SOME PROBLEMS IN FUZZY SET THEORY AND RELATED TOPICS

STUDY ON FUZZY ORDERED FUZZY TOPOLOGICAL SPACES

THESIS SUBMITTED TO THE
COCHIN UNIVERSITY OF SCIENCE AND TECHNOLOGY
FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

IN MATHEMATICS UNDER THE FACULTY OF SCIENCE

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AUGUST 1993

CERTIFICATE

This is to certify that this thesis is an authentic record of research work carried out by Sri. Sunny Kuriakose. A. under my supervision and guidance in the Department of Mathematics and Statistics, Cochin University of Science and Technology, and no part of it has previously formed the basis for the award of any other degree in any other university.

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CHAPTER 0

INTRODUCTION

The principal endeavour of this thesis is to introduce the notion of fuzzy ordered fuzzy topological space and to study its properties.

Lotfi A. Zadeh's classic paper of 1965 opened up a new area in modern mathematics, namely, Fuzzy Set Theory. The roots of fuzzy sets can be traced back to the second half of the nineteenth century, more precisely, to the well known controversy between G. Cantor and L. Kronecker on the mathematical meaning of infinite sets. Cantor was for infinite sets and Kronecker refused to accept the concept of infinite sets. R. Dedekind reacted in favour of Cantor. A compromise between Kronecker's and Dedekind's points of view could be described thus: A set S is completely determined if and only if there is a decision procedure specifying whether an element is a member of S or not. Using naive set theory, this approach leads to characteristic functions in the context of binary logic, whereas in the case of many-valued logic this leads to the concept of membership functions introduced by Zadeh. Therefore, Kronecker's rejection of infinite sets and Dedekind's defence of Cantor's set might have resulted in the advent of fuzzy set theory.

Another source of fuzzy sets lies in the inherent imprecision in human decision making, which was Zadeh's main motivation. Since the very inception of the theory, several people all over the world have explored its various facets and a large number of results have been generated.

Fuzzy set theory has now become a major area of interest for modern scientists. According to S. Mac Lane- " Math Intelligencer Vol.5. nr. 4, 1983"

" ... The case of fuzzy sets is even more striking. The original idea was an attractive one ... . Someone then recalled (pace Lowere) that all mathematics can be based on set theory; it followed at once that all mathematics could be rewritten so as to be based on fuzzy sets. Moreover, it could be based on fuzzy sets in more than one way, so this turned out to be a fine blue print for the publication of lots and lots of newly based mathematics."

Fuzzy set theory offers wider applications than ordinary set theory. Besides, it provides sufficient motivation to researchers to review various concepts and theorems of mathematics in the broader frame work of fuzzy setting. The manifold applications of fuzzy set theory have permeated almost all spheres of human activity like artificial intelligence and robotics, image processing and speech recognition, biological and medical sciences, applied operations research, control, economics and geography, sociology, psychology, linguistics, semiotics and quantum mechanics.

Binary relations play a vital role in pure mathematics. The notions of equivalence and ordering relations are used practically in all fundamental mathematical constructions. They are being applied in modelling various concepts in the field of psychology, sociology, linguistics, art, etc. Many important models in decision making and measurement theories are based on binary relations.

The theory of Ordered Sets is a rapidly developing branch of mathematics, and ordered sets are abundant in all branches of mathematics. Axioms defining the concept of an ordered set are found in the work of C.S. Piere on the algebra of logic in 1880. In 1890 such axioms were studied systematically by Schroeder. These studies, also, were carried out from the point of view of the needs of logic.

It was R. Dedekind who first observed the frequent occurrence of ordered sets in mathematics. In 1897 he suggested that the theory of ordered sets has to be treated as an independent autonomous subject. This suggestion was later backed up by many an eminent mathematician like, Hausdorff (Foundation of Set Theory), Emmy Noether (Algebra), L. Nachbin and S. Purish. Several others also studied ordered sets rigorously and the theory has been enriched further by introducing intrinsic topologies, i.e., topologies defined purely in terms of order relations.

Order topology has been extensively studied by Vaidyanathaswamy, G. Birkhoff, L. Nachbin, J. Van Dalen, David J. Lutzer, H.R. Bennet, S. Eilenberg, M.E. Rudin, S.A. Gaal, etc. Gaal has studied the continuity properties of functions whose domain and range are totally ordered and endowed with the order topologies. In [Ga] he discussed various properties of a totally ordered topological space with least upper bound property.

The approach of fuzzy sets provides a very natural basis for generalising the concept of order relations. The theory of fuzzy order relations was initiated by Zadeh [Z_1]. In [Z_2] he gave various aspects of fuzzy binary relations, particularly of similarity relations and fuzzy orderings. He defined the notion of similarity as a generalization of the notion of equivalence relation. Fuzzy ordering on a set X was defined as a subset of X x X by generalising the notions of reflexivity, antisymmetry and transitivity. Since then, several authors have studied fuzzy relations and orderings. Among them, Sergei Ovchinnikov [O_1–O_4], M.K. Chakraborty, M. Das [Ch-D_1 - Ch-D_3], S. Sarkar [Ch-S], P. Venugopalan [Ve], A.K. Katsaras [Kat], V. Murali [Mu_1, Mu_2], S.K.Bhagat, P. Das [Bh-D] and Marc Roubens [O-R] deserve special mention.

In order to introduce the properties of fuzzy binary relations as in the classical case, we have to model basic logical connectives as operations on the unit interval [0,1]. The general method is based on MAX. and MIN. operations as models for logical connectives OR and AND and the negation is represented by $x \longmapsto 1-x$. Recently, logical connectives and operations on fuzzy sets have been defined by means of triangular norms and conorms and general negation functions. Fuzzy interval orderings and fuzzy orderings of fuzzy numbers are of special interest in fuzzy set theory and its applications.

The introduction of the idea of metric spaces by Frechet in 1906 marked the beginning of a new discipline called Set Topology. The works of people like Hausdorff, Kuratowski, A. Tychonoff, A.H. Stone and Dieudonné, were pioneering contributions to this area.

Fuzzy topology was initiated by C.L. Chang [Chan] in 1968. After he introduced fuzzy set theory into topology, C.K. Wong [WO_1-WO_3], R. Lowen [LO_1-LO_6], Bruce Hutton [Hut], Hu Cheng-Ming [Hu], Goguen J.A. [Go], Gottwald [Got], A.K. Srivastava [Sri-D, Sri-L] etc. have studied different aspects of fuzzy topology. Here a fuzzy topological space is defined as a crisp subset of the fuzzy power set of a non empty set (crisp), which is closed for finite intersection and arbitrary union operations and contains the largest and the smallest elements. This fuzzy topology is generally called Chang's topology. However, in [Lo-W] Lowen has defined fuzzy topology by including all constant functions to the subset considered in Chang's definition. This topology is termed as Lowen's topology. Recently Hazra R.N., Samanta, S.K. and Chattopadhyay [Ha] introduced the idea of gradation of openness (closedness) of fuzzy subsets and proposed a new definition of fuzzy topology.

In this study we combine the notions of fuzzy order and fuzzy topology of Chang and define fuzzy ordered fuzzy topological space. Its various properties are analysed. Product, quotient, union and intersection of fuzzy orders are introduced. Besides, fuzzy order preserving maps and various fuzzy completeness are investigated. Finally an attempt is made to study the notion of generalized fuzzy ordered fuzzy topological space by considering fuzzy order defined on a fuzzy subset.

