REFERENCE ONLY

MAPS PRESERVING CERTAIN ASPECTS OF LINEAR OPERATORS

THESIS SUBMITTED TO
THE COCHIN UNIVERSITY OF SCIENCE AND TECHNOLOGY
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY
UNDER THE FACULTY OF SCIENCE

By

RANI MARIA THOMAS

DEPARTMENT OF MATHEMATICS AND STATISTICS
COCHIN UNIVERSITY OF SCIENCE AND TECHNOLOGY
COCHIN-682 022, INDIA

MARCH 1994

CERTIFICATE

Certified that the thesis entitled " MAPS PRESERVING CERTAIN ASPECTS OF LINEAR OPERATORS " is a bona fide record of work done by Rani Maria Thomas under my guidance in the Department of Mathematics and Statistics, Cochin University of Science and Technology and that no part of it has been included any where previously for the award of any degree or title.

Dr. M.N. NARAYANAN NAMBOODIRI
Lecturer in Mathematics
Department of Mathematics and Statistics
Cochin University of Science and Technology
COCHIN 682 022

Cochin 682 022
March 1, 1994

CONTENTS

CHAPTER O

INTRODUCTION

0.1. GENERAL INTRODUCTION

Let $M_n(\mathbb{C})$ denote the set of all n x n matrices over the field $\mathbb{C}$ of complex numbers and let Φ be a linear transformation on it. In 1959 Marcus and Moyl's [20] proved the following elegant theorem

MARCUS AND MOYL'S THEOREM

Let $\Phi : M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$ be a linear transformation. Then Φ preserves eigen values and their multiplicities if and only if there exists a non singular matrix A in $M_n(\mathbb{C})$ such that

$$\Phi(T) = ATA^{-1} \text{ for all } T \text{ in } M_n(\mathbb{C})$$

In 1959 itself Marcus and Purves [21] characterised invertibility preserving linear maps on $M_n(\mathbb{C})$. Their characterisation is as follows:

MARCUS AND PURVES THEOREM

Let $\Phi : M_n(\mathbb{C})$ to $M_n(\mathbb{C})$ be a linear transformation.

Then Φ preserves invertibility of matrices in $M_n(\mathbb{C})$ if and only if Φ is a Jordan homomorphism; that is

$$\Phi(T_1T_2+T_2T_1) = \Phi(T_1)\ \Phi(T_2) + \Phi(T_2)\ \Phi(T_1)$$

for all T_1, T_2 in $M_n(\mathbb{C})$.

These two results created a lot of research activity. Since $M_n(\mathbb{C})$ can be identified with the Banach algebra of all linear operators on a finite dimensional Hilbert space, various attempts were made to generalize these results to the algebras of operators. So the general problem studied by various Mathematicians can be stated as follows:

PROBLEM

Let $\mathfrak{A}$ and $\mathfrak{B}$ be Complex Banach algebras with identity and let $\Phi: \mathfrak{A} \longrightarrow \mathfrak{B}$ be a linear map. When does Φ preserves the spectrum of elements of $\mathfrak{A}$.

This problem when $\mathfrak{A}$ and $\mathfrak{B}$ are C*-algebras were studied by Russo [25], Gleason [11], Kahane and Zelasko [15], Bernard Aupetit [2], M-D Choi, D.Hadwin,E. Nordgren,H. Radjavi, P.Rosenthal [6]etc.

The general problem remains open even now. But when $\mathcal{A} = \beta(X), \mathcal{B} = \beta(Y)$, the Banach algebras of all bounded linear operators on Complex Banach spaces X and Y, considerable progress have been made by Mathematicians like Ali A. Jafarian and A.R. Sourour [14], Hou Jin-Chuan [13].

Motivated by these developments, Mathematicians started studying linear maps between operator algebras preserving other properties like positivity, hermiticity, commutativity, ranks of operators, trace of operators etc. In this direction significant contributions were made by Heydar Radjavi [26], Bernard Russo [24], Roy B. Beasley [17], [18], G.H. Chan and M.H.Lim [5], Marvin Marcus [19], Raphael Loewy [16], Roger A. Horn, Chi-Kwongli and Nam-Kiu Tsing [12], C.K. Fong and A.R. Sourour [10], HOU Jin-Chuan [13], etc.

This thesis is an attempt to continue the work on similar problems.

0.2. DEFINITIONS AND NOTATIONS

Let $\mathbb{C}$ denote the set of all complex numbers. All the vector spaces considered in this thesis are over $\mathbb{C}$. Also it is assumed that all the topological vector spaces considered here are Hausdorff.

C*-ALGEBRA

A <u>C*-algebra</u> is a uniformly closed subalgebra of the set $\beta(H)$ of all bounded linear operators on a complex Hilbert space H, which is closed under the adjoint operation * .

POSITIVITY, COMPLETE POSITIVITY

An element T in a C*-algebra $\mathcal{A}$ is said to be positive and written $T \geq 0$ if $T = V^*V$ for some V in $\mathcal{A}$

A linear map $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are C*-algebras is called positive if

$$\Phi(T) \geq 0 \text{ whenever } T \in \mathcal{A} \text{ and } T \geq 0.$$

Let $\mathcal{A}$ be a C*-algebra and A_n denote the C*-algebra of all n x n matrices with entries from $\mathcal{A}$.

Let $\Phi^{(n)}: \mathfrak{A}_n \longrightarrow \mathfrak{B}_n$ be defined as follows;

$$\Phi^{(n)}([a_{ij}]) = [\Phi(a_{ij})], \ [a_{ij}] \varepsilon \mathfrak{A}_n.$$

If $\Phi^{(n)}$ is positive for n =1,2,..., then Φ is called <u>completely positive</u>.

* REPRESENTATION, IRREDUCIBLE REPRESENTATION

A * <u>representation</u> of a C^*-algebra $\mathfrak{A}$ on a Hilbert space is a homomorphism of $\mathfrak{A}$ into $\beta(H)$ which preserves involution * in $\mathfrak{A}$.

A * representation π of $\mathfrak{A}$ on H is called irreducible if the only closed subspaces of H invariant under $\pi(\mathfrak{A})$ are H and $\{0\}$.

It is well known that every C*-algebra $\mathfrak{A}$ has an irreducible representation [1].

0.2.1. STINESPRINGS THEOREM [27]

Let Φ be a completely positive linear map from a C*-algebra $\mathfrak{A}$ to a C*-algebra $\mathfrak{B}$ on H. Then there exists a *-representation π of $\mathfrak{A}$ on K and a bounded linear map $V: H \longrightarrow K$ such that

$$\Phi(T) = V^* \pi(T) V \text{ for all } T \text{ in } \mathfrak{A}.$$

0.2.2. DENSITY THEOREM OF VON NEUMANN [1]

Let $\mathfrak{A}$ be a self adjoint algebra of operators which has trivial null space. Then $\mathfrak{A}$ is dense in the second commutant $\mathfrak{A}''$ of $\mathfrak{A}$.

0.2.3. KAPLANSKI'S DENSITY THEOREM [27]

Let $\mathfrak{A}$ be a self adjoint algebra of operators and let $\mathfrak{A}_s$ be the closure of $\mathfrak{A}$ in the strong operator topology. Then every self adjoint element in the ball of $\mathfrak{A}_s$ can be strongly approximated by self adjoint element in the ball of $\mathfrak{A}$.

CALKIN ALGEBRA

Let H be a Hilbert space and K(H) denote the two sided ideal (* closed) of all compact operators on H. Then the quotient B*-algebra (i.e., a Banach * algebra B such that $\|x^*x\| = \|x\|^2$ for all x in B). $\beta(H)/K(H)$ is called the CALKIN ALGEBRA.

0.2.4. THEOREM

If H is separable, then $K(H)$ is the only non trivial two sided ideal in $\beta(H)$ which is closed under norm topology and adjoint operation.

Consequently every non trivial * representation of $\beta(H)/K(H)$ is one-one, when H is separable.

0.3. SUMMARY OF THE THESIS

This thesis is devoted to the study of mappings between algebras of operators on locally convex topological vector spaces and their characterisations when they preserve various aspects of operators like spectrum, eigen values,hermiticity,positivity etc. Apart from the introductory chapter, the thesis is divided into three chapters.

In chapter I, spectrum preserving linear mappings from $\beta(X)$ to $\beta(Y)$ are studied where X and Y are locally convex topological vector spaces. Theorems 1.1.6, 1.1.7 and 1.2.1 are the main results proved in this chapter. Theorem 1.1.6 and 1.1.7 are generalization of the corresponding results of Jafarian and Sourour to the set up of locally convex topological vector spaces. It is observed in Remarks 1.1.9 that the proof of Theorem 1.1.6 given here is simpler than that of Jafarian and Sourour [14]. Remark 1.2.3 is another observation regarding essential spectrum preserving linear maps between $\beta(X)$ and $\beta(Y)$ when X and Y are Complex Banach spaces.

In chapter II elementary operators on $\beta(X)$ are considered. The well-known notions of hermiticity of operators on Complex Banach spaces, do not share many properties of Hilbert space adjoint. So we select a class $\widetilde{H}_L$ of operators on X which coincides with the class of self adjoint bounded linear operators on X when X is a Hilbert space. This is done in Definition 2.1.1. Then certain types of elementary operators on $\beta(X)$ which leaves $\widetilde{H}_L$ invariant are characterised. This is given in Theorem 2.1.7. In section 2.2 elementary operators on $\beta(H)$, when H is a complex separable Hilbert space are studied. Elementary operators on $\beta(H)$ which preserves essential self adjointness and essential positivity (i.e. positivity and self adjointness modulo compact operators) are characterised in theorems 2.2.3 and 2.2.8. In section 2.3 the transformation Δ_∞ on $\beta(H)$

is introduced. This transformation is an infinite series analogue of elementary operators. Theorem 2.3.7 characterises certain class of operators of the form Δ_∞, which preserves self adjointness of operators in $\beta(H)$. Finally in Remarks 2.3.10 it is observed that in the proof of JIN-CHUAN'S theorem, spectral representation of hermitian matrices may be used, instead of the explicit usage of diagonalisation. This approach may be helpful in dealing with Δ_∞ because diagonalisation of the scalar matrix (a_{ij}) may not be possible for a large class of maps of the type Δ_∞. The details are not supplied.

The third and final chapter is extremely short. There, some properties of non linear maps on $\beta(H)$ are studied, when H is a Complex Hilbert space.

CHAPTER I

SPECTRUM PRESERVING LINEAR MAPS

In this chapter the structure of spectrum preserving linear maps between $\beta(X)$ and $\beta(Y)$ is studied, where X and Y are locally convex topological vector spaces over the field $\mathbb{C}$ of complex numbers. This is a generalisation of the work of Ali A Jafarian and A.R. Sourour [14], where they considered spectrum preserving linear maps on $\beta(X)$ where X is a complex Banach space. Section 1.1 deals with this. In Section 1.2, spectrum preserving linear maps on $\beta(X)$ which preserves eigen values of operators in $\beta(X)$ are studied.

1.1. SPECTRUM PRESERVING LINEAR MAPS ON $\beta(X)$

THEOREM (ALI A JAFARIAN and A.R. SOUROUR)

Let X and Y be Banach spaces and $\Phi:\beta(X) \longrightarrow \beta(Y)$ be a spectrum preserving surjective linear mapping. Then either

(i) there is a bounded invertible operator $A:X \longrightarrow Y$ such that $\Phi(T) = ATA^{-1}$ for all T in $\beta(X)$, or

(ii) there is a bounded invertible operator B from X^* (the dual of X) to Y such that

$$\Phi(T) = BT^*B^{-1} \text{ for every } T \in \beta(X).$$

As specified we establish the same result, when X and Y are locally convex topological vector spaces over $\mathbb{C}$. Since the method adopted is the same as in [14], we start with generalising various technical results proved in [14].

Even though the proofs are exactly similar, we supply the details. Through out this section X and Y denote locally convex topological vector spaces over $\mathbb{C}$ and $\beta(X)$ the set of all continuous linear mappings on X.

LEMMA 1.1.1

Let A be in $\beta(X)$. Then $\sigma(T+A) \subseteq \sigma(T)$ for every T in $\beta(X)$ if and only if $A=0$.

PROOF

$$A = 0 \Longrightarrow \sigma(T+A) = \sigma(T) \quad \text{for all } T.$$

Now assume that $\sigma(T+A) \subseteq \sigma(T)$ for all T.

To show that $A = 0$.

Let if possible $A \neq 0$. Then there exists $x \in X$, $x \neq 0$ such that $A(x) = y \neq 0$. By Hahn Banach theorem in locally convex topological vector spaces, there exists $f \in X^*$ (X^*- the dual of X) such that

$$f(x) = 1 \quad \text{and} \quad f(y) \neq 0$$

Let α be a nonzero complex number and let

$$T = (\alpha x-y) \otimes f, \text{ where}$$

$$T(z) = ((\alpha x-y) \otimes f)(z) = f(z)\ (\alpha x-y),\ z \in X$$

Continuity of T follows from the continuity of f.

Now

$$(T+A)(x) = T(x) + A(x) = \alpha x-y+y = \alpha x$$

Hence α is an eigen value of $T+A$.

