STUDIES ON CONVEX STRUCTURES WITH EMPHASIS ON CONVEXITY IN GRAPHS

Thesis submitted to the
Cochin University of Science and Technology
for the award of the degree of
DOCTOR OF PHILOSOPHY
under the Faculty of Science

By

K S. PARVATHY

DIVISION OF MATHEMATICS
SCHOOL OF MATHEMATICAL SCIENCES
COCHIN UNIVERSITY OF SCIENCE AND TECHNOLOGY
KOCHI - 682 022

November 1995

CERTIFICATE

*This is to certify that the thesis entitled "**STUDIES ON CONVEX STRUCTURES WITH EMPHASIS ON CONVEXITY IN GRAPHS**" submitted to the Cochin University of Science and Technology by Smt. K.S. Parvathy for the award of the degree of Doctor of Philosophy in the Faculty of Science is a bonafide record of studies done by her under my supervision. This report has not been submitted previously for considering the award of any degree, fellowship or similar titles elsewhere.*

Cochin 682 022
November 16, 1995.

DR. A.VIJAYAKUMAR
LECTURER
DIVISION OF MATHEMATICS
SCHOOL OF MATHEMATICAL SCIENCES
COCHIN UNIVERSITY OF SCIENCE AND TECHNOLOGY.

CONTENTS

I

INTRODUCTION

The concept of *convexity* which was mainly defined and studied in R^n in the pioneering works of Newton, Minkowski and others as described in [18], now finds a place in several other mathematical structures such as vector spaces, posets, lattices, metric spaces and graphs. This development is motivated by not only the need for an abstract theory of convexity generalising the classical theorems in R^n due to Helly, Caratheodory etc., but also to unify geometric aspects of all these mathematical structures. In the course of the development it is found that the properties of convex sets have been analyzed mainly in three ways, qualitatively, quantitatively and combinatorially and finds its applications in problems of pattern recognition, optimization, etc. [68].

The theory of *graphs* which originated in the solution of the famous Königsberg bridge problem during 1736 by Leonard Euler, now finds quite a lot of applications in many other branches of science, engineering and social science. See [5], [6], [10] for details.

This thesis is an attempt to study mainly some combinatorial problems of convexity spaces and graphs, following the footsteps of Levi, Jamison, Sierksma, Soltan, Duchet and others.

1.1 DEFINITIONS AND PRELIMINARIES

In this section, we consider some basic definitions and concepts mainly from [2], [7], [8] and [12]. For notations and terms not mentioned here, we follow [7], [8] and [12].

By a graph G = G(V,E) = G(p,q) we generally mean a finite connected graph without loops and multiple edges, with vertex set V, edge set E, of order p and size q. The symbol <S> means the subgraph induced by S.

Definition 1.1. Let G = (V,E) be a graph. d(u,v), the distance between u and v in V(G) is the length of the shortest path connecting u and v, the eccentricity of the vertex u, $e(u) = \max\{d(u,v): v \in V(G)\}$, $diam(G) = \max\{e(u):u \in V(G)\}$, $rad(G) = \min\{e(u):u \in V(G)\}$, $C(G) = \{u:e(u) = rad(G)\}$ the center of G and a graph G is called *self centered* if $C(G) = V(G)$.

Definition 1.2. Let $G_1= (V_1,E_1)$, $G_2= (V_2,E_2)$ be two graphs. The *Cartesian product* $G_1 x G_2$ of G_1 and G_2 is defined as the graph G where $V(G) = V_1 x V_2$ and (u_1,v_1) is adjacent to (u_2,v_2) if either $u_1= u_2$ and $v_1v_2 \in E_2$ or $u_1u_2 \in E_1$ and $v_1= v_2$. The *join* G_1+G_2 is obtained by joining all the vertices of G_1 to all the vertices of G_2. The *sequential join* $G_1+G_2+\ldots+G_n$ of $G_1,G_2,\ldots,G_n$ is obtained by joining all vertices of G_i to all vertices of G_{i+1} for $i = 1,2,\ldots,n-1$. The graph $S_{m,n} \simeq \bar{K}_m+K_1+K_1+\bar{K}_n$ is called a double star.

Definition 1.3. A chord of a cycle C is an edge connecting non consecutive vertices of C. A graph G is *chordal* if every cycle of length at least four has a chord.

Definition 1.4. A graph G is *Ptolemaic* if for any $u,v,w,x \in V(G)$,

$$d(u,v).d(w,x) \leq d(u,w).d(v,x)+d(u,x).d(v,w).$$

Definition 1.5. The size of the maximum clique in G is the clique number $\omega(G)$ of G. $S \subset V(G)$ is said to *separate* u,v in V(G) if u and v lie in different components of $G \setminus S$. S is a *clique separator* whenever S induces a clique in G.

Definition 1.6. Let X be a set. Then $I: X \times X \longrightarrow X$ is an *interval* function on X if the following conditions hold.

(a) $a,b \in I(a,b)$ - Extensive law.

(b) $I(a,b) = I(b,a)$ - Symmetry law.

Definition 1.7. Let G = (V,E) be a graph. $S \subseteq V$ is *geodesically convex* if for all x,y of S, $I(x,y) = \{z : z$ is on some shortest x-y path$\} \subseteq S$. These convex sets are also called distance convex (d-convex) sets. $S \subseteq V$ is *minimal path convex* (m-convex) if for all x,y of S, $I(x,y) = \{z : z$ is on some chordless x-y path$\} \subseteq S$.

Definition 1.8. For a graph G, V(G), ϕ and $S \subseteq V(G)$ whose induced subgraphs are isomorphic to K_n for n >0 are called trivial convex sets. For any integer $k \geq 0$ a graph G is k-convex if it has exactly k nontrivial convex sets. A (k,ω)-convex graph is a k-convex graph with clique number ω. (0,2)-convex graphs are called *distance convex simple (d.c.s)* if the convexity is geodesic convexity and *m-convex simple (m.c.s.)* if the convexity is m-convexity. When k=1 the k-convex graphs are called *uniconvex* graphs.

Definition 1.9. A graph is *convex simple* if it is either d.c.s or m.c.s.

Definition 1.10. A graph G is *interval monotone* if I(u,v) is convex for each pair of vertices u and v of G. It is *totally non interval monotone* (t.n.i.m.) if no nontrivial interval is convex. Here, the trivial intervals are those I(a,b) for which a=b, a adjacent to b or I(a,b) = V(G).

Definition 1.11. Let G = (V,E) be a graph and $S \subseteq V(G)$. Then the closure of S, (S) = {x:x is on some shortest path connecting vertices of S}. Then, define S^k as follows. $S^1 = (S)$, $S^k = (S^{k-1})$. If $S^k = S^{k+1}$ then S^k is convex. The geodetic iteration number gin(S) is the smallest number n such that $S^n = S^{n+1}$. The *geodetic iteration number* gin(G) is defined as the maximum value of a gin S over all $S \subseteq V(G)$.

Definition 1.12. A family $\mathscr{C}$ of subsets of a nonempty set X is called a *convexity* on x if

1) ϕ, X $\in \mathscr{C}$
2) $\mathscr{C}$ is stable for intersection, and
3) $\mathscr{C}$ is stable for nested union.

(X,$\mathscr{C}$) is called a *convexity space* and members of $\mathscr{C}$ are called convex sets. The smallest convex set containing a set A is called *convex hull* of A, denoted by Co(A).

Definition 1.13. A convexity space X is an *interval convexity space* if its convexity is induced by an interval.

Definition 1.14. A convexity space is of *arity* $\leq n$ if its convex sets are determined by n-polytopes. That is, a set C is convex if and only if Co(F) $\subseteq$ C for each subset F of cardinality at most n.

Definition 1.15. A convexity space X is a *matroid* if it satisfies the exchange axiom A $\subseteq$ X and p, q $\in$ X$\setminus$ Co(A), then p $\in$ Co({q} $\cup$ A) implies that q $\in$ Co({p} $\cup$ A) and is an *antimatroid* (convex geometry) if it satisfies the antiexchange law, $A \subseteq X$, $p, q \in X \setminus Co(A)$ then, $p \in Co(\{q\} \cup A)$ implies that $q \notin Co(\{p\} \cup A)$.

Definition 1.16. A subset H of X is called a half space if both H and $X \setminus H$ are convex. A convexity space X is said to have *separation property*

S_1: if all singletons are convex.

S_2: if any two distinct points are separated by half spaces. That is, if $x_1 \neq x_2 \in X$ then there is a half space H of X such that $x_1 \in H$ and $x_2 \notin H$.

S_3: if any convex set and any singleton not contained in C can be separated by half spaces. That is, if $C \subseteq X$ is convex and if $x \in X \setminus C$, then there is a half space H of X such that $C \subseteq H$ and $x \notin H$.

S_4: if any two disjoint convex sets can be separated by half spaces. That is if $C_1, C_2 \subseteq X$ are disjoint convex sets then there is a half space H of X such that $C_1 \subseteq H$ and $C_2 \subset X \setminus H$.

Definition 1.17. A subset S of an interval space X is *star shaped* at a point $p \in S$ provided for every $x \in S$,

$I(x,p) \subseteq S$. The *star center* of S is the set of all points at which S is star shaped. X is said to have the *Brunn's property* if the star center of each subset of X is convex. The star center is also called the kernel of S, denoted by Ker(S).

Definition 1.18. Let X be convexity space then,

1. The *Helly number* of X is the smallest `n' such that for each finite set $F \subset X$ with cardinality at least n+1, $\cap \{Co(F \setminus \{a\}) : a \in F\} \neq \phi$
   (that is, F is Helly (H-) dependent).

2. The *Caratheodory number* of X is the smallest number `n' such that for each $F \subset X$ with cardinality at least n+1, $Co(F) \subset \cup \{co(F \setminus \{a\}) : a \in F\}$
   (that is, F is Caratheodory (C-) dependent).

3. The *Radon number* of X is the smallest number `n' such that each $F \subset X$ with cardinality at least n+1, can be partitioned into two sets F_1 and F_2 such that $Co(F_1) \cap Co(F_2) \neq \phi$
   (that is, F is Radon (R-) dependent).

4. The exchange number of X is the smallest number n such that for each $F \subset X$ of cardinality at least n+1 and for each $p \in F$, $Co(F \setminus \{p\}) \subset \cup \{Co(F \setminus \{a\}) : a \in F, a \neq p\}$ (that is, F is exchange (E-) dependent).

These numbers are called convex invariants, denoted by, h,c,r and e respectively.

Definition 1.19. A convexity space X is said to be *join hull commutative* (JHC) if for any convex set C and any $p \in X$,

$$Co(C \cup \{p\}) = \cup\{Co(\{c,p\}) : c \in C\}.$$

Definition 1.20. An interval convexity space X is said to have the

1. *Pasch property* if for any a,b,p of X, $a' \in I(a,p)$ and $b' \in I(b,p)$ implies that $I(a,b') \cap I(a',b) \neq \phi$.

2. *Peano property* if for any a,b,c,u,v of X such that $u \in I(a,b)$, $v \in I(c,u)$, there is a v' in I(b,c) such that $v \in I(a,v')$.

If X is having both the properties it is called *Pasch-Peano space* (PP space).

Definition 1.21. Let V be a vector space over R. Let $\mathcal{F}$ be a nonempty family of a linear functionals on V . Then, $\mathcal{S} = \{f^{-1}(-\infty,a]:F \in \mathcal{F}\}$ generates a convexity $\mathcal{C}$ on V called the *H convexity* generated by $\mathcal{F}$. If $-f \in \mathcal{F}$ whenever $f \in \mathcal{F}$, it is called the symmetric H-convexity.

1.2. BACKGROUND OF THE WORK

Convexity is a very old topic whose origin can be traced back at least to Archimedes. This extremely simple and natural notion was however systematically studied by Minkowski during 1911. Bonnesen and Fenchel [1], Valentine [11] and many others also discuss the early development of the theory.

Among the different aspects of convex analysis, such as quantitative, qualitative and combinatorial, our concern will be the last one, where in the classical theorems of convexity in R^n of combinational type play a significant role.

It is well known that, a subset A of a real vector space is convex if and only if it contains with each pair x and y of its points, the entire line segment joining them. It immediately follows that the intersection of any family of convex sets is again a convex set, though the intersection may be empty. The classical theorem due to Edward Helly (1913) sets the condition under which this intersection cannot be empty. Helly's theorem and the theorems due to Caratheodory (1907) and Radon (1921) made a tremendous impact in the development of combinatorial convexity theory and has been studied, applied and generalised by many other authors [21], [31], [72], [74] since 1950s. These theorems in R^n states as follows [8].

Helly's theorem: Let $B = \{B_1, B_2, \ldots, B_r\}$ be a family of r convex sets in R^n with $r \geq n+1$. If every subfamily of n+1 sets in B has a nonempty intersection then $\bigcap_{i=1}^{r} B \neq \phi$.

Caratheodory's theorem: If S is a nonempty subset of R^n, then every x in the convex hull of S can be expressed as a convex combination of n+1 or fewer points.

Radons theorem: Let $S = \{x_1, x_2, \ldots, x_r\}$ be any set of finite points in R^n. If $r \geq n+2$, then S can be partitioned in to two disjoint subsets S_1 and S_2 such that $Co(S_1) \cap Co(S_2) \neq \phi$.

Not only to generalise these classical theorems of R^n, but also to unify the properties of a variety of mathematical structures such as vector spaces, ordered sets, lattices, metric spaces and graphs, an axiomatic foundation of convexity was laid down by Levi[51].

Let $(X,\mathscr{C})$ be a 'Convexity Space'(convex structure, aligned space, algebraic closure systems [31]). The members of $\mathscr{C}$ are called convex sets and $Co(A) = \cap \{C: A \subseteq C \in \mathscr{C}\}$, the convex hull of A. CO(F), with F finite is called a polytope. A polytope which can be spanned by n or less points (where n > 0) will be refered to as an n-polytope. The empty set is a 0-polytope. A 2-polytope $Co(\{a,b\})$ is also called a segment joining a and b. Aconvex structure(or, its convexity) is of arity $\leq$ n provided its convex sets are precisely the sets C with the property that $Co(F) \subseteq C$ for each subset F with cardinality atmost

n. That is, a convexity of arity n is "determined by its n-polytopes".

The standard convexity of a vector space, the order convexity of a poset, convexity in a lattice, semilattice and the convexity in a metric space [12] are examples of convexity spaces of arity 2. The study of H-convexity in a real vector space has been made in [19] and [20].

For a convexity space X there exists four numbers $h(x)$, $c(X)$, $r(X)$, $e(X) \in \{0,1,2,...\}$ called the Helly number, the Caratheodory number, the Radon number and the exchange number (Sierksma number), See Definition 1.18. It may be noted that many authors define the Radon number to be one unit larger, which is defined as the first n such that each set with at least n points has a Radon partition. However, we prefer the Definition 1.18.

Let f be a function defined on the class of all convex structures, and ranging into the set $\{0,1,2,...\}$. Then f is called a convex invariant provided that isomorphic convex structures have equal f-values. Obviously, each of the above defined functions h,c,r,e is a convex invariant. Such functions allow for a classification of convex structures according to their combinatorial properties. The function h,c,r go back to traditional topics in the combinatorial geometry of Euclidean space, and they are therefore called classical convex invariants. Attempts to find the interrelation between these invariants were made by Levi [51], Sierksma [71] and Jamison [45]. We shall mention some of these important results.

Levi's theorem [51]. Let $(X,\mathscr{C})$ be a convex structure. Then the existence of r implies the existence of h and $h \leq r$.
Eckhoff-Jamison inequality [45]. If c and h exists for a convexity space, then r exists and $r \leq c(h-1)+1$ if $h \neq 1$, or $c < \infty$.

Sierksma's theorem [71]. $e-1 \leq c \leq \max\{h,e-1\}$.

There are many other inequalities between these invariants. The different cases regarding the existence or otherwise of c,h,r and e is analysed in [12]. Kay and Womble [46] has shown that Levi's theorem is the only one possible if we assume the finiteness of exactly one of the numbers. Study of generalized Helly and Radon numbers [48],[49], extension of Radon theorem due to Tverberg [74], etc. are also found in literature.

The survey paper by Danzer et al. [31], has considerably stimulated the investigations on various aspects of convexity spaces. In the pioneering paper of Ellis [35]., the condition of join hull commutativity (JHC) was considered though the term was introduced by Kay and Womble [46]. It is known [12] that a JHC space is of arity ≤ 2. Products of convexity spaces were studied by Sierksma [70] and proved that JHC property is productive.

The concept of half space familiar in vector space has been generalized to a convexity space [42]. Four separation axioms (Definition 1.16) were introduced by Kay and Womble [46] and Jamison [42]. Under the assumption of S_1, it is an easy observation that $S_4 \longrightarrow S_3 \longrightarrow S_2$.

It is known that, a convex structure is S_3 if and only if it is generated by half spaces and that a lattice is S_4 if and only if it is distributive.

We shall now consider the important concept of interval operators (Definition 1.6) introduced by Calder [22] in 1971 which provide a natural method of constructing convex structures. The segment operator of a convex structure $(u,v) \longrightarrow Co\{u,v\}$ is an interval operator.

Conversely, if I is an interval operator, define a subset C of X to be interval convex provided $I(x,y) \subseteq C$ for all x,y in C, we get a convexity space, called the interval convexity space. If Co denotes the segment operator of $\mathscr{C}$, then for any a,b in X, $I(a,b) \subseteq Co\ \{a,b\}$. The two operators need not be equal. It is an important observation that, though the standard intervals and order intervals are convex, the metric interval $\{z \in X: d(x,z)+d(z,y) = d(x,y)\}$ [52] need not be convex. Also, a convexity space is induced by an interval operator if and only if it is of arity ≤ 2. Another important property of interval convexity which is of interest to us is Pasch-Peano property (Definition 1.20). These properties are known to hold for vector spaces. Some interesting results in this direction are,

Theorem 1.1 [22]. A convexity space of arity two is JHC if and only if its segment operator satisfies the Peano property.

Theorem 1.2 [35]. A convexity space of arity two is S_4 if and only if the segment operator of X has the Pasch property.

Another interesting concept is that of starshapedness (Definition 1.17). It was proved by Brunn in 1913 [47] that for R^n with standard convexity, the star center of each set is convex.

Several other aspects of convexity theory has been studied by many authors. The prominent among them include the theory of convex geometries [34], ramification property due to Calder [22] and Bean [17], Prenowits [9] theory of join spaces linking up with the theory of ordered geometry, the theory Bryant-Webster spaces [21] and Eckoff's partition conjecture [45].

Since 1950s the theory of convexity spaces has branched and grown into several related theories. An elegant survey has been done by Van de vel [12] whose work has been acclaimed as remarkable.

Attempts were also made by Changat, M and Vijayakumar, A [28] to evaluate the convex invariants of order and metric convexities of Z^n and Onn [58] has studied the Radon number of integer lattice.

Regarding the application part of convexity theory, interesting problems attempted include the determination of computational complexity of the construction of convex hulls and computational complexity of the evaluation of convex invariants. A bibliography on digital and computational convexity has been prepared by Ronse [68].

CONVEXITY IN GRAPHS

It is natural that the concept of convexity could be introduced in graphs also, via its intrinsic metric. Convexity problems in graphs is an emerging line of research in metric graph theory and has proved to be quite successful with respect to applications also, such as facility location problems, dynamic researching in graphs etc. [54]. Several convexities can be defined in a graph, most widely discussed being the geodesic convexity [73] and the minimal path convexity [33] (Definition 1.7). It is obvious that any m-convex set is d-convex. Introducing the notion of an interval function of a graph, Mulder [53] observed that geodesic interval in a graph need not be convex. He called a graph to be interval monotone if all its intervals are convex.

Edelman and Jamison [34] studied the convexity spaces satisfying the antiexchange law (Definition 1.15) and are called the convex geometries or antimatroids. It was observed that antimatroids are precisely convex structures satisfying the Krein-Milman property that, every convex set is the convex hull of its extreme points. They investigated this property for graphs also and proved that,

Theorem 1.3 [38]. G is chordal if and only if the minimal path of convexity is a convex geometry.

Theorem 1.4 [38]. G is a disjoint union of Ptolemaic graphs if and only if the geodesic convexity is a convex geometry.

Theorem 1.5 [44]. G is a connected block graph if and only if the connected alignment is a convex geometry.

Bandelt [14] studied separation properties in graphs and Chepoi [29] gave a characterization of S_3, S_4 and JHC in a bipartite graphs. The geodesic convexity and the m-convexity being defined in terms of intervals, they have some interesting properties.

Theorem 1.6 [12]. A connected graph with Pasch property is interval monotone.

Theorem 1.7 [12]. If a connected graph is S_3 with respect to geodesic convexity, then it is interval monotone.

Theorem 1.8 [12]. Ptolemaic graphs with respect to geodesic convexity are interval monotone.

Theorem 1.9 [14]. The geodesic convexity of a bipartite graph G is S_3 if and only if G embeds isometrically in a hypercube.

Considerable attempts have been made by Bandelt [14], [15], Duchet [32] and Farber-Jamison [38] to evaluate the convexity parameters in graphs. Some interesting results in this context are,

Theorem 1.10 [32] Caratheodory number of any graph with respect to m-convexity is atmost 2.

Theorem 1.11 [33]. Let G = (V,E) be a connected graph with at least two vertices and suppose the maximum size of a clique in G is ω . Denote by h(G) and r(G) respectively the Helly number and the Radon number of the minimal path convexity of G. Then

$$r(G) = \omega$$

$$r(G) = \omega + 1, \text{if } \omega \geq 3$$

$$r(G) = 4 \quad , \text{if } \omega \leq 2$$

It is also proved that the Radon number of the minimal path convexity in a triangle free graph G is 3 if and only if the block graph of G is a path. It is known that the Helly number of a graph with respect to d-convexity is bounded from below by $\omega(G)$. Generalizing the results for chordal graphs and distance hereditary graphs due to Chepoi [29], Duchet [32] and others, Bandelt and Mulder [15] proved that $h(G) = \omega(G)$ for a dismantlable graph (Pseudomodular graph). For other related results, see [26] [36] [37] and [69].