Our approach is distinct from those of the earlier authors like A.K. Katsaras [Kat], R. Lowen [Lo_6] and P. Venugopalan [Ven]. Katsaras has defined a fuzzy topology on a crisp ordered set and investigated its various properties analogous to the work of Nachbin [N]. Lowen studied ordered fuzzy topology on the real line. He put forward a new definition of fuzzy real line and observed that it was the order of $\mathbb{R}$, and not the topology, which determined the fuzzy real line. Venugopalan's definition of fuzzy order was different from ours. He considered a special type of transitivity and introduced the fuzzy interval topology on a fuzzy ordered set (P,μ) generated by the fuzzy sets $P \setminus \downarrow e$, $P \setminus \uparrow d$ for e,d fuzzy points of P as subbasic open sets, where $e = x_\lambda$ is a fuzzy point of P.

$$\downarrow e(y) = [\mu(y,x) + \lambda - 1] \vee 0$$

and

$$\uparrow e(y) = [\mu(x,y) + \lambda - 1] \vee 0$$

We now give the summary of each chapter.

The thesis comprises seven chapters and an introduction to the subject.

Chapter 1

Preliminary definitions of the terms, like fuzzy topology and fuzzy ordering, required for the later chapters are given in chapter 1. The valuation set of every fuzzy set is taken as the unit interval [0,1] and the Chang's definition of fuzzy topology is followed throughout. Also we stick to strict partial ordering R on X satisfying irreflexivity, i.e., for $x \neq y$; $R(x,y) \neq R(y,x)$, and max-min transitive, i.e., $R(x,z) \geq \bigvee_y [R(x,y) \wedge R(y,z)]$, $x,y,z \in X$. Besides, the algebra of fuzzy sets and various other definitions of reflexivity, antisymmetry and transitivity are also discussed.

Chapter 2

The notion of fuzzy ordered fuzzy topological space (X,F_R) on a foset (X,R) is introduced in the first section. It is proved that every fuzzy order R defined on a set X determines a total order $\prec$ as $x \prec y$ iff $R(x,y) > R(y,x)$ and the corresponding order topology on X is denoted by $T_{\prec}$. It is found that the associated topology $\ell(F_R)$ of F_R contains $T_{\prec}$. An example in which this inclusion is strict is provided. Also the fuzzy topological space (X,F) defined by taking all lower semi continuous functions and (X,F_R), the fuzzy ordered fuzzy topological space are compared.

In section 2 we recall several definitions of interval topologies in the crisp sense and their fuzzy analogues are proposed.

Chapter 3

Chapter 3 begins with the product of strong fuzzy orders. An example showing that the product of fuzzy orders, if they are not strict, need not be a fuzzy order is given. It is shown that the product of crisp orders induced by fuzzy orders is the same as the induced crisp order of the product of fuzzy orders.

Quotient spaces are analysed in section 2. Certain maps from X to $X/\sim$, $(X,T_{\leq})$ to $(X/\sim , T_{\underset{\sim}{\leq}})$ and $(X, \ell(F_R))$ to $(X/\sim , \ell(F_{\underset{\sim}{R}}))$ are proved to be quotient.

In section 3 union of fuzzy orders is considered and it is proved that the union of strong fuzzy orders R_i, $i \in \Lambda$ defined on X is a strong fuzzy order iff $R_i(x,y) \wedge R_j(y,x) = 0$, $i \neq j$, $x \neq y$. Intersection of fuzzy orders is also mentioned.

Finally, certain aspects of fuzzy ordered subspaces are discussed in section 4. Let Y be a subspace of a foset (X,R). Then a fuzzy order R_Y is defined on Y and it is proved that $F_{R_Y} = F_R \wedge Y$ and $\ell(F_{R_Y}) = \ell(F_R) \cap Y$; where F_R and F_{R_Y} are the fuzzy ordered fuzzy topological spaces on X and Y respectively and $\ell(F_R)$ and $\ell(F_{R_Y})$ are their corresponding associated topologies.

Chapter 4

Certain separation properties of the fuzzy ordered fuzzy topological space are discussed in chapter 4. It is proved that the fuzzy ordered fuzzy topological space

(X,F_R) is fuzzy T_1. Also, when R is strong (X,F_R) is found to be fuzzy Hausdorff. If R is not strong the notion of weak fuzzy Hausdorffness is introduced and (X,F_R) is found to be weak fuzzy Hausdorff.

Chapter 5

This chapter consists of a brief analysis of fuzzy order preserving maps. Two types of fuzzy order preserving maps between fuzzy ordered fuzzy topological spaces are defined. It is proved that a bijective strict order preserving map is a fuzzy homeomorphism. Also the set Y^X of all maps from X to Y, when Y is a strong fuzzy ordered set is made a strong fuzzy ordered set. Natural monoid homomorphisms between $\hat{Y}$, $\widehat{\widehat{(X, Y)}}$ and between $\hat{X} \times \hat{Y}$, $\widehat{\widehat{(X, Y)}}$ are obtained.

Chapter 6

In this chapter various characterizations of fuzzy ordered completeness are considered. In particular, least upper bound property, greatest lower bound property, Didekind completeness and Cantor completness are discussed.

Chapter 7

An extension of the definition of fuzzy order defined on a crisp set to fuzzy order defined on a fuzzy set is the chunk of the final chapter. The notion of the generalised fuzzy ordered fuzzy topological space, defined by means of a generalised fuzzy order is introduced as well.

The author does not claim that the study made in this thesis is a complete exposition in all respects-rather, there are various problems connected with the work done here, worth investigating. As is often found, any investigation opens up new areas for further exploration.

CHAPTER 1

PRELIMINARIES

1.0 Introduction

In this chapter we give a concise account of the preliminary definitions and results required for the forthcoming chapters.

Let X be a set and A be a subset of X. A is completely determined by the elements belonging to it. This belongingness or membership of elements can be described by means of the characteristic functions $\chi_A : X \longrightarrow [0,1]$ defined by

$$\chi_A(x) = 1 \quad \text{if } x \in A$$
$$= 0 \quad \text{otherwise}$$

i.e., $A = \{(x, \chi_A(x)) \mid x \in X\}$

Here $\{0,1\}$ is called the valuation set or membership set. For a classical or crisp set the membership (non membership) of element is abrupt and its boundary is rather rigid.

It is worth considering the membership of elements to be gradual rather than abrupt. This can be achieved by extending the valuation set $\{0,1\}$ to the unit interval $I = [0,1]$. This is the basic characteristic of a fuzzy set.

1.1 Fuzzy Set

We now give a precise definition of fuzzy set and discuss the basic operations of fuzzy sets.

1.1.1 Definition

Let X be a set and I = [0,1]. Then a fuzzy set μ in X is a member of I^X, the family of all functions from X to I.

$\mu(x)$ is the membership value of the membership function at x.

Obviously, a fuzzy subset is a generalised subset of a crisp set.

1.1.2 Note

If an arbitrary set L is taken as the valuation set instead of I, we get an L-fuzzy set [Go].

If L has a given structure, such as lattice or group structure, then L^X, the family of all L-fuzzy sets in X will also have this structure.

If $(\mu_i)_{i \in \wedge}$ is a family of fuzzy sets in X then $\sup_{i \in \wedge} \mu_i$ (or $\bigvee_{i \in \wedge} \mu_i$) is a fuzzy set μ in X defined by

$$\mu(x) = \sup_{i \in \Lambda} \{ \mu_i(x) ; x \in X \}$$

The fuzzy set $\inf_{i \in \Lambda} \mu_i$ (or $\bigwedge_{i \in \Lambda} \mu_i$) is defined analogously.

However, unless otherwise stated the valuation set is taken as $I = [0,1]$ for defining fuzzy sets throughout this thesis.

1.1.3 Definition

If μ is a fuzzy set in X then the crisp set $\mu_o = \{ x \in X | \mu(x) > 0 \}$ is called the support of μ. Also, a point $p \in X$ belongs to a fuzzy set μ if $\mu(p) = 1$.

