TOPICS IN RING THEORY

A TORSION THEORETIC APPROACH TO ORE LOCALISATION AND LINKS

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CERTIFICATE

Certified that the work reported in the present thesis is based on the bonafide work done by Smt.Santha A., under my guidance in the Department of Mathematics and Statistics, Cochin University of Science and Technology, and has not been included in any other thesis submitted previously for the award of any degree.

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CHAPTER ONE

INTRODUCTION AND PRELIMINARIES

INTRODUCTION

In 1921, Emmy Noether proved that a commutative ring has the ascending chain condition on ideals if and only if all ideals are finitely generated. Such rings, now called commutative Noetherian rings, were extensively studied from the 1920s onwards because of their importance in algebraic geometry.

The noetherian condition is very natural in commutative ring theory, since it holds for the rings of integers in algebraic number fields and the co-ordinate rings crucial to algebraic geometry.

The first important result in the theory of non-commutative Noetherian rings was Goldie's theorem (1958) which gives an analogue of the familiar result that every commutative domain can be embedded in its quotient field. Since then, Noetherian ring theory has steadily gathered strength, partly from its own impetus and partly through feedback from neighbouring areas in which Noetherian ideas found applications. By now, various methods and results from the theory of commutative Noetherian rings have been adapted to non-commutative Noetherian rings.

In commutative ring theory, we have the elementary but powerful technique of localisation at a prime ideal. If R is a commutative ring and P is a prime ideal in R, then the set $S = R \setminus P$ is multiplicatively closed, and the localisation of R at P is got by considering the set $R \times S$ and defining an equivalence relation $\sim$ on it by $(a,b) \sim (c,d)$ if and only if $(ad - bc)\, e = 0$ for some $e \in S$. This gives the ring of fractions R_P. This is the generalisation of the formulation of the field of fractions of a commutative integral domain (in that case, $S = R \setminus \{0\}$).

We can reduce questions on arbitrary rings and modules over such rings to the case of local rings via localisation at prime ideals. In many important instances, a result will be valid for a ring R, if it holds for every localised ring R_P (where P is a prime ideal in R). For a non-commutative ring, such a localisation is not, in general, possible, even at the zero ideal of an integral domain. Ore (1930) characterised those non-commutative domains which have right rings of fractions that are division rings. For years, mathematicians worked to find a procedure which would enable one to localise non-commutative Noetherian rings at prime ideals. The standard procedure that emerged took the commutative situation and the situation in Goldie's theorem as models and attempted to use Ore's method to localise Noetherian rings at semiprime ideals.

In the 1970s and 1980s, Jategaonkar, Mueller and others worked on the problem of localisation at a prime ideal. They found that there exist "links" between prime ideals and that these links "obstruct" localisation. But in the case of Noetherian rings satisfying the "second layer condition", Jategaonkar has found that it is possible to describe localisation at a prime (or a collection of primes) under certain conditions.

Goodearl (1988) defined links between uniform injective right modules over a right Noetherian ring. He observed that links between "tame" injectives correspond to prime ideal links, while, there exist other injective module links which provide more obstructions to Ore localisations than prime ideal links do.

For a right Noetherian ring, there is a one-to-one correspondence between uniform injectives and prime torsion classes. Because of this connection, we have tried to study Ore localisation using the torsion-theoretic approach.

Before proceeding further, we take a look at the preliminary definitions and results required in the rest of the thesis.

PRELIMINARY DEFINITIONS AND RESULTS

Most of the material in this section is taken from [G4], [GW], [J], [MR] and [S1].

CONVENTIONS

All rings are assumed to be associative with 1 and all modules are unital. We denote the fact that M is a right R-module, by writing M_R. The set of all right R-modules is denoted by $\mathcal{Mod}$-R. We use the notations $\leq$, $<$, $\nleq$ for inclusions among submodules or ideals. In particular, if A is a module, the notation $B \leq A$ means that B is a submodule of A and the notation $B < A$ means that B is a proper submodule of A. An ideal refers to a two-sided ideal. One sided ideals will be referred to as such. This convention applies to other one-sided properties also.

THE NOETHERIAN CONDITION

A collection $\mathscr{A}$ of subsets of a set A satisfies the *ascending chain condition* (*ACC*) if there does not exist a properly ascending infinite chain $A_1 < A_2 < \ldots\ldots$ of subsets from $\mathscr{A}$. A set $B \in \mathscr{A}$ is said to be *maximal* in $\mathscr{A}$, if there does not exist a set in $\mathscr{A}$ which properly contains B.

PROPOSITION 1.1: Let R be a ring and A be a right R-module. The following conditions are equivalent:

a) A has ACC on submodules.

b) Every non-empty family of submodules of A has a maximal element.

c) Every submodule of A is finitely generated.

A right R-module A is said to be *Noetherian* if and only if the above equivalent conditions are satisfied. A ring R is *right*(*left*) *Noetherian* if and only if the right R-module R (left R-module R) is Noetherian. R is *Noetherian* if it is both right and left Noetherian.

If B is a submodule of A, then A is Noetherian if and only if B and A/B are Noetherian. Any finite direct sum of

Noetherian modules is Noetherian. If R is a Noetherian ring, then all finitely generated right R-modules are Noetherian.

PRIME IDEALS

A proper ideal P in a commutative ring R is said to be *prime* if whenever we have two elements a and b of R such that $ab \in P$, it follows that $a \in P$ or $b \in P$, equivalently, P is a prime ideal if and only if the factor ring R/P is a domain. We need a non-commutative analogue of a prime ideal. An ideal P in a ring R is said to be *completely prime* if R/P is an integral domain.. Thus, if R is commutative, P is prime if and only if it is completely prime.

There are non-commutative rings, however, in which there are not many completely prime ideals, and sometimes none. For example, in a simple Artinian ring, the only proper ideal is the zero ideal. Also, we would like every maximal ideal to be prime. The following definition, proposed by Krull in 1928, satisfies this property, and reduces to the familiar one in commutative rings: P is *prime* if for any ideals I and J, $IJ \leq P \Rightarrow I \leq P$ or $J \leq P$. The set of prime ideals of R is denoted by $\mathcal{S}pec\ R$. If O is a prime ideal, we say that R is a *prime ring*. If O is a completely prime ideal, R is a *domain*.

PROPOSITION 1.2 : For a proper ideal P in a ring R, the following are equivalent:

a) P is a prime ideal.

b) R/P is a prime ring.

c) If x, $y \in R$ with $xRy \leq P$, either $x \in P$ or $y \in P$.

d) If I and J are any right ideals of R such that $IJ \leq P$, either $I \leq P$ or $J \leq P$.

e) If I and J are any left ideals of R such that $IJ \leq P$, either $I \leq P$ or $J \leq P$.

f) If I, J are right ideals of R, such that $I \cap J \subseteq P$, then either $I \leq P$ or $J \leq P$. This is a lattice theoretic condition.

It immediately follows that if P is a prime ideal in a ring R and $J_1, \ldots, J_n$ are right ideals of R such that $J_1 J_2 \ldots J_n \leq P$ then some $J_i \leq P$.

By a *maximal ideal* in a ring, we mean an ideal which is a maximal element in the collection of proper ideals. Then, every maximal ideal M of a ring R is a prime ideal.

SEMIPRIME IDEALS

A *semiprime ideal* in a ring R is any ideal of R which is an intersection of prime ideals.

PROPOSITION 1.3: For an ideal I in a ring R, the following are equivalent:

a) I is a semiprime ideal.

b) If J is any ideal of R such that $J^2 \leq I$, then $J \leq I$.

c) If $x \in R$ with $xRx \leq I$, then $x \in I$.

A *semiprime ring* is any ring in which 0 is a semiprime ideal. The *prime radical* of a ring R is the intersection of all the prime ideals of R. A ring R is semiprime if and only if its prime radical is zero. In any ring R, the prime radical equals the intersection of the minimal prime ideals of R

ANNIHILATORS

If M is a right R-module, the *annihilator* of M, written ann M (or $\text{ann}_R M$) is the set $\{r \in R : mr = 0 \text{ for all } m \in M\}$. If M is a right R-module and S is a subset of R, then the *annihilator of S in M*, written $\text{ann}_M S$ is $\{x \in M : xS = 0\}$. If S is a left ideal of R, then $\text{ann}_M S$ is a submodule of M. If N is any subset of M, the *annihilator of N* is ann $N = \{ r \in R : Nr = 0 \}$. Ann N is a right ideal of R, and if N is a submodule of M, then ann N is a two-sided ideal. In particular, this defines the right annihilator r-ann S of a subset S of R:

r-ann $S = \{ r \in R : sr = 0 \ \forall \ s \in S \}$. The left annihilator l-ann S of S is defined similarly.

A right R-module M is said to be *faithful* if ann $M = 0$. M is *fully faithful* if ann $N = 0$ for every non-zero submodule N of M.

ESSENTIAL SUBMODULES

A submodule M' of M is said to be *essential in* M, denoted $M' \leq_e M$, if $N \neq 0 \Rightarrow N \cap M' \neq 0$ for any submodule N of M. If $M' \leq_e M$, then M is called an *essential extension* of M'. If R is considered as a right (or left) R-module, we obtain *essential right* (or *left*) *ideals*. A module M is *uniform* if all its non-zero submodules are essential

A ring R is *right bounded* if every essential right ideal of R contains an ideal which is essential as a right ideal. A ring R is *right fully bounded* if every prime factor ring of R is right bounded. A *right* (*left*) *FBN ring* is any right (left) fully bounded right (left) Noetherian ring. An *FBN ring* is any right and left FBN ring.

ASSASSINATORS AND PRIMARY MODULES

Let R be a right Noetherian ring. and let V be a uniform right R-module. Then the set of the annihilator ideals of non-zero submodules of V has a unique largest member, say P.

Then P is a prime ideal of R, and is called the *assassinator* of V, denoted ass V. For any non-zero submodule W of V, we have ass $W = P$. Moreover, setting $W = \text{ann}_V P$, we have $W = 0$ and ass $W = \textit{ann}\ W = P$.

For an arbitrary right R-module, the set

$$\{ \textit{ass}\ V : V \text{ is a uniform submodule of } M \}$$

is called the *assassinator of* M, and is denoted as ass M. The members of ass M are often referred to as the *assassinator prime ideals* of M.

A non-zero right module M over a right Noetherian ring is called a *primary module* if ass M is a singleton set. If P is the sole member of ass M, the module M is called a *P-primary module*. For any prime ideal P in a right Noetherian ring R, the class of all P-primary modules is closed under non-zero submodules, essential extensions and arbitrary direct sums.

Let S be a semiprime ideal in a right Noetherian ring R. A right R-module M is called an *S-primary module* if ass $M \subseteq \text{ass}(R/S)$.

INJECTIVE MODULES

A right R-module A is *injective* provided that for any right

R-module B and any submodule C of B, all homomorphisms $C \longrightarrow A$ extend to homomorphisms $B \longrightarrow A$. Given $M \in$ *Mod*-R is called an *injective envelope (injective hull)* of M, if E is a minimal injective module containing M. Alternatively, an injective hull for M turns out to be a maximal essential extension of A.

PROPOSITION 1.4:

i) Every module has an injective envelope, unique upto isomorphism and denoted by $E(M)$.

ii) A right R-module M is injective if and only if $M = E(M)$.

iii) If $M \leq_e N$, then $E(M) = E(N)$.

iv) If M is injective and $M \leq N$, then M is a direct summand of N.

v) If $\bigoplus_{\alpha \in A} E(M_\alpha)$ is injective, (for instance, if A is finite), then $E(\bigoplus_{\alpha \in A} M_\alpha) = \bigoplus_{\alpha \in A} E(M_\alpha)$.

vi) Direct products and direct summands of injective modules are injective.

vii) A non-zero module M is uniform if and only if $E(M)$ is indecomposable.

viii) If E is an indecomposable injective module, then E is the injective hull of every non-zero submodule of E.

If M, M' are right R-modules such that $E(M) = E(M')$, we say that M and M' are *similar*.

SIMPLE AND SEMISIMPLE MODULES

A right R-module A is said to be *simple* if A has no proper submodules. A ring R is *simple* if it has no proper ideals. The *socle* of a right R-module A is the sum of all simple submodules of A and is denoted by soc A. This is the direct sum of some simple submodules of A. A is *semisimple* if A = soc A if and only if A is a direct summand of any module containing it.

ARTINIAN MODULES

A module A is *Artinian* if A satisfies the descending chain condition (DCC) on submodules, i.e., there does not exist a properly descending infinite chain of submodules of A. A ring R is called *right(left) Artinian* if the right R-module R (left R-module R) is Artinian. If both conditions hold, R is called an *Artinian ring*. A right R-module A is Artinian if and only if A/B and B are Artinian where B is a submodule of A. Any finite direct sum of Artinian modules is Artinian, If R is a right Artinian ring, all finitely generated right

R-modules are Artinian. If R is a right Artinian ring, then R is also right Noetherian. If R is a non-zero right or left Artinian ring, then all prime ideals in R are maximal.

SEMISIMPLE ARTINIAN RINGS

In a ring R, the following sets coincide:

a) The intersection of all maximal right ideals.

b) The intersection of all maximal left ideals.

This intersection is called the *Jacobson radical* $J(R)$ of R.

PROPOSITION 1.5: For any ring R, the following conditions are equivalent:

a) R is right Artinian and semiprime.

b) R is left Artinian and semiprime.

c) All right R-modules are semisimple.

d) All left R-modules are semisimple.

e) R_R is semisimple.

f) ${}_RR$ is semisimple.

g) R is right Artinian and $J(R) = 0$.

h) R is left Artinian and $J(R) = 0$.

i) All right R-modules are injective.

j) All left R-modules are injective.

k) $R = M_{n_1}(D_1) \times M_{n_2}(D_2) \ldots \times M_{n_k}(D_k)$ for some positive integers $n_1, n_2, \ldots, n_k$ and division rings $D_1, \ldots, D_k$.

A ring satisfying the above conditions is called a *semisimple Artinian ring*.

PROPOSITION 1.6: For a ring R, the following conditions are equivalent

a) R is prime and right Artinian.

b) R is prime and left Artinian.

c) R is simple and right Artinian.

d) R is simple and left Artinian.

e) R is simple and semisimple Artinian.

f) $R \cong M_n(D)$ for some positive integer n and some division ring D.

The rings characterised above are referred to as *simple Artinian rings*.

RINGS OF FRACTIONS

In the theory of commutative rings, localisation plays a very important role. Most basic is the idea of a quotient field, without which one cannot imagine studying integral domains. Next comes the idea of localisation at a prime ideal, which reduces many problems to the study of local rings and their maximal ideals.