As an attempt towards the classification of graphs according to the number of nontrivial convex sets, considerable study has been made by Hebbare [13],[39], [41], Rao and Hebbare [66] and Batten [16]. They called, the empty set, singletons, vertices inducing a complete subgraph and V(G) to be trivial convex sets. A graph is called (k,ω)-convex if it has exactly k nontrivial convex sets and has clique number ω. The (0,2) convex graphs with respect to the geodesic convexity were called distance convex simple (d.c.s) graphs [41] and such graphs with respect to m-convexity were called m-convex simple (m.c.s) graphs by Changat, M [26]. It is easy to observe that every d.c.s graph of order $p \geq 4$ is a triangle free block. When k=1, (k,w)-convex graphs are called uniconvex graphs [40]. Several other interesting results on planar d.c.s graph, o-convex graphs, (0,3) convex graphs, (1,2) convex graphs are in [41]. Changat, M [26] while studying m-convex simple graphs, has proved that, a connected graph $G \ncong P_3$, having no nontrivial cliques is m.c.s if and only if G is m-self centroidal. Also, a connected graph G is m.c.s. if and only if G has no nontrivial cliques or clique separator. In [27] he has proved that a graph G has geodesic iteration number 1 if and only if G is interval monotone which has Caratheodory number 2. Also, a graph G is interval monotone with respect to m-convexity if and only if the minimal path iteration number of G, min(G) is 1. Some other results are in [24] and [25].

We have thus given a survey of results on the theory of convexity spaces and convexity in graphs, related to the results mentioned in this thesis.

1.3 GIST OF THE THESIS

This thesis consists of five chapters including this introductory one, where in we have given some basic definitions and a survey of results on the theory of abstract convexity spaces and convexity in graphs.

In the second chapter, we study the properties of convex simple graphs, interval monotone graphs and totally non interval monotone graphs. It is observed that, two necessary conditions given by Hebbare [41] are not sufficient. Some of the important observations included in this chapter are,

1. It is obvious that d.c.s. graphs are triangle free and t.n.i.m. But, the converse is not true. We have given two different methods of constructing a triangle free t.n.i.m. graph having exactly k non trivial convex sets.

2. Regarding the separation properties of d.c.s and t.n.i.m graphs, it is found that they are half space free.

3. For d.c.s graph, the convex invariants are,

$h(G) = c(G) = r(G) = 2$ and $e(G) = 3$.

4. Chordal graphs with m-convexity has Brunn's property, though it is not true in general.

5. There is no uniconvex graphs with respect to m-convexity.

6 In difference with the observation mentioned in 1, with m-convexity, for a triangle free, 2-connected graph to be k-convex, it is necessary that there is an 'n' such that $(n-1)(n+2)/2 \leq k \leq 2^n-2$.

7. For any graph with geodesic convexity,if its geodetic iteration number is 1 then it is interval monotone and JHC. Converse need not be true. But, if G is a JHC, interval monotone graph, we can give a bound for gin(S) for $S \subset V(G)$. In fact, $gin(S) \leq k$ where $k-1 < \frac{\log |S|}{\log 2} \leq k$.

8. If G is a geodetic, JHC graph then gin(G) = 1

The third chapter deals mainly with the concept of *solvable trees*, which was introduced to answer the problem, of finding the smallest d.c.s. graph containing a given tree of order atleast four. We say that a tree T is solvable if there is a planar d.c.s. graph G such that T is isomorphic to a spanning tree of G. We prove that,

9. Any tree of order atmost nine is solvable. The bound for the order is sharp. We note that there are graphs of

order 10 which are not solvable.

10. Trees of diameter three, five and trees of diameter four whose central vertex has even degree are solvable. There are trees of diameter six which are not solvable.

A similar problem was posed, with respect to m-convex simple graphs and found that,

11. The size of the smallest m-convex simple graph containing a tree T satisfies, $p-1+m/2 \leq q \leq p+m-2$ where $p = |V(T)|$ and m is the number of pendent vertices of T.

We further study the convexity properties of product of graphs and have,

12. If G_1 and G_2 are d.c.s. graphs then $G_1 \times G_2$ is not so.

13. If G_1 and G_2 are connected, triangle free graphs, $G_i \neq K_1$ or K_2 for i = 1,2, then $G_1 \times G_2$ is m-convex simple.

14. If G_1 is m.c.s. and G_2 is any triangle free graph, then $G_1 \times G_2$ is m.c.s.

We conclude this chapter with a discussion on the centers of d.c.s. graphs.

15. If G is a planar d.c.s. graph, then G is self centered if diam (G) = 2 and diam(G) = 2rad(G) or 2rad(G)-1, if diam(G) > 2, C(G) is isomorphic to $\bar{K}_2$ or C_4 according as diam(G) = 2rad(G) or 2rad(G)-1.

In the fourth chapter, we initiate the study of convexity for the edge set of a graph, which is less studied earlier. We define $S \subset E(G)$ to be cyclically convex if it contains all edges comprising a cycle whenever it contains all but one edges of this cycle. This convexity space $(G,\mathscr{C})$ satisfies the exchange law also and hence is a matroid. Further,

16. The arity of $(G,\mathscr{C})$ is 1 if G is a tree and is one less than the size of the largest chordless cycle in G, otherwise.

Thus, $(G,\mathscr{C})$ is not an interval convexity space in general. The convex invariants have also been evaluated.

17. If G is a connected graph of order p, Helly number h(G) = p-1.

18. Caratheodory number $C(G) = 1$ if G is a tree

$= circ(G)-1$, otherwise.

19. Radon number of $(G,\mathscr{C})$, $r(G) = p-1$.

20 For a connected graph G, the exchange number,

$e(G) = 2$ if G is a tree or a cycle.

$= \max \{circ(G-v)/v \in V(G)\}$, otherwise.

By generalizing the Pasch-Peano properties to any convexity space, we have obtained a forbidden subgraph characterization also.

21. The convexity space $(G,\mathscr{C})$ is a Pasch space if and only if K_4-x is not an induced subgraph of G.

22 The convexity space $(G,\mathscr{C})$ is a Peano space if and only if G does not contain K_4-x as a subgraph.

Though, for a matroid the Peano property implies Pasch, the converse need not be true by the observations made above.

The last chapter deals with some problems on the H-convexity of R^n. The motivation for this study is the problem posed in [12]. A symmetrically generated H-convexity need not be JHC or S_4. Van de Vel asked as to whether each symmetric H-convexity of R^n $(n>2)$ is of arity two ? We have obtained

23. The arity of the H-convexity in R^3 symmetrically generated by a family of linear functionals corresponding to a family of planes intersecting in a line, is two.

24. An example of an H-convexity in R^3 of infinite arity.

25. The H-convexity symmetrically generated by a family $\mathcal{F}$ of linear functionals from $R^3 \rightarrow R$, is S_4 if and only if for any two intersecting convex straight lines, the plane dittermined by these lines is convex.

26. An example of an H-convexity which is neither JHC nor S_4 but is Pasch and Peano and hence not of arity two.

The study initiated in thesis is definitely far from being complete. The last section of this chapter is a list of problems that remains to be tackled, which include some interesting problems posed by others also.

We have included as an appendix, a counter example to a conjecture of Chang [23] on the centers of chordal graphs.

CHAPTER II

II

CONVEX SIMPLE GRAPHS AND INTERVAL MONOTONICITY

In this chapter, we focus on the properties of convex simple graphs. Though any distance convex simple graph is totally non interval monotone, the converse is not true. We give two methods of constructing a triangle free t.n.i.m. graph having exactly k non trivial convex sets. It is also observed that d.c.s graphs and t.n.i.m. graphs are halfspace free. However, with respect to minimal path convexity it is seen that there are no uniconvex graphs and that, values of k for which a k-convex graph exists should satisfy certain conditions. We further concentrate on the iteration number of an interval monotone, JHC graph and also a geodesic, JHC graph.

2.1 DISTANCE CONVEX SIMPLE GRAPHS AND TOTALLY NON INTERVAL MONOTONE GRAPHS

Let us first consider the two necessary conditions for a graph G of order at least five to be distance convex simple.

Theorem 2.1 [41]. A d.c.s graph G of order at least five satisfies the following conditions.

C1. For any 2-path u-v-w in G, there is an x in V such that < {u,v,w,x} > is a chordless 4-cycle of G.

C2. For any 4-cycle u-v-w-x-u in G there is a y in G such that y is adjacent to either u and w or v and x.

Q_3-graph of the 3-cube satisfies C1 but is not d.c.s. We first observe that C1 and C2 are not sufficient conditions. The graph in Fig.2.1 satisfies both the conditions but is not d.c.s.

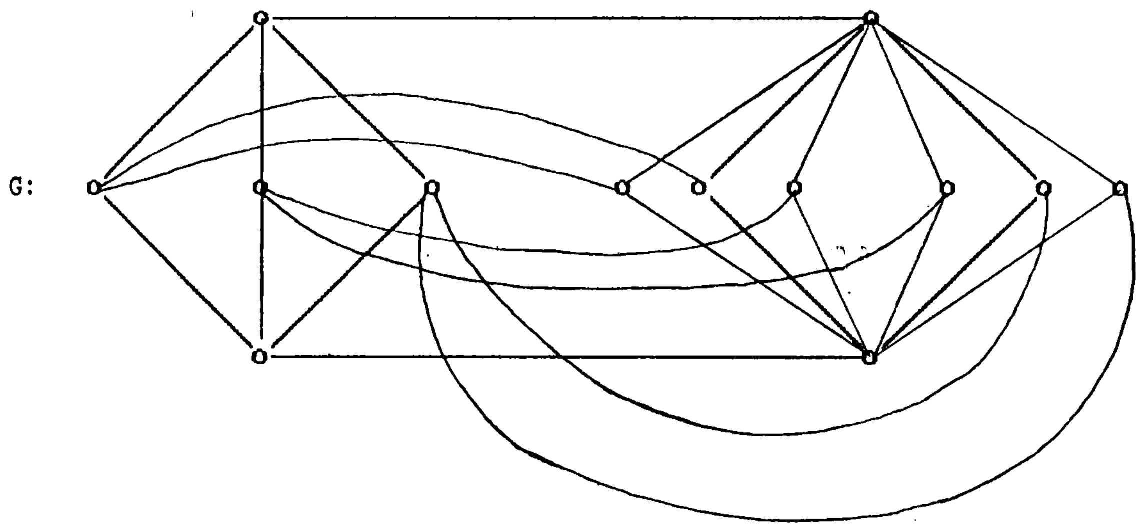


Fig. 2.1

In G, {a,b,c,d,e} is a convex set.

All connected graphs of order atmost three, $K_{m,n}$ for m,n > 1. $\bar{K}_{n_1}+\bar{K}_{n_2}+\ldots+\bar{K}_{n_r}, n_i > 2$ for i = 1,2,...,r are examples of d.c.s graphs.

The following theorem gives another class of d.c.s graphs.

Theorem 2.2. [13] Let G be a triangle free graph. Then the graph $D_\lambda(G)$ obtained by taking λ copies, $G_1, G_2, \ldots, G_\lambda$ of G and joining each vertex u_i in G_i to the neighbours of the corresponding vertex u_j in G_j for $i,j = 1,2,\ldots,\lambda$, is a d.c.s graph for $\lambda > 1$.

The graph $D_2(C_5)$ is shown in Fig. 2.2.

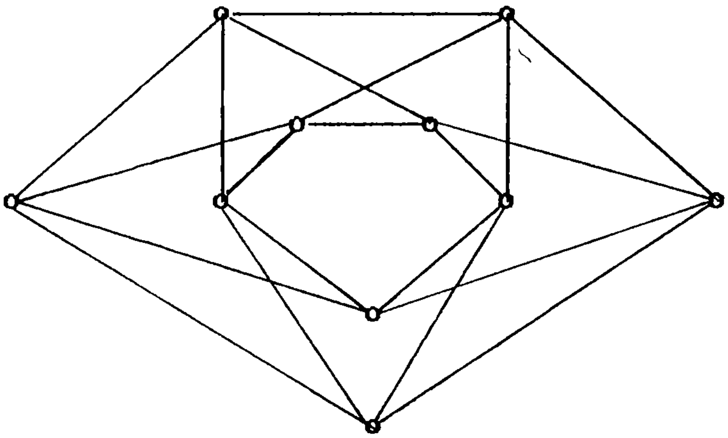


Fig. 2.2

The following theorems from [41] are of much use to us.

Theorem 2.3. Let G be a planar connected graph of order at least four. Then the following are equivalent.

1. G is d.c.s.

2. G is a block without an induced subgraph isomorphic to a cycle C_3, C_n for $n > 4$ or a 6-cycle with exactly one bichord.

3. For each vertex u of degree at least three, there is a unique vertex u' in G such that $N(u) = N(u')$.

Two such vertices u and u' are called partners.

Theorem 2.4. A d.c.s graph G(p,q) is planar if and only if $q = 2p-4$.

Theorem 2.5. [66] Let G be a connected, planar graph of order $p \geq 4$ and $G \neq Q_3$. Then G is a d.c.s graph if and only if it satisfies Cl.

Interval monotone graphs [53] are those for which all its intervals are convex. Trees, hypercubes, Ptolemaic graphs are examples of interval monotone graphs. A graph is totally noninterval monotone (t.n.i.m) if no nontrivial geodesic interval is convex. It is clear that I(a,b) is convex whenever a = b, a adjacent to b or I(a,b) = V(G). These are called the trivial geodesic intervals.

Note 2.1. A t.n.i.m. graph satisfies the conditions C1 and C2. Otherwise, if u-v-w is a 2-path in G such that there is not an x adjacent to u and w, then I(u,w) = {u,v,w} will be a convex interval. Similarly, if C2 is not satisfied, then the cycle u-v-w-x-u gives the convex interval

I(u,w) = {u,v,w,x}.

However, the conditions C1 and C2 are not sufficient for a graph to be t.n.i.m. In the graph of Fig. 2.1, I(a,e) = {a,b,c,d,e} is a convex interval.

It is clear that d.c.s graphs are triangle free and t.n.i.m. But the converse is not true. The graph G of Fig.2.3 is a triangle free t.n.i.m. graph which is not d.c.s.

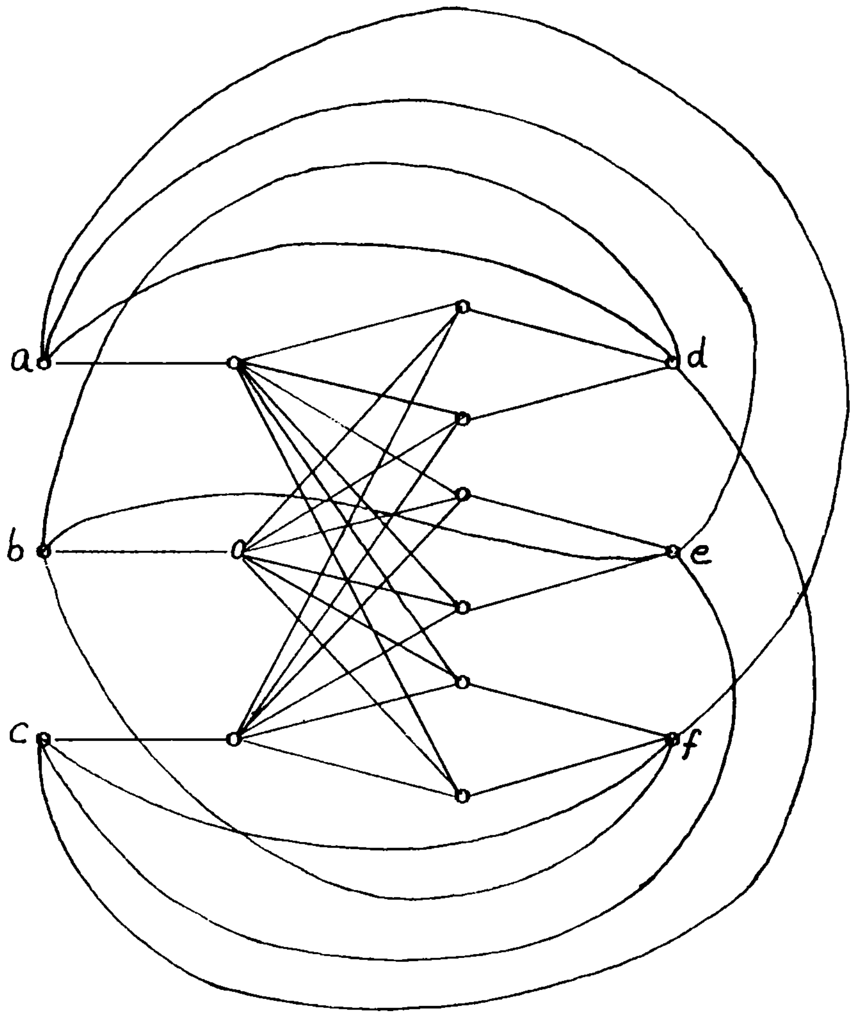


Fig 2.3

In G, the only nontrivial convex set is {a,b,c,d,e,f} and it is not an interval. That is, G is a uniconvex graph in which no nontrivial interval is convex.

Since any connected graph of order atmost five which satisfies C1 and C2 can be expressed as an interval

(C_4 and $K_{2,3}$, which are the only such graphs, can be expressed as interval) a convex set in a trianglefree t.n.i.m. graph will contain at least six vertices.

However, for a triangle free planar graph G, the following theorem holds.

Theorem 2.6. Let G be a triangle free planar graph. Then G is d.c.s if and only if it is t.n.i.m.

Proof: If G is d.c.s then it is t.n.i.m trivially. Now, let it be a triangle free planar t.n.i.m graph. Then $G \not\cong Q_3$ (the 3-cube) because Q_3 is not t.n.i.m. Now by theorem 2.4, G is d.c.s..

We shall now give two methods of constructing a triangle free, t.n.i.m graph, having exactly k non trivial convex sets.

CONSTRUCTION 1. Let G be a d.c.s graph with $I(a,b) \neq V(G)$ for any $a,b \in V(G)$ and let G_1, G_2 and G_3 be three copies of G. Join each vertex of G_1 to the corresponding vertices of G_2 and G_3 and each vertex of G_2 to the neighbours of corresponding vertices of G_3. The resulting graph is denoted by G^1.

Remark 2.1 G^1 can also be obtained by taking $K_2 \times G$ and then multiplying all the vertices of the copy of G corresponding to one of the vertices of K_2. Also if $u.v \in G$ and u_i, v_i are the vertices corresponding to u and v, for i = 1,2,3. Then

$$d(u_i, v_i) = d(u,v) \text{ for } i = 1,2,3$$

$$d(u_1, v_2) = d(u_1, v_3) = d(u_1, v_1) + 1$$

The graphs induced by $G_1 \cup G_2$ and $G_1 \cup G_3$ are isomorphic to $G \times K_2$ and that induced by $G_2 \cup G_3$ is $D_2(G)$.

Claim: G^1 is having exactly one convex set and it is $V(G_1)$.

It is enough to prove that $Co(\{u,v\}) = V(G_1)$ whenever $u,v \in V(G_1)$ and $Co(\{u,v\}) = V(G^1)$ if one of u and v is in G_2 or G_3.

Case 1: Let $u_1, v_1 \in V(G_1)$ be non adjacent vertices. Let $w \in G$. Then $d(u_1, w_2) = d(u_1, w_1) + 1$ where w_i is the corresponding vertex of w in G_i for i=1,2,3.

Also $d(v_1, w_2) = d(v_1, w_1)+1$. Hence,

$d(u_1,v_1) \leq d(u_1,w_1)+d(v_1,w_1) < d(u_1,w_2)+d(v_1,w_2)+d(v_1,w_2)$. Hence, w_2 is not on a u_1-v_1 shortest path. Now, because G is d.c.s no nontrivial subset of G_1 is convex. Hence, $Co(\{u_1,v_1\}) = V(G_1)$.

Case 2. If $u,v \in G_2 \cup G_3$, then by theorem 1.2, $G_2 \cup G_3$ induce a d.c.s graph and hence $V(G_1)$, $V(G_2) \subset Co\ \{u,v\}$. Now, for any $w \in G$, $w_2,w_3 \in Co\ (\{u,v\})$, where w_2,w_3 are copies of w in $G_2$and G_3. w_1 is on a shortest w_2-w_3 path and hence $w \in Co(\{w_2,w_3\}) \subset Co(\{u.v\})$. Therefore $Co(\{u,v\}) = V(G^1)$.

Case 3. Let $u_1 \in G_1$ and $v_2 \in G_2$ (similarly when $v_3 \in G_3$). Then $u_2,v_1 \in Co(\{u_1,v_2\})$. Now, since $N(u_2) = N(u_3)$, u_3 is on a shortest u_1-v_2 path. That is $u_2;u_3 \in Co(\{u_1,v_2\})$. Then, as in case 2, $Co\ (\{u_1,v_2\}) = V(G^1)$.

Now, since $V(G_1)$ cannot be expressed as an interval, G^1 is t.n.i.m. Taking G^1 in the place of G, construct G^2 in which $V(G_1^1)$ and $V(G_1)$ are the only convex sets. Proceeding like this we get G^k in which $V(G_1)$, $V(G_1^1)$, $V(G_1^2),\ldots,$ $V(G_1^{k-1})$ are the only convex sets.

CONSTRUCTION 2. Let G be a d.c.s graph in which $I(a,b) \neq V(G)$ for any $a,b \in V(G)$. replace each vertex of a star $K_{1,k}$ by a copy of G. Join each vertex of the copy G_u of G corresponding to the center of $K_{1,k}$ to the corresponding vertices of the other copies. Now, replace each vertex of G_u by a pair of nonadjacent vertices. The graph G so obtained is a triangle free t.n.i.m graph with exactly k convex sets.

Remark 2.2. In general, the k-convex graphs obtained by Construction 1 and Construction 2 are not isomorphic. In Construction 1 the convex sets of G^k form an ascending chain $V(G_1) \subset V(G_1^1) \subset \ldots \subset V(G_1^{k-1})$. But in Construction 2, the k convex sets are disjoint. However, when k=1 both the constructions give the same graph.

We shall now discuss the separation properties (Definition 1.16) of d.c.s graphs. Any graph trivially satisfies S_1 property. The graphs in Fig.2.4 indicate that there are graphs satisfying S_i but not S_{i+1}, for i = 1,2,3.