1.1.4 Algebra of fuzzy sets

Let μ_1 and μ_2 be two fuzzy sets in X. Then,

i) $\mu_1 \subseteq \mu_2$ iff $\mu_1(x) \leqslant \mu_2(x)$ for every x in X ... (inclusion)

ii) $\mu_1 = \mu_2$ iff $\mu_1(x) = \mu_2(x)$ for every x in X ... (equality)

iii) Union of μ_i; $i \in \Lambda$ is the fuzzy set $\bigcup_{i \in \Lambda} \mu_i$ where $\bigcup_{i \in \Lambda} \mu_i(x) = \bigvee_{i \in \Lambda} \{ \mu_i(x) | x \in X \}$

iv) Intersection of μ_i; $i \in \Lambda$ is the fuzzy set $\bigcap_{i \in \Lambda} \mu_i$

where $\bigcap_{i \in \Lambda} \mu_i(x) = \bigwedge_{i \in \Lambda} \{\mu_i(x) \mid x \in X\}$

v) The complement of μ is the fuzzy set μ'

where $\mu'(x) = 1-\mu(x)$ for every x in X.

1.1.5 Definition

If $\mu_1(x) + \mu_2(x) > 1$, then μ_1 is quasi coincident with μ_2.

1.2 Fuzzy Topology

According to C.L. Chang [Chan] the definition of fuzzy topology is as follows.

1.2.1. Definition

Let F be a family of fuzzy sets in X, satisfying the following axioms.

i) $0, 1 \in F$

ii) If $\mu_i \in F$ for every $i \in \Lambda$ then $\sup_{i \in \Lambda} \mu_i \in F$

iii) If μ_1 and $\mu_2 \in F$ then $\mu_1 \wedge \mu_2 \in F$,

then F is called a fuzzy topology for X and the pair (X,F) is a fuzzy topological space (fts for short).

Members of F are called F-open fuzzy sets, (or simply open fuzzy sets) and their complements are F-closed fuzzy sets (or closed fuzzy sets).

In this thesis, Chang's definition of fuzzy topology is followed.

1.2.2. Definition

Let (X,F) be a fts. A fuzzy set $\mu \in F$ is a neighbourhood of a fuzzy set β in X iff there exists $\delta \in F$ such that $\beta \subseteq \delta \subseteq \mu$.

Clearly, μ is open iff for each fuzzy set β contained in μ, μ is a neighbourhood of β.

1.2.3 Definition

Let (X,F) be a fts. The closure $\bar{\mu}$ of a fuzzy set μ in X is defined by

$$\bar{\mu} = \bigcap \{\beta \mid \beta \supset \mu, \ \beta' \in F\}$$

The interior μ^o of μ is defined by

$$\mu^o = \bigcup \{\beta \mid \beta \subset \mu, \ \beta \in F\}$$

1.2.4 Definition

Let F be a fuzzy topology. A subfamily $\mathcal{B}$ of F is a base for F iff each member of F can be expressed as the union of some members of $\mathcal{B}$.

A subfamily $\mathcal{S}$ of F is a subbase for F iff the family of finite intersections of members of $\mathcal{S}$ forms a base for F.

1.2.5 Definition

A mapping $\alpha:(X,F_1) \longrightarrow (Y,F_2)$ is called fuzzy continuous iff for each $\mu \in F_2$ we have $\alpha^{-1}(\mu) \in F_1$, where $\alpha^{-1}(\mu) = \mu \circ \alpha$.

1.2.6 Note

In a fts (X,F), a subset of X is either open or not open. Recently a new definition for fuzzy topology has been proposed [Ha] assigning different grades of openness to the fuzzy sets in X. It is as follows.

1.2.7 Definition

Let X be a nonempty set and $\tau : I^X \longrightarrow I$. Then (X,τ) is called a fts provided

i) $\tau(0) = \tau(1) = 1$

ii) $\tau(\mu_i) > 0; i = 1,2$ implies $\tau(\mu_1 \cap \mu_2) > 0$

iii) $\tau(\mu_i) > 0;\ i \in \Lambda$, implies $\tau(\bigcup_{i \in \Lambda} \mu_i) > 0$.

Apparently, Chang's topology is a special case of this topology.

1.2.8 Definition

Let (X,T) be a topological space and $I = [0,1]$. A function $f: (X,T) \longrightarrow I$ is lower semicontinuous (lsc for short) iff for each $a \in I$, $f^{-1}(a,1] \in T$.

1.2.9 Note

Constant functions are lsc and the characteristic function of a set A in X is lsc iff A is open.

In [Lo_1] R. Lowen gives a natural association between fuzzy topology and a given topology which is put in the following definition.

1.2.10 Definition

Given a fts (X,F), let $\ell(F)$ be the smallest topology on X such that every member μ in F is lsc. Then $\ell(F)$ is called the associated topology of F.

Conversely, given a topological space (X,T) if F_T denotes the set of all lsc functions on X with regard to T, then (X,F_T) is a fts.

1.2.11 Note

If a given fuzzy topology F is the same as the one obtained by taking lsc functions with regard to some topology T for X, then, we say that F is topologically generated.

1.3 Fuzzy Orderings

1.3.1 Definition

Given a nonempty set X, a binary relation '$\prec$' on X is called a strict partial order on X if it has the following properties.

i) irreflexivity i.e., the relation $x \prec x$ never holds

ii) antisymmetry i.e., for $x \neq y$, $x \prec y$ and $y \prec x$ together never hold

iii) transitivity i.e., $x \prec y$ and $y \prec z$ then $x \prec z$

In addition to these conditions if the fourth condition

iv) comparability i.e., for every x and y for which $x \neq y$, $x \prec y$ or $y \prec x$ is also satisfied, '$\prec$' is called a total (or linear) ordering on X.

1.3.2. Remark

Let '<' be a strict partial order on the set X. Associated with this strict partial order, we define '$\leqslant$' on X as $x \leqslant y$ if either $x<y$ or $x=y$. Then the relation '$\leqslant$' is called a partial order. Thus every partial order determines and is determined by a strict partial order. There are many authors preferring to deal with strict partial orderings rather than partial orderings.

In this study we stick to strict partial orderings satisfying irreflexivity, antisymmetry and transitivity unless otherwise stated.

In $[Z_2]$ Zadeh had defined a fuzzy order R on a crisp set X as follows.

1.3.3 Definition

A fuzzy binary relation R from $X \times X \longrightarrow I$ is a fuzzy order on X if R satisfies the following:

i) reflexivity i.e., $R(x,x) = 1$ for every $x \in X$

ii) antisymmetry i.e., For $x \neq y$, $R(x,y) \neq R(y,x)$

iii) Max-min transitivity i.e.,

$$R(x,z) \geqslant \bigvee_{y} [R(x,y) \wedge R(y,z)], \quad x,y,z \in X.$$

1.3.4 Note

A number of other definitions for reflexivity, antisymmetry and transitivity can be seen in the literature. Mentioning a few of them,

1.3.5 Reflexivity

A fuzzy binary relation R defined on X is

a) reflexivie iff $R(x,x) > 0$ for every $x \in X$, [Ven]

b) ε-reflexive iff $R(x,x) \geqslant \varepsilon$ for every $x \in X$, [Ye]

c) Weakly reflexive iff $R(x,x) \geqslant R(x,y)$, for every $x,y \in X$ [Ye]

d) irreflexive iff $R(x,x)=0$ for every $x \in X$ [Z_2].

1.3.6 Antisymmetry

A fuzzy binary relation R defined on X is antisymmetric iff

a) for $x \neq y$, $R(x,y) = R(y,x)$ implies that
$R(x,y) = R(y,x) = 0$ [Kau]

b) $R(x,y) + R(y,x) > 1$ implies $x=y$ [Ven]

c) $\text{Min}\{R(x,y), R(y,x)\} = 0$ for every $x,y \in X$ [O-R], and

d) perfectly antisymmetric iff for $x \neq y$,
$R(x,y) > 0$ implies $R(y,x) = 0$; $x,y \in X$ [Z_2].