However, this is not the case with non-commutative rings. Although the set of non-zero elements is a multiplicative set in any domain, we have examples of domains which do not possess a division ring of quotients. It was in 1930, that Ore characterised those non-commutative domains which possess division rings of fractions. In 1962, Gabriel gave the necessary condition for a multiplicative set in a ring to have a right (left) ring of fractions.

A subset C of a ring R is a *multiplicatively closed set* if $1 \in C$ and $c_1, c_2 \in C \Rightarrow c_1c_2 \in C$. A multiplicatively closed subset C of R is a *right (left) Ore set* if, given $r \in R$, $c \in C$, there exist $s \in R$ and $d \in C$ such that $rd = cs$ ($dr = sc$). If C is a right and left Ore set, it is called an *Ore set*. C is a *right reversible set* if $r \in R$, $c \in C$ with $cr = 0$ in R implies $rd = 0$ for some $d \in C$. A right Ore, right reversible set is called a *right denominator set*. In a right Noetherian ring, every right Ore set is right reversible.

Let C be a multiplicative set in a ring R. A *right quotient ring* (or a *right ring of fractions* or *right Ore localisation*) of R relative to C is a pair (Q,f), where Q is a ring and f is a ring homomorphism from R to Q such that

a) $f(C)$ is a unit of Q for all $c \in C$.

b) Each element of Q has the form $f(r)f(c)^{-1}$ for some $r \in R$, $c \in C$.

c) Ker $f = \{ r \in R : rc = 0$ for some $c \in C \}$.

By abuse of notation, we usually refer to Q as the right ring of fractions and we write elements of Q in the form rc^{-1} for $r \in R$, $c \in C$.

THEOREM 1.7: Let C be a multiplicative set in a ring R. Then there exists a right ring of fractions for R with respect to C if and only if C is a right denominator set.

If C is the set of regular elements of R and if the right quotient ring $Q(R)$ of R relative to C exists, we say that R is a *right order* in $Q(R)$.

A ring R is a domain if it has no zero divisors. The non-zero elements in a domain form a multiplicative set and if $C = R \setminus \{0\}$, then we have the following corollary to the above theorem:

COROLLARY 1.8: A domain R has a right division ring of fractions if and only if C is a right Ore set if and only if the intersection of any two non-zero right ideals is non-zero.

A domain which satisfies this condition is called a *right Ore domain*.

GOLDIE'S THEOREMS

A very useful technique in commutative ring theory is to pass from a commutative ring R to a prime factor ring R/P. In the non-commutative case we could ask whether it is possible to pass to a factor ring from which a division ring may be built from fractions. Since non-commutative Noetherian rings need not have any factor rings which are domains, this is rather restrictive. Instead we look for factor rings from which simple artinian rings can be built using fractions. The main result is Goldie's theorem which says that if P is a prime ideal in a noetherian ring, then the factor R/P has a ring of fractions. It turns out to be no extra work to investigate rings from which semisimple rings of fractions can be built.

A *regular element* in a ring R is any non-zero-divisor,i.e., any element $x \in R$ such that $r\text{-ann}(x) = 0$ and $l\text{-ann}(x) = 0$.

A *right(left)annihilator* in a ring R is any right(left) ideal of R which equals the right(left)annihilator of some subset of R.

We say that a right R-module M has *finite Goldie dimension* if M does not contain a direct sum of an infinite number of non-zero submodules. A ring R is said to have *finite right Goldie dimension* if R has finite Goldie dimension as a right R-module.

PROPOSITION 1.9: If M has finite Goldie dimension, then there is a largest positive integer r such that M contains a direct sum of n non-zero submodules. This is called the *Goldie dimension* of M

A *right Goldie ring* is any ring R that has finite right Goldie dimension and ACC on annihilators. For example, every right Noetherian ring is right Goldie.

PROPOSITION 1.10 (Goldie): Let R be a semiprime right Goldie ring, and let I be a right ideal of R. Then I is an essential right ideal if and only if I contains a regular element.

THEOREM 1.11 (Goldie): A ring R is a right order in a semisimple ring if and only if R is a semiprime right Goldie ring.

THEOREM 1.12 (Goldie, Lesieur-Croisot): A ring R is a right order in a simple artinian ring if and only if R is a prime right Goldie ring.

Let R be a semiprime right Goldie ring. Any semisimple ring Q in which R is a right order is called a *right Goldie quotient ring of R*. An important property of Q_R is that it is an injective hull of R_R.

TORSION CLASSES

It is often convenient to think of localisation in the broader context of torsion classes. We can characterise the right Ore condition on a multiplicative set in terms of the associated torsion class. In this subsection we define right torsion classes and other torsion theoretic terms which we use later.

A *right torsion class* σ for a ring R is a non-empty class of right R-modules satisfying the following two conditions:

i) The direct sum of any family of modules in σ is also in σ.

ii) For any exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ of right R-modules, M belongs to σ if and only if M' and M'' both belong to σ.

It follows that a torsion class is closed under submodules and homomorphic images. The set of all torsion classes over R is denoted by $\mathcal{T}ors$-R. Over a commutative domain, the modules which are torsion in the usual sense form a right torsion class.

We define the notions that are usually associated with 'torsion'. Let σ be a right torsion class for a ring R. For any right R-module M, the unique largest submodule of M belonging to σ is called the *σ-torsion submodule* of M and is denoted as $\sigma(M)$. M is called a *σ-torsion module* if $\sigma(M) = M$ and a *σ-torsion-free module* if $\sigma(M) = 0$. The class of σ-torsion-free modules is closed under submodules, injective hulls, direct products and isormorphic copies. Let N be a submodule of M. Then N is said to be *σ-dense in M* if M/N is σ-torsion and *σ-closed in M* if M/N is σ-torsion-free. A σ-dense (σ-closed) submodule of R_R is called a *σ-dense (σ-closed) right ideal of R.*

A module $B \in \mathcal{Mod}\text{-}R$ is σ-torsion-free if and only if $\mathrm{Hom}(A,B) = 0$ or every σ-torsion module $A \in \mathcal{Mod}\text{-}R$. A module $A \in \mathcal{Mod}\text{-}R$ is σ-torsion if and only if $\mathrm{Hom}(A,B) = 0$ for every σ-torsion-free module $B \in \mathcal{Mod}\text{-}R$.

The set $\mathcal{Tors}\text{-}R$ is partially ordered under inclusion. Under this partial order, $\mathcal{Tors}\text{-}R$ is a complete lattice in which meet and join of any collection of torsion classes exist.

Given $A \in$ *Mod-R*, *the torsion class* $\chi(A)$ *cogenerated by* A is the greatest torsion class σ such that A is σ-torsion-free.

PROPOSITION 1.13:

(a) A right R-module B is $\chi(A)$-torsion if and only if $\mathrm{Hom}(C,A) = 0$ for all submodules C of B if and only if $\mathrm{Hom}(B,E(A)) = 0$. where $E(A)$ is the injective envelope of A.

(b) $\chi(0)$ is the largest element of *Tors-R*.

(c) If $A \in$ *Mod-R*, and $B \leq_{e} A$, then $\chi(A) = \chi(B)$.

(d) For every $\sigma \in$ *Tors-R*, there is an injective module E with $\sigma = \chi(E)$.

(e) If E, E' are injective right R-modules, then $\chi(E) \geq \chi(E')$ if and only if E' can be embedded in a product of copies of E.

Corresponding to the notion of a prime ideal in the lattice of two-sided ideals of a ring (a prime ideal is $\cap$-irreducible by proposition 1.2), Simmons[S1] has defined a prime element in the lattice of torsion classes.

A *point* of *Tors-R* is a $\wedge$-irreducible element, i.e.,a point is an element $\pi \in$ *Tors-R* such that $\pi \neq \chi(0)$ and $\sigma \wedge \tau \leq \pi$ implies $\sigma \leq \pi$ or $\tau \leq \pi$ for each $\sigma, \tau \in$ *Tors-R*. *Pt-R* denotes the set of points of *Tors-R*.

EXAMPLE 1.14: $\chi(A)$ is a point for each uniform module A over R, and $\chi(R/P)$ is a point for each prime ideal P of R.

Let $\tau \in \mathcal{T}ors\text{-}R$. A non-zero module $M \in \mathcal{M}od\text{-}R$ is τ-*critical* if M is τ-torsion-free and every non-zero submodule N of M is τ-dense in M. For example, a simple right R-module is τ-critical for every τ relative to which it is torsion-free. A non-zero right R-module is *critical* it is $\chi(M)$-critical.

Let $\tau \in \mathcal{T}ors\text{-}R$. If $\tau = \chi(M)$ for some critical right R-module M, then we say that τ is *prime* [G3]. For example, if M is a simple right R-module, then $\chi(M)$ is prime. The set of all prime torsion classes of $\mathcal{T}ors\text{-}R$ is denoted by $sp\text{-}R$. Every prime torsion class is a point. In a right Noetherian ring R, every point is a prime and hence $sp\text{-}R = pt\text{-}R$. The map $\phi : \mathcal{S}pec\text{-}R \longrightarrow pt\text{-}R$ is an injection, where $\phi(P) = \chi(R/P)$ for $P \in \mathcal{S}pec\text{-}R$.

If R is a commutative Noetherian ring or an FBN ring, then $pt\text{-}R = sp\text{-}R = \{ \chi(R/P) : P \in \mathcal{S}pec\ R \}$.

PROPOSITION 1.15:

(i) If $\sigma \in sp\text{-}R$, then there is a uniform injective right R-module E such that $\sigma = \chi(E)$.

(ii) If $\sigma \in \mathcal{Sp}\text{-}R$ and M, M' are σ-critical uniform injectives, then $E(M) = E(M')$.

(iii) Let R be a right Noetherian ring. If $\tau \in \mathcal{Tors}\text{-}R$ and $\tau \neq \chi(0)$, then

$\tau = \wedge \{ \chi(M) : M$ is a τ-critical right R-module $\}$.

A point π is a *principal point* if there is an ideal Q such that if I is a two-sided ideal of R, then I is π-dense if and only if $I \not\subseteq Q$. Then Q is the union of all the ideals of R that are not π-dense, and is Q is prime. We write $Q = \psi(\pi)$ and say that π is *Q-principal*. Every prime torsion class is a principal point.

PROPOSITION 1.16:

(i) If E is a uniform injective right R-module, then, $\psi(\chi(E)) = \text{ass } E$.

(ii) If π is a principal point and I is a two-sided ideal of R, then R/I is π-torsion if and only if $I \not\subseteq \psi(\pi)$.

C-TORSION AND C-TORSION-FREE MODULES

Given a multiplicative set C in a ring R, there is a torsion class ρ_C associated with it: A right R-module M is said to be *ρ_C-torsion* (or *C-torsion*) if, for every $m \in M$, there is $c \in C$

such that $mc = 0$. M is ρ_C*-torsion-free* if $\rho_C(M) = 0$, where $\rho_C(M)$ is the ρ_C-torsion submodule of M. If C is a right Ore set, then M is ρ_C-torsion-free if and only if, given $m \in M$, there is no $c \in C$ such that $mc = 0$.

The right Ore condition on C can be characterised in terms of C, as follows:

PROPOSITION 1.17: A multiplicative set C in a ring R is right Ore if and only if R/cR *is a* ρ_C-torsion module for every $c \in C$ if and only if for any $M \in$ *Mod*-R, $\rho_C(M) = \{ m \in M : mc = 0$ for some $c \in C \}$.

If C is the set of regular elements of R, we use the term '*torsion*' for 'C-torsion' and '*torsion-free*' for 'C-torsion-free'.

For any ideal I of R, we denote by $\mathscr{C}(I)$, the multiplicative set of elements of R that are regular modulo I, i.e., $\mathscr{C}(I) = \{r \in R : r+I$ is regular in $R/I \}$.

PROPOSITION 1.18 [LM]: If R is a right Noetherian ring, then $\rho_{\mathscr{C}(S)} = \chi(R/S)$ for any semiprime ideal S of R.

THE UNIFORM INJECTIVE MODULE E_P

Let P be a prime ideal in a right Noetherian ring R. We use the notation E_P to denote the right R-injective hull of a uniform right ideal of R/P. Upto an R-isomorphism, the indecomposable right R-injective module E_P is uniquely determined by P. If n denotes the Goldie dimension of R/P, then $E(R/P) \cong E_P^n$. Then $\mathrm{ass}(E_P) = \mathrm{ass}(E(R/P) = P$ and $\chi(R/P) = \chi(E(R/P) = \chi(E_P)$.

TAME MODULES AND WILD MODULES

Let V be a uniform right module over a right Noetherian ring R. Set $P = \mathrm{ass}\ V$, $W = \mathrm{ann}_V P$, and $R' = R/P$. Then P is a prime ideal of R, and the uniform right R'-module W has no non-zero unfaithful submodules. Moreover, as a module over the prime right Noetherian ring R', W is either a torsion module or a torsion-free module but not both.

If the R'-module W is torsion then we call the R-module V a *wild module* or a *P-wild module*, if we wish to convey that P is the assassinator of V. If the R'-module W is torsion-free then we call the R-module V a *tame module* or a *P-tame module*. W is torsion-free over R' $\Leftrightarrow$ $E(W)_R$ is a direct summand of

$E(R/P)_R$. Hence, a uniform right R-module V over a right Noetherian ring is P-tame if and only if $E(V) \cong E_P$. Thus a P-tame uniform module is uniquely determined by P upto similarity.

EXAMPLE 1.19: Uniform modules over commutative Noetherian rings and over right Artinian rings are tame. A uniform module over a simple Noetherian ring is tame if and only if it is torsion-free.

A SUMMARY OF THE THESIS

In this thesis, we study Ore localisation and related ideas from the point of view of torsion classes. Hence we have tried to get torsion-theoretic versions of various definitions and results of Jategaonkar, Goodearl etc.. In the case of commutative rings, for a prime ideal P, the set $R \setminus P$ is a right Ore multiplicative set. The localisation of R at P, which is the localisation of R at the set $R \setminus P$, always exists. If R is not commutative, then $R \setminus P$ is not necessarily a multiplicative set. The counterpart of $R \setminus P$ in this case is

$$\mathscr{C}(P) = \{ r \in R : r+P \in R/P \text{ is regular } \},$$

which is a multiplicative set and is equal to $R \setminus P$ if R is commutative. The localisation of R at $\mathscr{C}(P)$, called the *localisation of R at P* exists if and only if $\mathscr{C}(P)$ is a right denominator set. Hence it is important to find when $\mathscr{C}(P)$ is right Ore. In [G4] Goodearl considers, for a right module E over a right Noetherian ring R, the multiplicative set

$$\mathcal{N}(E) = \{ r \in R : \mathrm{ann}_E\, r = 0 \}.$$

By [J, proposition 3.1.4], if R is a prime ideal in a right Noetherian ring R, then $\mathscr{C}(P)$ is right Ore if and only if $\mathscr{C}(P) \subseteq \mathcal{N}(E(R/P))$. In chapter two, we get a generalisation of this result for an arbitrary multiplicative set C, by defining, for a torsion class $\tau \in \mathcal{T}ors\ R$, a multiplicative set

$$C_\tau = \{ r \in R : R/rR \text{ is } \tau\text{-torsion} \}.$$

Then C is right Ore if and only if $C \subseteq C_{\rho_C}$. We see that if E is an injective right R-module, then $C_{\chi(E)} = \mathcal{N}(E)$. Using torsion classes, we get some situations when $\mathscr{C}(P)$ is right Ore, for $P \in \mathcal{S}pec\ R$.

