STUDIES IN RINGS

GENERALISED UNIQUE FACTORISATION RINGS

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CERTIFICATE

Certified that the work reported in this thesis is based on the bona fide work done by Sri. K.P. Naveena Chandran, under my guidance in the Department of Mathematics and Statistics, Cochin University of Science and Technology, and has not been included in any other thesis submitted previously for the award of any degree.

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C O N T E N T S

Chapter-1

INTRODUCTION AND PRELIMINARIES

The evolution of non-commutative ring theory spans a period of about one hundred years beginning in the second half of the 19th century. This period also saw the development of other branches of algebra such as group theory, commutative ring theory, etc. However, the non-commutative Noetherian ring theory has been an active area of research only for the last thirty years, eversince Alfred W. Goldie proved some fundamental results in the late fifties of this century.

The concept of commutative Noetherian Unique Factorisation Domains has been extended to rings which are not necessarily commutative, in different ways. A.W. Chatters [1,2], D.A. Jordan [2] are the forerunners in this direction. A.W. Chatters in [1] defined Non-Commutative Noetherian Unique Factorisation Domains [NUFDs]. Although the rings of this class have many properties of commutative UFDs, there are not many non-commutative rings in this class. In [2], A.W. Chatters and D.A. Jordan extended the concept of NUFD to Non-Commutative Unique Factorisation Rings [NUFR].

REMINDER OF THE COMMUTATIVE CASE

A commutative domain R is a UFD if every non zero element of R is a unit or is a product of irreducible elements which are unique except for their order and multiplication by units. Examples include the ring of polynomials in a finite number of indeterminates over a field or the integers; the Gaussian integers, etc. I. Kaplansky [3] has proved that a commutative domain R is a UFD if and only if every non zero prime ideal of R contains a principal prime ideal, equivalently every height one prime ideal is principal (a height one prime in these circumstances being a prime ideal, minimum with respect to not being zero). Note that if R is a commutative UFD then so also is the polynomial ring in an indeterminate x over R and also R is integrally closed.

NON COMMUTATIVE UNIQUE FACTORISATION DOMAINS

In [1] Chatters considered only Noetherian domains which are not necessarily commutative. An element p in such a ring R is called prime if $pR = Rp$ and R/pR is a domain (which implies that if p divides ab then p divides a or p divides b). The letter C is used to denote the elements of R which are regular (non-zero divisors) modulo all height 1 prime ideals (i.e., if p is a height 1 prime ideal and $cd \in P$ for some $c \in C$ and $d \in R$, then $d \in P$).

Definition 1.1.

A ring R is a NUFD if every height 1 prime ideal of R is of the form pR for some prime element p equivalently if every non zero element of R is of the form $cp_1 \dots p_n$ where $c \in C$ and p_i are prime elements.

Even though this non-commutative analogue is the exact extension of UFD, it lacks some properties, for example, the polynomial ring R[x] over the UFD, R, is a UFD, in the commutative case. With these unpleasant consequences of this extension in mind Chatters and Jordan defined Noetherian Unique Factorisation Rings [2].

Instead of Noetherian domains, they considered the more general prime Noetherian rings and used the characterisation of UFDs by Kaplansky in this definition.

Definition 1.2.

Let R be a prime Noetherian ring. Then R is a Noetherian Unique Factorisation Ring (NUFR) if every non zero prime ideal of R contains a non zero principal prime ideal.

Since every domain is a prime ring, this class of rings contains the class of NUFDs. The rings of this class have almost all properties of UFDs but the factorisation can be done only for those elements p with pR = Rp.

Before entering into more details of the material of this thesis we give a brief review of the preliminary materials.

PRELIMINARIES

Conventions

All the rings in this thesis are assumed to be associative and they have identity elements unless it is otherwise mentioned. To emphasize the order theoretic nature, we use the notations of inequalities $\leq$, $<$, $\nleq$ for 'contained in', 'properly contained in' and 'not contained in' respectively.

We begin with the basic equivalent conditions which are abbreviated by "Noetherian" honoring, E. Noether, who first demonstrated the importance and usefulness of these conditions. Recall that a collection $\mathcal{A}$ of subsets of a set A satisfies the ascending chain condition (or ACC) if there does not exist a properly ascending infinite chain $A_1 < A_2 < \dots$ of subsets from $\mathcal{A}$. Recall also that a subset $B \in \mathcal{A}$ is said to be maximal of $\mathcal{A}$, if these does not exist a subset in $\mathcal{A}$ which properly contains B.

Proposition 1.3.

Let R be a ring and A_R be a right R-module. The following conditions are equivalent.

(a) A_R has ACC on submodules

(b) Every non-empty family of submodules of A_R has a maximal element.

(c) Every submodule of A_R is finitely generated.

Definition 1.4.

A module A_R is said to be Noetherian if and only if the equivalent conditions of Proposition 1.3 are satisfied.

Definition 1.5.

A ring R is right (left) Noetherian if and only if the right R-module R_R (left R-module ${}_RR$) is Noetherian.

If both conditions hold, R is said to be Noetherian.

Example 1.6.

It is easy to observe that the 2 x 2 matrices of the form $\begin{bmatrix} a & b \\ o & c \end{bmatrix}$ where $a \in Z$ and $b, c \in Q$, make, a ring which is right Noetherian but not left Noetherian.

Proposition 1.7.

Let B be a submodule of A. Then A is Noetherian if and only if B and A/B are both Noetherian.

Corollary 1.8.

Any finite direct sum of Noetherian modules is Noetherian.

Corollary 1.9.

If R is a Noetherian ring, all finitely generated right R-modules are Noetherian.

Definition 1.10.

Given a ring R and a positive integer n, we use $M_n(R)$ to denote the ring of all n x n matrices over R. The standard n x n matrix units in $M_n(R)$ are the matrices e_{ij} (for $i,j = 1,2,\ldots,n$) such that e_{ij} has 1 as the $i\text{-}j^{th}$ entry and 0 elsewhere.

Proposition 1.11.

Let R be a right Noetherian ring and let S be a subring of $M_n(R)$. If S contains the subring

$$R' = \{\text{diagonal } (r,r \ldots,r) \mid r \in R\}$$ of all

scalar matrices, then S is right Noetherian. In particular $M_n(R)$ is a right Noetherian ring.

Proof

It is obvious that R is isomorphic to R' and $M_n(R)$ is generated as a right R' module by the standard n x n matrix units. Since R' is right Noetherian and the number of e_{ij}'s is finite, $M_n(R)$ is a Noetherian R'-module, by corollary 1.9. As all right ideals of S are also right R'-submodules of $M_n(R)$, we conclude that S is right Noetherian.

PRIME IDEALS

It is well known that the prime ideals are the 'building blocks' of ideal theory in commutative rings. We recall that a proper ideal P in a commutative ring is said to be prime if whenever we have two elements a and b in R such that $ab \in P$, it follows that either $a \in P$ or $b \in P$; equivalently P is prime if and only if R/P is a domain.

In non-commutative rings, it turns out that it is not a good idea to concentrate on prime ideals P such that R/P is a domain (ab $\in$ P implies a $\in$ P or b $\in$ P). In fact, there are many non-commutative rings with no factor rings which are domains. Thus the desirable thing is to give a more relaxed definition for prime ideals. The key is to change the commutative definition by replacing products of elements by products of ideals which was first proposed by Krull in 1928.

Definition 1.12.

A prime ideal in a ring R is a proper ideal P of R such that whenever I and J are ideals of R with IJ $\leq$ P, either I $\leq$ P or J $\leq$ P, P is said to be a completely prime ideal, if whenever a,b $\in$ R such that ab $\in$ P, either a $\in$ P or b $\in$ P. A prime ring is a ring in which 0 is a prime ideal and a domain is a ring in which 0 is a completely prime ideal.

From part (c) of the following proposition it follows that in the commutative case the prime ideals and the completely prime ideals coincide with the usual prime ideals and in non-commutative setting, every completely prime ideal is a prime ideal.

Proposition 1.13.

For a proper prime ideal P in a ring R, the following are equivalent.

(a) P is a prime ideal

(b) R/P is a prime ring

(c) If $x, y \in R$ with $xRy \leq P$, either $x \in P$ or $y \in P$

(d) If I and J are any two right ideals of R such that $IJ \leq P$, either $I \leq P$ or $J \leq P$

(e) If I and J are any two left ideals such that $IJ \leq P$, either $I \leq P$ or $J \leq P$.

It follows immediately (by induction) from the above proposition that if P is a prime ideal in a ring R and $J_1, \ldots, J_n$ are right (or left) ideals of R such that $J_1 J_2 \ldots J_n \leq P$, then some $J_i \leq P$.

Proposition 1.14.

Every maximal ideal M of a ring R is a prime ideal.

Definition 1.15.

A minimal prime ideal in a ring R is any prime ideal which does not properly contain any other prime ideal .

For instance, if R is a prime ring, then 0 is a minimal prime ideal.

The next two propositions guarantee the existence of minimal prime ideals in a ring R and their connection with the ideal O in a right Noetherian ring.

Proposition 1.16.

Any prime ideal P in a ring R contains a minimal prime ideal.

Proposition 1.17.

In a right Noetherian ring R, there exist only finitely many minimal prime ideals, and there is a finite product of minimal prime ideals (repetitions allowed) equal to zero.

Remark 1.18.

Given an ideal I in a right Noetherian ring R, we may apply proposition 1.16 to the ring R/I to get a finite number of minimal prime ideals $Q_1/I, Q_2/I, \ldots Q_n/I$ of R/I such that their product is O. Since Q_i/I is a minimal prime ideal of R/I for each i, each Q_i, $i=1,2,\ldots,n$ is a prime ideal of R containing I and the minimality of Q_i's assures that they are minimal among the prime ideals containing I. Thus in a right Noetherian ring, given any ideal I, there exist a finite number of prime ideals minimal among all prime ideals of R containing I, such that their product is contained in I. Such prime ideals are called minimal prime ideals over I.

SEMIPRIME IDEALS

Definition 1.19.

A semiprime ideal in a ring R is any ideal of R which is an intersection of prime ideals. A semiprime ring is any ring in which 0 is a semiprime ideal.

For example, the proper semiprime ideals of Z are of the form nZ, where n is a square-free integer. In fact, in a commutative ring R, an ideal I is semiprime if and only if whenever $x \in R$ and $x^2 \in I$, it follows that $x \in I$. The example of a matrix ring over a field shows that this criterion fails in the noncommutative case. However, we have an analogous criterion.

Proposition 1.20.

An ideal I in a ring R is semiprime if and only if whenever $x \in R$ with $xRx \leq I$, then $x \in I$.

Corollary 1.21.

For an ideal I in a ring R, the following conditions are equivalent.

(a) I is a semiprime ideal

(b) If J is any ideal such that $J^2 \leq I$, then $J \leq I$.

<u>Corollary 1.22.</u>

Let I be a semiprime ideal in a ring R, J be any left or right ideal of R such that $J^n \leq I$ for some positive integer n, then $J \leq I$.

<u>Definition 1.23.</u>

A right or left ideal J in a ring R is <u>nilpotent</u> provided $J^n = 0$ for some positive integer n. More generally, J is nil provided every element of J is nilpotent.

<u>Definition 1.24.</u>

The <u>prime radical</u> of a ring R is the intersection of all prime ideals of R.

It is easy to observe that the prime radical of any ring is nil and R is semiprime if and only if its prime radical is zero.

<u>Proposition 1.25.</u>

In any ring R, the prime radical equals the intersection of all minimal prime ideals.

In Noetherian rings we have the following important result.

<u>Proposition 1.26</u>.

In a right Noetherian ring, the prime radical is nilpotent and contains all the nilpotent right or left ideals.

PRIMITIVE IDEALS

<u>Definition 1.27</u>.

Let R be a ring and S be a subset of a right R-module A. The <u>annihilator of S</u> is defined as $\{r \in R \mid sr = 0 \text{ for all } s \in S\}$. If S is a subset of R, r(S), the right annihilator of S is defined as $\{r \in R \mid sr = 0 \text{ for all } s \in S\}$ and left annihilator $\ell(S)$ is defined as $\{r \in R \mid rs = 0 \text{ for all } s \in S\}$. A module A is said to be <u>faithful</u> if annihilator of A = 0.

<u>Definition 1.28</u>.

An R-module A is said to be <u>simple</u> if A has no proper submodules. A ring R is said to be simple if it has no proper ideals.

Definition 1.29.

An ideal P in a ring R is said to be right (left) primitive provided P = ann $_{R}A$ for some simple right (left) R-module A. A right (left) primitive ring is any ring in which 0 is a primitive ideal, ie. any ring with a faithful, simple right (left) R-module.

Proposition 1.30.

In any ring R, the following sets coincide:

(a) The intersection of all maximal right ideals.

(b) The intersection of all maximal left ideals.

(c) The intersection of all right primitive ideals.

(d) The intersection of all left primitive ideals.

Definition 1.31.

A ring R is semiprimitive (Jacobson Semisimple) if and only if the Jacobson radical J(R) of R is equal to zero where J(R) is the intersection defined in proposition 1.30.

SEMISIMPLE RINGS

Vector spaces, when viewed module theoretically, are distinguished by many nice properties. For instance, every vector space is a direct sum of one-dimensional subspaces. We view simple modules as analogous to one dimensional spaces, and the corresponding analogoues to higher dimenaional vector spaces are the semisimple modules; modules which are direct sums of simple submodules.

Definition 1.32.

The socle of an R-module A is the sum of all simple submodules of A and is denoted by soc A. A is semisimple if A = soc A.

In any ring R, it is easy to observe that soc (R_R) is an ideal of R. Similarly soc $({}_R R)$ is an ideal of R, but these two socles need not coincide in general. However, there are rings in which these two coincide. For instance $R = M_n(D)$, where n is a positive integer and D is a division ring. In case n = 2, the right ideals $I_1 = \begin{bmatrix} D & D \\ 0 & 0 \end{bmatrix}$, $I_2 = \begin{bmatrix} 0 & 0 \\ D & D \end{bmatrix}$ are the simple right ideals and $M_2(D) = I_1 \oplus I_2$. Similarly $M_2(D) = J_1 \oplus J_2$, where $J_1 = \begin{bmatrix} D & 0 \\ D & 0 \end{bmatrix}$, $J_2 = \begin{bmatrix} 0 & D \\ 0 & D \end{bmatrix}$ are the simple left ideals. We state a proposition.

Proposition 1.33.

For any ring R, the following conditions are equivalent.

(a) All right R-modules are semisimple

(b) All left R-modules are semisimple

(c) R_R is semisimple

(d) ${}_RR$ is semisimple

Definition 1.34.

A ring satisfying the conditions of Proposition 1.33 is called a semisimple ring.

Definition 1.35.

A module A is Artinian provided A satisfies the descending chain condition (DCC) on submodules, i.e., there does not exist a properly descending infinite chain of submodules of A. A ring R is called right (left) Artinian if and only if the right R-module R_R (left R-module ${}_RR$) is Artinian. If both conditions hold, R is called an Artinian ring.

Remark 1.36.

As in the case of Noetherian structures it is easy to observe that A is Artinian if and only if A/B and B are

Artinian where B is a submodule of the module A and that any finite direct sum of Artinian modules is Artinian. Also, if R is an Artinian ring, so is every finitely generated R-module.

Now we state the celebrated theorems on simple and semisimple rings, due to Weddernburn and Artin.

<u>Proposition 1.37</u>.

For a ring R, the following conditions are equivalent.

(a) R is right Artinian and $J(R) = 0$

(b) R is left Artinian and $J(R) = 0$

(c) R is semisimple

(d) $R = M_{n_1}(D_1) \times M_{n_2}(D_2) \times \dots \times M_{n_k}(D_k)$ for some positive integers $n_1, n_2, \dots, n_k$ and division rings $D_1, \dots, D_k$.

Hopkins and Levitzki have proved the significant result that if R is a right Artinian ring then R is also right Noetherian, and J(R) is nilpotent. The following proposition is a consequence of this result.

Proposition 1.38.

For a ring R, the following conditions are equivalent.

(a) R is simple left Artinian

(b) R is simple right Artinian

(c) R is simple and semisimple

(d) $R = M_n(D)$, for some positive integer n and some division ring D.