1.3.7 Transitivity

Instead of max-min transitivity, several other types of transitivities; - *, where x*y is given by

i) $x \wedge y = \max(o, x+y-1)$ — (bold intersection)

ii) $x \,\square\, y = \frac{1}{2}(x+y)$ — (arithmetic mean)

iii) $x \vee y = \max(x,y)$ — (union)

iv) $x \hat{+} y = x+y-xy$ — (probabilistic sum)

have been used [B-H].

1.3.8 Remark

According to M.K. Chakraborty, [Ch-S] a fuzzy binary relation R is called fuzzy weak ordering if R is reflexive, transitive (as given in [1.4.3]) and for every $x,y \in X$; $x \neq y$, $R(x,y) \vee R(y,x) > 0$ and fuzzy strong ordering if R is also antisymmetric. (i.e., for $x \neq y$ $R(x,y) \neq R(y,x)$, $x,y \in X$).

However, we define a fuzzy order and a strong fuzzy order as given below:

1.3.9 Definition

Let X be a nonempty crisp set. A fuzzy order R defined on X is a fuzzy binary relation, i.e., a fuzzy subset of X x X which is

i) irreflexive i.e., $R(x,x) = 0$ for every $x \in X$

ii) antisymmetric i.e., for $x \neq y$ $R(x,y) \neq R(y,x)$

iii) transitive i.e., $R(x,z) \geq \bigvee_{y} [R(x,y) \wedge R(y,z)]$

$x,y,z \in X$.

1.3.10 <u>Definition</u>

R is a strong fuzzy order if it is irreflexive, transitive and perfectly antisymmetric (i.e., $R(x,y) > 0$ implies $R(y,x) = 0$, for $x \neq y$). Clearly every perfectly antisymmetric relation is antisymmetric and thus every strong fuzzy order is a fuzzy order.

CHAPTER 2

FUZZY ORDERED FUZZY TOPOLOGICAL SPACE

2.0 Introduction

This chapter is devoted mainly to introduce the notion of fuzzy ordered fuzzy topological space defined on a fuzzy ordered set, and to study its various properties. It is proved that the associated topology of the fuzzy ordered fuzzy topology contains the order topology defined by the crisp order induced by the fuzzy order. Moreover, an example is provided in which this inclusion is strict. Besides, some other results regarding the fuzzy ordered fuzzy topology are also obtained. Also, several fuzzy interval topologies are proposed in the second section.

2.1 Fuzzy ordered fuzzy topological space

Let us now prove that every fuzzy order defined on a set determines a crisp order.

2.1.1 Result

Every fuzzy order R defined on a nonempty set X determines a crisp order '$\prec$' on X as

$$x \prec y \text{ iff } R(x,y) > R(y,x);\ x,y \in X.$$

<u>Proof</u>

i) Clearly $\prec$ is irreflexive

ii) $\prec$ is antisymmetric, as if $x \prec y$ and $y \prec x$ hold simultaneously then,

$$R(x,y) > R(y,x) \text{ and } R(y,x) > R(x,y)$$

which is impossible.

iii) $\prec$ is transitive.

For, if $x \prec y$ and $y \prec z$ then,

$$R(x,y) > R(y,x) \text{ and } R(y,z) > R(z,y)$$

Since R is transitive,

$$R(x,y) > R(x,z) \wedge R(z,y)$$

$$R(x,z) > R(x,y) \wedge R(y,z)$$

$$R(y,x) > R(y,z) \wedge R(z,x)$$

$$R(z,x) > R(z,y) \wedge R(y,x)$$

$$R(z,y) > R(z,x) \wedge R(x,y)$$

Also, $R(x,y) \neq R(y,x)$

$$R(x,z) \neq R(z,x)$$

$$R(y,z) \neq R(z,y)$$

and $R(x,x) = R(y,y) = R(z,z) = 0$

Simple calculations show that

$$R(x,z) > R(z,x)$$

Therefore $x \prec z$ and hence $\prec$ is transitive and

$\prec$ is a crisp order on X.

2.1.2 Remark

$\prec$ is clearly a total order. The order topology induced by $\prec$ on X is denoted by $T_{\prec}$.

Fuzzy ordered fuzzy topological space is defined on a foset as follows.

2.1.3 Definition

Let (X,R) be a foset. For each $a \in X$ define the fuzzy sets μ_a and ${}_a\mu$ as;

$$\mu_a(x) = R(a,x)$$

$${}_a\mu(x) = R(x,a), \text{ for every } x \in X.$$

If X does not possess either the largest element or the smallest element, then the fuzzy topology on X with

$$\{\mu_a, {}_a\mu, 1-\mu_a-\chi_a, 1 - {}_a\mu - \chi_a; \text{ where a varies in } X\} \quad (*)$$

as subbasis is called the fuzzy ordered fuzzy topology induced by R on X, and is denoted by F_R.

The pair (X,F_R) is called the fuzzy ordered fuzzy topological space.

When X has the largest element ℓ (in the induced order $\prec$), we include $(P_\ell \vee {}_\ell\mu)$ and $(1-\mu_\ell)$ to the element in (*) with $a \neq \ell$.

When X has the smallest element s (in the induced order $\prec$), $(P_s \vee \mu_s)$ and $(1 - {}_s\mu)$ are included to the elements in (*) with $a \neq s$. Recall that P_ℓ is the fuzzy point with value 1 and support ℓ.

2.1.4 <u>Theorem</u>

Let (X,F_R) be a fuzzy ordered fuzzy topological space and $\ell(F_R)$ be its associated topology. Let R induce a crisp order $\prec$ on X as in [2.1.2] and $T_\prec$ be the order topology defined by $\prec$. Then, $\ell(F_R)$ contains $T_\prec$.

i.e., Given

$$\begin{array}{ccc} (X,F_R) & \longrightarrow & (X,\prec) \\ \downarrow & & \downarrow \\ (X, \ell(F_R)) & \supseteq & (X,T_\prec) \end{array}$$

<u>Proof</u>:

A typical open set in $T_\prec$ is one of the forms (a,b), $[s,b)$ where s is the smallest element in X or $(a,\ell]$ where ℓ is the largest element in X and $a,b \in X$.

Now,

$$
\begin{aligned}
(a,b) &= \{x \in X \mid a \prec x \prec b;\ a,b \in X\} \\
&= \{x \in X \mid R(a,x) > R(x,a) \text{ and } R(x,b) > R(b,x)\} \\
&= \{x \in X \mid \mu_a(x) > {}_a\mu(x) \text{ and } {}_b\mu(x) > \mu_b(x)\} \\
&= \{x \in X \mid \mu_a(x) > {}_a\mu(x)\} \cap \{x \in X \mid {}_b\mu(x) > \mu_b(x)\} \\
&= \bigcup_t [\{x \in X \mid \mu_a(x) > t\} \cap \{x \in X \mid t > {}_a\mu(x)\}] \\
&\quad \cap \bigcup_t [\{x \in X \mid {}_b\mu(x) > t\} \cap \{x \in X \mid t > \mu_b(x)\}] \\
&= \bigcup_t [\{x \in X \mid \mu_a(x) > t\} \cap \{x \in X \mid (1 - {}_a\mu - \chi_a)(x) > 1-t\}] \\
&\quad \cap \bigcup_t [\{x \in X \mid {}_b\mu(x) > t\} \cap \{x \in X \mid (1 - \mu_b - \chi_b)(x) > 1-t\}] \\
&\quad \in \mathcal{L}(F_R)
\end{aligned}
$$