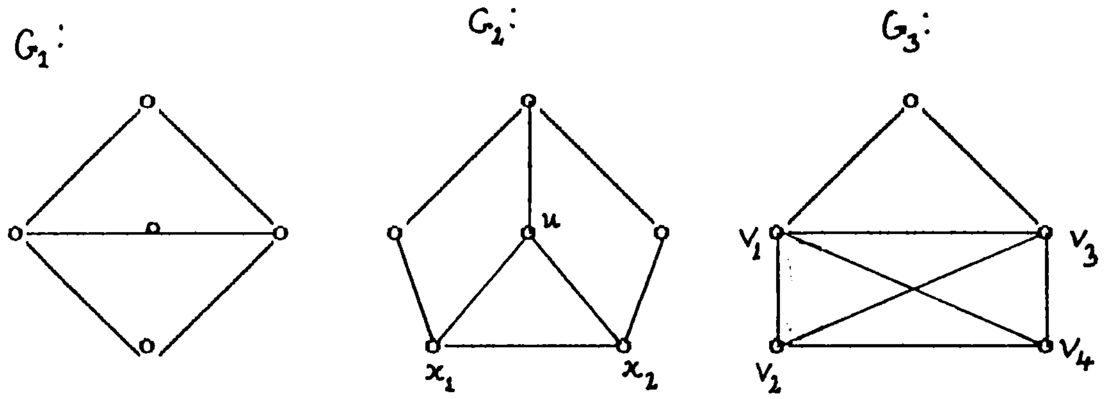


Fig 2.4

G_1 is not S_2 . G_2 is S_2 but not S_3 . Here, there is no halfspace separating the convex set $\{x_1, x_2\}$ and the vertex u. G_3 is S_3 but not S_4 . The convex sets $\{v_1, v_2\}$ and $\{v_3, v_4\}$ are disjoint convex sets which cannot be separated by halfspaces.

There are graphs for which $V(G)\backslash C$ is not convex for any convex set C. We make the following.

Definition 2.1. A graph G is halfspace free if no subset of V(G) is a halfspace.

Theorem 2.7. A connected triangle free graph G of order at least five is halfspace free if it satisfies the conditions C1 and C2.

Proof. Let G be a connected triangle free graph satisfying C1 and C2. Let $C \subset V(G)$ be a convex subset. To prove that $V(G)\backslash C$ is not convex.

Let $u \in V(G)\backslash C$, $v \in V(G)\backslash C$ and $uv \in E(G)$.

Let $w \in V(G)$, $w \neq v$ and $wu \in E(G)$. Note that such a vertex exist because G is of order at least five and it satisfies C1. Now w-u-v is a 2-path and by C1 there is an x in V(G) which is adjacent to w and v (see Fig.2.5)

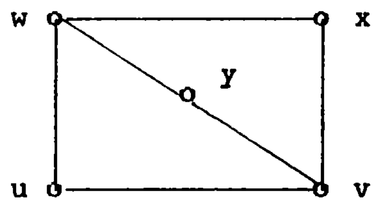


Fig. 2.5.

Now $x \in V(G)\backslash C$ because C is convex, $v \in V(G)\backslash C$ and $v \in Co(\{u,x\})$.

If $w \in V(G)\backslash C$, it is not convex because $u \in Co(\{w,v\})$, but $u \in C$. So let $w \in C$ and $x \in V(G)\backslash C$.

Now, w-u-v-x-w is a 4-cycle in G and by C_2, there is a vertex y adjacent to either w and v or u and x.

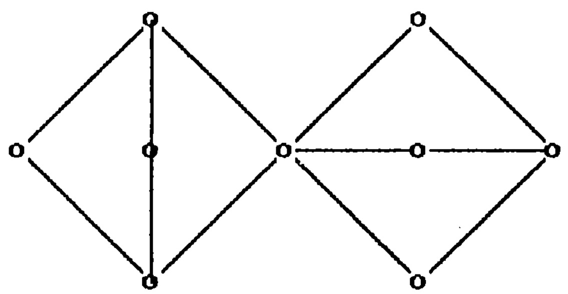
Let y be adjacent to w and v. Then $y \notin C$ because in that case $v \in Co(\{u,y\}) \subset C$ which is a contradiction. Hence, $v \in V(G) \setminus C$. Then, since $w \in Co\{y,x\}$ and $w \in C$, $V(G) \setminus C$ is not convex.

Similar is the case when y is adjacent to u and x. Hence, for any convex set C, $V(G) \setminus C$ is not convex. That is, there is no halfspace in G.

Corollary. Distance convex simple graphs and t.n.i.m graphs are halfspace free.

Note 2.2. Neither C1 nor C2 is necessary for a graph to be halfspace free. The graph G_1 of Fig.2.6 does not satisfy C1, and G_2 of Fig.2.6 does not satisfy C2, but both are halfspace free.

G_1:



G_2:

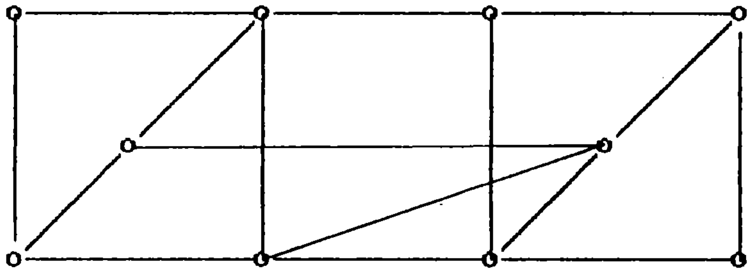


Fig 2.6

The convex invariant could easily be determined for d.c.s graphs. If G is a d.c.s graph, then any set S of three vertices contains a pair u.v of non adjacent vertices and Co({u,v}) = V(G). This observation leads to

Theorem 2.8. For a d.c.s graph G, h(G) = c(G) = r(G) = 2 and e(G) = 3.

It is interesting to observe the star center (Definition 1.17) of a d.c.s graph. It is known that

Theorem 2.9. [12]. A convex structure with Caratheodory number 2 is JHC.

Theorem 2.10 [12]. A JHC convex structure has the Brunn's property.

By theorems 2.8, 2.9 and 2.10 it follows that d.c.s graphs satisfies Brunn's property with respect to the convex hull operator. But when we consider the geodesic interval operator, this will not be true.
For example, the graph G in Fig.2.7 is d.c.s.

G:

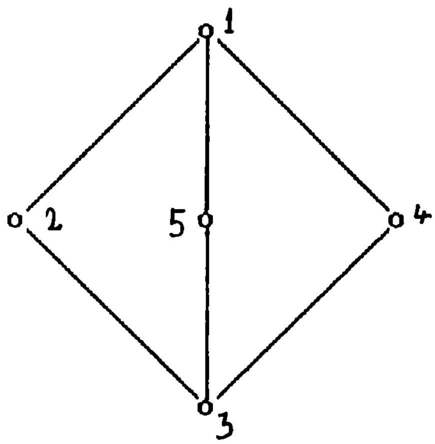


Fig.2.7

Let S = {1,2,3,4}, Ker(S)is the set {2,4} which is disconnected.

2.2. MINIMAL PATH CONVEXITY AND m-CONVEX SIMPLE GRAPHS

In this section by convex sets we mean only m-convex sets and by intervals, only minimal path intervals. It is known (Theorem 1.10) that for any graph G, c(G) is at most 2 and has JHC property. Hence, by theorem 2.10, G has the Brunn's property. But, if Ker(S) is taken with respect to the minimal path interval operator, this is not true. Consider the graph G in Fig.2.8

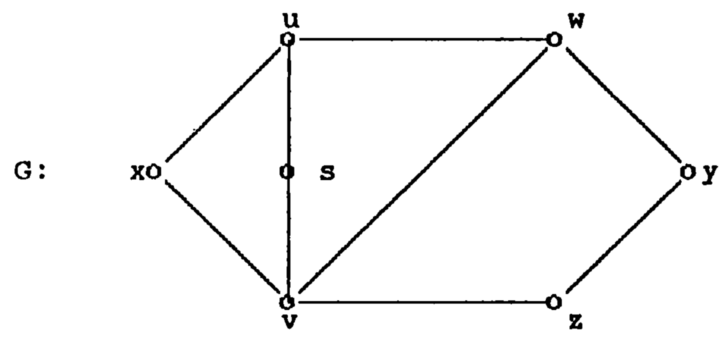


Fig. 2.8

In G, let $S = \{x,u,v,w,z,y\}$. It can be seen that $x,y \in Ker(S)$. But u, which is on a chordless x-y path is not in Ker(S).

However, the following theorem gives a class of graphs for which the Brunn's property holds with respect to minimal path interval.

Theorem 2.11. Let G be a chordal graph and let,
$Ker(S) = \{z \in S: I(z,s) \subset S\}$ for every $s \in S\}$ for $S \subset V(G)$.
Then Ker(S) is convex.

Proof: Let $x,y \in Ker(S)$, and z is on some x-y chordless path where $S \subset V(G)$.

To prove that $I(x,y) \subset Ker(S)$ where,

$I(x,y) = \{z: z \text{ is on some chordless x-y path}\}$

Since $x,y \in Ker(S)$, $I(x,s) \subset S$, $I(y,s) \subset S$, for every $s \in S$. Let $z \in I(x,y)$. To prove that $I(z,s) \subset S$ for every s $s \in S$. Assume without loss of generality that z is adjacent to x. Let $P_1 = z\text{-}a_1\text{-}a_2\text{-}\ldots\text{-}a_n\text{-}s$ be an z-s chordless path and

$P_2 = x\text{-}z\text{-}b_1\text{-}b_2\text{-}\ldots\text{-}b_k = y$ be an x-y chordless path.

If $x\text{-}z\text{-}a_1\text{-}a_2\text{-}\ldots\text{-}a_n\text{-}s$ is chordless, then clearly $z,a,\ldots,a_n,s \in S$.

Similarly when $y\text{-}b_k\text{-}\ldots\text{-}b_1\text{-}z\text{-}a_1\text{-}\ldots\text{-}a_n\text{-}z$ is chordless path. So assume that these are having chords. If ℓ is such that a_ℓ is adjacent to x, (Note that one end vertex of any chord of this path is x, because $z\text{-}a_1\text{-}a_2\text{-}\ldots\text{-}a_n\text{-}s$ is chordless). Then $x\text{-}a_\ell\text{-}a_{\ell-1}\text{-}\ldots\text{-}a_1\text{-}z\text{-}x$ is a cycle in G. If $\ell > 1$ this is a cycle of length at least four and hence has a chord. Thus we can see that x is adjacent to a_1. Similarly if b_i is adjacent to a_m for some m=1, we can see that a_1 adjacent to b_1 (see Fig.2.9).

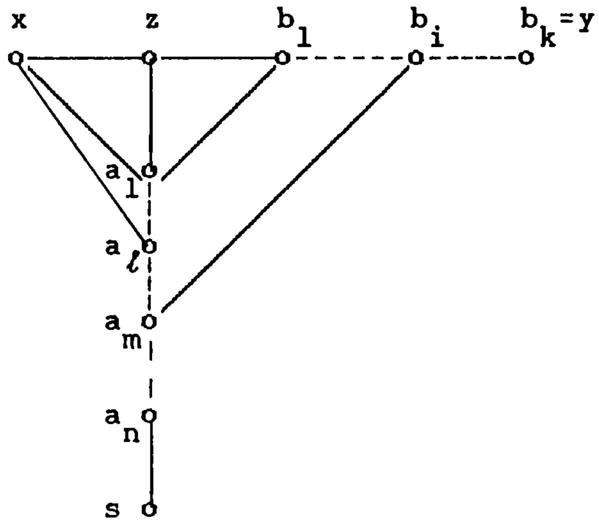


Fig 2.9

Now if $P_3 = x_a-a_1-\ldots-a_n-s$ is a chordless path or $P_4 = y-b_{k-1}\ldots b_1-a_1\ldots a_n-s$ is a chordless path, $a_1,\ldots,a_n \in S$. As above, if x is adjacent to a_ℓ for some $\ell > 1$ then x is adjacent to a_2. Also if b_i is adjacent to a_m for some $m > 1$, a_2 will be adjacent to some vertex on $b_1-b_2\ldots b_i$. Let b_j be the first vertex on $b_1-b_2-\ldots-b_i$ which is adjacent to a_2. (see Fig.2.10).

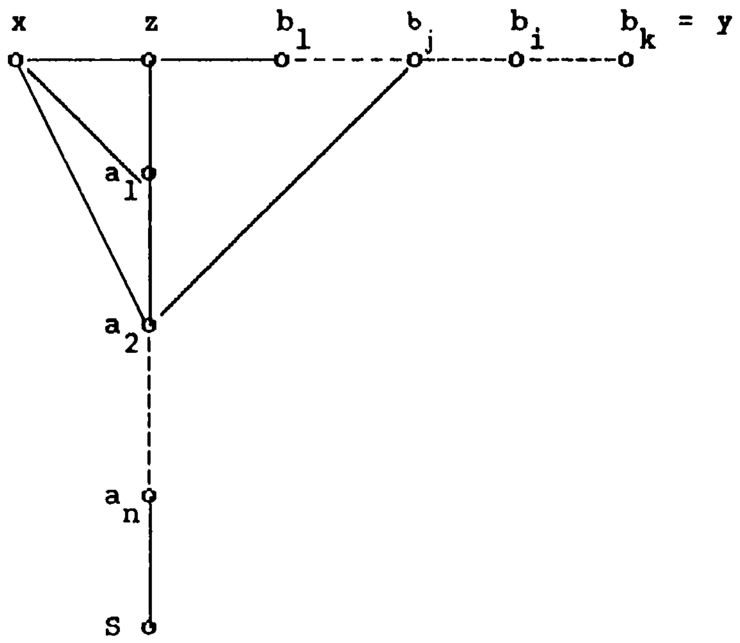


Fig. 2.10.

Then $x-z-b_1-b_2-\ldots-b_j-a_2-x$ will be a chordless cycle of length at least four. Hence either P_3 or P_4 is chordless. Hence $l(z,s) \subset S$ and therefore $z \in Ker(S)$. □

m-convex simple (m.c.s) graphs are those whose only nontrivial convex subsets are the null set, singletons, pairs of adjacent vertices and the whole set V(G). The following theorem gives a necessary and sufficient condition for a graph to be m.c.s.

Theorem 2.12. [26]. A graph is m.c.s if and only if it has no nontrivial clique or clique separator.

It is clear that d.c.s graphs are m.c.s. But the converse is not true. For example, the graph in Fig.2.11 is an m.c.s graph which is not d.c.s.

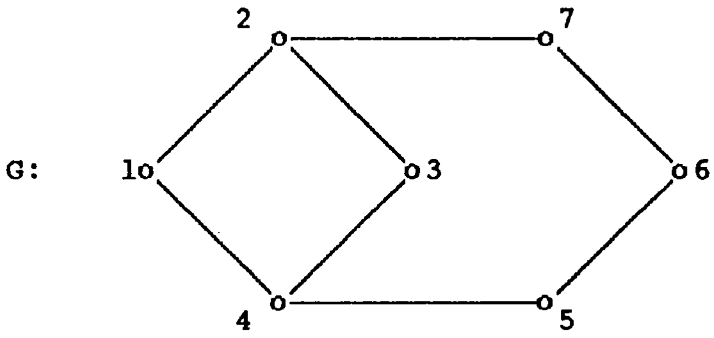


Fig. 2.11

By theorem 2.12 it is clear that G is an m.c.s graph. But it is not d.c.s because {5,6,7} is a nontrivial d-convex set. The question as to whether there exist for any given k, a k-convex graph which is triangle free and totally non interval monotone, with respect to m-convexity also, lead us to following theorems.

Theorem 2.13. There is no uniconvex graph.

Proof: Let G be a graph having a nontrivial convex subset. Then by theorem 2.12, G contains a clique separator S. Let

$C_1, C_2, \ldots, C_n$ be components of $G \backslash S$. Clearly $n \geq 2$. Then $C_i \cup S$ is convex in G because any chordless path connecting vertices of $C_i \cup S$ will be contained in $\langle C_i \cup S \rangle$. Note that since S is complete, any path containing a vertex not in $C_i \cup S$ will have a chord. Thus the number of convex sets is at least two. □

We call a convex set C to be a minimal nontrivial convex subset if no proper subset of C of cardinality at least three is convex.

The following theorem specify the condition on k which is necessary for a graph to be k-convex.

Theorem 2.14. Let G be a k-convex, triangle free, 2-connected graph. Then there is an 'n' such that $(n-1)(n+2)/2 \leq k \leq 2^n-2$.

Proof: Let $C_1, C_2, \ldots, C_n$ be minimal nontrivial subsets of G. Hence $C_i \cap C_j$ contains at most two vertices for $i \neq j$. Otherwise $C_i \cap C_j$ will be a nontrivial convex set which is a proper subset of C_i. Let $C_i \cap C_j = S$ with $|S| = 2$.

Claim 1. S is a clique separator.

Let $S = \{x,y\}$. Then,

$Co(S) \subset C_i \cap C_j$. If x is not adjacent to y, Co(S) will be a nontrivial convex subset properly contained in C_i. Hence x is adjacent to y, that is S is a clique.

Now to prove that G\S is disconnected. If not, each pair of vertices in G\S is connected by a path. In particular, each vertex of $C_i \setminus S$ is connected to each vertex of $C_j \setminus S$ by some path in G\S. Let $c_i \in C_i \setminus S$ and $c_j \in C_j \setminus S$. Let $c_i - u_1 - u_2 \ldots u_\ell - c_j$ be a chordless $c_i - c_j$ path in G\S. Assume without loss of generality that c_i is so chosen that $u_k \notin C_i$, for $k = 1 \ldots \ell$. Since G is triangle free c_i is not adjacent to at least one vertex of S. Let it be x. Consider the path joining c_i and x which contain c_j on it. It is clear that some subset of this will induce a chordless c_i- x path containing a vertex in $C_j \setminus S$. This is not possible because C_i is convex. Hence G\S is disconnected. Therefore, if $C_i \cap C_j = S$, any clique of size at least two, then S is a separator set.

Now, let H be a graph with,

$V(H) = \{C_1, C_2, \ldots, C_n\}$ and C_i is adjacent to C_j, if $C_i \cap C_j$ is a clique separator.

Claim: H is a block graph.

If not there will be a block B in H and $C_i, C_j \in B$ such that C_i is not adjacent to C_j in H. Assume without loss of generality that $d(C_i, C_j) = 2$ and let i = 1, j = 3. Let C_1-C_2-C_3 be a path and since these are vertices of a block, there will be another path C_3-C_4-...-C_i-C_1 connecting C_3 and C_1.

Let $C_1 \cap C_2 = S_1$, $C_2 \cap C_3 = S_3 \ldots C_i \cap S_1 = S_i$. Note that $S_1 \neq S_2$. Otherwise $S_1 = S_2 \subset C_1 \cap C_3$ and hence C_1 will be adjacent to C_3 which is a contradiction. Now, since G is triangle free, we get an $x \in S_1, y \in S_2$ such that x is not adjacent to y. Note that $x, y \in C_2$. Assume without loss of generality that $S_1 \neq S_i$ and $S_2 \neq S_3$.

[If $S_1 = S_i = S_{i-1} = ,,, = S_3$, $C_1 \cap C_2 = S_3$, $C_3 \cap C_4 = S_3$ and C_1 will be adjacent to C_3. Similarly when $S_2 = S_3 = \ldots\ldots = S_i$.]

Now because $C_1 \cap C_i \neq \phi$, $C_i \cap C_{i-1} \neq \phi$, ... $C_3 \cap C_2 \neq \phi$ and $x \in C_1$ and $y \in C_2$, we get an x-y path through C_i, C_{i-1},.... .., C_3, C_2 and hence a chordless path joining x and y containing vertices of these sets. That is C_2 is not convex, which is a contradiction. Hence H is a block graph.

Now observe that if C_1 and C_2 are convex and $C_1 \cap C_2 = S$, a clique separator, $C_1 \cup C_2$ is convex. Hence, the convex sets of G are those corresponding to the connected subsets of H. It is known that the number of connected sets of a block graph is minimum when it is a path and is a maximum when it is a complete graph. The number of connected sets other than the whole set is $(n-1)(n+2)/2$ when it is a path and it is 2^n-2 when it is a complete graph. Hence the number of connected sets in H lies between $(n-1)(n+2)/2$ and 2^n-2. Therefore, G is a k-convex graph implies that there is an n such that $(n-1)(n+2)/2 \leq k \leq 2^n-2$ □.

Illustration: If n = 1, then k = 0 and G is an m.c.s. graph.

If $n = 2$, then $k=2$. So, there is no uniconvex graph.

If $n = 3$, then $5 \leq k \leq 6$, so there is no 3-convex graph or 4-convex graph.

If $n = 4$, then $9 \leq k \leq 14$, so there is no 7-convex or 8-convex graphs.

Remark 2.3. In the theorem 2.14, for any $C_i \in H$, if $N(C_i)$ consists of m pairwise nonadjacent vertices, then the subgraph of G induced by C_i consists of at least m-edges. This is because if $C_1 \ldots C_n$ are the neighbours of C_i which are pairwise nonadjacent, then in G,

$C_i \cap C_k \neq S_k \simeq K_2$ for $k = 1,\ldots,m$ and $S_k \neq S_\ell$ for $k \neq \ell$

COROLLARY: Let H be a block graph of order p. Then there is a t.n.i.m graph G' such that G' is k-convex where k is the number of convex subsets of H other than the null set and the whole set.

Proof: Let $G \simeq K_{n,n}$ $n \geq 3$. Take n to be sufficiently large so that if $C_1, C_2, \ldots, C_m$ are the vertices of H as in the Remark 2.3, then $m \leq n^2$. Let $V(H) = \{C_1, C_2, \ldots, C_p\}$. Now form G' as follows. Let $G_1, G_2, \ldots, G_p$ be p copies of G. Identify an edge of G_i with the corresponding edge of G_j if and only if C_i is adjacent to C_j in H. That is, $G_i \cap G_j \simeq K_2$ in G' if C_i is adjacent to C_j in H. Now, the nontrivial convex sets of G' are those corresponding to the convex sets of H different from the null set and the whole set.

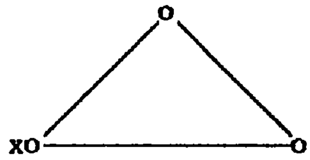
Now we prove that none of these is an interval.