RING OF FRACTIONS

In the theory of commutative rings, localisation at a multiplicative set plays a very important role. Most important is the idea of a quotient field, without which one can hardly imagine the study of integral domains. A very useful technique in commutative theory is the localisation at a prime ideal, which reduces many problems to the study of local rings and their maximal ideals.

However, this is not the case with non-commutative rings. Although the set of nonzero elements is a multiplicative set in any domain, we have examples of domains which do not possess a division ring of quotients. It was in 1930, that O. Ore characterised those non-commutative domains which possess division rings of fractions. In fact, Ore has proved a more general result by classifying the multiplicative sets in a ring R, at which the right (left) ring of quotients (fractions) of R exists.

Definition 1.39.

Let R be any ring. A multiplicative set D in R is said to satisfy the right (left) Ore condition if given $r \in R$, $s \in D$ there exist $r' \in R$ and $s' \in S$ such that $rs' = sr'$ ($s'r = r's$). In this case D is said to be a right (left) Ore set. If D satisfies both right and left conditions, D is simply called an Ore set.

Property 1.40.

We have a very useful property in a right Ore set known as the right common multiple property.

If D is a right Ore set in R, then given any $d_1, d_2, \ldots, d_n \in D$, there exist $d \in D$ and $r_1, r_2, \ldots, r_n$ in R such that $d = d_1 r_1 = d_2 r_2 = \ldots = d_n r_n$. The left common multiple property is defined like wise.

Definition 1.41.

A multiplicative set D in a ring R is said to be right reversible in R, if for any $d \in D$, $r \in R$ with $dr = 0$, there exists $d' \in D$ such that $rd' = 0$. D is defined to be a <u>right denominator</u> set if D is right Ore and right reversible.

<u>Proposition 1.42</u>.

In a right Noetherian ring every right Ore set is right reversible.

<u>Definition 1.43</u>.

Let D be a multiplicative set in a ring R. <u>A right quotient ring of R relative to D is a pair</u> (Q,f) where Q is a ring and f is a homomorphism from R to Q satisfying the following conditions.

(a) For any $d \in D$, $f(d)$ is a unit in Q.

(b) For every $q \in Q$, there exist $r \in R$ and $d \in D$ such that $q = f(r)\, f(d)^{-1}$.

(c) $\ker f = \{r \in R \mid rd = 0 \text{ for some } d \in D\}$.

<u>Remark 1.44</u>.

A right localisation of a ring R with respect to a multiplicative set D is a ring $RD^{-1} = \{rd^{-1} \mid r \in R,\ d \in D\}$ such that

(a) $(d1^{-1})(dl^{-1}) = (dl^{-1})(d1^{-1}) = 1$ in RD^{-1} for all $d \in D$.

(b) The map $r \longrightarrow rl^{-1}$ is a ring homomorphism from R to RD^{-1}.

(c) For $r,s \in R$ and $d \in D$ $rd^{-1} = sd^{-1}$ if and only if $rc = sc$ for some $c \in D$. The c occurs because D may contin zero divisors. If D consists of non zero divisors, then $rd^{-1} = sd^{-1}$ if and only if $r=s$.

It can be easily seen that the definitions of a right quotient ring in 1.43 and the right localisation in remark 1.44 are equivalent.

Now we state Ore's theorem.

Theorem 1.45.

Suppose D is a multiplicative set in a ring R. A right localisation of R relative to D exists if and only if D is a right Ore right reversible set.

Remark 1.46.

Let us write an element of RD^{-1} as a/s where $a \in R$, $s \in D$ and call 'a' the numerator and 's' the denominator of this expression. Then it can be interpreted that two fractions are equal if and only if when they are brought to a common denominator, their numerators agree. It follows from the right common multiple property of D that any two expressions can be brought to a common denominator. So we can define the addition of two fractions by the rule $(a/s)+(b/s) = (a+b/s)$. Here it can be easily verified that the expression in the right depends only on a/s and b/s and not on a,b and s. To define the product of a/s and b/t we determine $b_1 \in R$ and $s_1 \in D$ such that $bs_1 = sb_1$ and then put $(a/s)(b/t)=(ab_1/ts_1)$. Again it is easy to check that this product is well defined.

A ring R is said to be a domain if, it is without zero divisors. It is obvious that the nonzero elements in a domain form a multiplicative set and if $D = R-O$, D trivially satisfies the right and left reversibility conditions. From this fact we get the following corollary of Ore's theorem.

<u>Corollary 1.47</u>.

A domain R has a right division ring of fractions (right quotient division ring) if and only if D is a right Ore set if and only if the intersection of any two nonzero right ideals is nonzero.

<u>Definition 1.48</u>.

A domain which satisfies the condition of Corollary 1.47 is called a <u>right Ore domain.</u> <u>Left Ore</u> domains are defined analogously.

Ore's theorem, though proved in 1930, was only a theoretical curiosity for a long time until Alfred Goldie proved some results, nowadays known as Goldie's theorems, in this direction in 1958. The importance of Goldie's theorems is that it paved the way to many new investigations and answered many questions posed on non-commutative ring theory. We have seen that there are many non-commutative domains which do not possess a right or left division ring of fractions and there are many rings which do not have any factor rings which are right or left Ore domains. Instead of looking for Ore domains and division rings of fractions, we look for rings from which Simple Artinian rings can be built using fractions. Goldie's main result states that if R is a Noetherian ring with O a prime ideal (P a prime ideal), then R has (R/P has) a simple Artinian ring of fractions. It turns out to be no extra work to investigate rings from which semisimple ring of fractions can be built. We begin with some definitions.

Definition 1.49.

A regular element in R is any non-zero divisor, i.e., any $x \in R$ such that $r(x) = 0$ and $\ell(x) = 0$.

Note that if $R \leqslant Q$ are rings and x is any element of R which is invertible in Q, then x is a regular element in R.

Definition 1.50.

Let I be an ideal of R. An element $x \in R$ is said to be regular modulo I provided the coset x+I is regular in R/I. The set of such x is denoted by C(I). Thus the set of regular elements in R may be denoted by $C_R(0)$. Often we use the notation $C_R(I)$ for C(I).

Definition 1.51.

A right (left) annihilator ideal in a ring R is any right (left) ideal of R which equals the right (left) annihilator of some subset X.

Definition 1.52.

A ring R is said to be of finite right (left) rank if R_R $({}_RR)$ contains no infinite direct sum of submodules.

Definition 1.53.

A ring R is said to be right (left) Goldie if R_R $({}_RR)$ has finite rank and R has ACC on right (left) annihilators.

Definition 1.54.

Let A be a right R-module and B a submodule of A. B is said to be an essential submodule of A if $B \cap C \neq 0$. for every non zero submodule C of A.

Definition 1.55.

Let Q be a ring. A right order in Q is any subring $R \leqslant Q$ such that

(a) Every regular element of R is invertible in Q

(b) Every element of Q has the form ab^{-1} for some $a \in R$ and some regular element b in R.

It is clear that the ring Q in the definition 1.55 and the localization of the ring R at the multiplicative set $C_R(0)$ are same.

Remark 1.56.

A right Goldie ring is any ring R, such that R has finite right rank and ACC on right annihilators. Thus every right Noetherian ring is right Goldie.

Remark 1.57.

Goldie has proved that in a semiprime right Goldie ring every essential right ideal contains a regular element and that the right ideal generated by a right regular element is right essential. As a consequence, in such rings right regular elements are regular. Also it can be seen easily that any ideal in a prime right Goldie ring is essential as a right ideal and as a left ideal and so it contains regular elements.

Theorem 1.58 (Goldie)

A ring R is a right order in a semisimple Artinian ring Q if and only if R is a semiprime right Goldie ring.

Theorem 1.59 (Goldie)

A ring R is a right order in a simple Artinian ring Q if and only if R is a prime right Goldie ring.

Remark 1.60.

The ring Q, as in theorem 1.58, is called a right Goldie quotient ring of R. Analogous results exist for left semiprime (prime) Goldie rings. When both left Goldie quotient ring and right Goldie quotient ring exist they can be identified and called the Goldie quotient ring. An important property of Q_R is that it will be the injective hull of R_R.

Goldie's theorems give the structure of a semiprime (prime) right Goldie ring, as it is a right order in a semisimple Artinian (simple Artinian) ring which in turn is the finite direct product of matrix rings over division rings (matrix ring over a division ring). Thus, in particular we get the structure theorems for semiprime (prime) right Noetherian rings.

ARTINIAN QUOTIENT RINGS

In the previous section we have seen that every right Noetherian semiprime ring (every right Noetherian prime ring) is a right order in a semisimple (simple) Artinian ring. Now we see the more general case, i.e., when Q, the quotient ring, is simply an Artinian ring. Some times we call Q the total quotient ring as it consists of all quotients with denominators varying over the regular elements.

<u>Proposition 1.61</u>.

Let R be a ring which has a right quotient ring Q which is right Artinian and let A be an ideal of R, then AQ is an ideal of Q.

Proposition 1.62.

Let R be a right Noetherian ring with N, the prime radical and $P_1, P_2, \ldots, P_n$ the minimal prime ideals of R. Then,

(1) The right regular elements are regular modulo N

(2) $C_R(N) = C_R(P_1) \cap C_R(P_2) \cap \ldots \cap C_R(P_n)$

(3) Let $a, c \in R$ with c right regular, then there exists $b \in R$ and $d \in C_R(N)$ such that $ad = bc$.

(4) R has a right Artinian quotient ring if and only if, $C_R(O) = C_R(N)$.

We state a result, a characterisation of Noetherian rings which are orders in Artinian rings, proved by P.F. Smith [4].

Proposition 1.63.

A Noetherian ring R is an order in an Artinian ring if and only if

$$\{P \in \text{Spec } R \mid P \cap C_R(O) = \emptyset\} \leq \text{Minimal (Spec R)}$$

where Spec R denotes the collection of all prime ideals of R.

SCOPE OF THE THESIS

In this thesis we define and study the properties of a particular class of Noetherian rings namely, Generalised Unique Factorisation Rings (GUFR). First of all the class of GUFRs is a subclass of the class of Noetherian rings with over rings. A GUFR R is defined as a Noetherian ring with an over ring S such that every non-minimal prime ideal of R contains a principal ideal (i.e., there exists a $p \in R$ such that $pR = Rp$) which is so called S-invertible ideal.

It can be seen that every commutative Noetherian integral domain is a GUFR. Further it is easy to see that every ideal of the form $pR = Rp$ in a prime Noetherian ring is Q-invertible, where Q is the simple Artinian quotient ring of R. Thus one way to look at GUFRs is as a generalisation of NUFRs [2]. The class of GUFRs is quite larger than the class of NUFRs. A natural example of a GUFR which is not an NUFR is given in the thesis. Many examples of non commutative Noetherian rings are constructed by twisting polynomials, using derivations and automorphisms, over well known Noetherian (Commutative and non-Commutative) rings. Using this tool of twisting of polynomials it could be seen that there are even some prime Noetherian rings which are not NUFRs but are GUFRs.

It is found that the elements 'p' which give rise to S-invertible ideals in the definition of GUFR are regular elements.

The fact that every commutative domain has a field of fractions and every NUFR has a simple Artinian quotient ring (by Goldie's theorem) generalises to the result that every GUFR has a classical ring of quotients which is Artinian.

Just as every principal ideal domain is a UFD, every principal ideal ring with an Artinian quotient ring is a GUFR. From this we get a characterisation of commutative GUFRs.

The polynomial ring over a GUFR is studied. It could be proved that R[x] is a prime GUFR, when R is so. The general case, when R is not prime, is investigated.

Hereditary Noetherian Prime rings (HNP rings) constitute a rich class of Prime Noetherian rings. We recall that a ring R is a right hereditary ring if every right ideal is projective. Left hereditary rings are defined analogously. An HNP ring is a Noetherian prime ring which is both left and right hereditary. We refer the reader to Chatters [5], Chatters and Hajarnavis [6], Faith [7] and Eisenbud and Robson [8] for details.

Right bounded HNP rings are HNP rings in which every essential right ideal contains a two sided ideal. Lenagan [9] has shown that right bounded HNP rings are rings with enough invertible ideals, i.e, in such rings every nonzero prime ideal contains invertible ideals.

Thus, a second way to look at prime GUFRs is through their connection with prime Noetherian rings with enough invertible ideals. It can be seen that if a prime Noetherian ring R with enough invertible ideals is such that all its invertible ideals are principal, then R is a prime GUFR. In particular, right bounded HNP rings in which each invertible ideal is principal, are also prime GUFRs.

After proving all the above mentioned results in Chapter 2, we move over to Chapter 3 in which we study different extension rings of GUFRs.

A finite central extension [10 (pp. 343-77)] ring S of a GUFR R is shown to be a GUFR if the regular elements of R are also regular elements in S. As a consequence the n x n matrix ring $M_n(R)$ over any GUFR, R, is found to be a GUFR. $R[x, \alpha]$, the ring of polynomials, twisted by an automorphism, over a GUFR [11] and $R[x, \delta]$, the ring of polynomials, twisted by a derivation δ over a GUFR [12] are investigated.

The concept of a ring with few zero divisors [13] in the commutative case is generalised to the non-commutative case and the idea of weakly invertible elements is introduced. Some analogous results of quasi-valuation rings [13] in the non-commutative case have been proved. We conclude Chapter 3 with a discussion of integral closure [14], [15] of a GUFR.

The technique of localisation at a prime ideal in Commutative Noetherian rings, cannot be brought into non commutative Noetherian rings as it is. This is because of the general behaviour of prime ideals in non-Commutative rings. This is a major problem (i.e., under what conditions, can a Noetherian ring be localised at its prime ideals?) still confronting the study of non-Commutative Noetherian rings. At present, a theory has emerged as the correct one. Jategaonkar [16], Muller [17] etc. are some of the forerunners in this study. We give a detailed discussion of this recent development in the localisation at prime ideals in Noetherian rings in Chapter 4 and identify some prime ideals and cliques of prime ideals at which the localisation is possible in GUFRs.

In chapter 5, we discuss some problems that arose in the thesis which are to be investigated. Also, a possible extension of the concept of GUFR to non-Noetherian case is given.

The preliminary materials of this chapter have been taken from [18], [19], [20] and [21].

Chapter 2

GENERALISED UNIQUE FACTORISATION RINGS

INTRODUCTION

In commutative ring theory I. Kaplansky [3] classified the UDFs as those integral domains in which every non zero prime ideal contains a principal prime ideal.

The unique factorisation concept, in non-commutative rings, has been investigated by several mathematicians in different contexts. A.W. Chatters [1] was one of the forerunners in this direction. In [1] Chatters called an element p, in a non-commutative Noetherian domain R, (henceforth called Noetherian domain) a prime element if $pR = Rp$ and R/pR is a domain. This is analogous to the definition of a prime element in a commutative Noetherian integral domain. He defined a Noetherian Unique Factorisation Domain [NUFD] as a Noetherian domain in which every non zero element is of the form $cp_1 \dots p_n$, where p_is are prime elements and c is a regular element in R. Equivalently R is a NUFD if every height 1 prime P of R is of the form $pR=Rp$. Examples include all commutative Noetherian UFDs and the universal enveloping algebra $U(L)$ of any solvable or semisimple Lie algebra in the non-commutative case. Several basic facts about commutative UFDs are extended to NUFDs by Chatters in [1]. M.P.Gillchrist and M.K. Smith have proved that NUFDs are often principal ideal domains (in one of their papers).

In 1986 Chatters and Jordan [2] investigated unique factorisation in prime Noetherian rings. They defined a Noetherian unique factorisation ring by analogy with the characterisation of UFDs by Kaplansky. They called a prime Noetherian ring a Noetherian unique factorisation ring (NUFR) if every non zero prime ideal contains a principal prime ideal.

In this chapter we define generalised unique factorisation rings and study the properties of these rings.

BASIC DEFINITION AND EXAMPLES.

Definition 2.1.

Let R be any ring and S an over-ring of R. An ideal I of R is said to be S-invertible, if the R-bimodule S contains an R-subbimodule I^{-1} such that $II^{-1}=I^{-1}I=R$.

<u>Definition 2.2</u>.

