SOME PROBLEMS IN TOPOLOGY, CATEGORY THEORY
AND THEIR APPLICATIONS

SOME PROBLEMS ON LATTICES OF FUZZY TOPOLOGIES AND RELATED TOPICS

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CERTIFICATE

Certified that the work reported in this thesis is based on the bona fide work done by Sri. Johnson T.P., under my guidance in the Department of Mathematics and Statistics, Cochin University of Science and Technology, and has not been included in any other thesis submitted previously for the award of any degree.

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CONTENTS

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INTRODUCTION

Fuzzy set theory after its introduction by L.A. Zadeh [39] has become important with application in almost all areas of mathematics, of which one is the area of topology. Zadeh took the closed unit interval $[0,1]$ as the membership set. J.A.Goguen [12] considered different ordered structures for the membership set. He considered a fuzzy subset as a generalized characteristic function. Thus the ordinary set theory is a special case of the fuzzy set theory where the membership set is $\{0,1\}$. Goguen suggested that a complete and distributive lattice would be a minimum structure for the membership set.

The theory of general topology is based on the set operations of union, intersection and complementation. Fuzzy sets do have the same kind of operations. It is therefore natural to extend the concept of point set topology to fuzzy sets resulting in a theory of fuzzy topology. The study of general topology can be regarded as a special case of fuzzy topology, where all fuzzy sets in question take values 0 and 1 only. C.L.Chang [7] was the first to define a fuzzy topology. Since then an extensive study of fuzzy topological spaces has been carried out by many researchers. Many mathematicians, while developing fuzzy topology have used different lattice structures for the membership set like (1) completely distributive lattice with 0 and 1 by T.E.Gantner, R.C.Steinlage and R.H.Warren [11], (2) complete and completely distributive lattice equipped with order reversing involution by Bruce Hutton and Ivan Reilly [21], (3) complete and completely distributive non-atomic Boolean algebra by Mira Sarkar [32], (4) complete chain by Robert Badarad [2] and F. Conard [8], (5) complete Brouwerian lattice with its dual also Brouwerian by Ulrich Höhle [18], (6) complete Boolean algebra by Ulrich Höhle [19], (7) complete and distributive lattice by S.E.Rodabaugh [31]. R.Lowen [24,25] modified the definition of fuzzy topology given by C.L.Chang and obtained a fuzzy version of Tychnoff theorem, but he lost the concept that fuzzy topology generalizes topology. We take the definition of fuzzy topology in the line of Chang with closed interval [0,1] as membership lattice in the first three chapters. In the last chapter we replace [0,1] by an arbitrary complete and distributive lattice.

In topology, the concept group of homeomorphisms of a space has been studied by various authors. Many problems relating the topological properties of a space and the algebraic properties of its group of homeomorphisms were investigated. In 1959 J. de Groot [14] proved that any group is isomorphic to the group of homeomorphisms of a topological space. A related problem is to determine the subgroup of the group of permutations of a fixed set X, which can be represented as the group of homeomorphisms of a topological space (X,T) for some topology T on X. The Ph.D. thesis of P.T. Ramachandran [29] is a study on these and related problems. Ramachandran proved that no non-trivial proper normal subgroup of the group S(X) of all permutations of a fixed set X can be represented as the group of homeomorphisms of a topological space (X,T) for any topology T on X. (see also [30]) Also he has proved that the group generated by cycles cannot be represented as group of homeomorphisms.

In this thesis, a major part is an attempt to have an analogous study in fuzzy topological spaces. In the first chapter we investigate some relations between group of fuzzy homeomorphisms of a fuzzy topological space X and the group of permutations of the ground set X. It is observed that non-homeomorphic fuzzy topological spaces can have isomorphic group of fuzzy homeomorphisms. In contrast to the results in the topological context, we prove in this chapter that the subgroups generated by cycles and proper normal subgroups of S(X) can be represented as groups of fuzzy homeomorphisms for some fuzzy topology on X. The relation between group of fuzzy homeomorphisms and group of homeomorphisms of the associated topology are discussed and it is proved that for topologically generated fuzzy topological spaces both the groups are isomorphic.

In 1970 R.E. Larson [23] found a characterization for completely homogeneous topological spaces. A method to construct completely homogeneous fuzzy topological spaces is also given in the first chapter.

Čech closure spaces is a generalization of the concept of topological spaces. Eduard Čech, J. Novak, R. Fric, Ramachandran and many others have studied this concept and many topological concepts were extended to the Čech closure spaces. In 1985 A.S. Mashhour and M.H. Ghanim [27] defined Čech fuzzy closure spaces and they extended Čech proximity to fuzzy topology. In the second chapter we investigate the group of fuzzy closure isomorphisms and extend some results discussed for the fuzzy topological spaces in the first chapter to Čech fuzzy closure spaces. Also in this chapter an attempt is made to study the lattice structure of the set of all Čech fuzzy closure operators on a fixed set X.

In 1936, G. Birkhoff [4] described the comparison of two topologies on a set and proved that the collection of all topologies on a set forms a complete lattice. In 1947, R. Vaidyanathaswamy [36] proved that this lattice is atomic and determined a class of dual atoms. In 1964, O. Fröhlich [9] determined all the dual atoms and proved that the lattice is dually atomic. In 1958, Juris Hartmanis [16] proved that the lattice of topologies on a finite set is complemented and raised the question about the complementation in the lattice of topologies on an arbitrary set. H. Gaifman [10] proved that the lattice of topologies on a countable set is complemented. Finally in 1966, A.K. Steiner [34] proved that the lattice of topologies on an arbitrary set is complemented. Van Rooij[37] gave a simpler proof independently in 1968. Hartmanis noted that even in the lattice of topologies on a set with three elements, only the least and the greatest elements have unique complements. Paul S. Schnare [33] proved that every element in the lattice of topologies on a set X, except the least and the greatest elements have at least n-1 complements when X is finite such that $|X| = n \geq 2$ and have infinitely many complements when X is infinite.

In 1989, S. Babusundar [1] proved that the collection of all fuzzy topologies on a fixed set forms a complete lattice with the natural order of set inclusion. He introduced t-irreducible subsets in the membership lattice and proved that the existence of minimal t-irreducible subsets in the membership lattice is a necessary and sufficient condition for the lattice of fuzzy topologies to have ultra fuzzy topologies and solved complementation problem in the negative.

Lattice structure of the set of all fuzzy topologies on a given set X is further explored in the third chapter. For a given topology t on X, we have studied properties of the lattice $\mathcal{F}_t$ of fuzzy topologies defined by families of lower semicontinuous function with reference to t on X. We obtain from the lattice $\mathcal{F}_t$ that the lattice of fuzzy topologies forms a complete atomic lattice. This lattice is not complemented and it has no dual atoms. Also it is proved that topologically generated fuzzy topologies and crisp topologies have complements.

Some of the results in the previous chapters raise the question of how far the results do hold good when the membership lattice is not [0,1] but an arbitrary complete and distributive lattice. The results of chapter 4 are in this direction.

Chapter-1

GROUP OF FUZZY HOMEOMORPHISMS*

In this chapter we investigate, for a fuzzy topological space X, the relation between the group of fuzzy homeomorphisms and the group S(X) of all permutations of the ground set X. In chapter-1 of [29], P.T. Ramachandran attempted to determine subgroups of the group of all permutations of a fixed set X, which can be represented as the group of homeomorphisms of a topological space (X,T) for some topology T on X. In [29] it is proved that (1) (Theorem 1.1.5) For a finite set $X = \{a_1, a_2, \ldots, a_n\}$ $n \geqslant 3$ the group of permutations of X generated by the cycle $p = (a_1, a_2, \ldots, a_n)$ cannot be represented as the group of homeomorphisms for any topology T on X. (2) For a finite set X, $|X| \geqslant 3$ there is no topology T on X such that the group of homeomorphisms of (X,T) is the alternating group of permutation of X (Theorem 1.1.6). (3) For an infinite set X no non-trivial proper normal subgroup of the group of all permutation of X, can be represented as the group of homeomorphisms of (X,T) for any topology T on X (Theorem 1.2.10).

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* Some of the results presented in this chapter will appear in the Journal Fuzzy Sets and Systems [22].

In contrast to these results in the topological situation, we prove that the groups generated by cycles and proper normal subgroups of S(X) can be represented as groups of fuzzy homeomorphisms. The relation between the group of fuzzy homeomorphisms and the group of homeomorphisms of the associated topology are discussed and it is proved that for a topologically generated fuzzy topological space, both the groups are isomorphic.

1.1 Preliminaries

Definition 1.1.1

A fuzzy set a in a set X is defined as (characterized by) a function from X to [0,1].

