On s-Numbers and Semi - pseudo - s-Numbers of Operators

Thesis submitted to

THE COCHIN UNIVERSITY OF SCIENCE AND TECHNOLOGY
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY
UNDER THE FACULTY OF SCIENCE

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MARCH -1999

CERTIFICATE

This is to certify that the work done by Chithra.A.V. which is reported in the synopsis of her thesis entitled " **On s-Numbers and Semi - pseudo - s - Numbers of Operators**" is original and has not been submitted elsewhere for the award of a degree.

Cochin - 22
11- 3- 1999

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CONTENTS

CHAPTER I

INTRODUCTION

The concept of s-numbers of operators originated in the study of integral operators by E.Schmidt in 1907 and F.Smithies in 1937. Let T be a compact operator on a complex Hillbert space H and $(T^*T)^{1\backslash 2}$ be the positive square root of T^*T.
Let $\{ \lambda_n ([T^* T]^{1\backslash 2})\}$ be the sequence of eigen values of $(T^*T)^{1\backslash 2}$ written in the decending order, counting multiplicity. Then the n^{th} singular value of T denoted by $s_n(T)$ is $\lambda_n ([T^*T]^{1\backslash 2})$.

It is well known that $s_n(T)$ can be computed using Min-Max principle .An important usage of singular values for compact operators is the singular value decomposition [15] .

In the finite dimensional case, the singular value decomposition leads to the following factorisation of a given $n \times n$ matrix A ;

$$A = U A_d V \qquad \text{Where}$$

$$A_d = \begin{bmatrix} s_1 & & 0 \\ & s_2 & \\ 0 & & \ddots s_n \end{bmatrix}$$,U and V are $n \times n$ unitary matrices and

$s_1, s_2, \ldots s_n$ are the singular values of A.

In the infinite dimensional case the singular value decomposition of a compact operator A on a Hilbert space H, leads to the following factorisation of A.

$A = U A_d V$ where A_d is the diagonal operator with $s_n(A)$ as the n$^{\text{th}}$ diagonal entry, $U: H \to l_2$ and $V: l_2 \to H$ are bounded linear operators such that $U U^*$ and V^*V are identity operators on l_2 .

APPROXIMATION NUMBERS

DEFINITION .

Let T be a bounded linear transformation from a Banach space X to another Banach space Y. then the n^{th} approximation number $a_n(T)$ of T is defined as

$$a_n(T) = \inf \{ \|T - L\| : L \in B(X,Y), rank L < n \}$$

where $B(X,Y)$ denote the class of all bounded linear transformation of X to Y.

When X and Y are complex Hilbert spaces, approximation number measures the compactnes of T in $B(X,Y)$ in the following sense.

The compact if and only if $\lim_{n\to\infty} a_n(T) = 0$. If S and T are compact operators and if $a_n(S) \to 0$ faster than $\{a_n(T)\}$, then one could say that S is more compact than T.

KOLMOGOROV NUMBERS

For every operator $S \in B(E,F)$ the Kolmogorov numbers are defined by

$$d_n(S) = \inf\{\|Q_N^F S\| : \dim(N) < n\}$$

where Q_N^F is the canonical map of F onto the quotient space E/M.

GELFAND NUMBERS

For every operator $S \in B(E,F)$ the Gelfand numbers are defined by

$$c_n(S) = \inf\{\|SJ_M^E\| : co\dim(M) < n\}$$

where J_M^E is the embedding map of a subspace M into E.

These are some of the well known s-numbers. In 1974 Albrecht Pietsch [29] developed an axiomatic theory of s-numbers. The axiomatic definition is as follows.

Let T be in $B(X,Y)$ and let $(s_n(T))$ be a unique sequence of numbers associated with T such that

1) $\|T\| = s_1(T) \geq s_2(T) \geq \ldots \geq s_n(T) \geq \ldots$

2) $s_n(S+T) \leq s_n(S) + \|T\|$, T, S in B(X,Y)

3) $s_n(RST) \leq \|R\| s_n(S) \|T\|$, where $T \in B(X_0, X), S \in B(X,Y)$ and $R \in B(Y, Y_0)$.

where X_0 and Y_0 are Banach spaces.

4) Rank$(T) < n$ implies $s_n(T)$ 0.

5) Dimension $X \geq n$ implies $s_n(I) = 1$.

Ultimately it is known that if X and Y are Hilbert spaces then every s-numbers concides with the approximation numbers[31].

When X=Y=H a Hilbert space , the following description of approxiation numbers is well known.

ESSENTIAL SPECTRUM

For T in $B(H)$ the essential spectrum $\sigma_e(T)$ is $\bigcap_{k\in K(H)} \sigma(T+K)$ where $K(H)$ denotes the set of all compact operators on H.

For T in B(H) with T*=T let $\mu_1, \mu_2, \ldots \mu_N$ be the eigenvalues of finite multiplicity above $\sigma_e(T)$. Then

$$a_n(T) = \mu_n, n = 1,2,3 \ldots N$$

$$= \mu_N, n \geq N+1 \text{ when N is finite.}$$

Otherwise $a_n(T) = \mu_n, n = 1,2,3,\ldots$

In fact it is known that $\mu = \lim_{n\to\infty} \mu_n$ is the least upper bound of $\sigma_e(T)$ [15].

This description turns out be very important spectral theory point of view.

DEGREE OF A BOUNDED LINEAR OPERATORS[1]

Definition.

Let $\{H_n\}$ be an increasing sequence of finite dimensional subspaces of a complex Hilbert space H such that $\cup H_n$ is dense in H. For T in B(H) degree of T, denoted by deg(T) is defined as

$$\deg(T) = \sup_n rank(TP_n - P_nT)$$

ARVESONS CLASS

Let A denote the class of all T in B(H) such that

$$T = \sum_1^\infty A_k, \text{ where } A_k \in B(H) and \deg(A_k) < \infty \text{ such that}$$

$$\|T\|_k = \sum_1^\infty (1 + \deg(A_k)^{\frac{1}{2}})\|A_k\| < \infty.$$

Then Arvesson shows that if A is in $\mathcal{A}$ and self adjoint then the essential spectrum of A can be computed linear algebrically [1].This work of Arveson is used in chapter III to find lower bounds for certain types of positive operators on Hilbert spaces.

A.Pietsch [30] introduced the concept of pseudo-s-function axiomatically, which satisfies only the first three axiom of an s-function. The so called entropy numbers are the prime examples of pseudo-s-function. A.Pietsch has contributed enormously to the theory of entropy numbers in connection with the theory of operator ideals [30].

SUMMARY OF THE THESIS

In the second chapter the concept of semi-pseudo-s-numbers is introduced axiomatically. This is motivated from the study of operators on the space of operators especially elementary operators on B(H) when H is a complex Hilbert space. Just like approximation numbers, the so called V-numbers are introduced in this chapter measures the strength of compactness of elementary operators. Other examples based on concepts like index, degree, trace, nullity and co-rank are also given in this chapter.

The third chapter is devoted to computation of approximation numbers. This leads determination of bounds for essential spectra of certain types positive operators in $B(H)$ where H is a separable Hilbert over C. Through a diagrom it is illustrated that how the computation can be implemented algorithemically.

The fourth and final chapter deals with closed linear operators between complex Banach spaces.

The aim is to extend the notion of s-numbers to a class of closed linear operators which includes the bounded ones, preferably to the whole class of closed linear operators. This chapter is divided into two sections. In the first section the so called β and β' numbers are introduced using Kato's notion of gap of operators. In the second section s' numbers are studied for a class of closed linear transformation using the well known relative boundedness of Kato [20].

Finally s-number sets are defined for every closed linear transformation, again using relative boundedness of operators. It is observed that for bounded linear operators, the corresponding s-number sets are singleton sets consisting of approximation numbers.

CHAPTER II

SEMI-PSEUDO-s-NUMBERS

The concept of semi-pseudo-s-numbers of bounded linear operators between complex Banach spaces is introduced, axiomatically. This concept arise naturally when the Banach space under consideration is the Banach space $B(X)$ of all bounded linear operators on a Banach space X, with supreum norm. More specifically when one approximate bounded linear operators on $B(X)$, by bounded linear operators Λ_n on $B(X)$ such that rank(Δ_n)< n and rank(Δ_n (T))< n for all T and estimate the error involved in it, one gets semi-pseudo-s-numbers. Of course this is the prime example that is studied in this chapter. Various examples based on concepts like index, degree, trace, nullity etc. are also given.

Let us recall the definition [Chapter I]

Definition. A map s which assigns to every bounded linear operator T from a complex Banach space X to a complex Banach space Y a unique sequence of numbers denoted by $\{s_n(T)\}_{n=1,2,3...}$ such that

1. $\|T\| = s_1(T) \geq s_2(T) \geq \ldots$; and

2. $s_n(S+T) \leq s_n(S) + \|T\|$ for every S, T in $B(X,Y)$

is called a semi-pseudo-s-function.

It is to be mentioned that this is an extension of the pseudo-s-function introduced by A. Pietsch [30], which is a generalisation of the abstract s-function introduced by Pietsch himself. It is also clear from axiom (2) that the semi-pseudo-s-function is continous with respect to the norm topology of operators. Throughout this chapter X and Y will denote complex Banach spaces and $B(X,Y)$ the class of all bounded linear transformations from X to Y. Now what follow are various examples and their properties.

Examples

2.1. V-numbers

For each Φ in $B(B(X),B(Y))$, let

$$V_n(\Phi) = \inf\{\|\Phi - L\| : L \in B(B(X),B(Y)), rank(L) < n \quad and \quad rankL(T) < n \ \ for$$

all T in $B(Y)\}$.

Theorem 2.1.1.

The map $\Phi \to \{V_n(\Phi)\}$ is a semi-pseudo-s-function on $B(B(X),B(Y))$.

Proof.

$$V_1(\Phi) = \|\Phi\|$$

$$V_{n+1}(\Phi) = \inf\{\|\Phi - L\| : rankL < n+1 \, and \, rank \, L(T) < n+1 \forall T \}$$

$$\leq \inf\{\|\Phi - L\| : rankL < n \ and \ rankL(\mathrm{T}) < n \forall T \}$$

$$= V_n(\Phi)$$

Now

$$V_n(\Phi + \Psi) = \inf\{\|\Phi + \Psi - L\| : rankL < n \ and \ rankL(T) < n \}$$

$$\leq \inf\{\|\Phi - L\| : rankL < n \, and \, rankL(T) < n \} + \|\Psi\|$$

$$\leq V_n(\Phi) + \|\Psi\|, \Phi, \Psi \ \text{ in } B(B(X),B(Y))$$

This completes the proof.

Proposition 2.1.2.

The map $\Phi \to \{V_n(\Phi)\}$ is not a pseudo-s-function.

Proof.

Let R, S, Q be in $B(B(H))$ be as follows. $Q = I$, the identity operator. Let L in $B(B(H))$ be such that rank$L < n$ and let P be a projection of rank $< n$. Now define $S(T) = PL(T)P$, T in $B(H)$.