Now one can easily show that, for $\beta \in \mathbb{C}$, $T-\beta I$ is not invertible in $\beta(X)$ if and only if either $\beta = 0$ or $\beta = f(\alpha x-y)$. Therefore $\sigma(T) = \{0,\ f(\alpha x-y)\}$

Since $f(\alpha x-y) = \alpha-f(y) \neq \alpha$, we have

$$\sigma(T+A) \nsubseteq \sigma(T)$$

This proves our assertion.

LEMMA 1.1.2.

Let Φ: $\beta(X) \longrightarrow \beta(Y)$ be a spectrum preserving linear mapping. Then Φ is injective.

PROOF

$$\sigma(T) = \sigma(\Phi(T)) \text{ for all } T \text{ in } \beta(X).$$

Suppose $\Phi(A) = \Phi(B)$, $A,B \in \beta(X)$.

To show that $A = B$

$$\begin{aligned} \sigma(T+A-B) &= \sigma\ (\Phi(T+A-B)) \\ &= \sigma\ (\Phi(T)) \\ &= \sigma\ (T) \end{aligned}$$

for all T in $\beta(X)$. Hence by lemma 1.1.1, $A-B = 0$.

LEMMA 1.1.3.

If Φ: $\beta(X)$ to $\beta(Y)$ is a spectrum preserving surjective mapping, then $\Phi(I_X) = I_Y$, where I_X (or I_Y) denote the identity mapping on X (or Y respectively).

PROOF

Let $\Phi(S) = I_Y$

For T in $\beta(X)$,

$$\begin{aligned}\sigma(T+S-I_X) &= \sigma(\Phi(T-I_X+S))\\ &= \sigma(\Phi(T)-\Phi(I_X)+\Phi(S))\\ &= \sigma(\Phi(T))\\ &= \sigma(T)\end{aligned}$$

Hence by lemma 1.1.1, $S = I_X$.

LEMMA 1.1.4.

Let X be a locally convex topological vector space and K(X) denote the set of all compact operators on X. Let A be in $\beta(X)$ and C be in K(X). If $\lambda \in \sigma(A)$ is not an eigen value of A, then $\lambda \in \sigma(A+C)$.

PROOF

Let if possible, $A+C-\lambda.I_X$ is invertible in $\beta(X)$. Therefore,

$$A-\lambda.I_X = (A+C-\lambda.I_X)[I_X-(A+C-\lambda I_X)^{-1}C]$$

Since $(A+C-\lambda I_X)^{-1}$ is continuous and C is compact,

$(A+C-\lambda I_X)^{-1}.C$ is compact [9]

Case 1

$I_X - (A+C-\lambda\cdot I_X)^{-1}$ C is invertible.

In this case $A-\lambda\cdot I_X$ is invertible.

Case 2

$I_X-(A+C-\lambda\cdot I_X)^{-1}$ is not invertible.

Here, since $(A+C-\lambda\cdot I_X)^{-1}C$ is compact, 1 is an eigen value of $(A+C-\lambda I_X)^{-1}C$ [9].

Hence, there exists a non zero vector x such that

$$I_X - (A+C-\lambda\cdot I_X)^{-1}C\ (x) = 0.$$

Therefore $(A-\lambda\cdot I_X)\ (x) = 0.$

In either case the conclusions are contradictions to the assumption that $\lambda \varepsilon\ \sigma(A)$ but not an eigen value of A.

This completes the proof.

LEMMA 1.1.5.

For T in $\beta(X)$, $x \varepsilon X$, $f \varepsilon X^*$ and λ not in $\sigma(T)$ we have $\lambda\ \varepsilon\ \sigma(T+ x\otimes f)$ if and only if $f((\lambda I_X-T)^{-1}(x))=1$.

PROOF

Assume that $f(\lambda \cdot I_X - T)^{-1}(x) = 1$

Then $(x \otimes f)\ ((\lambda \cdot I_X - T)^{-1}(x)) = (f(\lambda \cdot I_X - T)^{-1}(x)).x$

$$= x$$

Hence

$$(T + x \otimes f)(\lambda \cdot I_X - T)^{-1}(x)$$

$$= T((\lambda \cdot I_X - T)^{-1}(x)) + (x \otimes f)(\lambda\, I_X - T)^{-1}(x)$$

$$= T(\lambda \cdot I_X - T)^{-1}(x) + x$$

$$= (T(\lambda \cdot I_X - T)^{-1} + I_X)(x)$$

$$= (T + \lambda \cdot I_X - T)(\lambda \cdot I_X - T)^{-1}(x)$$

$$= \lambda(\lambda \cdot I_X - T)^{-1}(x)$$

Therefore λ is an eigen value of $T + x \otimes f$.

Conversely assume that $\lambda \in \sigma(T + x \otimes f)$. Then by lemma 1.1.4, λ is an eigen value of $T + x \otimes f$. Hence there exists a non zero vector u in X such that

$$(T + x \otimes f)(u) = \lambda\, u$$

i.e., $T(u) + f(u).x = \lambda\, u$

since $\lambda \notin \sigma(T)$, $f(u) \neq 0$

$\therefore \quad (\lambda I_X - T)^{-1}(x) = \frac{u}{f(u)}$

i.e., $f((\lambda I_X - T)^{-1}(x)) = 1$

THEOREM 1.1.6.

Let $A \in \beta(X)$, $A \neq 0$. Then the following conditions are equivalent.

(1) A has rank 1

(2) $\sigma(T+A) \cap \sigma(T+cA) \subseteq \sigma(T)$ for every T in $\beta(X)$ and every $c \neq 1$.

PROOF

Assume that A is of rank 1. Hence there exists $x \in X$ and $f \in X^*$ such that

$$A = x \otimes f.$$

Now let T be in $\beta(X)$ and λ not in $\sigma(T)$. Then by lemma 1.1.5 λ is in $\sigma(T+cA)$ if and only if $f((\lambda I_X - T)^{-1}(x)) = 1$. Hence λ does not belong to $\sigma(T + cA)$ for two distinct values of c. Hence (1) implies (2).

Now to show that (2) implies (1).

Assume that rank $A \geq 2$.

Case 1

$A = \alpha . I_X$ for some nonzero scalar α.

Let T in $\beta(X)$ be such that $\sigma(T) = \{0, \alpha\}$.
It is enough to take $T = y \otimes g$ for suitable $y \in X$ and g in X^*.

Then

$$\sigma(T+A) = \{\alpha, 2\alpha\} \text{ and}$$
$$\sigma(T+2A) = \{2\alpha, 3\alpha\}.$$

Therefore $\sigma(T+A) \cap \sigma(T+2A) = \{2\alpha\}$ which is not contained in $\sigma(T)$. This completes the proof of case 1.

Case 2.

$A \neq \alpha I_X$ for any α in C and rank $A \geq 2$.

Case 2'.

There exists a vector $u \in X$ such that $\{u, Au, A^2u\}$ is linearly independent. Let U be the linear span of $\{u, Au, A^2u\}$ and V be a closed complement of U in X. It is enough to take

$$V = \ker f_{Au} \cap \ker f_{A^2u} \cap \ker f_u$$

where fu, f_{Au}, f_{A^2u} are bounded linear functionals on X such that

$$f_{A^m u}(A^n u) = \delta_{m,n}, \quad m,n = 0,1,2,\ldots$$

where,

$$\delta_{m,n} = 0 \quad \text{if } m \neq n$$
$$= 1 \quad \text{if } m = n$$

Put

$$Nu = u - Au$$
$$N(Au) = Au - 2A^2u$$

$N(A^2u) = -\frac{u}{2} + \frac{3Au}{2} - 2A^2u$ and

$N_v = 0$ for all v in V and extend it linearly.

Clearly $N \varepsilon \beta(X)$ and $N^3 = 0$, $(N+A)(u)=u$ and $(N+2A)(Au) = Au$.

Therefore, $1 \varepsilon \sigma(N+A) \cap \sigma(N+2A)$

But $\sigma(N) = \{0\}$.

Thus $\sigma(N+A) \cap \sigma(N+2A)$ is not contained in $\sigma(N)$.

This establishes case 2'.

Case 2"

$\{u, Au, A^2u\}$ are linearly dependent for every u in X.

Let $A^2u = \alpha u + \beta Au$ for some scalars α and β.

First we assume that $\alpha \neq 0$.

Let $N(x) = x$ for every $x \varepsilon V$, and

$N(u) = -Au$

$N(Au) = A^2u = \alpha u + \beta Au$

Clearly N is invertible.

Also $(N+A)(u) = 0$ and $(N-A)(Au) = 0$

Thus $0 \varepsilon \sigma(N+A) \cap \sigma(N-A)$ whereas 0 is not in $\sigma(N)$.

If $A^2(u) = \beta(u)Au$ for every u, we get,

$$A^2(u-v) = A^2u - A^2v = \beta(u)Au - \beta(v)Av$$
$$= \beta(u-v)Au - \beta(u-v)Av$$

$$\Longrightarrow \beta(u) = \beta(v) = \beta(u-v) \text{ since rank } A \geq 2.$$

Thus $A^2 = \beta A$ for a fixed scalar β.

i.e., $P(A) = 0$ where $P(t) = t^2 - \beta.t$

Now rank $(A) \geq 2$. Also 0 and β are eigen values of A. Hence there exists three linearly independent vectors x, y, Az such that

$$A(x) = 0$$
$$A(y) = \beta y, \text{ and}$$
$$A(z) \neq 0$$

Let W be the three dimensional space generated by x,y and A(z). Then $A(W) \subseteq W$ and $\begin{bmatrix} \beta & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is the matrix of A/W with respect to $\{A(z), y, x\}$. Now we define a nilpotent operator N as follows. Let Z be a complement of W in X and let $N(Z) = \{0\}$. Let N/W has the matrix representation

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2\beta & 2\beta \\ 0 & -2\beta & -2\beta \end{bmatrix} \text{ with respect to } \{A(z), y, x\}.$$

Then N is nilpotent. One can easily see that 2β is an eigen value of $(N+A)$ and $(N+2A)$. Since $\beta \neq 0$ we get

$$\sigma(N+A) \cap \sigma(N+2A) \not\subseteq \sigma(N).$$

Now we prove the main theorem of this section.

THEOREM 1.1.7.

Let $\Phi: \beta(X) \longrightarrow \beta(Y)$ be a spectrum preserving surjective linear mapping. Then either

(i) there is an invertible linear operator $A: X \longrightarrow Y$ such that $\Phi(T) = ATA^{-1}$ for every T in $\beta(X)$ for which there is an unbounded sequence in $C-\sigma(T)$

or

(ii) there is an invertible linear operator $B: X^* \longrightarrow Y$ such that $\Phi(T) = BT^*B^{-1}$ for every T in $\beta(X)$, for which there is an unbounded sequence in $\mathbb{C}-\sigma(T)$.

PROOF

Let x and f be nonzero elements in X and X^* respectively. Let L_x and R_f be linear subspaces of $\beta(X)$ defined by

$$L_x = \{x \otimes h : h \in X^*\} \text{ and}$$

$$R_f = \{u \otimes f : u \in X\}$$

First we prove the following. Corresponding to each x in X there is a $y \in Y$ such that

$$\Phi(L_x) = L_y$$

or corresponding to each x in X, there is a $g \in X^*$ such that

$$\Phi(L_x) = R_g$$

Also if $\Phi(L_x) = L_y$ for some $x \in X$, then

$$\Phi(L_u) \neq R_g \text{ for any } u \in X$$

This follows from the following observations.

(i) By lemma 1.1.2 and theorem 1.1.6, if R is of rank one, $\Phi(R)$ is of rank one.

(ii) $L_y \cap R_g$ is one dimensional where as $L_u \cap L_v$ has dimension 0 or dimension X^*.

(iii) If $\Phi(L_u) = L_y$ for some $u \in X$, then

$$\Phi(L_v) \neq R_g \text{ for any } v \text{ in } X.$$

For,

$$\Phi(L_u \cap L_v) = \Phi(L_u) \cap \Phi(L_v) = L_y \cap R_g$$

Since Φ is one-one and onto, dimension $L_u \cap L_v$ should equal dimension $\Phi(L_u \cap L_v) = \dim L_y \cap R_g$ which is not possible. This leads to two cases.

Case 1.

$\Phi(L_x) = L_{y(x)}$ for every $x \in X$. Put $y(x)=y$ for brevity.

Therefore,

$$\Phi(x \otimes f) = y \otimes g \quad \text{for some } g \in X^*$$

Now let,

$C_x \colon X^* \longrightarrow Y^*$ be defined by

$C_x(f) = g$. Clearly C_x is linear.

Claim

The set, $\{C_x : x \in X\}$ is one dimensional. Let if possible, there exists two linearly independent transformations C_{x_1} and C_{x_2}, where

$$\Phi(x_1 \otimes f) = y_1 \otimes C_{x_1}(f) \quad \text{and}$$

$$\Phi(x_2 \otimes f) = y_2 \otimes C_{x_2}(f)$$

Now,

$$\Phi((x_1+x_2) \otimes f) = \Phi(x_1 \otimes f) + \Phi(x_2 \otimes f)$$

Since $x_1+x_2 \otimes f$ is of rank 1,

$$\Phi(x_1+x_2)\otimes f = y\otimes g$$

for some $y \in X$ and $g \in X^*$. Hence we get

$$C_{x_1}(f)(u).y_1 + C_{x_2}(f)(u).y_2 = g(u).y$$

for every u in X. Since C_{x_1} and C_{x_2} are linearly independent, y_1 and y_2 should be linearly dependent.