For a point x in X, a(x) is called the membership value of x in the fuzzy subset a. Let a and b be fuzzy sets in X. Then we define

$a = b \iff a(x) = b(x)$ for all x in X

$a \leqslant b \iff a(x) \leqslant b(x)$ for all x in X

$c = a \vee b \iff c(x) = \sup\{a(x), b(x)\}$ for all x in X

$d = a \wedge b \iff d(x) = \inf\{a(x), b(x)\}$ for all x in X

$e = a' \iff e(x) = 1 - a(x)$ for all x in X.

Also for $\{a_\alpha\}_{\alpha \in A}$, we define

$$c = \bigvee_{\alpha \in A} a_\alpha = \sup a_\alpha \iff c(x) = \sup\{a_\alpha(x) : \alpha \in A\} \text{ for all } x \text{ in } X$$

and

$$d = \bigwedge_{\alpha \in A} a_\alpha = \inf a_\alpha \iff d(x) = \inf\{a_\alpha(x) : \alpha \in A\} \text{ for all } x \text{ in } X.$$

Definition 1.1.2.

A subset F of I^X is called a fuzzy topology [7] on X if

(i) the constant functions 0 and 1 $\in$ F

(ii) $a, b \in F$ implies $a \wedge b \in F$

(iii) $a_j \in F$ for all $j \in J$ implies $\vee a_j \in F$

The fuzzy sets in F are called open fuzzy sets. A fuzzy set $a \in I^X$ is called closed if a' is open. A subset $G \subset F$ is a base for F if and only if for every $f \in F$ $\exists \{a_j\}_{j \in J} \subset G$ such that $f = \bigvee_{j \in J} a_j$.

A subset $G' \subset F$ is a subbase for F if and only if the family of finite infima of members of G' is a base for F.

A fuzzy point in a set X is a fuzzy set in X which is zero every where except at one point say x where it takes the positive value say r less than or equal to 1. (Some authors make a distinction between fuzzy point and fuzzy singleton- for the former, $r \in (0,1)$ and for the latter $r \in (0,1]$.)

Definition 1.1.3 [7]

Let θ be a function from X to Y and f be a fuzzy set in Y. Then the inverse of f written as $\theta^{-1}(f)$ is a fuzzy set in X whose membership function is given by $\theta^{-1}(f)(x) = f(\theta(x))$ for all x in X. Conversely let g be a fuzzy set in X. Then the image of g, written as $\theta(g)$ is a fuzzy set in Y whose membership function is given by

$$\theta(g)(y) = \begin{cases} \text{Sup} \{g(z) \mid z \in \theta^{-1}(y) & \text{if } \theta^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

where $\theta^{-1}(y) = \{x \in X \mid \theta(x)=y\}$

Definition 1.1.4 [7]

A function θ from a fuzzy topological space (X,T) to a fuzzy topological space (Y,U) is fuzzy continuous if and only if the inverse image of every U-open fuzzy set is

T-open. A fuzzy homeomorphism is a fuzzy continuous one-to-one map of a fuzzy topological space X on to a fuzzy topological space Y such that the inverse of the map is also fuzzy continuous.

1.2. Group of fuzzy homeomorphisms

For any fuzzy topological space (X,F), the set of all fuzzy homeomorphisms of (X,F) on to itself forms a group.

Theorem 1.2.1.

If (X,F) is a discrete or an indiscrete fuzzy topological space, then the group of fuzzy homeomorphisms is the group of all permutations of X.

Proof: Trivial.

The following result shows that the converse of 1.2.1 is not true.

Theorem 1.2.2.

Let X be any set and F be the fuzzy topology generated by all fuzzy points with the same membership value. Then the group of fuzzy homeomorphisms of (X,F) is the group of all permutations of X.

Proof:

Let θ be any permutation of X. Then the image and the inverse image of any fuzzy point under θ are fuzzy points. Hence the result.

Note 1.2.3.

In [29], it is proved that for a finite set $X = \{a_1, a_2, \ldots, a_n\}$ $n \geq 3$ the group of permutations of X generated by the cycle $p = (a_1, a_2, \ldots, a_n)$ cannot be represented as the group of homeomorphisms for any topology T on X. But in fuzzy topology we have.

Theorem 1.2.4.

Let X be any set such that cardinality of $X \leq C$. Let S(X) be the group of all permutations of X. Then the subgroup of S(X) generated by a finite cycle can be represented as the group of fuzzy homeomorphisms for some fuzzy topology F on X.

Proof:

Consider a finite subset $\{a_1, a_2, \ldots, a_n\}$ of X. Let H be the subgroup of S(X) generated by the cycle $p = (a_1, a_2, \ldots, a_n)$. Then clearly $H = \{p, p^2, \ldots, p^n\}$.

Consider a fuzzy subset g of X such that distinct elements in X have distinct membership values. This is possible since $|X| \leqslant C$. Assume that the membership values of $a_1, a_2, \ldots, a_n$ respectively are $\alpha_1, \alpha_2, \ldots, \alpha_n$ where we can assume without loss of generality that $\alpha_1 < \alpha_2 < \ldots < \alpha_n$. Define fuzzy subsets $g_1, g_2, \ldots, g_n$ by $g_m(x) = g(p^m(x))$ for every x in X. Let F be the fuzzy topology generated by $S = \{g_1, g_2, \ldots, g_n\}$ as a subbasis. Now we claim that the group of fuzzy homeomorphisms of (X,F) is H. In fact, every element of H is a fuzzy homeomorphism since

$$p^{m^{-1}}(g_i) = g_i p^m = gp^i.p^m = g(p^i.p^m) = gp^k = g_k$$

for some k, $1 \leqslant k \leqslant n$, and

$$p^m(g_i) = g_i p^{m^{-1}} = gp^i p^{m^{-1}} = g(p^i.p^{m^{-1}}) = gp^\ell = g_\ell \quad ,$$

for some ℓ , $1 \leqslant \ell \leqslant n$.

Conversely if θ is different from p^m, then $\theta^{-1}(g_i) = g_i\theta = g(p^i\theta) = h$, a fuzzy subset which has distinct membership values for distinct points and $h \notin S$. We prove that h is not open. Let $f = g_1 \wedge g_2$, then

$$
\begin{aligned}
f(a_n) &= g_1(a_n) \wedge g_2(a_n) \\
&= g(p(a_n)) \wedge g(p^2(a_n)) \\
&= g(a_1) \wedge g(a_2) \\
&= \alpha_1 \wedge \alpha_2 = \alpha_1 \qquad \text{by our assumption } \alpha_1 < \alpha_2
\end{aligned}
$$

$$
\begin{aligned}
\text{and } f(a_{n-1}) &= g_1(a_{n-1}) \wedge g_2(a_{n-1}) \\
&= g(p(a_{n-1})) \wedge g(p^2(a_{n-1})) \\
&= g(a_n) \wedge g(a_1) \\
&= \alpha_n \wedge \alpha_1 = \alpha_1
\end{aligned}
$$

That is f has two elements with the same membership value α_1. Arguing in a similar way, we get that the meet of any pair of elements in S have two elements with the same membership value α_1. Also the join of any pair of elements in S have two elements with the same membership value α_n. Repeating the process, we see that any member of F, the fuzzy topology generated by S which is not in S, has at least two elements with the same membership value, while $\Theta^{-1}(g_i)$ has all values different. Thus $\Theta^{-1}(g_i) \notin F$. Thus the group of fuzzy homeomorphisms of (X,F) is H.

Definition 1.2.5. [15]

An infinite cycle is a permutation π on an infinite set of letters x_i; $i \in \mathbb{Z}$ such that $\pi(x_i) = x_{i+1}$.

Theorem 1.2.6.

Let X be any infinite set such that cardinality of $X \leqslant C$. Then the group generated by an infinite cycle can be represented as the group of fuzzy homeomorphisms for some fuzzy topology on X.