An element a in a ring R is said to be a <u>normal element</u>, if aR = Ra = I. In this case we call the ideal I, a <u>normal ideal</u>.

<u>Definition 2.3</u>.

Let R be a Noetherian ring with an over-ring S. Then R is called a <u>Generalised Unique factorisation ring</u> (GUFR), if every non-minimal prime ideal of R contains a normal S-invertible ideal.

<u>Examples 2.4</u>.

(1) In any commutative Noetherian domain D every nonzero prime ideal contains Q-invertible principal ideals, where Q is the quotient field of D. Thus every commutative Noetherian integral domain is a GUFR.

(2) A Noetherian unique factorisation ring, as defined in [2] is a prime Noetherian ring R in which every non zero prime ideal contains a normal prime ideal. Taking S = Q(R), the simple Artinian quotient ring of R, it can be seen that every normal element in R is invertible in S and thus every normal prime ideal is S-invertible. So R is a prime GUFR.

(3) We give an example of a GUFR which is neither a commutative Noetherian domain nor a NUFR.

Let k be a field and $T = k[x_1 \ldots x_n]$. Let $I = \sum_{i=1}^{k} x_i^2 T$ where $k \leq n$. Set $R = T/I$, then $P = \sum_{i=1}^{k} \bar{x}_i R$ is the unique minimal prime ideal of R, where $\bar{x}_i = x_i + I$ for $i=1,2,\ldots,k$. Localise R at P and let the localised ring be R_P. Now it is easy to see that R_P is an over-ring of R and that P contains no R_P-invertible principal ideals. But every non-minimal prime ideal of R strictly contains P and thus contains elements of the complement of P, i.e., units in R_P, which in turn lead to R_P-invertible principal ideals in non-minimal prime ideal. Thus R is a commutative GUFR.

Since R can be embedded in R_P, $M_2(R)$ can be embedded in $M_2(R_P)$. Because of the order preserving bijection between the prime ideals of R and that of $M_2(R)$, $M_2(P)$ is the unique minimal prime ideal of $M_2(R)$. None of the elements of $M_2(P)$ is invertible in $M_2(R_P)$, therefore $M_2(P)$ contains no $M_2(R_P)$-invertible normal ideals.

Let N be a non-minimal prime ideal of $M_2(R)$, then $N \neq M_2(P)$. Let $N = M_2(Q)$, where Q is a prime ideal of R. Then $Q \neq P$ and hence there exists at least one element 'a' in Q such that $a \notin P$. Then the scalar matrix X with non zero entries 'a' is in $M_2(Q) = N$. Put $I = X\, M_2(R) = M_2(R)X$, then $I \leq N$. Furthermore $X^{-1} \in M_2(R_p)$, since X^{-1} is the scalar matrix, with non zero entries a^{-1} which is in R_p and thus $I^{-1} = X^{-1}M_2(R) = M_2(R)X^{-1}$ is contained in $M_2(R_p)$ and $II^{-1} = I^{-1}I = M_2(R)$. Therefore $M_2(R)$ is a GUFR which is not prime.

<u>Remark 2.5</u>.

(1) The principal ideal theorem for a right Noetherian ring asserts that the minimal prime ideals over any normal ideal has height atmost 1. Thus in a GUFR even though every non-minimal prime ideal contains normal ideals, each normal ideal is contained in either a minimal prime ideal or in a prime ideal of height 1.

(2) If R is Noetherian ring satisfying descending chain condition on prime ideals, then R is a GUFR with the over ring S if and only if every height 1 prime ideal of R contains an S-invertible principal ideal.

(3) By Proposition 1.16, if R is a GUFR with over ring S, then every prime ideal contains a normal S-invertible ideal if and only if every minimal prime ideal contains a normal S-invertible ideal.

Let R be a GUFR with over-ring S. We shall make use of a certain partial quotient ring of R. Let $C = \{a \in R / aR = Ra \text{ is S-invertible}\}$. We prove C consists of regular elements and C is a (right and left) Ore set.

QUOTIENT RINGS

Theorem 2.6.

Let R be a GUFR with the over-ring S. Then C contains only regular elements and C is an Ore set.

Proof

Let $a \in C$, we prove $\ell_R(a) = r_R(a) = 0$. Since $a \in C$, $aR = Ra$ is S-invertible and so there exists an R-subbimodule I^{-1} of S such that $(aR)I^{-1} = I^{-1}(aR) = R$. Thus we can find elements $r_i \in R$, $s_i \in I^{-1} \subseteq S$ for $i=1,2,\dots,n$, such that $\sum_{i=1}^{n} (ar_i)s_i = 1$. i.e $a \sum_{i=1}^{n} r_i s_i = 1$ which implies $\ell_S(a \sum_{i=1}^{n} r_i s_i) = \ell_S(1) = 0$ and consequently $\ell_R(a) \leq \ell_S(a) \leq \ell_S(a \sum_{i=1}^{n} r_i s_i) = \ell_S(1) = 0$. Similarly $r_R(a) = 0$.

For the second part of the theorem, let $a, b \in C$, then $aR = Ra$ and $bR = Rb$. Now $abR = a(Rb) = Rab$. Since aR and Rb are S-invertible ideals, there are R-bimodules I^{-1} and J^{-1} such that $(aR)I^{-1} = (bR)J^{-1} = R$. If we write $K^{-1} = J^{-1}I^{-1}$, it is easy to see that $K^{-1}(abR)=(abR)K^{-1}=R$. Thus $a,b \in C$ implies $ab \in C$, i.e. C is a multiplicative set. To prove that C is an Ore set, let $a \in C$ and $r \in R$, then $ra \in Ra = aR$ and so $ra = ar'$ for some $r' \in R$. Thus C satisfies right Ore condition. Similarly C satisfies left Ore condition.

Theorem 2.7.

Let R be a GUFR with the over-ring S. Let $T = RC^{-1} = C^{-1}R$ be the localised ring of R at C. Then T has atmost a finite number of maximal ideals.

Proof

Since C is a right and left Ore set, by proposition 1.42 and theorem 1.45, $T = RC^{-1} = C^{-1}R$ exists and the homomorphism from R to RC^{-1} $(r \longrightarrow r1^{-1})$ is a monomorphism, since C has only regular elements. Thus T is an over-ring of S.

To prove that T has only finite number of maximal ideals, we use the correspondence $P \longrightarrow PT$ which is a bijection from $\{P \in \text{Spec } R / P \cap C = \emptyset\}$ to Spec T. Let $P_1, \ldots, P_n$ be the minimal prime ideals of R such that

$P_i \cap C = \emptyset$ for $i = 1,2,\ldots,n$. Then P_iTs are prime ideals of T for $i=1,2,\ldots,n$. Let J be an ideal of T such that $P_iT < J$ for each $i = 1,2,\ldots,n$. Then $P_iT \cap R < J \cap R = I$ for $1 \leq i \leq n$. Let $P_1', P_2', \ldots, P_m'$ be the minimal primes over I. Then it is obvious that $P_i \neq P_j'$ for $i = 1,2,\ldots,n$, $j = 1,2,\ldots,m$ and thus each P_j' contains elements of C. Therefore the product $P_1'P_2'\ldots P_m'$ also contains elements of C. But $P_1'P_2'\ldots P_m' \leq I$, consequently I contains an element C, i.e., I contains a unit of T. Also we have $IT = (J \cap R)T \leq J$. Hence J contains a unit of T. Thus J = T and we proved that $P_1T, P_2T, \ldots, P_nT$ are maximal ideals of T.

Further, if M is any maximal ideal, then $M = P_iT$ for some $i = 1,2,\ldots,n$. For, if $M \neq P_iT$ for all $i = 1,2,\ldots,n$. Then $M \cap R$ is not contained in P_i for any $i = 1,2,\ldots,n$. Thus, as above, it can be seen that $(M \cap R) \cap C \neq \emptyset$, which implies that M contains a unit of T, contradicting the maximality of M. This completes the proof.

In an NUFR, the minimal prime ideal not containing a normal Q(R)-invertible ideal is 0, and so, 0T = 0 is a maximal ideal of T. Thus we obtain, Corollary 2.8.

If R is an NUFR, then T is a simple ring.

Artinian rings are generally regarded as generalisation of semisimple Artinian rings. Goldie's theorem gives a characterisation of those rings which are orders in semisimple Artinian rings. This result naturally gives rise to the question: Which rings can be orders in Artinian rings?. The importance of Artinian quotient rings is that they will be useful in the study of localisation at a prime ideal in Noetherian rings and in the study of finitely generated torsion free modules over Noetherian rings. It is seen that there are Noetherian rings which lack Artinian quotient rings. However, if R is a GUFR, R always have an Artinian quotient ring. We prove this next.

Theorem 2.9.

Every GUFR has an Artinian quotient ring.

Proof

From the definition of a GUFR, every non-minimal prime ideal contains normal invertible ideals. The generators of these normal invertible ideals are in $C_R(0)$ (the set of regular elements of R), by theorem 2.6. Now the theorem follows from proposition 1.63, which states that R is a

Noetherian order in an Artinian ring if and only if $\{P \in \text{Spec } R / P \cap C_R(0) = \emptyset\} \leq \text{Min Spec } R$.

Remark 2.10.

Thus, even though in the definition of GUFR, we are not assuming that it has an Artinian quotient ring, it turns out that GUFRs always have Artinian quotient rings. It is also obvious that the Artinian quotient ring is an over-ring of the GUFR, and the so called S-invertible ideals are invertible with respect to this Artinian quotient ring also. Hence the terminologies, over-ring S and S-invertible ideals, can be avoided in the definition of a GUFR.

Definition 2.11.

Let X be a right denominator set in a ring R. If I is a right ideal of RX^{-1}, the set $\{a \in R / a1^{-1} \in I\}$ is called the contraction of I to R and is denoted by I^c. If J is a right ideal of R, then $\{cx^{-1} / c \in I, x \in X\}$ in RX^{-1} is called the extension of J in RX^{-1} and is denoted by J^e.

Proposition 2.12.

Let X be a right denominator set in a right Noetherian ring. Then

(1) RX^{-1} is a right Noetherian ring.

(2) For any ideal I of RX^{-1}, I^c is an ideal of R.

(3) For any ideal I of R, I^e is an ideal of RX^{-1}.

(4) For any ideal I of RX^{-1}, $I = (I^c)^e$.

(5) An ideal I of RX^{-1} is prime (semiprime) if and only if I^c is prime (semiprime) in R.

(6) Let P be a prime (semiprime) ideal of R. Then $P = Q^c$ for some prime (semiprime) ideal if and only if $X \leq C(P)$.

<u>Proof</u>:

As in [19, theorem 9.20].

<u>Remark 2.13</u>.

We look at T, the partial quotient ring of R at C. Since $C \leq C_R(0)$, it is obvious that $T \leq Q(R)$, the Artinian quotient ring of R formed by localising R at $C_R(0)$. Now T has the following properties.

<u>Theorem 2.15</u>.

Let R be a GUFR and T be the partial quotient ring of R at C. Then

(1) T is a GUFR

(2) T has an Artinian quotient ring

(3) $C(T) = \{t \in T / \ tT = Tt$ is $Q(R)$-invertible$\}$ has only units of T.

Proof

Since R is a Noetherian ring and C is a right and left Ore set, C is a right and left denominator set by proposition 1.42. Now $T = RC^{-1} = C^{-1}R$ is a Noetherian ring by proposition 2.12(1).

By theorem 2.7, T has only a finite number of maximal ideals. We prove that they are the minimal prime ideals of T. Let M be a maximal ideal of T. If possible assume P is a prime ideal of T strictly contained in M. Then P^c is strictly contained in M^c, (otherwise by (4) of proposition 2.12, $P = (P^c)^e = (M^c)^e = M$). But M is the extension, in T, of some minimal prime ideal P_1 (say) of R. Since R has an Artinian quotient ring $C_R(O) = \bigcap_{i=1}^{n} C_R(P_i)$, by proposition 1.62, where $P_1, P_2, \ldots, P_n$ are the minimal prime ideals of R. Thus we have $C \leq C_R(O) \leq C_R(P_1)$ and so, by proposition 2.12(6), there exists a prime ideal Q of T such that $P_1 = Q^c$. Therefore $M = P_1^{\,e} = (Q^c)^e$, i.e., $M = (M^c)^e = (Q^c)^e = Q$. Consequently we have $P^c < M^c = Q^c = P_1$. Also P^c is a prime ideal of R by proposition 2.12 (5). This violates the minimality of P_1. Thus M does not contain any prime ideal properly. Hence the maximal ideals of T are the minimal prime ideals which implies that T has no non minimal prime ideals and thus T is obviously a GUFR.

(2) Follows immediately from theorem 2.9.

(3) Follows from [1, theorem 2.7]. Still for completion we sketch it. Let $t \in C(T) < T$. Then $t = ac^{-1}$, for some $a \in R$ and $c \in C$. Thus $a = tc$, where c is a unit of T. Since c is a unit of T, we have $T = cT = Tc$ and so $c \in C(T)$, so that $a \in C(T)$. Now $a \in C$ follows from the fact that $a \in R$. Thus 'a' is also a unit in T. Consequently $t = ac^{-1}$ is a unit in T.

<u>Definition 2.15</u>.

An ideal P in a ring R is said to be <u>right localisable</u>, if $C(P) = \{x \in R / x+P \text{ is regular in } R/P\}$ is a right reversible set in R.

<u>Definition 2.16</u>.

A ring R is said to have a <u>right quotient ring</u>, if $C_R(O)$ is a right reversible set. R is said to have <u>quotient ring</u>, if $C_R(O)$ is a right and left reversible set.

For instance, every GUFR has a quotient ring.

<u>Lemma 2.17</u>.

Let R be a Noetherian ring with a quotient ring Q. Let $P = pR = Rp$ be a normal prime ideal of R with p regular. Then P is localisable.

In the proof of lemma 2.17 we have to make use of the well known AR property and some other localisation techniques which we have not yet discussed in this thesis. When we discuss the localisation at a prime ideal in chapter 4, we will give a proof of this lemma.

Lemma 2.18.

Let R be a GUFR and P = pR = Rp be a non-minimal prime ideal of R. Then p is regular and P is localisable.

Proof

Since P is a non minimal prime ideal of R, from the definition of GUFR and by theorem 2.6, P contains a regular normal element e(say). Therefore $e = pr_1 = r_2p$ for some $r_1, r_2 \in R$. Now the regularity of p follows from the regularity of e. The second part of the lemma follows from lemma 2.17 and from theorem 2.9.

Lemma 2.19.

Let R be a GUFR and P be minimal prime ideal of R. Then P cannot contain any normal invertible ideal.

Proof

Suppose if possible that P contains a normal invertible ideal aR = Ra (say). Then $a \in C_R(0)$ by theorem 2.6. Since R has an Artinian quotient ring, $C_R(0) = \bigcap_{i=1}^{n} C_R(P_i)$, where $P_1, P_2, \ldots, P_n$ are the distinct minimal prime ideals of R, so that $P = P_i$ for some i, $1 \leq i \leq n$. Thus $a \in C_R(0) \leq C_R(P_i) = C_R(P)$ contradicting the fact that $aR = Ra \leq P$.

Remark 2.20.

We consider a special case of GUFRs, i.e. GUFRs with all height 1 primes are of the form $pR = Rp$. Then, by lemma 2.18, each p is a regular element in R and so each $pR = Rp$ is invertible (in $Q(R)$) and thus $p \in C$. Further it can be seen that, each $c \in C$ can be written as $up_1 \ldots p_n$, for some unit u in R and for some positive integer n, and p_is are such that $p_iR = Rp_i$ is a height 1 prime ideal of R for $i = 1,2,\ldots,n$. Thus the ring T, localised ring of R at C, coincides with the partial quotient ring of R with respect to the multiplicative set generated by the elements p of R such that $pR = Rp$ is a height 1 prime.

Theorem 2.21.

Let R be a GUFR and every height one prime ideal is of the form $pR = Rp$ for some $p \in R$. Then the following are equivalent.

(1) $C_R(P) \leq C_R(O)$ for each height one prime ideal P of R.

(2) $R = T \cap (\cap R_p)$, where T is the partial quotient ring of R at C and $\cap R_p$ is the intersection of all localised rings of R at its height one primes.