Hence $L_{y_1} = L_{y_2}$

$\therefore \quad \Phi(L_{x_1}) = \Phi(L_{x_2})$

Since Φ is one-one we have $L_{x_1}=L_{x_2}$. Therefore x_1 and x_2 are linearly independent. Then C_{x_1} and C_{x_2} are linearly dependent. This is a contradiction.

$\therefore \quad$ dimension $\{C_x : x \in X\} = 1$

Hence there is a linear operator C such that

$$\{C_x : x \in X\} = \{\lambda C : \lambda \in \mathbb{C}\}$$

Therefore,

$$\Phi(x\otimes f) = y\otimes Cf \text{ where } y \text{ depends on } x.$$

Put $Ax=y$. Hence $\Phi(x\otimes f) = Ax \otimes Cf$. Since Φ is bijective both A and C are bijective linear mappings. Now let $T\in \beta(X)$ be such that there is an unbounded sequence in $\mathbb{C} -\sigma(T)$.

$$\Phi(T+x\otimes f) = \Phi(T) + Ax\otimes Cf$$

Let λ be not in $\sigma(T)$. Hence by lemma 1.1.5 we have

$$f((\lambda -T)^{-1}x)= 1 \text{ if and only if } \lambda \in \sigma(\Phi(T)+Ax\otimes Cf)$$

and

$$\lambda \in \sigma(\Phi(T)+ Ax\otimes Cf) \text{ if and only if}$$
$$Cf(\lambda I_Y- \Phi(T))^{-1} A(x) = 1$$

Thus for λ not in $\sigma(T)$, we get

$$f((\lambda I_X-T)^{-1}(x)) = Cf((\lambda I_Y-\Phi(T))^{-1} Ax)$$

Replacing λ with $\frac{1}{z}$ and using similar argument as in [14] we get,

$$f((I_X-zT)^{-1}(x)) = Cf(I_Y-z\Phi(T))^{-1}(y), \text{ where } A(x)=y$$

That is

$$f((I_X-zT)^{-1} A^{-1}(y)) = Cf(I_Y-z\Phi(T))^{-1}(y)$$

Since $\mathcal{O} - \sigma(T)$ contains an unbounded sequence, by taking the limit as $z \longrightarrow 0$ we get,

$$f(A^{-1}(y)) = Cf(y)$$

Again $$\frac{f[(I_X - zT)^{-1} A^{-1}(y) - A^{-1}(y)]}{z}$$

$$= \frac{Cf[(I_Y - z\Phi(T))^{-1}(y) - y]}{z}$$

Now letting z tend to 0 we get,

$$f(TA^{-1}(y)) = Cf(\Phi(T)y) \text{ for all } f \in X^*$$

But we already have,

$$f(A^{-1}(y)) = Cf(y)$$

Combining these two we get,

$$Cf(\Phi(T)(y)) = f(A^{-1}\,\Phi(T)(y)),$$

$$Cf(\Phi(T)(y)) = f(TA^{-1}(y))$$

and $$Cf(\Phi(T)(y)) = A^{-1}(\Phi(T)(y))$$

Thus we get,

$$f(TA^{-1}(y)) = f(A^{-1}\Phi(T)y) \text{ for every } f \text{ in } X^*.$$

That is, $TA^{-1}(y) = A^{-1}\,\Phi(T)(y)$ for all $y \in Y$

That is, $ATA^{-1} = \Phi(T)$

Case 2

Let $x \in X$ and $\Phi(L_x) = R_g$ for some $g \in Y^*$. As in case 1, we can show that for each $x \in X$ and $f \in X^*$,

$$\Phi(x \otimes f) = Bf \otimes Ax$$

where $B: X^* \longrightarrow Y$ is linear.

As before for $T \in \beta(X)$, $x \in X$, $f \in X^*$ and $\lambda \notin \sigma(T)$, $\lambda \in \sigma(T+x\otimes f)$ if and only if $f((\lambda I_X-T)^{-1}(x)) = 1$ and finally for every $x \in X$, $f \in X^*$ and $\lambda \notin \sigma(T)$

$$f((\lambda I_X-T)^{-1}(x)) = A(x)\ ((\lambda I_Y-\Phi(T))^{-1}\ (Bf))$$

Now for $T \in \beta(X)$ such that $\mathbb{C}-\sigma(T)$ contains an unbounded sequence, identical arguments leads to the conclusion

$$f(T(x)) = A(x)\ (\Phi(T)\ B(f))$$

$$f(x) = A(x)\ (B(f))$$

Therefore,

$$A(x)\ \Phi(T)\ B(f) = f(T(x)) = AT(x)\ (B(f)) = A(x)(BT^*f)$$

Hence we get

$$g\,\Phi(T)\,(B(f)) = g(BT^*(f)) \text{ for all } g \text{ in } Y^*$$

Therefore,

$$\Phi(T)\,B(f) = BT^*f \text{ for all } f \text{ in } X^*$$

Thus $\Phi(T) = BT^*B^{-1}$.

REMARK 1.1.8.

One does not know whether the operators A or B obtained in Theorem 1.1.7 is continuous or not. But when X and Y are Frechet spaces, using closed graph theorem, continuity of A and B can be established [9].

REMARK 1.1.9.

It is to be observed that Theorem 1.1.6 is a generalisation of the corresponding theorem of Jafarian and Sourour [14]. Though the proof goes along the same line as in [14], our proof is simpler in the following sense. By considering one more simple case, we are able to arrive at the quadratic polynomial $P(t) = t(t-\beta)$ such that $P(A) = 0$ directly without using any existence theorems. Also the other forms of minimal quadratic polynomials are not needed at all.

1.2. EIGEN VALUE PRESERVING LINEAR MAPS

In section 1.1 we analysed spectrum preserving surjective linear maps of $\beta(X)$ to $\beta(Y)$, where X and Y are locally convex topological vector spaces over $\mathbb{C}$. In this section we characterise spectrum preserving linear maps which preserves eigen values when X and Y are complex Banach spaces with Schauder basis.

THEOREM 1.2.1.

Let X and Y be Complex Banach spaces with Schauder basis and $\Phi: \beta(X) \longrightarrow \beta(Y)$ be a spectrum preserving, surjective linear mapping. Then Φ preserves eigen values if and only if it is of the form $\Phi(T) = ATA^{-1}$ for every T in $\beta(X)$ where $A:X \longrightarrow Y$ is a bounded invertible linear operator.

PROOF

Let $\Phi(T) = ATA^{-1}$, for all T in $\beta(X)$, $A:X \longrightarrow Y$ an invertible bounded linear map. Let $\lambda \varepsilon \sigma(T)$. Then

$T(x) = \lambda x$ for some nonzero x in X.

Let $y = A(x)$. Thus $TA^{-1}(y) = \lambda A^{-1}(y)$

i.e., $ATA^{-1}(y) = \lambda y$

Therefore λ is an eigen value of T.

From theorem 1.1.7, either

(i) $\Phi(T) = ATA^{-1}$, $A:X \longrightarrow Y$ a bounded invertible linear map, or

(ii) $\Phi(T) = BT^*B^{-1}$, where $B: X^* \longrightarrow Y$ is a bounded invertible linear map.

We show that if Φ takes the form (ii), Φ will not preserve eigen values for all T.

Let if possible λ is an eigen value of T implies λ is an eigen value of BT^*B^{-1}. Then there exists a nonzero x in X such that $T(x) = \lambda\cdot x$.

i.e., $(T-\lambda\cdot I_X)$ is not one-one.

Since λ is an eigen value of BT^*B^{-1},

$$BT^*B^{-1}(y) = \lambda\cdot y \quad \text{for some non zero } y \text{ in } X.$$

Therefore $T^*B^{-1}(y) = \lambda B^{-1}(y)$

i.e., λ is an eigen value of T^*.

Let $f = B^{-1}(y)$. Then we get,

$$(T^*f)(z) = \lambda f(z) \text{ for all } z \text{ in } X.$$

Hence,

$$f((T-\lambda I)(z)) = 0 \text{ for all } z \in X.$$

Since f is a non zero continuous linear functional on X, its null space is a proper closed subspace of X. Therefore Range $(T-\lambda I)$ is not dense in X.

Now we show that there is a bounded linear operator S on X which is not one-one but onto. In this case, 0 is an eigen value of S but it is not an eigen value of $\Phi(S)$.

Let $\{x_1, x_2, \ldots, x_n\}$ be a Schauder basis in X.

For z in X,

$$z = \sum_{n=1}^{\infty} \alpha_n(z) x_n, \text{ where } \alpha_n(z) \in \mathbb{C} \qquad n=1,2,\ldots$$

Put $$S(z) = \frac{\alpha_2(z)}{2^2} x_1 + \frac{\alpha_3(z)}{3^2} x_2 + \ldots + \frac{\alpha_{n+1}(z)}{(n+1)^2} x_n + \ldots$$

Then $S \in \beta(X)$, $S(x_1) = 0$.

Now we show that Range (S) is dense in X.

Let $$y = \sum_{1}^{\infty} \alpha_n(y) x_n$$

Let $$x_n = 0.x_1 + 2^2.\alpha_1(y)x_2 + 3^2\alpha_2(y)x_3 + \ldots + (n+1)^2 \alpha_n(y)\, x_{n+1}$$

Then

$$S(x_n) = \alpha_1(y)x_1 + \alpha_2(y)x_2 + \ldots + \alpha_n(y)x_n$$

Therefore,

$$S(x_n) \longrightarrow y \text{ as } n \longrightarrow \infty$$

i.e., Range(S) is dense in Y.

REMARK 1.2.3.

Let X and Y be complex Banach spaces and let $\Phi: \beta(X) \longrightarrow \beta(Y)$ be a linear mapping. If

$$\Phi(T) = ATA^{-1} + K_1TK_2,$$

where $A:X \longrightarrow Y$ a bounded invertible linear map, $K_1:X \longrightarrow Y$ compact linear mapping and $K_2: Y \longrightarrow X$ a compact linear mapping, one can easily see that Φ preserves the essential spectrum of T, for every T in $\beta(X)$.

Now let $\Phi(T) = BT^*B^{-1} + K_1TK_2$, where $B:X^* \longrightarrow Y$ a bounded, invertible linear map, and K_1, K_2 as above. One can easily prove that $\sigma_e[\Phi(T)] \subseteq \sigma_e(T)$ for all T in $\beta(X)$, where $\sigma_e(.)$ denote the essential spectrum. The inclusion may be proper as every compact operator K on X* need not be dual of some compact operator on X.

The following is an example for that

EXAMPLE 1.2.4.

Let $X = \ell_1$, the Banach space of all summable sequences of complex numbers with ℓ_1 norm. Then $\ell_1 \subset \ell_\infty$ and the closure $\overline{\ell_1}$ of ℓ_1 under the ℓ_∞ norm is properly contained in ℓ_∞. Hence there is a non zero bounded linear functional F on ℓ_∞ such that $F(x) = 0$ for all x in ℓ_1. Let f in $\ell_\infty - \overline{\ell_1}$ be such that $F(f)=1$.

Now let,

$$\tau = f \otimes F, \text{ where } f \otimes F(g) = F(g).f, \; g \in \ell_\infty.$$

Then τ is a compact linear operator on ℓ_∞. We show that $\tau \neq T^*$ for any T in $\beta(\ell_1)$.

Let if possible $\tau = T^*$ for some T in $\beta(\ell_1)$.

Therefore,

$$\tau(g)(u) = T^*(g)(u) \text{ for all } g \text{ in } \ell_\infty \text{ and for all } u \text{ in } \ell_1$$

i.e., $F(g).f(u) = g(Tu)$ for all u in ℓ_1

Hence $0 = g(T(u))$ for all g in ℓ_1.

Now let $h \in \ell_1$ be arbitrary, and let

$T(h) = (\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_n \ldots) \in \ell_1$. Let $g = (u_1, u_2, \ldots u_n \ldots) \in \ell_1$.

Then $0 = g(T(h)) = \sum_1^\infty |u_n|^2 \Longrightarrow u_n = 0$ for all n.

Hence $T(h) = 0$. But h is arbitrary. Hence $T = 0$.

Therefore $T^* = 0$ which is not true. Hence $\tau \neq T^*$ for any T in $\beta(\ell_1)$.

CHAPTER II

ELEMENTARY OPERATORS

In this chapter, the study of elementary operators on the Banach algebra of all bounded linear operators on a complex Banach space is carried out. These observations are generalisations of some recent work of HOU JIN-CHUAN [13]. This is also based on the work of C.K. FONG and A.R. SOUROUR [10]. Section 2.1. contains these general versions.

Throughout this chapter, $\beta(X)$ will denote the Banach algebra of all bounded linear operators on a complex Banach space X. For doubly infinite sequences $\{A_i\}$ and $\{B_i\}$ in $\beta(X)$, the transformation Δ_∞ on $\beta(X)$ defined by

$$\Delta_\infty(T) \quad = \quad \sum_{-\infty}^{\infty} A_i T B_i \qquad T \in \beta(X)$$

is studied in section 2.3.