If s is the smallest element,

$$
\begin{aligned}
[s,b) &= \{x \in X \mid s \preceq x \prec b\} \\
&= \{x \in X \mid s \preceq x\} \cap \{x \in X \mid x \prec b\} \\
&= [\{s\} \cup \{x \in X \mid s \prec x\}] \cap \{x \in X \mid x \prec b\} \\
&= [\{s\} \cup \{x \in X \mid R(s,x) > R(x,s)\}] \\
&\quad \cap [\{x \in X \mid R(x,b) > R(b,x)\}]
\end{aligned}
$$

But $\{s\} \cup \{x \in X \mid R(s,x) > R(x,s)\}$

$$= \{s\} \cup \{x \in X \mid \mu_s(x) > {}_s\mu(x)\}$$

$$= \{s\} \cup \left(\bigcup_t [\{x \in X \mid \mu_s(x) > t\} \cap \{x \in X \mid (1-{}_s\mu)(x) > 1-t\}]\right)$$

$$= \bigcup_t [\{x \in X \mid (P_s \vee \mu_s)(x) > t\} \cap \{x \in X \mid (1-{}_s\mu)(x) > 1-t\}]$$

$$\in \ell(F_R).$$

If ℓ is the largest element,

$$(a,\ell] = \{x \in X \mid a \prec x \preceq \ell\}$$

$$= \{x \in X \mid a \prec x \prec \ell\} \cup \{\ell\}$$

$$= [\{x \in X \mid a \prec x\} \cap \{x \in X \mid x \prec \ell\}] \cup \{\ell\}$$

$$= \{x \in X \mid R(a,x) > R(x,a)\}$$

$$\cap \ [\{x \in X \mid R(x,\ell) > R(\ell,x)\} \cup \{\ell\}] \quad \text{since } R(a,\ell) > R(\ell,a)$$

But $\{x \in X \mid R(x,\ell) > R(\ell,x)\} \cup \{\ell\}$

$$= \{x \in X \mid {}_\ell\mu(x) > \mu_\ell(x)\} \cup \{\ell\}$$

$$= \bigcup_t [\{x \in X \mid {}_\ell\mu(x) > t\} \cap \{x \in X \mid t > \mu_\ell(x)\}] \cup \{\ell\}.$$

$$= \bigcup_t [\{x \in X \mid (P_\ell \vee {}_\ell\mu)(x) > t\} \cap \{(1-\mu_\ell)(x) > 1-t\}]$$

$$\in \ell(F_R).$$

Therefore, $\ell(F_R)$ contains $T_\prec$.

The following is a nontrivial example illustrating the case where $\ell(F_R) = T_\prec$.

2.1.5 <u>Example</u>

Let $X = (0,1)$

$$R(x,y) = 1 \quad \text{if} \quad y > x + \tfrac{1}{2}$$

$$= \tfrac{1}{2} \quad \text{if} \quad x < y < x + \tfrac{1}{2}$$

$$= \tfrac{1}{4} \quad \text{if} \quad x - \tfrac{1}{2} < y < x$$

$$= 0 \quad \text{if } y \leqslant x - \tfrac{1}{2} \quad \text{or } x = y$$

The induced order $\prec$ on X is given by

$x \prec y$ iff $R(x,y) > R(y,x)$.

$R(x,y) > R(y,x)$ iff

i) $R(x,y) = 1$ and $R(y,x) = \frac{1}{2}$

or ii) $R(x,y) = 1$ and $R(y,x) = \frac{1}{4}$

or iii) $R(x,y) = 1$ and $R(y,x) = 0$

or iv) $R(x,y) = \frac{1}{2}$ and $R(y,x) = \frac{1}{4}$

or v) $R(x,y) = \frac{1}{2}$ and $R(y,x) = 0$

or vi) $R(x,y) = \frac{1}{4}$ and $R(y,x) = 0$.

Now,

i) holds iff $y > x + \frac{1}{2}$ and $y < x \leqslant y + \frac{1}{2}$ which is not possible.

ii) holds iff $y > x + \frac{1}{2}$ and $y - \frac{1}{2} < x < y$
which is also not possible.

iii) holds iff $y > x + \frac{1}{2}$ and $x=y$ or $x \leqslant y - \frac{1}{2}$,
i.e., $x + \frac{1}{2} < y$.

iv) holds iff $x < y \leqslant x + \frac{1}{2}$ and $y - \frac{1}{2} < x < y$
i.e., $x < y < x + \frac{1}{2}$.

v) holds iff $x < y \leqslant x + \frac{1}{2}$ and $x \leqslant y - \frac{1}{2}$ or $x=y$
i.e., $y = x + \frac{1}{2}$.

vi) holds iff $x - \frac{1}{2} < y < x$ and $x \leqslant y - \frac{1}{2}$
or $x=y$ which is not possible.

Hence $x \prec y$ iff $x + \frac{1}{2} < y$ or $x < y < x + \frac{1}{2}$ or $y = x + \frac{1}{2}$.

The induced order topology $T_{\prec}$ has a typical open set,

$$((a,b)) = \{x \in X \mid a \prec x \prec b\}$$

$$= \{x \in X \mid a \prec x\} \cap \{x \in X \mid x \prec b\}$$

$$= \{x \in X \mid a < x < a + \tfrac{1}{2}\} \cap \{x \in X \mid x < b < x + \tfrac{1}{2}\}$$

or $$\{x \in X \mid x = a + \tfrac{1}{2}\} \cap \{x \in X \mid b = x + \tfrac{1}{2}\}$$

or $$\{x \in X \mid a + \tfrac{1}{2} < x\} \cap \{x \in X \mid x + \tfrac{1}{2} < b\}.$$

$$= \{(a, a+\tfrac{1}{2}) \cap (b-\tfrac{1}{2}, b)\} \cup \{(a, a+\tfrac{1}{2}) \cap \{b-\tfrac{1}{2}\}\}$$

$$\cup \{(a, a+\tfrac{1}{2}) \cap [0, b-\tfrac{1}{2})\} \cup \{\{a+\tfrac{1}{2}\} \cap (b-\tfrac{1}{2}, b)\}$$

$$\cup \{\{a+\tfrac{1}{2}\} \cap \{b-\tfrac{1}{2}\}\} \cup \{\{a+\tfrac{1}{2}\} \cap [0, b-\tfrac{1}{2})\}$$

$$\cup \{(a+\tfrac{1}{2}, 1] \cap (b-\tfrac{1}{2}, b)\} \cup \{(a+\tfrac{1}{2}, 1] \quad \{b-\tfrac{1}{2}\}\}$$

$$\cup \{(a+\tfrac{1}{2}, 1] \cap [0, b-\tfrac{1}{2})\}.$$

$$= (b-\tfrac{1}{2}, a+\tfrac{1}{2}) \cup \{b-\tfrac{1}{2}\} \cup (a, b-\tfrac{1}{2}) \cup \{a+\tfrac{1}{2}\}$$

$$\cup (b-\tfrac{1}{2}, 1] \cup (a+\tfrac{1}{2}, b-\tfrac{1}{2})$$

Here $\{\frac{1}{2}\}$ is not open and all other points are discrete.