Proof

By lemma 2.18 each height 1 prime ideal is localisable and so R_p exists for each height 1 prime P of R.

Assume 1 . Then every element in $C_R(P)$ is a regular element and so the homomorphism $(r \longrightarrow r1^{-1})$ from R to R_p is a monomorphism and hence $R \leq R_p$ for each height one prime. Also $R_p \leq Q(R)$ for each height one prime. Thus $R \leq T \cap (\cap R_p)$. Now let $q \in T \cap (\cap R_p)$, then $q = r(up_1, .. p_n)^{-1} \in T$, where u is a unit of R and $p_iR = Rp_i$, for $i = 1,2,...,n$, is a height one prime, by remark 2.20. Since $q \in \cap R_p$, $q \in R_{p_i}$ for each $i = 1,2,...,n$, where P_is are the height one primes $p_iR=Rp_i$ for $i = 1,2,...,n$ and so there are $s_i \in R$ and $c_i \in C(P_i)$ for $i=1,2,...,n$ such that $q = s_ic_i^{-1}$ for each $i=1,2,...,n$. Therefore

$$q = r(up_1...p_n)^{-1} = s_ic_i^{-1} \text{ for } i = 1,2,...,n$$

i.e., $qup_1...p_n = r \in R$.

We have $c_n \in C_R(P_n)$ and $up_1 \ldots p_n \in R$, so there exists $t \in C_R(P_n)$ and $s \in R$ such that $up_1 \ldots p_n t = c_n s$, since $C_R(P_n)$ is a right Ore set. Then

$c_n s = up_1 \ldots (p_n t) = up_1 \ldots p_{n-1}(t' p_n) \in Rp_n = p_n R = P_n$,

where t' is such that $p_n t = t' p_n$. Now $c_n \in C_R(P_n)$ and $c_n s \in P_n$ implies that $s \in P_n = p_n R = Rp_n$ and so $s = p_n r_1 = r_1' p_n$ for some $r_1, r_1' \in R$. Thus $rt = qup_1 \ldots p_n t = qc_n s = qc_n r_1' p_n \in P_n$. Again $t \in C_R(P_n)$ implies $r \in P_n$ and so $r = p_n r_2 = r_2' p_n$ for some $r_2, r_2' \in R$.

$$\text{Therefore } q = r(up_1 \ldots p_n)^{-1} = r_2' p_n (p_n^{-1} p_{n-1}^{-1} \ldots u^{-1})$$
$$= r_2'(p_{n-1}^{-1} \ldots p_1^{-1} u^{-1}) = r_2(up_1 \ldots p_{n-1})^{-1}.$$

Repeating the argument n-1 times, we get $q = mu^{-1}$ where $m \in R$ and u is a unit R. Hence $q \in R$ and it follows that $T \cap (\cap R_P) \leq R$. This completes the proof of 1 implies 2 .

Conversely assume 2 . Then $R \leq R_P$ for each height one prime P of R and so the homomorphism $f, (r \longrightarrow r1^{-1})$ from $R \longrightarrow R_P$ is one-one (remark 1.44). Let $d \in C_R(P)$ and assume $sd = 0$ for some $0 \neq s \in R$, then $sdd^{-1} = s1^{-1} = 0$. Thus $s \in \ker f = 0$. This contradiction enables us to conclude that d is left regular. Now d is right regular follows from the fact that R has an Artinian quotient ring and so every left regular element of R is regular in R. (Proposition 1.62).

Remark 2.22.

Prime GUFRs are the prime Noetherian rings in which every non zero (non minimal) prime ideal contains a normal Q(R)- invertible ideal, where Q(R) is the simple Artinian quotient ring of R. Thus every NUFR is a prime GUFR. Examples of prime GUFRs which are not NUFRs are given in Chapter 3. As in corollary 2.8, it can be seen that, if R is a prime GUFR, T is a simple Noetherian ring.

Noetherian rings in which every prime ideal contains a normal invertible ideal are a generalisation of GUFRs. But we show that there is nothing to be gained by this extension as such rings turn out to be prime GUFRs.

Theorem 2.23.

Let R be a GUFR in which every prime ideal contains normal invertible ideals. Then R is a prime GUFR.

Proof

First we prove, in GUFRs with every prime ideal contains normal invertible ideals, every non zero ideal contains a normal invertible ideal.

Let I be a non zero ideal of R. Let $P_1, P_2 \ldots P_m$ be the minimal primes over I (remark 1.18). Then the product $P_1 P_2 \ldots P_m \leq I$. By hypothesis, for $1 \leq i \leq m$, there exists $a_i \in R$ such that $a_i R = Ra_i \leq P_i$ and each $a_i R = Ra_i$, for $1 \leq i \leq m$, is invertible. Then

$a_1 a_2 \ldots a_m R = Ra_1 \ldots a_m \leq P_1 \ldots P_m \leq I$ and $a_1 \ldots a_m R$ is invertible.

Now let I and J be two non zero ideals of R. It follows by the above paragraph that there exists $a \in I$ and $b \in J$ such that aR=Ra and bR=Rb and they are invertible. Thus, by lemma 2.6, both a and b are regular. Consequently $0 \neq ab \in IJ$ and we have $IJ \neq 0$. Therefore the product of two nonzero ideals of R is non zero, which implies R is prime.

<u>Definition 2.24</u>.

A ring R is said to be a sub direct product of the rings $\{S_i / i \in J\}$, if there is a monomorphism

$K: R \longrightarrow S = \prod_{i \in J} S_i$ (the direct product of S_is) such

that $\pi_i oK$ is surjective for all i, where $\pi_i : S \longrightarrow S_i$ is the natural projection.

<u>Proposition 2.25</u>.

R is a sub direct product of S_i, $i \in I$, if and only if S_i is isomorphic to R/K_i, where K_is are ideals of R with $\bigcap_{i \in I} K_i = 0$.

Proof:

As in [22, Proposition 2.10].

Theorem 2.26.

Every semiprime GUFR is a sub direct product of prime GUFRs.

Proof:

Let $P_1 \ldots P_n$ be the minimal prime ideals of R. Then $\bigcap_{i=1}^{n} P_i = 0$, since R is semi prime. Thus, if we show that R/P_i, for $1 \leq i \leq n$, is a prime GUFR, then the theorem follows from proposition 2.25. It is clear that R/P_i is a prime Noetherian ring, for $1 \leq i \leq n$. Let P/P_i be a non zero prime ideal of R/P_i. Then P is a non-minimal prime ideal of R with $P_i \subseteq P$. So there exists an element $a \in P$ such that $aR=Ra$ is invertible. By lemma 2.19, $a \notin P_i$ and thus $\bar{a}$ is a non zero element of R/P_i with $\bar{a}(R/P_i) = (R/P_i)\bar{a}$, where $\bar{a} = a+P_i$. Also $\bar{a}(R/P_i) = (R/P_i)\bar{a}$ is $Q(R/P_i)$-invertible, where $Q(R/P_i)$ is the simple Artinian quotient ring of R/P_i. Now R/P_i is a prime GUFR follows from the fact that $\bar{a} \in P/P_i$.

PRINCIPAL IDEAL RINGS

It is well known that every commutative principal ideal domain is a UFD. We prove an analogous result for GUFRs.

Recall that a right (left) regular element in a ring R is any element x such that $xy = 0$ implies $y = 0$ ($yx = 0$ implies $y = 0$).

Lemma 2.27.

Let R be a Noetherian ring in which every left regular element is regular. Suppose $aR = Rb$ for some $a \in R$ and b regular in R. Then $aR = Ra$ and a is regular.

Proof:

Since $aR = Rb$, we have $a = ub$ and $b = av$, for some u and v in R. Thus $bv = av^2 \in aR = Rb$ and $b = av = ubv = u(bv) = u(av^2) \in u(aR) = uRb$, therefore there exists an element p in R such that $b = upb$, i.e. $(1-up)b = 0$. By regularity of b, $up = 1$, so that u is left regular and from hypothesis u is regular. Now using the regularity of u and the equation $up = 1$, it can be seen that $up = pu = 1$. Consequently

$$aR = Rb = Rpub = (Rp)ub = Rub = Ra$$

Now the regularity of a follows from the regularity of b and the fact that $Ra = Rb$.

Theorem 2.28.

Let R be a Noetherian ring in which the principal left ideals generated by regular elements are also principal right ideals. Then R is a GUFR if and only if R has an Artinian quotient ring.

Proof:

Assume that R has an Artinian quotient ring Q(R). Let P be a non-minimal prime ideal of R. Then $P \cap C_R(0) \neq \emptyset$, by proposition 1.63. Let $b \in P \cap C_R(0)$, then by hypothesis there exists an element $a \in R$ such that Rb = aR. Since R has an Artinian quotient ring, every left regular element is regular (Proposition 1.62). Hence, by lemma 2.27, we have $a \in C_R(0)$ and aR = Ra. Since $a^{-1} \in Q(R)$, aR = Ra is Q(R)-invertible. Also P contains aR = Ra. This completes the proof of the sufficient part.

The necessary part of the theorem follows from theorem 2.9.

Corollary 2.29.

Suppose R is a Noetherian ring in which every regular element is normal. Then R is a GUFR if and only if R has an Artinian quotient ring.

The next theorem characterises the GUFRs in commutative case.

Theorem 2.30.

If R is a commutative Noetherian ring, then R is a GUFR if and only if R has an Artinian quotient ring.

Proof:

Follows from theorem 2.28, as, in every commutative ring the principal left ideals are two sided ideals.

POLYNOMIAL RINGS

Remark 2.31.

It is obvious that when P is a prime ideal of R[x], $P \cap R$ is a prime ideal of R. If P is a prime ideal of R, the map $a_o + a_1x + \ldots + a_nx^n \longrightarrow (a_o+P)+(a_1+P)x + \ldots +(a_n+P)x^n$ is clearly a surjective homomorphism from R[x] to (R/P)[x] with kernel PR[x]. Consequently $\frac{R[x]}{PR[x]}$ is isomorphic to (R/P)[x], and thus (R/P) is isomorphic to a subring R' of (R[x]/PR[x]).

First we consider the case when R is a prime GUFR. By a central element in a ring R, we mean any element x in R such that xr = rx for all $r \in R$.

Lemma 2.32.

Let R be a prime GUFR and T be the partial quotient ring of R at C. Then every non zero prime ideal of T[x] can be generated by a central element in T[x].

Proof:

Although the proof is similar to the proof given in [2], we give it. Let P be a non zero prime ideal of T[x]. Since R is prime GUFR, as in corollary 2.8, it can be seen that T is a simple Noetherian ring. Let f be a non-zero polynomial of P of least degree, deg f = n (say). The subset of T consists of the leading coefficients of the polynomials of P of degree n, together with zero, is a non zero ideal of T and this equal to T, since T is simple. Thus 1 is an element of that ideal and hence without loss of generality we can assume that f is a monic polynomial. Let $g \in P$, using division algorithm $g = fq+r$, where q and r are in T[x] and deg r < deg f or $r = 0$, but $r = g-fq \in P$ and f is a polynomial of least degree in P, which implies $r = 0$. Hence $g = fq \in fT[x]$. i.e., $P \leq fT[x]$ and so $P = fT[x]$. Furthermore $x.f = f.x$ and $sf-fs \in P$ for all $s \in T$, and its degree < degree f. Thus $sf-fs = 0$ and we get $sf = fs$ for sll $s \in T$ and consequently $fT[x] = T[x]f$.

Theorem 2.33.

Let R be a prime GUFR, then so is R[x].

Proof:

Since R is a prime Noetherian ring, so is R[x]. Let Q(R[x]) be the simple Artinian quotient ring of R[x]. Let P be a non zero prime ideal of R[x]. Then

Case-1

$P \cap R \neq 0$. Since $P \cap R$ is a non zero prime ideal of R and R is a prime GUFR, $P \cap R$ contains an element 'a' such that aR = Ra. So aR[x] = R[x] a is contained in P and is Q(R[x])- invertible.

Case-II

$P \cap R = 0$. Then $PT[x] \neq T[x]$, for, if PT[x] = T[x], then for any $a \in C$, $a = a1^{-1} = \frac{a}{1} \in T[x] = PT[x]$, which implies $a \in P$ and thus $0 \neq a \in P \cap R = 0$. But PT[x] is a proper prime ideal of T[x] and so, by lemma 2.32, fT[x] = T[x]f = PT[x], for some $f \in T[x]$. Therefore, by using common multiple property of C, $f = ga^{-1}$ for some $a \in C$ and $g \in P$. So we have g = fa and gR = faR = fRa = Rfa = Rg. This together with $g.x = x.g$ implies that gR[x] = R[x]g and that gR[x] = R[x]g is contained P. Since R[x] is prime Noetherian, g is a regular element of R[x] and so gR[x] is Q(R[x])-invertible. Thus P contains a normal invertible ideal.

Therefore, in both cases we have proved that P contains a normal invertible ideal and so R[x] is a prime GUFR.

Remark 2.34.

Let P be a prime ideal of R. Then PR[x] is a prime ideal of R[x]. We write

$$E_p = \left\{ f \in R[x] \Big/ (f+PR[x])(R[x]/PR[x]) = (R[x]/PR[x])(f+PR[x]) \right\}$$

Since R[x]/PR[x] is prime,

(f+PR[x]) (R[x]/PR[x]) = (R[x]/PR[x]) (f+PR[x]) implies that f+PR[x] is regular in R[x]/PR[x] and that $f \in C_{R[x]}(PR[x])$. Therefore $E_p \leq C_{R[x]}(PR[x])$ for each prime ideal P of R. Also we write

$$C' = \left\{ f \in R[x] \Big/ fR[x] = R[x]f \right\} .$$

If R[x] is a GUFR, clearly $C \leq C'$.

Theorem 2.35.

Let R be a GUFR and suppose $E_P \leq C_{R[x]}(0) \cap C'$, for every minimal prime ideal P of R. Then R[x] is a GUFR.

Proof:

Since R is a GUFR, R has an Artinian quotient ring and so R[x] has an Artinian quotient ring [23, theorem 3.6]. We denote the quotient ring of R[x] by Q(R[x]).

Let P_1 be a non-minimal prime ideal of $R[x]$. Then,

<u>Case-I</u>.

$P_1 \cap R$ is a non minimal prime ideal of R. Then, from the definition of GUFR, $P_1 \cap R$ contains an element 'a' such that $aR = Ra \leq P_1 \cap R$ and $aR = Ra$ is invertible. Hence $aR[x] = R[x]a \leq P_1$ and it is easy to see that $aR[x] = R[x]a$ is $Q(R[x])$-invertible.

<u>Case-II</u>.

$P_1 \cap R$ is a minimal prime ideal of R. Let $P = P_1 \cap R$. Then $PR[x]$ is a minimal prime ideal of $R[x]$ and so $PR[x] \neq P_1$ i.e., $PR[x] < P_1$. By lemma 2.19, P contains no normal invertible ideals and as in the proof of theorem 2.26 (R/P) is a prime GUFR.

Since (R/P) is a prime GUFR, $(R/P)[x]$ is a prime GUFR by theorem 2.33 and hence so is $(R[x]/PR[x])$. Since $PR[x] < P_1$ there is a $g \in P_1$ such that $(g+PR[x])\,(R[x]/PR[x]) = (R[x]/PR[x])\,(g+PR[x])$ is contained in P_1' (where P_1' is the copy of P_1 in $R[x]/PR[x]$), i.e., $g \in E_P \leq C_{R[x]}(0) \cap C'$ and thus g is regular in $R[x]$ and $gR[x] = R[x]g$. Consequently $gR[x] = R[x]g$ is contained in P_1 and is $Q(R[x])$-invertible.

Thus in both cases P_1 contains $Q(R[x])$-invertible, normal ideals. Hence $R[x]$ is a GUFR.

RINGS WITH ENOUGH INVERTIBLE IDEALS

We recall that a ring R is said to be right (left) hereditary, if all of its right (left) ideals are projective. If R is Noetherian, then R is left hereditary if and only if R is right hereditary [6, corollary 8.18] and in this case R is called hereditary.

Definition 2.36.

If every essential right ideal of a ring R contains a non zero ideal, then R is said to be right bounded. By symmetry we define left bounded rings and R is said to be bounded if it is both left and right bounded.