2.1. HERMITICITY PRESERVING ELEMENTARY OPERATORS

HOU JIN-CHUAN'S THEOREM.

Let $\{A_i\}_{i=1}^{n}$ and $\{B_i\}_{i=1}^{n}$ be operators in $\beta(H)$, where H is a complex Hilbert space. Then the elementary operator Δ on $\beta(H)$ defined by

$$\Delta(T) = \sum_{i=1}^{n} A_i T B_i$$

is self adjointness preserving if and only if there exists operators $D_1, D_2, \ldots, D_n$ in $\beta(H)$ such that

$$\Delta(T) = \sum_{i=1}^{\ell} D_i T D_i^* - \sum_{i=\ell+1}^{n} D_i T D_i^*$$

for every T.

We wish to consider similar characterisation problems when X is a complex Banach space. There are several notions of hermiticity of operators in $\beta(X)$. Among them, the well known notions are due to LUMER [23] and STAMPFLI [23]. Recall [3,4] that these notions of hermiticity do not possess some well known properties of selfadjoint operators in Hilbert spaces. For example square of a hermitian operator need not be hermitian in the Lumer's sense. Similarly sum of two hermitian operators need not be hermitian in the Stampfli's sense [3, 4]. So we introduce a new class $\widetilde{H}_L$ as follows and designate an operator hermitian if it belongs to the class $\widetilde{H}_L$.

DEFINITION 2.1.1.

Let H_L denote the class of all operators in $\beta(X)$ which are hermitian in the Lumer's sense. Then $\widetilde{H}_L$ is the largest linear subspace of H_L over the field of real numbers such that

(1) $T \in \widetilde{H}_L$ implies $T^2 \in \widetilde{H}_L$

(2) $T_1, T_2 \in \widetilde{H}_L$ implies $i(T_1T_2 - T_2T_1) \in \widetilde{H}_L$, $i = \sqrt{-1}$

(3) $\widetilde{H}_L + i\,\widetilde{H}_L$ contains all rank one operators on X.

REMARK 2.1.2.

When X is a complex Hilbert space, Lumer's hermiticity and Stampfli's hermiticity coincides with usual Hilbert space self adjointness of operators.

In general $\widetilde{H}_L$ is a proper subset of H_L.

Now let $\{A_i\}_{i=1}^{n}$ and $\{B_i\}_{i=1}^{n}$ be operators from the class $\widetilde{H}_L$. We wish to characterise the associated elementary operator which leaves $\widetilde{H}_L$ invariant. To achieve this, the following Lemmas are needed.

LEMMA 2.1.3.

Let A_i and B_i $(1 \leq i \leq n)$ be bounded linear operators on a Banach space X, where $B_1, B_2, \ldots, B_n$ are linearly independent. Then $\Delta(T) = 0$ for all T in $\widetilde{H}_L$ if and only if $A_i = 0$, for $i = 1,2,\ldots,n$.

PROOF

Assume that $\Delta(T) = 0$ for all T in $\widetilde{H}_L$. Since $B_1, B_2, \ldots, B_n$ are linearly independent, there exists [10] vectors $x_1, x_2, \ldots, x_r$ in X and linear functionals $f_1, f_2, \ldots, f_r$ in X* such that

$$\sum_{k=1}^{r} f_k (B_j x_k) = 0 \quad \text{if} \quad j=2,\ldots,n$$
$$= 1 \quad \text{if} \quad j=1.$$

Now let $x \in X$ and $T_j = f_j \otimes x$, $j = 1,2,\ldots,r$.

Since $T_j \in \widetilde{H}_L + i\,\widetilde{H}_L$, $\Delta(T_j) = 0$, $j=1,2,\ldots,r$.

Then we have

$$0 = \sum_{j=1}^{r} \Delta(T_j)\,(x_j)$$

$$= \sum_{j=1}^{r} \sum_{i=1}^{n} A_i\, T_j\, B_i(x_j)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{r} A_i\, T_j\, B_i(x_j)$$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{r} f_j(B_i\, x_j)\right) A_i(x)$$

$$= A_1(x)$$

Thus $A_1 = 0$, since x is arbitrary.

Similarly one can show that $A_i = 0$ for all $i = \;\; 2,\ldots,n$.

LEMMA 2.1.4.

Let A_i and B_i $(1 \leq i \leq n)$ be in $\beta(X)$ where $\{B_1, B_2, \ldots, B_m\}$ $m<n$ form a maximal linearly independent subset of $\{B_1, B_2, \ldots, B_n\}$. Then $\Delta(T) = 0$ for all T in $\widetilde{H}_L$ if and only if

$$A_k = - \sum_{j=m+1}^{n} a_{kj} A_j \quad (1 \leq k \leq m)$$

where

$$B_j = \sum_{k=1}^{m} a_{kj} B_k \quad (m+1 \leq j \leq n)$$

PROOF

If $A_k = - \sum_{j=m+1}^{n} a_{kj} A_j$ $(1 \leq k \leq m)$, then one can easily see that $\Delta(T) = 0$ for all T in $\widetilde{H}_L$ by substituting for A_k and then rearranging the expression.

Conversely assume that $\Delta(T) = 0$ for all T in $\widetilde{H}_L$. Since $B_1, B_2, \ldots, B_m$ is a maximal linearly independent subset of $B_1, B_2, \ldots, B_n$, there exists constants a_{kj} $(1 \leq k \leq m$ and $m+1 \leq j \leq n)$ such that

$$B_j = \sum_{k=1}^{m} a_{kj} B_k$$

Substituting this in $\Delta(T) = 0$ we get,

$$0 = \Delta(T) = \sum_{k=1}^{m} (A_k + \sum_{j=m+1}^{n} a_{kj} A_j) TB_k$$

Since $\{B_1, B_2, \ldots, B_m\}$ is linearly independent by

Lemma 2.1.3 we must have $A_k + \sum_{j=m+1}^{n} a_{kj} A_j = 0$.

LEMMA 2.1.5.

Let A_i and B_i $(1 \leq i \leq n)$ be bounded linear operators on a Banach space X where $A_1, A_2, \ldots, A_n$ are linearly independent. Then $\Delta(T) = 0$ for all T in $\widetilde{H}_L$ if and only if $B_i = 0$ for all $i = 1,2,\ldots,n$.

PROOF

Assume that $\Delta(T) = 0$ for all T in $\widetilde{H}_L$. Since $A_1, A_2, \ldots, A_n$ are linearly independent, there exists [10] vectors $x_1, x_2, \ldots, x_r$ in X and linear functionals $f_1, f_2, \ldots, f_r$ in X^* such that

$$\sum_{k=1}^{r} f_k (A_j x_k) \begin{array}{ll} = 0 & \text{if } j = 2,\ldots,n \\ = 1 & \text{if } j = 1. \end{array}$$

Now let $T_j = f \otimes x_j$, $f \in X^*$, $j = 1,2,\ldots,r$

Since $T_j \in \widetilde{H}_L + i\,\widetilde{H}_L$, we have $\Delta(T_j) = 0$ for all j.

Therefore $\sum_{j=1}^{r} \Delta(T_j)^* (f_j)(x) = 0$ for all x in X.

But $\sum_{j=1}^{r} \Delta(T_j)^* (f_j)(x)$

$$= \sum_{k=1}^{n} \left(\sum_{j=1}^{r} f_j(A_k x_j) \right) f(B_k x)$$

$$= f(B_1 x)$$

Since x and f are arbitrary, $B_1 = 0$. Similarly we can prove that $B_i = 0$ for $i = 2,\ldots,n$.

LEMMA 2.1.6.

Let A_i and B_i $(1 \leq i \leq n)$ be in $\beta(X)$ where $\{A_1, A_2, \ldots, A_m\}$ $m<n$ is a maximal linearly independent subset of $\{A_1, A_2, \ldots, A_n\}$. Then $\Delta(T) = 0$ for all T in $\widetilde{H}_L$ if and only if

$$B_k = - \sum_{j=m+1}^{n} a_{kj} B_j \qquad (1 \leq k \leq m)$$

where $$A_j = \sum_{k=1}^{m} a_{kj} A_k \qquad (m+1 \leq j \leq n)$$

The proof is quite similar to the proof of Lemma 2.1.4 and hence omitted.

Now we prove the main theorem of this section.

THEOREM 2.1.7.

Let $\{A_j\}_{j=1}^{n}$ and $\{B_j\}_{j=1}^{n}$ be operators from the class $\widetilde{H}_L$. Then $\Delta(\widetilde{H}_L) \subseteq \widetilde{H}_L$, where $\Delta(T) = \sum_{j=1}^{n} A_j T B_j$ if and only if there are operators $D_1, D_2, \ldots, D_n$, in $\widetilde{H}_L$ such that

$$\Delta(T) = \sum_{j=1}^{\ell} D_j \, T D_j - \sum_{j=\ell+1}^{n} D_j \, T \, D_j$$

for every T in $\beta(X)$ where X is a complex Banach space such that $\widetilde{H}_L + i\,\widetilde{H}_L$ contains all rank one operators on X.

PROOF

Suppose $\Delta(T) = \sum_{j=1}^{\ell} D_j \, T \, D_j - \sum_{j=\ell+1}^{n} D_j \, T \, D_j$ for every T in $\beta(X)$ where $D_1, D_2, \ldots, D_n$ are in $\widetilde{H}_L$. To show that $\Delta(T)$ belongs to $\widetilde{H}_L$ whenever T belongs to $\widetilde{H}_L$.

It is enough to prove that A,B belongs to $\widetilde{H}_L$ implies ABA belongs to $\widetilde{H}_L$.

We have AB + BA belongs to $\widetilde{H}_L$.

Therefore $A(AB+BA) + (AB + BA)A$

$$= A^2B + BA^2 + 2ABA \text{ belongs to } \widetilde{H}_L.$$

This implies ABA belongs to $\widetilde{H}_L$. Using this we can see that $\Delta(T)$ belongs to $\widetilde{H}_L$, whenever T belongs to $\widetilde{H}_L$.

Conversely assume that $\Delta(T)$ belongs to $\widetilde{H}_L$ for all T in $\widetilde{H}_L$.

CLAIM

$$\sum_{j=1}^{n} A_j \, T \, B_j = \sum_{j=1}^{n} B_j T A_j \text{ for every } T \text{ in } \widetilde{H}_L.$$

Since the identity operator I belongs to $\widetilde{H}_L$,

$$\sum_{j=1}^{n} A_j \, B_j \text{ belongs to } \widetilde{H}_L.$$

$$\sum_{j=1}^{n} B_j \, A_j = \sum_{j=1}^{n} (B_j \, A_j + A_j \, B_j) - \sum_{j=1}^{n} A_j \, B_j$$

$$\therefore \quad \sum_{j=1}^{n} B_j \, A_j \text{ belongs to } \widetilde{H}_L.$$

$$\therefore \quad \sum_{j=1}^{n} A_j \, B_j - \sum_{j=1}^{n} B_j \, A_j \text{ belongs to } \widetilde{H}_L.$$

But $i(A_j B_j - B_j A_j)$, $i = \sqrt{-1}$ belongs to $\widetilde{H}_L$ for $j = 1,2,\ldots,n$.

$$\therefore \quad i \sum_{j=1}^{n} (A_j B_j - B_j A_j) \text{ belongs to } \widetilde{H}_L.$$

Hence by a result in [3] it follows that

$$\sum_{j=1}^{n} (A_j B_j - B_j A_j) = 0.$$

Now let $T \in \widetilde{H}_L$.

$$i(A_j T - TA_j) \text{ belongs to } \widetilde{H}_L$$

$$\therefore \quad i(A_j T - TA_j)B_j + B_j . i(A_j T - TA_j) \text{ belongs to } \widetilde{H}_L.$$

i.e. $i(A_j TB_j - B_j TA_j)+i(B_j A_j T - TA_j B_j)$ belongs to $\widetilde{H}_L$

for $j = 1,2,\ldots,n$.

adding we get,

$$i \sum_{j=1}^{n} (A_j TB_j - B_j TA_j) + i \sum_{j=1}^{n} (B_j A_j T - TA_j B_j)$$

belongs to $\widetilde{H}_L$.

But $i \sum_{j=1}^{n}(B_j A_j T - TA_j B_j) = i((\sum_{j=1}^{n} B_j A_j)T - T(\sum_{j=1}^{n} A_j B_j))$

This is in $\widetilde{H}_L$ since $\sum_{j=1}^{n} A_j B_j = \sum_{j=1}^{n} B_j A_j$

Therefore $i \sum_{j=1}^{n}(A_j T B_j - B_j T A_j)$ is in $\widetilde{H}_L$.

We have $(A_j + B_j) T (A_j + B_j) \varepsilon \widetilde{H}_L$.

i.e., $A_j T A_j + A_j T B_j + B_j T A_j + B_j T B_j$ belongs to $\widetilde{H}_L$.

This implies $B_j T A_j$ belongs to $\widetilde{H}_L$.

$\therefore \quad \sum_{j=1}^{n}(A_j T B_j - B_j T A_j)$ is in $\widetilde{H}_L$.

Hence by a result in [3]

$$\sum_{j=1}^{n} A_j T B_j = \sum_{j=1}^{n} B_j T A_j \tag{2.1}$$

for every T in $\widetilde{H}_L$. This proves our claim. Now assume that $\{A_j\}$, $\{B_j\}$, j=1,2,...,n are linearly independent.