Given a multiplicative set C in a ring R, it is known that there is a right Ore set contained in C, which contains all right Ore sets contained in C. Using torsion classes, we construct this largest right Ore subset.

Let R be a right Noetherian ring. To study the regularity of an element of R at different prime ideals, it is convenient to put a topology on $\mathcal{S}pec\ R$. Two such topologies are the Zariski topology and the patch topology. In [S1], Simmons has generalised the Zariski topology to prime torsion theories. In the case of prime ideals, the patch topology and the "generic regularity condition" are important in the study of localisation. Hence we find it appropriate to get a torsion-theoretic version of these concepts. We discuss some properties of the patch topology and see that if R is Artinian, then the patch topology on $sp\text{-}R$ is the discrete topology. We also see some collections of prime torsion classes that satisfy the generic regularity condition.

As we have already mentioned, Jategaonkar has defined links between prime ideals and Goodearl has generalised these links between uniform injective right modules over a right Noetherian ring. In chapter four, we define links between prime torsion classes in such a way that an injective (Goodearl) link between two uniform injectives implies a generalised injective link between the prime torsion classes cogenerated by them. An example shows that these links provide more obstructions to Ore localisation than injective links do. We also see some sets that are "right stable" under these links.

The construction of the largest right Ore subset of a multipicative set has motivated us to define new links between prime torsion classes. In chapter five, we define these links (Ore links) and observe that they provide obstructions to Ore localisations in the following sense: If a multiplicative set C in R is a right Ore set, then, whenever $C \subseteq C_{\tau}$, we should have $C \subseteq C_{\sigma}$ for prime torsion classes σ and τ such that σ is Ore-linked to τ.

The following result of Jategaonkar is important in characterising localisable sets of prime ideals: Let X be a non-empty set of prime ideals in a right Noetherian ring R. If X is "right stable" and satisfies the "right second layer condition" and the "right intersection condition", then $\mathscr{C}(X) = \bigcap_{P \in X} \mathscr{C}(P)$ is a right Ore set.

We define a intersection condition for a set of uniform injectives (or, equivalently, the prime torsion classes cogenerated by them), analogous to Jategaonkar's condition, using $\mathscr{N}(E)$ instead of $\mathscr{C}(P)$ and obtain a version of the above result, for Ore links, without assuming the right second layer condition.

We also discuss the behaviour of Ore links in various cases and obtain some situations when the set $\bigcap \{ C_\tau : \tau \in$ "rt cl σ" $\}$ is right Ore.

We conclude by discussing the scope for further work and by mentioning certain problems that arose in the torsion-theoretic study of Ore localisation and links.

Propositions 2.1., 2.4, 2.6, 4.12, 4.14 and example 4.16 were included in my M.Phil. dissertation. They are mentioned here for the sake of completeness.

CHAPTER TWO

TORSION CLASSES AND MULTIPLICATIVE SETS

INTRODUCTION

For a ring R, there is a bijection of σn-R into the collection of all isomorphic classes of uniform injective right R-modules, given by

$$\tau \longmapsto \left\{ M \in \mathcal{M}od\text{-}R : E \cong E(M) \text{ for some } \tau\text{-critical right } R\text{-module } M \right\}$$

This map is well-defined, since, if M, M' are τ-critical, then $E(M) = E(M')$. If R is right Noetherian, this map is a bijection [G1]. This fact induces us to study localisation from the point of view of torsion theories. Since this approach seems to be promising, we have tried to generalise various results of Jategaonkar, Goodearl etc. to torsion classes.

In this chapter, we define, for a torsion theory τ, a corresponding multiplicative set C_{τ} as the set of elements of

R that generate τ-dense right ideals. We see its connection with the multiplicative set $\mathscr{C}(P)$ of elements regular modulo a prime ideal P and the set $\mathscr{N}(E)$ of elements that act regularly on an injective right R-module E. We obtain some results concerning the right Ore condition for these sets. We also see a new proof of the fact that every multiplicative set S has a largest right Ore subset (i.e., one that contains every right Ore subset of S). Several of these results were published in [SC].

THE MULTIPLICATIVE SET C_τ

PROPOSITION 2.1: Let $\tau \in \mathscr{T}ors$-R. Then the set

$$C_\tau = \{r \in R : R/rR \text{ is } \tau\text{-torsion in } R\}$$

is multiplicatively closed.

PROOF: Clearly, $1 \in C_\tau$. If $r_1, r_2 \in C_\tau$, then r_1R/r_1r_2R is a homomorphic image of R/r_2R , i.e., r_1r_2R is τ-dense in r_1R. So r_1r_2R is τ-dense in R. i.e., $r_1r_2 \in C_\tau$.

NOTE:

(i) If $\tau = \{(0)\}$, the smallest torsion class, then $C_\tau = \{1\}$.

(ii) $C_\tau = R \Leftrightarrow 0 \in C_\tau \Leftrightarrow \tau = \mathscr{M}od$-$R$, the largest torsion class.

NOTE 2.2: Let $\mathcal{M}$ be the class of all multiplicative sets in R. Define $f : \mathcal{M} \longrightarrow \mathcal{T}ors\text{-}R$ with $f(C) = \rho_C$ for $C \in \mathcal{M}$ and $g : \mathcal{T}ors\text{-}R \longrightarrow \mathcal{M}$ with $g(\tau) = C_\tau$ for $\tau \in \mathcal{T}ors\text{-}R$. Then both f and g are order-preserving.

PROPOSITION 2.3: If $\tau \in \mathcal{T}ors\text{-}R$, then $\rho_{C_\tau} \leq \tau$.

PROOF: If M is a right R-module which is ρ_{C_τ}-torsion, then for every $x \in M$, there is $c \in R$ such that R/cR is τ-torsion and $xc = 0$. So xR is a homomorphic image of R/cR. Hence xR is τ-torsion for all $x \in M$, i.e., M is τ-torsion.

PROPOSITION 2.4: If C is a multiplicative set, then $C \subseteq C_{\rho_C}$ if and only if C is a right Ore set.

PROOF: By proposition 1.17 we know that C is a right Ore set if and only is R/cR is ρ_C-torsion for every $c \in C$, i.e., if and only if $c \in C_{\rho_C}$ for every $c \in C$.

COROLLARY 2.5: If C_τ is a right Ore set for some $\tau \in \mathcal{T}ors\text{-}R$, then $C_\tau = C_{\rho_{C_\tau}}$.

PROOF: By proposition 2.4, $C_T \leq C_{\rho_{C_T}}$. By proposition 2.3 and note 2.2, $C_{\rho_{C_T}} \leq C_T$.

PROPOSITION 2.6: If C is a right Ore set, then C_{ρ_C} is a right Ore set but the converse is not true.

PROOF: We have

$C_{\rho_C} = \{ r \in R : R/rR \text{ is } \rho_C\text{-torsion} \}$

$= \{ r \in R : \text{given } s \in R, \text{ there is } c \in C \text{ such that } sc \in rR \}$

By proposition 2.4, $C \leq C_{\rho_C}$. Hence, if $r \in C_{\rho_C}$ and $s \in R$, there is $c \in C_{\rho_C}$ such that $sc \in rR$.

To see that the converse is not true, let k be a field, and let R be the ring of 2x2 upper triangular matrices over k. Then R is an Artinian ring with two prime ideals P nd Q, where

$$R = \begin{bmatrix} k & k \\ 0 & k \end{bmatrix}, \quad P = \begin{bmatrix} k & k \\ 0 & 0 \end{bmatrix}. \text{ and } Q = \begin{bmatrix} 0 & k \\ 0 & k \end{bmatrix}$$

Then $\mathscr{C}(P) = R \setminus P$ and $\mathscr{C}(Q) = R \setminus Q$.

We compute $C_{\rho_{\mathscr{C}(P)}}$.

i) For $a, b \in k$, $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \notin C_{\rho_{\mathscr{C}(P)}}$, for, taking $\begin{bmatrix} d & e \\ 0 & 1 \end{bmatrix} \in R$, if there is $\begin{bmatrix} c_1 & c_2 \\ 0 & c_3 \end{bmatrix} \in R$ and $\begin{bmatrix} g_1 & g_2 \\ 0 & g_3 \end{bmatrix} \in \mathcal{G}(P)$ $(g_3 \neq 0)$ such that

$$\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 & c_2 \\ 0 & c_3 \end{bmatrix} = \begin{bmatrix} d & e \\ 0 & 1 \end{bmatrix} \begin{bmatrix} g_1 & g_2 \\ 0 & g_3 \end{bmatrix}, \text{ then}$$

$$\begin{bmatrix} ac_1 & ac_2+bc_3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} dg_1 & dg_2+eg_3 \\ 0 & dg \end{bmatrix}, \text{ i.e., } g_3 = 0,$$

which is a contradiction.

ii) If $a \in k$, then $\begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} \notin C_{\rho_{\mathcal{G}(P)}}$, for if $a = 0$, then clear.

If $a \neq 0$, take $\begin{bmatrix} 0 & 1 \\ 0 & f \end{bmatrix} \in R$. If $\begin{bmatrix} c_1 & c_2 \\ 0 & c_3 \end{bmatrix} \in R$ and

$\begin{bmatrix} g_1 & g_2 \\ 0 & g_3 \end{bmatrix} \in \mathcal{G}(P)$ $(g_3 \neq 0)$ such that

$$\begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} c_1 & c_2 \\ 0 & c_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & f \end{bmatrix} \begin{bmatrix} g_1 & g_2 \\ 0 & g_3 \end{bmatrix},$$

then, $\begin{bmatrix} 0 & 0 \\ 0 & ac_3 \end{bmatrix} = \begin{bmatrix} 0 & g_3 \\ 0 & fg_3 \end{bmatrix}$, i.e., $g_3 = 0$, which is false.

iii) If $a, b \in k$, then $\begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} \notin C_{\rho_{\mathscr{G}(P)}}$, for, if $a = 0$, then we have the proof by case (ii). If $a \neq 0$, consider

$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in R$. If there is $\begin{bmatrix} c_1 & c_2 \\ 0 & c_3 \end{bmatrix} \in R$ and $\begin{bmatrix} g_1 & g_2 \\ 0 & g_3 \end{bmatrix} \in \mathscr{G}(P)$ $(g_3 \neq 0)$, such that

$$\begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} \begin{bmatrix} c_1 & c_2 \\ 0 & c_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} g_1 & g_2 \\ 0 & g_3 \end{bmatrix},$$ then

$\begin{bmatrix} 0 & ac_3 \\ 0 & bc_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & g_3 \end{bmatrix}$. Thus $ac_3 = 0$ and $bc_3 = g_3$. Since $a \neq 0$, we have $c_3 = 0$. So $g_3 = 0$, which is a contradiction.

iv) If $a, b, c \in k$ such that $a \neq 0$, $c \neq 0$, then

$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in C_{\rho_{\mathscr{G}(P)}}$, for, given $\begin{bmatrix} e & f \\ 0 & g \end{bmatrix} \in R$, we have

$\begin{bmatrix} a^{-1}e & a^{-1}(e+f-bc^{-1}g) \\ 0 & c^{-1}g \end{bmatrix} \in R$ and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in \mathscr{G}(P)$ such that

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \begin{bmatrix} a^{-1}e & a^{-1}(e+f-bc^{-1}g) \\ 0 & c^{-1}g \end{bmatrix} = \begin{bmatrix} e & f \\ 0 & g \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{----(A)}$$

Cases (i), (ii), (iii), (iv) together cover all the elements of R and hence we get

$$C_{\rho_{\mathcal{C}(P)}} = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in k, a \neq 0, c \neq 0. \right\}$$

$$= \mathcal{C}(P) \cap \mathcal{C}(Q).$$

Now by case (iv), $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in C_{\rho_{\mathcal{C}(P)}}$ and hence by equation (A), we see that $C_{\rho_{\mathcal{C}(P)}}$ is right Ore. But $\mathcal{C}(P)$ is not right Ore, since,

$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in \mathcal{C}(P)$ and $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in R$ such that if $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in R$ and $\begin{bmatrix} d & e \\ 0 & f \end{bmatrix} \in \mathcal{C}(P)$ $(f \neq 0)$, then

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}$$

whereas $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} d & e \\ 0 & f \end{bmatrix} = \begin{bmatrix} 0 & f \\ 0 & 0 \end{bmatrix}$ and these two cannot be equal for $f \neq 0$.

NOTE: Following Stenstrom [S2], we say that a multiplicative set C in a ring R satisfies property SO if, for a, $b \in R$, $ab \in R$ implies $a \in C$.

PROPOSITION 2.7: If a multiplicative set C in a ring R satisfies SO, then $C_{\rho_C} \subseteq C$.

PROOF: Let $r \in C_{\rho_C}$. Since $1 \in R$, there are $c \in C$, $d \in R$ such that $1.c = r.d$, i.e., $rd \in C$. By property SO, $r \in C$.

NOTE 2.8: If R/P is a right Goldie ring, then $\mathscr{C}(P)$ satisfies SO, for, let $ab \in \mathscr{C}(P)$. If $\bar{a} = a+P \in R/P$, then $\bar{a}\bar{b} \in \mathscr{C}(0) \subseteq R/P$. Thus $\bar{a}\bar{b}$ is invertible in $Q(R/P)$, i.e., $\bar{a}\bar{b}Q(R/P) = Q(R/P)$. Also $\bar{a}\bar{b}Q(R/P) \subseteq Q(R/P)$. Thus, $\bar{a}Q(R/P) = Q(R/P)$, i.e, $\bar{a}$ is invertible in $Q(R/P)$, i.e., $a \in \mathscr{C}(P)$.

NOTE 2.9: If E is a right R-module, then the set $\mathscr{N}(E)$ defined as $\mathscr{N}(E)=\{ r \in R: \text{ann}_E r = 0 \}$ is a multiplicative set [G4]. The set $\mathscr{N}(E)$ satisfies SO, for, if $ab \in \mathscr{N}(E)$, then, for $x \in E$, if $xab = 0$, then $x = 0$. Now, if $xa = 0$, then $xab = 0$ and so $x = 0$, i.e., $a \in \mathscr{N}(E)$.