Definition 2.37.

If every non zero ideal of a ring R contains an invertible ideal, then R is said to be a ring with enough invertible ideals.

We state some results that are given in [8].

Lemma 2.38.

If R is a right bounded hereditary Noetherian prime Ring, then R has enough invertible ideals.

Lemma 2.39.

If R has enough invertible ideals, then R is bounded or premitive.

Lemma 2.40.

If R is a prime GUFR, then R is a ring with enough invertible ideals.

Proof:

Since R is a prime GUFR, every non-zero prime ideal contains a normal invertible ideal. Now, if I is any non zero prime ideal of R and $P_1, P_2 \ldots P_n$ are the prime ideals minimal over I, the product $P_1 \ldots P_n \leq I$ and is obvious that $P_1 P_2 \ldots P_n$ contains an invertible ideal as each P_i contains normal invertible ideal for $1 \leq i \leq n$.

Lemmas 2.39 and 2.40 together gives

Theorem 2.41.

If R is a prime GUFR, then R is either bounded or primitive.

Definition 2.42.

If an ideal I of a ring R contains a regular element, then I is called an integral ideal.

GUFRs can be characterised using their integral ideals.

Theorem 2.43.

Let R be a Noetherian ring with an Artinian quotient ring. Then R is a GUFR if and only if every integral ideal contains a normal invertible ideal.

Proof

Assume that R is a GUFR. Let I be an integral ideal and $P_1, P_2 \ldots P_n$ be the prime ideals minimal over I. Suppose if possible that P_i is minimal for some i. Since I is an integral ideal, there is an element $x \in I \cap C_R(O)$ and $x \in I$ implies $x \in P_i$. Let $Q_1, Q_2 \ldots Q_m$ be the minimal primes of R. Then $C_R(O) = \bigcap_{j=1}^{m} C_R(Q_j)$, since R has an Artinian quotient ring. Now the minimality of P_i implies that $P_i = Q_j$ for some j, $1 \leq j \leq m$. Thus

$x \in C_R(O) = \bigcap_{j=1}^{m} C_R(Q_j)$. Consequently $x \in C_R(Q_j)$ and

$x \in P_i = Q_j$. From this contradiction, we conclude that each P_i is non-minimal for $1 \leq i \leq n$. Thus each P_i, for $1 \leq i \leq n$, contains a normal invertible ideal and now the proof follows as in the proof of lemma 2.40.

Conversely, since R has an Artinian quotient ring, every non minimal prime ideal of R contains a regular element, i.e., each non-minimal prime ideal is an integral ideal. Thus each of them contains a normal invertible ideal by assumption. This completes the proof.

Theorem 2.44.

If R is a right bounded hereditary Noetherian prime ring in which every invertible ideal is normal. Then R is a prime GUFR.

Proof:

Since every invertible ideal is normal, the theorem follows from lemma 2.38.

COMPLETELY FAITHFUL MODULES

Definition 2.45.

Let R be a ring and M be a right R-module. Then M is

(1) Unfaithful if it is not faithful.

(2) Completely faithful if A/B is faithful for all submodules A > B of M.

(3) Locally unfaithful provided every finitely generated submodule is unfaithful.

(4) Locally Artinian provided every finitely generated submodule is Artinian.

From theorem 2.40 and the results [23, lemma 2.4, theorem 2.6], we get the following theorems for a prime GUFR.

Theorem 2.46.

Let R be a prime GUFR and M be a cyclic R-module. If N is a submodule of M such that

(1) N is completely faithful and M/N unfaithful, or

(2) N is unfaithful and M/N completely faithful.
Then N is a direct summand of M.

Proof:

As in [24, lemma 2.3].

Theorem 2.47.

Let R be a prime GUFR and A,B,C right R-modules. Then the exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ splits provided any one of the following statements holds.

(1) A is completely faithful and C is locally unfaithful,

(2) A is unfaithful and C is completely faithful,

(3) A is locally unfaithful and C is completely faithful.

Proof:

As in [24, theorem 2.4].

Remark 2.48.

For any module M, it can be proved, using Zorn's lemma, that there exists a unique maximal completely faithful submodule C(M), which contains every completely faithful submodule of M. [24, lemma 2.2].

Theorem 2.49.

Let R be a prime GUFR and let M be a locally Artinian right R-module. Then there exists a locally unfaithful submodule N of M such that $M = C(M) \oplus N$.

Proof

As in [24, theorem 2.6].

Chapter-3

EXTENSIONS AND RINGS WITH MANY NORMAL ELEMENTS

INTRODUCTION

In this chapter, we discuss rings which are extensions of GUFRs namely the finite centralising extensions, Ore extension and the ring of polynomials twisted by a derivation. Also we introduce the concept of rings with many normal elements.

We show that any finite centralising extension of a GUFR is a GUFR. As a corollary of this result, $M_n(R)$, the n x n matrix ring, over a GUFR R is a GUFR. A sufficient condition for the Ore extension, over a Noetherian ring with Artinian quotient ring, to be a GUFR is obtained. The Noetherian rings with Artinian quotient rings such that the Ore extensions over them are prime GUFRs are characterised. The skew polynomial rings over some special Noetherian rings are investigated.

We extend the concept of rings with few zero divisors[13] in the commutative case to rings with many normal elements in the non-commutative case. By introducing the concept of weakly invertible elements, we study some properties of Noetherian rings with many normal elements. Also, we prove some results on the integral closure of Noetherian rings, in this chapter.

CENTRALISING EXTENSIONS

Definition 3.1.

Let R and S be rings with $R \leq S$. If the R-module S_R has a finite set of generators $\{z_i/i=1,2,\ldots,n\}$ each of which normalises (i.e., $z_iR = Rz_i$ for each i, $1 \leq i \leq n$). Then S is called a finite normalising extension of R. If $z_ir = rz_i$, for all $i = 1,2,\ldots,n$ and for all $r \in R$; then R is called a finite centralising extension of R.

Definition 3.2.

Let S be a finite centralising extension of R. If, $I \in \text{Spec } S$ has the property that $(J\cap R)/(I\cap R)$ is essential as an ideal of $R/(I\cap R)$ for each ideal J of S with $I < J$, then we say that S satisfies essentiality at I. If this holds for every $I \in \text{Spec } S$, then we say that **S** satisfies essentiality.

Lemma 3.3.

Let S be a finite normalising extension of R and $P \in \text{Spec } S$. Then $P\cap R$ is a semiprime ideal of R. However, if S is a finite centralising extension, then $P \cap R$ is prime.

Proof:

As in [10, theorem 10.2.4.]

Lemma 3.4.

If S is a finite normalising extension of R and S is right Noetherian, then S satisfies essentiality.

Proof:

As in [10, proposition 10.2.12].

Theorem 3.5.

Let R be a GUFR and S be a finite centralising extension of R. Also suppose that $C \leq C_S(0)$, where $C = \{a \in R/aR=Ra \text{ is invertible}\}$. Then S is a GUFR.

Proof:

Since S is a finitely generated left and right R module, S is Noetherian. Now we prove C is an (left and right) Ore set in S. Let $a \in C$ and $s \in S$, then

$$a.s = a(z_1r_1 + z_2r_2 + \dots + z_nr_n) = a(r_1z_1 + \dots + r_nz_n)$$

$$= r_1'z_1a + r_2'z_2a + \dots + r_n'z_na = (r_1'z_1 + \dots + r_n'z_n)a = s'a.$$

Here we are assuming that $\{z_1, z_2, \dots, z_n\}$ is a centralising set of generators of S over R and we used the property that for each $r \in R$ $ar = r'a$; for some $r' \in R$. Thus for any $s \in S$, there exists $s' \in S$ such that $as = s'a$, i.e. $aS \leq Sa$. Similarly we have $Sa \leq aS$, it follows that C is an Ore set as in theorem 2.6.

Since $C \leq C_S(0)$, $T = SC^{-1} = C^{-1}S$ is an over-ring of S and since every centralising extension is a normalising extension, by lemma 3.4, S satisfies essentiality. Let P be a non minimal prime ideal of **S**. Then, by lemma 3.3, $P \cap R$ is a prime ideal of R and it is a non minimal prime ideal of R. For, if $P \cap R$ is minimal, then $P_o \cap R = P \cap R$ for any minimal prime ideal P_o of S with $P_o < P$ (P_o necessarily exists as P is non-minimal), which is a violation to the **ess**entiality of S (because under these circumstances $(P \cap R)/(P_o \cap R)$ should be essential in $R/P_o \cap R$ and so $(P \cap R)/(P_o \cap R)$ should be non zero in $R/P_o \cap R$). Consequently there exists an element $0 \neq a \in C$ such that $aR=Ra \leq P \cap R$ and hence $aS=Sa \leq P$ is T-invertible. This completes the proof.

<u>Corollary 3.6</u>.

If R is a GUFR, then so is $M_n(R)$.

<u>Proof</u>:

Clearly we can identify R with the subring of scalar matrices, in $M_n(R)$. Then $M_n(R)$ is a finite centralising extension of R, with generators, the matrix units, $\left\{ c_{ij} / \begin{array}{l} i=1,2,\ldots,n. \\ j=1,2,\ldots,n. \end{array} \right\}$. It is also easy to see that every regular element of R is regular in $M_n(R)$. Now the corollary follows from theorem 3.5.

TWISTED POLYNOMIALS

In this section we study some aspects of the relationship between a ring R, where R is a Noetherian ring an automorphism α, and the Ore extension ring $R[x,\alpha] = S$. The elements of S are polynomials in x with coefficients from R written on the left of x. We define $xr = \alpha(r)x$ for all $r \in R$. A typical element of S has the form, $f(x) = a_o + a_1x + \dots + a_nx^n = a_o + x\alpha^{-1}(a_1) + \dots + x^n\alpha^{-1}(a_n)$, where $n \geq 0$ and $a_i \in R$. The automorphism α on R can be extended to S by setting $\alpha(x) = x$ so that

$$\alpha(f(x)) = \alpha(a_o) + \alpha(a_1)x + \dots + \alpha(a_n)x^n$$

Definition 3.7

An <u>α-ideal</u> I of a ring R with an automorphism α is any ideal I of R with $\alpha(I) \leq I$. An <u>α-prime ideal</u> of R is an α-ideal P such that if X and Y are two α-ideals with $XY \leq P$, then either $X \leq P$ or $Y \leq P$. R is said to be an α-prime ring, if 0 is an <u>α-prime</u> ideal.

Remark 3.8.

It is easy to see that, if R is a Noetherian ring and I an α-ideal, then $\alpha(I) = I$ and if R is a prime ring, then R is an α-prime ring. Further, if $S = R[x,\alpha]$ is an Ore extension of R, then $xS = Sx$ and $x \in C_S(0)$.

We need some lemmas from [11].

Lemma 3.9.

Let R be a Noetherian ring and $S = R[x,\alpha]$ be an Ore extension of R. Then

(1) S is Noetherian

(2) An α-prime ideal P of S (or R) is a semiprime ideal.

(3) If A is an α-ideal of S, $S/(A \cap R)S$ is isomorphic to $(R/(A \cap R))\,[x,\alpha]$.

Lemma 3.10.

Let P be a prime ideal of S. Then either

(1) $x \in P$ and $P = P \cap R + xS$, or

(2) $x \in C_S(P)$ and $\alpha(P) = P$.

<u>Lemma 3.11.</u>

Let P be a prime ideal of S, such that $\alpha(P) = P$; then $P \cap R$ is a prime ideal of R and $\alpha(P \cap R) = P \cap R$.

<u>Lemma 3.12.</u>

Let P be a prime ideal of S with $\alpha(P) = P$; then $(P \cap R)S$ is a prime ideal of S.

<u>Theorem 3.13.</u>

Let R be a Noetherian ring with an Artinian quotient ring and let α be an automorphism on R. If every non zero α-prime ideal of R contains a normal invertible α-ideal; then $S = R[x,\alpha]$ is a GUFR.

<u>Proof:</u>

By lemma 3.9, S is a Noetherian ring. Suppose that every non zero α-prime ideal of R contains a normal invertible α-ideal. We shall show that S has an Artinian quotient ring Q(S) and every non-minimal prime ideal of S contains a normal invertible ideal.

Assume that S is not prime. Let P be a non-minimal prime ideal of S. If $x \in P$, then $x \in C_S(0)$ by remark 3.8. If $x \notin P$, then $x \in C_S(P)$ and $\alpha(P) = P$ by lemma 3.10.

In this case $P \cap R$ is an α-prime ideal by lemma 3.11 and $P \cap R \neq 0$. For, if $P \cap R = 0$, then 0 is an α-prime ideal of R again by lemma 3.11, and hence $oS = 0$ is a prime ideal of S by lemma 3.12, which is not possible as S is not prime.

Since $P \cap R$ is a non zero α-prime ideal of R, by assumption, it contains a normal invertible α-ideal $aR = Ra$, it follows that $a \in C_R(0)$. Now we prove that $a \in C_S(0)$. . Let $f(x) \in S$. Then $f(x) = a_o + a_1x + \ldots + a_nx^n$, where n is a non-negative integer and a_i, for $1 \leq i \leq n$, are in R. Consider

$$f(x)\circ a = (a_o + a_1x + \ldots + a_nx^n)a$$
$$= a_oa + a_1\alpha(a)x + \ldots + a_n\alpha^n(a)x^n$$

and so if $f(x)a = 0$, then $a_i = 0$ for all $i=1,2,\ldots,n$, as α is an automorphism. i.e. $f(x) = 0$. Similarly, if $a \cdot g(x) = 0$ for some $g(x) \in S$, we get $g(x) = 0$ and thus a is regular in S. Thus in both cases (i.e., $x \in P$ and $x \notin P$) we proved that P contains a regular element. Therefore, every non-minimal prime ideal of S contains a regular element and hence S has an Artinian quotient ring Q(S) (say) by proposition 1.63.

Now, if $x \in P$, then $xS = Sx$ is contained in P and $xS = Sx$ is Q(S)-invertible. Otherwise, we prove that $aS = Sa$ is Q(S)-invertible, where a is as in the above paragraph. Let $g \in S$ and assume $g=c_o+c_1x+\ldots+c_mx^m$ where $c_i \in R$, for $0 \leq i \leq n$. Since $\alpha(aR) = aR$, it follows that $\alpha(a) = au$, for some unit u in R. Consider

$$g.a = (c_o + c_1x + \ldots + c_mx^m)a = c_oa + c_1xa + \ldots + c_mx^ma$$

$$= c_oa + c_1\alpha(a)x + \ldots + c_k\alpha^k(a)x^k + \ldots c_m\alpha^m(a)x^m$$

$$= c_oa + c_1ad_1x + \ldots c_kad_kx^k + \ldots c_mad_mx^m$$

$$\left[\text{where } d_i = \prod_{j=0}^{i-1} \alpha^j(u), \text{ where } \pi \text{ stands for product}\right]$$

$$= ac_o' + ac_1'd_1x + \ldots + ac_k'd_kx^k + \ldots ac_m'd_mx^m$$
$$= a(c_o' + c_1'd_1x + \ldots + c_m'd_mx^m)$$
$$= ag' \text{ where } g' = c_o' + c_1'd_1x + \ldots + c_m'd_m'x^m \in S$$

Thus $Sa \leq aS$ and similarly $aS \leq Sa$. Also $a \in P$ and the proof is complete as $aS = Sa$ is Q(S)-invertible (since $a \in C_S(0)$).

Next assume that S is prime. Then S has a simple Artinian quotient ring Q(S) by Goldie's theorem. In this case, the proof is similar to the proof given in [2, theorem 4.1], we sketch it.

Let P be a non-minimal prime ideal of S. If $x \in P$, then $xS = Sx \leq P$ and is $Q(S)$-invertible. Otherwise, consider $E = \{a \in R/aR = Ra$ is an invertible α-ideal of $R\}$. It is easy to see that E is an Ore set in R. Let T be the localised ring of R at E. Then α can be extended to an automorphism β on T such that $\beta(ac^{-1}) = \alpha(a)c^{-1}$ for all $ac^{-1} \in T$. Thus $T^* = T[x,\beta]$ is an Ore extension of T and T is β-simple, i.e., T and 0 are the only β-ideals of T.