If $a,b \in G_i$ for some i, then, I(a,b) cannot be convex. Assume that $a \in G_1$, $b \in G_2$. Then any path connecting a and b contain the vertices of a clique separator S where $S \subset V(G_1)$. Let V_1 and V_2 be the bipartition of V(G). Let $V_{1,1}$ and $V_{1,2}$ be the corresponding sets in $V(G_1)$. Let $a \in V_{1,1}$ (similarly when $a \in V_{1,2}$).
Let $a_1 \in V_{1,1} \setminus (S \cup \{a\})$. Such a vertex exist because $S \cap V_{1,1}$ is a singleton and $|V_{1,1}| \geq 3$.

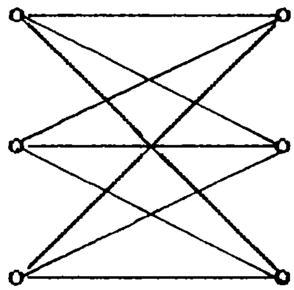
Claim: $a_1 \notin I(a,b)$.
If $a-b_1-a_1-b_2-a_2\ldots-b$ is an a-b path then $a-b_2$ is a chord. Hence, there does not exist a chordless a-b path containing a_1. Hence no nontrivial interval is convex.

Illustration:

H:



G:



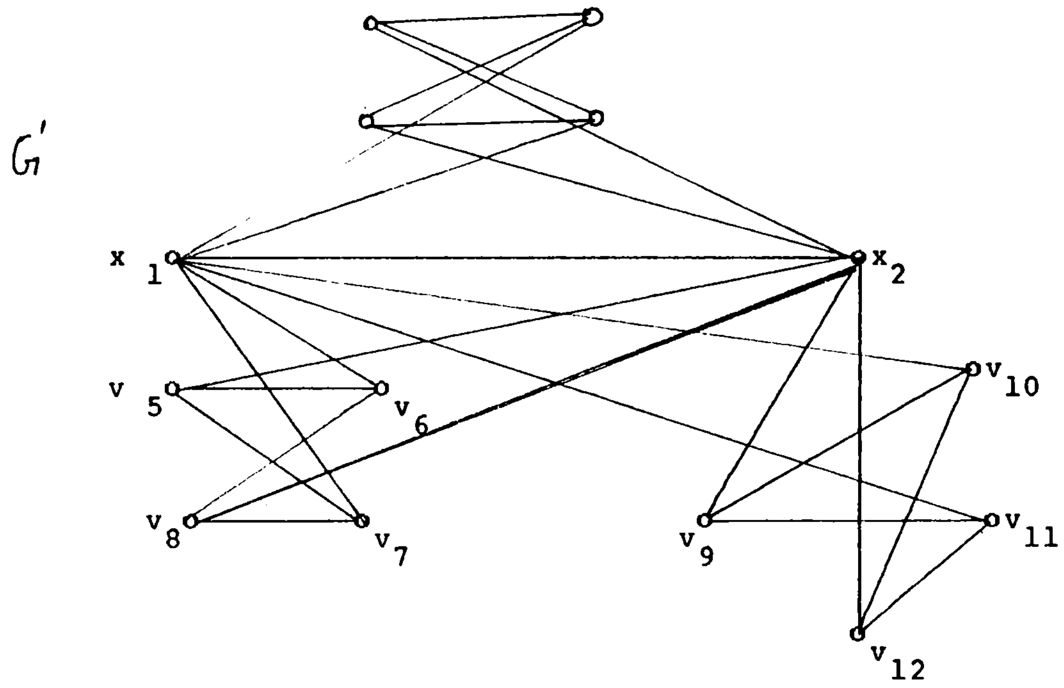


Fig. 2.12

2.3. ITERATION NUMBER

Minimal path iteration number of a graph G, min(G) [27] is a concept analogous to geodetic iteration number (Definition 1.11). It is obtained in a similar manner by replacing the geodetic interval operator by the minimal path interval operator.

It can be observed that for any given k, the sequential join of k+1 copies of $\bar{K}_2$, is a graph which is both d.c.s and m.c.s and both its minimal path iteration number and geodetic iteration number is k.

We know that the Caratheodory number of any graph with m-convexity is atmost 2 and hence JHC. In addition, if G is interval monotone with respect to m-convexity, min(G) = 1 and conversely.

However, in the case of graphs with geodesic convexity it is necessary that G should be interval monotone and JHC in order that gin(G) = 1. But, it is not sufficient (See Fig.2.13).

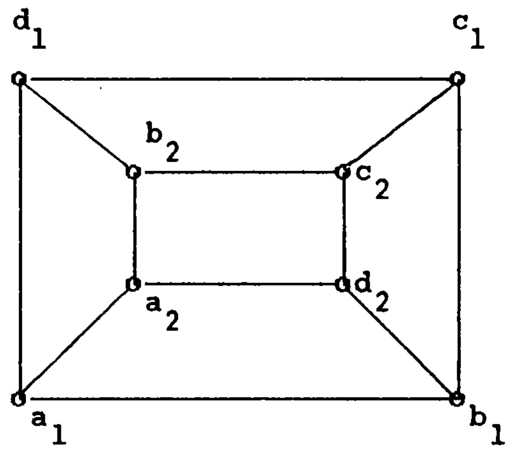


Fig. 2.13.

Let $S = \{a_2, b_1, d_1\}$. Then,

$$S^1 = \{a_1, b_1, c_1, d_1, a_2, b_2, d_2\} \text{ and } S^2 = V(G).$$

Hence gin $(G) \neq 1$.

If G is interval monotone but not JHC, there are graphs G and $S \subset V(G)$ with $|S| = 3$ such that gin(S) is large. However, we have,

Theorem 2.15. Let G be a JHC, interval monotone graph and let $S \subset V(G)$. Then g in $(S) \leq k$, where k is such that

$$k-1 < \frac{\log |S|}{\log 2} \leq k.$$

Proof: Let $S = \{a_1, \ldots, a_n\}$.

Let $C_1 = Co(\{a_1, \ldots, a_{\lceil n/2 \rceil}\})$ and

$$C_2 = Co(\{a_{\lceil n/2 \rceil + 1}, \ldots, a_n\})$$

Then $Co(S) = CO(C_1 \cup C_2) = \bigcup\{Co(\{c_1, c_2\}): c_1 \in C_1, c_2 \in C_2\}$, since G is JHC.

$= \bigcup \{I(c_1, c_2): c_1 \in C_1, c_2 \in C_2\}$ because G is interval monotone.

Hence $Co(S) = \bigcup \{I(c_1,c_2): c_1,c_2 \in C_1 \cup C_2\}$

$$= (C_1 \cup C_2)^1.$$

Now let $C_{11} = Co(\{a_1,a_2,\ldots,a_{\lceil n/4\rceil}\})$

$$C_{12} = Co(\{a_{\lceil n/4\rceil+1}, \ldots , a_{\lceil n/2\rceil}\})$$

$$C_{21} = Co(\{a_{\lceil n/2\rceil+1}, \ldots , a_{\lceil 3n/4\rceil}\})$$

$$C_{22} = Co(\{a_{\lceil 3n/4\rceil+1}, \ldots , a_n\})$$

Then $C_1 = Co(C_{11} \cup C_{12})$ and $C_2 = Co(C_{21} \cup C_{22})$. Then as above, $C_1 = (C_{11} \cup C_{12})^1$ and $C_2 = (C_{21} \cup C_{22})^1$

Hence $Co(S) = ((C_{11} \cup C_{12})^1 \cup (C_{21} \cup C_{22})^1)^1$

$$= (C_{11} \cup C_{12} \cup C_{21} \cup C_{22})^2.$$

$$= (Co(\{a_1,\ldots, a_{\lceil n/4\rceil}\}) \cup Co(\{a_{\lceil n/4\rceil+1},\ldots,a_{\lceil n/2\rceil}\}) \cup \ldots\}^2$$

$$= (Co(\{a_1,\ldots,a_{\lceil n/2^2\rceil}\}) \cup Co(\{a_{\lceil n/2^2\rceil+1},\ldots,a_{\lceil n/2\rceil}\}) \cup \ldots\}^2$$

Proceeding like this,

$$Co(S) = (Co\{a_1,\ldots,a_{\lceil n/2^k\rceil}\} \cup Co\{a_{\lceil n/2^k\rceil+1}\},\ldots a_{\lceil n/2^{k-1}\rceil} \cup \ldots \cup \{a_{\lceil(2^k-1)n/2^k\rceil+1}, a_n\})^k$$

Now, When $\lceil n/2^k\rceil = 1$

$2^{k-1} < n \le 2^k$ and $Co(a_1 \ldots a_{\lceil n/2^k\rceil}) = Co(a_1) = \{a_1\}$

$$Co(S) = (\{a_1\} \cup \{a_2\} \ldots \cup \{a_n\})^k$$

$$= (\{a_1 \ \ldots \ a_n\})^k = S^k$$

Hence, gin $S \le k$, where $2^{k-1} < n < 2^k$. That is $k-1 < (\log n / \log 2) \le k$. That is $k-1 < \frac{\log |S|}{\log 2} \le k$

□

The following discussion illustrates that there are graphs G and $S \subset V(G)$ such that gin(S) = k where $2^{k-1} < S \le 2^k$.

Let k be any integer and $n = 2^k$. Let Q_n be the n-cube, vertices labelled with (0,1) valued n-tuples.

Let $\delta_i = (x_1,\ldots,x_n)$ where $x_i = 1$ and $x_j = 0$ for $j \ne i$ and $\delta_0 = (0,0,\ldots,0)$. Then, $d(\delta_i,\delta_j) = 2$, for $i \ne 0$.

Let $S = \{\delta_1, \delta_2, \ldots, \delta_n\}$.

If $\delta_{i,j} = (x_1, \ldots, x_n)$ where $x_i = x_j = 1$ and $x_k = 0$ for $k \neq i,j$, then $\delta_{i,j}$ is adjacent to δ_i and δ_j.

Hence, $S^1 = \{\delta_0\} \cup S \cup N_2(\delta_0)$

Now if $\delta_{i,j}\ \delta_{k,\ell} \in N_2(\delta_0)$ be such that $i,j \neq k,\ell$, then $d(\delta_{i,j}, \delta_{k,\ell}) = 4$ and if $A = \{i,j,k,\ell\}$ and $\delta_A = (x_1, \ldots, x_n)$ where $x_i = x_j = x_k = x_\ell = 1$ and $x_m = 0$ for $m \notin A$, then, $\delta_A \in I(\delta_{i,j}, \delta_{k,1})$.

Hence, $S^2 = \{\delta_0\} \cup S \cup N_2(\delta_0) \cup N_3(\delta_0) \cup N_4(\delta_0)$

$$= \{\delta_0\} \cup S \cup N_2(\delta_0) \cup N_3(\delta_0) \cup N_{2^2}(\delta_0)$$

Similarly, $S^3 = \{\delta_0\} \cup S \cup \{N_2(\delta_0) \cup \ldots \cup N_{2^3}(\delta_0)$, and

$$S_k = \{\delta_0\} \cup S \cup N_2(\delta_0) \cup \ldots \cup N_{2^k}(\delta_0) = V(Q_n).$$

Hence, $gin(S) = k$.

Note 2.2. If n is such that $2^{k-1} < n < 2^k$, in the above example,

$$S^{k-1} = \{\delta_0\} \cup S \cup N_2(\delta_0) \cup \ldots \cup N_{2^{k-1}}(\delta_0)$$

and $S^k = \{\delta_0\} \cup S \cup N_2(\delta_0) \cup \ldots \cup N_{2^{k-1}}(\delta_0) \ldots \cup N_n(\delta_0)$.

Therefore if n is such that $2^{k-1} < n \leq 2^k$

$$g\ in\ (S) = gin\ (\{\delta_i\}) = k.$$

If an interval monotone, JHC graph has the additional property that the geodesic intervals are decomposable [12], then g in (G) = 1. Also, we observe that the class of graphs with decomposable intervals are nothing but the class of geodetic graphs. Hence we have,

Theorem 2.16. If G is a geodetic, JHC graph, then gin(G) = 1.

Proof: Since G is geodetic, it is interval monotone. Because G is JHC also, the geodesic interval operator satisfies the Peano property by Theorem 1.3. We denote by ab the shortest path connecting a and b.

Now, let $a,b,c \in V(G)$, $u \in ab$, and $v \in cu$. It is enough to prove that v is in one of the intervals $I(a,b)$, $I(b,c)$ or $I(a,c)$. Because G is geodetic $I(a,b) = ab$.

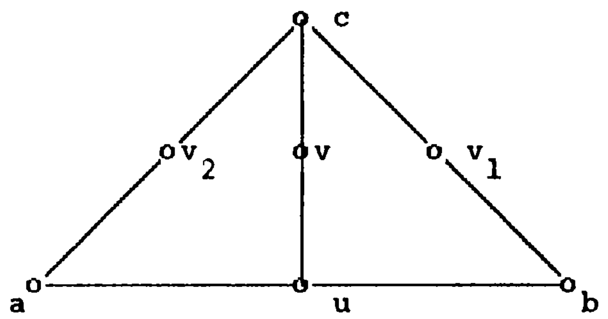


Fig. 2.14.

Assume without loss of generality that $d(c,v) = 1$. Let $d(a,c) = \ell_1$ and $d(b,c) = \ell_2$. Now, by the Peano property, there are vertices $v_1 \in bc$, $v_2 \in ac$ such that $v \in av_1 \cap bv_2$.

Now because $d(a,c) = \ell_1$, $d(a,v) \geq \ell_1 - 1$.

If $d(a,v) = \ell_1 - 1$ then $d(a,c) = d(a,v) + 1 = d(a,v) + d(a,c)$ and hence $v \in ac$.

So assume $d(a,v) \geq \ell_1$.

If $d(a,v) > \ell_1$ then $d(a,v_1) = d(a,v) + d(v,v_1) > \ell_1 + d(v,v_1)$

That is $d(a,v_1) > d(a,c) + d(v,v_1)$

Now $d(a,v_1) \leq d(a,c) + d(c,v_1)$

Therefore $d(v,v_1) < d(c,v_1)$ and

$$\ell_2 - d(c,v_1) < \ell_2 - d(v,v_1)$$

$$d(b,v_1) < \ell_2 - d(v,v_1)$$

$d(b,v_1) + d(v,v_1) < \ell_2$ and so $d(b,v) \le \ell_2 - 1$ and $d(b,v) < \ell_2 - 1$ is not possible and hence $d(b,v) = \ell_2 - 1$ and in this case $v \in bc$.

Now assume that $d(a,v) = \ell_1$.

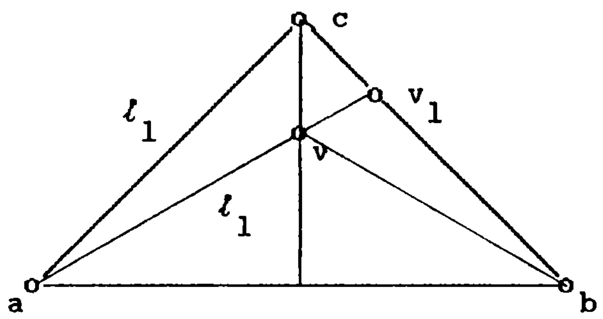


Fig.2.15

In this case $d(a,v_1) = \ell_1 + d(v,v_1)$

Now, if $d(c,v_1) > d(v,v_1)$, then

$$d(b,v_1) + d(v,v_1) \le \ell_2 - 1 \text{ and hence } v_1 \in bc.$$

So let $d(c,v_1) \leq d(v,v_1)$. But $d(c,v_1) < d(v,v_1)$ is not possible because av_1 is a shortest path containing v.

Therefore $d(c,v_1) = d(v,v_1)$.

But this is again a contradiction because these give two distinct shortest paths connecting a and v_1. □

CHAPTER III

III

CONVEX SIMPLE GRAPHS AND SOLVABILITY

In this chapter, we continue the study of properties of convex simple graphs. Motivated by a problem posed in [41], we define the notion of solvability and make an interesting observation that, all trees of order at most nine are solvable and that the bound is sharp. All trees of diameter three, five, and those with diameter four whose central vertex has even degree are also solvable. However, a characterization of solvable trees is yet to be obtained. A problem of similar type with respect to m-convexity is also discussed. We then discuss about the center of d.c.s graphs. We conclude this chapter with the study of the convexity properties of product of graphs. Some results of this chapter are in [60].

3.1 SOLVABLE TREES

In this section, we introduce the notion of solvable trees associated with a d.c.s graph, to answer the following,

PROBLEM [41] Describe the smallest distance convex simple graph containing a given tree of order at least four.

$K_{2,n}$ is such a graph for $K_{1,n}$. For a tree T which is not a star, let V_1 and V_2 be the bipartition of V(T) with $|V_1|=m, |V_2|=n$, then $K_{m,n}$ is a d.c.s graph containing a tree isomorphic to T. However, to find the smallest d.c.s. graph, we note by theorem 2.4. that, for any d.c.s. graph $q \geq 2p-4$ and the lower bound is attained if and only if it is planar. So, for a given tree T if there exists a planar d.c.s. graph containing T as a spanning subgraph, then that will be the smallest d.c.s. graph containing T. This observation leads us to,

Definition 3.1. A tree T is *solvable* if there is a planar distance convex simple graph G such that T is isomorphic to a spanning tree of G.

From the remarks made above, it is clear that $K_{1,n}$ is not solvable. Hence, in the following discussions we consider only trees which are not stars.

A USEFUL GRAPH OPERATION:

We shall now describe an operation frequently used in this section. Let u and v $\in V(G)$. Join u to all the vertices in $N(v)$ and v to all the vertices in $N(u)$. The resulting graph is denoted by $G^*(u,v)$ and in this graph $N(u) = N(v)$.

Remark 3.1 If G is planar and if G can be embedded so that $u, v, N(u)$ and $N(v)$ are all contained in the same face, then $G^*(u,v)$ is planar. Also, if u and v are partners then $G^*(u,v) \simeq G$.

Lemma 3.1. Any path of length at least four is solvable.

Proof: Let P be a path of length at least four and let $u \in C(P)$. Then $N_i(u)$ consists of two non-adjacent vertices for $i=1,2,\dots r-1$ and $N_r(u)$ is either a pair of non adjacent vertices or a Singleton according as $C(P) \simeq K_1$ or K_2, where r is the radius of P.
Now, the graph $G = \langle u\rangle+\langle N(u)\rangle+\dots+\langle N_r(u)\rangle$ is a planar d.c.s. graph containing P. □

Theorem 3.2. Any tree of order atmost nine is solvable.

Proof. If T is a path then it is solvable by the lemma 3.1. Suppose that T is not a path. Let u be a vertex of T such that $d(u) \geq 3$ and let $N(u) = \{a_1, a_2, \ldots a_n\}$, $n \geq 3$.

Case I. Any vertex in $N_2(u)$ is of degree one.

Assume that $d(a_1) = \min\{d(a_i) : a_i \in N(u)\}$. Choose $u' \in N_2(u)$ such that $N_2(u) \cap N(a_1) \backslash \{u'\} = \phi$. Construct

$G \simeq T*(u,u')*(a_1,a_2)*\ldots*(a_{n-1},a_n)$ if n is even and

$G \simeq T*(u,u')*(a_2,a_3)*\ldots*(a_{n-1},a_n)$ if n is odd.

Using theorem 2.3 and the remark 2.3, it follows that G is a planar d.c.s. graph which contains T.

Case II. There is a vertex in $N_2(u)$ of degree at least two. Choose $u' \in N_2(u)$ such that $d(u') = \max\{d(v) : V \in N_2(u)\}$ and let $N(u') = \{v_1, v_2, \ldots, v_m\}$. Let $N = N(u) \cup N(u')$. Note that, $m > 3$. Since $|V(T)| \leq 9$, $N(v_i) - \{u,u'\} = \phi$ for at least one value of i.

Sub case 1. $N[u] \cup N[u'] = V(T)$. Then $T*(u,u') \simeq K_{2,p-2}$ is such a planar d.c.s. graph.

Sub case 2. $N[u] \cup N[u'] \neq V(T)$, but

$$N[u] \cup N[u'] \cup \left(\bigcup_{i=1}^{m} N(v_i)\right) = V(T).$$

Without loss of generality assume that $N(v_1)\setminus\{u,u'\} = \phi$.

Then the required graph is $T*\langle u,u'\rangle*(v_1,v_2)* \ldots *(v_{m-1},v_m)$ if m is even and $T*(u,u')*(v_2,v_3)*\ldots*(v_{m-1},v_m)$ if m is odd.

Sub case 3. $N[u] \cup N[u'] \cup \left(\bigcup_{i=1}^{m} N(v_i)\right) \neq V(T)$. but

$$N[u] \cup N[u'] \cup \left[\bigcup_{i=1}^{m} N(v_i)\right] \cup \left[\bigcup_{i=1}^{m} N_2(v_i)\right] = V(T).$$

Here, note that $N(v_i)\setminus\{u,u'\} \neq \phi$ for at most two values of i, say 1 and 2. Let $w_1 \in N(v_i)\setminus \{u,u'\}$ be such that $d(w_1) \geq 2$. Since $|V(T)| \leq 9$, $d(w_1)$ can not exceed three. If $d(w_1) = 3$, by the choice of u', we can see that $w_1 \in N_4(u)$ in T and let $u\text{-}v_2\text{-}u'\text{-}v_1\text{-}w_1$, be the $u\text{-}w_1$ path in T (That is, $v_1 \notin N(u)$ and $v_2 \in N(u')$).

Now, $G \simeq T*(u,w_1)*(v_1,v_2)$ is the required planar d.c.s. graph.

If $d(w_1) = 2$, let $w_2 \in N(w_1) \setminus \{v_1\}$, then $T*(u,u')*(w_2,v_1)*(v_2,v_3)$ is the required graph.

Sub case 4. $N[u] \cup N[u'] \cup (\bigcup_{i=1}^{m} N(v_i)) \cup (\bigcup_{i=1}^{m} N_2(v_i)) \neq V(T)$.