PROPOSITION 2.10: Let $\pi = \chi(E)$, where E is an injective right R-module. Then

$$\mathcal{N}(E) = \bigcap \left\{ R \setminus I : I \text{ is a right ideal of } R,\ I \neq R \text{ and } R/I \text{ is } \pi\text{-torsion-free} \right\}.$$

PROOF: Denote the left hand side of the above expression by Y. Let $c \in \mathcal{N}(E)$, and let I be a right ideal of R, $I \neq R$, such that R/I is π-torsion-free. Then $\chi(E(R/I)) \geq \pi = \chi(E)$, i.e., $E(R/I)$ can be embedded in say, E', a product of copies of E. Then $\mathcal{N}(E(R/I)) \supseteq \mathcal{N}(E') = \mathcal{N}(E)$. Hence $c \in \mathcal{N}(E(R/I))$. Now if $c \in I$, then there is $1+I \in E(R/I)$ such that $(1+I)c = c+I = 0$. But since we have $c \in \mathcal{N}(E(R/I))$, this means $1+I = 0$, which is false. So $c \in R \setminus I$. Hence $\mathcal{N}(E) \subseteq$ Y.

Next, let $c \in$ Y, and $0 \neq x \in E$. Since xR is a submodule of E, xR is π-torsion-free. So ann x is one of the I's in the definition of Y. Hence $c \in R \setminus$ ann x, i.e., $xc \neq 0$. Thus $c \in \mathcal{N}(E)$, i.e., Y $\subseteq \mathcal{N}(E)$. This completes the proof.

NOTE: By the above proposition, it is clear that if E, E' are injective right R-modules, with $\chi(E) = \chi(E')$, then $\mathcal{N}(E) = \mathcal{N}(E')$.

PROPOSITION 2.11: Let C be a multiplicative set in a ring R. If E is an injective right R-module, then, $C_{\chi(E)} = \mathcal{N}(E)$.

PROOF: Let $r \in C_{\chi(E)}$, i.e., R/rR is $\chi(E)$-torsion. Let I be a proper right ideal of R such that R/I is $\chi(E)$-torsion-free. If $r \in I$, then rR is a submodule of I and hence R/I is $\chi(E)$-torsion. Thus, R/I is $\chi(E)$-torsion-free and $\chi(E)$-torsion, which is false, since $I \neq R$. Hence $r \in R \setminus I$. By Proposition 8, $r \in \mathcal{N}(E)$, i.e., $C_{\chi(E)} \subseteq \mathcal{N}(E)$.

Next, let $c \in \mathcal{N}(E)$. Suppose there is a homomorphism $f : R/cR \longrightarrow E$ such that $f(1+cR) = x$ (say). Then $xc = f(1+cR)c = 0$. Since $c \in \mathcal{N}(E)$, we get $x = 0$, i.e., $f(1+cR) = 0$. Hence $f = 0$. Thus R/cR is $\chi(E)$-torsion. Thus $\mathcal{N}(E) \subseteq C_{\chi(E)}$.

NOTE 2.12: If P is a prime ideal in a ring R, then $\chi(R/P)$ is the largest of all P-principal points, for, let π be a P-principal point. Then, $\psi(\pi) = P$. Now, for a two-sided ideal I of R, R/I is π-torsion if and only if $I \nsubseteq \psi(\pi)$. Hence R/P is not π-torsion. By [J, proposition 5.4.2], R/P is π-torsion-free, i.e., $\chi(R/P) \geq \pi$.

PROPOSITION 2.13: If E is an injective right R-module over a right Noetherian ring R, such that $\chi(E)$ is a P-principal point for some prime ideal P of R, then $\mathcal{N}(E) \subseteq \mathcal{C}(P)$.

PROOF: By note 2.12, $\chi(R/P) \geq \chi(E)$. Hence $C_{\chi(E)} \subseteq C_{\chi(R/P)}$. So, $\mathcal{N}(E) \subseteq \mathcal{N}(E(R/P)) = C_{\chi(R/P)} = C_{\rho_{\mathscr{C}(P)}} \subseteq \mathscr{C}(P)$, by proposition 2.11, proposition 2.7 and note 2.8.

COROLLARY 2.14: If R is a right Noetherian ring and E is a uniform injective with ass $E = P$, then $\mathcal{N}(E) \subseteq \mathscr{C}(P)$.

PROOF: Since E is a uniform injective right R-module, we have $\psi(\chi(E)) =$ ass E by proposition 1.16.

COROLLARY 2.15: If R is a right Noetherian ring and P is a prime ideal in R, then $\mathcal{N}(E(R/P)) = \mathscr{C}(P)$ if and only if $\mathscr{C}(P)$ is right Ore.

PROOF: Follows from propositions 2.4 and 2.13.

THE RIGHT ORE CONDITION ON $\mathscr{C}(P)$

In the next few propositions, we see some situations where $\mathscr{C}(P)$ is right Ore (for a prime ideal P), using torsion classes.

PROPOSITION 2.16: If R is a right duo ring (i.e., a ring in which every right ideal is two sided), then $\mathscr{C}(P)$ is right Ore for every prime ideal P in R.

PROOF: By assumption, if $r \in R$, then $rR = RrR$. Now $r \in C_{\rho_{\mathcal{C}(P)}}$ if and only if R/rR is $\mathcal{C}(P)$-torsion if and only if R/RrR is $\mathcal{C}(P)$-torsion if and only if $RrR \leq P$ if and only if $r \in R \setminus P$. Since $\mathcal{C}(P) \subseteq R \setminus P$, we have $\mathcal{C}(P) \subseteq C_{\rho_{\mathcal{C}(P)}}$. So, by proposition 2.4, $\mathcal{C}(P)$ is right Ore.

PROPOSITION 2.17: Let R be a right Noetherian ring and P be a prime ideal of R. If $(R/P)_R$ is injective, then $\mathcal{C}(P)$ is right Ore.

PROOF: Since $(R/P)_R$ is injective, we have

$$\mathcal{N}(E(R/P)) = \mathcal{N}(R/P) = \{ r \in R : \text{ann}_{R/P} = 0 \}$$
$$= \{ r \in R : xr = 0 \Rightarrow x = 0 \text{ for any } x \in R/P \}$$
$$= \{ r \in R : sr \in P \Rightarrow s \in P \text{ for any } s \in R \}$$
$$\supseteq \mathcal{C}(P).$$

Thus, by proposition 2.11, $\mathcal{C}(P) \subseteq C_{\rho_{\mathcal{C}(P)}}$. By proposition 2.4, $\mathcal{C}(P)$ is right Ore.

COROLLARY 2.18: If R is semisimple Artinian and P is any prime ideal of R, then $\mathcal{C}(P)$ is right Ore.

PROOF: Over a semisimple Artinian ring, any module is injective.

COROLLARY 2.19: If R is semisimple Artinian, and E is any simple right R-module, then $\mathcal{N}(E) = \mathcal{C}(P)$, where $P = \text{ass } E$.

PROOF: Since R is semisimple Artinian, E is tame and so $E = E_P$. By corollary 2.18, $\mathcal{C}(P)$ is right Ore and so by corollary 2.15, we have $\mathcal{C}(P) = \mathcal{N}(E(R/P)) = \mathcal{N}(E)$.

NOTE 2.20: In a general right Noetherian ring R, if E and E' are uniform injectives with ass E = ass E', then $\mathcal{N}(E)$ need not be equal to $\mathcal{N}(E')$, for, let E be a uniform injective right module over a simple right Noetherian ring R. Then $\mathcal{C}(0)$ is right Ore and so by corollary 2.15, $\mathcal{C}(0) = \mathcal{N}(E_0)$.

Now, suppose $c \in \mathcal{C}(0)$ such that $cR \neq R$. Then cR is an essential right ideal and so R/cR is torsion. Let E be a uniform submodule of R/cR. Then E is torsion. Hence, given $x \in E$, there is $r \in \mathcal{C}(0)$ such that $xr = 0$, i.e., $r \notin \mathcal{N}(E)$. So $\mathcal{C}(0) \nsubseteq \mathcal{N}(E)$.

THE LARGEST RIGHT ORE SUBSET OF A MULTIPLICATIVE SET

So far, we have seen many situations when multiplicative sets of interest to us are right Ore. But we know that there are cases when sets are not right Ore. Now, given a multiplicative set C we give a new proof that there exists a right Ore set contained in C which contains all right Ore subsets of C.

This fact has been known for a long time. A proof is given in [GW, Exercise 9F]. However, our proof will lead to a characterisation of this subset as an intersection of right cliques, as conjectured in [G4], in the case $C = C(P)$.

To prove the next theorem, we define, for any multiplicative subset C of R, a sequence of subsets C_α, for every ordinal α. Let $C_0 = C$, $C_1 = C_{\rho_C} \cap C$, and for any successor ordinal α, let $C_{\alpha+1} = (C_\alpha)_1$. For a limit ordinal α, let $C_\alpha = \cap_{\beta<\alpha} C_\beta$. Then the C_α's form a descending chain of multiplicative sets in R.

LEMMA 2.21: If T is a right Ore subset of a multiplicative set C, then $T \subseteq C_1$.

THEOREM 2.22: Let C be a multiplicative set in a ring R. Then C has a right Ore subset which contains every right Ore subset of C.

PROOF: The map $\alpha \mapsto C_\alpha$,from the class of ordinals to the power set of C, cannot be one-one since the ordinals do not form a set. Hence for some α, $C_\alpha = C_{\alpha+1} = \cap \{ C_\beta : \beta$ is an ordinal number $\}$. By proposition 2.4, C_α is a right Ore set. By lemma 2.21, it contains every right Ore subset of C.

CHAPTER THREE

THE PATCH TOPOLOGY

INTRODUCTION

Let R be a right Noetherian ring and $\mathcal{S}pec\ R$ denote the set of prime ideals of R. To consider regularity of an element of R at different prime ideals, it is convenient to put a topology on $\mathcal{S}pec\ R$. One such topology is the Patch topology introduced by Hochster in 1969. In 1986, Goodearl defined the generic regularity condition for subsets of $\mathcal{S}pec\ R$, and this helps us to clarify the discussion of various continuity results on $\mathcal{S}pec\ R$.

In this chapter, we give an analogue of the Patch topology for prime torsion classes and discuss its properties. We also define the generic regularity condition for prime torsion classes. In the case of prime ideals this condition has an important role in the study of localisation. Though we study patch topology and generic regularity condition on prime torsion classes for their own sake, we hope that they can be used in the torsion theoretic approach to localisation.

Let R be a ring with ACC on ideals. Then the sets $V(I) \cap W(J)$ (I, J ideals of R) form a base for the open sets of the *patch topology* on $\mathcal{S}pec\ R$, where

$$V(I) = \{ P \in \mathcal{S}pec\ R : P \geq I \}$$

$$W(J) = \{ P \in \mathcal{S}pec\ R : P \ngeq J \}.$$

THE PATCH TOPOLOGY ON $\mathcal{P}t\text{-}R$

Let R be a ring in which all points are principal, i.e., a point principal ring (for example, a right Noetherian ring).

DEFINITION 3.1: For each ideal I in R, define

$\delta(I) = \{ \pi \in pt\text{-}R : R/I \text{ is } \pi\text{-torsion} \}$

$\delta'(I) = \{ \pi \in pt\text{-}R : R/I \text{ is not } \pi\text{-torsion} \} = pt\text{-}R \setminus \delta(I)$

Then, for ideals I_1, I_2 of R,

$\delta(I_1) \cup \delta(I_2) = \delta(I_1 + I_2)$

$\delta'(I_1) \cap \delta'(I_2) = \delta'(I_1 + I_2)$ and

$\delta(I_1) \cap \delta(I_2) = \delta(I_1 \cap I_2)$ so that for ideals I_1, I_2, J_1, J_2 of R, we have

$(\delta'(I_1) \cap \delta(J_1)) \cap (\delta'(I_2) \cap \delta(J_2)) = \delta'(I_1+I_2) \cap \delta(J_1 \cap J_2)$.

Hence, the sets of the form $\delta'(I) \cap \delta(J)$ (I, J ideals of R), form the base of open sets of a topology (the *patch topology on* $\mathcal{S}pec\ R$).

By proposition 1.16(ii), for an ideal I and prime ideal P of R, $I \leq P$ if and only if R/I is not $\chi(R/P)$-torsion. Hence we have the following result.

NOTE 3.2: For $P \in \mathcal{S}pec\, R$, $P \in V(I) \cap W(J)$ if and only if $\chi(R/P) \in \delta'(I) \cap \delta(J)$.

PROPOSITION 3.3: Let R be a ring with ACC on ideals. The map $\phi : \mathcal{S}pec\, R \longrightarrow rt\text{-}R$ with $\phi(P) = \chi(R/P)$ (where $P \in \mathcal{S}pec\, R$) is a topological embedding, where $\mathcal{S}pec\, R$ and $sp\text{-}R$ are given the coresponding patch topologies.

PROOF: Let $\delta'(I) \cap \delta(J)$ be a basic patch open set in $rt\text{-}R$ where I, J are ideals of R. Then we have
$\phi^{-1}[\delta'(I) \cap \delta(J)] = \delta'(I) \cap \delta(J) \cap \phi(\mathcal{S}pec\, R)$, which is open in $\phi\ (\mathcal{S}pec\, R)$. Hence ϕ is continuous. Now,

$$\phi(V(I) \cap W(J)) = \{ \phi((P) : P \in V(J) \cap W(I) \}$$
$$= \delta'(I) \cap \delta(J) \cap \phi(\mathcal{S}pec\, R),$$

which is open in $\phi(\mathcal{S}pec\, R)$. Hence ϕ is an open map.

PROPOSITION 3.4: Let $\sigma \in rt\text{-}R$. Then the patch neighbourhoods $\delta'(\psi(\sigma)) \cap \delta(J)$ form a base for the patch neighbourhoods of σ, where J is an ideal properly containing $\psi(\sigma)$.

PROOF: Any patch neighbourhood of a point $\sigma \in rt\text{-}R$ must contain a neighbourhood of the form $\delta'(I) \cap \delta(J) \ni \sigma$. Since

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$\sigma \in \delta'(\psi(\sigma))$ and $\delta'(\psi(\sigma)) \subseteq \delta'(I)$, we may replace $\delta'(I)$ by $\delta'(\psi(\sigma))$. Now, $\delta'(\psi(\sigma)) \cap \delta(J) = \delta'(\psi(\sigma)) \cap \delta(\psi(\sigma) + J)$, and so we replace J by $\psi(\sigma) + J$. Thus, every patch-neighbourhood of σ contains a neighbourhood of the form $\delta'(\psi(\sigma)) \cap \delta(J)$, where J is an ideal properly containing $\psi(\sigma)$

PROPOSITION 3.5: If R is a ring with ACC on ideals, then $\mathcal{M}$-R with the patch topology is a compact space.