For a nonzero continuous linear functional ϕ on $B(H)$, let $R(T) = \phi(T).I$, T in $B(H)$ where I is the identity operator on the Hilbert space H.

Observe that, rank $RS = 1$, rank $R(S(T)) = +\infty$

Hence $V_n(RS) \neq 0$, but $V_n(S) = 0$ for n >1

Hence $V_n(RSQ) \nleq \|R\| V_n(S) \|Q\|$

Thus $\Phi \rightarrow \{ V_n(\Phi) \}_{n=1,2,...}$ is not a pseudo-s-function.

Remark 2.1.3.

The above theorem shows that operators on the spaces of operators have to be treated seperatily and deserves a special status. The well - known theory of completely positive maps and the theory of elementary operators suggest the importance of studying operators on operators [26].

Recall that if $\{a_n(T)\}$ is the sequence of approximation numbers for T in $B(X,Y)$ then $a_n(T) = 0$ if and only if rank $(T) < n$. Also, if X and Y are separable Banach spaces with Schauder basis, then T is compact if and only if $\lim_{n\to\infty} a_n(T) = 0$

Analogously, the following observation can be made for V-numbers also. Clearly $V_n(\Phi) = 0$ if and only if rank $(\Phi) < n$ and rank$(\Phi(T)) < n$. As before, $\lim_{n\to\infty} V_n(\Phi) = 0$ implies that Φ is compact and $\Phi(T)$ is compact for every T in $B(X,Y)$.

[26] Recalling the definition of elementary operator, a linear map Δ: $B(X) \rightarrow B(X)$ is called elementary if there are $2n$ operators $A_1, A_2, \ldots A_n, B_1, B_2, \ldots B_n$ in $B(X)$ such that $\Delta(T) = \sum_1^n A_i T B_i$, $T \in B(X)$. It is known that Δ is compact if and only if $A_1, A_2, \ldots A_n$, $B_1, B_2, \ldots B_n$ are all compact, provided $\{A_1, A_2, \ldots A_n\}$ and $\{B_1, B_2, \ldots B_n\}$ are linearly independent sets. Thus when Δ is compact $\Delta(T)$ is also compact for every T in $B(X)$. Hence when X is a complex Banach space with schauder basis, by approximating coefficient operators by finite rank bounded linear operators, one gets the following result.

Theorem 2.1.4.

Let X be a complex Banach space with Schauder basis. Then an elementary operator Δ on $B(X)$ is compact if and only if $\lim_{n\to\infty} V_n(\Delta) = 0$

The following example shows that the above semi-pseudo-s-function doesn't satisfy the fourth axiom of s-function. That is, rank $(\Phi) < n$ doesn't imply that $V_n(\Phi) = 0$.

Example 2.1.5.

Let ϕ be a nonzero bounded linear functional on $B(X)$. For T in $B(X)$, put $\Phi(T) = \phi(T).I$, where I is the identity operator on X. Then for $n > 1$, rank $\Phi < n$, but $V_n(\Phi) \neq 0$.

Theorem 2.1.6.

V- numbers satisfy the fifth axiom of s- function namely dimension $B(X) \geqslant n$ implies that $V_n(I) = 1$ where I is the identity operator on $B(X)$.

Proof.

Clearly $V_n(I) \leq 1$. Now $V_n(I) < 1$ implies the existence of an operator Φ on $B(X)$ such that rank $\Phi < n$ and rank $\Phi(T) < n$ for all T in $B(X)$ and $\|I - \Phi\| < 1$.

But this means that Φ is invertible which is not true. Hence the result.

Remark 2.1.7.

It is trivial to see that $V_n(\lambda\Phi) = |\lambda|\ V_n(\Phi)$ for every complex number λ and Φ in $B(B(X),B(Y))$. Now a study of some of the properties of approximation numbers like additivity, injectivity and surjectivity is carried out for V-numbers. The proof of the following proposition is exactly the same as that of approximation numbers [30].

Proposition 2.1.8.[30]

V-numbers are additive. That is, for every pair of positive integers

$V_{m+n-1}(\Phi) \leq V_m(\Phi) + V_n(\Phi)$, Φ in $B(B(X),B(Y))$

Next recall the definition of metric injection and the associated injectivity of s-function.

Definition 2.1.9. [30]

J in $B(X,Y)$ is called a metric injection if $\|J(x)\| = \|x\|$. Semi-pseudo-s-function s is called injective if $s_n(JT) = s_n(T)$ for all J, metric injection J in $B(X,Y)$ and for all T in $B(X_0,X)$.

The following example shows that V-numbers are not injective.

Example 2.1.10.

Consider the Banach spaces X_0, X_1 and X_2 defined as follows.

$X_0 = X$, $X_2 = Y \oplus Y$ and $X_1 = Y$ where X and Y are Hilbert spaces. Here $Y \oplus Y$ is given the maximum norm namely,

$$\|x \oplus y\| = \max\{(\|x\|, \|y\|), x, y \in Y\}$$

Let $L : B(X_0) \to B(X_1)$ be a bounded linear operator with rank$< n$, and P an orthogonal projection on Y where rank$< n$.

Let $\Phi : B(X_0) \to B(X_1)$ be defined by

$$\Phi(T) = PL(T)P,\ T \in B(X_0).$$

Then rank $\Phi < n$ and rank $\Phi(T) < n$ for every T in $B(X_0)$

For a bounded linear functional (nonzero) on $B(Y)$ such that $\|\phi\| \le 1$ let

$$J(S) = S \oplus \phi(S)I,\ S \in B(X_1)$$

where I is the identity operator on Y.

Then J is an injection.

But $V_n(J\Phi) \neq 0$

But $V_n(\Phi) = 0$

Hence V is not injective

Definition 2.1.11 [30]

A surjection $Q\in B(X,Y)$ is an operator which maps X onto Y. In this case $\|y\|_Q = \inf\{\|x\| : x \in X, Qx = y\}$ for all $y\in Y$ defines an equivalent norm on Y. If, in addition, we have $\|\cdot\| = \|\cdot\|_Q$, then Q is said to be a metric surjection.

Definition 2.1.12.[30]

A semi-pseudo-s-function s is surjective if, given any metric surjection $Q\in B(X_0,X)$ $s_n(S) = s_n(SQ)$ for all $S\in B(X,Y)$.

Proposition 2.1.13.

V - numbers are surjective.

Proof.

From the definition of metric surjection we get

$$\|S-L\| = \|(S-L)Q\|$$

$$V_n(S) = \inf\{\|S-L\| : rankL < n \ \ and \ \ rankL(T) < n \ for \ all \ T\}.$$

$$= \inf\{\|(S-L)Q\| : rankL < n \ \ and \ rankL(T) < n \ for \ all \ T\}$$

$$= \inf\{\|SQ-LQ\| : rankLQ < n \ and \ rankLQ(T_0) < n \ for \ all \ T_0\}$$

$$= V_n(SQ)$$

Lemma 2.1.14.

Let (L_i) be a bounded family of operators $L_i \in B(B(X_i),B(Y_i))$ be such that rank$L_i<n$ and rank $L_i(T_i) < n$. Then rank$L_i < n$ implies rank $((L_i)_\upsilon) < n$.

Proof.

Using the same technique as in Lemma 11.10.9. [30].

Lemma 2.1.15.

Let $(L_i(T_i))$ be a bounded family of operators $L_i(T_i) \in B(Y_i)$ be such that rank $L_i(T_i) < n$. Then rank $L_i(T_i) < n$ implies rank $((L_i(T_i))_\upsilon) < n$.

Let us recall the definition of $((S_i)_\upsilon)$ [30]

Let (E_i) be a family of Banach spaces and suppose that υ is given on the index set I. The Banach space of all bounded families (x_i), where $x_i \in E_i$ for $i \in I$, is denoted by $l_\infty(E_i, I)$. Moreover, put

$$c_\upsilon(E_i, I) = \{(x_i) \in l_\infty(E_i, I) : \lim_\upsilon \|x_i\| = 0 \}.$$

We now form the quotient space $(E_i)_\upsilon = \dfrac{l_\infty(E_i, I)}{c_\upsilon(E_i, I)}$.. If $x = (x_i)_\upsilon$ denote the equivalence class corresponding to (x_i), then the norm of x can be computed $\|x\| = \lim_\upsilon \|x_i\|$.

The Banach space $(E_i)_\upsilon$ obtained in this way is called the ultraproduct of the Banach spaces E_i with respect to the ultrafilter υ.

Let (E_i) and (F_i) be families of Banach spaces.Suppose that (S_i) is a bounded family of operators $S_i \in B(E_i, F_i)$. By setting

$$(S_i)_\upsilon (x_i)_\upsilon = (S_i x_i)_\upsilon$$

Definition 2.1.16.[30]

A semi – pseudo-s-function s is called ultrastable if $(s_n(S_i)_\upsilon) \leq \lim_\upsilon s_n(S_i)$ for every bounded family (S_i) of operators $S_i \in B(X_i, Y_i)$ and every ultrafilter υ.

Proposition 2.1.17.

V– numbers are ultrastable.

Proof.

Let (S_i) be a bound family of operators $S_i \in B(B(X_i), B(Y_i))$. Given $\varepsilon > 0$, we choose $L_i \in B(B(X_i), B(Y_i))$ such that rank $L_i < n$, rank $L_i(T_i) < n$ and $\|S_i - L_i\| \leq (1+\varepsilon)V_n(S_i)$

$$\|S_i(T_i) - L_i(T_i)\| \leq (1+\varepsilon)V_n(S_i)$$

It follows from

$$\|L_i\| \le \|S_i - L_i\| + \|S_i\|$$
$$\le (1+\varepsilon)V_n(S_i) + \|S_i\|$$
$$\le (2+\varepsilon)\|S_i\|$$
$$\|L_i(T_i)\| \le \|S_i(T_i) - L_i(T_i)\| + \|S_i(T_i)\|$$
$$\le (1+\varepsilon)V_n(S_i) + \|S_i(T_i)\|$$
$$\le (2+\varepsilon)\|S_i\|$$

that the families (L_i) and $(L_i(T_i))$ are also bounded. Hence rank $((L_i)_\upsilon) < n$ and rank $((L_i(T_i)_\upsilon) < n$

We have $(V_n(S_i)_\upsilon) \le \|(S_i)_\upsilon - (L_i)\upsilon\|$

$$= \lim_\upsilon \|S_i - L_i\|$$
$$\le (1+\varepsilon)\lim_\upsilon V_n(S_i)$$

Letting $\varepsilon \to 0$, we get

$$V_n((S_i)\upsilon) \le \lim_\upsilon V_n(S_i)$$

2.2 δ-numbers

For every $S \in B(X,Y)$ and $n = 2,3,\ldots$ the n^{th} δ- number is defined by

$\delta_n(S) = \inf\{\|S - L\| : L \in B(X,Y), -n \le indL \le n\}$, where ind L = dim ker$L$ – codim(ImL).

Put $\delta_1(S) = \|S\|$.

Theorem 2.2.1.

The map $\delta : S \to (\delta_n(S))$ is a semi-pseudo-s-function.

Proof.