Proof:

Let p be the infinite cycle $(\ldots x_{-2}, x_{-1}, x_o, x_1, x_2, \ldots)$ and H be the group generated by p. Then $H = \{ p^n | n \in Z\}$. Consider a fuzzy subset g of X such that distinct elements in X have distinct membership values. Consider a countable collection $\{\alpha_o, \alpha_1, \alpha_2, \ldots \}$ of points in [0,1] where we can assume without loss of generality that $\alpha_o = 0$, $\alpha_n = \frac{1}{n}$ $\forall$ n, $g(x_o) = \alpha_1$, $g(x_n) = \alpha_{2n-2}$ and $g(x_{-n}) = \alpha_{2n+1}$. Corresponding to each element $p^n \in H$ we define fuzzy subset g_n by $g_n(x) = g(p^n(x))$ for every x in X. Let F be the fuzzy topology generated by $S = \{ g_n | n \in Z\}$ as a subbasis. Now we claim that the group of fuzzy homeomorphisms of (X,F) is H. For, every element of H is a fuzzy homeomorphism since

$$p^{n^{-1}}(g_m) = g_m p^n = gp^m.p^n = gp^{m+n} = g_{m+n}$$

and

$$p^n(g_m) = g_m p^{n^{-1}} = gp^m p^{-n} = gp^{m-n} = g_{m-n}$$

Conversely if θ is a bijection different from p^n, $n \in Z$, then $\theta^{-1}(g_m) = g_m\theta = g(p^m\theta) = h$, a fuzzy subset which has distinct membership values for distinct points and $h \notin S$. We prove that h is not open. Let $k = g_1 \wedge g_2$, then

$$\begin{aligned} k(x_o) &= g_1(x_o) \wedge g_2(x_o) \\ &= g(p(x_o)) \wedge g(p^2(x_o)) \\ &= g(x_1) \wedge g(x_2) \\ &= \alpha_o \wedge \alpha_2 \\ &= \alpha_o \end{aligned}$$

and

$$\begin{aligned} k(x_{-1}) &= g_1(x_{-1}) \wedge g_2(x_{-1}) \\ &= gp(x_{-1}) \wedge g(p^2(x_{-1})) \\ &= g(x_o) \wedge g(x_1) \\ &= \alpha_1 \wedge \alpha_o \\ &= \alpha_o \end{aligned}$$

That is k has two elements with the same membership value α_o. Similarly the meet of every pair of elements in S have two elements with the same membership value α_o. Also the join of any pair of elements in S have two elements with the same membership value α_1. Repeating the process we see that any member in F, the fuzzy topology generated by S which does not belong to S, has at least two elements with the same membership value, while $\theta^{-1}(g_m)$ has all values different. That is $\theta^{-1}(g_m) \notin F$. Thus the group of fuzzy homeomorphisms of (X,F) is H.

Theorem 1.2.7.

Let X be any set such that cardinality of X $\leqslant$ C. Let p be a permutation which is a product of disjoint cycles such that the lengths of the cycles are bounded. Then the group generated by p can be represented as the group of fuzzy homeomorphisms for some fuzzy topology on X.

Proof.

Let the lengths of the cycles be bounded by k and $p = p_2 \circ p_3 \cdot p_4 \ldots p_k$ where p_r represents the product of cycles of length r. Let N be the least common multiple of 2,3, ..., k. Then the elements of the group H generated by p are $p, p^2, p^3, \ldots, p^N$. Let g be a fuzzy subset of X such that distinct elements in X have distinct membership values. Let the membership values of points belonging to 2-cycles be in the interval $[0, \frac{1}{2k}]$, those to the 3-cycles be in the interval $[\frac{1}{k}, \frac{1}{k} + \frac{1}{2k}]$ ... those to the k-cycles be in the interval $[\frac{k-1}{k}, \frac{k-1}{k} + \frac{1}{2k}]$. Corresponding to each element p^m in H we define fuzzy subset g_m by $g_m(x) = g(p^m(x))$ for every x in X. Let F be the fuzzy topology generated by $S = \{g_1, g_2, \ldots, g_N\}$ as a subbasis. We claim that the group of fuzzy homeomorphisms of (X,F) is H. For, every element p^m of H is a fuzzy homeomorphism since

$$p^{m^{-1}}(g_n) = g_n p^m = gp^n . p^m = gp^\ell = g_\ell \text{ for some } \ell,\ 1 \leqslant \ell \leqslant N$$

and

$$p^m(g_n) = g_n p^{m^{-1}} = gp^n . p^{m^{-1}} = gp^q = g_q \text{ for some } q,\ 1 \leqslant q \leqslant N.$$

Conversely if θ is a bijection different from p^n, then $\theta^{-1}(g_m) = g_m\theta = gp^m\theta = h$, a fuzzy subset which has distinct membership value and $h \notin S$. We claim that h is not open. Let $f = g_m \wedge g_n$ ($m \neq n$). Then p^m and p^n are distinct for at least one factor from the collection

$\{p_2, p_3, p_3^2, p_4, p_4^2, p_4^3, \ldots, p_k, p_k^2, \ldots, p_k^{k-1}\}$.

Let that factor or product of smaller factors involves $a_1, a_2, \ldots, a_r$ [may be $(a_1, a_2, \ldots, a_r)$ or $(a_1, a_2)(a_3, \ldots, a_r)$ etc. ...], $2 \leqslant r \leqslant k$ and $\alpha_1, \alpha_2, \ldots, \alpha_r$ be the membership values of $a_1, a_2, \ldots, a_r$ respectively in g where we can assume without loss of generality that $\alpha_1 < \alpha_2 \ldots < \alpha_r$. Then f has two points with the same membership value α_1. Similarly $g_m \vee g_n$ has two points with the same membership value α_r. Thus the join or meet of any pair of members from S have at least two points with the same membership value. That is any member different from subbasic open fuzzy set in F, the fuzzy topology generated by S, has at least two points with the same membership value, while $\theta^{-1}(g_m)$ has all values different. Therefore $\theta^{-1}(g_m) \notin F$. This completes the proof.

Theorem 1.2.8.

Let X be any set such that $|X| \leqslant C$. Let p be a permutation which is a product of disjoint finite cycles such that the lengths of the cycles in it is unbounded. Then the group generated by p can be represented as the group of fuzzy homeomorphisms for some fuzzy topology on X.

Proof:

Let $p = p_2 \cdot p_3 \cdot p_4 \ldots p_k \ldots$ where p_k represents the product of cycles of length k. Let H be the group generated by p, then $H = \{p^n; n \in Z\}$. Choose a collection $\{A_\alpha | \alpha \in Z_+\}$ of disjoint subsets of [0,1] each containing an uncountable number of points and each of which has a smallest and largest element. Let g be a fuzzy subset of X such that distinct elements in X have distinct membership values and such that the membership values of points belonging to the 2-cycles be in the set A_2, those to the 3-cycles be in the set A_3 and so on. Corresponding to each element $p^n : n \in Z$ in H we define fuzzy subset g_n by $g_n(x) = g(p^n(x))$ for every x in X. Let F be the fuzzy topology generated by $S = \{g_n | n \in Z\}$ as subbasis. Then as in the previous cases, it can be proved that the group of fuzzy homeomorphisms of (X,F) is H.

Notations 1.2.9.

Let p be a permutation of X. Let $M(p) = \{x \in X: p(x) \neq x\}$. A(X) denotes the group of all permutation p of X such that M(p) is finite and p can be written as product of an even number of transpositions. If α is any cardinal number let $H_\alpha = \{p \in S(X): |M(p)| < \alpha\}$ We use the following lemma proved by Baer [3].

Lemma 1.2.10.

The normal subgroups of the group S(X) of all permutations of X are precisely the trivial subgroup, A(X), S(X) and the subgroups of S(X) of the form H_α for some infinite cardinal number $\alpha \leqslant |X|$.

Note 1.2.11.

In [29], using the concept of homogenity, it is proved that for a finite set X, ($|X| \geqslant 3$) there is no topology T on X such that the group of homeomorphisms of (X,T) is the alternating group of permutation of X. Also it is proved that for an infinite set X no non-trivial proper normal subgroup of S(X) can be represented as the group of homeomorphisms of (X,T) for any topology T on X. In contrast to these results in topology, we prove the following.

Theorem 1.2.12.

Let X be any set such that cardinality of $X \leqslant C$. Then the group A(X)of all even permutation of X can be represented as the group of fuzzy homeomorphisms for some fuzzy topology on X.

Proof:

Consider a fuzzy subset g of X such that distinct elements in X have distinct membership value. We can well-order the set of even permutations of X by using some ordinal number δ, where $\delta \leq \Omega$. Corresponding to each even permutation e_α, $\alpha < \delta$ define fuzzy subset g_α by $g_\alpha(x) = g(e_\alpha(x))$ for every x in X. Let F be the fuzzy topology generated by $S = \{g_\alpha : \alpha < \delta\}$. Now we claim that the group of fuzzy homeomorphisms of (X,F) is the group of all even permutation of X. If θ is an even permutation, then $\theta = e_\alpha$ for some $\alpha < \delta$, it is a fuzzy homeomorphism since $e_\alpha^{-1}(g_\beta) = g_\beta e_\alpha = ge_\beta . e_\alpha = ge_\gamma = g_\gamma$ for some $\gamma < \delta$ and $e_\alpha(g_\beta) = g_\beta e_\alpha^{-1} = ge_\beta e_\alpha^{-1} = ge_\eta = g_\eta$ for some $\eta < \delta$.