PROOF: Let X be a family of patch-open sets covering $\mathcal{M}$-R. Suppose no finite subfamily of X covers $\mathcal{M}$-R. Then, since $\mathcal{M}$-$R = \delta'(0)$, we may use ACC on ideals to choose an ideal Q maximal with respect to the property that no finite subfamily of X covers $\delta'(Q)$. If A and B are ideals properly containing Q, there must be a finite subfamily Y of X that covers $\delta'(A)$ and $\delta'(B)$. Then, since $\delta'(AB) \subseteq \delta'(A) \cup \delta'(B)$, Y covers $\delta'(AB)$. Hence $\delta'(AB) \not\supseteq \delta'(Q)$. So, $AB \not\leq Q$, i.e., Q is a prime ideal in R.

Choose U in X such that $\chi(R/Q) \in U$. Then $\chi(R/Q)$ must have a patch-neighbourhood $\delta'(\psi(\chi(R/Q))) \cap \delta(J)$, for some ideal $J \geq \psi(\chi(R/Q))$, such that $\delta'(\psi(\chi(R/Q))) \cap \delta(J) \subseteq U$, i.e., $\chi(R/Q)$ must have a patch-neighbourhood $\delta'(Q) \cap \delta(J)$, where $J > Q$, such that $\delta'(Q) \cap \delta(J) \subseteq U$.

Now, by maximality of Q, $\delta'(J)$ can be covered by some finite subfamily Y' of X. But $\delta'(Q) \setminus \delta'(J) = \delta'(Q) \cap \delta(J) \subseteq U$. So $\delta'(Q)$ can be covered by $Y' \cup \{U\}$ contrary to our choice of Q. Thus there must be a finite subfamily of X which covers pt-R.

Recall that there is a surjection ψ: $\mathcal{P}t$-$R \longrightarrow \mathcal{S}pec\ R$ given by $\pi \mapsto \psi(\pi)$.

DEFINITION 3.6: A point-principal ring is a *T-ring* if the map ψ: pt-$R \longrightarrow \mathcal{S}pec\ R$ is injective.

PROPOSITION 3.7: $\mathcal{P}t$-R with the patch topology is Hausdorff if and only if R is a T-ring.

PROOF: Suppose R is a T-ring. Let σ, π be distinct points in pt-R. Then either $R/\psi(\sigma)$ is π-torsion or $R/\psi(\pi)$ is σ-torsion (For, if not, then $\psi(\sigma) = \psi(\pi)$, i.e., $\sigma = \pi$, since R is a T-ring), say, $R/\psi(\sigma)$ is π-torsion. Then, since $\delta(R) = \mathit{pt}$-R, $\delta'(\psi(\sigma)) \cap \delta(R)$ is a patch-neighbourhood of σ and $\delta'(\psi(\pi)) \cap \delta(\psi(\sigma))$ is a patch neighbourhood of π and these are disjoint since $\delta'(\psi(\sigma))$ and $\delta(\psi(\sigma))$ are disjoint. Thus, if R is a T-ring, then pt-R with the patch topology is Hausdorff.

Now, suppose R is not a T-ring. Then, there is a prime torsion class σ and a prime ideal P such that $\psi(\sigma) = P$, but $\sigma \neq \pi = \chi(R/P)$. By note 2.12, $\sigma \leq \pi$. Now, any neighbourhood of σ contains a neighbourhood of the form $\delta'(\psi(\sigma)) \cap \delta(J) \ni \sigma$ where $J > P$, i.e., $\sigma \in \delta'(P) \cap \delta(J)$. Since R/J is σ-torsion, R/J is π-torsion, i.e., $\pi \in \delta(J)$. Now, $\pi \in \delta'(\psi(\pi))$ and $\psi(\sigma) = \psi(\pi)$, and so $\pi \in \delta'(\psi(\sigma))$, i.e., $\pi \in \delta'(\psi(\sigma)) \cap \delta(J)$.

Thus, any neighbourhood of σ contains π also, i.e., we cannot find a neighbourhood N of σ such that $\pi \notin N$, i.e., $\mathcal{R}t\text{-}R$ with patch topology is not even T_1 if R is not a T-ring.

PROPOSITION 3.8.: $\mathcal{R}t\text{-}R$ with patch topology is totally disconnected.

PROOF: Since $\delta'(0) = \delta(R) = \mathcal{R}t\text{-}R$, for an ideal I, we have $\delta'(I) = \delta'(I) \cap \delta(R)$ and $\delta(I) = \delta'(0) \cap \delta(I)$, both of which are patch-open and patch-closed. So, the basic open sets $\delta'(I) \cap \delta(J)$ for ideals I, J of R are all patch-closed. Hence the patch topology on $\mathcal{R}t\text{-}R$ has a basis of open sets which are also closed.

PROPOSITION 3.9: If R is a right Artinian ring, then the patch topology on $\mathcal{R}t\text{-}R$ is the discrete topology.

PROOF: Since R is Artinian, we have

$$pt\text{-}R = \{ \chi(R/P) : P \in \mathcal{S}pec\ R \}.$$

If $P \in \mathcal{S}pec\ R$, then $\delta(P) = \{ \pi \in pt\text{-}R : R/P$ is π-torsion $\}$. Let $\pi \in pt\text{-}R$, say, $\pi = \chi(R/Q)$ for some $Q \in \mathcal{S}pec\ R$. Since all prime ideals of R are maximal, we have $Q \nsubseteq P$ if $Q \neq P$. Hence $\chi(R/P) \nleq \chi(R/Q)$, i.e., R/P is not $\chi(R/Q)$-torsion-free, i.e., by [J, proposition 5.4.2], R/P is $\chi(R/Q)$-torsion, i.e., $\chi(R/Q) \in \delta(P)$. Thus we have $\delta(P) = pt\text{-}R \setminus \{\chi(R/P)\}$ for any prime ideal P in R, i.e., $\delta'(P) = \{\chi(R/P)\}$. Hence all singletons are open in the patch-topology on $pt\text{-}R$, i.e., when R is Artinian, the patch topology on $pt\text{-}R$ is the discrete topology.

NOTE 3.10: For a torsion class π in $\mathcal{T}ors\text{-}R$, let E be an injective right R-module such that $\pi = \chi(E)$. Then we denote $\mathcal{N}(\pi) = \mathcal{N}(E)$. This is well-defined by proposition 2.10.

THE GENERIC REGULARITY CONDITION ON $\mathcal{P}t\text{-}R$

DEFINITION 3.11: Let $X \subseteq sp\text{-}R$. We define generic regularity condition as follows. If, for any $\pi \in sp\text{-}R$ and any $c \in \mathcal{N}(\pi)$, there is a patch-open neighbourhood U of π such that $c \in \mathcal{N}(\sigma)$ for any $\sigma \in U \cap X$, then we say that X satisfies the ***generic regularity condition***.

PROPOSITION 3.12: If R is right Artinian, then any $X \subseteq sp\text{-}R$ satisfies the generic regularity condition.

PROOF: By proposition 3.9, the patch topology on $sp\text{-}R$ is the discrete topology. Hence given $\pi = \chi(R/P) \in sp\text{-}R$ and any $c \in \mathcal{N}(\pi)$, $\delta'(P) = \{ \chi(R/P) \}$ is a patch-open-neighbourhood of π.

PROPOSITION 3.13: If R is a right duo ring, then any $X \subseteq sp\text{-}R$ satisfies the generic regularity condition, where a right duo ring is as defined in proposition 2.17.

PROOF: If $\sigma \in sp\text{-}R$, then $c \in \mathcal{N}(\sigma)$ if and only if R/cR is σ-torsion if and only if R/RcR is σ-torsion if and only if $\sigma \in \delta(RcR)$. Now, $\delta(RcR)$ is an open set and hence is an open neighbourhood of σ. Hence, given $\tau \in sp\text{-}R$ and $c \in \mathcal{N}(\tau)$, there is a patch-open neighbourhood $U = \delta(RcR)$ of τ such that $c \in \mathcal{N}(\sigma)$ for any $\sigma \in U$ (and hence for any $\sigma \in U \cap X$).

CHAPTER FOUR

GENERALISED INJECTIVE LINKS

INTRODUCTION

In the theory of commutative Noetherian rings, several fundamental results are obtained by using the procedure of localisation at prime ideals. In the non-commutative case, localisation at a prime ideal is not always possible and it has been found that if we wish to localise at one prime, we have to look at a whole bunch of primes "linked" to the first one. In the 1970s and 1980s, Jategaonkar, Mueller and others worked on this problem.

There is a large class of Noetherian rings that satisfy a certin condition called the "second layer condition" by Jategaonkar in which it is possible to describe localisation at a prime (or a collection of primes) under conditions that apply widely. However, there are important classes of rings that do not satisfy this condition. A study of localisation in such rings was started by Goodearl (1988). He found a closer connection between prime ideal links and the second layer and used it to define links between uniform injective right modules over a right Noetherian ring. He observed that links between tame injectives correspond precisely to links between prime ideals, while, in general, other links exist, which provide more obstructions to Ore localisations than prime ideal links do.

In our endeavour to study localisation using torsion classes we have defined links between prime torsion classes in such a way that an injective link between two uniform injectives (as defined by Goodearl) implies a torsion-theoretic link between the prime torsion classes cogenerated by them. Some of the results of this chapter are in [CS].

PRELIMINARIES

Most of the material in this section is taken from [J] and [G4].

DEFINITION 4.1: Let R be a right Noetherian ring and let P, Q be prime ideals in R. We say that Q *is linked to* P (via the ideal $A < Q \cap P$), denoted $Q \rightsquigarrow P$, if $QP \leq A \leq Q \cap P$ such that the right R/P-module $(Q \cap P)/A$ is torsion-free, and the left R/Q-module $(Q \cap P)/A$ has no non-zero unfaithful submodules [J]

DEFINITION 4.2: Let $X \subseteq \mathcal{S}pec\ R$. We say X is *right stable* if, whenever $P \in X$, $Q \in \mathcal{S}pec\ R$, and $Q \rightsquigarrow P$, we have $Q \in X$. We say X is *stable* if $Q \rightsquigarrow P$ implies either both $Q, P \in X$ or both $Q, P \notin X$. If $P \in \mathcal{S}pec\ R$, the *right clique of P*, denoted rt cl (P) is the smallest right stable subset of $\mathcal{S}pec\ R$ containing P, i.e., rt cl P is the smallest set of primes containing P and all prime ideals Q in R such that $Q \rightsquigarrow Q_n \rightsquigarrow Q_{n-1} \rightsquigarrow \ldots\ldots \rightsquigarrow Q_1 \rightsquigarrow P$, where Q_i $(1 \leq i \leq n)$ are prime ideals in R. The *clique* of $P \in \mathcal{S}pec\ R$ is the smallest stable subset containing P.

PROPOSITION 4.3 : Let R be a right Noetherian ring, P a prime ideal of R and C a right Ore set in R. If $C \subseteq C(P)$, then $C \subseteq C(Q)$ for all $Q \in$ rt cl (P).

EXAMPLE 4.4: For the ring in proposition 2.6, the only prime ideals are Q and P and the only link is $Q \rightsquigarrow P$. Then we have rt cl $Q = \{ Q \}$ and rt cl $P = \{ P, Q \}$

DEFINITION 4.5: Let S be a semiprime ideal in a right Noetherian ring R and let M be an S-primary right R-module. Then the *first layer* of M is defined as the module $\mathrm{ann}_M S$. This is defined by M alone (independently of S).

Consider the module $E(M/\mathrm{ann}_M\ S)$. This module can be decomposed as $E(M/\mathrm{ann}_M\ S) = \oplus_{i\in I} E_i^{(\mu_i)}$ where $\langle E_i : i \in I \rangle$ is a family of pairwise non-isomorphic indecomposable injectives, $\langle \mu_i : i \in I \rangle$ is a family of non-zero cardinals, and $E_i^{(\mu_i)}$ denotes the direct sum of a family of copies of E_i that is indexed by a set of cardinality μ_i. The family $\langle E_i : i \in I \rangle$ is uniquely determined by M upto permutation and isomorphism, and, for $i \in I$, μ_i is uniquely determined by E_i.

The *second layer* of M is defined as the set of the similarity classes of the indecomposable injectives E_i, $i \in I$. Then, the second layer of M is just the set of the similarity classes of uniform submodules of $M/\mathrm{ann}_M\ S$. Often, we loosely treat a set of the representatives of the second layer of M as if it were the second layer of M.

DEFINITION 4.6: A prime ideal P in a right Noetherian ring R is said to satisfy the *right second layer condition* if every uniform module in the second layer of $(E_P)_R$ is tame. A prime ideal P in a Noetherian ring is said to satisfy the *second layer condition* if P satisfies the right and left second layer condition.

A set X of prime ideals in a (right) Noetherian ring R is said to satisfy the (right) second layer condition if every member of X does. Finally, the ring R is said to satisfy the (right) second layer condition if $\mathcal{S}pec\ R$ satisfies it. For, example, FBN rings, having no wild modules, satisfy the second layer condition.

DEFINITION 4.7: If $X \subseteq \mathcal{S}pec\ R$, we say X satisfies the *right intersection condition* if any right ideal of R that has non-empty intersection with $\mathcal{C}_R(P)$ for every $P \in X$ also has non-empty intersection with $\mathcal{C}_R(X) = \bigcap_{P \in X} \mathcal{C}(P)$.

PROPOSITION 4.8 [J, lemma 7.1.4]**:** Let X be a non-empty set of prime ideals in a right Noetherian ring R. Assume X is right stable and that it satisfies the right second layer condition as well as the right intersection condition. Then $\mathcal{C}(X)$ is a right Ore set in R.

Goodearl [G4] studies the influence of injective module structure on localisation questions for non-commutative Noetherian rings.

DEFINITION 4.9: If F, E are uniform injective right modules over a right Noetherian ring R, we say that there is a *link* *from F to E* , written $F \rightsquigarrow E$, if F is isomorphic to a direct summand of the injective hull of $E/\mathrm{ann}_E(\mathrm{ass}E)$, i.e., $F \rightsquigarrow E$, if and only if the isomorphism class of F belongs to the second layer of E.

The *right clique of E* consists of E and all those uniform injective right R-modules F such that

$$F \rightsquigarrow E_n \rightsquigarrow E_{n-1} \rightsquigarrow \dots\dots \rightsquigarrow E_1 \rightsquigarrow E$$

for some uniform injective R-modules E_i $(1 \le i \le n)$.

Injective module links provide obstructions to Ore localisations in R, in the following sense.

PROPOSITION 4.10 [G4, proposition 1.2]: Let C be a right Ore set in a right Noetherian ring R, and let E be a uniform injective right R-module. If $C \subseteq \mathcal{N}(E)$, then $C \subseteq \mathcal{N}(F)$ for all F in the right clique of E.