Then,

$$N[u] \cup N[u'] \cup (\bigcup_{i=1}^{m} N(v_i)) \cup (\bigcup_{i=1}^{m} N_2(v_i)) \cup (\bigcup_{i=1}^{m} N_3(v_i)) = V(T).$$

Note that, $N(v_i) \setminus \{u,u'\} \neq \phi$, for only one value of i, there is only one vertex w_1 in it and there are two vertices w_2 and w_3 such that w_1w_2 and $w_2w_3 \in E(T)$.

Then, $T*(u,u')*(v_1,w_2)$ is the required graph. □

Remark 3.2. In theorem 3.2 the upper bound for the order of T is sharp. Consider the tree T of order 10,

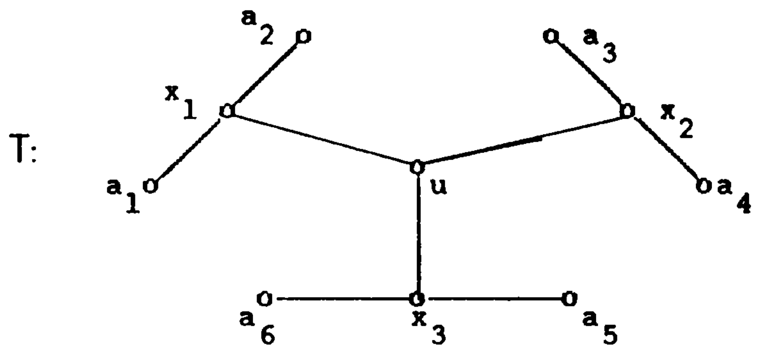


Fig. 3.1

A non- solvable tree of order 10 and diameter 4.

Here, $d(x_i) > 2$ in T and hence also in G. So, by theorem 2.3, for each x_i there is a unique partner x_i' in V(T). Now, $x_i' \neq a_j$ or u because $G^*(x_i, a_j)$ and $G^*(x_i, u)$ will contain a triangle for i=1,2,3 and j=1,2,...,6. Hence x_i' can only be x_j for some $j \neq i$. Then there will be one x_i for which there is no partner.

Theorem 3.3. The following classes of trees are solvable.

(a) Trees of diameter three.

(b) Trees of diameter four whose central vertex has even degree.

(c) Trees of diameter five.

Proof:(a) Since T is of diameter three, $T \simeq S_{m,n}$ (Definition 1.2.), for $m,n > 0$.

Let c_1 and c_2 be the central vertices and let $N(c_1) = \{a_1, a_2, \ldots, a_m\}$ and $N(c_2) = \{b_1, b_2, \ldots, b_n\}$. Then $T^*(b_1, c_1)^*(a_1, c_2)$ is a planar d.c.s. graph containing T as a spanning tree.

(b) Let $\text{diam}(T) = 4$ and the central vertex c has even degree.

Let $N(c) = \{a_1, a_2, \ldots, a_n\}$ and $c' \in N_2(c)$. Then $T*(c,c')*(a_1,a_2)*\ldots*(a_{n-1},a_n)$ is the required graph.

(c) Let diam(T) = 5. Then T will be as in Fig. 3.2.

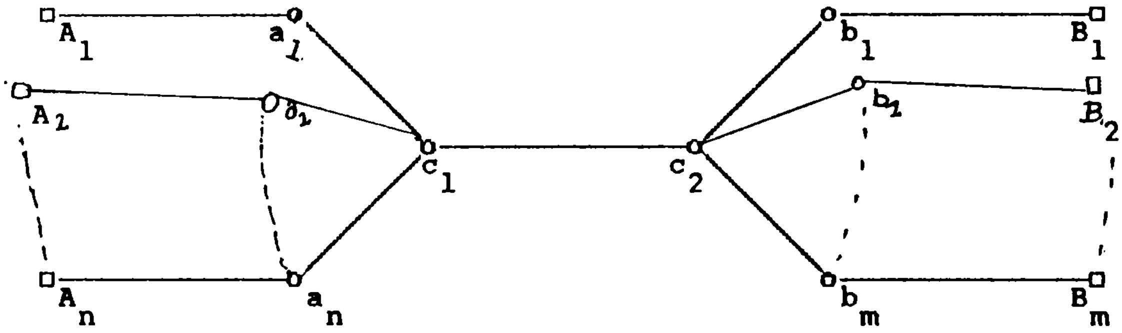


Fig. 3.2

Clearly A_i and B_j are independent sets and are nonemtpy for at least one value each of i and j, $i = 1,2,\ldots,n$ and $j=1,2,\ldots,m$.

Case 1. Both m and n are even.

Then $\{c_1, a_1, \ldots, a_n\} \cup (\bigcup_{i=1}^{n} A_i)$ and $\{c_2, b_1, \ldots, b_n\} \cup (\bigcup_{j=1}^{m} B_j)$ induce trees say T_1 and T_2 respectively. Note that $\text{diam}(T_i) < 5$ for $i = 1,2$.Choose a c_1' from some A_i and a c_2' from some B_j. Then

$$G_1 \simeq T_1*(c_1,c_1')*(a_1,a_2)*\ldots*(a_{n-1},a_n), \text{ and}$$

$$G_2 \simeq T_2*(c_2,c_2')*(b_1,b_2)*\ldots*(b_{m-1},b_m) \text{ are planar d.c.s}$$

graphs containing T_1 and T_2 respectively. Now, embed G_1 and G_2 so that c_1, c_2, c_1', c_2' lie in the exterior face. Then, join c_1 and c_1' to c_2 and c_2'. Note that the resulting graph G is planar and for each vertex of degree at least 3 there is a partner u'. Hence G is d.c.s.

Case 2. m is even and n is odd.

Obviously, $d(c_1) = n+1$, which is even and

$$\{c_1, c_2, a_1, \ldots, a_n, b_1, \ldots, b_m\} \cup \left(\bigcup_{i=1}^{n} A_i\right)$$

form a tree, say T' of diameter four and $C(T') = \{c_1\}$. Choose a vertex a_i^1 from some A_i. Now,

$$T*(a_i^1, c_i)*(a_1, c_2)*(a_2, a_3)*\ldots*(a_{n-1}, a_n)*(b_1, b_2)*\ldots(b_{m-1}, b_m)$$

is a planar d.c.s graph containing T.

Case 3. Both m and n are odd.

Here T is a spanning tree of the planar d.c.s graph,

$$T*(c_1, b_1)*(c_2, a_1)*(a_2, a_3)*\ldots*(a_{n-1}, a_n) *(b_2, b_3)*\ldots*(b_{m-1}, b_m).$$

□

Remark 3.3. (i) In (b), if the central vertex has odd degree, the result need not be true, as seen in Fig 3.1.
(ii) There exists non solvable trees of diameter six. Also, if V_1 and V_2 are the bipartition of $V(T)$ such that $|V_1|$ is odd and each vertex of V_1 is of degree greater than 2, then T is not solvable.

We ask a problem similar to the problem discussed earlier.

PROBLEM: Find the smallest m.c.s. graph containing a given tree T, $|T| \geq 4$.

If $T = K_{1,n}; n \geq 3$, $K_{2,n}$ is such a graph and its size is 2n.

Theorem 3.4. The size of the smallest m-convex simple graph containing a tree $T \neq (K_{1,n})$ satisfies,
$p-1+(m/2) \leq q \leq p+m-2$, where $|V(T)| = p$ and m is the number of pendent vertices.

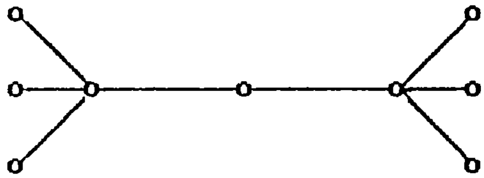
Proof. Let u_1 be a pendent vertex of T and v be the vertex adjacent to u_1. Let $u_2, u_3, \ldots, u_k$ be the other pendent vertices adjacent to v. Let $v_1, v_2, \ldots v_\ell$ be the pendent vertices other than u_is. Add edges to T such that $\{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_\ell\}$ induce a tree in which $\{u_1, u_2, \ldots, u_k\}$ and $\{v_1, v_2, \ldots, v_\ell\}$ is a bipartition. This is possible by taking a spanning tree of $K_{k,\ell}$. The resulting graph is triangle-free and neither a vertex nor an edge can separate G. So, by theorem 2.12, G is an m.c.s. graph and size of G is $p-1+\ell+k-1 = p+m-2$ where m is the number of pendent vertices of T. So size q of the smallest m.c.s. graph is atmost $p+m-2$.

Now, note that m.c.s. graphs are triangle free blocks and hence all vertices are of degree at least two. Therefore, to make T a block, the degree of each pendent vertex is to be increased by at least one. So, at least $\lceil \frac{m}{2} \rceil$ edges are to be added and hence

$q \geq p-1+\lceil \frac{m}{2} \rceil > p-1+\frac{m}{2}$. □

The following example illustrate that there are trees attaining both the bounds. Consider the tree T_1 in Fig 3.4.

T_1:



Here p=9, m=6

Fig. 3.4

The graph G in Fig 3.5 is an m.c.s. graph of size

$q = 11 = p-1+\frac{m}{2}$, containing T.

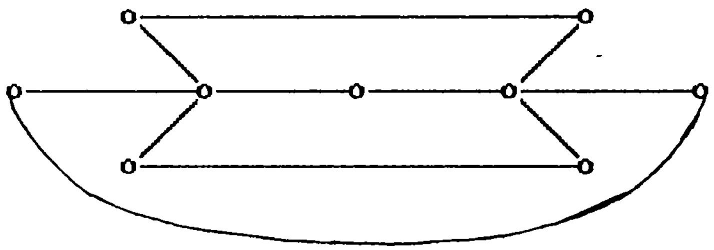


Fig. 3.5

Consider the tree T_2 of Fig 3.6. In T_2, $\{x_1, x_2\}$ is a clique such that $T_2 \setminus \{x_1, x_2\}$ is totally disconnected. So, to get an m.c.s. graph at least five edges are to be added. So, $q = 13 = p+m-2$.

T_2:

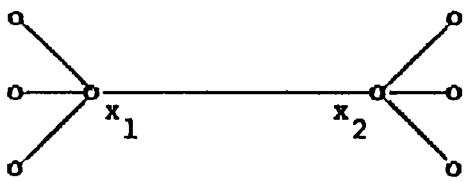


Fig. 3.6

3.2. CENTER OF DISTANCE CONVEX SIMPLE GRAPH

In this section, we determine the center of d.c.s graphs. Properties of centers of various type of graphs have been discussed by Chang [23],. Chepoi [30], Nieminen [55], Prabir Das [63] and Proskurowski [64].

Theorem 3.5. If G is a planar d.c.s. graph of order at least four, then,

(1) G is self centered if diam(G) = 2.

(2) diam(G) = 2rad(G) or 2rad(G)-1, if diam(G) > 2, C(G) is isomorphic to $\bar{K}_2$ or C_4 according as diam(G) = 2rad(G) or 2rad(G)-1.)-1.

Proof: (1) Let G be a planar d.c.s. graph with diam(G) = 2. It follows from C1 of Theorem 2.1 that rad(G) > 1. So rad(G) = diam(G) and hence C(G) = V(G).

(2) Suppose diam(G) > 2.

Case I: diam(G) = 2k, k > 1

Let $u,v \in V(G)$ be such that d(u,v) = 2k and $u = a_0-a_1-a_2-\ldots-a_{2k} = v$ be a shortest u-v path. Then by

C1 and theorem 2.3, we get another u-v path $u = a_0 - a'_1 - \ldots a'_{2k-1} - a_{2k} = v$ where a'_i and a_i are partners for $i = 1,2,\ldots, 2k-1$. Note that $e(a_k) \geq rad(G) \geq k$. Let w be a vertex such that $d(a_k,w) = e(a_k)$. If w = u or v then $e(a_k) = k$, which implies $e(a_k) = rad(G)$. Note that $e(a_k) = e(a'_k)$.

If $w \neq u$, v suppose that I(v,w) contains a_k or a'_k (note that if I(v,w) contains a_k it will contain a'_k also). Then $d(v,w) = d(v,a_k) + d(a_k,w) = k + e(a_k) \leq 2k$. This imply that $e(a_k) = k$. Similarly for I(u,w). Hence in these two cases $e(a_k) = e(a'_k) = rad(G)$. If neither I(u,w) nor I(v,w) contains these vertices, consider a shortest u-w path and shortest v-w path. Then using C1 and theorem 2.3 it can be observed that there is a subgraph homeomorphic to $K_{3,3}$. Hence $e(a_k) = e(a'_k) = k = rad(G)$, that is $\{a_k, a'_k\}$ is contained in C(G).

Now, we prove that these are only central vertices. If there is some other vertex, say c, in C(G) then $d(c,u) \leq rad(G)$ and $d(c,v) \leq rad(G)$. But, since

$d(u,v) = 2rad(G)$, $d(c,u) = d(c,v) = rad(G)$. Thus we get a u-v path which is different from the two paths mentioned earlier. Now it can be observed that a subgraph homeomorphic to $K_{3,3}$ is contained in G.
Hence $C(G) = \{a_k, a'_k\}$.

Case II: diam (G) = 2k+1 for some k > 0.
As in the case I, if u and v are such that $d(u,v) = 2k+1$ and $u = a_0 - a_1 - \ldots - a_{2k} - a_{2k+1} = v$ and $u = a_0 - a'_1 - \ldots - a'_{2k} - a_{2k+1} = v$ are the two distinct paths then $rad(G) = k+1$ and $C(G) = \{a_k, a'_k, a_{k+1}, a'_{k+1}\}$ which will induce subgraph isomorphic to C_4. □

Remark 3.4. Planar d.c.s. graphs resembles trees in its radius-diameter relation and center-diameter relation. For a tree $T, C(T) \simeq K_1$ or K_2 according as diam(T) is 2rad(T) or 2rad(T)-1. For a planar d.c.s. graph G also, $C(G)$ is $\bar{K}_2 \simeq D_2(K_1)$ or $C_4 = D_2(K_2)$ according as diam(G) is 2rad(G) or 2rad(G)-1. □

3.3. CONVEXITY PROPERTIES OF PRODUCT OF GRAPHS

In this section, it is proved that the property of being distance convex simple is not productive. However, m.c.s graphs behave nicely.

Theorem 3.6. Let $G_1(p_1,q_1)$ and $G_2(p_2,q_2)$ be two distance convex simple graphs. Then $G_1 x\ G_2$ has exactly $p_1+p_2+q_1+q_2+q_1q_2$ non trivial d-convex subsets.

Proof: Let $G_1(p_1,q_1)$ and $G_2(p_2,q_2)$ be two d.c.s graphs. Let A be a convex subset of $V(G_1\ x\ G_2)$.

Claim: $A = A_1 x\ A_2$ where $A_1 = \{u:(u,v) \in A\}$ and $A_2 = \{v:(u,v) \in A.\}$. To prove that $A_1 x\ A_2 \subset A$.

Let $u \in A_1$, $v \in A_2$. Then there is a $u_0 \in A_1$ and $v_0 \in A_2$ such that $(u_0,v) \in A$ and $(u,v_0) \in A$.

Let $u_0-u_1-...-u_\ell-u$ be a shortest u_0-u path in G_1 and $v_0-v_1,...,v_k-v$ be a shortest v_0-v path in G_2. Then $(u_0,v)-(u_1,v)-(u_2,v)...(u_\ell,v),(u,v)-(u,v_k)...(u,v_1)-(u,v_0)$ is a $(u_0,v)-(u,v_0)$ path. Hence $(u,v) \in A$. Therefore, $A = A_1xA_2$.

Now, even if A_i is a trivial convex set in G_i for $i = 1,2$, $A_1 x A_2$ need not be trivial. Thus the non trivial convex subsets are $\{x\} \times V(G_2)$, where x is in $V(G_1)$, $V(G_1) \times \{y\}$ where y is in $V(G_2)$, $\{x_1, X_2\} \times V(G_2)$ where $x_1x_2 \in E(G_1)$, $V(G_1) \times \{y_1,y_2\}$ where $y_1y_2 \in E(G_2)$ and $\{x_1,x_2\} \times \{y_1,y_2\}$ where $x_1x_2 \in E(G_1)$ and $y_1y_2 \in E(G_2)$. Number of such convex sets are p_1,p_2,q_1,q_2 and q_1q_2 respectively. Hence $G_1 x G_2$ is k-convex where $k = p_1+p_2+q_1+q_2+q_1q_2$. □

Theorem 3.7 Let G_1 and G_2 be connected, triangle free graphs. $G_i \not\simeq K_1$ or K_2 for $i = 1,2$. Then $G_1 x G_2$ is m-convex simple.

Proof: Let $G_i \simeq K_1,K_2$ be connected, triangle free graphs. Note that, if $u_1-u_2-...-u_n$ and $v_1-v_2-...-v_m$ are chordless paths in G_1 and G_2 respectively, then

$$(u_1,v_1)-(u_1,v_2)-...-(u_1,V_m)-(u_2,v_m)-...-(u_n,v_m)$$

is a chordless $(u_1,v_1)-(u_n,v_m)$ path in $G_1 x G_2$.

To prove that $G_1 x G_2$ is m.c.s, it is enough to prove that any (u,v) in $V(G_1 x\ G_2)$ is in the m-convex hull of any two nonadjacent vertices (u_1,v_1) and (u_2,v_2). Now, it can be easily seen that (u_1,v_2) and (u_2,v_1) lie on a chordless (u_1,v_1)-(u_2,v_2) path.

Assume without loss of generality that (u,v) is adjacent to (u_1,v_1).

Let u be adjacent to u_1 and $v = v_1$.

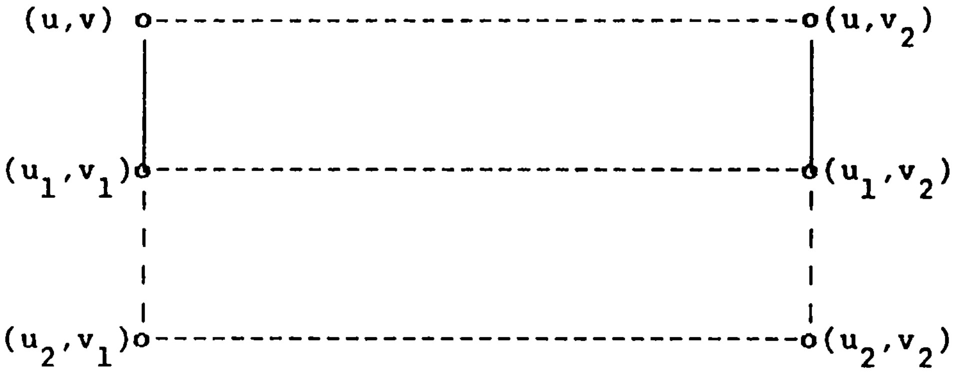


Fig. 3.7.

If u is on any chordless u_1-u_2 path say u_1-u-$a_1 \ldots a_n = u_2$ then (u_1,v_1)-(u,v_1)-$(a_1,v_1) \ldots (u_2,v_1) \ldots (u_2,v_2)$ is a chordless (u_1,v_1)-(u_2,v_2) path containing $(u,v_1) = (u,v)$.

So assume that u is not on any chordless path connecting u_1 and u_2 (See Fig. 3.7).

Case 1. $v_1 \neq v_2$ and v_1 is not adjacent to v_2.

Then $(u_1,v_1)-(u,v_1)\ldots(u,v_2)(u_1v_2)\ldots(u_2,v_2)$ is a chordless $(u_1,v_1)-(u_2,v_2)$ path containing $(u,v_1) = (u,v)$.

Case 2. v_1 is adjacent to v_2.

Then there is vertex v_3 in G_2 different from v_1 and v_2 because $G_2 \neq K_2$. Assume v_3 to be adjacent to v_1. Then,

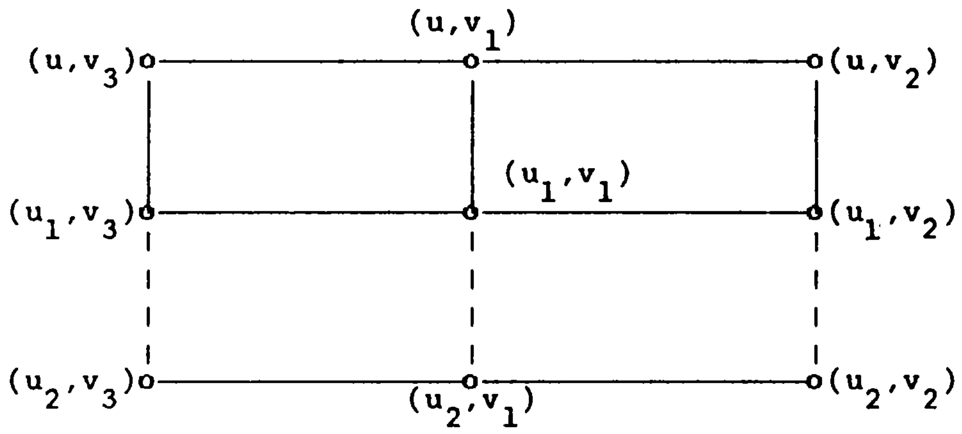


Fig 3.8

$(u_2,v_1)-(u_2,v_3)\ldots(u_1,v_3)-(u,v_3)-(u,v_1)-(u,v_2)-(u_1,v_2)$ is a chordless $(u_2,v_1)-(u_1,v_2)$ path containng (u,v) (See Fig 3.8).

That is

$(u,v) \in Co(\{(u_2,v_1),(u_1,v_2)\}) \subset Co(\{(u_1,v_1),(u_2,v_2)\})$.

If V_3 is adjacent to v_2, then

$(u_1,v_1)-(u,v_1)-(u,v_2)-(u,v_3)-(u_1,v_3)\ldots(u_2,v_3)-(u_2,v_2)$

is a chordless $(u_1,v_1)-(u_2,v_2)$ path containing $(u,v_1) = (u,v)$.

Case 3. $v_1 = v_2$. Then $(u_2,v_2) = (u_2,v_1)$ and u_1 is not adjacent to u_2. Since $G_2 \not\cong K_1,K_2$ there are two vertices v_3 and v_4 in G such that $\langle\{v_1,v_3,v_4\}\rangle$ is connected.