Now as X has neither the largest nor the smallest element, the fuzzy ordered fuzzy topology F_R has the subbasis $\{\mu_a, {}_a\mu, (1-\mu_a-\chi_a)\ (1-{}_a\mu-\chi_a) \mid a \in X\}$.
The corresponding associated topology $\ell(F_R)$ is generated by

$$\{\mu_a^{-1}(0,1],\ \mu_a^{-1}(\tfrac{1}{4},1],\ \mu_a^{-1}(\tfrac{1}{2},1],\ {}_b\mu^{-1}(0,1]$$

$${}_b\mu^{-1}(\tfrac{1}{4},1],\ {}_b\mu^{-1}(\tfrac{1}{2},1],\ (1-\mu_a-\chi_a)^{-1}(0,1]$$

$$(1-\mu_a-\chi_a)^{-1}(\tfrac{1}{4},1],\ (1-\mu_a-\chi_a)^{-1}(\tfrac{1}{2},1]\ (1-{}_b\mu-\chi_a)^{-1}(0,1]$$

$$(1-{}_b\mu-\chi_b)^{-1}(\tfrac{1}{4},1],\ (1-{}_b\mu-\chi_b)^{-1}(\tfrac{1}{2},1];\ a,b \in X\}$$

Here also $\{\frac{1}{2}\}$ is not open and all other points are discrete.

Hence $\ell(F_R)$ is the same as $T_\measuredangle$.

However the following example shows that $\ell(F_R)$ is not always the same as $T_\measuredangle$.

2.1.6 Example

$X = \mathbb{N} \cup \{0\}$

R is defined as

$R(n,o) = 1$

$R(o,n) = 0$ for every $n = 2m+1$, $(m=0,1.2,\ldots,)$

$R(2m,2n+1) = 0$ $\qquad$ $R(o,2m) = 1$

$R(2m,o) = 0$

$R(2n+1,2m) = \frac{1}{2}$ for every $m = 1,2,\ldots,$

$R(2m+1, 2n+1) = 0$

$R(2n+1,2m+1) = \frac{1}{2}$ for every $n<m$

$R(2n,2m) = 0$

$R(2m,2n) = \frac{1}{2}$ for every $n<m$

$R(m,m) = 0$ for every $m \in X$

The induced order $\prec$ is given by

$$x \prec y \quad \text{iff} \quad R(x,y) > R(y,x)$$

Thus, $1 \prec 3 \prec \ldots \prec 0 \prec \ldots \prec 6 \prec 4 \prec 2$

In the induced order topology $T_{\prec}$, $\{0\}$ is not discrete and all other singletons are discrete.

Now, in the associated topology $\ell(F_R)$

$$\mu_1^{-1}(\tfrac{1}{2},1] = \{x \in X \mid R(1,x) = 1\}$$
$$= \{0\}$$

i.e., $\{0\}$ is open in $\ell(F_R)$

Hence the $\ell(F_R)$ is different from $T_{\prec}$.

2.1.7 <u>Remark</u>

However if X is a finite set then $\ell(F_R)$ coincides with $T_{\prec}$ as the discrete topology.

2.1.8 <u>Remark</u>

A necessary and sufficient condition for these two topologies to be equal is yet to be obtained.

In the following result we compare the fuzzy topologies.

2.1.9 <u>Result</u>

Given a foset (X,R), let the fuzzy order R define a crisp order $\prec$ as in [2.1.2]. Let F be the set of all lsc functions on X with regard to the order topology $T_{\prec}$. Then the fuzzy ordered fuzzy topology F_R contains F.

i.e., Given

$$
\begin{array}{ccccc}
(X,R) & & \longrightarrow & & (X,\prec) \\
\downarrow & & & & \downarrow \\
(X,F_R) & \supseteq & (X,F) & \longleftarrow & (X,T\ )
\end{array}
$$

<u>Proof</u>:

Let $\mu \in F$. Then μ is lsc with regard to $T_{\prec}$.

i.e., $\mu^{-1}(t,1] = \cup B_i, \quad B_i \in T_{\prec}$

where $B_i = (a_i,b_i)$ or $[s,b_i)$ or $(a_i\ell]$

s,ℓ being the smallest and the largest elements of X (if any) respectively.

Now, $\{x \in X | x \in (a_i,b_i) = \bigcup\limits_{\lambda\in(0,1)} [\{x \in X | \mu_{a_i}(x) > \lambda\}$

$$\cap \{x \in X | (1-\mu_{b_i} - \chi_{b_i})(x) > 1-\lambda\}]; \; a_i, b_i \in X$$

$\{x \in X \mid s \preceq x \prec b_i\} = [\{x \in X \mid s \prec x\} \cup \{s\}]$

$\cap \{x \in X \mid x \prec b_i\}$

$= \{x \in X \mid (P_s \vee \mu_s)(x) > t\} \cap \{x \in X \mid (1 - {}_s\mu)(x) > 1 - t\}$

$\cap \bigcup_{\lambda} [(P_s \vee \mu_s)(x) > \lambda\} \cap \{x \in X \mid (1 - \mu_{b_i} - \chi_{b_i}) > 1 - \lambda\}]$

and $\{x \in X \mid a_i \prec x \preceq \ell\} = \{x \in X \mid a_i \mid a_i \prec x\} \cap [\{x \in X \mid x \prec \ell\} \cup \{\ell\}]$

$= [\bigcup_{\lambda} \{x \in X \mid \mu_{a_i}(x) > \lambda\} \cap \{x \in X \mid (1 - {}_{a_i}\mu - \chi_{a_i})(x) > 1 - \lambda\}]$

$\cap [\{x \in X \mid (P_\ell \vee \mu_\ell)(x) > \lambda\} \cap \{x \in X \mid P_\ell \vee (1 - \mu_\ell)(x) > 1 - \lambda\}]$

In each case it is open in (X, F_R).

Hence (X, F_R) contains (X, F).

Next we observe that the trivial fuzzy order R defined on a linearly ordered set $(X, <)$ as

$R(x,y) = 1$ if $x < y$

$= 0$ otherwise, determines the $\ell(F_R)$

same as the induced crisp order topology.

2.1.10 <u>Result</u>

Given a linearly ordered set $(X, <)$

define $R : X \times X \longrightarrow [0,1]$ by

$$R(x,y) = 1 \text{ if } x<y$$
$$= 0 \text{ otherwise}$$

Then R is trivially a fuzzy order and $\mathcal{L}(F_R)$ coincides with $T_<$.

i.e., Given

$$\begin{array}{ccc} (X,<) & \longrightarrow & (X,R) \\ \downarrow & & \downarrow \\ (X,T_<) = (X, \mathcal{L}(F_R)) & \longleftarrow & (X,F_R) \end{array}$$

<u>Proof</u>: Trivial.

Also it can be proved that for this trivial fuzzy order, the fuzzy ordered fuzzy topology is the same as the fuzzy topology defined by taking all lsc functions on X, with regard to the order topology $T_<$ on X.

2.1.11 <u>Result</u>

Given an order $<$ on X and let it determine a fuzzy order as in [2.1.10]. Let F be the fuzzy topology on X defined by taking all lsc functions on X. Then (X,F) coincides with (X, F_R).

i.e.,

Given

$$\begin{array}{ccccc} (X,<) & & \longrightarrow & & (X,R) \\ \downarrow & & & & \downarrow \\ (X,T_<) & \longrightarrow & (X,F) & = & (X,F_R) \end{array}$$

<u>Proof</u>: From [2.1. 9] we get $F \subseteq F_R$

Now, to prove $F_R \subseteq F$.

F = all lsc functions on $(X, T_<)$

= all lsc functions on $(X, \mathcal{L}(F_R))$; since $T_< = \mathcal{L}(F_R)$

by [2.1.10]

$\mathcal{L}(F_R)$ is the smallest topology such that all functions in F_R are lsc and F is the set of all lsc functions with regard to $\mathcal{L}(F_R)$.

Therefore, $F_R \subseteq F$

Hence equality.

2.2 Fuzzy Interval Topologies

Interval topologies have been defined in several ways in the crisp case. In this section, we recall some of them and propose the corresponding fuzzy translations. However, in our later discussions we pursue only the fuzzy ordered fuzzy topology that has been introduced in [2.1.3].