Conversely, if θ is not an even permutation then $\theta^{-1}(g_\alpha) = g_\alpha . \theta = h$, a fuzzy subset which has distinct membership values for distinct points and $h \notin S$. We prove that h is not open. Let, as an illustration, $e_\alpha = (a_1,a_2)(a_3,a_4)$ and $e_\beta = (a_5,a_6)(a_7,a_8)$ be two even permutation of X. (The method is similar when there are more transpositions in either or both of the even permutations).

Then the fuzzy subset g_α and g_β are obtained from g by moving the membership values corresponding to e_α and e_β. Let $p = g_\alpha \wedge g_\beta$. Then $p(x) = g_\alpha(x) = g_\beta(x)$ for all x different from $a_1, a_2, \ldots, a_8$;

$$p(a_1) = g_\alpha(a_1) \wedge g_\beta(a_1) = g(a_2) \wedge g(a_1)$$

and

$$p(a_2) = g_\alpha(a_2) \wedge g_\beta(a_2) = g(a_1) \wedge g(a_2)$$

That is $p(a_1) = p(a_2)$. Similarly $p(a_3) = p(a_4)$, $p(a_5)=p(a_6)$ and $p(a_7) = p(a_8)$. Thus the meet of any pair of members of S have at least two elements with the same membership value. Similarly join of any pair of members of S have at least two elements with the same membership value. Repeating this process we see that any member different from subbasic open fuzzy set in F, the fuzzy topology generated by S, has at least two elements with the same membership value, while $\theta^{-1}(g_\alpha)$ has all values different. That is $\theta^{-1}(g_\alpha) \notin F$. Thus the group of fuzzy homeomorphisms of (X,F) is the group of all even permutation of X.

Theorem 1.2.13.

Let X be any set such that $|X| \leqslant C$, then $H_{\aleph}$ can be represented as the group of fuzzy homeomorphisms for some fuzzy topology on X.

Proof:

We can prove the theorem by the same argument as in the theorem 1.2.12.

Note 1.2.14.

The normal subgroup of S(X) for a finite set X such that $|X| = 3$ or $|X| \geq 5$ are precisely the trivial subgroup, the alternating group and S(X) itself.

When X is the set $\{a,b,c,d\}$ with four elements the normal subgroup of S(X) are precisely the trivial subgroup, $\{I, (a,b)(c,d), (a,c)(b,d), (a,d)(b,c)\}$, A(X) the alternating group and S(X) where I is the identity map on X. It is known [29] that neither A(X) nor $\{I, (a,b)(c,d), (a,c)(b,d), (a,d)(b,c)\}$ can be represented as group of homeomorphisms of any topological space on X. In contrast to this result in topology, we prove

Theorem 1.2.15.

Let $X = \{a_1, a_2, a_3, a_4\}$. Consider the normal subgroup $H = \{I, (a_1,a_2)(a_3,a_4), (a_1,a_4)(a_2,a_3), (a_1,a_3)(a_2,a_4)\}$ of S(X). Then H can be represented as the group of fuzzy homeomorphisms for some fuzzy topology on X.

Proof:

Let $H = \{p_1, p_2, p_3, p_4\}$ where $p_1 = (a_1, a_2)(a_3, a_4)$, $p_2 = (a_1, a_4)(a_2, a_3)$, $p_3 = (a_1, a_3)(a_2, a_4)$ and $p_4 = I$, identity permutation. Consider a fuzzy subset g of X such that distinct elements in X have distinct membership value. Define fuzzy subset g_m by $g_m(a_j) = g(p_m(a_j))$, $1 \leqslant j \leqslant 4$ for $m = 1,2,3,4$. Let F be the fuzzy topology generated by $S = \{g_1, g_2, g_3, g_4\}$ as a subbasis. Now we claim that the group of fuzzy homeomorphisms of (X,F) is H. Every element of H is a fuzzy homeomorphism since

$$p_i^{-1}(g_m) = g_m p_i = g p_m p_i = g p_k = g_k \quad \text{and}$$

$$p_i(g_m) = g_m p_i^{-1} = g p_m p_i^{-1} = g p_k = g_k \text{ for } k=1,2,3,4.$$

Conversely if θ is different from p_i, then $\theta^{-1}(g_m) = g_m\theta = g(p_m\theta) = h$, a fuzzy subset which has different membership values for distinct points and $h \notin S$. The meet of any pair of members of S have two pairs of elements with the same membership value. Similarly the join of any pair of members of S have two elements with the same membership value. This shows that any member of F, the fuzzy topology generated by S, which is different from the members of S, has at least two elements with the same membership value, while $\theta^{-1}(g_m)$ has all values different. So $h \notin F$. This completes the proof.

Definition 1.2.16 [24]

For a topology T on X let W(T) be the set of all lower semicontinuous function from (X,T) to [0,1]; W(T) turns out to be a fuzzy topology on X. A fuzzy topology of the form W(T) is called topologically generated. For a fuzzy topology F on X, i(F) be the weak topology on X induced by all functions $f:X \longrightarrow I_r$ where $f \in F$ and $I_r = [0,1]$ with subspace topology of the right ray topology on R.

Theorem 1.2.17.

Let (X,F) be any fuzzy topological space. Then the group of fuzzy homeomorphisms of (X,F) is a subgroup of the group of homeomorphisms of (X i(F)).

Proof:

Let θ be any fuzzy homeomorphism of (X,F). Then both $\theta(f)$ and $\theta^{-1}(f)$ belong to F for every $f \in F$. That is $f\theta^{-1}$ and $fo\theta \in F$ which implies that $(fo\theta^{-1})^{-1}$ $(\alpha,1]$ and $(fo\theta)^{-1}$ $(\alpha,1] \in i(F)$, $0 < \alpha \leq 1$. That is $\theta(f^{-1}(\alpha,1])$ and $\theta^{-1}(f^{-1}(\alpha,1]) \in i(F)$. Thus θ is a homeomorphism of (X, i(F)).

Remark 1.2.18.

The following example shows that the subgroup mentioned in theorem 1.2.17 can be a proper subgroup.

Let $X = \{a,b,c\}$. Let F be the fuzzy topology generated by $\{g_1, g_2, g_3\}$ where

$$g_1: \begin{array}{l} a \longrightarrow \frac{1}{2} \\ b \longrightarrow \frac{1}{3} \\ c \longrightarrow 1 \end{array} \qquad g_2: \begin{array}{l} a \longrightarrow \frac{1}{3} \\ b \longrightarrow 1 \\ c \longrightarrow \frac{1}{2} \end{array} \qquad g_3: \begin{array}{l} a \longrightarrow 1 \\ b \longrightarrow \frac{1}{2} \\ c \longrightarrow \frac{1}{3} \end{array}$$

Then the associated topology (X, i(F)) is the discrete topology on X. Therefore group of homeomorphisms is the group of all permutation of X. But the group of fuzzy homeomorphisms of (X,F) is the group of all even permutation of X.

But for a topologically generated fuzzy topological space we have

Theorem 1.2.19.

If (X,F) is topologically generated, then the group of fuzzy homeomorphisms of (X,F) and group of homeomorphisms of (X, i(F)) are the same.

Proof:

By theorem 1.2.17, group of fuzzy homeomorphisms is a subgroup of group of homeomorphisms of $(X, i(F))$. Conversely let θ be any homeomorphism of $(X, i(F))$. For every $f \in F$, $\theta(f) \in F$ since $\theta(f) = f o \theta^{-1}$ and $(f o \theta^{-1})^{-1} (\alpha,1] = \theta(f^{-1}(\alpha,1]) \in i(F)$. Also $\theta^{-1}(f) \in F$ since $\theta^{-1}(f) = f o \theta$ and $(f o \theta)^{-1}(\alpha,1] = \theta^{-1}(f^{-1}(\alpha,1]) \in i(F)$, $0 < \alpha \leqslant 1$. Thus θ is a fuzzy homeomorphism of (X,F), it being topologically generated.

Theorem 1.2.20.

If (X,F) is any finite fuzzy topological space such that the group of fuzzy homeomorphisms is the group of all even permutations of X, then the group of homeomorphisms of $(X, i(F))$ is the group of all permutations of X.

Proof:

Let G be the group of fuzzy homeomorphisms of (X,F) and H be the group of homeomorphisms of $(X, i(F))$. By theorem 1.2.17, G is a subgroup of H. But H cannot be equal to G since there is no topology T on X such that group of homeomorphisms is the group of all even permutation of X (Theorem 1.1.6 of [29]). Therefore H is the group of all permutations of X.

Definition 1.2.21.

A fuzzy topological space (X,F) is homogeneous if for any x,y in X, there exist a fuzzy homeomorphism Θ of (X,F) onto itself such that $\Theta(x) = y$.