Identity (2.1) is equivalent to the following

$$\sum_{j=1}^{2n} A_j T B_j = 0 \qquad (2.2)$$

for every T in $\widetilde{H}_L$, where

$$A_{n+j} = -B_j \quad j = 1,2,\ldots,n$$
$$B_{n+j} = A_j \quad j = 1,2,\ldots,n$$

If $\{A_j, B_j\}$ $j = 1,2,\ldots,n$ form a linearly independent set we have $A_j = B_j = 0$ for all j. Otherwise we may assume that $\{A_1, A_2, \ldots, A_n\}$ form a maximal linearly independent subset of $\{A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_n\}$. Therefore there exists real scalars a_{ji} such that

$$B_j = \sum_{i=1}^{n} a_{ji} A_i \qquad (2.3)$$

Substituting (2.3) in (2.1) we have,

$$\sum_{j=1}^{n} A_j T \left(\sum_{i=1}^{n} a_{ji} A_i \right) = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} a_{ji} A_i \right) T A_j$$

i.e. $\displaystyle\sum_{j=1}^{n} \left(\sum_{i=1}^{n} (a_{ij} - a_{ji}) A_i \right) T A_j = 0$ for all T in $\widetilde{H}_L$.

Hence by Lemma 2.1.3 and the linear independence of $\{A_j\}$ we get $a_{ij} = a_{ji}$ for all $i,j = 1,2,\ldots,n$. Thus the matrix $A = (a_{ji})_{i,j=1}^{n}$ is a non singular $n \times n$ symmetric matrix. Therefore there exists a unitary matrix $U = (u_{ij})$ such that

$$U^*AU = \begin{bmatrix} d_1 & & & & & \\ & d_2 & & & \bigcirc & \\ & & & & & \\ & & & d_\ell & & \\ & \bigcirc & & & -d_{\ell+1} & \\ & & & & & \ddots \\ & & & & & & -d_n \end{bmatrix}$$

where $d_1, d_2, \ldots, d_\ell$ are the positive eigen values of A and $-d_{\ell+1}, \ldots, -d_n$ are the negative eigen values of A. We may assume that the entries of U are all real.

Now as in [13] put

$$[C_1, C_2, \ldots, C_n] = [A_1, A_2, \ldots, A_n]\,U$$

ie. $$\begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{bmatrix} = U^T \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix} = U^* \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix} = U^{-1} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix}$$

ie. $$\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix} = U \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{bmatrix}$$

Therefore

$$\triangle(T) = \sum_{j=1}^{n} A_j T B_j$$

$$= \sum_{j=1}^{n} A_j \ T \ \left(\sum_{i=1}^{n} a_{ji} \ A_i \right)$$

$$=[A_1T, A_2T, \ldots, A_nT] \begin{bmatrix} a_{11}A_1 + a_{12}A_2 + \ldots + a_{1n}A_n \\ a_{21}A_1 + a_{22}A_2 + \ldots + a_{2n}A_n \\ \vdots \\ a_{n1}A_1 + a_{n2}A_2 + \ldots + a_{nn}A_n \end{bmatrix}$$

$$=[A_1,A_2,\dots,A_n]\begin{bmatrix} T & & & 0 \\ & T & & \\ & & \ddots & \\ 0 & & & T \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix}$$

$$=[C_1,C_2,\dots,C_n]U^*\begin{bmatrix} T & & & 0 \\ & T & & \\ & & \ddots & \\ 0 & & & T \end{bmatrix}(a_{ji})\,U\begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{bmatrix}$$

$$=[C_1,C_2,\dots,C_n]\begin{bmatrix} T & & & 0 \\ & T & & \\ & & \ddots & \\ 0 & & & T \end{bmatrix}\begin{bmatrix} d_1 & & & & & 0 \\ & d_2 & & & & \\ & & \ddots & & & \\ & & & d_\ell & & \\ & & & & -d_{\ell+1} & \\ 0 & & & & & -d_n \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{bmatrix}$$

$$= \sum_{j=1}^{\ell} C_j \, T \, d_j \, C_j - \sum_{j=\ell+1}^{n} C_j \, T \, d_j \, C_j$$

$$= \sum_{j=1}^{\ell} D_j \, T \, D_j - \sum_{j=\ell+1}^{n} D_j \, T \, D_j$$

where $D_j = \sqrt{d_j}\, C_j$. It is also clear that C_j belongs to $\widetilde{H}_L$ for all j. Therefore D_j is in $\widetilde{H}_L$ for all j. This completes the proof.

2.2. ELEMENTARY OPERATORS PRESERVING SELF ADJOINTNESS MODULO COMPACT OPERATORS

Let $A_1, A_2, \ldots, A_n$; $B_1, B_2, \ldots, B_n$ be bounded linear operators on a complex separable Hilbert space H and let

$$\Delta(T) = A_1TB_1 + A_2TB_2 + \ldots + A_nTB_n, \quad T \in \beta(H)$$

C.K.FONG and A.R. SOUROUR [10] has obtained the following result.

THEOREM **2.2.1.** (C.K. FONG and A.R. SOUROUR)

Let $A_1, A_2, \ldots, A_n; B_1, B_2, \ldots, B_n$ be as above where $B_1, B_2, \ldots, B_m$ $(m \leq n)$ are linearly independent modulo the compacts and there are constants C_{kj}, $1 \leq k \leq m$ and $m+1 \leq j \leq n$, such that

$$B_j = \sum_{k=1}^{m} C_{kj} B_k \text{ modulo the compacts } (m+1 \leq j \leq n)$$

Then $\Delta(T)$ is compact for each T in $\beta(H)$ if and only if

$$A_k = -\sum_{j=m+1}^{n} C_{kj} A_j \text{ modulo the compacts } (1 \leq k \leq m)$$

Now we prove the following variant of HOU JIN-CHUAN'S Theorem. To do this we need the following concepts which are well known.

DEFINITION 2.2.2.

Let H be a complex Hilbert space and K(H) denote the ideal of all compact operators on H. An operator T in $\beta(H)$ is called essentially self adjoint if T-T* belongs to K(H), where T* is the Hilbert space adjoint of T.

THEOREM 2.2.3.

Let H be a separable Hilbert space and $A_1, A_2, \ldots, A_n$; $B_1, B_2, \ldots, B_n$ be operators in $\beta(H)$, where $A_1, A_2, \ldots, A_n$, $B_1, B_2, \ldots, B_n$ are linearly independent modulo K(H). Then the elementary operator $\Delta(T) = \sum_{i=1}^{n} A_i T B_i$ preserves essential self adjointness if and only if there are operators $D_1, D_2, \ldots, D_n$ in $\beta(H)$ such that

$$\Delta(T) = \sum_{i=1}^{\ell} D_i T D_i^* - \sum_{i=\ell+1}^{n} D_i T D_i^* + K(T)$$

for every T, where K(T) is a compact operator on H depending on T.

PROOF

Sufficiency is clear.

Even though the proof of necessity is quite similar to that of Lemma 2.1 in [13] we supply the details. Suppose that $T-T^* \in K(H)$ implies $\Delta(T)-\Delta(T)^* \in K(H)$. Therefore,

$$\sum_{i=1}^{n} A_i T B_i - \sum_{i=1}^{n} B_i^* T A_i^* \in K(H) \tag{2.4}$$

for every T in $\beta(H)$.

Now assume that $A_1^*, A_2^*, \ldots, A_n^*$ form a maximal linearly independent modulo compact subset of $\{B_1, B_2, \ldots, B_n, A_1^*, A_2^*, \ldots, A_n^*\}$. Therefore there exist a matrix $(a_{ij})_{i,j=1}^{n}$ such that

$$B_i = \sum_{j=1}^{n} a_{ij} A_j^* + K_i \tag{2.5}$$

Substituting (2.5) in (2.4) and on applying theorem 2.2.1 we can see that $(a_{ij})_{i,j=1}^{n}$ is a hermitian matrix. Also it is nonsingular.

Let U denote an $n \times n$ unitary matrix such that

$$U^*(a_{ij})U = \begin{bmatrix} d_1 & & & & & \\ & d_2 & & & \bigcirc & \\ & & \ddots & & & \\ & & & d_\ell & & \\ & \bigcirc & & & -d_{\ell+1} & \\ & & & & & \ddots \\ & & & & & & -d_n \end{bmatrix}$$

where $d_1, d_2, \ldots, d_\ell$, $-d_{\ell+1}, \ldots, -d_n$ are the positive and negative eigen values of $A = (a_{ij})$ respectively.

As in [13] define $[C_1, C_2, \ldots, C_n] = [A_1, A_2, \ldots, A_n]U$

Then we have

$$\begin{bmatrix} C_1^* \\ C_2^* \\ \vdots \\ C_n^* \end{bmatrix} = (U^T)\begin{bmatrix} A_1^* \\ A_2^* \\ \vdots \\ A_n^* \end{bmatrix} = U^*\begin{bmatrix} A_1^* \\ A_2^* \\ \vdots \\ A_n^* \end{bmatrix} = U^{-1}\begin{bmatrix} A_1^* \\ A_2^* \\ \vdots \\ A_n^* \end{bmatrix}$$

Now $\triangle(T) = \sum\limits_{i=1}^{n} A_i T B_i$

$$= \sum_{i=1}^{n} A_i T \left(\sum_{j=1}^{n} a_{ij} A_j^* + K_i \right)$$

$$= \sum_{i=1}^{n} A_i T \left(\sum_{j=1}^{n} a_{ij} A_j^* \right) + K(T)$$

$$= [A_1, A_2, \ldots, A_n] \begin{bmatrix} T & & & \\ & T & & \\ & & \ddots & \\ & & & T \end{bmatrix} (a_{ij}) \begin{bmatrix} A_1^* \\ A_2^* \\ \vdots \\ A_n^* \end{bmatrix} + K(T)$$

$$= [C_1, C_2, \ldots, C_n] \begin{bmatrix} T & & & \\ & T & & \\ & & \ddots & \\ & & & T \end{bmatrix} \begin{bmatrix} d_1 & & & & & & \\ & d_2 & & & & & \\ & & \ddots & & & & \\ & & & d_\ell & & & \\ & & & & -d_{\ell+1} & & \\ & & & & & \ddots & \\ & & & & & & -d_n \end{bmatrix} \begin{bmatrix} C_1^* \\ C_2^* \\ \vdots \\ C_n^* \end{bmatrix} + K(T)$$

$$= \sum_{i=1}^{\ell} C_i T d_i C_i^* - \sum_{i=\ell+1}^{n} C_i T d_i C_i^* + K(T)$$

$$= \sum_{i=1}^{\ell} D_i T D_i^* - \sum_{i=\ell+1}^{n} D_i T D_i^* + K(T)$$

where $D_i = \sqrt{d_i}\, C_i$.

Now we prove a theorem which characterises essential positivity preserving elementary operators.

DEFINITION 2.2.4.

Let H be a complex Hilbert space and π be the canonical homomorphism of $\beta(H)$ onto the Calkin algebra $\beta(H)/K(H)$. An operator T in $\beta(H)$ is called essentially positive if $\pi(T)$ is a positive element in $\beta(H)/K(H)$.

Let $A_1, A_2, \ldots, A_n$, $B_1, B_2, \ldots, B_n$ be operators in $\beta(H)$. HOU JIN-CHUAN introduced elementary operators $\Delta^{(K)}$ for each positive integer K by

$$\Delta^{(K)}(T) = \sum_{i=1}^{n} A_i^{(K)} T B_i^{(K)}$$

$T \in \beta(\overbrace{H \oplus H \oplus \cdots \oplus H}^{K \text{ copies}})$ and $A_i^{(K)} = \overbrace{A_i \oplus \cdots \oplus A_i}^{K \text{ copies}}$,

$B_i^{(K)} = \overbrace{B_i \oplus \cdots \oplus B_i}^{K \text{ copies}}$. HOU JIN-CHUAN has proved the following theorem in [13].

THEOREM 2.2.5.

$\Delta^{(K)}$ are positivity-preserving for all positive integers K if and only if there are bounded linear

Operators $D_1, D_2, \ldots, D_\ell$ in $\beta(H)$ such that

$$\Delta(.) = \sum_{i=1}^{\ell} D_i(.)D_i^* .$$

We wish to characterise those elementary operators Δ, such that $\Delta^{(K)}$ preserves essential positivity for all positive integers K.

In the following remark we observe that positivity of $\Delta^{(K)}$ for all K is equivalent to the well known complete positivity

REMARK 2.2.6.

Let $\Phi : \mathfrak{A} \longrightarrow \mathfrak{B}$ be a positivity preserving linear map from $\mathfrak{A}$ to $\mathfrak{B}$ where $\mathfrak{A}$ and $\mathfrak{B}$ are C* sub algebras of $\beta(H)$ for some complex Hilbert space H. Recall that Φ is completely positive if the map

$$\Phi^{(K)} : \mathfrak{A} \otimes M_k \longrightarrow \mathfrak{B} \otimes M_k$$

defined by

$$\Phi^{(K)}(a_{ij})_{k \times k} = (\Phi(a_{ij}))_{k \times k}$$

where $\mathfrak{A} \otimes M_k = \{k \times k \text{ matrices } (a_{ij}) \text{ over } \mathfrak{A}\}$ is positive for all k.

