

If the replenishment is just after the n^{th} demand epoch, then Z_n may be 1, 2 or 3. The expressions (8.43) and (8.44) are derived on the same lines as (8.22) and (8.25). Now consider a replenishment epoch ending with the n^{th} arrival where $Z_n = 3$. If the replenishment has happened due to the falling of inventory level of C_1 to s_1 or the levels of both C_1 and C_2 together to (s_1, s_2), then its distribution is given by (8.46), otherwise by (8.47).

If w_i be the holding cost for one unit of C_i per unit time (i = 1,2),then the expected holding cost per unit time,

$$E(H^2) = \sum_{(i_1,j_1,k)\in E^{*2}} (iw_1 + jw_2)\,\pi^2(i_1+s_1, j_1+s_2, k) + s_1w_1 + s_2w_2. \qquad (8.48)$$

Therefore $E(H^2)$ is minimum for $s_1 = s_2 = 0$.

The quantity ordered at each replenishment epoch is not fixed in the present model. Let M^1 and M^2 represent the random replenishment quantities of C_1 and C_2 respectively. Since each arrival can demand utmost one unit of C_1, from (8.42),

$$\begin{aligned}
E(M^1) = & \sum_{k=1}^{M_1} \sum_{\substack{n_1,n_2,\ldots,n_k=0 \\ \sum n_j + k = M_1}}^{M_2-1} \sum_{\substack{r_{1,0},r_{1,1},\ldots,r_{1,n_1},r_{2,0},\ldots,r_{k,n_k}=0 \\ \sum r_{j,i} + \sum n_j < M_2}}^{M_2-1} b_{n_1}^{r_{1,i}} b_{n_2}^{r_{2,i}} \ldots\ldots b_{n_k}^{r_{k,i}} M_1 \\
& + \sum_{k=1}^{M_2} \sum_{\substack{n_1,n_2,\ldots,n_k=0 \\ \sum n_j + k = M_2}}^{M_1-1} \sum_{\substack{r_{1,0},r_{1,1},\ldots,r_{1,n_1},r_{2,0},\ldots,r_{k,n_k}=0 \\ \sum r_{j,i} + \sum n_j < M_1}}^{M_1-1} d_{n_1}^{r_{1,i}} d_{n_2}^{r_{2,i}} \ldots d_{n_k}^{r_{k,i}} \{\sum r_{j,i} + \sum n_j\} \\
& + \sum_{\substack{k=1 \\ k \le M_2}}^{M_1} \sum_{\substack{n_1,n_2,\ldots,n_k=0 \\ \sum n_j + k = M_1}}^{M_1-k} \sum_{\substack{r_{1,0},r_{1,1},\ldots,r_{1,n_1},r_{2,0},\ldots,r_{k,n_k}=0 \\ \sum r_{j,i} + k \le M_2}}^{M_2-k} h_{n_1}^{r_{1,i}} h_{n_2}^{r_{2,i}} \ldots\ldots h_{n_k}^{r_{k,i}} M_1 \\
& + \sum_{\substack{k=1 \\ k < M_1}}^{M_2} \sum_{\substack{n_1,n_2,\ldots,n_k=0 \\ \sum n_j + k < M_1}}^{M_1-k-1} \sum_{\substack{r_{1,0},r_{1,1},\ldots,r_{1,n_1},r_{2,0},\ldots,r_{k,n_k}=0 \\ \sum r_{j,i} + k = M_2}}^{M_2-k} h_{n_1}^{r_{1,i}} h_{n_2}^{r_{2,i}} \ldots h_{n_k}^{r_{k,i}} \{k + \sum_{j=1}^{k} n_j\}
\end{aligned}$$

(8.49)

An expression for $E(M^2)$ can be obtained similarly from (8.42).

Let K_1, K_2 be the fixed ordering costs for C_1, and C_2 respectively and K be the joint fixed ordering cost. Then

$$K_1 \le K \le K_1 + K_2 \quad \text{and} \quad K_2 \le K \le K_1 + K_2 \tag{8.50}$$

Though the replenishment policy is to order for both C_1 and C_2 there may be replenishment periods without a single demand for C_2. The probability for this event is

$$\beta_1 = \alpha_1 \, p_{11}^{M_1 - 1} \tag{8.52}$$

Similarly the probability for a replenishment period without a single demand for C_1 is

$$\beta_2 = \alpha_2 \, p_{22}^{M_2 - 1} \tag{8.53}$$

Therefore the expected fixed joint ordering cost is

$$E(K^*) = K(1 - \beta_1 - \beta_2) + K_1\beta_1 + K_2\beta_2 \tag{8.54}$$

The expected replenishment cycle time, $E(T)$ can be derived from (8.42). If c_i is the unit procurement cost for C_i (i = 1, 2), then the total expected cost (TEC^2) for the system per unit time is

$$TEC^2 = \frac{E(K^*) + c_1 E(M^1) + E(M^2)}{E(T)} + E(H^2) \tag{8.55}$$

Since $E(T)$, $E(M^1)$, $E(M^2)$ are dependent only on M_i (i = 1, 2), and not on s_i, TEC^2 is minimum when $s_1 = s_2 = 0$.