Theorem 1.2.22.

If X is any set with $|X| \geq 4$ and F be a fuzzy topology on X such that the group of even permutations is a subgroup of the group of fuzzy homeomorphisms of (X,F), then (X,F) is homogeneous.

Proof:

Let $x_1, y_1 \in X$. Choose two more points $x_2, y_2 \in X$. Consider $p = (x_1,y_1)\ (x_2,y_2)$. It is a fuzzy homeomorphism since it is an even permutation of X and it maps x_1 to y_1. Hence (X,F) is homogeneous.

Definition 1.2.23.

A fuzzy topological space (X,F) is completely homogeneous if the group of fuzzy homeomorphisms of (X,F) coincides with the group of all permutations of X.

Theorem 1.2.24.

If X is any infinite set and F_α is a fuzzy topology defined by $F_\alpha = \{0\} \cup \{f \subset I^X \mid \text{Card } (X \setminus \sup(f) < \alpha\}$ for some infinite cardinal number α, $\alpha \leqslant |X|$. Then the group of fuzzy homeomorphisms of (X, F_α) is the group of all permutations of X.

Proof:

Let Θ be any permutation of X and $f \in F_\alpha$. Then clearly $\text{card}(X \setminus \sup(\Theta(f)) < \alpha$. That is $\Theta(f) \in F_\alpha$. Similarly we can show that $\Theta^{-1}(f) \in F_\alpha$. Thus Θ is a fuzzy homeomorphism.

Theorem 1.2.25.

Every completely homogeneous space is hereditarily homogeneous.

Proof:

Every completely homogeneous space is homogeneous, and every subspace of completely homogeneous space is completely homogeneous. Hence the result.

Remark 1.2.26.

If (X,F) is a fuzzy topological space such that the group of fuzzy homeomorphisms is the group of all even permutations of X, then (X,F) is hereditarily homogeneous.

Remark 1.2.27.

If X is any set such that card X $\leqslant$ C, then corresponding to each fuzzy subset g such that distinct elements in X have distinct membership values, we can generate completely homogeneous fuzzy topological space by taking subbasis as follows. Let $\{e_\alpha : \alpha \in \Lambda\}$ be the group of all permutations of X. Define fuzzy subset g_α for $\alpha \in \Lambda$ by $g_\alpha(x) = g(e_\alpha(x))$ for all x in X. Then $S = \{g_\alpha : \alpha \in \Lambda\}$ is a subbasis for a completely homogeneous fuzzy topology on X.

Remark 1.2.28.

Remark 1.2.27 gives a method to construct completely homogeneous fuzzy topological spaces. But, the problem of characterizing completely homogeneous fuzzy topological space remains yet to be solved.

Chapter-2

ČECH FUZZY CLOSURE SPACES

In this chapter we investigate some problems related to Čech fuzzy closure spaces. In chapter-3 of [29], Ramachandran determined completely homogeneous Čech closure spaces. He proved that a closure space (X,V) is completely homogeneous if and only if V is the closure operator associated with a completely homogeneous topology on X. In section 2 of this chapter we determine completely homogeneous Čech fuzzy closure spaces and extend some of the results discussed for the fuzzy topological spaces in the first chapter to Čech fuzzy closure spaces.

Lattice structure of the set of all Čech fuzzy closure operators on a fixed set X is discussed in section 3 of this chapter. We prove that the set of all Čech fuzzy closure operators forms a complete lattice and this lattice is not complemented.

2.1. Preliminaries

Notation 2.1.1.

I^X denotes the set of all fuzzy subsets of a set X.

Definition 2.1.2 [27]

A Čech fuzzy closure operator on a set X is a function $\gamma : I^X \longrightarrow I^X$ satisfying the following three axioms.

(1) $\gamma(o) = 0$

(2) $f \leqslant \gamma(f)$ for every f in I^X

(3) $\gamma(f \vee g) = \gamma(f) \vee \gamma(g)$

For convenience we call it a fuzzy closure operator on X. Also (X,γ) is called a fuzzy closure space.

Definition 2.1.3.

In a fuzzy closure space (X,γ), a fuzzy subset f of X is said to be closed if $\gamma(f) = f$. A fuzzy subset f of X is open if its complement is closed in (X,γ). The set of all open fuzzy subsets of (X,γ) forms a fuzzy topology on X called the fuzzy topology associated with the fuzzy closure operator γ.

Let F be a fuzzy topology on a set X. Then a function γ from I^X into I^X defined by $\gamma(f) = \overline{f}$ for every f in I^X, where $\overline{f}$ is the fuzzy closure of f in

(X,F), is a fuzzy closure operator on X called the fuzzy closure operator associated with the fuzzy topology F.

A fuzzy closure operator on a set X is called fuzzy topological if it is the fuzzy closure operator associated with a fuzzy topology on X.

Remark 2.1.4.

Note that different fuzzy closure operators can have the same associated fuzzy topology.

2.2. Group of fuzzy closure isomorphisms

Definition 2.2.1.

Let (X, Ψ) and $(Y, \emptyset)$ be fuzzy closure spaces. Then a one-one function Θ from X on to Y is called a fuzzy closure isomorphism if $\Theta(\Psi(f)) = \emptyset(\Theta(f))$ for every f in I^X.

Note that if Θ is a fuzzy closure isomorphism, then Θ^{-1} also is one such.

Also, the fuzzy closure isomorphisms of a fuzzy closure space (X, Ψ) onto itself form a group under the operation of composition of function which is called the group of fuzzy closure isomorphisms of the fuzzy closure space.

Definition 2.2.2.

A fuzzy closure space (X, Ψ) is called completely homogeneous if the group of fuzzy closure isomorphisms of (X, Ψ) coincides with the group of all of permutations of X.

Theorem 2.2.3.

A fuzzy topological closure space (X, Ψ) is completely homogeneous if and only if Ψ is the fuzzy closure operator associated with a completely homogeneous fuzzy topology on X.

Proof:

Necessary

Suppose (X, Ψ) is completely homogeneous we prove that the associated fuzzy topology F on X is completely homogeneous. Let θ be any permutation of X and g be a closed fuzzy set in (X,F). Since (X, γ) is completely homogeneous, θ is a fuzzy closure isomorphism and so we have $\theta(\gamma(f)) = \gamma(\theta(f))$ and $\theta^{-1}(\gamma(f)) = \gamma(\theta^{-1}(f))$ for every f in I^X. Then $\theta(g) = \theta(\gamma(g)) = \gamma(\theta(g))$ and $\theta^{-1}(g) = \theta^{-1}(\gamma(g)) = \gamma(\theta^{-1}(g))$ which implies that both $\theta(g)$ and $\theta^{-1}(g)$ are closed. Thus θ is a fuzzy homeomorphism of (X,F).

Sufficiency

Suppose γ is the fuzzy closure operator associated with a completely homogeneous fuzzy topology F on X. We claim that (X, γ) is completely homogeneous. Let θ be any permutation of X. Since θ is a fuzzy homeomorphism of (X,F) for any fuzzy subset f of X we have $\theta(\overline{f}) \leqslant \overline{\theta(f)}$ [28]

i.e., $\theta(\gamma(f)) \leqslant \gamma(\theta(f))$ (1)

Also $\theta^{-1}(\overline{\theta(f)}) \leqslant \overline{\theta^{-1}(\theta(f))}$

i.e., $\theta^{-1}(\gamma(\theta(f)) \leqslant \overline{f} = \gamma(f)$

i.e., $\gamma(\theta(f)) \leqslant \theta(\gamma(f))$ (2)

By (1) and (2) $\Theta(\gamma(f)) = \gamma(\Theta(f))$ for every f in I^X. Thus Θ is a fuzzy closure isomorphism.

Remark 2.2.4.

The following example shows the existence of non fuzzy topological completely homogeneous fuzzy closure spaces.

Let X be any set and $\alpha \in (0,1)$. Define $\gamma_\alpha : I^X \longrightarrow I^X$ by $\gamma_\alpha(o) = 0$, $\gamma_\alpha(f) = \alpha$ for every fuzzy subset $f \dot{<} \alpha$ and $\gamma_\alpha(g) = 1$ for any other fuzzy subset g. It can be easily checked that (X, γ_α) is non fuzzy topological and completely homogeneous.

Theorem 2.2.5.

The group of fuzzy closure isomorphisms of a fuzzy topological closure space (X,γ) is the group of all even permutations of X if and only if γ is a fuzzy closure operator associated with a fuzzy topology whose group of fuzzy homeomorphisms is the group of all even permutations of X.