Let v_3 be adjacent to v_1 and v_4. Then

$$(u_1,v_1)-(u,v_1)-(u,v_3)-(u,v_4)-(u_1,v_4)\ldots(u_2,v_4)-(u_2,v_3)-(u_2,v_1)$$

is a chordless $(u_1,v_1)-(u_2,v_1)$ path containing $(u,v_1) = (u,v)$.

Now, let v_1 be adjacent to v_3 and v_4. (See Figure 3.9.)

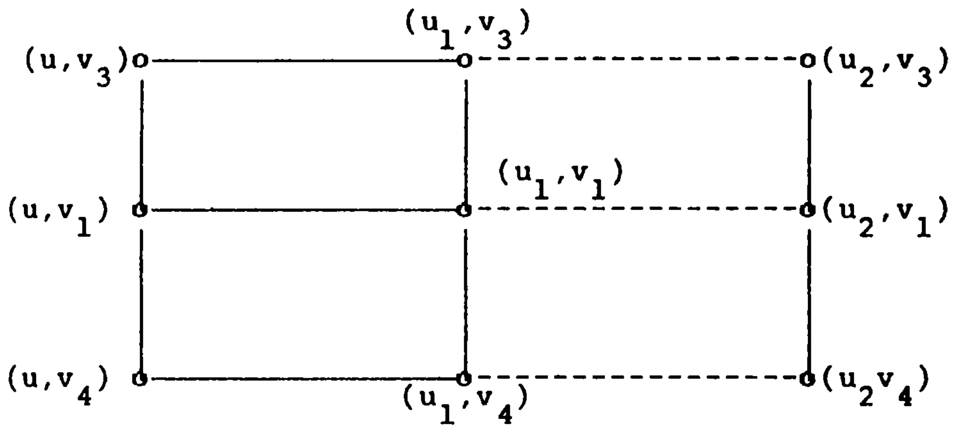


Fig 3.9

From Fig.3.9. it is clear that (u_1,v_3) and (u_1,v_4) lies on some chordless path connecting (u_1,v_1) and (u_2,v_1) because u_1 is not adjacent to u_2. Then

$$(u_1,v_3)-(u,v_3)-(u,v)-(u,v_4)-(u_1,v_4)$$

is a chordless $(u_1,v_3)-(u_1,v_4)$ path.

Hence $(u,v) \in Co(\{(u_1,v_3),(u_1,v_4)\}) \subset Co(\{(u_1,v_1),(u_2,v_1)\})$.

Hence, in any case $(u,v) \in Co(\{(u_1,v_1)(u_2,v_2)\})$ and so $G_1 x\ G_2$ is m.c.s. □

Theorem 3.8. Let G_i for $i = 1,2$ be connected triangle free graphs, where G_1 is 2-connected, $G_2 \neq K_1$, then $G_1 x\ G_2$ is m.c.s

As in the proof of theorem 2.16, let (u_1,v_1) and (u_2,v_2) be two non adjacent vertices of $G_1 \times G_2$.

Let $(u,v) \in G_1 \times G_2$. Assume (u,v) to be adjacent to (u_1,v_1). Let $u = v_1$ and u_1 is adjacent to u.

If v_1 is not adjacent to v_2, then as in the above theorem $(u,v) \in Co(\{(u_1,v_1)(u_2,v_2)\})$.

Case I. v_1 adjacent to v_2.

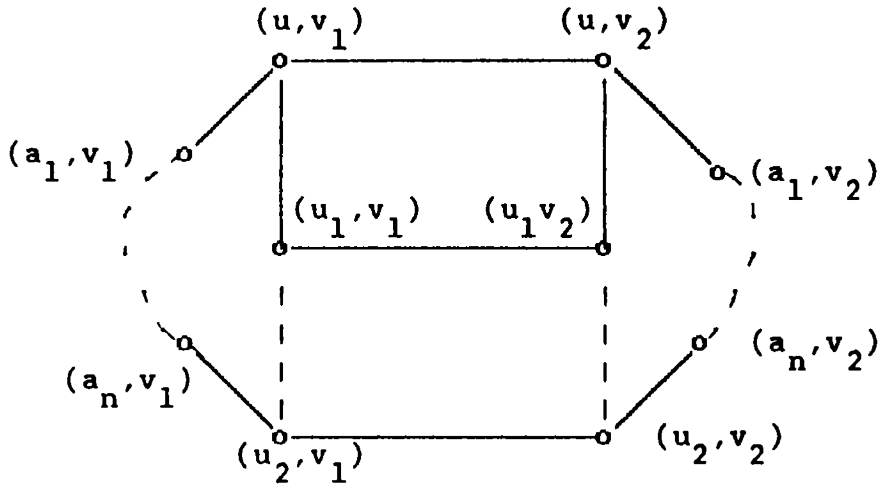


Fig. 3.10.

Since G_1 is 2-connected, there is a path connecting u and u_2 distinct from the path u-u_1-...-u_2.

Let it be $u\text{-}a_1\text{-}a_2 \ldots a_n = u_2$. Then

$$(u_1,v_1)-(u,v_1)-(u,v_2)-(a_1,v_2)\ldots(a_n,v_2)-(u_2,v_2)$$

is a chordless path.

Case II $v_1 = v_2$. Then $(u_2,v_2) = (u_2,v_1)$ and u_1 is not adjacent to u_2. Since G_2 K, there is a vertex v_2 adjacent to v_1. Then

$(u_1,v_1)-(u,v_1)-(u,v_2)-(a_1,v_2)\ldots(a_n,v_2)-(u,v_2)-(u_2,v_1)$ is a chordless $(u_1,v_1)-(u_2,v_1)$ path. □

Now let v be adjacent to v_1 and $u = u_1$.

Then if $v = v_2$, then, $(u_1,v_1)-(u_1,v_2)-(u_2,v_2)$ is a chordless path containing $(u_1,v_2) = (u\ ,v)$.

If $v \neq v_2$, then v, v_1 and v_2 are distinct vertices of G_2 and hence $G_2 \neq K_2$ or K_1. Then the theorem holds as in Theorem 3.7. Now if $v_1 = v_2$ and v is adjacent to v_1, then u_1 is not adjacent to u_2. (See Fig. 3.11).

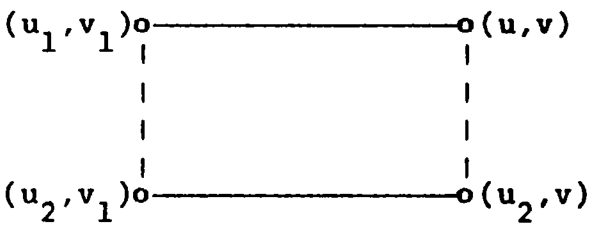


Fig. 3.11.

In this case $(u_1,v_1)-(u_1,v)\dots(u_2,v)-(u_2,v_1)$ is a $(u_1,v_1)-(u_2,v_1)$ path containing $(u_1,v) = (u,v)$. □

Remark 3.5. The condition that G_1 is 2-connected is necessary. For, taking G_2 be K_2 and G_1 to be a graph having a cut point c, then $G_1 \times G_2$, the copy of K_2 corresponding to c will be a clique separator for $G_1 \times G_2$ and hence $G_1 \times G_2$ will not be m.c.s.

Theorem 3.9. If G_1 is an m.c.s graph and G_2 is any connected triangle free graph, then $G_1 \times G_2$ is m.c.s

Proof: If $G_2 \simeq K_1$, then $G_1 \times G_2 \simeq G_1$ and hence is m.c.s.

If $G_2 \cong K_1$, then using theorem 2.17, $G_1 \times G_2$ is m.c.s. □

CHAPTER IV

IV

CONVEXITY FOR THE EDGE SET OF A GRAPH

In this chapter we introduce a notion of convexity for the edge set of a connected graph. This definition is motivated by the concept of edge lattice of a graph discussed in [4]. Though there is a vast literature concerning different aspects of convexity for the vertex set of a graph, little work is done on similar lines for the edge set.

We first observe that this convexity on E(G) in addition satisfies the exchange law and hence is a matroid (Definition 1,15). Also, its arity is not in general two and hence the convexity is not induced by an interval. It is known that the Caratheodory number of a convex structure is an upper bound for its arity.

In this chapter, we have evaluated the convex invariants of this convex structure. The Pasch Peano properties (Definition 1.20) are also discussed and also a forbidden subgraph characterization. Some results of this chapter are in [61].

4.1 CYCLIC CONVEXITY

Definition 4.1 Let G = (V,E) be a graph with $E \neq \phi$. $S \subseteq E$ is cyclically convex if it contains all edges comprising a cycle whenever it contains all but one edge of this cycle.

Equivalently if S is convex and if $a_1a_2, a_2a_3, \ldots, a_{n-1}a_n \in S$ and $a_na_1 \in E$ then a_na_1 also will be in S where a_ia_{i+1} is an edge of G for i = 1,2,...,n-1.

If $\mathscr{C}$ denotes the collection of all such convex subsets of E, then $(G,\mathscr{C})$ is convexity space. For convenience, the cyclic convexity on E will also be referred to as convexity.

Example : (a) For a tree T, every subset of E(T) is trivially convex.

(b) In the graph G of Fig 4.1,

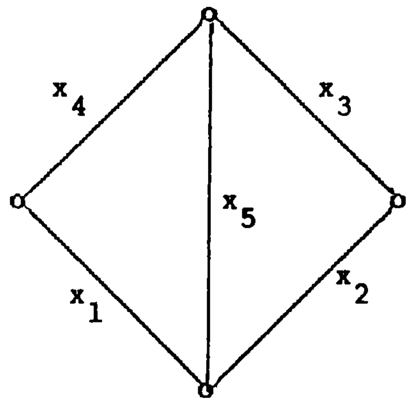


Fig 4.1

$\{x_1\}, \{x_1, x_2\}$ are convex but $\{x_1, x_4\}$ is not.

Now, we shall consider a generalization of the notion of geodetic iteration number (Definition 1.11) of an interval convexity space to a convexity space of arity greater than two.

Definition 4.2. Let X be a convexity space of arity n (n>2) and $S \subset X$. The closure of S, denoted by (S) is defined as $(S) = \bigcup\{Co(F) : F \subset X, |F| \leq n\}$. S^m is recursively defined as, $S^1 = (S)$, $S^m = (S^{m-1})$. The smallest positive integer m such that $S^m = S^{m+1}$ is called the iteration number of S. The iteration number of X is defined to be max{iteration number of S : S ⊆ X} if it exists.

Lemma 4.1. For the convexity space (G,$\mathscr{C}$), the iteration number is equal to 1.

Proof: We shall prove that for $S \subset E$, $S^2 = S^1$. It is obvious that $S^1 \subset S^2$. Let $e \in S^2 = (S^1)$. Then, there is a sequence of edges say $e_1 = a_1a_2$, $e_2 = a_2a_3, \ldots, e_{n-1} = a_{n-1}\,a_n$ in S^1 such that $\{e_1, e_2, \ldots, e_{n-1}, e\}$ forms a cycle in G. Then, for each $i = 1,2,\ldots,n-1$, we get,

$$S_i = \{e_{i1} = a_i a_i^1,\ e_{i2} = a_i^1\ a_i^2, \ldots, e_{ik_i} = a_i^{k_i}\ a_{i+1}\} \subset S$$

such that $\{e_i, e_{i1}, \ldots, e_{ik_i}\}$ comprise a cycle in G. Now, observe that $\bigcup S_i$ is a sequence in S_i which contains a subsequence $e^1, e^2, \ldots, e^m$ forming a path joining a_1 and a_n. Hence, $\{e, e^i, \ldots, e^m\}$ forms a cycle in G and so $e \in S^1$. Thus, $S^2 = S^1$. □

Theorem 4.2 The arity of (G,$\mathscr{C}$) is 1 if G is a tree and is one less than the size of the largest minimal cycle in G, otherwise.

Proof: If G is a tree, then the lemma is trivially true. So assume G to be a graph having a cycle. Let k be the size of the largest minimal cycle. Let $S \subset E(G)$ is such that $Co(F) \subset S$ for $F \subset S, |F| \leq k-1$. Let $e \in Co(S)$. Then by lemma 4.1, there is a sequence $\{e_1, e_2, \ldots, e_t\}$ of edges in S such that $\{e, e_1, \ldots, e_t\}$ comprise a cycle. If $\{e_1, e_2, \ldots, e_t\}$ comprise a minimal path, then, $\{e, e_1, \ldots, e_t\}$ comprise a minimal cycle and hence $t \leq k-1$. Hence,

$e \in Co(\{e_1, e_2, \ldots, e_t\}) \subset S$. If $e_1, \ldots, e_t$ comprise a path having a chord, assume that, $Co(F) \subset S$ for $|F| < t$. Let e_0 be a chord of this path such that there is a sequence $e_{i_1}, e_{i_2}, \ldots, e_{i_j}$, $i_1, \ldots, j_i \in \{1, 2, \ldots, t\}$ and $\{e_0, e_{i_1}, \ldots, e_{i_j}\}$ comprise a minimal cycle.

Then $e_j \leq k-1$ and $e_0 \in Co(e_{i_1}, \ldots, e_{i_j}\}) \subset S$.

Now, $\{e, e_0, e_1, \ldots, e_t\} \setminus \{e_{i_1}, \ldots, e_{i_j}\}$ comprises a cycle of length less than t+1 and $e \in Co(\{e_0, e_1, \ldots, e_t\}) \subset S$ by induction hypothesis. Hence, arity of $(G, \mathfrak{F}) \leq k-1$.

Now, if C_k is some largest minimal cycle in G, then let $S = E(C_k) \setminus x$, where x is an edge in the cycle. Then S is with the property that $Co(F) \subset S$ for each subset of cardinality at most k-2, but S is not convex. Hence, the arity of $(G,\mathscr{C})$ is one less than the length of some largest minimal cycle in G. Hence arity $A(G,\mathscr{C}) = k-1$. □

We shall now consider the concept of rank of a matroid. For a convex structure X, a nonempty subset $F \subseteq X$ is convexly independent provided $x \notin Co(F \setminus \{x\})$ for each $x \in F$. Further, if X is a matroid (Definition 1.15) there exists a maximal independent subset of X and such a set is called a basis of the matroid. The cardinality of the basis is called the rank.

Theorem 4.3[12]. In a matroid X the hull of a basis equals X and all bases of X have the same cardinality.

Now we prove the following,

Theorem 4.4. If G is a connected graph, $(G,\mathscr{C})$ is a matroid of rank p-1, where $p = |V(G)|$.

Proof: $(G,\mathscr{C})$ is a matroid follows from the fact that if

$\{p,q,x_1,\ldots,x_n\}$ comprise a cycle, $p \in Co(\{q,x_1,\ldots,x_n\})$ and $q \in Co(\{p,x_1,\ldots,x_n\})$.

Now, we have to prove that rank (G) = p-1. Let T be a spanning tree of G and let F = E(T). Then each pair of vertices in G is connected by a path in T. Now if $e \in E(G)$ such that $e \notin E(T)$ then there is a sequence $e_1,\ldots,e_n$ of edges in F connecting the end vertices v_1 and v_2. That is $\{e_1,e_2,\ldots,e_n\}$ comprise a cycle. Hence E(G) = Co(F). Hence, rank (G) $\leq$ p-1.

Now, let $F \subset E$, be such that $|F| < p-1$. Then there are two vertices v_1 and v_2 in G such that it is not connected by a path comprised by edges in F. If $e_1,\ldots,e_k$ are those edges in G which comprise a path joining v_1 and v_2 and if $e_1,\ldots,e_k \subset Co(F)$ then by lemma 4.1 there is a sequence of edges in F which comprise a v_1-v_2 path which is a contradiction □

Corollary: If G is disconnected, then $(G,\mathcal{C})$ is a matroid of rank p-k where k is the number of components of G. □

4.2 CONVEX INVARIANTS

The convex invariants (Definition 1.18) of (G,γ) will be denoted by $h(G)$, $c(G)$, $r(G)$ and $e(G)$ respectively.

Theorem 4.5. If G is a connected graph of order p, the Helly number of (G,γ) is p-1.

Proof: Let T be a spanning tree of G and $F = E(T)$. We shall prove that F is H-independent.

Let $e^1 \in Co(F \setminus \{e\})$ for every e in F. Then by the lemma 4.1, there is a sequence of edges, $e_1 = a_1a_2$, $e_2 = a_2a_3$, $e_{n-1} = a_{n-1}a_n$ in $F \setminus \{e\}$ such that $e^1 = a_1a_n$, for some e in F. Then $e_1 \in F$ and $e^1 \in Co(F \setminus \{e_1\})$ also. Again using lemma 4.1, we get another sequence $e_{1,1} = a_1a_{1,2}, e_{1,2} = a_{1,2}a_{1,3}, \ldots, e_{1,k} = a_{1,k}a_n$. This is contradiction, since $e_1, e_1, \ldots, e_{n-1}$ and $e_{1,1}, \ldots, e_{1,k}$ will then comprise two distinct paths joining a_1 and a_n of T. Hence, $\bigcap \{Co(F-\{a\})/a \in S\}$ is empty and so $h(G) \geq p-1$.

Now, we prove that any subset F of cardinality at least p is H-dependent. In this case F contains a subset,

$C = \{e_1, e_2, \ldots, e_k\}$, comprising a cycle in G and $e_i \in Co(F\setminus \{e\})$ for each e in F and i = 1,2,...,k. Hence, $\bigcap \{Co(F\setminus \{e\})/e \in F\}$ is not empty. Therefore, F is H-dependent and so $h(G)<p$. Thus, $h(G) = p-1$. □

Theorem 4.6. The Caratheodory number of $(G,\mathcal{X})$ is given by

$$c(G) = \begin{cases} 1 \text{ if } G \text{ is a tree} \\ Circ(G)-1, \text{ otherwise, where } Circ(G) \text{ is the circumference of } G. \end{cases}$$

Proof: If G is a tree, then every subset of E is convex. Hence, for each $F\subset E$ with cardinality at least two, we have, $Co(F) = F \subset \bigcup_{e \in F}(F-\{e\}) = \bigcup_{e \in F} Co(F\setminus \{e\})$. Hence, $c(G) = 1$.

Now, let C be a longest cycle in G of length k and $S= E(C) = \{a_1a_2, a_2a_3, \ldots, a_{k-1}a_k, a_ka_1\}$. Then $a_ia_{i+1} \in Co(S\setminus \{a_ia_{i+1}\})$ for each $i = 1,2,\ldots,k$.
Let $S_i = (S-\{a_ia_{i+1}\})$.

Claim: $a_ia_{i+1} \notin Co(S_i\setminus \{e_i\})$ for $e_i \in S_i$.
If $a_ia_{i+1} \in Co(S_i\setminus\{e_i\})$, by the lemma 4.1 we get a sequence $e_1, e_2, \ldots, e_{k_i}$ of edges in $S_i-\{e_i\}$ such that $\{a_ia_{i+1}, e_1, \ldots, e_{K_i}\}$ comprise a cycle in G. This is not

possible because $S-\{e_i\}$ consists of the edges of a path only. Hence, $Co(S_i) \subset \bigcup Co(S_i \setminus \{e_i\} : e_i \in S_i\}$ and so $c(G) \geq k-1$.

Now, let S be a subset of E of cardinality at least k. Let $e \in Co(S)$. If $e \in S$, $e \in S-\{e^1\} \subset Co(S \setminus \{e^1\})$, for some $e^1 \neq e$ in S. If $e \notin S$, there is a sequence $e_{1,1},\ldots,e_{1,\ell}$ in S such that $\{e,e_{1,1},\ldots,e_{1,\ell}\}$ comprise a cycle in G. Also, $S \neq \{e_{1,1},\ldots,e_{1,\ell}\}$ because of the maximality of C. Let $e^1 \in S - \{e_{1,1},\ldots,e_{1,\ell}\}$. Then $e \in Co(S \setminus \{e^1\})$ and so $Co(S) \subset \bigcup Co(S \setminus \{e^1\} / e \in S)$ and $c(G) \leq k-1$. Hence, $c(G) = k-1$. □

Theorem 4.7. If G is a connected graph of order p, the Radon number of $(G,\mathscr{E})$ is p-1.

Proof. Let T be a spanning tree and let F=E(T). Then if F can be partitioned into F_1 and F_2 such that $Co(F_1) \cap Co(F_2) \neq \phi$ and if $e \in Co(F_1) \cap Co(F_2)$, then there is a sequence of edges $e_{11},\ldots,e_{1\ell}$ in F_1 and $e_{21},\ldots,e_{2m}$ in F_2 such that $e,e_{11},\ldots,e_{1\ell}$ and $e,e_{21},\ldots,e_{2,m}$ comprise cycles. Then $e_{11},\ldots,e_{1\ell}$ and $e_{21},\ldots,e_{2m}$ are paths connecting the end vertices of e and hence $\{e_{11},\ldots,e_{1\ell}, e_{21},\ldots,e_{2m}\}$ contains a sequence comprising a cycle, which is not possible. So **F cannot have** a Radon partition. Hence, $r(G) \geq p-1$.

Now, let $F \subset E(G)$ be of cardinality greater than $p-1$. Then it contains a subsequence $\{e_1, \ldots, e_s\}$ comprising a cycle C. Then for $e \neq e_i$, $e_i \in Co(F \setminus \{e\})$ for $i = 1, \ldots, S$. Also $e_i \in Co(E(C) \setminus \{e_i\}) \subset Co(F \setminus \{e_i\})$. Now, let $F = F_1 \cup F_2$ be such that $E(C) \setminus \{e_i\} \subset F_1$ and $\{e_i\} \subset F_2$. Then $e_i \in Co(F_1) \cap Co(F_2)$. Hence, $r(G) \leq p-1$. Therefore, Radon number $r(G) = p-1$. □

Theorem 4.8. For a connected graph G, the exchange number is given by $e(G) = 2$ if G is a tree or a cycle

$= \max \{Circ(G-v): v \in V(G)\}$, otherwise.

Proof:

Case I: Let G be a tree. In this case, every subset F of E(G) is convex. If $|F| \leq 2$ then let $F = \{e_1, e_2\}$. Then $F \setminus \{e_1\} \subset F \setminus \{e_2\}$, hence F is E-independent. If $|F| \geq 3$, let $F = \{e_1, \ldots, e_n, p\}$, $n \geq 2$. Then,

$$Co(F \setminus \{p\}) = F \setminus \{p\} = \{e_1, \ldots , e_n\}$$

$$= \{e_1,\ldots,e_{n-1}\} \cup \{e_1,\ldots,e_{n-2},e_n\} \cup$$

$$\ldots \cup \{e_1,e_3,\ldots,e_n\} \cup \{e_2,\ldots,e_n\}$$

$$\subset \cup \{ F \setminus \{e_i\}) : i=1,..,n\}.$$

Hence, $Co(F\setminus\{p\} \subset \cup \{CoF \setminus \{e\}) : e \neq p, e \in F\}$.