As in the proof of lemma 2.32, it can be seen that for any prime ideal P of T^* with $x \notin P$, there exists a central element $f \in T^*$ such that $P = fT^* = T^*f$.

If $P \cap R \neq 0$, the proof is as in the general case. If $P \cap R = 0$, $x \notin P$, it can be seen that $x \notin PT^*$ and PT^* is a non zero prime ideal of T^* and thus $PT^* = fT^* = T^*f$ for some $f \in T^*$, by the above observation. It is obvious that $xf = fx$ and $Rf = fR$. By the common multiple property of E, we have $f = gd^{-1}$ where $g \in P$ and $d \in E$. Using the fact $\alpha(d) = du$ (since $\alpha(dR) = dR$) for some unit u in R and $dR = Rd$, we get

$$Rg = Rfd = fRd = fdR = gR \quad \text{and}$$

$$xg = xfd = fxd = f\alpha(d)x = fdux = fdxu^{-1} = gxu^{-1},$$

so that $Sg = gS \leq P$. Since S is prime $g \in C_S(O)$ and hence $gS = Sg$ is $Q(S)$-invertible. Thus when S is prime every non zero prime ideal P contains a normal invertible ideal and so S is a GUFR.

Remark 3.14.

In the proof of the above theorem, we have not used the non-minimality of P. Thus in S every (non zero) prime ideal contains a normal invertible ideal. Hence S is a prime GUFR by theorem 2.23, and R is an α-prime ring by lemma 3.11. However, in this case we have the following characterisation.

Theorem 3.15.

Let R be a Noetherian ring with an Artinian quotient ring and let α be an automorphism on R. Then $S = R[x,\alpha]$ is a prime GUFR if and only if R is an α-prime ring in which every non-zero α-prime ideal contains a normal invertible α-ideal.

Proof:

Sufficient part of the theorem follows from theorem 3.13.

<u>Necessity</u>

Since S is a prime ring it follows that R is α-prime. Let P be a non-zero α-prime ideal of R. Because PS is a non-zero prime ideal of S, there is a non-zero element g of PS such that gS = Sg is Q(S)-invertible, where Q(S) is the simple Artinian quotient ring of S. Clearly $x \notin$ PS. For, if $x \in$ PS, then

$$x = \sum_{i=o}^{n} r_i f_i, \text{ where } r_i \in P \text{ and } f_i \in S, \text{ for } 0 \leq i \leq n. \tag{1}$$

Equating the coefficient on both sides of (1) we get

$$r_o a_{o1} + r_1 a_{11} + r_2 a_{21} + \dots + r_n a_{n1} = 1$$

and the remaining coefficients in R.H.S of (1) vanish. (Here we are assuming that $f_i = a_{io} + a_{i1}x + \dots + a_{ik_i}x^{k_i}$ for each i and k_i is a non-negative integer). But each $r_i \in P$, for $0 \leq i \leq n$, implies that $1 \in P$. Thus $g \neq x$, and without loss of generality we may assume that $g = c_o + c_1 x + \dots + c_n x^n$, where $c_i \in P$ for each i. Since gS = Sg, we have gR = Rg and since g is regular in S, g is regular in R. Now $c_i R = R c_i$ for each i. For, let $r \in R$, then $\alpha^{-i}(r) \in R$. Since gR = Rg, there is an $r' \in R$ such that

$$(c_o+c_1x+ \ldots + c^ix^i + \ldots +c_nx^n)\alpha^{-i}(r)$$
$$= r'(c_o+c_1x + \ldots + c_nx^n).$$

Equating the i^{th} coefficients, we get

$c_ir = r'c_i$, which implies $c_iR \leq Rc_i$ and similarly $Rc_i \leq c_iR$. This observation together with the fact that g is regular in R implies that c_i is a regular element of R, for some i, $0 \leq i \leq n$.

Next we prove $c_iR = Rc_i$ is an α-ideal of R. We consider $\alpha(r)x\ (c_o+c_1x + \ldots + c_nx^n) = (c_o+c_1x+\ldots+c_nx^n)r'x$ for some $r' \in R$. Equating the i^{th} term coefficients of this expression we get $\alpha(r)\ \alpha(c_i) = c_i\alpha(r')$, i.e. $\alpha(rc_i) = c_i\alpha(r')$ and hence $\alpha(c_iR) = \alpha(Rc_i) \leq c_iR$.

Thus the non zero α-prime ideal P contains at least one regular element c_i such that $c_iR = Rc_i$ is an α-ideal and $c_iR = Rc_i$ is Q(R)-invertible, since $c_i \in C_R(0)$. This completes the proof.

Definition 3.16.

If R is a ring and $\delta : R \longrightarrow R$ is a derivation on R, then the extension $S = R[x, \delta]$ of R is the ring of skew polynomials with coefficients written on the left of x. Here xa is defined as, $ax + \delta(a)$, for all $a \in R$.

Definition 3.17.

An ideal I of R is said to be a δ-ideal if $\delta(I) \leq I$. A δ-ideal I of R is said to be δ-prime, if for all δ-ideals A,B of R such that $AB \leq I$, either $A \leq I$ or $B \leq I$.

The following lemma relates the ideals R and those of $S = R[x, \delta]$

Lemma 3.18.

(1) If J is an ideal of S, then $J \cap R$ is a δ-ideal of R.

(2) If I is a δ-ideal of R, then IS is an ideal of S.

(3) If P is a prime ideal of S, then $P \cap R$ is a δ-prime ideal of R.

(4) If Q is a δ-prime ideal of R, then QS is a prime ideal of S.

(5) If R is Noetherian, then so is S.

Proof:

As in [12, lemma 1.3].

Lemma 3.19.

Let R be a Noetherian ring and $\delta : R \longrightarrow R$ a derivation. Then the following are equivalent.

(1) $R[x, \delta]$ is prime

(2) R is δ-prime

(3) The prime radical N of R is a prime ideal of R and $\bigcap_{i=1}^{k} \delta^{-i}(N) = 0$ for some integer k.

Proof:

As in [12, theorem 2.2].

Remark 3.20.

If a Noetherian ring R with a derivation satisfies any one of the above equivalent conditions, then $C_R(0)=C_R(N)$ so that R has an Artinian quotient ring.

Theorem 3.21.

Let R be a δ-prime Noetherian ring such that every non-zero δ-prime ideal contains a normal invertible δ-ideal. Then $R[x, \delta]$ is a prime GUFR.

Proof:

$S = R[x, \delta]$ is Noetherian by lemma 3.18(5). S is prime by lemma 3.19 and both R and S have Artinian quotient rings by remark 3.20 and Goldie's theorem respectively.

Let $E = \{ b \in R \,/\, bR = Rb$ is an invertible δ-ideal$\}$. Then E is an Ore set consisting regular elements and the localised ring T, of R at E, is an η-simple ring, i.e., O and T are the only η-ideals of T, where η is the extension of δ to T defined by $\eta(ac^{-1}) = \delta(a)c^{-1}$ for all $ac^{-1} \in T$. If P is a non-minimal prime ideal of S, then as in the pattern of the proof of the theorem 3.13, we have, $PT[x,\delta] = fT[x,\delta] = T[x,\delta]f$, for some $f \in T[x,\delta]$ and we get $f = gd^{-1}$ for some $g \in P$ and $d \in E$ and we have $gR = Rg$. Also, $xg = xfd = fxd = f(dx + \delta(d)) = fdx + f\delta(d)$

$$= fdx + fdu, \text{ for some } u \in R.$$

Thus $xg = fd(x+u) = g(x+u)$ and it follows that $Sg \leq gS$ and similarly $gS \leq Sg$. Thus $gS = Sg$ is contained in P and is Q(S)-invertible, since $g \in C_S(O)$.

RINGS WITH MANY NORMAL ELEMENTS

In this section we introduce the concept of many normal elements, which is a generalisation of GUFRs. By a normal element, we mean a normal regular element in this section.

Definition 3.22.

Let R be any ring. Then R is called a ring with many normal elements if R has only a finite number of prime ideals not containing any normal elements.

Examples 3.23

(1) In [13], a commutative ring with few zero divisors are defined as any commutative ring with only a finite number of maximal O-ideals, where a maximal O-ideal is an ideal maximal with respect to not containing non-zero divisors. Since every maximal O-ideal is a prime ideal, it follows that in the commutative case, rings with few zero-divisors are rings with many normal elements.

(2) If R is a GUFR, then every non-minimal prime ideal contains a normal element. Since the number of minimal prime ideals in any Noetherian ring is finite, it follows that every GUFR is a ring with many normal elements.

Remark 3.24.

Let R be a Noetherian ring with many normal elements and $C = \{a \in R / aR = Ra \text{ is normal}\}$. Then as in theorem 2.7, it can be proved that C is an Ore set and the localised ring $T = RC^{-1} = C^{-1}R$ has only a finite number of maximal ideals, precisely the extensions of the prime ideals of R not containing normal elements. Also T is an over-ring, since C has only regular elements.

We state a theorem known as the "prime avoidance" theorem [20, proposition 2.12.7].

Theorem 3.25

If A and B are two ideals of a ring R and $\Delta = \{P_1 \dots P_t\}$ is a collection of prime ideals of R with $A \sim B \leq \bigcup_{i=1}^{t} P_i$, then either $A \leq B$ or $A \leq P_i$ for some i.

Theorem 3.26

Let Δ and Δ' be finite collections of non zero prime ideals in a ring R with neither $P \lneq Q$ nor $Q \lneq P$ for any $P \in \Delta$ and $Q \in \Delta'$. Then there exist at least one element $u \in \bigcap_{P \in \Delta} P$ such that $u \notin \bigcup_{Q \in \Delta'} Q$.

Proof:

Let $\Delta = \{P_1 \dots P_n\}$ and $\Delta' = \{Q_1 \dots Q_m\}$. Then $P_1 \nleq \bigcup_{i=1}^{m} Q_i$. For, if $P_1 \leq \bigcup_{i=1}^{m} Q_i$, $P_1 \sim 0 < \bigcup_{i=1}^{m} Q_i$. Thus by "prime avoidance" either $P_1 = 0$ or $P_1 \leq Q_j$ for some j, which is impossible. Similarly $P_2 \nleq \bigcup_{i=1}^{m} Q_i$. Denote $\bigcup_{i=1}^{m} Q_i = U$. Then there exists $0 \neq p_1 \in P_1$ such that $p_1 \notin U$ and $0 \neq p_2 \in P_2$ such that $p_2 \notin U$. Now $p_1Rp_2 \neq 0$. For, if $p_1Rp_2 = 0$, then $p_1Rp_2 \leq Q_j$ for each j and thus $p_1 \in Q_j$ or $p_2 \in Q_j$ for each j which is not possible. An argument using "prime avoidance" theorem again shows that $Rp_1 Rp_2R \nsubseteq U$. Thus there exists at least one element $p \in Rp_1 Rp_2R$ such that $p \notin U$. Since $p \in Rp_1 Rp_2R$, $p \in P_1 \cap P_2$. Next consider P_3 and proceed as above, we get an element $p' \in Rp\, Rp_3R$ such that $p' \in U$, also $p' \in Rp\, Rp_3R$ such that $p' \in U$, also $p' \in P_1 \cap P_2 \cap P_3$. Continue the process until all the P_i's exhausted, we get an element

$$u \in \bigcap_{i=1}^{n} P_i \text{ such that } u \notin \bigcup_{j=1}^{m} Q_j.$$

Definition 3.27.

Let R be a ring and S an over-ring of R. Then a weakly S-invertible element in R is any element a in R such that $1 = \sum_{i=1}^{n} a_i a b_i$ for some a_i, b_i in S, for $1 \leq i \leq n$. Equivalently the ideal SaS = S.

Examples 3.28.

(1) Every unit in a ring R is weakly R-invertible.

(2) If R is a prime Noetherian ring with the simple Artinian quotient ring Q(R), Q(R)aQ(R)=Q(R) for any $0 \neq a \in R$. Thus every non zero element in R is weakly Q(R)-invertible.

Theorem 3.29.

Let R be a Noetherian ring with many normal elements and T be the partial quotient ring of R at $C=\{a \in R/a \text{ is normal}\}$. Then for any z in T and $x \in C$, there is a an element u such that z + ux is weakly T-invertible.

Proof:

By remark 3.24, T is an over-ring of R and T has only a finite number of maximal ideals.

Let $\Delta = \{M/M \text{ is a maximal ideal of T with } z \in M\}$

and $\Delta' = \{M/M \text{ is a maximal ideal of T with } z \notin M\}$

Then Δ and Δ' are finite collections of prime ideals and neither $M \lneq M'$ nor $M' \lneq M$ for any $M \in \Delta$ and $M' \in \Delta'$, as they are maximal ideals. Thus by theorem 3.26, there exists an element $u \in M$, for all $M \in \Delta'$ and $u \notin M$ for any $M \in \Delta$. Then the element $z+ux \notin M$ for any maximal ideal of T. For, if $z+ux \in M$ for some $M \in \Delta$, then $z+ux-z = ux \in M$. But $x \in C$ and so $x^{-1} \in T$, it follows that $u = u1^{-1} = uxx^{-1} \in M$; if $z + ux \in M$ for some $M \in \Delta'$, then $z+ux-ux = z \in M$. Further, if $\Delta = \emptyset$, then take $u = 0$ and if $\Delta' = \emptyset$, then take $u = 1$.

Thus the ideal $T(z+ux)T$ is not contained in any maximal ideal of T and hence $(z+ux)$ is weakly T-invertible.

Theorem 3.30.

Let R be a Noetherian ring with many normal elements and I be a one sided ideal of R containing a normal element. Then I can be generated by a set of weakly T-invertible elements.

Proof:

Suppose I is a left ideal. Let $\{z_1, z_2, \ldots, z_n\}$ be a generating set of I and x be a normal element in I. Consider z_1, by theorem 3.29, there exists an element u_1' such that $z_1+u_1'x$ is weakly T-invertible. Since $u_1' = u_1 c_1^{-1}$ for some $u_1 \in R$ and $c_1 \in C$, we have $u_1 = u_1 1^{-1} = u_1' c_1$ does not belong to the maximal ideals not containing u_1' and u_1 belongs to all maximal ideals, which contains u_1'. Thus as in the proof of theorem 3.29, $z_1 + u_1 x$ is a weakly T-invertible element in R. Similarly we get a collection $\{z_i+u_i x\}$ of weakly T-invertible elements for each z_i. Since x is normal, $x \in C$ and so x is invertible in T. Thus $\{z_i+u_i x, x\}$ is a collection of weakly T-invertible elements in I. We prove this is generating set for I.

Let $r \in I$. Then $r = r_1z_1+r_2z_2+ \ldots + r_nz_n$

where $r_i \in R$, for $1 \leq i \leq n$.

$$\text{i.e., } r = r_1z_1 + r_1u_1x + r_2z_2 + \ldots r_nz_n + r_nu_nx - (r_1u_1 + r_2u_2 + \ldots + r_nu_n)x$$

$$= r_1(z_1+u_1x) + r_2(z_2+u_2x) + \ldots + r_n(z_n+u_nx) - (r_1u_1 + r_2u_2 + \ldots + r_nu_n)x.$$

Thus r can be generated by $\{z_i+u_ix, x\}$. This completes the proof.

Theorem 3.31.

Let R be a Noetherian ring with many normal elements. Also assume that for any pair of weakly T-invertible elements x and y, either $Rx \leq Ry$ or $Ry \leq Rx$. Then

$$\mathcal{A} = \{I / I \text{ is a left ideal of R containing a normal element}\}$$

is linearly ordered.

Proof:

Let I and J be two elements of $\mathcal{A}$. Suppose if possible that $I \nleq J$ and $J \nleq I$. Then, by theorem 3.30, there exists at least one weakly T-invertible element b (say) in the generating set of I such that $b \notin J$ and similarly there is a weakly T-invertible element $c \in J$ such that $c \notin I$. Since $Rb \leq I$ and $Rc \nleq I$, we have $Rc \nleq Rb$ and similarly $Rb \nleq Rc$, which contradicts the hypothesis. Thus either $I \leq J$ or $J \leq I$, and the proof is complete.

<u>Theorem 3.32</u>.