Proof:

Necessary

Suppose that the group of fuzzy closure isomorphisms is the group of all even permutations of X. Let F be the associated fuzzy topology. We claim that the group of fuzzy homeomorphisms of (X,F) is the group of all even permutations of X. Let θ be any even permutation of X. Then we have $\theta(\gamma(f)) = \gamma(\theta(f))$ and $\theta^{-1}(\gamma(f)) = \gamma(\theta^{-1}(f))$ for every f in I^X. Let f be a closed fuzzy set in F. Then $\theta(f) = \theta(\gamma(f)) = \gamma(\theta(f))$ and $\theta^{-1}(f) = \theta^{-1}(\gamma(f)) = \gamma(\theta^{-1}(f))$. This shows that θ is a fuzzy homeomorphism of (X,F). Conversely suppose θ is a fuzzy homeomorphism of (X,F). Then for any fuzzy subset f in I^X we have

$$\theta(\overline{f}) \leqslant \overline{\theta(f)}$$

i.e. $\theta(\gamma(f)) \leqslant \chi(\theta(f))$ (1)

Also $\theta^{-1}(\overline{\theta(f)}) \leqslant \overline{\theta^{-1}(\theta(f))}$

i.e. $\theta^{-1}(\gamma(\theta(f))) \leqslant \overline{f} = \chi(f)$

i.e. $\gamma(\theta(f)) \leqslant \theta(\gamma(f))$ (2)

By (1) and (2) θ is a fuzzy closure isomorphisms, and so by hypothesis, an even permutation.

Sufficiency

Suppose γ is a fuzzy closure operator associated with a fuzzy topology F on X whose group of fuzzy homeomorphisms is the group of all even permutations of X.

Let θ be any even permutation of X. We claim that θ is a fuzzy closure isomorphism. Since θ is a fuzzy homeomorphism we have

$$\theta(\overline{f}) \leq \overline{\theta(f)}$$

i.e., $$\theta(\gamma(f)) \leq \gamma(\theta(f)) \qquad (1)$$

Also $$\theta^{-1}(\overline{\theta(f)}) \leq \overline{\theta^{-1}(\theta(f))}$$

i.e., $$\theta^{-1}(\gamma(\theta(f))) \leq \overline{f} = \gamma(f)$$

i.e., $$\gamma(\theta(f)) \leq \theta(\gamma(f)) \qquad (2)$$

By (1) and (2) $\theta(\gamma(f)) = \gamma(\theta(f))$ for every f in I^X. Thus θ is a fuzzy closure isomorphism. Conversely if θ is a fuzzy closure isomorphism and f is a closed fuzzy set in (X,F). Then

$$\theta(f) = \theta(\gamma(f)) = \gamma(\theta(f))$$

and

$$\theta^{-1}(f) = \theta^{-1}(\gamma(f)) = \gamma(\theta^{-1}(f))$$

This shows that θ is a fuzzy homeomorphism of (X,F), and so by hypothesis, θ is an even permutation of X.

Theorem 2.2.6.

The group of fuzzy closure isomorphisms of a fuzzy topological closure space (X, ψ) is the group generated by a cycle on X if and only if ψ is a fuzzy closure operator associated with a fuzzy topology whose group of fuzzy homeomorphisms is the group generated by a cycle on X.

Proof:

We can prove the theorem by the same argument as in the theorem 2.2.5.

2.3. Lattice of fuzzy closure operators

Notation 2.3.1.

L(X) denotes the set of all fuzzy closure operators on a fixed set X.

Definition 2.3.2.

Let ψ_1 and ψ_2 be fuzzy closure operators on L(X). Then define $\psi_1 \leqslant \psi_2$ if and only if $\psi_2(f) \leqslant \psi_1(f)$ for every f in I^X.

This relation is a partial order in the set L(X).

Definition 2.3.3.

The fuzzy closure operator D on X defined by $D(f) = f$ for every f in I^X is called the discrete fuzzy closure operator.

The fuzzy closure operator I on X defined by

$$I(f) = \begin{cases} 0 & \text{if } f = 0 \\ 1 & \text{otherwise} \end{cases}$$

is called the indiscrete fuzzy closure operator.

Remark 2.3.4.

Note that D and I are the fuzzy closure operators associated with the discrete and the indiscrete fuzzy topologies on X respectively.

Moreover D is the unique fuzzy closure operator whose associated fuzzy topology is discrete.

Also I and D are the smallest and the largest elements of L(X) respectively.

Theorem 2.3.5.

L(X) is a complete lattice.

Proof:

It is enough to show that every subset of L(X) has the greatest lower bound in L(X).

Let $S = \{\psi_j : j \in J\}$ be a subset of L(X).

Then $\psi = \underset{j \in J}{\text{Sup}}\{\psi_j\}$ is the greatest lower bound of S in L(X). (Note that ψ as defined here is clearly a fuzzy closure operator).

Definition 2.3.6.

A fuzzy closure operator on X is called an infra fuzzy closure operator if the only fuzzy closure operator on X strictly smaller than it is I.

Theorem 2.3.7.

There is no infra fuzzy closure operator in L(X).

Proof:

Let ψ be any fuzzy closure operator other than I. Then there exist at least one fuzzy subset g of X such that $\psi(g) \neq 1$. Then there exist at least one point x in X such that $\psi(g)(x) = \alpha$ for some $\alpha \in (0,1)$.

Define $\varphi : I^X \longrightarrow I^X$ by $\varphi(o) = 0$,

$$\varphi(f)(y) = \begin{cases} 1 \text{ if } y \neq x \text{ and also if } y = x \text{ with } \psi(f)(x) > \alpha \\ \beta \text{ for some chosen } \beta > \alpha \text{ if } y = x \text{ and } \psi(f)(x) \leq \alpha \end{cases}$$

Then clearly $\xi \neq I$ is a fuzzy closure operator strictlyy less than ψ. Thus for any fuzzy closure operator other than I we can find a fuzzy closure operator strictly smaller than it. Therefore L(X) has no infra fuzzy closure operator.

Remark 2.3.8.

Let X be any set and $x \in X$. Define ψ_x and ξ_x from $I^X \longrightarrow I^X$ by

$$\psi_x(o) = 0, \quad \psi_x(f)(y) = \begin{cases} f(y) & \text{if } y \neq x \\ 1 & \text{if } y = x \end{cases}$$

$$\xi_x(o) = 0, \quad \xi_x(f)(y) = \begin{cases} 1 & \text{if } y \neq x \\ f(x) & \text{if } y = x \end{cases} \quad \text{for every } f \text{ in } I^X.$$

Then ψ_x and ξ_x are fuzzy closure operators on X.

Theorem 2.3.9.

ψ_x and ξ_x are complements to each other.

Proof:

Clearly $\psi_x \wedge \xi_x = I$. We claim that $\psi_x \vee \xi_x = D$. Suppose not. That is $\psi_x \vee \xi_x = \psi$, a fuzzy closure operator other than D. Then there exist at least one fuzzy subset f such that $\Psi(f) \neq f$. Then there exist at least one point y in X such that $\Psi(f)(y) > f(y)$. This is a contradiction to the assumption that $\Psi_x \vee \xi_x = \Psi$. Therefore $\Psi_x \vee \xi_x = D$.

Theorem 2.3.10.

The fuzzy closure operator Ψ_α defined in remark 2.2.4 has no complement.

Proof:

Clearly I and D are not complements of Ψ_α. Let Ψ be any fuzzy closure operator other than I and D. We claim that Ψ is not a complement of Ψ_α. Define $\xi : I^X \longrightarrow I^X$ by $\xi(f)(x) = \text{Sup}\{\Psi_\alpha(f)(x), \Psi(f)(x)\}$ for every f in I^X. Then ξ is a fuzzy closure operator, $\xi = \Psi_\alpha \wedge \Psi$ and $\xi \neq I$.

Chapter-3

LATTICE OF FUZZY TOPOLOGIES

In 1936 G. Birkhoff [4] proved that the family of all topologies on a set forms a complete lattice under the natural order of set inclusion. The least upper bound of a collection of topologies on a set X is the topology which is generated by their union and greatest lower bound is their intersection. The atoms in the lattice $\Sigma(X)$ of all topologies on a set X are of the form $\{\emptyset, A, X\}$, where $A \subset X$. In 1964, O. Frölich [9] characterized the dual atoms called ultratopologies as $\mathcal{P}(X \setminus \{a\}) \cup \mathcal{U}$, where $a \in X$ and $\mathcal{U}$ is an ultrafilter on X. Also, $\Sigma(X)$ is atomic as well as dually atomic, but not modular if $|X| > 2$. In 1966, A.K. Steiner [34] proved that the lattice of topologies on an arbitrary set is complemented.