Now one can identify $\beta(H \oplus \overset{K \text{ copies}}{\cdots} \oplus H)$ with $\beta(H) \otimes M_k$. Also one can see that $\Delta^{(K)}$ is positive for all K if and only if Δ is completely positive. Now we prove the following lemma.

LEMMA 2.2.7.

Let $A_1, A_2, \ldots, A_n$, $B_1, B_2, \ldots, B_n$ be bounded linear operators on a separable Hilbert space H and $\Delta(T) = \sum_{i=1}^{n} A_i T B_i$, T in $\beta(H)$. Assume that $\{A_1, A_2, \ldots, A_n\}$ and $\{B_1, B_2, \ldots, B_n\}$ are linearly independent modulo K(H). If $\Delta^{(K)}$ preserves essential positivity for every positive integer K, then there exists bounded linear operators $D_1, D_2, \ldots D_n$ in $\beta(H)$ such that

$$\Delta(T) = \sum_{i=1}^{\ell} D_i T D_i^* - \sum_{i=\ell+1}^{n} D_i T D_i^* + K(T)$$

where $K(T)$ is a compact operator depending on T. Also the map $\tilde{\Delta}$ on the Calkin algebra $\beta(H)/K(H)$ defined by

$$\tilde{\Delta}(\tilde{T}) = \sum_{i=1}^{\ell} \tilde{D}_i \tilde{T} \tilde{D}_i^* - \sum_{i=\ell+1}^{n} \tilde{D}_i \tilde{T} \tilde{D}_i^* ,$$

where for $T \in \beta(H)$, $\tilde{T} = T + K(H) \in \beta(H)/K(H)$ is completely positive.

PROOF

Since Δ preserves essential positivity, it preserves essential self adjointness. Therefore by Theorem 2.2.3, we can find operators $D_1, D_2, \ldots, D_n$ in $\beta(H)$ such that

$$\Delta(T) = \sum_{i=1}^{\ell} D_i T D_i^* - \sum_{i=\ell+1}^{n} D_i T D_i^* + K(T)$$

for some integer ℓ , $1 \leq \ell \leq n$.

Now to show that $\tilde{\Delta}$ is completely positive. For that let $(\tilde{a}_{ij})_{n \times n}$ be a positive element in $\beta(H)|K(H) \otimes M_n$. To show that $(\tilde{\Delta}(\tilde{a}_{ij}))_{n \times n}$ is positive. For that consider the mapping Φ from $\beta(H \overset{K \text{ copies}}{\oplus \cdots \oplus} H)/K(H \oplus \cdots \oplus H)$ to $\beta(H)/K(H) \otimes M_n$ defined by

$$\Phi\,(\widetilde{(a_{ij})}) = (\tilde{a}_{ij}) \text{ for } (a_{ij}) \text{ in } \beta(H \oplus \cdots \oplus H).$$

Thus Φ is a $*$ preserving isomorphism. One-oneness is proved using the fact that the map

$$(a_{ij})_{k \times k} : H \overset{K \text{ copies}}{\oplus \cdots \oplus} H \longrightarrow H \overset{K \text{ copies}}{\oplus \cdots \oplus} H$$

is compact if and only if $a_{ij} : H \longrightarrow H$ is compact for all i,j. Thus,

$(\tilde{a}_{ij})_{n \times n}$ is positive if and only if $\widetilde{(a_{ij})}_{n \times n}$ is positive. Since $\Delta^{(K)}$ is essential positivity preserving for all K, we find that $(\widetilde{\Delta(a_{ij})})_{n \times n}$ is positive. Therefore $(\tilde{\Delta}(\tilde{a}_{ij}))_{n \times n}$ is positive.

Now we prove the main theorem.

THEOREM 2.2.8.

Let $A_1, A_2, \ldots, A_n$, $B_1, B_2, \ldots, B_n$ be bounded linear operators in $\beta(H)$ such that $\{A_1, \ldots, A_n\}, \{B_1, \ldots, B_n\}$ are linearly independent modulo $K(H)$, where H is a separable Hilbert space. Then

$$\Delta^{(K)}(T) = \sum_{i=1}^{n} A_i^{(K)} T B_i^{(K)}, \quad T \in \beta(H \oplus \overset{K \text{ copies}}{\cdots} \oplus H)$$

preserves essential positivity for all positive integers K if and only if there exists bounded linear operators $D_1, D_2, \ldots, D_\ell$ in $\beta(H)$ such that

$$\Delta(T) = \sum_{i=1}^{\ell} D_i T D_i^* + K(T), \quad i \leq \ell \leq n,$$

where $K(T)$ is a compact operator on H depending on T.

PROOF

By lemma 2.2.7, there exists bounded linear operators $D_1, D_2, \ldots, D_n$ in $\beta(H)$ such that

$$\Delta(T) = \sum_{i=1}^{\ell} D_i T D_i^* - \sum_{i=\ell+1}^{n} D_i T D_i^* + K(T), T \in \beta(H), \ K(T) \in K(H).$$

Therefore,

$$\tilde{\Delta}(\tilde{T}) = \sum_{i=1}^{\ell} \tilde{D}_i \tilde{T} \tilde{D}_i^* - \sum_{i=\ell+1}^{n} \tilde{D}_i \tilde{T} \tilde{D}_i^*, \quad \tilde{T} \in \beta(H)/K(H)$$

Now let μ be an irreducible representation of the Calkin algebra $\beta(H)/K(H)$ on some Hilbert space H_μ. Since H is separable, μ is faithful and therefore $\beta(H)/K(H)$ and $\mu(\beta(H)/K(H))$ can be identified. Therefore the map $\Omega: \mu(\beta(H)/K(H)) \longrightarrow \mu(\beta(H)/K(H))$ defined by

$$\Omega(\mu(\tilde{T})) = \sum_{i=1}^{\ell} \mu(\tilde{D}_i)\mu(\tilde{T})\,\mu(\tilde{D}_i)^* - \sum_{i=\ell+1}^{n} \mu(\tilde{D}_i)\mu(\tilde{T})\,\mu(\tilde{D}_i)^*$$

is completely positive and continuous in the weak operator topology of $\beta(H_\mu)$. Moreover, since μ is irreducible, by Von Neumann density theorem [1], $\mu(\beta(H)/K(H))$ is dense in $\beta(H_\mu)$ under the weak operator topology.

Let $(a_{ij})_{k\times k}$ be a positive element in $\beta(H) \otimes M_k$. Using functional calculus and Kaplanski's density theorem [27] one can find a net $(a_{ij}^{(\alpha)})_{\alpha\in I}$ in $\mu(\beta(H)/K(H)) \otimes M_k$, which are positive such that

$$\lim_{\alpha} (a_{ij}^{(\alpha)}) = (a_{ij})$$

in the weak operator topology of $\beta(H_\mu)$. Since Ω is completely positive $(\Omega(a_{ij}^{(\alpha)}))$ are all positive. Since Ω is continuous in the weak operator topology it follows that

$$(\Omega(a_{ij})) = \sum_{i=1}^{\ell} \mu(\widetilde{D}_i)\ a_{ij}\ \mu(\widetilde{D}_i)^* - \sum_{i=\ell+1}^{n} \mu(\widetilde{D}_i) a_{ij}\ \mu(\widetilde{D}_i)^*$$

is positive.

Therefore the extended map, Ω,

$$\Omega(T) = \sum_{i=1}^{\ell} \mu(\widetilde{D}_i)\ T\mu(\widetilde{D}_i)^* - \sum_{i=\ell+1}^{n} \mu(\widetilde{D}_i)\ T\mu(\widetilde{D}_i)^*$$

$T \in \beta(H_\mu)$ is completely positive. Equivalently $\Omega^{(K)}$ is positive for all K.

Now the argument used in the proof of Theorem 2.1 in [13] shows

$\mu(\widetilde{D}_i) = 0$ for $i = \ell+1,\ldots,n$

i.e $\widetilde{D}_i = 0$ for $i = \ell+1,\ldots,n$

i.e D_i's are compact for $i = \ell+1,\ldots,n$.

$$\therefore\ \Delta(T) = \sum_{i=1}^{\ell} D_i\ T\ D_i^* + \overline{K}(T)$$

where $\overline{K}(T) \in K(H)$ for all T in $\beta(H)$.

2.3. THE TRANSFORMATION Δ_∞

Here we study the operator Δ_∞ on $\beta(H)$, where H is a complex Hilbert space. Even though Δ_∞ is not 'elementary' as per the definition of elementary operators, its form makes it elementary in the literal sense. Such transformations makes its appearance in the context of normal completely positive maps [7] and C*-algebraic approach to quantum mechanics [7]. We begin with the formal definition of Δ_∞ .

DEFINITION 2.3.1.

Let $\{A_n\}$, $\{B_n\}$, $n=0, \pm 1, \pm 2, \ldots$ be doubly infinite sequences of bounded linear operators in $\beta(X)$ such that $\sum_{n=-\infty}^{\infty} A_n T B_n$ belongs to $\beta(X)$ for all T in $\beta(X)$, where X is a complex Banach space. Then Δ_∞ is defined as

$$\Delta_\infty(T) = \sum_{n=-\infty}^{\infty} A_n T B_n$$

In this section we wish to characterise Δ_∞, which preserves self adjointness of operators. To do this job we require the following extensions of some results of C.K. FONG and A.R. SOUROUR, [10].

LEMMA 2.3.2.

Let $\{A_n\}$, $\{B_n\}$ $n = 0, \pm 1, \pm 2, \ldots$ be bounded linear operators in $\beta(H)$ such that

1. $\sum_{-\infty}^{\infty} \|A_n(x)\| + \|B_n(x)\| < \infty$ for all x in H.

2. $\{B_n\}$, $n=0, \pm 1 \ldots$ are linearly independent.

3. For each B_k, there exists a bounded set W_k of trace class operators on H such that for each finite subset F of $\{B_n : n = 0, \pm 1, \ldots\}$ not containing B_k, there exists $\tau \in W_k$ such that

$$\operatorname{trace}(B_k \tau) = 1 \quad \text{and}$$
$$\operatorname{trace}(B_m \tau) = 0 \quad \text{for all } B_m \in F.$$

Then $\Delta_\infty(T) = \sum_{-\infty}^{\infty} A_n T B_n = 0$ for all T in $\beta(H)$ if and only if $A_n = 0$ for all n.

PROOF

First we show that $A_o = 0$ and the same argument can be used to show that $A_n = 0$ for any n.

Consider $\{B_o, B_{\pm 1}, B_{\pm 2}, \ldots, B_{\pm n}\}$ for a fixed n. By assumption there exists a bounded set W_o of trace class operators on H such that

$$\text{trace } (B_o \tau) = 1, \quad \text{and}$$

$$\text{trace } (B_m \tau) = 0, \; m = \pm 1, \pm 2, \ldots, \pm n .$$

Let $\{y_i\}$ be a complete orthonormal set in H and $\tau y_i = x_i$. Let $T_k = x \otimes y_k$, where $x \in H$ is arbitrary

We have,

$$0 = \Delta_\infty(T_k)(x_k) = \sum_{n=-\infty}^{\infty} A_n \, T_k \, B_n(x_k)$$

$$= \sum_{n=-\infty}^{\infty} A_n(x \otimes y_k) \, B_n(x_k)$$

$$= \sum_{n=-\infty}^{\infty} \langle B_n \, x_k, y_k \rangle \, A_n(x)$$

Therefore,

$$0 = \sum_{k=1}^{\infty} \Delta_\infty(T_k) \, (x_k)$$

$$= \sum_{n=-\infty}^{\infty} \sum_{i=1}^{\infty} \langle B_n x_i, y_i \rangle \, A_n(x)$$

$$= \sum_{n=-\infty}^{\infty} \text{trace } (\tau B_n) \; A_n(x)$$

$$= A_o(x) + \sum_{k=\pm n+1, \pm n+2, \ldots} \text{trace}(B_k \tau) \; A_k(x)$$

Since B_k's are uniformly bounded and τ belongs to the bounded set W_o, there exists a positive constant M independent of n such that

$$\| A_o \, x \| \leq M \sum_{k=\pm n+1, \pm n+2 \ldots} \| A_k(x) \|$$

Therefore $A_o \, x = 0$ since $\sum_{k=\pm n+1, \pm n+2 \ldots} \|A_k(x)\|$ can be made arbitrarily small.

REMARK 2.3.3.

Condition 3 mentioned in Lemma 2.3.2 looks very strange. But atleast in the following special cases one can verify it.

Case 1

The collection $\{B_n: n=0, \pm 1, \ldots\}$ is finite. It is well known that the dual of the Banach space of all trace class operators $\mathcal{J}(H)$ on H is $\beta(H)$ [27]. Hence by a result in [8], there exists a trace class operator $\tau^{(K)}$ in $\mathcal{J}(H)$ such that trace $(B_k \tau^{(K)}) = 1$ and trace $(B_n \tau^{(K)}) = 0$ for $n \neq k$. In this case it is enough to take $W_k = \{\tau^{(K)}\}$.

Case 2.