The notion of linked uniform injectives, when restricted to tame injectives is equivalent to the notion of linked primes, as follows.

THEOREM 4.11: [G4, theorem 1.4]: Let R be a right Noetherian ring, and let $P, Q \in \mathcal{S}pec\ R$. THen $Q \rightsquigarrow P$ if and only if $E_Q \rightsquigarrow E_P$.

It can also happen that a Q-wild uniform injective is linked to a P-tame uniform injective. In this case, Q need not link to P. Thus, there can exist links between uniform injectives that do not correspond to links between the assassinator primes.

For a right Noetherian ring R, since there is a bijection of $\mathfrak{sp}\text{-}R$ onto the collection of all isomorphism classes of uniform injective right R-modules, we define links between prime torsion classes, in such a way that an injective link between two uniform injective modules implies a link between the prime torsion classes cogenerated by them.

GENERALISED INJECTIVE LINKS BETWEEN PRIME TORSION CLASSES

DEFINITION 4.12: Let R be a ring and σ, $\pi \in \mathfrak{sp}\text{-}R$. Let E be a uniform injective right R-module with $\pi = \chi(E)$. Put ass $E = P$ and $\mathrm{ann}_E(P) = L$. We say that σ is *linked to* π written $\sigma \rightsquigarrow \pi$ if $\sigma \geq \chi(E/L)$

NOTE 4.13: This definition is independent of the choice of E, i.e., if $\pi = \chi(E) = \chi(E')$ (where E, E' are injective right R-modules) with ass E = ass $E' = P$, $L = \mathrm{ann}_E(P)$, and $L' = \mathrm{ann}_{E'}(P)$, then we have $\chi(E/L) = \chi(E'/L')$, for, since $\chi(E') \geq \chi(E)$, by proposition 1.13(e), E' is embedded in

$\prod_{i\in I} E_i$, (where $E_i \cong E$ for every $i \in I$) say, $x \longmapsto (x_i)$. If $r \in R$, then $xr \longmapsto (x_i r)$. Let $L_i = \mathrm{ann}_{E_i} P$. Then $x \in L'$ if and only if $x_i \in L_i$ for every $i \in I$. So we have a map f: $E'/L' \longmapsto \prod_{i\in I} (E_i/L_i)$, with $x+L' \longmapsto (x_i+L_i)$. Then f is well-defined and is one-to-one, since if $(x_i+L_i) = 0$, then $x+L'=0$. So E'/L' can be embedded in a product of copies of E/L , and thus we have $\chi(E'/L') \geq \chi(E/L)$. Similarly, $\chi(E/L) \geq \chi(E'/L')$.

PROPOSITION 4.14: Let R be a right Noetherian ring, and E, F be uniform injective right R-modules. If $F \rightsquigarrow E$ as in definition 4.9, then $\chi(F) \rightsquigarrow \chi(E)$.

PROOF: Since $F \rightsquigarrow E$, F can be embedded in $E(E/\mathrm{ann}_E(\mathrm{ass}E))$. Hence $\chi(F) \geq \chi(E/\mathrm{ann}_E(\mathrm{ass}E))$ and so $\chi(F) \rightsquigarrow \chi(E)$.

The next proposition shows how torsion-theoretic links obstruct localisation.

PROPOSITION 4.15: Let π, $\sigma \in \mathcal{sp}\text{-}R$ and C be a right Ore set in R. If $C \subseteq \mathcal{N}(\pi)$, then $C \subseteq \mathcal{N}(\sigma)$, for every $\sigma \rightsquigarrow \pi$.

PROOF: Let $\pi = \chi(E)$, where E is a uniform injective right R-module. Let ass $E = P$ and $\mathrm{ann}_E P = L$. Then, since $\sigma \rightsquigarrow \pi$, we have $\sigma \geq \chi(E/L)$. Since $C \subseteq \mathcal{N}(\pi)$, we have $\rho_C \leq \rho_{\mathcal{N}(\pi)} \leq \pi$ by propositions 2.3 and 2.11. Hence, using [G4, lemma 1.1], we have $\rho_C \leq \chi(E/L)$, and hence $\rho_C \leq \sigma$. Since C is right Ore, using proposition 2.4, we have $C \subseteq C_{\rho_C} \subseteq C_\sigma$. By proposition 2.11, $C \subseteq \mathcal{N}(\sigma)$ for every $\sigma \rightsquigarrow \pi$.

The following example shows that generalised injective links provide more obstructions to Ore localisation than injective links do.

EXAMPLE 4.16: Let $R = \mathbb{Z}$, the ring of integers and let $E = \mathbb{Z}(p^{\infty}) = \{ a/p^n : n = 0, 1, 2, \ldots\ldots, 0 \leq a \leq b-1 \}$. We have, for a prime p,
$E(\mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}(p^{\infty}) \leq \mathbb{Q}/\mathbb{Z} = \{ a/b : b \neq 0, 0 \leq a \leq b-1\}$. So we have an embedding of $\mathbb{Z}/p\mathbb{Z}$ into $\mathbb{Z}(p^{\infty})$ with $i \longmapsto i/p$.

Now E is an indecomposable injective and ass $E = p\mathbb{Z}$. Let $L = \text{ann}_E(\text{ass}E) = \{ 0, 1/p, \ldots\ldots, (p-1)/p \}$. Then $E/L \cong E$. Also E is faithful as a right R-module, i.e., ann $E = 0$, i.e., for each $r \in R$, there is $x \in E$ such that $xr \neq 0$. Hence we can embed $R = \mathbb{Z}$ in a product of copies of E,
i.e., $\chi(R) \geq \chi(E) = \chi(E/L)$. So $\chi(R) > \chi(E)$.

But $E(E/\text{ann}_E(\text{ass}E)) = E(E/L) \cong E$, which is indecomposable.

So the right clique of E (as in definition 4.9) consists of E alone, i.e., R is not linked to E.

DEFINITION 4.17: Let $X \subseteq \mathfrak{sp}\text{-}R$. We say X is *right stable* if, whenever $\pi \in X$, $\sigma \in \mathfrak{sp}\text{-}R$ and $\sigma \rightsquigarrow \pi$, we have $\sigma \in X$. If $\pi \in \mathfrak{sp}\text{-}R$, the *right clique* of π, denoted rt cl π, is the smallest right stable subset of $\mathfrak{sp}\text{-}R$ containing π.

NOTE 4.18: By proposition 4.15, we have: If $\pi \in \mathfrak{sp}\text{-}R$ and C is a right Ore set in R and $C \subseteq \mathcal{N}(\pi)$, then $C \subseteq \mathcal{N}(\sigma)$ for every $\sigma \in$ rt cl π.

PROPOSITION 4.19: If C is a right Ore set disjoint from a prime ideal P in a right Noetherian ring R, then we have,
$\rho_C \le \wedge \{ \sigma \in \mathfrak{sp}\text{-}R : \sigma \in \text{rt cl } (\chi(R/P))\}$.

PROOF: Since C is disjoint from P, we have $C \subseteq C(P)$ by [B, Theorem 2.1(c)] and hence $\rho_C \le \rho_{C(P)} = \chi(R/P)$ by proposition 1.18. Hence, by note 4.18, $\rho_C \le \sigma$ for every $\sigma \in$ rt cl $\chi(R/P)$.

PROPOSITION 4.20: If R is a prime right Noetherian ring and E is a fully faithful uniform injective right R-module, then there is no prime torsion class linked to $\chi(E)$, i.e., the right clique of $\chi(E)$ consists of $\chi(E)$ alone.

PROOF: Since E is fully faithful, ass $E = 0$. Hence, we have $L = \mathrm{ann}_E(\text{ass } E) = E$. So, if $\sigma \in \text{sp-}R$ such that $\sigma \rightsquigarrow \pi$, then $\sigma \geq \chi(E/L) = \chi(E/E) = \chi(0)$ which is the largest torsion class, i.e., $\sigma = \chi(0)$, which is not a prime torsion class. Hence, rt cl $\chi(E) = \{ \chi(E) \}$.

PROPOSITION 4.21: If E is a simple uniform injective over a right Noetherian ring, then there is no prime torsion class linked to $\chi(E)$.

PROOF: Let ass $E = P$. Since E is simple, $L = \mathrm{ann}_E P = 0$ or E. If $L = E$, then if $\sigma \in \text{sp-}R$ and $\sigma \rightsquigarrow \chi(E)$, we have $\sigma \geq \chi(E/L) = \chi(0)$ which is not possible, as in proof of proposition 4.20. But $L \neq 0$, since P = ass E is the annihilator of some non-zero submodule of E. Hence there is no prime torsion class linked to $\chi(E)$ and
rt cl $\chi(E) = \{ \chi(E) \}$.

NOTE: Let $\pi = \chi(E)$, for a uniform injective E, P = ass E, $L = \mathrm{ann}_E P$. Then $\sigma \rightsquigarrow \pi \Leftrightarrow \sigma \geq \chi(E/L)$. Hence, if $\sigma \rightsquigarrow \pi$, then I is σ-closed $\Rightarrow$ I is $\chi(E/L)$-closed for any right ideal I of R
Thus,

$$\bigcup \{I : I \text{ is } \sigma\text{-closed and } \sigma \rightsquigarrow \pi \} \subseteq \bigcup \{ I : I \text{ is } \chi(E/L)\text{-closed} \}$$

Hence, we have,

$$\bigcap_{\substack{I \text{ is } \sigma\text{-closed} \\ \sigma \rightsquigarrow \pi}} (R \setminus I) \supseteq \bigcap_{I \text{ is } \chi(E/L)\text{-closed}} (R \setminus I)$$

Let us represent the left hand side of the above equation by A and the right hand side by B. Now, if $\chi(E/L)$ is a prime torsion class, then $\chi(E/L) \;\; >\pi$. So, then, A ≤ B and so we have A = B. Thus, if $\chi(E/L)$ is prime, then,

$\bigcap \{ \mathcal{N}(\sigma) : \sigma \rightsquigarrow \pi \} = \mathcal{N}(E(E/L))$

SOME RIGHT STABLE SETS

DEFINITION 4.22: Let R be a right Noetherian ring and $\sigma \in \mathcal{T}ors\text{-}R$. We define $\Gamma(\sigma) = \{ \pi \in sp\text{-}R : \pi \geq \sigma \}$.

PROPOSITION 4.23: Let R be a right Noetherian ring and let $X \subseteq sp\text{-}R$. If C is a multiplicative set in R such that $C \subseteq \bigcap \{ C_\tau : \tau \in X \}$, then $X \subseteq \Gamma(\rho_C)$.

PROOF: Let $\sigma \in X$. Then $C \subseteq C_\sigma$. Hence $\rho_C \leq \rho_{C_\sigma} \leq \sigma$ by proposition 2.3, i.e., $\sigma \in \Gamma(\rho_C)$, i.e., $X \subseteq \Gamma(\rho_C)$.

PROPOSITION 4.24: Let D be a right Ore set in a right Noetherian ring R. Then $\Gamma(\rho_D)$ is right stable under generalised injective links.

PROOF: Let $\pi \in \Gamma(\rho_D)$ and $\sigma \rightsquigarrow \pi$. Then $\pi \geq \rho_D$ and so $C_{\rho_D} \subseteq C_\pi$, i.e., $D \subseteq \mathcal{N}(\pi)$ by proposition 2.4. So, by proposition 4.15, $D \subseteq \mathcal{N}(\sigma)$, i.e., $\rho_D \leq \rho_{\mathcal{N}(\sigma)} \leq \sigma$, by propositions 2.3 and 2.11, i.e., $\sigma \in \Gamma(\rho_D)$.

NOTE: If P is a right localisable prime ideal in a right Noetherian ring R, then $\rho_{\mathscr{C}(P)} \in \Gamma(\rho_{\mathscr{C}(P)})$ (since $\rho_{\mathscr{C}(P)}$ is a prime torsion class by example 1.14 and proposition 1.18) and so, by proposition 4.24, rt cl $(\chi(R/P) \subseteq \Gamma(\chi(R/P)))$.

PROPOSITION 4.25: Let R be a right Noetherian ring. If C is a right Ore set in R such that ρ_C is a point, then, for $c \in C$, rt cl $\rho_C \subseteq \delta(RcR)$, where $\delta(I)$ (for an ideal I in R) is as in definition 3.1.

PROOF: Since C is right Ore, by proposition 2.4, $C \subseteq C_{\rho_C} = \mathcal{N}(\rho_C)$. Hence $c \in \mathcal{N}(\rho_C)$, i.e., R/cR is ρ_C -torsion. So R/RcR is ρ_C -torsion. Now, by note 4.18, $C \subseteq \mathcal{N}(\sigma)$ for every $\sigma \in$ rt cl ρ_C , i.e., $\sigma \in \delta(RcR)$ for every $\sigma \in$ rt cl ρ_C, i.e., rt cl $\rho_C \subseteq \delta(RcR)$.

COROLLARY 4.26: If R is a right Noetherian ring and $\mathscr{C}(P)$ is right Ore for some $P \in \mathcal{S}pec\ R$, then for $c \in \mathscr{C}(P)$, we have rt cl $\chi(R/P) \subseteq \delta(RcR)$, i.e., rt cl $\chi(R/P) \subseteq \bigcap_{c \in C} \delta(RcR)$.

PROOF: By proposition 1.18, $\rho_{\mathscr{C}(P)} = \chi(R/P)$ and by example 1.14, $\chi(R/P)$ is a point.The result now follows from proposition 4.25.

PROPOSITION 4.27: If R is a semisimple Artinian ring, then $\delta(I)$ is right stable under generalised injective links for any two-sided ideal I of R.

PROOF: Since R is semisimple Artinian, $sp\text{-}R = \{ \chi(R/P) : P \in \mathcal{S}pec R \}$ and so, if $c \in R$ and $\sigma \in \delta(RcR)$, then $\sigma = \chi(R/P)$ for some $P \in \mathcal{S}pec R$. By corollary 2.18, $\mathcal{C}(P) = \mathcal{N}(\rho_{\mathcal{C}(P)})$. Since $\sigma \in \delta(RcR)$, R/RcR is σ-torsion and hence, by proof of proposition 4.25, rt cl $\chi(R/P) \subseteq \delta(RcR)$, i.e., $\delta(RcR)$ is right stable under generalised injective links for any $c \in R$,

Now, let I be a two-sided ideal in R. If $c \in I$, then $RcR \leq I$, and so $\delta(RcR) \subseteq \delta(I)$. If $\sigma \in \delta(I)$, then $\sigma = \chi(R/P)$ for some prime ideal P in R. By proposition 1.16(ii), $I \not\leq P$, i.e., there is $c \in I$ such that $RcR \not\leq P$, i.e., $\chi(R/P) \in \delta(RcR)$. i.e., there is $c \in I$ such that $\sigma \in \delta(RcR)$, i.e., $\delta(I) \subseteq \bigcup_{c \in I} \delta(RcR)$. Thus $\delta(I) = \bigcup_{c \in I} \delta(RcR)$. Since $\delta(RcR)$ is right stable under links, for any $c \in C$, so is $\delta(I)$ for any ideal I of R.