2.2.1 Definition [Bi]

The interval topology on a partially ordered set X with bounds 0 and 1, is defined by taking all closed intervals [a,b] as a subbasis for closed sets.

2.2.2 Definition [Ga]

Let X be a linearly ordered set. The family of all intervals $[a,b)$; $a,b \in X$ and $[a, +\infty)$; $a \in X$ is a base for open sets of a topology called the right half open interval topology. Similarly the left half open interval topology has a base consisting of the left half open intervals $(a,b]$; $a,b \in X$, $(-\infty,b]$; $b \in X$.

2.2.3. Remark

The half open interval topologies are finer than the order topology with subbasis consisting of the intervals of the form $\{x \mid x>a\}$ and $\{x \mid x<a\}$; $a \in X$; and the order topology is finer than the interval topology given in (2.2.1).

For,

the open interval (a,b) can be expressed in the form,

$$(a,b) = \bigcup\{[\alpha,b) \mid a < \alpha < b\}$$
$$\bigcup\{(a,\beta] \mid a < \beta < b\}$$

Therefore, the half open interval topologies are finer than the order topology.

Also, all closed intervals [a,b] form a subbasis for the interval topology.

Now, the complement of $[a,b] = (-\infty,a) \cup (b,\infty)$. Therefore, the order topology is finer than the interval topology.

We now propose the fuzzy translations of these interval topologies.

The following definition is the fuzzy translation of the interval topology given in (2.2.1).

2.2.5 Definition

Let (X,R) be a foset, with bounds 0 and 1. The fuzzy interval topology of X is defined by taking $\{1-\mu_a,\ 1-{}_a\mu \mid a \in X\}$ as subbasis for fuzzy closed sets, where $\mu_a(x) = R(a,x)$, ${}_a\mu(x) = R(x,a)$; $x \in X$.

2.2.6 Remark

This definition of fuzzy interval topology is a special case of the definition of fuzzy interval topology proposed by P. Venugopalan as given below.

2.2.7 Definition [Ve]

Let (X,R) be a foset. The fuzzy interval topology on X is defined to be the fuzzy topology generated by the fuzzy sets $X \setminus {\downarrow}e$, $X \setminus {\uparrow}d$ as subbasic open sets. Where,

$$\downarrow e(y) = [R(y,x) + \lambda - 1] \vee 0$$

$$\uparrow d(y) = [R(x,y) + \mu - 1] \vee 0$$

where $e = x_\lambda$, $d = x_\mu$ are fuzzy points of X.

Note that when $\lambda = \mu = 1$, then, definitions (2.2.5) and (2.2.7) are the same.

The half open fuzzy interval topologies are defined now.

2.2.8 <u>Definition</u>

Let (X,R) be a foset. The fuzzy topology generated by $\{1-{}_a\mu,\ 1-\mu_a-\chi_a \mid a \in X\}$ is called the right half open fuzzy interval topology on X. Similarly, the left half open fuzzy interval topology is generated by

$$\{1-{}_a\mu-\chi_a\ ,\ 1-\mu_a \mid a \in X\}\ .$$

CHAPTER 3

PRODUCT AND QUOTIENT SPACES

3.0 Introduction

In this chapter we discuss product of fuzzy orders first. It is proved that the product of two fuzzy orders is again a fuzzy order if the orders are strong. Also it is shown that the induced crisp order of the product of strong fuzzy orders is the same as the product of the crisp orders induced by the strong fuzzy orders.

Secondly, let '$\sim$' be an equivalence relation defined on a foset (X,R). Then the quotient space with regard to the equivalence relation $\sim$ is made a fuzzy ordered set, provided R is compatible with $\sim$. Also various quotient maps are analysed.

Finally union and intersection of fuzzy orders are discussed and fuzzy order defined on a subset is considered.

3.1. Product of fuzzy orders

Let us recall the definition of a strong fuzzy order. Any fuzzy relation R defined on X is a strong fuzzy order if it is irreflexive, i.e., $R(x,x) = 0$ for every $x \in X$,

perfectly antisymmetric, i.e., for $x \neq y$,
$R(x,y) > 0 \Longrightarrow R(y,x) = 0$ and transitive. i.e.,
$R(x,y) \geq \bigvee_{z} [R(x,z) \wedge R(z,y)]$.

The product of fuzzy binary relations is defined as follows.

3.1.1 Definition

Let R_α be a fuzzy relation in X_α for each $\alpha \in \Lambda$. Then, the product of fuzzy relations R_α, $\prod R_\alpha$ is defined as the relation on $\prod X_\alpha$ by

$$\prod R_\alpha(\bar{x},\bar{y}) = \inf_{\alpha} R_\alpha(x_\alpha, y_\alpha)$$

where, $\bar{x} = (x_\alpha)_{\alpha \in \Lambda}$ and $\bar{y} = (y_\alpha)_{\alpha \in \Lambda}$

3.1.2 Remark

Note that this will in some sense coincide with the definition of the fuzzy subset $\prod R_\alpha$ of $\prod (X_\alpha \times X_\alpha)$ when the product $\prod \mu_\alpha$ of subsets μ_α of X_α is defined as $\prod \mu_\alpha(\bar{x}) = \bigwedge_{\alpha} \mu_\alpha(x_\alpha)$.

Throughout this section we let α vary in the indexed set Λ.

The following result shows that the product of strong fuzzy orders is also a strong fuzzy order.

3.1.3 Result

$\prod R_\alpha$ is a strong fuzzy order on $\prod X_\alpha$, where each R_α is a strong fuzzy order defined on X_α.

Proof:

i) $\prod R_\alpha$ is irreflexive

as $\prod R_\alpha(\bar{x},\bar{x}) = \bigwedge_\alpha R_\alpha(x_\alpha, x_\alpha) = 0$, for every $\bar{x} = (x_\alpha)_{\alpha\in\Lambda}$ element of $\prod X_\alpha$.

ii) $\prod R_\alpha$ is perfectly antisymmetric

For;

for $\bar{x} \neq \bar{y}$ in $\prod X_\alpha$

if $\prod R_\alpha(\bar{x},\bar{y}) > 0$

i.e., $R_\alpha(x_\alpha,y_\alpha) > 0$, for every α.

then $R_\alpha(y_\alpha,x_\alpha) = 0$, for every α

i.e., $\prod R_\alpha(\bar{y}, \bar{x}) = 0$

Therefore, $\prod R_\alpha(\bar{x},\bar{y}) > 0$ implies $\prod R_\alpha(\bar{y},\bar{x}) = 0$.

iii) $\prod R_\alpha$ is transitive, as,

$$\bigvee_{\bar{z}} [\prod R_\alpha(\bar{x},\bar{z}) \wedge \prod R_\alpha(\bar{z},\bar{y})]$$

$$= \bigvee_{\bar{z}} [\bigwedge_\alpha R_\alpha(x_\alpha,z_\alpha) \wedge \bigwedge_\beta R_\beta(z_\beta,y_\beta)]$$

$$= \bigvee_{\bar{z}} [\bigwedge_\alpha \{R_\alpha(x_\alpha,z_\alpha) \wedge R_\alpha(z_\alpha,y_\alpha)\}]$$

$$\leqslant \bigwedge_\alpha [\bigvee_z R_\alpha(x_\alpha,z_\alpha) \wedge R_\alpha(z_\alpha,y_\alpha)]$$

$$\leqslant \bigwedge_\alpha R_\alpha(x_\alpha,y_\alpha)$$

$$= \prod R_\alpha(\bar{x},\bar{y})$$

$$\therefore \prod R_\alpha(\bar{x},\bar{y}) \geqslant \bigvee_{\bar{z}} [\prod R_\alpha(\bar{x},\bar{z}) \wedge \prod R_\alpha(\bar{z},\bar{y})]$$

$\bar{x} = (x_\alpha)_{\alpha\in\Lambda}$, $\bar{y} = (y_\alpha)_{\alpha\in\Lambda}$ and $\bar{z} = (z_\alpha)_{\alpha\in\Lambda}$ are elements of $\prod X_\alpha$.