Let R be a Noetherian ring with many normal elements. Also assume that for any pair of weakly T-invertible elements x and y either $xR \leq yR$ or $yR \leq xR$. Then $= \left\{ I / I \text{ is a right ideal of } R \text{ containing a normal element} \right\}$ is linearly ordered.

<u>Proof</u>:

As in theorem 3.31.

INTEGRALLY CLOSED RINGS

<u>Definition 3.33.</u>

Let R be any ring and M be an R-module. Then M is said to be integrally closed if any endomorphism of any finitely generated submodule extends to an endomorphism of M. A ring R is said to be right (left) integrally closed if R_R $({}^R R)$ is integrally closed [14].

Definition 3.34

Let R be a subring of Q. We say that R is classicaly right (left) integrally closed in Q, if R contains any element y of Q for which there exist elements $a_o, a_1, \ldots, a_{n-1}$ of R such that $y^n = a_o + ya_1 + \ldots + y^{n-1}a_{n-1}$ $(y^n = a_o + a_1y + \ldots + a_{n-1}y^{n-1})$. R is classically integrally closed in Q if it is classically right and left integrally closed in Q [14].

Lemma 3.35.

Suppose the ring R is integrally closed and is an order in a ring Q. Then the following assertions are true.

(1) $bmb^{-1} \in R$ and $b^{-1}mb \in R$ for all $b \in C_R(O)$ and $m \in R$.

(2) If A is a finitely generated submodule of Q_R and $f \in \text{End } A$, then there exists $d \in R$ such that $f(a)=da$ for all $a \in A$.

Proof

As in [14, lemma 2.12].

Theorem 3.36.

Let R be a semiprime right Noetherian ring with the semisimple Artinian quotient ring Q. Suppose also that $f(1) \in R$ for all $f \in \text{End } Q_R$. Then R is classically right integrally closed in Q.

Proof

First we prove that R is right integrally closed. Let I be a right ideal of R such that $f_1 \in$ End I_R. Then $f_1 \in \mathrm{Hom}_R(I_R\ R_R)$. Since I_R and R_R are $C_R(O)$ torsion free submodules of R_R, f_1 can be extended to $f_2 \in \mathrm{Hom}_Q(IQ, RQ)$ [16, corollary 2.2.5], i.e., $f_2 \in \mathrm{Hom}_Q(IQ,Q)$. Since Q is a semisimple Artinian ring, every unitary right Q-module is injective, and so in particular Q_Q is injective. Thus $f_2 \in \mathrm{Hom}_Q(IQ,Q)$ can be extended to $f_3 \in$ End Q_Q and thus $f_3 \in$ End Q_R. Now $f_4 = f_{3/R} \in$ End R_R, which follows from the hypothesis that $f(1) \in R$ for all $f \in$ End R_R. Also $f_{4/I} = f_1$. Therefore $f_1 \in$ End I_R can be extended to $f_4 \in$ End R_R. This completes the proof that R_R is right integrally closed.

We prove R is right classically integrally closed in Q, as in [14]. Let $y \in Q$ such that $y^n=a_o+ya_1+\ldots+y^{n-1}a_{n-1}$ where $n \geq 0$ and $a_i \in R$ for $0 \leq i \leq n-1$. Let A be the right R submodule of Q generated by $1,y,y^2,\ldots,y^{n-1}$. Then $yA \leq A$ and $f(a) = y.a$ is an endomorphism of A. Thus by lemma 3.36, there exists $d \in R$ such that $f(a) = da$ for all $a \in A$. Since $1 \in A$, we have $f(1) = d$ and consequently $y = d \in R$ and the proof is complete.

Corollary 3.37.

Let R be a semiprime GUFR with the quotient ring Q. Also suppose that $f(1) \in R$ for every right and left R endomorphism f of Q. Then R is classically integrally closed in Q.

Examples 3.38

(1) Corollary 3.6 states that $M_n(R)$ is a GUFR, whenever R is a GUFR. Thus for any commutative Noetherian integral domain $R, M_n(R)$ is a prime GUFR.

(2) Let $R = k[t,y]$ be the polynomial ring in two commuting indeterminates over a field k of characteristic zero. Let be the derivation $2y \frac{\partial}{\partial t} + (y^2+t) \frac{\partial}{\partial y}$. Then R has only two -prime ideals, namely (y^2+t+1) and $tR + yR$. The only height 1 primes of $R[x, \delta]$ are the extensions of these two δ-prime ideals. It is easy to see that these extensions contain normal invertible ideals and so $R[x, \delta]$ is a GUFR. But $R[x, \delta]$ is not an NUFR [2, example 5.2.].

Chapter 4

LOCALISATION

INTRODUCTION

In this chapter, we investigate the localisation at prime ideals in GUFRs. Persuaded by the importance of localisation in commutative rings and its application in the study of modules over commutative rings, several mathematicians investigated localisation at prime ideals in non-commutative rings, in particular in Noetherian rings, after Goldie proved his theorems for prime and semiprime Noetherian rings.

But, because of the general behaviour of prime ideals in non-commutative rings, the complement of a prime ideal need not be a multiplicative set in general. Although the complement of every completely prime ideal in a Noetherian ring is a multiplicative set, there are some completely prime ideals, whose complements do not satisfy the Ore condition. Thus in general the localisation at the complement of a prime ideal can be ruled out in Noetherian rings.

So, instead of looking at the complement of a prime ideal P in a Noetherian ring R, if we look at the set $C_R(P) = \{r \in R / r+P \text{ is regular in } R/P\}$, then we can gain something. We say that a prime ideal in a

Noetherian ring is right localisable if $C_R(P)$ is a right Ore set. It is obvious that in commutative rings $C_R(P)$ coincides with the complement of P in R. But unlike in commutative rings, $C_R(P)$ need not be a right Ore set in many cases.

First we look at the obstacles to the localisation at a prime ideal in Noetherian rings and then discuss a newly developed technique of localisation at a collection of prime ideals in which the elements are related in a special manner.

We begin with an example. Most of the material in the preliminaries of this chapter is taken from [16], [25] and [26].

<u>Example 4.1.</u>

Let k be a field and let R be the 2 x 2 upper triangular matrices over k. Then R is an Artinian (and thus Noetherian) ring with two prime ideals, the ideal Q of matrices in R whose upper left corner is zero and the ideal P of matrices in R whose lower right corner is zero. Now R/P and R/Q are both isomorphic to k, and $QP = 0$. Also $PQ = P \cap Q = J$, the Jacobson radical of R. Note that Q and P are completely prime ideals and thus $C(Q) = R - Q$ and $C(P) = R - P$.

Now if $\begin{bmatrix} a & b \\ o & c \end{bmatrix} \in C(Q)$ and $\begin{bmatrix} d & e \\ o & f \end{bmatrix} \in R$,

then $\begin{bmatrix} a & b \\ o & c \end{bmatrix} \begin{bmatrix} a^{-1}d & o \\ o & o \end{bmatrix} = \begin{bmatrix} d & e \\ o & f \end{bmatrix} \begin{bmatrix} o & 1 \\ o & o \end{bmatrix}$, i.e. $C(Q)$ is

a right Ore set and thus Q is right localisable.

On the other hand, C(P) is not right Ore, since for the elements $\begin{bmatrix} o & o \\ o & 1 \end{bmatrix} \in C(P)$ and $\begin{bmatrix} o & 1 \\ o & o \end{bmatrix} \in R$.

$$\begin{bmatrix} o & o \\ o & 1 \end{bmatrix} \begin{bmatrix} a & b \\ o & c \end{bmatrix} = \begin{bmatrix} o & 1 \\ o & o \end{bmatrix} \begin{bmatrix} d & e \\ o & f \end{bmatrix} \text{ where } \begin{bmatrix} a & b \\ o & c \end{bmatrix} \in R$$

and $\begin{bmatrix} d & e \\ o & f \end{bmatrix} \in C(P)$ if and only if $f = c = 0$, but in that case $\begin{bmatrix} d & e \\ o & f \end{bmatrix} \notin C(P)$.

Definition 4.2.

Let D be a right Ore set in a ring R, and M be a right R-module. An element $m \in M$ is said to be torsion if $md = 0$ for some $d \in D$. $T(M) = \{m \in M / md = 0\}$ is called the torsion submodule of M. If $T(M) = M$, then M is said to be torsion module and if $T(M)=0$, then M is said to be torsion free.

Note that in example 4.1, R/P and R/Q are prime Noetherian rings and so the regular elements in these rings are Ore sets. Also note that $JP = QJ = 0$ and thus

J is faithful and torsion free as both right R/P and left R/Q module.

<u>Lemma 4.3</u>.

Let R be a right Noetherian ring, C is a right Ore set in R, and A and B be ideals of R with $A < B$. Suppose also that $r(B/A) = P$ and $\ell(B/A) = Q$, where P and Q are prime ideals of R, and that B/A is torsion-free as a right R/P-module. If $C < C(P)$, then also $C < C(Q)$.

The situation in example 4.1 is exactly the same as in lemma 4.3. Thus to localise the ring of example 4.1 at a right Ore set C inside $C_R(P)$, we must include $C_R(Q)$. Thus $C \leq C_R(P) \cap C_R(Q)$. But $C_R(P) \cap C_R(Q)$ = units of R. Since C has only units, we cannot localise R, at C, further.

<u>Definition 4.4</u>.

Let R be a Noetherian ring and Q,P be prime ideals of R. If there exists an ideal A of R with $QP \lneq A < Q \cap P$ such that $\ell(Q \cap P/A) = Q$ and $r(Q \cap P/A) = P$ and $Q \cap P/A$ is left R/Q, right R/P torsion free module, then we say that <u>Q is linked to P</u> (via $A < Q \cap P$) and denoted by $Q \rightsquigarrow P$.

Remark 4.5.

It can be seen that in Noetherian rings the torsion free condition in the definition of 4.4 is equivalent to the condition that every non zero sub-bimodule of $Q \cap P/A$ is faithful as a right R/P and as a left R/Q module.

Definition 4.6.

Let R be a Noetherian ring. Suppose Spec R denotes the collection of all prime ideals of R. We say a subset X of Spec R is right stable, if whenever $P \in X$, $Q \in$ Spec R, and $Q \rightsquigarrow P$, we have $Q \in X$. We say X is stable if $Q \rightsquigarrow P$ implies either both $Q,P \in X$ or both $Q,P \notin X$. If $P \in$ Spec R the right clique of P, denoted by rt cl(P), is the smallest right stable subset of Spec R containing P. The clique of $P \in$ Spec R is the smallest stable subset containing P.

Thus we have the following corollary of lemma 4.3.

Corollary 4.7.

If R is a right Noetherian ring and C is a right Ore set contained in C(P), then $C \leq \cap \{C_R(Q)/Q \in \text{rt cl } P\}$. If C is an Ore set and R is Noetherian, then $C \leq \cap \{C_R(Q)/Q \in \text{cl } P\}$.

Remark 4.8.

We note two things about links between prime ideals.

(1) $Q \rightsquigarrow P$ if and only if $Q/QP \rightsquigarrow P/QP$ in R/QP.

(2) If C is any right denominator set in R disjoint from Q and P, then $Q \rightsquigarrow P$ if and only if $QC^{-1} \rightsquigarrow PC^{-1}$ in RC^{-1} [27].

Examples 4.9.

(1) In example 4.1 the only prime ideals of R are Q and P and the rt cl P = $\{P,Q\}$.

(2) In a commutative Noetherian domain R, if P is any prime ideal rt cl P = $\{P\}$.

Definition 4.10.

Let P be a right localisable prime ideal in a ring R and R_P be the localised ring of R at C(P). If for any finitely generated right R_P-module M, containing a simple right R_P-submodule S, which is also essential in M, M is Artinian, or equivalently $MP^n = 0$ for some n, then P is said to be classically right localisable.

Definition 4.11.

A right R-module M is said to be uniform, if every non zero submodule of M is essential in M.

Remark 4.12.

Let P be a classically right localisable prime ideal in R, Q be a prime ideal of R with $Q < P$, and there exist a f.g uniform R-module M, with $\mathrm{ann}(M) = Q$, containing a copy U of a non zero right ideal of R/P. By passing to R/Q, we assume $Q = 0$. We can localise at $C_R(P)$ and get the simple R_P/PR_P-module $U \otimes R_P$ inside $M \otimes R_P$, so there is an n with $(M \otimes R_P) P^n R_P = 0$. This implies that MP^n is $C_R(P)$-torsion, so $MP^n \cap U = 0$. Thus $MP^n = 0$ and hence $P^n = 0$. This contradiction shows that apart from the links between prime ideals we have another obstruction to localisation at a prime ideal.

Definition 4.13.

A prime ideal P, in a ring R, satisfies the right second layer condition (s.l.c) if the situation of the above remark does not occur, i.e., no such Q exists. Left second layer condition is defined analogously.

Theorem 4.14.

Let R be a Noetherian ring and let P be a prime ideal of R. Then P is classically right localisable if and only if $\{P\}$ is right stable and P satisfies the right second layer condition.

Definition 4.15.

An ideal I of R is said to have the right Artin-Rees (AR) property if for any finitely generated right R module M containing an essential submodule L with LI = 0, there is a positive integer n such that $MI^n = 0$. In this case we call I a right AR ideal. Left AR property is defined analogously.

Remark 4.16.

(1) A prime ideal P with the right AR property always satisfies the right second layer condition.

(2) An ideal I of R is right AR if and only if for every right ideal K of R, there is a positive integer n such that $K \cap I^n < KI$.

Theorem 4.17.

If R is a Noetherian ring and P is a prime ideal with the right AR property, then P is classically localisable if and only if there is no prime ideal Q of R with $P < Q$ and $Q \rightsquigarrow P$.

Lemma 4.18.

Suppose an ideal in a right Noetherian ring R has the right AR property. If $Q \rightsquigarrow P$ in Spec R and if $I \lesssim P$, then $I \leq Q$.

Lemma 4.19.

An invertible ideal in a (right) Noetherian ring has the (right) AR property.

Let R be a Noetherian ring with a quotient ring Q. Let P = aR = Ra be a prime ideal with 'a' regular. Then R has a partial quotient ring S obtained by localising R at the Ore set $\{1, a, a^2 \ldots\}$ It is easy to see that P = aR = Ra is S-invertible and hence P has the (right and left) AR property. Using induction and regularity of a, it is easy to see that $C(P) \leq C(P^n)$ for every n. Now by [28, proposition 2.1], P is localisable which gives the proof of lemma 2.17.

Given a prime ideal P, any right localisation at P must be found by inverting a right Ore set $C \leq C_R(P)$. Thus, in fact, $C \leq \cap \{C_R(Q)/Q \in \text{rt cl } P\}$ by corollary 4.7. Let $X \leq \text{Spec } R$ and define $C(X) = \cap \{C_R(Q)/Q \in X\}$. If X is a right clique and if we want to localise at X, then C(X) must be a right Ore set. We also want some nice properties for the quotient ring.

Definition 4.20.

Let R be a Noetherian ring and $X \leq \text{Spec } R$. Then

X is classically right localisable if $C(X)$ is a right Ore set and the localisation $R_X = R\, C(X)^{-1}$ has the following properties.

(1) For every $P \in X$, the ring R_X/PR_X is Artinian.

(2) The only right primitive ideals are PR_X for $P \in X$.

(3) Every finitely generated R_X-module which is an essential extension of a simple right R_X-module is Artinian.

Definition 4.21.

Let $X \leq \operatorname{Spec} R$. Then X satisfies the right intersection condition if for any right ideal I of R such that $I \cap C_R(P) \neq \emptyset$ for every $P \in X$, the intersection $I \cap C(X)$ is non-empty. We say X satisfies right second layer condition if every prime ideal in X satisfies right second layer condition and we say X satisfies the incomparability condition if there do not exist prime ideals $P,Q \in X$ with $Q < P$.

Proposition 4.22.

If R is a Noetherian ring and X is a right stable subset of Spec R satisfying the right intersection condition and right second layer condition, then $C(X)$ is a right Ore set.

Theorem 4.23.

If R is a Noetherian ring and $X \leq \text{Spec } R$, then X is classically right localisable if and only if

(1) X is right stable,

(2) X satisfies the right second layer condition,

(3) X satisfies the right intersection condition, and

(4) X satisfies the incomparability condition.