CHAPTER FIVE

ORE LINKS

INTRODUCTION

In chapter four, we have defined torsion-theoretic links between prime torsion classes and we have seen that they are extensions of injective links in some sense. Thus generalised injective links are important, but, so far, only as a matter of theoretical interest. Meanwhile, the construction, in theorem 2.23 of the largest right Ore subset of a multiplicative set has motivated us to define entirely new links between uniform injectives (or, equivalently, between the prime torsion classes cogenerated by them). In this chapter, we define these links, which we call Ore links, and also a right intersection condition for uniform injectives. Using these, we see that we can obtain a version of proposition 4.8 using the sets $\mathcal{N}(E)$ instead of $\mathscr{C}(P)$, and without assuming the right second layer condition.

MOTIVATION FOR ORE LINKS

Let R be a right Noetherian ring and let S be a multiplicative set in R satisfying property SO (i.e., for $a, b \in R$, $ab \in S \Rightarrow a \in S$).

Let $S_0 = S$, $S_1 = \{ r \in R : R/rR$ is S_0 -torsion $\}$, and $S_{\alpha+1} = (S_\alpha)_1$ for any successor ordinal α. Then $S = S_0 \supseteq S_1 \supseteq S_2 \supseteq \ldots\ldots$ Now, by proposition 1.15(iii),
$\rho_{S_i} = \wedge \{ \chi(M) : M$ is an S_i-critical right R-module $\}$.

$$\ldots\ldots\ldots\ldots(1)$$

Then, we have

$S_{i+1} = \mathcal{N}(\rho_{S_i})$

$\quad = \bigcap \{ \mathcal{N}(E(M)) : M$ is an S_i-critical right R-module $\}$

Let $s \in S_i \setminus S_{i+1}$, where i is a successor ordinal. Then R/sR is S_{i-1} -torsion but not S_i -torsion. Hence, by equation (1) above, R/sR is not $\chi(M)$-torsion for some M which is S_i -critical. Then, M is not S_i -torsion, but for every non-zero submodule N of M, M/N is S_i - torsion. Now, since $S_i = \bigcap \{ \mathcal{N}(E(M')) : M'$ is an S_{i-1} -critical right R-module $\}$, there is an S_{i-1} -critical module M' which is not $\mathcal{N}(E(M'))$-torsion, but M/N is $\mathcal{N}(E(M'))$-torsion for every non-zero submodule N of M. Then, M is $\mathcal{N}(E(M'))$-torsion-free, for, if not, let the $\mathcal{N}(E(M'))$-torsion submodule of M be $M_1 \neq 0$. Then, by the above statement, M/M_1 is $\mathcal{N}(E(M'))$-torsion, but by the property of torsion submodules, M/M_1 is $\mathcal{N}(E(M'))$-torsion-free, which is a contradiction. Thus, M is $\mathcal{N}(E(M'))$-torsion-free, but M/N is $\mathcal{N}(E(M'))$-torsion for every non-zero submodule N of M, i.e., M is $\mathcal{N}(E(M'))$-critical. We say that $\chi(M)$ is *Ore-linked* to $\chi(M')$, since this link occurs while looking at the construction of the largest right Ore subset of a multiplicative set.

ORE LINKS BETWEEN PRIME TORSION CLASSES

DEFINITION 5.1: If $\sigma, \tau \in sp\text{-}R$, for a right Noetherian ring R, we say that σ is *Ore-linked* to τ, written $\sigma \overset{o}{\rightsquigarrow} \tau$ if and only if $\sigma = \chi(M)$ for some $\mathcal{N}(\tau)$-critical right R-module M

NOTE 5.2: Since $\sigma, \tau \in sp\text{-}R$, by proposition 1.15(iii) we can find uniform injective right R-modules F and E such that $\sigma = \chi(F)$ and $\tau = \chi(E)$. Then $\chi(F) \overset{o}{\rightsquigarrow} \chi(E)$ if and only if $\chi(F) = \chi(M)$ for some $\mathcal{N}(E)$-critical right R-module M. i.e., if and only if $F = E(M)$ for some $\mathcal{N}(E)$-critical right R-module M, by proposition 1.15(ii). Thus, we may loosely say that F is Ore-linked to E. We will use Ore links between uniform injective modules, or, equivalently between the prime torsion classes cogenerated by them, as the situation requires.

We note that, given any torsion class σ, there is always a σ-critical right R-module [G1]. Hence, given $\tau \in sp\text{-}R$, there is always a $\sigma \in sp\text{-}R$ such that $\sigma \overset{o}{\rightsquigarrow} \tau$.

NOTE 5.3: A right stable subset of $sp\text{-}R$ and a right clique are defined in analogy with definition 4.17.

In this chapter, unless otherwise mentioned, links, cliques and related terms will be with reference to Ore links.

We have the following proposition, which shows how Ore links obstruct localisation.

PROPOSITION 5.4: Let R be a right Noetherian ring. Let τ, $\sigma \in sp\text{-}R$ and C be a right Ore set in R. Then, if $C \subseteq C_{\tau}$, we have $C \subseteq C_{\sigma}$ for every $\sigma \in$ rt cl τ under Ore links.

PROOF: If $\sigma \overset{o}{\rightsquigarrow} \tau$, we have $\sigma = \chi(M)$ for some C_{τ} -critical right R-module M. Then, since M is $\rho_{C_{\tau}}$ - torsion-free, $\chi(M) \geq \rho_{C_{\tau}}$, i.e., $\sigma \geq \rho_{C_{\tau}}$. Since $C \subseteq C_{\tau}$, we have $\rho_C \leq \rho_{C_{\tau}} \leq \sigma$, i.e., $C_{\rho_C} \subseteq C_{\sigma}$. By proposition 2.4, $C \subseteq C_{\sigma}$, since C is right Ore.

We next note a result that we frequently use.

PROPOSITION 5.5 [G1, proposition 19.2]: If $\tau \in sp\text{-}R$, then a right R-module M is τ-critical of and only if $\tau = \chi(M)$.

PROPOSITION 5.6: If R is a right Noetherian ring and $\tau \in sp\text{-}R$ such that $\rho_{C_\tau} = \tau$, then τ has only a self-Ore link.

PROOF: If $\sigma \in sp\text{-}R$ such that $\sigma \overset{o}{\rightsquigarrow} \tau$, then $\sigma = \chi(M)$ for some C_τ-critical right R-module M. Then, by assumption, M is τ-critical and so, by proposition 5.5, $\tau = \chi(M)$, i.e., $\sigma = \tau$.

COROLLARY 5.7: If R is a commutative Noetherian ring, and σ, $\tau \in sp\text{-}R$, then $\sigma \overset{o}{\rightsquigarrow} \tau$ if and only if $\sigma = \tau$.

PROOF: Since R is commutative Noetherian, $\tau = \chi(R/Q)$ for some $Q \in \mathscr{S}pec\ R$ and $\mathscr{C}(Q) = \mathscr{N}(E(R/Q))$. So by propositions 2.11 and 1.18, we have $\rho_{C_\tau} = \rho_{\mathscr{N}(E(R/Q))} = \rho_{\mathscr{C}(Q)} = \chi(R/Q) = \tau$. Hence, by proposition 5.6, $\sigma = \tau$. Also, if $\sigma = \tau = \chi(E)$ where E is a uniform injective, then by proposition 5.5, E is τ-critical and so $\sigma \overset{o}{\rightsquigarrow} \tau$.

COROLLARY 5.8: If R is a semisimple Artinian ring and σ, $\tau \in sp\text{-}R$, then $\sigma \overset{o}{\rightsquigarrow} \tau$ if and only if $\sigma = \tau$.

PROOF: Since R is semisimple Artinian $\tau = \chi(R/Q)$ for some $Q \in \mathcal{S}pec\ R$ and by corollaries 2.18 and 2.15, $\mathcal{C}(Q) = \mathcal{N}(E(R/Q))$. Hence the result follows from the proof of corollary 5.7.

Next we see that generalised injective links and Ore links need not imply each other.

PROPOSITION 5.9: Let R be a right Noetherian ring. Then a prime ideal link $Q \rightsquigarrow P$ ned not imply an Ore link between $\chi(R/Q)$ and $\chi(R/P)$.

PROOF: Let $Q \rightsquigarrow P$ with $\mathcal{C}(P)$ right Ore. Then by propositions 2.11, 1.18, and corollary 2.15, we have

$\rho_{C_{\chi(R/P)}} = \rho_{\mathcal{N}(E(R/P))} = \rho_{\mathcal{C}(P)} = \chi(R/P)$.

Hence, if $\chi(R/Q) \overset{o}{\rightsquigarrow} \chi(R/P)$, by proposition 5.6, we have $\chi(R/Q) = \chi(R/P)$, i.e., $Q = P$. Thus, if $Q \rightsquigarrow P$ with $Q \neq P$ and $\mathcal{C}(P)$ right Ore, we cannot have $\chi(R/Q) \overset{o}{\rightsquigarrow} \chi(R/P)$.

PROPOSITION 5.10: Let R be a right Noetherian ring. Then an Ore link between two prime torsion classes need not imply a generalised injective link between them.

PROOF: Let R be semisimple Artinian and E a simple right R-module. Then, by proposition 4.21, there is no $\sigma \in \mathcal{s}p\text{-}R$

such that $\sigma \rightsquigarrow \chi(E)$ (generalised injective link) but, by corollary 5.8, $\chi(E)$ is Ore-linked to $\chi(E)$.

PROPOSITION 5.11: Let R be a right Noetherian ring and C be a right Ore set in R such that $\rho_C \in \mathit{sp}\text{-}R$. Then, under Ore links, for any $c \in C$, rt cl $\rho_C \leq \delta(RcR)$. In particular, if $\mathscr{C}(P)$ is right Ore and $c \in \mathscr{C}(P)$, then rt cl $\chi(R/P) \subseteq \delta(RcR)$, where $\delta(RcR)$ is as in definition 3.1.

PROOF: Analogous to the proof of proposition 4.25.

PROPOSITION 5.12: If R is semisimple Artinian, then for any two- sided ideal I of R, $\delta(I)$ is right stable under Ore links.

PROOF: Analogous to the proof of proposition 4.27.

PROPOSITION 5.13: Let R be a right Noetherian ring and $\tau \in \mathit{sp}\text{-}R$ such that $\rho_{C_\tau} \leq \tau$. Then there is $\sigma \in \mathit{sp}\text{-}R$ such that $\sigma \neq \tau$ and $\sigma \overset{o}{\rightsquigarrow} \tau$.

PROOF: Since $\rho_{C_\tau} \leq \tau$, by [G1, proposition 18.18], there is a ρ_{C_τ}-critical right R-module M that is τ-torsion. Then $\chi(M) \overset{o}{\rightsquigarrow} \tau$, but $\chi(M) \neq \tau$.

PROPOSITION 5.14: Let R be a right Noetherian ring and E be a simple injective right R-module. Then $\chi(E) \overset{o}{\rightsquigarrow} \chi(E)$.

PROOF: Since E is simple, E is τ-critical with respect to any torsion class for which it is torsion-free. Now, by proposition 2.3, $\rho_{C_{\chi(E)}} \leq \chi(E)$, i.e., by proposition 2.11, $\rho_{\mathcal{N}(E)} \leq \chi(E)$ and so E is $\mathcal{N}(E)$-critical, i.e., $\chi(E) \overset{o}{\rightsquigarrow} \chi(E)$.

PROPOSITION 5.15: Let R be a right Noetherian ring. Suppose $X = \{ \sigma, \tau \}$ is a stable set of prime torsion classes with $\sigma \overset{o}{\rightsquigarrow} \tau$ and $\tau \overset{o}{\rightsquigarrow}$ being the only links. Then $\sigma = \tau$.

PROOF: By definition of Ore links, $\sigma = \chi(M_1)$ for some ρ_{C_τ}-critical right R-module M_1 and $\tau = \chi(M_2)$ for some ρ_{C_σ}-critical right R-module M_2. By proposition 1.15(iii), we have

$\rho_{C_\tau} = \wedge \{ \chi(M) : M$ is a ρ_{C_τ}-critical right R-module $\}$.

Hence $\rho_{C_\tau} = \chi(M_1) = \sigma$ and $\rho_{C_\tau} \leq \tau$, i.e., $\sigma \leq \tau$. Similarly $\tau \leq \sigma$.

PROPOSITION 5.16: If R is a right Artinian ring, then for every $P \in \mathcal{S}pec\ R$, we have $\chi(R/P) \overset{o}{\rightsquigarrow} \chi(R/P)$.

PROOF: Suppose there is no Ore link $\chi(R/P) \overset{\circ}{\rightsquigarrow} \chi(R/P)$, then $\chi(R/P) \neq \chi(M)$ for any $\mathcal{N}(E(R/P))$-critical right R-module M. Then, we have,

$$\rho_{\mathcal{N}(E(R/P))} = \wedge\{\, \chi(M) : M_R \text{ is } \mathcal{N}(E(R/P))\text{-critical}\} ,$$

by proposition 1.15(iii). Now, by propositions 2.3 and 2.11, $\rho_{\mathcal{N}(E(R/P))} \leq \chi(R/P)$, and since R is Artinian, there are only finitely many points in sp-R. Since $\chi(R/P)$ is a point, there is an $\mathcal{N}(E(R/P))$-critical right R-module M_1 such that $\chi(M_1) \leq \chi(R/P)$, but by assumption $\chi(M_1) \neq \chi(R/P)$, i.e., $\chi(M_1) \leq \chi(R/P)$. But since R is Artinian, $\chi(R/P)$ is minimal in sp-R by [G1, proposition 19.16], i.e., we cannot have $\chi(M_1) \leq \chi(R/P)$. Hence there has to be an Ore link $\chi(R/P) \overset{\circ}{>} \chi(R/P)$.

PROPOSITION 5.17: Let R be a right Noetherian ring and let $\sigma \in \mathit{sp}$-R. Then the prime torsion classes Ore-linked to σ satisfy the incomparability condition, i.e., if $\tau \overset{\circ}{\rightsquigarrow} \sigma$, then any prime torsion class π, such that $\pi \leq \tau$ or $\pi \geq \tau$, cannot be Ore-linked to σ.