1) $\delta_{n+1}(S) = \inf\{\|S - L\| : L \in B(X,Y), -(n+1) \le indL \le n+1\}$

$$\le \inf\{\|S - L\| : L \in B(X,Y), -n \le indL \le n\}$$
$$= \delta_n(S)$$

Therefore,

$$\|S\| = \delta_1(S) \geq \delta_2(S) \geq \ldots \geq 0 \quad \text{for all } S \in B(X,Y).$$

2) $$\delta_n(S+T) = \inf\{\|S+T-L\| : L \in B(X,Y), -n \leq indL \leq n\}$$
$$\leq \inf\{\|S-L\| : L \in B(X,Y), -n \leq indL \leq n\} + \|T\|$$
$$= \delta_n(S) + \|T\|$$

Proposition 2.2.2.

The map $\delta : S \to (\delta_n(S))$ is not a pseudo-s-function.

Proof.

The following example shows that

$\delta_n(RST) \leq \|R\|\delta_n(S)\|T\|$, not true.

Let S be an invertible operator such that $\delta_n(S) = 0$

Put $R = S^{-1}$

Define $T: l_2 \to l_2$ by

$$T(x_1, x_2, \ldots) = (x_{n+2}, x_{n+3}, \ldots).$$

So ind $T = n + 1$

Hence $\delta_n(RST) \neq 0$, but $\delta_n(S) = 0$

Therefore, $\delta_n(RST) \nleq \|R\|\delta_n(S)\|T\|$

Remarks 2.2.3.

a) δ- numbers do not satisfy the fifth condition of s- function.

Proof.

Whatever be the dimension of X, $\delta_n(I_X) = 0$ always.

b) $\delta_n(S) = \delta_n(S+K)$, where K is a compact operator.

Proof.

We know that ind$(S+K)$ = ind S, where K is a compact operator.

c) $\delta_n(S) = \delta_n(S^*)$

Theore 2.2.4.

$\delta_n(S) = 0$ if and only if $-n \leq \text{ind } S \leq n$

Proof.

If $\delta_n(S) = 0$, there exist a sequence $\{L_k\}$, $-n \leq \text{ind } L_k \leq n$ such that

$$\lim_{k\to\infty}\|S - L_k\| = 0$$

Therefore, $$S = \lim_{k\to\infty} L_k$$

So $$\text{ind}\, S = \lim_{k\to\infty} indL_k$$

Therefore $-n \leq \text{ind } S \leq n$.

Converse part is trivial.

Proposition 2.2.5 .

$\lim_{n\to\infty}\delta_n(S) = 0$ if and only if S is the limit of a sequence of finite index operators.

Proposition 2.2.6.

δ - numbers are not injective.

Proof.

The example shown below illustrates that $\delta_n(S) \neq \delta_n(JS)$.

Let $S = I$ and hence $\delta_n(S) = 0$

Define $J: l_2 \to l_2$ by

$$J(x_1, x_2, \ldots) = (\underbrace{0, \ldots 0}_{n+1}, x_1, x_2, \ldots).$$

So $\text{ind}\, J = -(n+1)$

Therefore, $\delta_n(JS) \neq 0$, but $\delta_n(S) = 0$

Proposition 2.2.7.

δ - numbers are not surjective.

Proof.

This can be shown using the example given below.

Let $S = I$, so $\delta_n(S) = 0$

Define $Q: l_2 \rightarrow l_2$ by

$$Q(x_1, x_2, \ldots) = (x_{n+2}, x_{n+3}, \ldots)$$

So ind $Q = n + 1$

Therefore $\delta_n(SQ) \neq 0$

Proposition 2. 2.8.

δ - numbers are not additive.

Proof.

The following example shows that $\delta_{m+n-1}(S+T) \nleq \delta_m(S) + \delta_n(T)$, is true.

Let $S = I$ and hence $\delta_n(S) = 0$.

Define $T: l_2 \rightarrow l_2$ by

$$T(x_1, x_2, \ldots) = (2x_2, -x_3, -x_4, \ldots)$$

$$(I+T)(x_1, x_2, \ldots) = (2x_2+x_1, x_2-x_3, x_3-x_4, \ldots)$$

Therefore, $\delta_n(T) = 0$ for $n = 2,3,\ldots$,because ind $T = 1$

But $\delta_{m+n-1}(I+T) \neq 0$,because ind $(I+T) = \infty$.

Definition 2.2.9.[30]

A semi- pseudo-s-function s is called symmetric if $s_n(S) \geq s_n(S')$ for all $S \in B(X,Y)$.

Proposition 2.2.10.

δ - numbers are symmetric.

Proof.

Given $\varepsilon > 0$, we choose $L \in B(X,Y)$ such that $-n \leq$ ind $L \leq n$ and

$$\|S - L\| \leq (1+\varepsilon)\delta_n(S)$$

Then $-n \leq indL' \leq n$ and $\|S' - L'\| \leq (1+\varepsilon)\delta_n(S)$

Therefore, $\delta_n(S') \leq (1+\varepsilon)\, \delta_n(S)$

Definition 2.2.11.[30]

A semi-pseudo-s-function s is called completely symmetric if $s_n(S) = s_n(S')$ for all $S \in B(X,Y)$.

Proposition 2.2.12.

δ- numbers are completely symmetric.

Proof.

The proof of this proposition can be carried out in the same way as the proof of proposition 2.2.10.

Definition 2.2.13.[30]

A semi-pseud-s-function s is called regular if $s_n(S) = s_n(K_Y S)$ for all $S \in B(X,Y)$, where K_X is the evaluation map from X into X''.

Proposition 2.2.14.

δ- numbers are regular.

Proof.

It is trivial.

Now what follows is an example of a semi-pseudo-s-function based on a concept, due to William Arveson, called degree of operators. First recall the definition [1]

Definition 2.2.15.

Let $\{H_n\}$ be an increasing sequence of finite dimensional subspace of a Hilbert space H such that $\bigcup_{1}^{\infty} H_n$ is dense in H. For T in $B(H)$, the degree of T denoted by deg(T) is defined as deg(T) = sup rank $(P_nT - TP_n)$, P_n is the projection onto H_n.

2.3 f- numbers

For every $S \in B(H)$ and $n = 2,3,\ldots$ the n^{th} f-number is defined by

$f_n(S) = \inf\{\|S - L\| : L \in B(H), \deg L < n\}$. Put $f_1(S) = \|S\|$

Theorem 2.3.1.

The map $f : S \to ((f_n(S))$ is a semi-pseudo-s-function.

Proof.

The proof is quite similar to the proof of theorem 2.2.1.

Proposition 2.3.2.

The map $f : S \to ((f_n(S))$ is not a pseudo-s-function.

Proof.

The following example shows that $f_n(RST) \nleq \|R\| f_n(S) \|T\|$, is true.

Let S, $T = I$, so $f_n(S) = 0$

Define $R: l_2 \to l_2$ by

$R(x_1, x_2, \ldots) = (x_{n+1}, x_{n+2}, x_{n+3}, \ldots, x_{2n}, 0, 0, \ldots)$.

Therefore $\deg R = n$

Hence $f_k(RST) \neq 0$ if $1 < k < n$

Remarks 2.3.3.

a) $f_n(\lambda S) = |\lambda| f_n(S)$.

b) $f_n(S) = f_n(S^*)$.

Proposition 2.3.4.

$f_n(S) = 0$ if and only if $\deg S < n$.

Proof.

We know that deg is lower semi-continuous. Therefore $f_n(S) = 0$ if and only if $\deg S < n$.

Converse part is trivial.

Proposition 2.3.5.

$\lim_{n\to\infty} f_n(S) = 0$ if and only if S is the limit of a sequence of finite degree operators.

Proposition 2.3.6.

f - numbers are additive.

Proof.

From the definition of degree it is clear that $\deg(L_1 + L_2) \leq \deg L_1 + \deg L_2$

Proposition 2.3.7.

f-numbers are not injective.

Proof.

This can be shown using the example given below.

Let $S = I$, therefore $f_n(S) = 0$

Define $J : l_2 \to l_2$ by

$$J(x_1, x_2, \ldots) = (\underbrace{0, \ldots 0}_{n}, x_{2n}, \ldots x_1, x_{2n+1}, \ldots).$$

Therefore $\deg J > n$

Therefore $f_n(JS) \neq 0$

Proposition 2.3.8.

f - numbers are not surjective.

Proof.

The following example shows that $f_n(S) \neq f_n(SQ)$, is true.

Let $S = I$, and hence $f_n(S) = 0$.

Define $Q : l_2 \to l_2$ by

$$Q(x_1, x_2, \ldots) = (x_{n+2}, x_{n+3}, \ldots).$$

Therefore, $\deg Q = n + 1$

Hence $f_n(SQ) \neq 0$.

Proposition 2.3.9.

f- numbers are symmetric..

Proof.

Given $\varepsilon > 0$, we choose $L \in B(H)$ such that deg $L < n$ and $\|S - L\| \leq (1+\varepsilon) f_n(S)$.

Then deg $L' < n$ and $\|S' - L'\| \leq (1+\varepsilon)\, f_n(S)$. Therefore $f_n(S') \leq (1+\varepsilon) f_n(S)$

Proposition 2.3.10.

The f- numbers are completely symmetric.

Proof.

The proof is quite similar to the proof of proposition 2.3.9.

Proposition 2.3.11.

The f– numbers are regular.

Proposition 2.3.12.

f- numbers are ultrastable.

2.4 g – numbers

For every operator $S \in B(H)$ and $n = 2,3,\ldots$ the n^{th} g – number is defined by $g_n(S) = \inf\{\|S - L\| : L \in B(H), trL < n\}$, put $g_1(S) = \|S\|$, where tr(L) denotes the trace of L

Theorem 2.4.1.

The map $\boldsymbol{g} : S \to ((g_n(S))$ is a semi-pseudo-s-function.

Proof.

The proof is quite similar to the proof of theorem 2.2.1.

Proposition 2.4.2.

The map $\boldsymbol{g} : S \to ((g_n(S))$ is not a pseudo-s-function.

Proof.

The example shown below illustrates that $g_n(RST) \nleq \|R\| g_n(S) \|T\|$.

Let $T = I$, so $\|T\| = 1$.

Define $R, S : l_2 \to l_2$ by

$$R(x_1, x_2, \dots) = (-x_1, x_2, \dots, x_n, x_{n+1}, \dots),$$

$$S(x_1, x_2, \dots) = (-x_1, x_2, \dots, x_n, 0, \dots).$$

Hence tr $S = n$-1.

Therefore, $g_n(S) = 0$.

$$RS(x_1, x_2, \dots) = (x_1, x_2, \dots, x_n, 0, \dots)$$

So tr $RS = n$

Therefore, $g_n(RS) \neq 0$

Hence the result.

Proposition 2.4.3.

tr $S < n$ if and only if $g_n(S) = 0$.

Remark 2.4.4.

If dim $H \geq n$, then $g_n(I_H) \neq 0$.

Proposition 2.4.5.

g- numbers are additive.

Proof.