Thus we have characterised the classically right localisable subsets of Spec R in Noetherian rings. The same can be done for classically left localisable subsets by defining the left second layer condition, left intersection property and left stability etc. analogously.

We conclude this section of preliminaries with two theorems.

Theorem 4.24.

If R is a Noetherian ring and X is a right stable subset of Spec R satisfying the right second layer condition and the right intersection condition, then C(X) is a right Ore set.

<u>Theorem 4.25</u>.

If R is a right Noetherian ring and X is a finite subset of Spec R, then X satisfies the right intersection condition.

MINIMAL PRIMES IN GUFRs

We have seen in chapter 1 that every GUFR has an Artinian quotient ring and so $C_R(N) = \bigcap_{i=1}^{n} C_R(P_i)$, where $P_1, \ldots, P_n$ are the minimal primes of R, is a right Ore set. Also in chapter 1, we proved that the minimal primes cannot contain any normal invertible ideals, i.e., $P_i \cap C = \emptyset$, for each i, $1 \leq i \leq n$. Now we look at the right cliques of minimal prime ideals of a GUFR.

First we state some lemmas.

<u>Lemma 4.26</u>.

Let D be an Ore set in a prime Noetherian ring R. Then D consists of regular elements or $0 \in D$.

<u>Lemma 4.27</u>.

Let R be a prime Noetherian ring and C be an Ore set in R such that $0 \notin C$. Let M be a torsion free right R-module. Then MC^{-1} is a torsion free right RC^{-1} module.

Lemma 4.28.

Suppose P and Q are maximal ideals in a Noetherian ring R. Then $Q \rightsquigarrow P$ if and only if $QP \neq Q \cap P$.

Theorem 4.29.

Let R be a GUFR and $P, Q \in \text{Spec } R$ with $P \in \text{Min Spec } R$ and $Q \notin \text{Min (Spec } R)$. Then Q is not linked to P.

Proof:

Suppose $Q \rightsquigarrow P$. Since Q is not minimal, there exists a normal invertible ideal I of R such that $I \leq Q$. By lemma 4.19, I has (right and left) AR property. Thus we have a positive integer n such that $I^n \cap (P \cap Q) \leq I(P \cap Q)$ by the left AR property of I. Because of the link from Q to P, we have an ideal A of R with $QP \leq A < Q \cap P$ such that $r\left(\frac{Q \cap P}{A}\right) = P$ and $\ell\left(\frac{Q \cap P}{A}\right) = Q$. Thus,

$(Q \cap P)I^n \leq (Q \cap P) \cap I^n = I^n \cap (Q \cap P) \leq I(Q \cap P) \leq Q(Q \cap P) \leq A.$

i.e. $(Q \cap P)I^n \leq A$ and so $I^n \leq r\left(\frac{Q \cap P}{A}\right) = P$. Since P is prime $I \leq P$, which violates the assumption that P is minimal and contains no normal invertible ideals. Therefore $Q \not\rightsquigarrow P$.

Theorem 4.30.

Let R be a GUFR and $P, Q \in \text{Min (Spec } R)$. Then $Q \rightsquigarrow P$ if and only if $QP \neq Q \cap P$.

Proof:

If $Q \rightsquigarrow P$, it is obvious that there exists an ideal A with $QP \leq A < Q \cap P$ and thus $QP \neq Q \cap P$.

Conversely, assume $QP \neq Q \cap P$. Suppose if possible that $Q \not\rightsquigarrow P$. The set C of regular normal elements is an Ore set in R and $P \cap C = Q \cap C = \emptyset$, since P and Q are minimal primes. So by remark 4.8, $QC^{-1} \not\rightsquigarrow PC^{-1}$ in RC^{-1}. But QC^{-1} and PC^{-1} are maximal ideals of RC^{-1} by theorem 2.7 and hence by lemma 4.28, $QC^{-1}\, PC^{-1} = QC^{-1} \cap PC^{-1}$.

Now let $x \in Q \cap P$, then $\frac{x}{1} = x1^{-1} \in PC^{-1} \cap QC^{-1}$, i.e., $x \in (QC^{-1})(PC^{-1})$, thus there exist $a_i \in Q$, $b_i \in P$ and $c_i, d_i \in C$ for $i=1,2,\ldots,n$ such that

$$x = x1^{-1} = \frac{x}{1} = \sum_{i=1}^{n} (a_i c_i^{-1})(b_i d_i^{-1}) = \sum_{i=1}^{n} \frac{a_i}{c_i}\frac{b_i}{d_i} .$$ But

$$\frac{a_i}{c_i}\frac{b_i}{d_i} = \frac{a_i . b_i'}{d_i c_i'}$$ for each $i = 1,2,\ldots,n$, where $b_i' \in R$

and $c_i' \in C$ such that $b_i\, c_i' = c_i\, b_i'$ (remark 1.46).

Therefore $\frac{x}{1} = \sum_{i=1}^{n} \frac{a_i b_i'}{d_i c_i'}$ .Now $b_i \in P$, therefore

$c_i b_i' = b_i c_i' \in P$, for each $i = 1,2,\ldots,n$. i.e,

$R\, c_i b_i' \leq P$. Since $c_i \in C$, $R\, c_i = c_i\, R$, which implies $c_i\, R\, b_i' \leq P$ for $i = 1,2,\ldots,n$. Hence $b_i' \in P$ for each $i = 1,2,\ldots,n$, as $C \cap P = \emptyset$, $c_i \in C$ for each $i = 1,2,\ldots,n$

and P is prime. Consequently $a_i b_i' \in QP$ for $i = 1,2,\ldots n$ and so $\frac{x}{1} \in (QP)C^{-1}$. Since C has only regular elements, we have $x \in QP$, it follows that $Q \cap P \leq QP$, which contradicts the assumption that $Q \cap P \neq QP$ and we have $Q \rightsquigarrow P$.

Remark 4.31.

Let P be a minimal prime ideal in a GUFR. Define,

$X_o(P) = \{Q \in \text{Spec } R / Q \rightsquigarrow P\}$,

$X_1(P) = \{Q \in \text{Spec } R / Q \rightsquigarrow P_1 \text{ for some } P_1 \in X_o(P)\} \ldots$

$X_{j+1}(P) = \{Q \in \text{Spec } R / Q \rightsquigarrow P_j \text{ for some } P_j \in X_j(P)\}$ for $j > 1$.

By theorems 4.29 and 4.30 we have

$X_o(P) = \{Q \in \text{Min Spec } R / QP \neq Q \cap P\}$ and

$X_{j+1}(P) = \{Q \in \text{Min Spec } R / QP_j \neq Q \cap P_j \text{ for some } P_j \in X_j(P)\}$

for $j = 0,1,2,\ldots$. Thus we have the right clique of

$$P = \bigcup_{j=o}^{\infty} X_j(P) = X(P).$$

Theorem 4.32.

Let R be a GUFR and P a minimal prime ideal of R. Then, right clique of $P = X(P) = \bigcup_{j=o}^{\infty} X_j(P)$, where

$X_o(P) = \{Q \in \text{Min spec } R / QP \neq Q \cap P\}$ and

$X_{j+1}(P) = \{Q \in \text{Min Spec } R / QP_j \neq Q \cap P_j, \text{ for some } P_j \in X_j(P)\}$

for $j = 0,1,2,\ldots$.

Theorem 4.33.

For each minimal prime P in a GUFR, X(P) is a classically right localisable set.

Proof:

For each $j = 0,1,2,\ldots, X_j(P)$ is a subset of Min (Spec R) and so $X(P) = \bigcup_{j=0}^{\infty} X_j(P)$ is also a subset of Min (Spec R). Since Min (Spec R) is finite, X(P) is also finite and thus by theorem 4.25, X(P) satisfies right intersection condition. The elements of X(P) are minimal primes and so none of them properly contains any other prime ideal of R and so X(P) satisfies right second layer condition. Further X(P) has only incomparable elements as they are minimal, and X(P) is right stable as it is a right clique. Now the theorem follows from theorem 4.23.

HEIGHT 1 PRIME IDEALS IN A GUFR.

Now we look at the height 1 prime ideals of a GUFR. We state a lemma, the proof of which follows from [16, corollary 3.3.10].

Lemma 4.34.

Let R be a Noetherian ring. An ideal I of R has right AR property if and only if for every right R-module

M annihilated by I, $E(M) = \bigcup_{n=1}^{\infty} \operatorname{ann}_{E(M)} I^n$, where E(M) is the injective hull of M.

Theorem 4.35.

Let P be a height 1 prime ideal of a GUFR. Then P satisfies the right second layer condition.

Proof:

Assume that there exists a prime ideal Q of R such that $Q < P$ and $Q = \operatorname{ann} M$ for some finitely generated uniform right R-module M containing a copy U of a non zero right ideal of R/P. Since P is height 1 prime, Q is a minimal prime ideal of R and so Q contains no normal invertible ideals. Let I be the normal invertible ideal contained in P. Put $J = I+Q$. Then J/Q is an invertible ideal of R/Q and so it has the right AR property. Since M is an R/Q module, by the above lemma we have $E(M) = \bigcup_{n=1}^{\infty} \operatorname{ann}_{E(M)} (J/Q)^n$. But M is finitely generated and is contained in E(M). This together with the fact that $\{\operatorname{ann}_{E(M)}(J/Q)^n\}$ is an ascending chain of submodules of E(M) implies that there exists a positive integer k such that $M \leq \operatorname{ann}_{E(M)}(J/Q)^k$, i.e., $M(J/Q)^k=0$ which implies $(J/Q)^k \leq \operatorname{ann} M = Q$ and $J^k/Q < Q$. Hence

$J^k \leq Q$. Consequently $J \leq Q$, since Q is prime. Thus $I \leq Q$, which contradicts the selection of Q. Therefore P satisfies the right second layer condition.

Corollary 4.36.

Let P be a height 1 prime ideal of a GUFR such that $\{P\}$ is right stable. Then P is classically right localisable.

Proof:

This is an immediate consequence of theorem 4.14 and theorem 4.35.

A semiprime ideal S of a Noetherian ring is said to be classically right localisable if the finite set of prime ideals associated with S is classically right localisable. Thus we get another consequence of theorem 4.35 and theorem 4.23.

Corollary 4.37.

Let S be a semiprime ideal in a GUFR and assume that the associated prime ideals of S are height 1 prime ideals. Suppose also that the collection of associated primes is right stable. Then S is classically right localisable.

Definition 4.38.

Let R be a Noetherian ring. A subset X of Spec R is said to be a sparse subset if, given any $Q \in$ Spec R and given any $c \in C_R(Q)$, we have

$$Q \neq \cap\{P \in X / Q < P;\ c \notin C_R(P)\}.$$

Remark 4.39.

Let R be a GUFR and I be a normal invertible ideal of R. Put $X_I = \{P \in \text{Spec } R/ \text{ height } P = 1 \text{ and } I \leq P\}$ Then by principal ideal theorem $X_I \neq \emptyset$. Let $Q \in$ Spec R and $c \in C(Q)$. Then, if Q is minimal, I cannot be contained in Q, since R is a GUFR, whereas $\cap\{P \in X_I/ Q < P \text{ and } c \notin C_R(P)\}$ contains I. Further, if Q is nonminimal, then height of $Q \geq 1$ and so there exists no height 1 prime P such that $Q < P$ and so $\{P \in X_I/Q < P,\ c \notin C_R(P)\} = \emptyset$. Thus in both cases $Q \neq \cap\{P \in X_I/Q < P,\ c \notin C_R(P)\}$. Therefore X_I is a sparse set in R.

Theorem 4.40.

Let R be a GUFR and I be a normal invertible ideal of R. Also assume that for a prime ideal Q, $\{P \in X_I/Q \rightsquigarrow P\}$ is right stable. Then $\{P \in X_I/Q \rightsquigarrow P\}$ is a classically right localisable set.

<u>Proof</u>:

Put $X = \{P \in X_I / Q \rightsquigarrow P\}$. Since the elements of X are height 1 prime ideals, X satisfies the right second layer condition and the incomparability condition. The sparsity of X_I implies that X is finite [16, theorem 6.2.14] and so X satisfies the right intersection property. Now the result follows from theorem 4.23 and the hypothesis that X is **right** stable.

From Theorem 4.35 and [illegible] 4.25 it follows that

<u>Theorem 4.41</u>.

Every finite right stable set consists of height 1 prime ideals in a GUFR is right classically localisable.

Chapter-5

REMARKS

In this concluding chapter, we shall review some of the results given in the previous chapters and discuss the scope of further work.

In chapter 2, we have proved that every GUFR has an Artinian quotient ring, by proving that every non-minimal prime ideal contains a regular element, which gives rise to a normal ideal. Thus the definition of a GUFR can be reformed as a Noetherian ring in which every non-minimal prime ideal contains a normal regular element.

The theorem 2.30 that R is a commutative GUFR if and only if R has an Artinian quotient ring, leads to the relevant question; is every commutative Noetherian ring a GUFR? Or does every commutative Noetherian ring have an Artinian quotient ring? In particular cases of commutative Noetherian rings, it can be proved that they have Artinian quotient rings. For instance, if R is a commutative Noetherian irreducible (i.e., for any ideal A of R, $A < A_1 \cap A_2$, whenever $A < A_1$ and A_2) ring, then R has an Artinian quotient ring. In the general case, we can say only upto the extent that a commutative Noetherian ring R can be embedded in a commutative

Artinian ring, more precisely, R is a subdirect product of irreducible Noetherian rings, i.e. rings with Artinian quotient rings.

In theorem 2.35, to prove that R[x] is a GUFR, we assumed that $E_P < C_{R[x]}(0) \cap C'$ for every minimal prime ideal P of R. We do not know whether this condition can be relaxed. However, other than for prime rings R, no examples of R[x]s, with non minimal prime ideal. P, could be found out with the property that, ... $P \cap R$ is a minimal prime ideal in R.

We shall state a result given in [29, pp. 59-60].

<u>Lemma 5.1</u>.

Let R be a right order in Q and A_R a submodule of Q_R that contains a regular element of R. Then A_R is a projective if and only if there exist elements $y_1 \dots y_n$ in Q and $a_1 \dots a_n$ in A such that $y_i A \leq R$ for all i and $1 = a_1 y_1 + a_2 y_2 + \dots + a_n y_n$.

Now if R is a right bounded prime GUFR and I is an essential right ideal of R, then I contains an ideal J, which in turn contains a normal ideal aR = Ra (say) of R and so $a^{-1}I = R$ and $a^{-1}a = 1$. Thus by the lemma given above, in a right bounded prime GUFR, every essential right ideal is projective, which is a partial converse of theorem 2.44. We do not know whether every right ideal of a right bounded prime GUFR is projective.

Another question that arose in chapter 2 is about the integrally closed rings. We proved that the semi-prime GUFRs are integrally closed, if every right and left endomorphisms of Q over R takes the identity element of Q to R itself. The relevant question is: If R is a commutative Noetherian UFD, then, is every R endomorphism of Q takes the identity element of Q to R? (Here Q is the quotient field of R). The question is important because in the case when R is a commutative Noetherian UFD, it is always integrally closed.

In chapter 3, we proved that the finite centralising extension of a GUFR is a GUFR. The case of finite normalising extension of a GUFR is yet to be proved. The obstacle in this case is that we cannot connect the prime ideals of R with the prime ideals of a finite normalising extension directly. (lemma 3.3)

In chapter 4, theorem 4.35 assures the right second layer condition for height one prime ideals in a GUFR. The right second layer condition for a prime ideal of height ≥ 1 in a GUFR is yet to be discussed. Also, it is not yet investigated whether $\{P \in X_I / Q \rightsquigarrow P\}$ in theorem 4.40, is always right stable or not. However, from [26], it follows that, in a GUFR if every height 1 prime ideal is maximal, then each X_I is right stable, satisfies the right second layer condition (theorem 4.35) and the incomparability condition.

It may be possible to extend the concept of GUFRs to (non-Noetherian) rings with (left and right) Krull dimension [30]. The analogous nature of such rings with Noetherian rings is a major source of interest in them. The invertible ideals, in rings with (left and right) Krull dimension, also behave well. A study of invertible ideals in rings with Krull dimension is given in [31].

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