In this chapter we investigate the lattice structure of the set of all fuzzy topologies on a given set X. For a given topology t on X, we study properties of the lattice $\mathcal{F}_t$ of fuzzy topologies defined by families of lower semicontinuous functions with reference to t on X.

We prove that the set of all fuzzy topologies forms a complete atomic lattice under the natural order of set inclusion. This lattice is not complemented and has no dual atoms. Also we prove that topologically generated fuzzy topologies and crisp topologies have complements.

3.1. Preliminaries

Definition 3.1.1.

Let (X,T) be a topological space, a function $f:X \longrightarrow [0,1]$ is lower semicontinuous if $f^{-1}(\alpha,1]$ is open in X for $0 < \alpha \leq 1$ [38].

Notations 3.1.2.

If F is a fuzzy topology on the set X, let F_c denote the 0-1 valued members of F, that is F_c is the set of all characteristic maps in F. Then F_c is a fuzzy topology on X. Throughout this chapter if $A \subset X$, let μ_A denote the characteristic map of the subset A, that is, $\mu_A(x) = 1$ if $x \in A$ and $\mu_A(x) = 0$ if $x \in X \setminus A$. Define $F_c^* = \{A \subset X \mid \mu_A \in F_c\}$. The fuzzy space (X,F_c) is then the "same" as the topological space (X,F_c^*).

Definition 3.1.3.

A fuzzy space (X,F) is said to be an induced fuzzy space or an induced space [26], provided that F is the collection of all lower semicontinuous maps from (X,F_c*) into [0,1].

Remark 3.1.4.

Induced spaces are equivalent to topologically generated spaces of Lowen [1.2.16].

Definition 3.1.5.

A fuzzy space (X,F) is said to be a weakly induced fuzzy space (according to H.W. Martin [26]) provided that whenever $g \in F$, then $g: (X, F_c*) \longrightarrow [0,1]$ is a lower semicontinuous map.

It is observed that every induced space is weakly induced and every topological space, regarded as a fuzzy space via the fuzzy topology of all characteristic maps of the open sets, may be thought of as weakly induced space.

Theorem 3.1.6 (H.W. Martin [26])

A fuzzy space (X,F) is an induced space iff (X,F) is a weakly induced space such that every constant map from X into [0,1] belongs to F.

Remark 3.1.7.

For a given topology T on X, the set consisting of all constant functions from (X,T) to [0,1] and characteristic functions corresponding to open sets in T forms a subbasis for W(T) = $\{$ f|f is a lower semicontinuous from (X,T to [0,1]$\}$.

3.2. Lattice of fuzzy topologies

For a given topology t on X, the family $\mathcal{F}_t$ of all fuzzy topologies defined by families of lower semicontinuous function from (X,t) to [0,1] forms a lattice under the natural order of set inclusion. The least upper bound of a collection of fuzzy topologies belonging to $\mathcal{F}_t$ is the fuzzy topology which is generated by their union and greatest lower bound is that which is their intersection. The smallest and largest elements in $\mathcal{F}_t$ are denoted by 0_t and 1_t respectively.

Theorem 3.2.1.

$\mathcal{F}_t$ is complete.

Proof:

Let $S = \{ F_\lambda : \lambda \in \Lambda \}$ be a subset of $\mathcal{F}_t$ and $G = \bigcap_{\lambda \in \Lambda} F_\lambda$. Then $0,1 \in G$ since $0,1 \in F_\lambda, \forall \lambda \in \Lambda$. Let $a,b \in G$, then $a,b \in F_\lambda \ \forall \lambda \in \Lambda$. Then $a \wedge b \in F_\lambda \ \forall \lambda \in \Lambda$. Therefore $a \wedge b \in G$. Let $\{ a_j : j \in J \}$ be a subset of G. Then $\bigvee_{j \in J} a_j \in F_\lambda \ \forall \lambda \in \Lambda$. Therefore $\bigvee_{j \in J} a_j \in G$. Therefore $G \in \mathcal{F}_t$ and is the greatest lower bound of S in $\mathcal{F}_t$. Hence $\mathcal{F}_t$ is complete since it has the largest element consisting of all lower semi-continuous functions.

Theorem 3.2.2.

$\mathcal{F}_t$ is atomic.

Proof:

Atoms in $\mathcal{F}_t$ are of the form $T(a) = \{0,1,a\}$. Let S be any element of $\mathcal{F}_t$ different from 0_t. Then $S = \bigvee_{a \in S} T(a)$. Hence $\mathcal{F}_t$ is atomic.

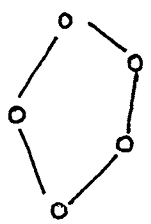
Theorem 3.2.3.

$\mathcal{F}_t$ is not modular.

Proof:

Choose distinct points α_1, α_2, α_3 $\in (0,1)$. Then fuzzy topologies $\{0,1\}$, $\{0,1,\alpha_1\}$, $\{0,1,\alpha_2\}$, $\{0,1,\alpha_2,\alpha_3\}$, $\{0,1,\alpha_1,\alpha_2,\alpha_3\}$ forms a sublattice of $\mathcal{F}_t$ isomorphic to

N_5 =



. Hence the result.

Theorem 3.2.4.

There is no dual atom in $\mathcal{F}_t$.

Proof:

Let $F \in \mathcal{F}_t$ such that $F \neq 1_t$; we prove that there is an element $G \in \mathcal{F}_t$ such that $F < G \neq 1_t$. By remark 3.1.7. F cannot contain at the same time all constant functions as well as all characteristic functions corresponding to open sets in t. Let S be the collection of lower semicontinuous functions which do not belong to F. First we claim that S is infinite. Two cases may arise.

Case-1.

S contains constant functions. Let constant function $\alpha \in S$. Then there exist a constant function $\beta < \alpha$ such that $\beta \in F$ and the elements of the set $K = \{\gamma \mid \beta < \gamma < \alpha\}$ of constant functions do not belong to F. Then by denseness property of real numbers S is infinite.

Case-II.

If S contains a characteristic function corresponding to an open set A in t, then there exists a lower semi-continuous function $f_A^{\alpha} \in F$ where f_A^{α} is defined by

$$f_A^{\alpha}(x) = \begin{cases} \alpha & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases} . \quad 0 \leqslant \alpha < 1$$ and the elements of the set $K = \left\{ f_A^{\beta} \mid \alpha < \beta < 1 \right\}$ of lower semicontinuous function do not belong to F. Again by denseness property of real numbers S is infinite.

Let g be a lower semicontinuous function belonging to S. Then $F < F \vee \{0, 1, g\} = G < 1_t$ where $g \notin F$.

3.3. Complementation in the lattice of fuzzy topologies

Theorem 3.3.1.

Atoms in $\mathcal{F}_t$ of the form $\{0,1,a\}$ where a is a lower semicontinuous function different from characteristic function have no complement in $\mathcal{F}_t$.

Proof:

We divide the proof into two parts.

Part-a.

Suppose $F = \{0,1,\alpha\}$ is an atom where α is a constant function, $\alpha \in (0,1)$. We claim that F has no complement. Clearly 1_t is not a complement of F. Let G be any fuzzy topology in $\mathcal{F}_t$ other than 1_t. If constant function $\alpha \in G$, then G is not a complement of F since $G \wedge F \neq 0_t$. Suppose constant function α does not belong to G. Then there exists a constant function $\beta < \alpha$ such that $\beta \in G$ and the set $K = \{\gamma \mid \beta < \gamma < \alpha\}$ of constant functions is not contained in G. Now $\mathcal{B} = \{f \wedge g \mid f \in F,\ g \in G\}$ is a base for $H = F \vee G$. Then K is not contained in H. Hence G is not a complement of F.

Part-b.

Suppose $F = \{0, 1, a\}$ is an atom where a is a lower semicontinuous function which is neither a constant function nor a characteristic function. We claim that F has no complement. Clearly 1_t is not a complement of F. Let G be any fuzzy topology in $\mathcal{F}_t$ other than 1_t. Then G cannot contain at the same time all constant functions as well as characteristic functions corresponding to open sets in t since they together generate 1_t (Remark 3.1.7). Two cases arise.

Case-1.

G does not contain all constant functions. If $a \in G$ then G is not a complement of F since $F \wedge G \neq 0_t$. Suppose $a \notin G$. Then there exists a constant function $\beta < \alpha$ such that $\beta \in G$ and the set $K = \{\gamma \mid \beta < \gamma < \alpha\}$ of constant functions is not contained in G. Then G is not a complement of F.

Case-II.