We know that tr$(A + B)$ = tr(A) + tr(B)

Theorem 2.4.6

$\lim_{n \to \infty} g_n(S) = 0$ if and only if S is the limit of a sequence of finite trace operators.

Proposition 2.4.7.

g - numbers are not injective.

Proof.

This can be shown using the example given below.

Define $J, S : l_2 \to l_2$ by

$$J(x_1, x_2, \dots) = (-x_1, x_2, \dots, x_n, x_{n+1}, x_{n+2}, \dots)$$

$$S(x_1,x_2,\dots) = (-x_1, x_2, \dots, x_n, 0, \dots)$$

$$JS(x_1,x_2,\dots) = (x_1, x_2, \dots, x_n, 0, \dots)$$

Therefore, tr $JS = n$,but tr $S = n-1$

Hence $g_n(JS) \neq 0$ and $g_n(S) = 0$

Proposition 2.4.8.

g- numbers are not surjective.

Proof.

The following example shows that $g_n(S) \neq g_n(SQ)$, is true.

Define $Q, S : l_2 \rightarrow l_2$ by

$$Q(x_1,x_2,\dots) = (-x_1, x_2, \dots, x_n, x_{n+1}, x_{n+2}, \dots)$$

$$S(x_1,x_2,\dots) = (-x_1, x_2, \dots, x_n, 0, \dots)$$

$$SQ(x_1,x_2,\dots) = (x_1, x_2, \dots, x_n, 0, \dots)$$

So tr $SQ = n$ and tr $S = n-1$

Therefore, $g_n(SQ) \neq 0$ and $g_n(S) = 0$

Remarks 2.4.9.

a) For each mapping $S \in B(H)$ and all numbers λ ($\lambda \neq 0$) $g_n(\lambda S) \neq |\lambda|\ g_n(S)$

Proof.

The example shown below illustrates that $g_n(\lambda S) \neq |\lambda|\ g_n(S)$

Define S: $l_2 \rightarrow l_2$ by

$$S(x_1,x_2,\dots) = (x_1, x_2, \dots, x_n, x_{n+1}, 0, 0, \dots)$$

So tr $S = n+1$

Therefore, $g_n(S) \neq 0$. Choose $\lambda = \dfrac{-1}{n+1}$ such that tr $\lambda S = -1$. Hence $g_n(\lambda S) = 0$

b) $g_n(S) = g_n(S^*)$

Proposition 2.4.10.

g - numbers are regular.

Proposition 2.4.11.

g- numbers are symmetric.

Proof.

We know that tr S = tr S'

Proposition 2.4.12.

g- numbers are completely symmetric.

2.5 θ- numbers

For every operator $S \in B(H)$ and n =2,3,..., the n^{th} θ - number is defined by $\theta_n(S) = \inf\{\|S - L\| : L \in B(H), nulL < n\}$,put $\theta_1(S) = \|S\|$

Theorem 2.5.1.

The map $\theta: S \to (\theta_n(S))$ is a semi-pseudo-s-function.

Proposition 2.5.2.

The map $\theta: S \to (\theta_n(S))$ is not a pseudo-s-function.

Proof.

The following example shows that $\theta_n(RST) \nleq \|R\|\theta_n(S)\|T\|$, is true.

Choose S and $T = I$ such that nul S = 0 and $\|T\| = 1$.

Therefore, $\theta_n(S) = 0$.

Define R: $l_2 \to l_2$ by

$$R(x_1,x_2,\dots) = (\underbrace{0,0,\dots,0}_{n},x_1,x_2,\dots).$$

So nul $R = n$.

Therefore, $\theta_n(RST) \neq 0$

Remarks 2.5.3.

a) θ- numbers do not satisfy the fifth condition of s-function.

Proof.

Whatever be the dimension of H, $\theta_n(I_H)$ 0 always.

b) $\theta_n(S) = \theta_n(S^*)$.

c) $\theta_n(\lambda S) = |\lambda|\ \theta_n(S)$.

Theorem 2.5.4.

$\theta_n(S) = 0$ if and only if nul $S < n$.

Proposition 2.5.5.

$\lim_{n\to\infty} \theta_n(S) = 0$ if and only if S is the limit of a sequence of finite nullity of operators.

Proposition 2.5.6.

θ- numbers are not additive.

Proof.

The following example shows that $\theta_{m+n-1}(S+T) \nleq \theta_m(S) + \theta_n(T)$, is true.

Define $S : l_2 \to l_2$ by

$$S(x_1, x_2, \ldots) = (-x_1, -x_2, \ldots, -x_{n+1}, x_{n+2}, \ldots).$$

Therefore, $\theta_n(S) = 0$.

Put $T = I$. Therefore, $\theta_n(T) = 0$.

Therefore, $(I + S)(x_1, x_2, \ldots) = (\underbrace{0, \ldots, 0}_{n+1}, x_{n+2}, \ldots)$

Put m $n = 2$.

Therefore, $\theta_{m+n-1}(S+T) = \theta_3(I + S) \neq 0$.

Because nul $(I+S) = n+1$

Proposition 2.5.7.

θ- numbers are injective.

Proof.

$$\theta_n(S) = \inf\{\|S - L\| : L \in B(H), nulL < n\}$$

$$= \inf\{\|JS - JL\| : L \in B(H), nulL < n\}$$

$$= \inf\{\|JS - JL\| : L \in B(H), nulJL < n\}$$

$$= \theta_n(JS).$$

Proposition 2.5.8.

θ- numbers are not surjective.

Proof.

The example shown below illustrates that $\theta_n(S) \neq \theta_n(SQ)$

Let $S = I$. Therefore, $\theta_n(S) = 0$

Define $Q : l_2 \to l_2$ by

$$Q(x_1, x_2, \dots) = (x_{n+2}, x_{n+3}, \dots).$$

Therefore, nul $Q = n+1$.

Therefore, $\theta_n(SQ) \neq 0$.

Proposition 2.5.9.

θ- numbers are not symmetric.

Proof.

The following example shows that $\theta_n(S) \neq \theta_n(S')$, is true.

Define $S : l_2 \to l_2$ by

$$S(x_1, x_2, \dots) = (\underbrace{0, \dots, 0}_{n}, x_1, x_2, \dots)$$

Therefore, $S^*(x_1, x_2, \dots) = (x_{n+1}, x_{n+2}, x_{n+3}, \dots)$

So nul $S = 0$. Hence $\theta_n(S) = 0$.

So nul $S^* \neq 0$. Hence $\theta_n(S^*) \neq 0$.

Proposition 2.5.10.

θ- numbers are regular.

Proof.

$$\theta_n(S) = \inf\{\|S - L\| : L \in B(H), nulL < n\}$$
$$= \inf\{\|K_H(S - L)\| : L \in B(H), nulL < n\}$$
$$= \inf\{\|K_H S - K_H L\| : L \in B(H), nulK_H L < n\}$$
$$= \theta_n(K_H S)$$

2.6 η-numbers

For every operator $S \in B(H)$ and n =2,3,..., the n^{th} η- number is defined by $\eta_n(S) = \inf\{\|S - L\| : L \in B(H), co-rankL < n\}$, put $\eta_1(S) = \|S\|$,where $co-rankL = \dim\ (RanL)^{\perp}$

Theorem 2.6.1.

The map $\eta : S \to (\eta_n(S))$ is a semi-pseudo-s-function.

Proposition 2.6.2.

The map $\eta : S \to (\eta_n(S))$ is not a pseudo-s-function.

Proof.

The following example shows that $\eta_n(RST) \nleq \|R\|\eta_n(S)\|T\|$.

Let S and $T = I$.

Define $R \cdot l_2 \to l_2$ by

$$R(x_1, x_2, \dots) - (\underbrace{0, \dots, 0}_{n}, x_1, x_2 \dots)$$

Remark 2.6.3.

η- numbers do not satisfy the fifth condition of s-function.

Proof.

Whatever be the dimension of H, $\eta_n(I_H) = 0$ always .

Theorem 2.6.4.

$\eta_n(S) = 0$ if and only if co-rank $S < n$.

Proposition 2.6.5.

$\lim_{n\to\infty} \eta_n(S) = 0$ if and only if S is the limit of a sequence of finite co-rank of operators.

Proposition 2.6.6.

η - numbers are not additive.

Proof.

The example shown below illustrates that $\eta_{m+n-1}(S \cdot T) \nleq \eta_m(S) \cdot \eta_n(T)$

Define $S : l_2 \to l_2$ by

$$S(x_1, x_2, \dots) = (-x_1, -x_2, \dots, -x_{n+1}, x_{n+2}, \dots).$$

Therefore, $\eta_n(S) = 0$.

Let $T = I$ be such that $\eta_m(T) \quad 0$

Therefore, $(I + S)(x_1, x_2, \dots) = (0, \dots, 0, x_{n+2}, \dots)$

Therefore, co-rank $(I+S) = n+1$.

Put $m = n = 2$.

Therefore, $\eta_{m+n-1}(S \cdot T) = \eta_3(S \cdot T) \neq 0$

Proposition 2.6.7.

η - numbers are not injective.

Proof.

The following example shows that $\eta_n(S) \neq \eta_n(JS)$

Put $S = I$. Define $J: l_2 \to l_2$ by

$$J(x_1, x_2, \dots) = (\underbrace{0, \dots, 0}_{n}, x_1, x_2, \dots).$$

Proposition 2.6.8.

η- numbers are surjective.

Proof.

$$\eta_n(S) = \inf\{\|S - L\| : L \in B(H), co-rank L < n\}$$

$$= \inf\{\|(S - L)Q\| : L \in B(H), co-rank L < n\}$$

$$= \inf\{\|SQ - LQ\| : L \in B(H), co-rank LQ < n\}$$

$$= \eta_n(SQ).$$

Proposition 2.6.9.

η- numbers are not symmetric.

Proof.

The example shown below illustrates that $\eta_n(S') \neq \eta_n(S)$

Define $S : l_2 \to l_2$ by

$S(x_1, x_2, \dots) = (x_{n+1}, x_{n+2}, x_{n+3}, \dots)$.

So co-rank $S = 0$. Therefore, $\eta_n(S) = 0$

$S^*(x_1, x_2, \dots) = (0, \dots, 0, x_1, x_2, \dots)$

So co-rank $S^* = n$, hence $\eta_n(S') = 0$.

Proposition 2.6.10.

η- numbers are regular.

2.7 $\bar{s}$ -numbers

Let $\{H_n\}$ be a decreasing sequence of closed subspace of H, $H_1=H$, n^{th} $\bar{s}$ -number of $S \in B(H)$ is defined by $\bar{s}_n(S) = \sup\{\|Sx\| : x \in H_n, \|x\| = 1\}$.

Proposition 2.7.1.

The map $\bar{s} : S \to (\bar{s}_n(S))$ is a semi-pseudo-s-function.

Proposition 2.7.2.

The map $\bar{s} : S \rightarrow (\bar{s}_n(S))$ is not a pseudo-s-function.

Proof.

The following example shows that $\bar{s}_n(RST) \nleq \|R\|\bar{s}_n(S)\|T\|$, is true.

Let $R = I$ and S be an orthogonal projection of H onto $H_n^{\perp}$.