PROOF: If $\tau \overset{\circ}{\rightsquigarrow} \sigma$, then $\tau = \chi(M)$ for some ρ_{C_σ} -critical right R-module M. But, by [G1, proposition 19.21], $\chi(M)$ is a minimal element of $\Gamma(\rho_{C_\tau}) = \{ \pi \in \mathit{sp}\text{-}R : \pi \geq \rho_{C_\tau} \}$, i.e., all prime torsion classes that are Ore-linked to τ are minimal elements of $\Gamma(\rho_{C_\tau})$ and hence cannot be greater than or lesser than one another.

PROPOSITION 5.18: If R is a right Noetherian ring and σ, $\tau \in \mathit{sp}\text{-}R$ such that $\sigma \overset{o}{\rightsquigarrow} \tau$, then σ cannot be strictly greater than τ.

PROOF: By definition of Ore links, $\sigma = \chi(M)$ for some ρ_{C_τ}-critical right R-module M. Hence $\sigma \geq \rho_{C_\tau}$. By proposition 2.3, $\rho_{C_\tau} \leq \tau$. As in proposition 5.17, since σ is a minimal element of $\Gamma(\rho_{C_\tau})$, we cannot have $\sigma > \tau$.

PROPOSITION 5.19: Let R be a right Noetherian ring and E be a uniform injective right R-module such that the only prime torsion class Ore-linked to $\chi(E)$ is itself. Then $\mathcal{N}(E)$ is right Ore.

PROOF: By proposition 1.15(iii), we have;
$\rho_{\mathcal{N}(E)} = \wedge \{ \chi(M) : M \text{ is } \mathcal{N}(E)\text{-critical} \}$. Now, since $\chi(E)$ is the only prime torsion class Ore-linked to $\chi(E)$, $\rho_{\mathcal{N}(E)} = \chi(E)$

by definition of Ore links. So $C_{\rho_{\mathcal{N}(E)}} = C_{\chi(E)} = \mathcal{N}(E)$, by proposition 2.11. Hence, by proposition 2.4, $\mathcal{N}(E)$ is right Ore.

Let us say that a prime torsion class $\sigma \in \mathit{sp}\text{-}R$ satisfies condition (A) if either

(a) there is only a self link to σ, or

(b) there is no self link to σ.

PROPOSITION 5.20: Let R be a right Noetherian ring and $\sigma \in \mathit{sp}\text{-}R$ such that for any $\tau \in$ rt cl σ (under Ore links),

(1) τ satisfies condition (A) and

(2) there are only finitely many prime torsion classes linked to τ then $\bigcap_{\tau \in \text{rt cl}\,\sigma} \mathcal{N}(\tau)$ is right Ore.

PROOF: Let $\sigma = \chi(E)$ for a uniform injective right R-module E, by proposition 1.15(i). By propositions 2.3 and 2.11, $\chi(E) \geq \rho_{\mathcal{N}(E)}$ and by proposition 1.15(iii), we have $\rho_{\mathcal{N}(E)} = \langle\, \chi(M) : M \text{ is } \mathcal{N}(E)\text{-critical} \,\rangle$. By assumption (1), there are only finitely many $\chi(M)$'s on the right hand side of this equation. If $\chi(E)$ does not have a self-link, then since $\chi(E)$ is a point, there is an $\mathcal{N}(E)$-critical right R-module M_1 such that $\chi(E) > \chi(M_1)$. If $\chi(E)$ has only a self-link, then the result follows form proposition 5.19. Similarly, considering the prime torsion classes Ore-linked to $\chi(M_1)$, we can find an $\mathcal{N}(M_1)$-critical right R-module M_2 such that $\chi(M_1) > \chi(M_2)$. Proceeding likewise, we get a descending chain $\chi(E) > \chi(M_1) > \chi(M_2) >$........ in rt cl $\chi(E)$ and this chain is finite since, by [G1, proposition 19.17], sp-R satisfies the descending chain condition. Thus we get a finite chain $\chi(E) > \chi(M_1) > \chi(M_2) > \ldots\ldots. > \chi(M_n)$, (for some n) and $\chi(M_n)$ will have only a self-link (since, otherwise, the chain will continue). By proposition 5.19, $\mathcal{N}(M_n)$ is right Ore and by note 2.2 and proposition 2.11, $\mathcal{N}(M_n) \leq \mathcal{N}(E)$, and so, by propositions 5.4 and 2.11, $\mathcal{N}(M_n) \leq \mathcal{N}(\tau)$ for every $\tau \in$ rt cl $\chi(E)$. Hence, we have $\bigcap_{\tau\in \mathrm{rtcl}\,\chi(E)} \mathcal{N}(\tau) = \mathcal{N}(M_n)$, which is right Ore.

PROPOSITION 5.21: If R is a right Noetherian ring and $\sigma \in \mathit{sp}$-R such that rt cl σ is finite and every $\tau \in$ rt cl σ satisfies condition (A), then $\bigcap_{\tau\in \mathrm{rtcl}\,\sigma} \mathcal{N}(\tau)$ is right Ore.

PROPOSITION 5.22: Let R be an FBN ring. If $\sigma \in \mathit{sp}$-R, such that every $\tau \in$ rt cl σ satisfies condition (A), then the set $\bigcap_{\tau\in \mathrm{rtcl}\,\sigma} \mathcal{N}(\sigma)$ is right Ore.

PROOF: Since R is FBN, sp-R = { $\chi(R/P)$: $P \in \mathit{Spec}\ R$ }. Let $\tau = \chi(R/Q)$. Then, if $\wedge_{\pi\in\Sigma}\, \pi \leq \chi(R/Q)$, there is $\pi_1 \in \Sigma$ such that $\pi_1 \leq \chi(R/Q)$, for any $\Sigma \subseteq \mathit{Tors}$-$R$, by [S1, lemma 2.4]. The proof is then similar to that proposition 5.20.

DEFINITION 5.23: Let R be a right Noethèrian ring and let X be a non-empty set of uniform injective right R-modules. We say that X satisfies the *intersection condition* if, whenever I is a right ideal of R which contains an element of $\mathcal{N}(E)$ for each $E \in X$, then I contains an element of $\mathcal{N}(X)$, where $\mathcal{N}(X) = \bigcap_{E\in X} \mathcal{N}(E)$.

The following is our counterpart of [J, theorem 7.1.4]

THEOREM 5.24: Let R be a right Noetherian ring and let X be a set of uniform injective right R-modules, right stable under Ore links and satisfying the right intersection condition. Then

(a) If N is an $\mathcal{N}(X)$-critical right R-module, then $E(N) \in X$.

(b) $\mathcal{N}(X)$ is right Ore.

PROOF:

(a) Suppose that for every $E \in X$, N is $\mathcal{N}(E)$-torsion. Then for every $n \in N$, there is $p \in \mathcal{N}(E)$ such that $np = 0$, i.e., r-ann$(n) \cap \mathcal{N}(E) \neq \emptyset$ for each $E \in X$. Then, by right intersection condition, r-ann$(n) \cap \mathcal{N}(X) \neq \emptyset$. So N is $\mathcal{N}(X)$-torsion, which is a contradiction, since N is $\mathcal{N}(X)$-critical. Hence, there is $E \in X$ such that N is not $\mathcal{N}(E)$-torsion. Now, if N_1 is a non-zero submodule of N, then N/N_1 is $\mathcal{N}(X)$-torsion, i.e., N/N_1 is $\bigcap_{E_1\in X} \mathcal{N}(E_1)$-torsion, i.e., for each $E_1 \in X$, N/N_1 is $\mathcal{N}(E_1)$-torsion. Hence, for the above E in X, N/N_1 is $\mathcal{N}(E)$-torsion for each non-zero submodule N_1 of N, but N is not $\mathcal{N}(E)$-torsion, i.e., N is $\mathcal{N}(E)$-critical, i.e., $N \overset{\circ}{\rightsquigarrow} E$. Now, $E \in X$ and X is right stable under Ore links, so $N \in X$.

(b) Let $c \in \mathcal{N}(X)$. Assume that R/cR is not $\mathcal{N}(X)$-torsion. Then let M be a submodule of R/cR maximal with respect to the property that $N = (R/cR)/M$ is not $\mathcal{N}(X)$-torsion. Then N is $\mathcal{N}(X)$-critical. By (a), there is $E \in X$ such that $E(N) \overset{\circ}{\rightsquigarrow} E$. Thus $E(N) \in X$. Since $c \in \mathcal{N}(X) = \bigcap_{F \in X} \mathcal{N}(F)$, we have $c \in \mathcal{N}(E(N))$, i.e., $\mathrm{ann}_{E(N)}\, c = 0$. Hence $\mathrm{ann}_N\, c = 0$. Now, N is a homomorphic image of R/cR, say, $f : R/cR \longrightarrow N$ such that f is a surjection. Let $f(1+cR) = x$. Since R/cR is cyclic, N is cyclic and is generated by x. Hence $x \neq 0$. Then $f(c+cR) = xc$, and $c+cR = 0$, but $xc \neq 0$ since $\mathrm{ann}_N\, c = 0$, which is a contradiction. Hence, by proposition 1.17, $\mathcal{N}(X)$ is right Ore.

PROPOSITION 5.25: Let R be a right Noetherian ring and let $X = \{\ \sigma,\ \tau\ \}$ be a stable subset of $\mathcal{S}\mathit{p}$-R such that the only link to τ is $\sigma \overset{\circ}{\rightsquigarrow} \tau$. Then $\mathcal{N}(X)$ is right Ore.

PROOF: Let $\tau = \chi(E_2)$ and $\sigma = \chi(E_1)$ where E_1, E_2 are uniform injective right R-modules. Then $\chi(E_2) = \chi(M_1)$ for some

$\mathcal{N}(E_1)$-critical right R-module M_1, and $\chi(E_1) = \chi(M_2)$ for some $\mathcal{N}(E_2)$-critical right R-module M_2. Then by propositions 1.15(iii), 2.3 and 2.11, $\rho_{\mathcal{N}(E_1)} = \chi(E_2) \le \chi(E_1)$. Hence $\mathcal{N}(E_2) \le \mathcal{N}(E_1)$. Then, $\mathcal{N}(X) = \mathcal{N}(E_1) \cap \mathcal{N}(E_2)$. Thus X satisfies the right intersection conditioon and so by proposition 5.24, $\mathcal{N}(X)$ is right Ore.

SCOPE FOR FURTHER WORK

So far, we have seen how torsion classes can be used in Ore localisation and related areas. Now, we discuss some problems that arose in the above study and which are still unsolved.

(a) As already mentioned, in the case of prime ideals, the generic regularity condition has an important role in the study of localisation. It would be interesting to see the connection between the generic regularity condition (definition 3.11) and Ore-localisation in the torsion-theoretic case.

(b) We have defined two types of links between prime torsion classes. Though the direct connection between the two is ruled out (propositions 5.9 and 5.10), it is quite possible that they are related to each other in some way.

(c) By proposition 5.16, we see that if R is an Artinian ring and $\sigma \in \mathcal{S}p$-R, then $\sigma \overset{o}{\rightsquigarrow} \sigma$. Now, let Q, $P \in \mathcal{S}pec\,R$ such that $\chi(R/Q) \overset{o}{\rightsquigarrow} \chi(R/P)$, i.e., $\chi(R/Q) = \chi(M)$ for some $\mathcal{K}(E(R/P))$ $(=\mathcal{K}(E_P))$- critical right R-module M. By note 5.2, $E_Q = E(M)$. Let $I = J(R)$ be the Jacobson radical of R. Then I is a semiprime ring and so R/I is a semisimple Artinian ring. Now, let $E' = \mathrm{ann}_{E_P} I$, $F' = \mathrm{ann}_{E_Q} I$, $M' = \mathrm{ann}_M Q$ and $\mathcal{K}' = \mathcal{K}(\mathrm{ann}_{E_P})_{R/I}$. Then, we have, $(F')_{R/I} = E(M')_{R/I}$, by [GW, exercise 4E]. Since E', F' are uniform injective modules over a semisimple Artinian ring, they are simple and so, $F' = M'$. Hence, $M'_{R/I}$ is either $\mathcal{K}'$-torsion or $\mathcal{K}'$-torsion-free. If $M'_{R/I}$ is $\mathcal{K}'$-torsion-free, then it is $\mathcal{K}'$-critical and hence $F' \overset{o}{\rightsquigarrow} E'$ as R/I-modules. By corollary 5.8, $F' = E'$, i.e., $\mathrm{ann}_{E_Q} I = \mathrm{ann}_{E_P} I$. By [J, proposition 4.4.3], $\mathrm{ann}_{E_Q} I = \mathrm{ann}_{E_Q} Q$ and $\mathrm{ann}_{E_P} I = \mathrm{ann}_{E_P} P$, i.e., $\mathrm{ann}_{E_Q} Q = \mathrm{ann}_{E_P} P$. Hence, $\mathrm{ann}(\mathrm{ann}_{E_Q} Q) = \mathrm{ann}(\mathrm{ann}_{E_P} P)$, i.e., $Q = P$ i.e., the Ore link $\chi(R/Q) \overset{o}{\rightsquigarrow} \chi(R/P)$ is a self-link. There is also the case when $M'_{R/I}$ is $\mathcal{K}'$-torsion. It would be interesting to know what happens then.

(d) If R is a right Noetherian ring, then any finite set of prime ideals satisfies Jategaonkar's right intersection condition [J, proposition 7.2.4]. From the proof of this proposition, it can be seen that in a semiprime right Noetherian ring, any finite set of tame injectives satisfy the right intersection condition as in definition 5.23. More generally, if X is a finite set of indecomposable injectives over a right Noetherian ring, then does X satisfy the right intersection condition?

(e) An ideal P in a ring R is *right primitive* if $P = \mathrm{ann}_R\, A$ for some simple right R-module A. If $X \subseteq \mathcal{S}pec\, R$, we say X is a ***classically right localisable set*** if $\mathcal{C}(X) = \bigcap_{P \in X} \mathcal{C}(P)$ is a right Ore set and the localisation $R_X = R\mathcal{C}(X)^{-1}$ has the following properties:

1) R_X / PR_X is a simple ring for all $P \in X$.
2) Every right primitive ideal of R_X has the form PR_X for some $P \in X$.
3) Every finitely generated right R_X -module which is an essential extension of a simple right R_X -module is Artinian.

A set X of prime ideals of R is said to satisfy the *incomparability condition* if the members of X are pairwise incomparable, i.e., no member of X is properly contained in any other member of X. Then, by [J, Theorem 7.1.5], we have

THEOREM: A non-empty set X of prime ideals in a (right) Noetherian ring is (right) classically localisable if and only if it is (right) stable and satisfies the (right) second layer condition, the (right) intersection condition and the incomparability condition.

It would be interesting to get an analogous definition of a classically localisable set of uniform injectives (or, equivalently, of prime torsion classes) and to get a torsion-theoretic version of the above theorem.

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