G contains all constant functions. Then at least one characteristic function μ_A corresponding to an open set A in t does not belong to G. If $a \in G$, then G is not a complement of F. Suppose $a \notin G$. Then we can find a set $K = \{ f_A^{\alpha} \mid \alpha < 1 \}$ of lower semi-continuous functions defined by $f_A^{\alpha}(x) = \begin{cases} \alpha & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$ such that K is not contained in G. Now consider the base $\mathcal{B} = \{ f \wedge g \mid f \in F,\ g \in G \}$ of $H = F \vee G$. Then K is not contained in H. Thus G is not a complement of F.

Remark 3.3.2.

When t = d, discrete topology on X, $\mathcal{F}_d$ becomes the lattice of all fuzzy topologies on X. Thus the family of all fuzzy topologies on a set forms a complete atomic lattice under the natural order of set inclusion. This lattice is not even modular. It is not complemented and has no dual atoms.

Let Σ denote the family of fuzzy topologies defined by lower semicontinuous function where each lower semi-continuous function is a characteristic function. Then Σ is a sublattice of $\mathcal{F}_d$ and is lattice isomorphic to the lattice of all topologies on X. The elements of Σ are the crisp topologies.

Theorem 3.3.3.

Every topologically generated or crisp topology has complement in $\mathcal{F}_d$.

Proof:

Let F be a topologically generated fuzzy topology. Then there exists a topology T on X such that W(T) = F. Since the lattice of topologies is complemented there exists at least one complement S of T. Then identify S with its characteristic function let it be denoted by F_S. Then F and F_S are complements of each other.

Remark 3.3.4.

The converse of theorem 3.3.3 is not true as shown by the following counter example. Let X be any set and F be the fuzzy topology which consists of constant function α where $\alpha \in [0 \; \frac{1}{2}]$ and the constant function 1. Let F' be the fuzzy topology generated by constant functions α where $\alpha \in (\frac{1}{2}, 1]$ together with all characteristic functions. Then F and F' are complements to each other.

Remark 3.3.5.

If we define an equivalance relation $\sim$ by $F_1 \sim F_2$ if and only if $i(F_1) = i(F_2)$. Then $(\mathcal{F}_d|\sim ,\leqslant)$ is lattice isomorphic to $(\Sigma, \leqslant)$.

Consider $Z = \bigcup_{t \in \Sigma} (t \times \mathcal{F}_t)$ and define $(t,F) \sim (s,G)$ if and only if $F = G$. Then

Theorem 3.3.6.

$Z|\sim$ is a complete lattice, isomorphic to the lattice of all fuzzy topologies on X.

Proof:

$$[(t,F)] \leqslant [(s,G)] \text{ if and only if } F \leqslant G$$

$$[(t,F)] \wedge [(s,G)] = [(d, F \wedge G)]$$

$$[(t,F)] \vee [(s,G)] = [(d, F \vee G)]$$

Then $(Z|\sim , \leqslant)$ is a lattice.

If S is any subset of $Z|\sim$ we can show that S has a greatest lower bound in $Z|\sim$. Therefore $Z|\sim$ is complete.

Define $\Theta : (\mathcal{F}_d, \leqslant) \longrightarrow (Z|\sim, \leqslant)$ by $\Theta(F) = [(d,F)]$.

Then Θ is one-one and on to.

$F_1 \leqslant F_2$ in $\mathcal{F}_d \Longrightarrow \Theta(F_1) \leqslant \Theta(F_2)$ in $Z|\sim$

$$\begin{aligned} \Theta(F_1 \vee F_2) &= [\ (d,\ F_1 \vee F_2)\ ] \\ &= [\ (d,F_1)\ ] \vee [\ (d,F_2)\ ] \\ &= \Theta(F_1) \vee \Theta(F_2) \end{aligned}$$

$$\begin{aligned} \Theta(F_1 \wedge F_2) &= [\ (d,\ F_1 \wedge F_2)\ ] \\ &= [\ (d,F_1)\ ] \wedge [(d,F_2)\ ] \\ &= \Theta(F_1) \wedge \Theta(F_2) \end{aligned}$$

Remark 3.3.7.

If we define $\sim$ in Z by $(t,F) \sim (s,G)$ if and only if $t = s$, then $Z|\sim$ is a complete lattice isomorphic to the lattice of all topologies on X.

Chapter-4

L-FUZZY TOPOLOGICAL SPACES

In this chapter we replace the membership lattice [0,1] by an arbitrary complete and distributive lattice L, and extend some of the results of chapters one and two to L-fuzzy topological spaces. The smallest and largest element of L are denoted by 0 and 1 respectively. The terminology used in this chapter are obvious analogies of the corresponding ones in the previous chapters. For example, (1) an L-fuzzy topology on X is a set of L-fuzzy subset of X which contains constant function 0 and 1 and is closed under finite infima and arbitrary suprema, (2) an L-fuzzy closure operator on a set X is a function $\Psi : L^X \longrightarrow L^X$ satisfying the following three axioms:

(i) $\Psi(0) = 0$

(ii) $f \leqslant \Psi(f)$ for every f in L^X

(iii) $\Psi(f \vee g) = \Psi(f) \vee \Psi(g)$

Proof of the results presented in this chapter are similar to those in the previous chapters. As a model we give the proof of first theorem.

4.1. Group of L-fuzzy homeomorphisms

For any L-fuzzy topological space (X,F), the set of all L-fuzzy homeomorphisms of (X,F) on to itself forms a group.

Theorem 4.1.1.

Let X be any set such that cardinality of X $\leq$ cardinality of L. Then the subgroup of S(X), the group of permutations of X, generated by any cycle can be represented as the group of L-fuzzy homeomorphisms for some L-fuzzy topology on X.

Proof:

Consider a finite subset $\{a_1, a_2, \ldots, a_n\}$ of X. Let H be the subgroup generated by the cycle $p = (a_1, a_2, \ldots, a_n)$. Then clearly $H = \{p, p^2, \ldots, p^n\}$. Consider an L-fuzzy subset g of X such that distinct elements in X have distinct membership values. Assume that the membership values of $a_1, a_2, \ldots, a_n$ respectively are $\alpha_1, \alpha_2, \ldots, \alpha_n$ where we can assume that $\alpha_1 = 0$ and $\alpha_2 = 1$. Define L-fuzzy subsets $g_1, g_2, \ldots, g_n$ by $g_m(x) = g(p^m(x))$ for every x in X. Let F be the L-fuzzy topology generated by $S = \{g_1, g_2, \ldots, g_n\}$ as a subbasis.

Now we claim that the group of L-fuzzy homeomorphisms of (X,F) is H. Every element of H is a fuzzy homeomorphism since

$$p^{m^{-1}}(g_i) = g_i p^m = gp^i p^m = gp^k = g_k \text{ for some } k, \quad 1 \leqslant k \leqslant n$$

and $$p^m(g_i) = g_i p^{m^{-1}} = gp^i\, p^{m^{-1}} = g(p^i\, p^{m^{-1}}) = gp^\ell = g_\ell \quad \text{for some } \ell\ ,\ 1 \leqslant \ell \leqslant n.$$

Conversely if θ is different from p^m, then $\theta^{-1}(g_i) = g_i\theta = g(p^i\theta) = h$, an L-fuzzy subset which has distinct membership values for distinct points and $h \notin S$. We prove that h is not open. Consider the meet of any pair of elements in S. Let $f = g_1 \wedge g_2$; then f has at least two points with the same membership value 0. Similarly the join of any pair of elements in S have at least two points with the same membership value 1. Repeating the process, we see that any member different from subbasic open fuzzy set in F, L-fuzzy topology generated by S, has at least two points with the same membership value while $\theta^{-1}(g_i)$ has all values different. That is $\theta^{-1}(g_i) \notin F$. Thus the group of L-fuzzy homeomorphisms of (X,F) is H.

Theorem 4.1.2.

Let X be any set such that Card X $\lneq$ Card L. Then the group generated by an infinite cycle can be represented as the group of L-fuzzy homeomorphisms for some L-fuzzy topology on X.

Definition 4.1.3.

An L-fuzzy closure space (X, Ψ) is called completely homogeneous if the group of L-fuzzy closure isomorphisms of (X, Ψ) coincides with the group of all permutations of X.

Theorem 4.1.4.

An L-fuzzy topological closure space (X, Ψ) is completely homogeneous if and only if Ψ is the L-fuzzy closure operator associated with a completely homogeneous L-fuzzy topology on X.

Theorem 4.1.5.

The group of L-fuzzy closure isomorphisms of an L-fuzzy topological closure space (X, Ψ) is the group generated by a cycle on X if and only if Ψ is an L-fuzzy closure operator associated with an

L-fuzzy topology whose group of L-fuzzy homeomorphisms is the group generated by a cycle on X.

<u>Remark</u>:

The results of chapter 3 also have similar analogous results in L-fuzzy set up- we are not stating them here.

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Section: B - Those which were consulted during the investigation.

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