Case II: Let G be a cycle. Then either F=E or F has no subsequence comprising a cycle.

If $F = E, Co(F\setminus \{e\}) = F$ for each e in F. If $F \neq E$, since F contains no sequence comprising a cycle, each proper subset of F is convex and so proof is as in the case of a tree. Hence for both the cases, the exchange number is 2.

Case III: G is a graph having a cycle `C' and a vertex v not in `C'.

Assume without loss of generality that `C' is the longest cycle with this property and let v be a vertex not in C. Let $C = a_1-a_2-a_3-\ldots a_n-a_1$, $a_i \in V$ for $i = 1,2,\ldots,n$. Let u be a vertex adjacent to v and let $S = \{a_1a_2,a_2a_3,\ldots,a_{n-1}a_n,uv\}$. Then it is clear that $a_na_1 \in Co(S)$.

Claim: $a_n a_1 \notin Co(S \setminus \{a_i a_{i+1}\})$ for any $i=1,2,\ldots,n$. If not, $(S \setminus \{a_i a_{i+1}\}) \cup \{a_n a_1\}$ contains a sequence comprising a cycle, which is not possible. Hence $a_n a_1 \notin Co(S \setminus \{a_i a_{i+1}\})$ for any i. Hence S is E-independent and the exchange number is at least the cardinality of S, which is equal to n.

Now, let S be a subset of cardinality atleast n+1, say $S = \{e_1, \ldots, e_m\}$, $m \geq n+1$.

Let $e \in Co(S \setminus \{e_i\})$ for some i.

To prove that $e \in Co(S \setminus \{e_j\})$ for some $j \neq i$.

Since $e \in Co(S \setminus \{e_i\})$ by lemma 4.1, we get a sequence $e_1', \ldots, e_k'$ in $S \setminus \{e_i\}$ Such that $e, e_1', \ldots, e_k'$, comprise a cycle.

If $S \setminus \{e_i\} = \{e_1', \ldots, e_k'\}$, then $S \cup \{e\} \setminus \{e_i\}$ comprise a cycle of length $m \geq n+1$ and it contradicts the maximality of C.

So, there is a subsequence of $S \setminus \{e_i\}$ say $f_1, f_2, \ldots, f_\ell$ such that $F = \{e, e_i, f_1, f_2, \ldots, f_\ell\}$ comprise a cycle. Let $f \in S \setminus F$, then $e \in Co(S \setminus \{f\})$. Hence S is E-dependent and so $e(G) < n+1$, Thus $e(G)=n$. □

These theorems are illustrated in Fig 4.2.

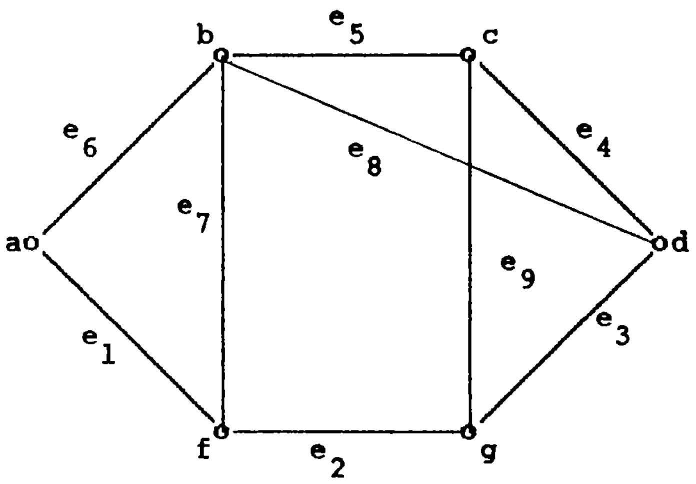


Fig. 4.2

In figure 4.2, Circ(G) = 6, max { Circ(G-v): v $\in$ G}=5.

Let F = $\{e_1,e_2,e_3,e_4,e_5\}$

Then Co(F) = E(G),

$Co(F \setminus \{e_1\}) = \{e_2,e_3,e_4,e_5,e_7,e_8,e_9\}$,

$Co(F \setminus \{e_2\}) = \{e_1,e_3,e_4,e_5,e_8,e_9\}$,

$Co(F \setminus \{e_3\}) = \{e_1,e_2,e_4,e_5,e_8\}$,

$Co(F \setminus \{e_4\}) = \{e_1,e_2,e_3,e_5\}$ and

$Co(F \setminus \{e_5\}) = \{e_1,e_2,e_3,e_4,e_9\}$.

Also $\bigcap$ { co F $\setminus$ $\{e_i\}$ | i=1,...,5} is empty.

So, F is an H-independent set. Actually, it is a maximal H-independent set and hence,

$h((G,\gamma)) = 5 = 6-1 = p-1$.

F is R-independent, because for any partition F_1 and F_2 of F, $Co(F_1) \cap Co(F_2) = \phi$. Hence F is an R-independent set and it is maximal. So r $((G,\mathscr{C})) = 5 = p-1$.

F is C-independent because $e_6 \in Co(F)$ and $e_6 \notin Co(F \setminus \{e_i\})$ for any i=1,2,3,4,5. Also F is maximal. Hence $C((G,\mathscr{C}))=5$.

F is E-independent because $e_7 \in Co(F \setminus \{e_1\})$ and $e_7 \notin Co(F \setminus \{e_i\})$ for i=2,3,4,5. Here also F is maximal. Hence $C((G,\mathscr{C})) = 5$.

Note 4.1. (a) In this example, we have h = c = r = e = 5.

(b) If the graph G is Hamiltonian, then h = c = r.

4.3 PASCH-PEANO PROPERTIES

In this section we shall consider the Pasch Peano properties (Definition 1.20). It is possible to express the Pasch Peano properties of a general convexity space by replacing the interval operator by the convex hull operator.

Here we discuss the Pasch Peano properties of $(G,\mathscr{C})$.

Definition 4.3. A convexity space X has Pasch property if, for $a,b,t,a^1,b^1 \in X$ such that $a^1 \in Co(\{a,t\})$, $b^1 \in Co(\{b,t\})$, then $Co(\{a,b^1\}) \cap Co(\{a^1,b\}) \neq \phi$ and X has Peano property if for a,b,d,u,v in X such that $u \in Co(\{a,b\})$, $v \in Co(\{d,u\})$, there is a `w' in $Co(\{b,d\})$ such that $v \in Co(\{a,w\})$.

We shall denote the edges of G by a,b,d,f and g.

Theorem 4.9. The convex structure $(G,\mathscr{C})$ is a Pasch space if and only if $K_4 - x$ is not an induced graph of G.

Proof: If $K_4 - X$ is a graph, let u,v,w,t be such that uv = a, vt = f, uw = d, vw = g and wt = b are in E and ut $\notin$ E (See Fig 4.3).

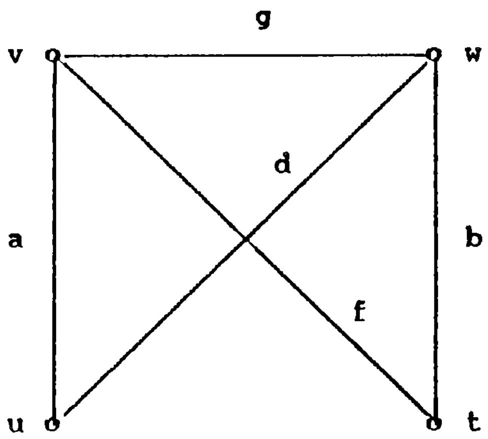


Fig 4.3

Then $d \in Co(\{a,g\})$, $f \in Co(\{b,g\})$ and

$Co(\{a,f\}) \cap Co(\{b,d\}) = \{a,f\} \cap \{b,d\} = \phi$.

Now assume that K_4 -x is not a subgraph. Let $a,b,g,d,f \in E$ be such that $d \in \in Co(\{a,g\})$, $f \in Co(\{a,g\})$.

If $d \neq a$, g: $f \neq b$, g then a,b,d,f and g will be as shown in the figure 2. Since K_4 - x is not an induced subgraph, $ut \in E$ and $ut \in Co(\{a,f\}) \cap Co(\{b,d\})$. If d=a (or if f=b), clearly $Co(\{b,d\}) \cap Co(\{a,f\}) \neq \phi$. Now if d=g, then $f \in Co(\{b,g\}) = Co(\{b,d\})$ and hence $Co(\{a,f\}) \cap Co(\{b,d\}) \neq \phi$. Hence the theorem. $(G,\mathscr{C})$ is Pasch if and only if K_4 - x is not an induced subgraph of G. □

Theorem 4.10. The convex structure $(G,\mathscr{C})$ is a Peano space if and only if G does not contain K_4 - x as a subgraph.

Proof: Let G contain K_4 - x as a subgraph. Then G contains a subgraph isomorphic to the graph in figure 4.4 .

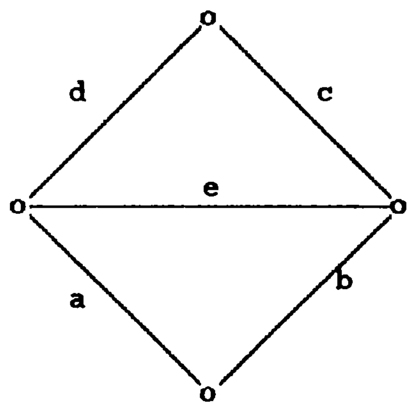


Fig. 4.4

In G, a,b,e,d,f are such that $e \in Co(\{a,b\})$, $f \in Co(\{e,d\})$. But it is not possible to find a `g' in $Co(\{b,d\}) = \{b,d\}$ such that $f \in Co(\{a,g\})$.

Now, let G be graph which contain no subgraph isomorphic to $K_4 - x$. Let a,b,d,e,f be as in the Peano condition.

Let $e \in Co(\{a,b\})$. If e=a or b, then the proof is trivial. So assume $e \neq$ a or b. If $Co(\{e,d\}) = \{e,d\}$, then f = e or d and belongs to $Co(\{a,b\})$ or $Co(\{a,d\})$. If $Co(\{e,d\}) \neq \{e,d\}$, there is an $f \neq e,d$ in $Co(\{e,d\})$. Then f is adjacent to e and d and so $\{a,b,d,e,f\}$ comprise a $K_4 - x$ which is not possible. Hence the theorem.

Note 4.2. It can be easily observed that for matroids Peano property implies the Pasch property. In particular, $(G,\mathscr{C})$ is a Peano space implies that it is a Pasch space. The converse is not true. $(K_4,\mathscr{C})$ is a pasch space which is not a Peano space, by theorem 4.9 and 4.10. □

CHAPTER V

V

SOME PROPERTIES OF H-CONVEXITY ON R^n.

In this chapter, we consider some problems posed by Van de Vel [12] on the H-convexity of R^n. This convexity on vectorspaces generated by linear functionals has been studied by Boltyanskii [19] and Bourguin [20] and has some interesting properties. In general, a symmetrically generated H-convexity need not be JHC or S_4. In the process of answering a Problem of Van de Vel ([12] and also on a recent private communication), as to whether each symmetric H-convexity is of arity two, we obtain a sufficient condition for a symmetrically generated H-convexity to be of arity two and give an example to illustrate that the arity could be infinite. A necessary and sufficient condition for the symmetrically generated H-convexity to be S_4, and an example of a PP space which is neither JHC nor S_4 and hence not of arity two are also obtained.

5.1 H-CONVEXITY

Let V be a vectorspace over a totally ordered field K and let $\mathcal{F}$ be a collection of linear functionals from $V \to K$. Then the family $\mathcal{S} = \{f^{-1}(-\infty,a] : a \in k, f \in \mathcal{F}\}$ generates a convexity $\mathcal{C}$ on V, coarser than the standard one. It is called an H-convexity. If $-f \in \mathcal{F}$ whenever $f \in \mathcal{F}$ then $\mathcal{C}$ is called a symmetric H-convexity. We usually omit one of f, -f and say that $\mathcal{F}$ symmetrically generate the convexity $\mathcal{C}$. The usual convexity in R^n is an H-convexity generated by the collection of all linear functionals from $R^n \to R$.

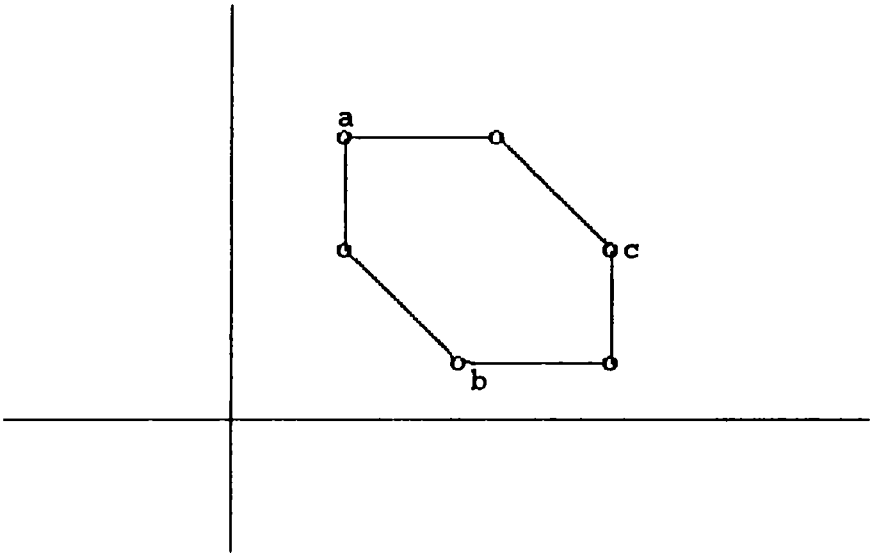


Fig.5.1

Figure 5.1 gives a typical polytope of R^2 generated by the co-ordinate projections and their sum in which {a,b,c} is a spanning set. Observe that the standard convex hull of {a,b,c} is the triangle with vertices a,b, and c and is contained in this polytope.

Let X and Y be two convexity spaces. A function $f: X \longrightarrow Y$ is a convexity preserving function (CP function) if for each convex set $C \subset Y$, $f^{-1}(C)$ is convex. A function f is convex to convex (C C function) if for each convex set $C \subset X$, $f(C)$ is convex.

If X is R^n with usual convexity and Y is R^n with an H-convexity then the identity mapping from $X \longrightarrow Y$ is a CP function.

A symmetrically generated H-convexity need not be JHC or S_4.

Example [12].

Let $\mathscr{C}$ be the H-convexity symmetrically generated by the co-ordinate projections f_i and their sum, defined on R^3. $\mathscr{F} = [f_1, f_2, f_3, f_4 = f_1+f_2+f_3]$.

Let $a = (o,3/4,1/4)$, $b = (1/2,1/4,0)$, $c =(0,0,1/2)$

$u = (1/2,1/4,1/4)$, $v = (1/2,0,1/2)$.

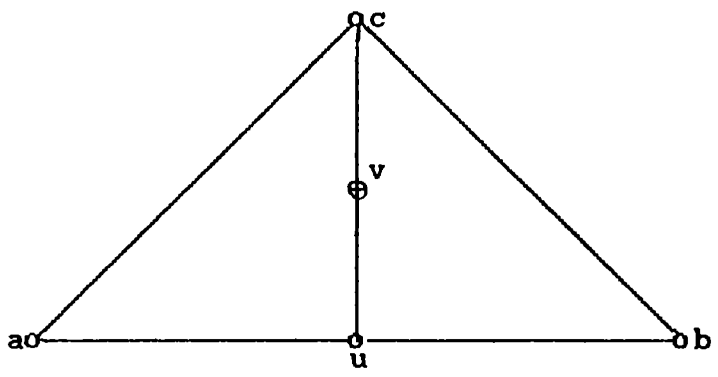


Fig. 5.2.

Then there does not exist a $v^1 \in Co\{a,c\}$ such that $v \in Co(\{b,v^1\})$. If such a v^1 exists, then

$0 \leq f_1(v^1) \leq 0$, hence $f_1(v^1) = 0$

$f_2(v^1) \leq 0 \leq 1/4$ hence $f_2(v^1) = 0$

$1/4 \leq f_3(v^1) \leq 1/2$, hence $f_3(v^1) \leq 1/2$

and therefore $f_4(v^1) \leq 1/2$.

But $f_4(v) = 1$, $f_4(b) = 3/4$. So there can not exists 'v^1' in $Co(\{a,c\})$ such that $v \in Co(\{b,v^1\})$.

That is, $\mathcal{E}$ does not satisfy the Peano property and hence is not JHC.

Example 5.2 Let $C_1 = \{(x,y,z): x \leq 0, y \leq 0\}$ and

$$C_2 = \{(x,y,z): z \leq -1, x+y+z \geq 0\}.$$

Then C_1 and C_2 are disjoint convex sets which cannot be separated by half spaces. That is, the H-convexity is not in general S_4. For another example, see [12].

From Van de Vel [12] we have the following theorems.

Theorem: 5.1 For a surjective C P function $f: X \longrightarrow Y$ the following are true.

(1) If $h(X) \geq h(Y)$ and $r(X) \geq r(Y)$

(2) If f is also C C then $C(X) = C(Y)$ and $c(X) \geq c(Y)$

Theorem 5.2. The following are equivalent for any convex structure

(1) If $h(X) \leq 3$ and if X is S_3 Then X is S_4.

(2) If $h(X) \leq 2$, and if X is S_2 then X is S_4.

Theorem 5.3. Let V be a finite dimensional vector space over the totally ordered field K, and let C be the

H-convexity on V generated symmetrically by a set $\mathcal{F}$ of linear functionals. If $\mathcal{F}$ is finite or if K = R, then, $h(V,\mathcal{E}) = md(\mathcal{F})$ where $md(\mathcal{F}) = \text{Sup}\{|\mathcal{F}_0| : \mathcal{F}_0 \subset \mathcal{F}$ and $\mathcal{F}_0$ is minimaly dependent$\}$ is the degree of minimal dependence of $\mathcal{F}$

We also have the following,

Theorem 5.4 [8]. Suppose H is a subset of R^n. Then H is a hyperplane if and only if there exists a non identically zero linear functionals f and a real constant δ such that $H = f^{-1}(\delta) = \{x \in R^n : f(x) = \delta\}$.

From these theorems, the following observations can be made.

1) The Helly number of any H-convexity on R^n is at most n+1

2) Any symmetric H-convexity on R^2 is S_4.

3) If $\mathcal{F}$ is a collection of linear functionals corresponding to a family of planes in R^3 whose intersection is a singleton and $|\mathcal{F}| \geq 4$, then the Helly number of the symmetrically generated H-convexity is 4.

5.2 A PROBLEM OF VAN DE VEL

In this section we consider a problem of

Van de Vel [12] and obtain some interesting results of the symmetrically generated H-convexity of R^3.

PROBLEM: Is each symmetric H-convexity of arity 2 ?

We studied the above problem and give an example of a symmetric H-convexity of infinite arity. We get a sufficient condition under which a family of linear functionals generates a symmetric H-convexity of arity 2.

Consider the vector space R^3 over R and let $\mathcal{F}$ be any collections of linear functionals over R^3. Let $\mathcal{C}$ be the H-convexity generated by $\mathcal{F}$. Then, for any $x_1, x_2 \in R^3$.

$$Co\{x_1, x_2\} = \bigcap \{f^{-1}[f(x_1), f(x_2)] : f \in \mathcal{F}\}.$$

By $[f(x_1), f(x_2)]$ we mean the set of all convex combinations of $f(x_1)$ and $f(x_2)$.

By theorem 5.4 each linear functional on R^3 corresponds to a plane in R^3. Now we prove,

Theorem 5.5. Let $\mathcal{F}$ be a family of linear functionals corresponding to a family of planes intersecting in a line, then the arity of the H-convexity symmetrically generated by $\mathcal{F}$ is two.

proof: Let $C \subset R^3$ have the property that $Co\{x_1, x_2\} \subset C$ whenever $x_1, x_2 \in C$. To prove that C is convex. Let $F \subset C$ where $|F| > 2$ and let $y \in Co(F)$. Let $f \in \mathcal{F}$. Then,

Claim: There are $x_1, x_2 \in F$ such that

$$f(x_1) \leq f(y) \leq f(x_2).$$

Otherwise, if $f(y) < f(x)$ for each $x \in F$ or $f(y) > f(x)$ for each x in F, then, $f^{-1}(-\infty, f(y)]$ or $f^{-1}[f(y)\ \infty)$ will be a half pace containing y and not intersecting with F. So $y \notin Co(F)$. Hence the claim.

Therefore, for each $f \in \mathcal{F}$, $f^{-1}(f(y))$ meets the standard convex hull of F. Since, $y \in f^{-1}(f(y))$ for each $f \in \mathcal{F}$, $\bigcap \{f^{-1}(f(y)): f \in \mathcal{F}\} \neq \phi$. Now, became $\mathcal{F}$ corresponds to the family of planes intersecting in a straight line, the set $\bigcap \{f^{-1}(f(y):f \in \mathcal{F}\}$ is a straight line. Let f and g be such that the angle between $f^{-1}(f(y))$ and $g^{-1}(g(y))$ is the maximum (see fig 5.3)

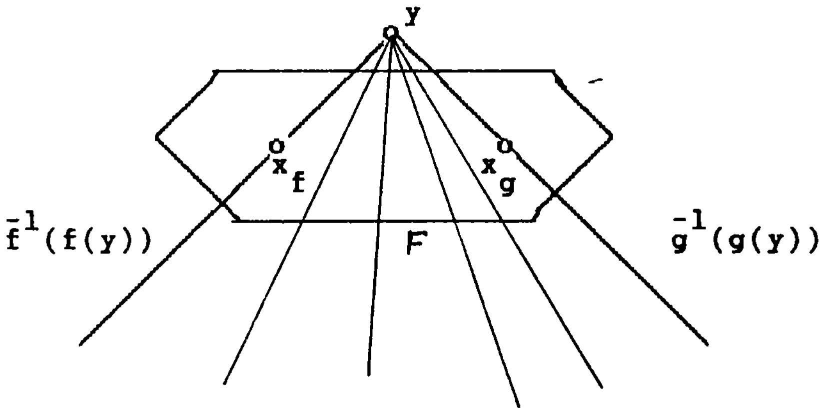


Fig. 5.3

Let $x_f \in f^{-1}\ f(y) \cap F_c$ and $x_g \in g^{-1}\ (g(y)) \cap F_c$ where F_c is the standard convex hull of F. Then $y \in Co(\{x_f\ ,\ x_g\}) \subset C$. Hence $Co(F) \subset C$ and therefore C is convex. Hence the H-convexity generated by $\mathcal{F}$ is of arity 2.