Therefore, $\bar{s}_n(S) = 0$

Define $T:H \rightarrow H$ such that $T(H_n) \neq 0$ and $T(H_n) \subseteq H_n^{\perp}$.

Therefore, $\bar{s}_n(ST) \neq 0$.

Remarks 2.7.3.

a) $\bar{s}$ -numbers do not satisfy the fourth condition of s-function.

b) If dim $H \geq n$, then $\bar{s}_n(I_H) = 1$.

c) $\bar{s}_n|\lambda S| = |\lambda|\bar{s}_n(S)$.

d) $\bar{s}_n(ST) \leq \|S\|\bar{s}_n(T)$.

e) $\bar{s}_n(S) = \bar{s}_n(S^*)$.

Proposition 2.7.4.

$\bar{s}$ -numbers are additive.

Proposition 2.7.5.

$\bar{s}$ -numbers are injective.

Proposition 2.7.6.

$\bar{s}$ -numbers are not surjective.

Proposition 2.7.7.

$\bar{s}$ -numbers are regular.

Remarks 2.7.8.

a) $a_n(SP_n) \le \bar{s}_n(S)$ where P_n is an orthogonal projection of H onto H_n.

b) $a_n(S) \le \bar{s}_n(S) + a_n(SP_n')$ where P_n' is an orthogonal projection of H onto $H_n^{\perp}$.

c) $\bar{s}_n(S) \le \bar{s}_n(SP) + a_n(S)$, where P is an orthogonal projection with rank $P <$ n.

Proof.

a) $a_n(SP_n) = \inf\{\|SP_n - L\| : rankL < n\}$

$$\le \|SP_n\|$$

$$\le \bar{s}_n(S).$$

b) Let $P_n' : H \to H_n^{\perp}$

$$(I - P_n') : H \to H_n$$

$$\bar{s}_n(S) \ge \|S(I - P_n')\|$$

$$S = S(I - P_n') + SP_n'$$

$$a_n(S) \le \|S(I - P_n')\| + a_n(SP_n')$$

$$\le \bar{s}_n(S) + a_n(SP_n')$$

Hence the result.

c)We know that $a_n(S) = \inf\{\|S - SP\| : P \in B(H), is$ an orthogonal projection with rank $P <$ n$\}$.

Therefore,

$$\|S - SP\| \le (1+\varepsilon)a_n(S)$$

$$\|(S - SP)x\| \le (1+\varepsilon)a_n(S)\,\|x\|$$

$$\|Sx\| \le \|SPx\| + (1+\varepsilon)a_n(S)\|x\|$$

$$\sup\{\|Sx\| : x \in H_n, \|x\| = 1\} \le \sup\{\|SPx\| : x \in H_n, \|x\| = 1\} + a_n(S)(1+\varepsilon)$$

Hence the result.

2.8 Relationships between *s*-numbers and semi-pseudo-s-numbers

Remarks 2.8.1.

a) $f_{2n}(S) \leq a_n(S)$

Proof.

Given $\varepsilon > 0$, we choose $L \in B(H)$ such that rank$L$<n and $\|S - L\| \leq (1+\varepsilon)a_n(S)$

Then deg$L< 2n$.

Therefore,

$$f_{2n}(S) \leq (1+\varepsilon)a_n(S).$$

b) $f_{2n}(S) \leq \lambda_n(S)$, $S \in K(H)$

c) $\delta_n(S) \leq a_n(S) + 1$

Proof.

Given $\varepsilon > 0$, we choose $L \in B(H)$ such that rank $L < n$ and $\|S - L\| \leq (1+\varepsilon)a_n(S)$

$$\delta_n(S) = \inf\{\|S - L\| : -n \leq indL \leq n\}$$

$$\leq \|S - (I + L)\|$$

$$\leq \|S - L\| + 1$$

$$\leq (1+\varepsilon)a_n(S) + 1$$

Hence the result.

d) $\lim\limits_{n\to\infty} a_n(T) \leq \lim\limits_{n\to\infty} \delta_n(T)$ if T is a compact operator.

e) $\lim\limits_{n\to\infty} f_n(T^m)^{\frac{1}{m}}$ $|\lambda_n(T)|$ if T is compact.

f). If T is compact and $T = TT^*T$, T^* T^*TT^* then $\delta_n(T) \leq a_n(T)$ and $f_{2n}(T) \leq 1$.

Lemma 2.8.2.

Let T be a continuous linear mapping from an arbitrary Hilbert space H into an $(n+1)$ dimensional Hilbert space F for which there is a mapping $S \in B(F,H)$ with $Tsy = y$

for $y \in F$. In the case of approximation numbers the inequality $a_n(T)\|S\| \geq 1$ holds. But this does not hold in the case of δ-numbers,f-numbers,g-numbers,θ-numbers and η-numbers.

Proof.

The following example. show that

$\delta_n(T)\|S\| \ngeq 1$, $f_n(T)\|S\| \ngeq 1$, $g_n(T)\|S\| \ngeq 1$, $\eta_n(T)\|S\| \ngeq 1$ $\theta_n(T)\|S\| \ngeq 1$, is true.

Define T: $l_2^{2n} \to l_2^{n+1}$ and $S: l_2^{n+1} \to l_2^{2n}$ by

$$T(x_1, x_2, \ldots x_{2n}) = (x_1, x_2, \ldots, x_{n+1})$$
$$S(x_1, x_2, \ldots x_{n+1}) = (x_1, x_2, \ldots x_{n+1}, \underbrace{0, \ldots 0}_{n-1})$$
$$TS(x_1, x_2, \ldots x_{n+1}) = (x_1, x_2, \ldots x_{n+1})$$

From this we get ,$\delta_n(T) = 0$,$f_n(T) = 0$, $\theta_n(T) = 0$,$\eta_n(T) = 0$ and $\|S\| = 1$.

Hence $\delta(T)\|S\| = 0$, $f_n(T)\|S\| = 0$,$\theta_n(T)\|S\| = 0$ and $\eta_n(T)\|S\| = 0$

Hence the result.

This chapter is concluded with the following remark.

Remark 2.8.3.

There exist one and only one s-function on the class of all bounded linear operators acting between Hilbert spaces. All s-number sequences coincide with the singular numbers of the operator namely, approximation numbers of the operator. But there are several semi-pseudo-s-functions on the class of all operators acting between Hilbert spaces. In the case of s-function on the class of all operators acting between complex Banach spaces approximation numbers are the largest s-function and Hilbert numbers are the smallest s-function. But in the case of semi-pseudo-s-numbers, the answer is not known.

CHAPTER III

COMPUTATION OF s-NUMBERS

This chapter aims at providing a computational method for finding singular values of Hilbert space operators. The results are given in two sections. The first section deals with the above mentioned computational method. Second section consists of an application of the observations of first section, to find lower bound of essential spectrum [algorithmically] for certain class of Hilbert space operators identified by William Arveson [1,2].

Of course the findings of the first section is motivated by the following Proposition. Let us recall the proposition[6].

Proposition.

Let E and F be Banach space and T in $B(E,F')$ where F' is the dual of F. Then $\hat{a}_n(T) = a_n(T)$, n =1,2... where $\hat{a}_n(T) = \sup\{a_n(TI_M^E) : M \subseteq E, \dim M < \infty\}$ (TI_M^E denotes the restriction of T to the finite dimensional subspace M).

3.1 Approximation of approximation numbers

Remarks 3.1.1.

1) It is well known that

$$s_n(T) = \inf\{\|T - A\| : A \in B(H), rank A < n\} \quad [29].$$

2) Also the following equivalent description is given in Gohberg, Goldberg and Kaashoek [15].

Let T be in $B(H)$ and let μ be the maximum of the essential spectrum $\sigma_e(T^*T)$ of T^*T. Let λ_1, λ_2,... λ_N be the eigen values of T^*T strictly above μ and arranged in the decreasing order.

Then $s_n(T) = \lambda_n^{\frac{1}{2}}$, $n = 1,2,3,\ldots$ if N is infinite.

Otherwise,

$$s_n(T) = \lambda_n^{\frac{1}{2}}, n = 1,2,3,\ldots N$$

$$= \mu^{\frac{1}{2}}, n = N+1, N+2, \ldots$$

Now let $\{e_1, e_2, \ldots\}$ be an orthonormal basis in H and let P_n denote the orthogonal projection of H onto the subspace H_n spanned by $e_1, e_2, \ldots e_n$. If $[T] = (a_{ij})$ is the matrix of T with respect to the above basis, then the matrix $[T]_n$ of $P_n T P_n$ can be identified with the $n \times n$ square matrix $(a_{ij})_{i,j=1,2,\ldots n}$. So whatever calculations we are going to do in the subsequent part of this chapter can be implemented in terms of $[T]$ and $[T]_n$ as Arveson does [1,2].

The main theorem of this section is as follows.

Theorem 3.1.2.

For each pair of positive integers (k,n), let $s_{n,k}(T)$ be the n^{th} s - number of $|TP_k|$. Then $\lim_{k\to\infty} s_{n,k}(T) = s_n(T)$ exists and $s_n(T)$ is the n^{th} s-number of T for each T in $B(H)$.

This theorem is a consequence of the following propositions.

Proposition 3.1.3.

For each T in $B(H)$, $s_n(T)$ exist for each n,

$s_1(T) = \|T\|$ and $s_n(S+T) \leq s_n(S) + \|T\|$ for all S and T in $B(H)$.

Proof.

Since $\{s_{n,k}(T)\}_{k\geq 1}$ is a bounded increasing sequence of numbers [3], $s_n(T)$ exists for each n.

Now $s_{1,k}(T) = \|TP_k\| \leq \|T\|$ for each k, where $|TP_k| = \sqrt{P_k T^* TP_k}$.

Also, $P_k T^* TP_k \to T^*T$ strongly as $k \to \infty$. Therefore, given $\varepsilon > 0$ there is a positive integer N such that

$$\|P_k T^* TP_k(x)\| \geq \|T\|^2 - \varepsilon \text{ for every } k \geq N \text{ and for some } x \text{ in } H \text{ with } \|x\| = 1.$$

From this it follows that

$$\lim_{k\to\infty} s_{1,k}(T) \geq \|T\|. \text{ Thus } s_1(T) = \|T\|.$$

Now to show that $s_n(S+T) \leq s_n(S) + \|T\|$. But this is an easy consequence of the fact that $s_{n,k}(T)$ is an s-number for each k and n.

Proposition 3.1.4.

$s_n(RST) \leq \|R\| s_n(S) \|T\|$ for each compact operator S in $B(H)$ and for every R, T in $B(H)$.

Proof.

For each j we have,

$$s_n(RSP_jT) = \lim_{k\to\infty} s_{n,k} |RSP_jTP_k|.$$

But $s_{n,k}|RSP_jTP_k|$ is the n[th] eigenvalue of $A_k = \sqrt{P_k T^* P_j S^* R^* RSP_jTP_k}$.