8.5 NUMERICAL ILLUSTRATIONS

In this section, we provide some numerical results for the models discussed. There are three sets of two problems each. The first set (Problem 8.1 and 8.2) contains problems related to Model I. The second set (Problems 8.3

and 8.4) illustrates the second model. The third set (Problems 8.5 and 8.6) compares the expected total cost of the two models for different values of K. Figure 8.1 shows that the second model is not preferable if $K \geq 53$. Similarly we can conclude from Table 8.6 that the first model is better than the second if $K \geq 70$. The probability of demanding C_2 is more than that of C_1 in Problems 8.1, 8.3, and 8.5 where as in the other problems it is reversed. In each table the optimum values of the pair (M_1, M_2) and the corresponding total expected cost are indicated.

Problem 8.1

$$P = \begin{bmatrix} .1 & .8 & .1 \\ .3 & .6 & .1 \\ .1 & .7 & .2 \end{bmatrix}$$

$K_1 = 40$, $K_2 = 30$, $c_1 = 5$, $c_2 = 4$, $w_1 = 1$, $w_2 = .8$, $\mu = 3$

Table 8.1
(TEC[1] of Problem 8.1)

M_1	M_2	$E(H_1)$	Or. Cost of C_1	Total cost of C_1	$E(H_2)$	Or. Cost of C_2	Total cost of C_2	TEC[1]
2	2	1.5	2.8782	4.3782	1.2	5.0319	6.2319	10.610095
1	3	1.0	5.2174	6.2174	1.6	3.6668	5.2668	11.484222
3	1	2.0	2.1050	4.1050	0.8	9.2727	10.0727	14.177761
2	3	1.5	2.8782	4.3782	1.6	3.6668	5.2668	9.645047
3	2	2.0	2.1050	4.1050	1.2	5.0319	6.2319	10.336912
3	**3**	2.0	2.1050	4.1050	1.6	3.6668	5.2668	**9.371865**

Problem 8.2

$$P = \begin{bmatrix} .6 & .1 & .3 \\ .7 & .2 & .1 \\ .8 & .1 & .1 \end{bmatrix}$$

$K_1 = 40$, $K_2 = 50$, $c_1 = 6$, $c_2 = 8$, $w_1 = 1.2$, $w_2 = 1.6$, $\mu = 3$

Table 8.2
(TEC^1 of Problem 8.2)

M_1	M_2	$E(H_1)$	Or. Cost of C_1	Total cost of C_1	$E(H_2)$	Or. Cost of C_2	Total cost of C_2	TEC^1
2	2	1.8	7.7037	9.5037	2.4	3.7993	6.1993	15.702962
1	3	1.2	13.6296	4.8296	3.2	2.8323	6.0323	20.861941
3	1	2.4	5.7284	8.1284	1.6	6.7246	8.3246	16.453034
2	3	1.8	7.7037	9.5037	3.2	2.8323	6.0323	15.536015
3	2	2.4	5.7284	8.1284	2.4	3.7993	6.1993	14.327653
3	**3**	2.4	5.7284	8.1284	3.2	2.8323	6.0323	**14.160706**

Problem 8.3

$$P = \begin{bmatrix} .1 & .8 & .1 \\ .3 & .6 & .1 \\ .1 & .7 & .2 \end{bmatrix}$$

$K = 60$, $K_1 = 40$, $K_2 = 30$, $c_1 = 5$, $c_2 = 4$, $w_1 = 1$,$w_2 = .8$, $\mu = 3$

Table 8.3
(TEC^2 of Problem 8.3)

M_1	M_2	$E(H)$	$E(M_1)$	$E(M_2)$	$E(T)$	TEC^2
2	2	2.903213	0.748667	1.875667	7.030000	11.198045
1	3	2.809675	0.748000	1.617333	6.360000	12.136090
3	1	3.549750	0.336667	0.998333	3.550000	16.125337
2	**3**	3.293567	1.099400	2.585000	9.902000	**10.155178**
3	2	3.767602	0.777200	1.981900	7.393000	11.772507
3	3	4.031526	1.202287	2.911327	11.051400	10.371303

Problem 8.4

$$P = \begin{bmatrix} .6 & .1 & .3 \\ .7 & .2 & .1 \\ .8 & .1 & .1 \end{bmatrix}$$

Table 8.4
(TEC^2 of Problem 8.4)