The collection $\{B_n : n=0, \pm 1, \pm 2, \ldots\}$ consists of compact operators on H and $\Sigma \alpha_n B_n = 0$ if and only if $\alpha_n = 0$ for all n. It is well known that the dual of K(H), the Banach space of all compact operators on H is $\mathcal{J}(H)$, the Banach space of all trace class operators on H. Since B_k does not belong to the closed linear span of the remaining B_n's, by Hahn Banach theorem there exists a $\tau \in \mathcal{J}(H)$ such that trace $(\tau B_k) = 1$ and trace $(\tau B_n) = 0$ for all $n \neq k$.

LEMMA 2.3.4.

Let $\{A_n\}$ and $\{B_n\}$, $n = 0, \pm 1, \pm 2, \ldots$ be two families of bounded linear operators in $\beta(H)$ such that

(1) $\sum_{-\infty}^{\infty} \| A_n(x) \| + \| B_n(x) \| < \infty$ for all $x \in H$.

(2) $\{B_n\}_{n=0,1,2,..}$ is a maximal linearly independent subset of $\{B_k : k = 0, \pm 1, \pm 2, \ldots\}$

(3) $\{B_n : n = 0,1,2,\ldots\}$ satisfy condition (3) of Lemma 2.3.2.

(4) $\Sigma\ \Sigma\ |a_{jk}| < \infty$, where

$$B_j = \sum_{k=o}^{\infty} a_{jk}\ B_k \text{ where } a_{jk} = 0 \text{ for all } k \geq N(j)$$

$$j = -1,\ -2,\ \ldots$$

Then $\Delta_\infty(T) = 0$ for all T in $\beta(H)$ if and only if

$$A_k = -\sum_{j=-1}^{-\infty} a_{jk}\ A_j,\ k = 0,1,2,\ldots$$

PROOF

Sufficiency is trivial.

Assume that $\Delta_\infty(T) = 0$ for all T in $\beta(H)$.

Therefore $0 = \Delta_\infty(T)$

$$= \sum_{k=o}^{\infty} A_k\ T\ B_k + \sum_{k=-1}^{-\infty} A_k\ T\ B_k$$

$$= \sum_{k=o}^{\infty} A_k\ T\ B_k + \sum_{j=1}^{\infty} A_{-j}\ T\ B_{-j}$$

$$= \sum_{k=o}^{\infty} A_k\ T\ B_k + \sum_{j=1}^{\infty} A_{-j} T\left(\sum_{k=o}^{N(-j)} a_{-jk} B_k\right)$$

$$= \sum_{k=o}^{\infty} (A_k + \sum_{j=1}^{\infty} a_{-jk} A_{-j}) T B_k$$

Now,

$$\sum_{k=o}^{\infty} \| A_k(x) + \sum_{j=1}^{\infty} a_{-jk} A_{-j}(x) \|$$

$$\leq \sum_{k=o}^{\infty} \|A_k(x)\| + (\sum_{j=1}^{\infty} |a_{-jk}|^2)^{1/2} (\sum_{j=1}^{\infty} \|A_{-j}(x)\|^2)^{1/2}$$

$$< \infty$$

Since $B_1, B_2, \ldots$ are linearly independent by Lemma 2.3.2 we have,

$$A_k + \sum_{j=1}^{\infty} a_{-jk} A_{-j} = 0, \quad k = 0,1,2,\ldots$$

i.e. $A_k = - \sum_{j=-1}^{-\infty} a_{jk} A_j$, $k=0,1,2,\ldots$

We state two more lemmas without proof. The proofs are exactly the same.

LEMMA 2.3.5.

Let $\{A_n\}$, $\{B_n\}$, $n=0, \pm 1, \pm 2, \ldots$ be two families of bounded linear operators in $\beta(H)$ such that

(1) $\sum_{-\infty}^{\infty} \|A_n(x)\| + \|B_n(x)\| < \infty$ for all x in H.

(2) $\{A_n\}$, $n = 0, \pm 1, \pm 2, \ldots$ are linearly independent.

(3) For each A_k, there exists a bounded set W_k of trace class operators on H such that for each finite subset F of $\{A_n : n=0, \pm 1, \pm 2, \ldots\}$ not containing A_k, there exists a τ in W_k such that

$$\text{trace } (A_k \tau) = 1 \quad \text{and}$$
$$\text{trace } (A_m \tau) = 0 \quad \text{for } A_m \in F.$$

Then $\Delta_\infty(T) = \sum_{-\infty}^{\infty} A_n T B_n = 0$ for all T in $\beta(H)$ if and only if $B_n = 0$ for all n.

LEMMA 2.3.6.

Let $\{A_n\}$ and $\{B_n\}$ be two families of bounded linear operators in $\beta(H)$ such that

(1) $\sum_{-\infty}^{\infty} \| A_n(x) \| + \| B_n(x) \| < \infty$ for all x in H

(2) $\{A_n\}$ $n=0,1,2,\ldots$ form a maximal linearly independent subset of $\{A_k : k=0, \pm 1, \pm 2, \ldots\}$

(3) $\{A_n: n=0,1,2,\ldots\}$ satisfy the condition prescribed in (3) of Lemma 2.3.5.

(4) $\sum_{j,k} |a_{jk}| < \infty$ where

$$A_j = \sum_{k=0}^{\infty} a_{jk} A_k, \ j = -1, -2, \ldots$$

where $a_{jk} = 0$ for $k > N(j)$

Then $\Delta_\infty(T) = 0$ for all T in $\beta(H)$ if and only if

$$B_k = -\sum_{j=-1}^{-\infty} a_{jk} B_j, \ k = 0,1,2,\ldots$$

Now we state and prove the main theorem of this section.

THEOREM 2.3.7.

Let $\{A_n\}$ and $\{B_n\}$, n=1,2,... be two sequences of bounded linear operators in $\beta(H)$ such that

(1) $\sum_{1}^{\infty} \| A_n(x) \| + \| B_n(x) \| < \infty$.

(2) $\{B_n\}$ form a maximal linearly independent subset of $\{B_n, A_n^*\}$.

(3) $\sum_i \sum_j |a_{ij}| < \infty$ where

$$A_i^* = \sum_{j=1}^{N(i)} a_{ij} B_j, \ i = 1,2, \ldots$$

(4) The collection $\{B_n\}$ satisfy condition (3) of lemma 2.3.2. Then the map $\Delta_\infty(T) = \sum_1^\infty A_n T B_n$ is self adjointness preserving if and only if there are bounded linear operators $\{U_n, V_n, n=1,2,\ldots\}$ in $\beta(H)$ such that

$$\Delta_\infty(T) = \sum_{n=1}^\infty U_n T U_n^* - \sum_{n=1}^\infty V_n T V_n^*$$

for every T in $\beta(H)$.

PROOF

Sufficiency is trivial.

Assume that Δ_∞ preserves self adjointness.

i.e. $\Delta_\infty(T) = \Delta_\infty(T)^*$ for all T in $\beta(H)$ such that $T=T^*$.

Therefore we have,

$$\sum_1^\infty A_n T B_n = \sum_1^\infty B_n^* T A_n^* \qquad (2.6)$$

for all T in $\beta(H)$.

Identity (2.6) is equivalent to the following:

$$\tilde{\Delta}_{\infty}(T) = \sum_{-\infty}^{\infty} A_n \, T \, B_n = 0 \qquad (2.7)$$

for all T in $\beta(H)$, where $A_n = A_n$, $B_n = B_n$, $A_{-n} = -B_n^*$, $B_{-n} = A_n^*$, for all $n \geq 1$.

By assumption there exists scalars a_{ij}, $j=1,2,\ldots N(i)$ such that

$$A_i^* = \sum_{j=1}^{N(i)} a_{ij} \, B_j, \; i = 1,2,\ldots \qquad (2.8)$$

Now consider the infinite matrix $A = (a_{ij})$, where $a_{ij} = 0$ for all $j \geq N(i)$ for each i. By condition (3) in the statement of the theorem, A is a compact matrix. Next we show that A is hermitian.

Using (2.8) substitute for A_i and A_i^* in (2.6) we get,

$$0 = \sum_{i=1}^{\infty} \Big(\sum_{j=1}^{N(i)} \overline{a_{ij}} \, B_j^* \Big) TB_i - \sum_{i=1}^{\infty} B_i^* \, T \Big(\sum_{j=1}^{N(i)} a_{ij} \, B_j \Big)$$

$$= \sum_{i=1}^{\infty} \left[\sum_{j=1}^{N(i)} (\overline{a_{ij}} - a_{ji}) B_j^* + \sum_{j=N(i)+1}^{\infty} a_{ji} \, B_j^* \right] TB_i$$

for every T in $\beta(H)$.

Now,

$$\sum_{i=1}^{\infty} \left\| \sum_{j=1}^{N(i)} (\overline{a_{ij}}-a_{ji})\ B_j^*(x) + \sum_{j=N(i)+1}^{\infty} a_{ji}\ B_j^*(x) \right\|$$

$$\leq \quad M \sum_{i=1}^{\infty} \sum_{j=1}^{N(i)} (|a_{ij}| + |a_{ji}|) + \sum_{j=N(i)+1}^{\infty} |a_{ji}|$$

where M is an upper bound for $\{B_j^*(x)\}$.

Hence by Lemma 2.3.2 we have

$$\overline{a_{ij}} = a_{ji} \quad \text{for all } j \leq N(i)$$

and $\quad a_{ji} = 0 \quad$ for all $j \geq N(i)+1$

Thus A is a hermitian matrix. Therefore there exists a unitary matrix $U = (u_{ij})$ such that

$$U^*\ A\ U = \begin{bmatrix} d_1 & & & \bigcirc \\ & d_2 & & \\ & & d_3 & \\ \bigcirc & & & \ddots \end{bmatrix}$$

where $d_1, d_2, \ldots$ are the eigen values of A.

Let $C_1, C_2, C_3, \ldots$ be a sequence of bounded linear operators in $\beta(H)$ defined by

$$[C_1, C_2, C_3, \ldots] = [B_1^*, B_2^*, \ldots]\ (U_{ij})$$

where the right side is the usual multiplication of matrices. We then have,

$$\begin{bmatrix} C_1^* \\ C_2^* \\ C_3^* \\ \vdots \end{bmatrix} = \overline{(U^T)} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ \vdots \end{bmatrix} = U^* \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ \vdots \end{bmatrix} = U^{-1} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ \vdots \end{bmatrix}$$

We have

$$\begin{aligned} \triangle_\infty(T) &= \sum_{i=1}^{\infty} A_i\ T\ B_i \\ &= \sum_{i=1}^{\infty} \Big(\sum_{j=1}^{N(i)} \overline{a_{ij}}\ B_j^* \Big)\ TB_i \\ &= \sum_{i=1}^{\infty} \Big(\sum_{j=1}^{N(i)} a_{ji}\ B_j^* \Big) TB_i \end{aligned}$$

$$= [B_1^*,\ B_2^*,\ B_3^*, \ldots]\ (a_{ij}) \begin{bmatrix} T & 0 & 0 & \cdots \\ 0 & T & 0 & \cdots \\ \vdots & \vdots & \ddots & \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ \vdots \end{bmatrix}$$

$$= [C_1, C_2, C_3, \ldots]\ U^*\ A\ U \begin{bmatrix} T & & 0 \\ & T & \\ 0 & & T \\ & & & \ddots \end{bmatrix} \begin{bmatrix} C_1^* \\ C_2^* \\ \vdots \end{bmatrix}$$

$$= [C_1, C_2, C_3, \ldots] \begin{bmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots \end{bmatrix} \begin{bmatrix} T & & 0 \\ & T & \\ 0 & & \ddots \end{bmatrix} \begin{bmatrix} C_1^* \\ C_2^* \\ \vdots \end{bmatrix}$$

$$= \sum_{i=1}^{\infty} C_i\ d_i\ T\ C_i^*$$

Now assume that $\{d_{i_1}, d_{i_2}, d_{i_3} \ldots\}$ and $\{d_{k_1}, d_{k_2}, \ldots\}$ are the positive and negative eigen values of A.

Then

$$\Delta_\infty(T) = \sum_{m=1}^{\infty} C_{i_m} d_{i_m} T C_{i_m}^* - \sum_{m=1}^{\infty} C_{k_m} d_{k_m} T C_{k_m}^*$$

$$= \sum_{m=1}^{\infty} D_{i_m} T D_{i_m}^* - \sum_{m=1}^{\infty} D_{k_m} T D_{k_m}^*$$

where $D_{i_m} = \sqrt{d_{i_m}}\, C_{i_m}$, $D_{k_m} = \sqrt{d_{k_m}}\, C_{k_m}$.

This completes the proof.

REMARK 2.3.8.

Obviously the whole analysis carried out here dwells upon a couple of lemmas proved at the beginning of this section. But the conditions prescribed are apparently strong. The problem of finding optimal conditions under which these results are valid remains open. However the following examples throw some light into this problem.

EXAMPLES 2.3.9.

Let H be a separable Hilbert space and $\{e_1, e_2, e_3, \ldots\}$ be a complete orthonormal set in H. Let P_i denote the one dimensional orthogonal projection to the subspace M_i generated by the vectors e_i.