□

The above theorem is not true for a family of functionals corresponding to a family of planes whose intersection is a singleton. The following example gives an example of a symmetrically generated H-convexity of infinite arity.

Let F be the linear functionals corresponding to the tangent planes of a cone. whose cross section is a circle parallel to the x-y plane. That is, $f \in \mathcal{F}$ corresponds to the planes making a constant angle with the x-y plane. Let us assume that this angle is $\pi/4$. That is,

$\mathcal{F} = \{f: f(x,y,z) = y \cos\alpha - x \sin\alpha - z,\ \alpha \in [0,2\pi)\}$

C:

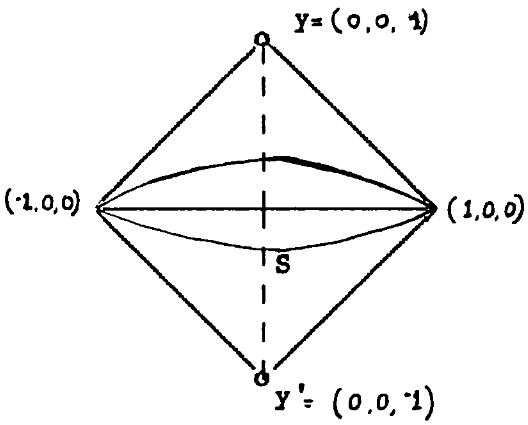


Fig. 5.4

Now the solid C which is the convex hull of S (See Fig 5.4) is a convex set.

Let $C_1 = C\backslash\{y,y'\}$.

It is clear that $y \in Co(C_1)$. Also C_1 is convex with respect to the standard convexity.

That is $f^{-1}(f(y)) \cap C_1 \neq \phi$ for each $f \in \mathcal{F}$.

But note that $f^{-1} f(y) \cap g^{-1}(g(y)) \cap C_1 = \phi$ if $f \neq g$.

hence corresponding to each f, we get $x_f \in C_1$

Such that $x_f \neq x_g$ whenever $f \neq g$. Now, since $\mathcal{F}$ is infinite $\{x_f: f \in \mathcal{F}\}$ is infinite.

Hence C_1 is with the property that $Co(F) \subset C_1$ for each finite set contained in C_1 but C_1 is not convex. Hence the convexity generated by $\mathcal{F}$ is of infinite arity. Further, it is of uncountable arity.

Remark 5.1. a). Since the above H-convexity is of arity greater than 2, it is not JHC.

b). For any n, if we replace the cone whose cross section is a circle by a Pyramid whose crossection is a regular 2n-gon, the H-convexity symmetrically genetated by the family of functionals corresponding to the family of tangent planes containing the lateral faces, is of arity n.

Remark 5.2. R^3 with the H-convexity generated by the family of functionals corresponding to the tangent planes of a cone, doesn't have the Peano property. For, let

$\mathcal{F} = \{f: f(x,y,z) = y \cos \alpha - x \sin \alpha - z, \alpha \in [0,2\pi)\}$.

Let $a = (-1,0,0)$, $b = (0,0,1)$, $c = (1/2 \ \ 0 \ \ 1/2)$ and $u = (1/2,0,0)$, $v = (1/2,1/4,1/4)$

Then u ∈ Co({a,b}). Also note that v ∈ Co({c,u})

(See fig 5.5)

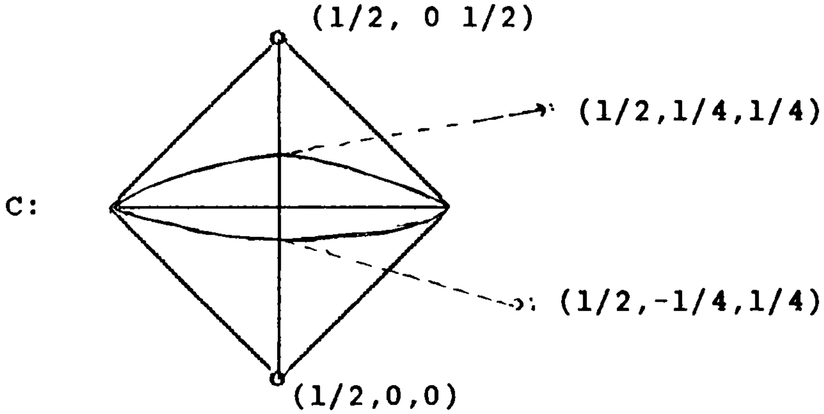


Fig. 5.5

Note that Co({c,u}) is the solid in fig 5.2, because any plane P making an angle $\pi/4$ with the x-y plane will either cut the ordinary segment cu or the solid C will be contained in one of the half spaces determined by P.

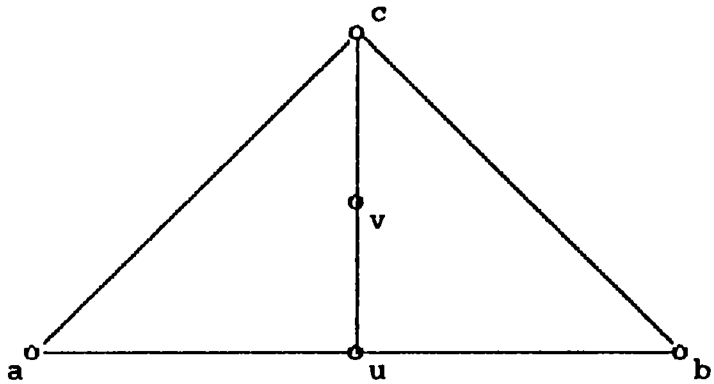


Fig 5.6

Define f:on R^3 as $f(x,y,z) = x-z$. Then $f \in \mathcal{F}$.

Then $f(x,y,z) \leq 0$ is a half plane containing both a and c.

But for $v = (1/2,1/4,1/4)$, $x-z > 0$.

Hence $v \notin Co(\{a,c\})$.

Note that $Co(\{b,c\}) =$ bc, the ordinary segment joining b and c, because it is the intersection of the solid C, the plane $x+z = 0$, and the convex set

$C_0 = \{(x,y,z) : 0 \leq x-z \leq 1\}$.

Now for any $v^1 \neq c$ in $Co\{b,c\}$,.

Let $v^1 = (x_0,y_0,z_0)$. Then $x_0 > 1/2$ and $z_0 < 1/2$.

In this case, $y_0+z_0 < 1/2$.

Define $g: R^3 \rightarrow R$ such that $g(x,y,z) = y + z$. Here $g \in \mathcal{F}$.

Then, the half space H: $f(x,y,z) < 1/2$ contain both v^1 and a but $v \notin H$.

Hence $v \notin Co(\{a,v^1\})$.

Remark 5.3. The H-convexity defined in the example is not S_4. For, the sets $\{(x,y,z) : z = 0, y = 0\}$ and $\{(x,y,z): z = 1, x = 0\}$ are convex sets which can not be separated.

Now we give a characterization for on H-convexity in R^3 to be S_4.

Theorem 5.6: The H-convexity symmetrically generated by a family of linear functionals $\mathcal{F}$ is S_4 if and only if for any two intersecting convex straight lines, the plane determined by these lines is convex. That is, $\mathcal{F}$ should contain the functionals corresponding to the plane determined by these lines.

Proof: Let ℓ_1 and ℓ_2 be any two intersecting convex lines. Then, ℓ_1 can be separated from a line ℓ which is parallel to ℓ_2 and which does not intersect with ℓ_1, only by a plane containing ℓ_1 and ℓ_2.

Now let $\mathcal{C}$ be the H-convexity on R^3 with the given condition and let C_1 and C_2 be disjoint convex sets. Since C_1 and C_2 are determined by half spaces, there are half spaces

Now, since the Helly number is at most four, the intersection of some four membered subfamily of the above family of half spaces is empty (see [12]).

If $H_i \cap K_{i,1} \cap K_{i,2} \cap K_{i,3} = \phi$

Then $H_i \cap C_2 = \phi$ and $C_1 \subset H_i$ and H_i is the required half space.

If $H_i \cap H_j \cap K_k \cap K_\ell = \phi$, let P_i, P_j, P_k, and P_ℓ be the corresponding planes.

Let $\ell_{i,j} = P_i \cap P_j$ and $\ell_{k,\ell} = P_k \cap P_\ell$.

Let P_0 be the plane determined by $\ell_{i,j}$ and the line ℓ which intersect with $\ell_{i,j}$ and which is parallel to $\ell_{k,\ell}$. Then P_o separates C_1 and C_2. □

Now the following example gives an H-convexity on R^3 which satisfies both Pasch and Peano properties but is neither JHC nor S_4.

Let $\mathcal{F} = \{f: f(x,y,z) = \tan\theta\ (y\cos\alpha - x\sin\alpha) - z:$

$\alpha \in [0,2\pi),\ \theta \in [\pi/4,\pi/2)\} \cup \{f: f(x,y,z) = ax+by,$

$a,b \in R\}$.

Then we observe that the H-convexity symmetrically generated by $\mathcal{F}$ has the following properties.

Property 1. Each straight line in R^3 is convex.

For this we prove that any straight line is contained in two distinct convex planes. If ℓ is perpendicular to the x-y plane, it is trivially true. Actually there are infinite number of convex planes by the choice of $\mathcal{F}$. Now for any ℓ, there is a plane perpendicular to the x-y plane, which contains ℓ. Assume without loss of generality that ℓ passes through (0,0,0). Then for any $(x_1,y_1,z_1) \in \ell \setminus \{(0,0,0)\}$.

Then $y_1x - x_1y = 0$ is a plane perpendicular to the x-y plane and containing ℓ.

Now if $\pi/4 \leq \theta < \pi/2$, then, by the choice of $\mathcal{F}$ we get an α such that, the plane,

$\tan\theta\ (y\cos\alpha - x\sin\alpha) - z = 0$, will contain ℓ.

Now let $0 \leq \theta \leq \pi/2$. Assume without loss of generality that the plane perpendicular to the x-y plane which contain ℓ is the x-z plane.

Let $(h,0,h+k) \in \ell$, where $k > 0$. Then

Let $\alpha = \sin^{-1}(-h/h+k)$. Then,

$y \cos \alpha - x \sin \alpha - z = 0$ is a convex plane containing ℓ.

Hence each straight line is convex.

Property 2. This is a Pasch- Peano space.

For any a,b,c,u,v such that, $u \in a\ b$, $v \in c\ u$ we get a v^1 on $b\ c$ such that $v \in a\ v^1$. This is because the convex hull of any two points is the ordinary segment joining those points. So this is having the Peano property. Using similar arguments we can prove that it is having the Pasch property. But this is neither JHc nor S_4, because any line on the x-y plane is convex but the plane is not convex. Therefore by theorems 1.1 and 1.2 this convexity is not of arity two.

5.3 CONCLUDING REMARKS AND SUGGESTIONS FOR FURTHER STUDY.

This thesis is an attempt to find out some properties of d.c.s. graphs, m.c.s. graphs, interval monotone graphs and totally non-interval monotone graphs. We have also introduced a new type of convexity to the edge set of graphs and its convex invariants and Pasch Peano properties are analysed. Also we discuss some properties of H-convexity.

The results of this thesis are far from being complete. We list some of the problems which we have either not attempted or found the answers to be difficult.

1. Characterize solvable trees.

2. Determine the size of the smallest d.c.s. graph containing a nonsolvable tree. Equivalently is it possible to express the size of the smallest d.c.s. graph containing any tree as a function of the order, diameter, radius and the degree?.

3. In the corollary of Theorem 2.14, is it possible to replace $K_{n,n}$ by any m.c.s. graph G of sufficiently large size with the property that $I(a,b) \neq V(G)$ for any pair $a,b \in V(G)$?

4. Characterize halfspce free graphs.

5. Characterize JHC graphs.

6. Since the study of edge convexity has been just initiated, properties of convexity in V(G) studied in detail by many authors can be attempted in this case also.

7. Characterize the H-convexity of arity two.

8. Characterize S_i graphs for i = 2,3 and 4.

REFERENCES

1. Bonnesen, T. — Theorie der Konvexen Korper
   Fenchel, W. — Springer Verlag (1934)

2. Buckley, F. — Distance in Graphs,
   Harary,F. — Addison-wesley (1990)

3. Cohn, P.M. — Universal Algebra,
   Harper and Row, NY (1965)

4. Crawley, P. — Algebraic Theory of Lattices,
   Dilworth, R.P. — Prentice Hall Inc. (1973)

5. Deo, N.S — Graph Theory with Applications
   to Engineering and computer Science,
   Prentice-Hall Inc. (1974).

6. Golumbic, M.C. — Algorithmic Graph Theory and Perfect
   Graphs,
   Academic Press (1980).

7. Harary, F — Graph Theory,
   Addison Wesley (1969).

8. Lay, S.R. — Convex sets and their applications,
   A Wiley Int. Pub. (1982).

9. Prenowitz, W. — A theory of convex sets and Linear
   Jantosciak,J. — Geometry,
   Springer- Verlag (1976).

10. Roberts, F.S. Discrete Mathematical Models: Applications to Social, Biological and Environmental Problems, Prentice Hall Inc.(1976).

11. Valentine, F.A. Convex sets, Mc Graw-Hill (1964).

12. Van-de-vel, M. Theory of convex structures, North Holland (1993).

13. Acharya, B.D. Hebbare, S.P.R. Varthak, M.N. Distance convex sets in Graphs, Proc. Symp. on optimization, design and graph theory, IIT Bombay (1986), 335-342.

14. Bandelt, H.J. Graphs with intrinsic S_3 convexities, J. Graph Theory, 13, (1989), 215-228.

15. Bandelt, H.J. Mulder, H.M. Helly theorems for dismantlable graphs and Pseudo-modular graphs. Topics in Combinatorics and Graph Theory Physica-Verlag Heidelberg, (1990), 65-71.

16. Batten, L.M. Geodesic Subgraphs,
J. Graph Theory, Vol. 7, (1983),
159-163.

17. Bean, P.W. Helly and Radon type theorems in
interval convexity,
Pacific J. Math, 51, (1974), 363-368.

18. Berger, M. Convexity
Amer. Math. Monthly. Vol.97, No.8,
(1990), 650-701.

19. Boltyanskii, V.G. Helly's theorem for H-convex sets.
Sov. Math. Dokl. 17, (1976),78-81.

20. Bourguin, P.G. Restricted separation of polyhedra,
Portugaliae Math. 11, (1952), 133-136

21. Bryant, V.W. Generalization of the theorems of
Webster, R.J. Radon, Helly and Caratheodory,
Monatsh. Math., 73, (1969), 309-315.

22. Calder, J. Some elementary properties of
interval convexities,
J. London. Math. Soc. 3, (1971)
422-428.

23. Chang, G.J. Centres of chordal graphs, Graphs and combinatorics, 7,(1991), 305-313.

24. Changat,M. on order and geodesic alignment of a connected bigraph, Czech. Math. J. 41, (116), (1991), 713-715.

25. Changat, M. Partition number for the Geodesic convexity in a generalized triangle, Utilitas Math.,(1992).

26. Changat, M. On Monophonically convex simple graphs, Graph Theory Notes of New York, XXV,(1993), 41-44.

27. Changat, M. On interval monotone graphs (communicated)

28. Changat, M. Vijayakumar, A On order and metric convexities in Z^n, Compositio Math., 81, (1992), 57-65.

29. Chepoi, V.D. Geometric properties of d-convexity in bipartite graphs,

Modelirovanic informacionnych sistem, (1986), 88-100.

30. Chepoi, V.D. Centres of triangulated graphs (Russian), Translated in Math. Notes, 43, (1988), 82-86.

31. Danzer, L
Grunbaum, B.
Klee, V.
Helly's theorem and its relatives, Proc. Symp. Pure Math. Amer. Math. Soc., (1963), 101-180.

32. Duchet, P. Convexity in combinatorial structures, Circ. Math. palermo No. 14, (1987), 261-293.

33. Duchet, P. Convex sets in Graphs, II-minimal path convexity, J. Comb. th. Ser. B. 44, (1988), 307-316.

34. Edelman, P.H.
Jamison, R.E.
The theory of convex geometries, Geometrial Dedicata, (1985), 247-270.

35. Ellis, J.W. A general set-separation theorem, Duke, Math. H.J., 19,(1952), 417-421.

36. Farber, M. Bridged graphs and Geodesic convexity, Discrete Math., 66,(1987), 249-257.

37. Farber, M. Jamison, R.E On local convexity in graphs, Discrete Math., 66, (1987), 231-247.

38. Farber, M. Jamison, R.E Convexity in graphs and hypergraphs, SIAM J. Alg. Disc. Methods, 7, (1986),433-444.

39. Hebbare, S.P.R. A class of distance convex simple graphs, Ars. Combinatoria, 7,(1979), 19-26.

40. Hebbare, S.P.R. Uniconvex graphs, Sankhya, Vol. 64,Series A (1991).

41. Hebbare, S.P.R. Two decades survey of geodesic convexity in graphs, Proc. Symp. in graph theory and combinatorics, Cochin Univ., (1991), 119-153.

42. Jamison, R.E. A perspective on abstract convexity, Dekker, New York, (1982), 113-150.

43. Jamison,R.E. Nowakowski, R. — A Helly theorem for convexity in graphs, Discrete Mathematics, 51(1984), 35-39.

44. Jamison,R.E. Waldner — Convexity in block graphs, Congressus Numerantium, Vol. 33, (1981), 129-142.

45. Jamison,R.E. Waldner — Partition number for trees and ordered sets, Pacific J. of Math. Vol. 96, No.1, (1981), 115-140.

46 Kay,D.C. Womble, E.W. — Axiomatic convexity theory and the relation-ship between the Caratheodory, Helly and Radon numbers, Pacific J. Math.,38,(1971),471-485.

47. Kolodziejczyk, K. — Star shapedness in convexity spaces, compositio Math., 56, (1985), 361-367.

48. Kolodziejczyk, K. — Two Constructional problems in aligned spaces, Zastoswania Matematyki, XIX, 3-4, (1987), 479-484.

49. Kolodziejczyk, K. Generalized Helly and Radon numbers, Bull. Austral. Math. Soc. Vol. 43, (1991),429-437

50. Laskar, R. Shier, D. On powers and centres of chordal graphs, Disc. Appld. Math. 6, (1983),139-147.

51. Levi, F.W. On Helly's theorem and axioms of convexity, J. Math.Soc.15, (1951), 65-76.

52. Menger. K. Untersuchungen uber allgemeine, Metrik Math. Ann., 100, (1928), 75-163.

53. Mulder, H.M. The interval function of a graph. Math. Centre tracts, 132, Amsterdam (1980).

54. Mulder, H.M . Triple convexities for graphs, Rostock. Math. Kolloq, 39, (1990), 35-52.

55. Nieminen, J. Distance centre and centroid of a median graph,

J.Franklin Inst. 323,,
(1987), 89-94.

56. Nieminen, J. — The center and distance center of ptolemaic graph, Operations Research Letters 7(2), (1988), 91-94.

57. Nieminen, J. — Boolean graphs, Comment. Math. Univ. Carolinae, 29, (1988), 387-392.

58. Onn, S. — On Radon number of the integer lattice, Proc. of conf. of Univ. of Waterloo, (1990), 385-395.

59. Parvathy, K.S., Remadevi, A., Vijayakumar, A. — About a conjecture on the centres of chordal graphs, Graphs and Combinatorics (1994). 269-270.

60. Parvathy, K.S., Vijayakumar, A. — Distance convex simple graphs and solvability, (communicated).

61. Parvathy, K.S., Vijayakumar, A. — On cyclic convexity of a graph (communicated).

62. Parvathy, K.S. Vijayakumar, A. — On some problems of H-convexity (communicated).

63. Prabir Das, Rao, S.B. — Center graphs of chordal graphs, Proc. of the seminar on combin. and Applns. ISI, (1982), 81-94.

64. Proskurowski, A. — Centres of 2-trees, Annals of Disc. Math. 9,(1980),1-5.

65. Proskurowski, A. — Centres of maximal outer planar graphs, J.Graph. Theory 4(2), (1980), 75-79.

66. Rao, S.B. Hebbare, S.P.R. — Characterization of planar distance convex simple graphs, Proc. Symp. of G.T., ISI (1976), 138-150.

67. Reay, J.R. — Open problems around Radon's theorem, Convexity and related combinatorial Geometry, Lecture notes in Pure and Applied Mathematics, 76, (1982), 151-172.

68. Ronse, C. — A bibliography on digital and

computational convexity (1961-1988), Transaction on pattern analysis and machine intelligence, Vol. 11, No. 2. (1989).

69. Sampath Kumar, E. Convex sets in graphs, Indian J.Pure and Applied Math. 15 (1984), 1065-1071.

70. Sierksma, G. Axiomatic convexity theory and the convex product space. Dissertation, University of Groningen, Netherlands, (1976).

71. Sierksma, G. Relationship between Caratheodory, Helly, Radon and exchange numbers of convexity spaces, Nieun Arch. Wisk. 25, (1977), 115-132.

72. Sierksma, G. Generalization of Helly's theorem, Convexity and related combinatorial geometry (norman, okla), Lecture notes in Pure and Applied Mathematics, 76, (1982), 173-192.

73. Soltan, V.P. d-convexity in graphs, Soviet Math. Dokl., 13, (1972), 975-978.

74. Tverberg, H. A generalization of Radon's theorem, J. London Math. Soc. 41, (1966), 123-128.

75. Van de Vel, M. Pseudo-boundaries and pseudo-interiors for topological convexities, Dissertations Math., 210, (1983), 1-72.