Now $$A_k \leq \|R\| \sqrt{P_k T^* P_j S^* SP_j TP_k}.$$

Therefore,

$$s_{n,k}(A_k) \leq \|R\| s_{n,k} \sqrt{P_k T^* P_j S^* SP_j TP_k}.$$

But spectrum $\sigma\left(\sqrt{P_k T^* P_j S^* S P_j T P_k}\right) = \sigma\left(\sqrt{S P_j T P_k P_k T^* P_j S^*}\right)$.

Since $\sigma(AB)=\sigma(BA)$, whenever A or B is compact [33], the above equation holds.

Therefore,

$$s_{n,k}(A_k) \le \|R\| s_{n,k} \sqrt{S P_j P_j S^*} \|T P_k\|.$$

Hence $\qquad s_n(A) \le \|R\| s_n(S P_j) \|T\|$.

Now since S is compact and since $P_j \to I_H$ strongly $SP_j \to S$ uniformly [14], as $j \to \infty$.

Since $s_n(T) \le \|T\|$ for all T, it follows that $s_n(SPj) \to s_n(S)$ as $j \to \infty$.

Thus $\qquad s_n(RST) \le \|R\| s_n(S) \|T\|$, whenever S is compact.

Proposition 3.1.5.

1) $s_n(T) = 0$ whenever rank $T < n$, and

2) $s_n(I_H) = 1$ whenever dimension of $H \ge n$.

Proof.

Follows easily from the definition of s_n.

Proposition 3.1-6.

$s_n(T) = a_n(T)$ for each compact operator T.

Proof.

We found that $s_n(.)$ satisfies all the axioms of an s-number whenever T is compact. Now, we may use the same proof as that of theorem 2.11.9 [32] to conclude that

$s_n(T) = a_n(T)$ for all compact T.

Proposition 3.1.7.

$s_n(T) = a_n(T)$ for all T in $B(H)$.

Proof.

Given $\varepsilon > 0$, let L in $B(H)$ be such that rank$(L) < n$, and $\|T - L\| \le (1+\varepsilon)a_n(T)$

Now $s_n(T) = s_n(T\text{-}L\text{+}L) \le \|T - L\| + s_n(L)$

$\le (1+\varepsilon)a_n(T)$

Therefore, $s_n(T) \le a_n(T)$.

Now we may use the same proof technique as that of theorem 2.11.9 [32], to conclude that

$a_n(T) \le s_n(T)$ for all T in $B(H)$.

Remark 3.1.8.

Thus theorem 3.1.2, which is a consequence of the above propositions, reveals that we may use matrix computations to find singular values of operators in $B(H)$.It is also clear that there is freedom in choosing suitable orthonormal basis.This is helpful computationally.

3.2 Application

In this section we use the observations made in section 1 to get a reasonable lower bound for the essential spectrum of positive operators belonging to the class of operators identified by W.Arveson [1].

First of all , let us recall the class of operators identified by Arveson [1] .

Definition 3.2.1.

Let A be the collection of all operators T in $B(H)$ such that

$$T=\sum_{k=1}^{\infty} T_k,\ \text{degree}\,(T_k)<\infty \text{ and}$$

$$\sum_{k=1}^{\infty}(1+(\deg T_k)^{\frac{1}{2}})\|T_k\|<\infty$$

If
$$\|T\|_{\infty}=\inf\sum_{k=1}^{\infty}(1+(\deg T_k)^{\frac{1}{2}})\|T_k\|$$

then A is a Banach algebra.

For T in A, Arveson shows that the essential spectrum $\sigma_e(T)$ coincides with the set of essential points [1].

Now we provide a systematic procedure for arriving at a reasonable lower bound of the essential spectrum $\sigma_e(T)$ of T, whenever T is in A and T is positive.

Let $\{s_{n,k},\ n=1,2,\dots k\}$, be the n[th] s-number of $|TP_k|$ for each $k\geq 1$. We arrange them in a triangular form as shown in the following figure (*).

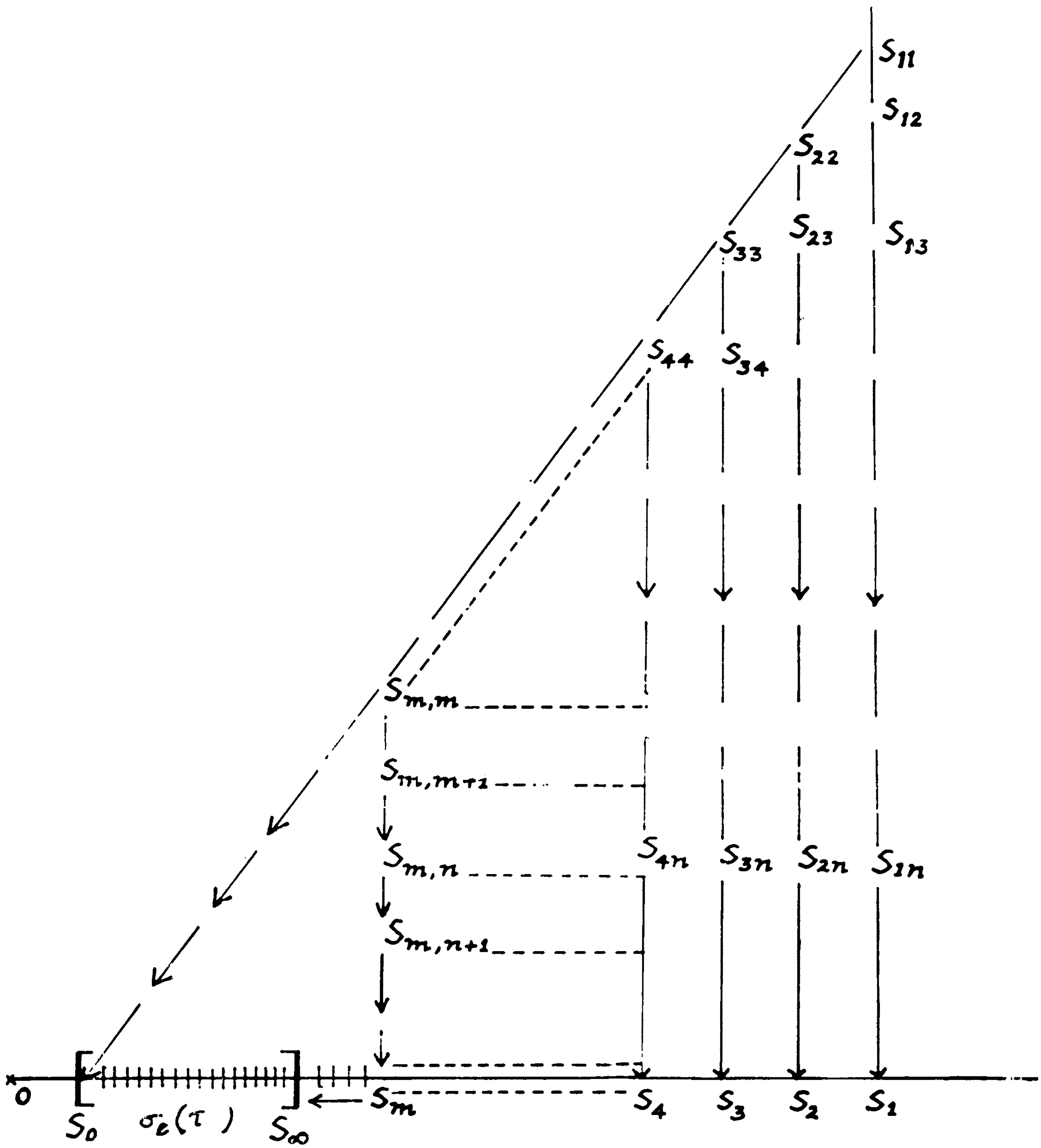
Proposition 3.2. 2.

Let $s_{n,n}$ be the n^{th} s-number of $|TP_n|$ as shown in the figure (*) and

let $s_0=\lim_{n\to\infty} s_{n,n}$ .Then s_0 is a lower bound for the essential spectrum of T whenever T is in A and T is positive.

Proof.

It is clear that $\lim_{n\to\infty} s_{n,n}=s_0$ exists. Let β be in $\sigma_e(T)$. Then by theorem 3.8[1] there is a sequence of spectral values β_n, $\beta_n\in\sigma_e[T]_n$ such that $\lim_{n\to\infty}\beta_n=\beta$. But $\beta_n\geq s_{n,n}$ for all n. Therefore $\beta\geq s_0$.



S_{11}
S_{12}
S_{22}
S_{23}
S_{13}
S_{33}
S_{44}
S_{34}
$S_{m,m}$
$S_{m,m+1}$
$S_{m,n}$
$S_{m,n+1}$
S_{4n}
S_{3n}
S_{2n}
S_{1n}
0
S_0
$\sigma_\varepsilon(\tau)$
S_∞
S_m
S_4
S_3
S_2
S_1

Remark 3.2.3.

Comparing figure(*) and the equivalent definition of s-numbers which is given in section 1, one finds that the limits along the vertical columns will never cross over the maximum value s_∞ of the essential spectrum and get inside $\sigma_e(T)$.So if at all one wants to compute the essential spectral values in (s_0, s_∞), one should consider sequences of the type $s_{n,f(n)}$, where f is a mapping on the set of positive integers such that $f(n) \neq n$. This is an easy consequence of Arveson's theory and our observations.

Again consider figure (*) for positive operators T in A. If s_0 is not in the essential spectrum $\sigma_e(T)$ of T (this can be checked by examing the sequence $\{s_{n,n}\}$, using Arvesons observations), one may consider $\{s_{n,n-1}\}$ and take the limit to get a better lower bound for $\sigma_e(T)$. This process can be repeated till we arrive at the best lower bound.

CHAPTER IV

SINGULAR NUMBERS FOR UNBOUNDED OPERATORS

In this chapter an attempt is made to extend the concept of s-numbers to a class of linear operators between Banach spaces which contains the class of bounded linear operators and some unbounded operators. In the case of bounded operators this coincides with the classical s-numbers.

Let X and Y be Banach spaces. Let $B(X,Y)$, $C(X,Y)$ and $B_T(X,Y)$ denote the class of all bounded linear operators, the class of all closed linear operators and the class of all T-bounded operators from X to Y respectively, where T is a linear transformation from X to Y.

This chapter comprises of 2 sections of which the first one deals with β and β' numbers which are defined using Kato's notion of gap of operators, second one deals with s' – numbers which is defined using Kato's notion of relative boundedness of operators. The second section deals with $\tilde{s}$ - numbers.

4.1 β and β' numbers

Let us recall Kato's notion of gap of operators.

Definition 4.1.1 [20].

For every $S, T \in B(H)$, $\hat{\delta}(T,S)$ is defined by

$$\hat{\delta}(T,S) = \max\,[\delta(G(T), G(S)), \delta(G(S), G(T))]$$

$$\delta(G(T), G(S)) = \sup_{u \in S_{G(T)}} \text{dist}\,(u, G(S))$$

where $S_{G(T)} = \{\, u \in G(T) : \|u\| = 1 \,\}$

$G(T)$, $G(S)$ are subspaces of the product space $H \times H$. $\hat{\delta}(T,S)$ is called the gap between T and S.