(1) Let $A_n = I$, the identity operator on H for every n and $B_1 = I-P_1$, $B_2 = -P_2, \ldots B_n = -P_n$, ...
Now consider the associated Δ_∞ defined by

$$\Delta_\infty(T) = \sum_{n=1}^{\infty} A_n \, T \, B_n, \quad T \text{ in } \beta(H).$$

For $x \in H$,

$$\sum_{n=1}^{\infty} A_n \, T \, B_n(x) = \lim_{N\to\infty} \sum_{n=1}^{N} T \, B_n(x)$$

$$= \lim_{N\to\infty} T(B_1(x)+B_2(x)+ \ldots + B_N(x))$$

$$= \lim_{N\to\infty} T(I-(P_1+P_2+\ldots+P_N)(x))$$

$$= 0 \text{ for all } T \in \beta(H).$$

But $A_n \neq 0$ for any n. We observe the following facts regarding $\{A_n\}$ and $\{B_n\}$.

(1) $\sum_{n=1}^{\infty} \|A_n(x)\| + \|B_n(x)\| = \infty$ for all $x \neq 0$

(2) $\{B_n\}$ form a linearly independent set.

(3) The collection $\{B_n\}$ satisfy the third condition of lemma 2.3.2.

Let $W = \{P_i,\ P_i - P_j :\ i,j = 1,2, \ldots\}$

Then W is a bounded set consisting of trace class operators.

Case 1.

$B_1 = I - P_1$

Let $F = \{B_{i_1}, B_{i_2}, \ldots B_{i_m}\}$ be a finite set in $\{B_n\}$ not containing B_1. Let $B_k = P_k \neq P_1$ be such that $P_k \neq P_i$ or P_j whenever P_i or P_j or $P_i - P_j$ are in F.

Put $\tau = B_k$. Then $\tau \in W$, and

trace $(B_1 \tau) = 1$

trace $(B_m \tau) = 0$ for all $B_m \in F$.

Case 2.

Consider $B_k \neq B_1$.

Let $F = \{B_{i_1}, B_{i_2}, \ldots, B_{i_m}\} \subset \{B_n\}$ be a finite set such that $B_k \notin F$. Now let $B_\ell \neq B_k$ be in F^c and $\tau = B_k - B_\ell$.

Then

$$\text{trace } (B_k \tau) = 1 \quad \text{and}$$

$$\text{trace } (B_{i_j} \tau) = 0, \text{ for } j = 1,2,\ldots,m.$$

(2) Let $A_n = \dfrac{I}{2^{n-1}}$, $n = 1,2,\ldots$

$$B_1 = \frac{P_1}{2^2} + \frac{P_2}{2^4} + \frac{P_3}{2^6} + \ldots + \frac{P_n}{2^{2n}}$$

$$B_2 = \frac{-P_1}{2} , \; B_3 = \frac{-P_2}{2^2} \; \ldots \; B_{n+1} = \frac{-P_n}{2^n}$$

Then $\Delta_\infty(T) = \sum_{n=1}^{\infty} A_n T B_n = 0$ for all $T \in \beta(H)$.

But none of the A_n's are zero.

We observe the following:

(1) $\sum_{n=1}^{\infty} \| A_n(x) \| + \| B_n(x) \| < \infty$

(2) $\{B_n\}$ is a linearly independent set

(3) The condition (3) of lemma 2.3.2 is not satisfied.

REMARK 2.3.10.

In the proof of JIN-CHUAN'S Theorem, one of the crucial point is the observation that the scalar matrix (a_{ij}) is hermitian and hence it can be diagonalised. But in the infinite situation we had to put extremely strong conditions to get compactness and hermiticity of (a_{ij}) so that it can be diagonalised. At least in the proof of JIN-CHUAN'S theorem, instead of using diagonalisation explicitly, we can use the spectral representation of (a_{ij}) and get the result. So in the infinite case, just demand that (a_{ij}) is symmetric (not necessarily compact) and bounded, and then use the spectral, integral representation of (a_{ij}) to get an integral representation of the map Δ_{∞}.

CHAPTER III

NONLINEAR MAPS ON $\beta(X)$

In this short chapter an attempt is made to study certain type of nonlinear maps on $\beta(X)$ where X is a complex Banach space.

Let $\Phi: \beta(X) \longrightarrow \beta(X)$ be a transformation. The problem is to find conditions under which there exist bounded linear operators A and B in $\beta(X)$ such that, $\Phi(T) = AT^2B$ for all T in $\beta(X)$.

PROPOSITION 3.1.1.

Let $\Phi: \beta(X) \longrightarrow \beta(X)$ be a map such that

(1) $\dfrac{\Phi(T_1+T_2) + \Phi(T_1-T_2)}{2} = \Phi(T_1) + \Phi(T_2)$ for all T_1, T_2 in $\beta(X)$.

(2) Rank $\Phi(T) \leq 1$ whenever rank $T=1$ and Rank $T \leq 1$, $\sigma(T) = 0$ implies $\Phi(T) = 0$

(3) $\Phi(\alpha T) = \alpha^2 \Phi(T)$ for all $T \in \beta(X)$ and for all $\alpha \in \mathbb{C}$. Then either,

(a) $\Phi(L_x) \subseteq L_{y(x)}$ for every x in X, or

(b) $\Phi(L_x) \subseteq R_{f(x)}$ for every x in X

PROOF

Let M_x be the vector space generated by $\Phi(L_x)$.

Case 1

Dimension $M_x=1$. If dim $(M_x) = 0$, there is nothing to be proved. If the dimension is 1, then

$$M_x = \{\alpha\, \Phi(x \otimes f_o) : \alpha \in \mathbb{C}\} \text{ for some } f_o \text{ in } X^*$$

$$= \{\alpha\, (y_o \otimes g_o) : \alpha \in \mathbb{C}\} \text{ for some } y_o \text{ in X and } g_o \text{ in } X^*$$

Since $\Phi(x \otimes f_o)$ is of rank 1. Hence $\Phi(L_x) \subseteq L_{y_o}$.

Case 2

dim $(M_x) \geq 2$.

Let if possible, there exist an x_o in X and f_1, f_2 in X* linearly independent, such that

$$\Phi(x_o \otimes f_1) = x_1 \otimes g_1 \neq 0$$

$$\Phi(x_o \otimes f_2) = x_2 \otimes g_2 \neq 0$$

where $\{x_1, x_2\}$ and $\{g_1, g_2\}$ are linearly independent sets in X and X* respectively.

Now by (1) we have,

$$\frac{\Phi(x_o \otimes f_1+f_2) + \Phi(x_o \otimes f_1-f_2)}{2} = \Phi(x_o \otimes f_1) + \Phi(x_o \otimes f_2)$$

But $\Phi(x_o \otimes f_1+f_2) = y_o \otimes g_o$

$\Phi(x_o \otimes f_1-f_2) = z_o \otimes h_o$

for some y_o, z_o in X and g_o, h_o in X*; by condition (3).

Thus,

$$\Phi(x_o \otimes f_1+f_2) = y_o \otimes g_o$$

$$= 2x_1 \otimes g_1 + 2x_2 \otimes g_2 - z_o \otimes g_o$$

By multiplying f_2 by a suitable scalar if necessary, we may assume that $f_1(x_o) = f_2(x_o)$ so that

$$\sigma(x_o \otimes f_1-f_2) = \{0\}$$

Therefore $\Phi(x_o \otimes f_1-f_2) = 0 = z_o \otimes h_o$

Thus we get,

$$\Phi(x_o \otimes f_1+f_2) = 2x_1 \otimes g_1 + 2x_2 \otimes g_2$$

This would imply that $\Phi(x_o \otimes f_1+f_2)$ is a rank 2 operator, since $\{x_1, x_2\}$, $\{g_1, g_2\}$ are linearly independent. This is contradictory to the assumption (3).

Hence $\Phi(L_x) \subset L_y$ for some y in X

and $\Phi(L_x) \subset R_g$ for some g in X*

Now we show that either $\Phi(L_x) \subset L_{y(x)}$ for every $x \in X$.

or $\Phi(L_x) \subseteq R_{f(x)}$ for every $x \in X$.

Let $M = \{x \in X \mid \Phi(L_x) \subseteq L_{y(x)}\}$ and

$N = \{x \in X \mid \Phi(L_x) \subseteq R_{f(x)}\}$

We found that $M \cup N = X$ and $M \cap N = \emptyset$. So assuming that $M \neq \emptyset$, it is enough to establish that $N = \emptyset$. Let if possible $x_1 \in N$ and let x_o be in M.

Also put,

$$\Phi(x_o \otimes f) = y_o \otimes g_o \quad \text{and}$$

$$\Phi(x_1 \otimes f) = y_1 \otimes g_1$$

We choose an f in X* so that y_o and y_1 are linearly independent. Also by multiplying x_o with a suitable scalar, if necessary, we may assume that $f(x_o)=f(x_1)$.

Now, as before, we get

$$\Phi(x_o+x_1 \otimes f) + \Phi(x_o-x_1) \otimes f$$

$$= 2\,\Phi(x_o \otimes f) + 2\,\Phi(x_1 \otimes f)$$

Thus, since $\underline{\Phi}((x_o - x_1) \otimes f) = 0$, we get

$$\underline{\Phi}(x_o + x_1) \otimes f = 2\,\underline{\Phi}(x_o \otimes f) + 2\underline{\Phi}(x_1 \otimes f)$$

But left side is of rank 1 and right is of rank 2 which is impossible. Hence $N = \emptyset$ if $M \neq \emptyset$. Similarly we can show that if $N \neq \emptyset$, then $M = \emptyset$.

—

REFERENCES

[1] Arveson William, An Invitation to C*-algebra, Graduate Texts in Mathematics No.39, Springer-Verlag 1976.

[2] Aupetit Bernard Properties Spectrales des Algebres de Banach, Lecture Notes No.735, Springer-Verlag 1979.

[3] Bonsall F.F. and John Duncan, Numerical Ranges of Operators on Normed Spaces and of Elements of Normed Algebras, London Mathematical Society Lecture Note, Series 2, Cambridge University Press 1971.

[4] Bonsall F.F. and John Duncan, Complete Normed Algebras, Springer-Verlag 1973.

[5] Chan G.H. and M.H. Lim, Linear transformation on Symmetric Matrices that preserve commutativity, Linear Algebra and its Applications, 47: 11-22 (1982).

[6] Choi M.D., D. Hadwin, E. Nordgren, H. Radjavi and P. Rosenthal, On Positive Linear Maps Preserving Invertibility, Journal of Functional Analysis, 59: 462-469(1984).

[7] Davies E.B., Quantum Theory of Open Systems, Academic Press, 1976.

[8] Dunford N and J.T. Schwartz, Linear Operators, Part I, Interscience/Wiley, New York 1957.

[9] Edwards R.E., Functional Analysis, Holt, Rinehart and Winston, New York, 1965.

[10] Fong C.K. and A.R. Sourour, On the Operator Identity $\Sigma A_k X B_k \equiv 0$, Canadian Journal of Mathematics, 31: 845-857(1979).

[11] Gleason A.M., A Characterisation of maximal ideals, J. Anal. Math., 19: 171-172(1967).

[12] Horn R.A., Chi-Kwong Li and Nam-Kiu Tsing, Linear Operators Preserving Certain Equivalence Relations on Matrices, SIAM J. Matrix. Anal. Appl., 12:195-204(1991).

[13] Hou Jin-Chuan, Rank-Preserving Linear Maps on $\beta(X)$, Science in China (Series A) 32:929-940(1989).

[14] Jafarian A.A. and A.R. Sourour, Spectrum-Preserving Linear Maps, Journal of Functional Analysis, 66: 255-261 (1986).

[15] Kahane J.P. and W. Zelazko, A Characterisation of maximal ideals in commutative Banach algebras, Studia. Math., 29:339-340 (1968).

[16] Loewy Raphael, Linear Maps which Preserve a Balanced Nonsingular inertia class, Linear Algebra and its Applications, 134: 165-179 (1990).

[17] Le Roy B. Beasley, Linear Transformations which preserve Fixed Rank, Linear Algebra and its applications, 40: 183-187(1981).

[18] Le Roy B. Beasley, Rank K-preservers and preservers of sets of Ranks, Linear Algebra and its applications, 55:11-17(1983).

[19] Marcus M, All linear operators leaving the Unitary group invariant, Duke Math. J.26: 155-163(1959).

[20] Marcus M and B.N. Moyls, Linear transformations on algebras of matrices, Canad. J. Math. 11:61-66 (1959).

[21] Marcus M and R. Purves, Linear transformations on algebras of matrices: The invariance of the elementary symmetric functions, Canad. J. Math. 11:383-396(1959).

[22] Martin Mathieu, Elementary Operators and Applications, World Scientific 1992.

[23] Nagel R, Schlotterbeck U and M.P.H. Wolff, Aspects of positivity in Functional Analysis, North Holland Mathematics Studies, 122, 1986.

[24] Russo B, Trace preserving mappings of Matrix Algebras, Duke Math.J., 36:297-300(1969).

[25] Russo B, Linear Mappings of Operator Algebras, Proc. Amer. Math. Soc. 17:1019-1022(1966).

[26] Radjavi Heydar, Commutativity-Preserving Operators on Symmetric Matrices, Linear Algebra and its Applications, 61: 219-224(1984).

[27] Takesaki Masamichi, Theory of Operator Algebras, Springer-Verlag 1979.