M_1	M_2	E(H)	E(M_1)	E(M_2)	E(T)	TEC^2
2	2	4.660237	1.893333	0.711667	6.490000	6.902662
1	3	5.799283	0.994667	0.348000	3.480000	23.210777
3	1	4.314512	1.874667	0.748000	6.360000	17.388726
2	3	6.129279	1.975467	0.741800	6.786000	18.040154
3	**2**	5.163030	2.708433	1.029600	9.221000	**15.314576**
3	3	6.496425	2.924533	1.105983	9.981300	16.129946

Problem 8.5

$$\mathbf{P} = \begin{bmatrix} .1 & .8 & .1 \\ .3 & .6 & .1 \\ .1 & .7 & .2 \end{bmatrix}$$

$K_1 = 40$, $K_2 = 30$, $c_1 = 5$, $c_2 = 4$, $w_1 = 1$,$w_2 = .8$, $\mu = 3$

Figure 8.1
(Comparison of TEC^1 and TEC^2 in problem 8.5 when $M_1 = 2$ and $M_2 = 3$)

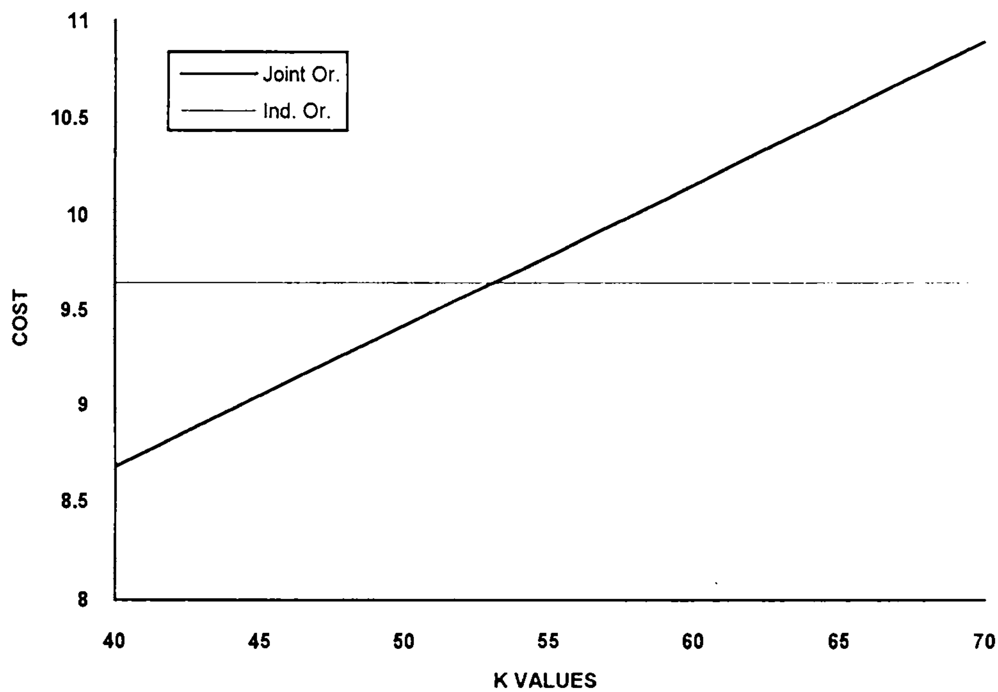

Table 8.5
(TEC[1] and TEC[2] for different values of K in Problem 8.5)

M_1	M_2	TEC^1	TEC^2 (K=40)	TEC^2 (K=45)	TEC^2 (K=50)	TEC^2 (K=60)
2	2	10.610095	9.595390	9.996054	10.396717	11.198045
1	3	11.484222	10.307998	10.765021	11.222044	12.136090
3	1	14.177761	14.444585	14.864773	15.284961	16.125337
2	3	9.645047	**8.678035**	**9.047321**	**9.416606**	**10.155178**
3	2	10.336912	10.207964	10.599100	10.990236	11.772507
3	3	**9.371865**	9.020645	9.358309	9.695974	10.371303

Problem 8.6

$$\mathbf{P} = \begin{bmatrix} .6 & .1 & .3 \\ .7 & .2 & .1 \\ .8 & .1 & .1 \end{bmatrix}$$

$K_1 = 40$, $K_2 = 50$, $c_1 = 6$, $c_2 = 8$, $w_1 = 1.2$,$w_2 = 1.6$, $\mu = 3$

Table 8.6
(TEC[1] and TEC[2] for different values of K in Problem 8.6)

M_1	M_2	TEC^1	TEC^2 (K=50)	TEC^2 (K=60)	TEC^2 (K=70)	TEC^2 (K=80)
2	2	15.702962	14.344880	15.197474	16.050068	16.902662
1	3	20.861941	20.670548	21.517291	22.364034	23.210777
3	1	16.453034	14.489355	15.455812	16.422269	17.388726
2	3	15.536015	15.499629	16.346471	17.193312	18.040154
3	2	14.327653	**12.967759**	**13.750031**	**14.532303**	**15.314576**
3	3	**14.160706**	13.897772	14.641830	15.385888	16.129946

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