We define β and β' numbers as follows.

Definition 4.1.2.

Let $X = Y = H$ be a Hilbert space. For every operator $T \in B(H)$ the n^{th} βeta and βeta prime numbers are defined by

$$\beta_n(T) = \frac{b_n(T)}{\sqrt{1 - bn(T)^2}}$$

$$\beta'_n(T) = \frac{b'_n(T)}{\sqrt{1 - b'_n(T)^2}}$$

where
$$b_n(T) = \inf\{\hat{\delta}\ (T,L)\ : rank\ L < n, \|L\| \leq 1\ \}$$

$$b'_n(T) = \inf\ \{\hat{\delta}\ (T - L, 0)\ : rank\ L < n\}.$$

Proposition 4.1.3.

Let H be a Hilbert space. Then for T in $B(H)$,

$$\|T\| = \beta_1(T) \geq \beta_2(T) \geq \ldots \geq 0.$$

Proof.

It is clear that $\beta_n(T) \geq \beta_{n+1}(T)$ for all n. So we prove that $\beta_1(T) = \|T\|$. For that it is enough to show that $b_1(T) = \dfrac{\|T\|}{\sqrt{1 + \|T\|^2}}$

By definition

$$b_1(T) = \hat{\delta}(T,0)$$

$$= \max\{\delta(G(T), G(0)), \delta(G(0), G(T))\ \}$$

But

$$\delta(G(T), G(0)) = \sup_{u \in S_{G(T)}} dist(u, G(0))$$

where

$$S_{G(T)} = \{(x, Tx) : \|x\|^2 + \|Tx\|^2 = 1\ \}$$

Now

$$dist((x,Tx),G(0))^2 = \inf\{\|x-y\|^2 + \|Tx\|^2\}$$

$$= \|Tx\|^2$$

Therefore,

$$dist((x,Tx),G(0)) = \|Tx\|$$

Let x' in H be that $\|x'\| = 1$

Put

$$x = \frac{x'}{\sqrt{1+\|Tx'\|^2}}$$

Therefore,

$$\sup_{u\in S_{G(T)}} dist\,(u,G(0)) \geq \frac{\|Tx'\|}{\sqrt{1+\|Tx'\|^2}} \geq \frac{\|Tx'\|}{\sqrt{1+\|T\|^2}}$$

Therefore,

$$\sup_{u\in S_{G(t)}} dist\,(u,G(0)) \geq \frac{\|T\|}{\sqrt{1+\|T\|^2}}$$

So

$$\delta(G(T),G(0)) \geq \frac{\|T\|}{\sqrt{1+\|T\|^2}}$$

Now

$$\delta((G(0),G(T)) = \sup_{u\in S_{G(0)}} dist(u,G(T))$$

where $\qquad S_{G(0)} \quad \{\,(x,0): \quad \|x\|^2 = 1\,\}$

$$dist(u,G(T))^2 = \inf\{\|x-y\|^2 + \|Tx\|^2\}$$

$$\leq \inf_{0\leq t\leq 1} dist(u,(tx,T(tx)))^2$$

$$\leq \inf_{0\leq t\leq 1}\{\|x-tx\|^2 + t^2\|Tx\|^2\}$$

$$\leq \inf_{0\leq t\leq 1}\{(1-t)^2 + t^2\|Tx\|^2\}$$

One can see that the infimum is attained when $t = \dfrac{1}{1+\|Tx\|^2}$ and the infimum equals $\dfrac{\|Tx\|^2}{1+\|Tx\|^2}$

Hence

$$\delta(G(0),G(T)) \le \frac{\|T\|}{\sqrt{1+\|T\|^2}}$$

Thus

$$\hat{\delta}(T,0) \ge \frac{\|T\|}{\sqrt{1+\|T\|^2}}$$

Therefore,

$$b_1(T) \ge \frac{\|T\|}{\sqrt{1+\|T\|^2}} \textit{ for every } T \textit{ in } B(H).$$

For u (x,Tx) in $S_{G(T)}$, let

$$z = \frac{x}{\sqrt{1-\|Tx\|^2}}$$

Then

$$\|Tx\|^2 = \frac{\|T(z)\|^2}{1+\|Tz\|^2} \le \frac{\|T\|^2}{1+\|T\|^2} \text{ since } \|z\| = 1$$

Therefore,

$$\delta(G(T),G(0)) \le \frac{\|T\|}{\sqrt{1+\|T\|^2}}$$

Thus using the previous inequalities we get

$$\hat{\delta}(T,0) \le \frac{\|T\|}{\sqrt{1+\|T\|^2}}$$

Hence

$$b_1(T) = \frac{\|T\|}{\sqrt{1+\|T\|^2}}.$$

Proposition 4.1.4.

$\beta_n(T) = 0$ if and only if rank $T < n$ and $\|T\| \leq 1$

Proof.

Assume that $\beta_n(T) = 0$.

Hence $\quad 0 = b_n(T) = \inf\{\hat{\delta}(T,L) : rank\ L < n, \|L\| \leq 1\}$

Therefore there exists a sequence $\{L_k\}$ in $B(H)$ such that rank $L_k < n$ and $\|L_k\| \leq 1$

such that

$$\lim_{k\to\infty} \hat{\delta}(T,L) = 0$$

That is,
$$\lim_{k\to\infty} \delta\,(G(T), G(L_k)) = 0 \qquad (4.1) \text{ and}$$

$$\lim_{k\to\infty} \delta\,(G(L_k), G(T)) = 0 \qquad (4.2)$$

Now (4.1) implies that, given $\varepsilon > 0$, there exists a positive integer N

such that

$$\sup_{\|x\|^2+\|Tx\|^2=1} \|x - y_x^k\|^2 + \|Tx - L_k y_x^k\|^2 < \frac{\varepsilon^2}{4}, \forall k \geq N \qquad (4.3)$$

Now let $x' \in H$ be such that $\|x'\| = 1$.

Now put $\qquad x = \dfrac{x'}{1+\|Tx'\|^2}$

Now (4.3) implies

$$\begin{aligned}\|Tx - L_k x\| &\leq \|Tx - L_k y_x^k + L_k y_x^k - L_k x\| \\ &\leq \|Tx - L_k y_x^k\| + \|L_k\|\|y_x^k - x\| \\ &< \varepsilon, \ \textit{for all } k \geq N\end{aligned}$$

Hence

$$\sup_{\|x'\|=1}\|T(x') - L_k(x')\| < (1+\|T\|^2)\varepsilon \textit{ for all } k\geq N$$

Since rank $L_k < n$ *for all k*, rank $T < n$.

Conversely rank $T < n$ and $\|T\| \leq 1$ implies that $\beta_n(T) = 0$

Proposition 4.1.5.

$\beta_n(I) = 1$ for all n.

Proof.

We show that $bn(I) = \frac{1}{\sqrt{2}}$ for all n.

$$b_n(I) = \inf\{\hat{\delta}(T, L) : rank L < n, \|L\| \leq 1 \}$$

Now

$$\hat{\delta}(I,L) = \max\{\delta(G(I), G(L)), \delta(G(L), G(I)) \}$$

Let $u = (x,x) \in G(I)$, $\|x\|^2 + \|x\|^2 = 1$

Then

$$d(u, G(L))^2 = \inf\left\{\|x-y\|^2 + \|x - Ly\|^2 : \|x\| = \frac{1}{\sqrt{2}} \right\}$$

$$\geq \inf\{\|x-y\|^2 : \|x\| = \frac{1}{\sqrt{2}}\} + \inf\left\{\|x - Ly\|^2 : \|x\| = \frac{1}{\sqrt{2}} \right\}$$

$$\inf\left\{\|x\|^2 + \|Ly\|^2 - 2\operatorname{Re}\langle x, Ly\rangle : \|x\| = \frac{1}{\sqrt{2}}\right\}$$

$$\frac{1}{2} + \inf\left\{\|Ly\|^2 - 2\operatorname{Re}\langle x, Ly\rangle : \|x\| = \frac{1}{\sqrt{2}} \right\}$$

Now choose x such that $L^*(x) = 0$.

Then

$$d(u, G(L))^2 \geq \frac{1}{2}$$

Hence

$$\delta((G(I), G(L)) \geq \frac{1}{\sqrt{2}}$$

Therefore,

$$\hat{\delta}(I, L) = \max[\ \delta(G(I), G(L)), \delta(G(L), G(I)) \geq \frac{1}{\sqrt{2}}$$

Hence

$$b_n(I) \geq \frac{1}{\sqrt{2}}$$

It is trivial to see that $b_n(I) \leq \frac{1}{\sqrt{2}}$, since $b_n(I) \leq b_1(I) = \frac{1}{\sqrt{2}}$.

Hence $\beta_n(I) = 1$.

Proposition 4.1.6.

β numbers are continuous with respect to gap convergence.

Proof.

Let $\{T_n\}$ be a sequence in $CL(H)$ such that $\hat{\delta}(T_n, T) \rightarrow 0$ as $n \rightarrow \infty$. For each positive integer k let L be any operator with rank $L < k$. Then

$$\hat{\delta}(T_n, L) \leq \hat{\delta}(T_n, T) + \hat{\delta}(T, L).$$

Therefore,

$$\inf_L \hat{\delta}(T_n, L) \leq \hat{\delta}(T_n, T) + \inf_L \hat{\delta}(T, L)$$

That is,

$$b_k(T_n) \leq b_k(T) + \hat{\delta}(T_n, T)$$

Similarly

$$b_k(T) \leq b_k(T_n) + \hat{\delta}(T_n, T)$$

Therefore

$$|b_k(T_n) - b_k(T)| \leq \hat{\delta}(T_n, T)$$

Thus $b_k(T_n) \rightarrow b_k(T)$ as $n \rightarrow \infty$ for each k.

Hence β_n is continuous.

Now some properties of β'_n numbers are considered. Its connection with the classical approximation numbers is established.

Proposition 4.1.7.

$\beta'_n(T) = a_n(T)$ for each bounded linear operator T.

Proof.

By definition

$$b'_n(T) = \inf\{\hat{\delta}(T-L,0) : rankL < n\}$$

$$= \inf\left\{\frac{\|T-L\|}{\sqrt{1+\|T-L\|^2}} : rankL < n\right\}$$

Now

$$\frac{a_n(T)}{\sqrt{1+a_n^2(T)}} \leq \frac{\|T-L\|}{\sqrt{1+\|T-L\|^2}}$$

Then

$$\frac{a_n(T)}{\sqrt{1+a_n^2(T)}} \leq \inf\left\{\frac{\|T-L\|}{\sqrt{1+\|T-L\|^2}} : rankL < n\right\}$$

$$\leq \frac{a_n(T)}{\sqrt{1+a_n^2(T)}}$$

Therefore ,

$$b'_n(T) = \frac{a_n(T)}{\sqrt{1+a_n^2(T)}}$$

Hence $\beta'_n(T)$ $a_n(T)$ for each bounded linear operator T on H.

Proposition 4.1.8.

Since $\hat{\delta}(T,0)$ 1 whenever T is unbounded $\hat{\delta}(T\text{-}L,0)$ 1 for any finite rank operator which is bounded and therefore $\beta'_n(T) = \infty$ for any unbounded operator. It is clear that $\beta_1(T)$ is also infinite whenever T is unbounded.

Now we introduce the so called s'- numbers in an attempt to associate a sequence of numbers to any operator belonging to a class of operators which includes the bounded ones also. Here we use Kato's notion of relative boundedness of operators.

4.2 s'- numbers

Let us first recall Kato's notion of relative boundedness.

Definition 4.2.1.

Let X and Y be normed linear spaces and T and A be linear operators from X to Y such that Domain(T) $\subseteq$ Domain(A) .If there exists non negative real numbers a and b such that

$$\|Au\| \leq a\|u\| + b\|Tu\| \text{ for every } u \text{ in } D(T).$$

($D(T)$ is the domain of T), then A is said to be relatively bounded with respect to T.

Definition 4.2.2.

For each T in $B(X,Y)$. Let A be in $B_T(X,Y)$ and let L be a bounded operator with rank $L < k$. Let a_L be the least positive number such that

$$\|(A-L)u\| < a_L\|u\| + b_L\|Tu\|$$

Where b_L is the relative bound of A-L with respect to T. Then the k^{th} s'_k number is defined as,

$$s'_k(T) = \inf_L a_L$$

Proposition 4.2.3.

Let S and R be in $B_T(X,Y)$ be such that R is bounded. Then

$$s'_n(S+R) \leq s'_n(S) + \|R\|$$

Proof.

Let rank $L < n$

Then

$$\|(S+R-L)u\| \leq \|(S-L)u\| + \|Ru\|$$

$$\leq \|(S-L)u\| + \|R\|\|u\|$$

$$\leq (a_L + \|R\|)\|u\| + b_L(S)\|Tu\|$$

Where $b_L(S)$ denote the relative bound of $S - L$ with respect to T.

Hence

$$b_L(S+R) \leq b_L(S)$$

Since R is any bounded linear transformation we get

$$b_L(S) = b_L(S+R-R) \leq b_L(S+R)$$

Thus

$$b_L(S+R) = b_L(S)$$

Hence by definition

$$a_L(S+R) \leq a_L(S) + \|R\|$$

Therefore,

$$s'_n(S+R) \leq s'_n(S) + \|R\|$$

Remark 4.2.4.

s' - numbers do not satisfy the third axiom of s-numbers.

Proof.

The following simple example shows that $s'_n(PQ) \nleq \|P\| s'_n(Q)$ even if P is bounded.

Let $X = Y = C[0,1]$

$$P(u)(t) = \int_0^t u(x)dx,\ u \in C[0,1] \quad \text{and}$$

$$Q(u)(t) = u'(t),\ \ u \in D(Q) = \{u : u' \ \text{exists and}\ u(0) = 0\} \text{and}$$

$$T(u)(t) = u'(t),$$

Then

$$\|(PQ)(u)\| = \|u\|$$

$$\|Q(u)\| = 0.\|u\| + 1.\|Q(u)\|$$

Therefore,

$$s_1'(PQ) = 1, \ \text{But} \quad s_1'(Q) = 0$$

Hence the proof.

Now the following simple result says that s_n'-numbers coincides with approximation numbers for bounded linear operators.

Proposition 4.2.5.

Let S be in $B(X,Y)$. Then $s_n'(S) = a_n(S)$.

Proof.

$$s_n'(S) = \inf\{a_L : rankL < n\}$$

When

$$\|(S - L)u\| \le a_L\|u\| + b_L\|Tu\|$$

where a_L is the minimum associated with T bound b_L of S L. But $b_L = 0$.

$$\inf\{a_L : rankL < n\} = \inf\{\|S - L\| : rankL < n\} = a_n(S).$$

Remark 4.2.6.

Thus we find that s_{nT}' - numbers is an extension of the classical approximation numbers to the class $B_T(X,Y)$.

Next some examples are considered. They are taken from [20], with some minor modifications occasionally.

Example 4.2.7

Let $X = L_p[a,b]$, $T(u) = u'$, $A(u) = u(c)$ where $c \in [a,b]$. Here domain $(T) = \{u: u' \in L_p[a,b]\}$ and domain $A = C[a,b]$

Then

$$\|Au\| \le \frac{\|u\|_1}{b-a} + \|u'\|_1 \quad u \in domain(T)$$

It is known that when $c = a$ or b, T-bound of A is exactly 1. Hence if $c = a$ or_b,

$$s'_T(A) \le \frac{1}{b-a}$$

Now take $u(x) = 1$ for every x.

Then

$$A(u) = 1,\ u' = 0,\ \frac{\|u\|_1}{b-a} = 1$$

This shows that

$$s'_1(A) = \frac{1}{b-a}$$

Example 4.2.8.

Let $X = L_p[a,b]$, $T(u) = u'$, $A(u) = u'(c)$

Let

$$g(x) = \frac{(x-a)^{n+1}}{(b-a)(c-a)^n},\ h(x) = \frac{-(n+1)n(x-a)^{n-1}}{(b-a)(c-a)^n} \quad a \le x \le c$$

$$g(x) = \frac{-(b-x)^{n+1}}{(b-a)(b-c)^n},\ h(x) = \frac{-(n+1)n(b-x)^{n-1}}{(b-a)(b-c)^n} \quad c < x \le b$$

Then

$$u'(c) = \langle u'', g\rangle + \langle u, h\rangle$$

When

$$\langle u'', f\rangle = \int_a^b u''(t)f(t)dt$$

The above identity is taken from Kato[20].

It is also known that

$$|u'(c)| \leq \|g\|_q \|u''\|_p + \|h\|_q \|u\|_p$$

$$\leq \left(\frac{(b-a)}{nq+q+1}\right)^{1/q} \|u''\|_p + \frac{(n+1)n}{(b-a)^{1+\frac{1}{p}}(nq-q+1)^{\frac{1}{q}}}\|u\|_p \ \left(\frac{1}{p}+\frac{1}{q}=1\right)$$

If $p>1$, then the coefficient of $\|u''\|_p$ on the right side of the above inequality can be made arbitrarily small by taking n- large. Hence $s'_T(A) = +\infty$.

If $p = 1$, then $q = +\infty$ and the above inequality reduces to

$$|u'(c)| \leq \|u''\|_1 + \frac{2\|u\|_1}{(b-a)^2}$$

It is known that the $T-$ bound of A is exactly 1.

Take $u(t) = t - a$

Then

$$|u'(c)| = 1, \ \|u''\|_1 = 0 \text{ and } \|u\|_1 = 1$$

This shows that

$$s'_T(A) = \frac{2}{(b-a)^2}$$

Example 4.2.9.

Let $X = L_1[a,b]$ $Y = C[a,b]$

$Tu = u'$, and $Au = u$, A:domain $(A) \to C[a,b]$, when domain $A = C[a,b]$ with L_1-norm.

Then

$$s'_T(A) = \frac{1}{b-a}$$

Example 4.2.10

The following example illustrates the fact that, when A and T are unbounded and if the $T-$ bound of A is 0, then $s'_{1T}(A) = +\infty$

$$X = L_p\,[a,b]$$

$$A(u) = P_1(x)\ u'$$

$$T(u) = P_0(x)\ u'' + P_1(x)\ u' + P_2(x)u,$$

where P_i is a polynomial for $i = 0, 1$ *and* 2. It is proved in Kato [20] that the $T-$ bound of A is zero. Hence $s'_{1T}(A) = +\infty$

Example 4.2.11

Let H be a separable Hilbert space and $\{e_1, e_2, \ldots\}$ be an orthonormal basis for H. For any $u \in H$, $u = \sum \langle u, e_k \rangle e_k$

Define

$$Tu = \sum_{k=1}^{\infty} k \langle u, e_k \rangle e_k,$$

$$Au = \sum_{k=1}^{\infty} k^{1/2} \langle u, e_k \rangle e_k$$

$$\|Au\|^2 = \sum_{k=1}^{\infty} k |\langle u, e_k \rangle|^2$$

$$\|Tu\|^2 = \sum_{k=1}^{\infty} k^2 |\langle u, e_k \rangle|^2$$

To find the smallest posssible value for β such that

$$\|Au\| \le \alpha\|u\| + \beta\|Tu\| \text{ for all u in } D(T)$$

Assume that

$$\|Au\| \le \alpha\|u\| + \beta\|Tu\|$$

Therefore, the following inequality holds;

$$\|Au\|^2 \le \alpha^2\|u\|^2 + \beta^2\|Tu\|^2 + 2\alpha\beta\|u\|\|Tu\| \qquad (4.4)$$

If $\|Au\|^2 \le \alpha^2\|u\|^2 + \beta^2\|Tu\|^2$, then (4.4) holds.

If $k \le \alpha^2 + \beta^2 k^2$, then also (4.4) holds. Thus whenever α, β satisfies the inequality

$$\beta^2 k^2 - k + \alpha^2 \ge 0 \qquad \text{(4.5), for all real k.}$$

Then (4.4) holds.

Now (4.5) holds if $1-4\beta^2\alpha^2 \le 0$

Therefore whenever $\alpha\beta \ge \frac{1}{2}$ (4.4) is satisfied.

If β tends to zero then α tends to ∞.

Therefore $s_{T}(A) = \infty$.

s-number set 4.2.12.

The mission of this chapter is to assign sequence of numbers to closed linear transformations between Hilbert spaces in such a way that sequences associated with bounded linear transformation are classical approximation numbers.

It is partially achieved by associating the so called s – number sets.

4.2.13

Definition. Let $CL(X,Y)$ be the class of all closed, densly defined linear transformations between two Hilbert spaces X and Y. For each $A \in CL(X,Y)$, let K_A denote the set of all linear transformations in $CL(X,Y)$ with which A is relatively bounded. Now the s – number set $\tilde{s}_n(A)$ is defined for each positive integer n as follows;

$$\tilde{s}_n(A) = \{s'_{nT}(A) : T \in K_A\}$$

Proposition 4.2.14.

For A in $BL(X,Y)$, $\widetilde{s}_n(A)$ is the singleton set $\{a_n(A)\}$, consisting of the n^{th} approximation number.

Proof.

It has been shown in the previous section that $s'_{nT}(A) = a_n(A)$ for any T in K_A.

Remark 4.2.15.

For each unbounded element A of $CL(X,Y)$ 0 is in $\widetilde{s}_n(A)$. This is because $s'_{n,A}(A) = 0$, and $+\infty$ is in $\widetilde{s}_n(A)$ since $s'_{nT}(A) = +\infty$, whenever T is $BL(X,Y)$, *and* $T \in K_A$.

We conclude this chapter with the following remark.

Remark 4.2.16.

The above observations shows that the set $\widetilde{s}_n(S)$ gives a measure of unboundedness.

The following questions are of interest in the above respect

Q.1.When is $\widetilde{s}_n(S) = [0,\infty)$?

Q.2. When is $\widetilde{s}_n(S)$ is discrete?

Q.3.Given the positive real number α, what conditions on S guarantee that $\alpha \in \widetilde{s}_n(S)$?

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