3.1.4 <u>Remark</u>

However, if R and S are just two fuzzy orders then R x S need not be a fuzzy order.

3.1.5 Example

Let R and S be two fuzzy orders defined on $X = \{A,B\}$ and $Y = \{a,b\}$ respectively as given below.

Table-1

R	A	B
A	0	0.8
B	0.2	0

S	a	b
a	0	0.2
b	0.8	0

R x S is defined on X x Y as

Table-2

RxS	(A,a)	(A,b)	(B,a)	(B,b)
(A,a)	0	0	0	0.2
(A,b)	0	0	0.8	0
(B,a)	0	0.2	0	0
(B,b)	0.2	0	0	0

Here (R x S) is not antisymmetric, as

$(R \times S)((A,a),(B,b)) = 0.2$

and $(R \times S)((B,b),(A,a)) = 0.2$.

In the following result we prove that product of crisp orders induced by fuzzy orders is the same as the induced crisp order of the product of fuzzy orders.

3.1.6 <u>Result</u>

Let R_α be a strong fuzzy order defined on X_α. Let $<_{R_\alpha}$ be the crisp order induced by R_α on X_α as in [2.1.2] and let the product fuzzy order $\prod R_\alpha$ induce $<_{\prod R_\alpha}$ or $\prod X_\alpha$ by

$$\bar{x} <_{\prod R_\alpha} \bar{y} \quad \text{iff} \quad \prod R_\alpha(\bar{x},\bar{y}) > \prod R_\alpha(\bar{y},\bar{x})$$

where $\bar{x} = (x_\alpha)_{\alpha \in \Lambda}$ and $\bar{y} = (y_\alpha)_{\alpha \in \Lambda}$ are points of $\prod X_\alpha$.

Then, $\quad <_{\prod R_\alpha} = \prod <_{R_\alpha}$.

<u>Proof</u>:

Assume that $\bar{x} <_{\prod R_\alpha} \bar{y}$

Then, $\quad \prod R_\alpha(\bar{x},\bar{y}) > \prod R_\alpha(\bar{y},\bar{x})$

i.e., $\quad \bigwedge_\alpha R_\alpha(x_\alpha,y_\alpha) > \bigwedge_\beta R_\beta(y_\beta,x_\beta)$

$\therefore \quad \bigwedge_\alpha R_\alpha(x_\alpha,y_\alpha) > 0$ in which case

each $\quad R_\alpha(x_\alpha,y_\alpha) > 0$ so that $R_\alpha(y_\alpha,x_\alpha) = 0$

i.e., $\quad R_\alpha(x_\alpha,y_\alpha) > R_\alpha(y_\alpha,x_\alpha)$ for every α.

i.e., $x_\alpha <_{R_\alpha} y_\alpha$ for every α.

$\therefore$ $\bar{x} \; \Pi <_{R_\alpha} \bar{y}$.

Retracing the steps we have

$\bar{x} \; \Pi <_{R_\alpha} \bar{y}$ implies $\bar{x} <_{\Pi R_\alpha} \bar{y}$. Hence equality.

3.2 Quotient Spaces

Let (X,R) be foset and '$\sim$' be an equivalence relation defined on X (i.e., $\sim$ is reflexive, i.e., $x \sim x$ for every $x \in X$, $\sim$ is symmetric, i.e., $x \sim y \Rightarrow y \sim x$, and $\sim$ is transitive. i.e, $x \sim y$ and $y \sim z \Rightarrow x \sim z$, $x,y,z \in X$). First we forward a necessary and sufficient condition for the compatibility of R with $\sim$ and make the quotient space $X/\sim$, fuzzy ordered.

3.2.1 Definition

Let (X,R) be a foset in which an equivalence relation '$\sim$' is defined. Then R is said to be compatible with $\sim$ iff $a \sim c$ and $b \sim d \Rightarrow R(a,b) = R(c,d)$, a,b,c,and $d \in X$.

3.2.2 Result

Let X be a nonempty set and '$\sim$' be an equivalence relation defined on X. Let $X/\sim$ be the quotient space consisting of the equivalence classes [x] for every $x \in X$, where $[x] = \{y \in X \mid x \sim y\}$. Let R be a fuzzy order defined on X compatible with $\sim$. Define a binary fuzzy relation $\underset{\sim}{R} : X/\sim \times X/\sim \longrightarrow [0,1]$ as

$$\underset{\sim}{R}([a],[b]) = R(a,b) \text{ for every } [a],[b] \in X/\sim .$$

Then $\underset{\sim}{R}$ is a fuzzy order.

Proof: Trivial.

Let this fuzzy order $\underset{\sim}{R}$ induce the crisp order $\lessdot$ on $X/\sim$

$$\text{as } [a[\lessdot [b] \text{ iff } \underset{\sim}{R}([a],[b]) > \underset{\sim}{R}([b],[a])$$
$$\text{for every } [a], [b] \in X/\sim$$

Then the following results follow immediately.

3.2.3 Result

The quotient map $q: X \longrightarrow X/\sim$ is order preserving both ways (Order preserving maps will be discussed in detail in Chapter 5).

Proof:

$$[a] \lessdot [c] \iff \underset{\sim}{R}([a],[c] > \underset{\sim}{R}([c],[a])$$
$$\iff R(a,c) > R(c,a)$$
$$\iff a < c.$$

Hence, q is order preserving both ways.

3.2.4 Result

Let $T_{<}$ and $T_{\underset{\sim}{<}}$ be the crisp order topologies induced by R and $\underset{\sim}{R}$ on X and $X/\sim$ respectively. Then $q: (X, T_{<}) \longrightarrow (X/\sim, T_{\underset{\sim}{<}})$ is a quotient map. [A surjective map $f: (X, T_1) \longrightarrow (Y, T_2)$ is a quotient map if $f^{-1}(V) \in T_1$ iff $V \in T_2$].

Proof:

Let $q^{-1}(A) \in T_{<}$ for some $A \subset X/\sim$.

Then $q^{-1}(A) = \bigcup A_i$ where A_i is a basic open set in $(X, T_{<})$.

Now,

$$A = q(\bigcup A_i) = \bigcup q(A_i)$$

$$A_i = (a_i, b_i) = \{x \in X \mid a_i < x < b_i\}$$

then $a_i < x$ and $x < b_i$

so $R(a_i, x) > R(x, a_i)$ and

$R(x, b_i) > R(b_i, x)$

i.e., $\underset{\sim}{R}([a_i],[x]) > \underset{\sim}{R}([x],[a_i])$ and

$\underset{\sim}{R}([x],[b_i] > \underset{\sim}{R}([b_i],[x])$

i.e., $[a_i] \underset{\sim}{<} [x]$ and $[x] \underset{\sim}{<} [b_i]$

i.e., $[a_i] \underset{\sim}{<} [x] \underset{\sim}{<} [b_i]$.

i.e., $q(A_i) \in T_{\underset{\sim}{<}}$

$\therefore$ $A = \bigcup q(A_i) \in T_{\underset{\sim}{<}}$

Hence q is a quotient map.

3.2.5 <u>Definition</u>

Let (X,F) be a fts and $q:(X,F) \longrightarrow X/\sim$, be a quotient map. Then the fuzzy topology $\underset{\sim}{F}$ on $X/\sim$ is defined by $\mu \in \underset{\sim}{F}$ iff $q^{-1}(\mu)$ is open in (X,F) is called the quotient fuzzy topology and $(X/\sim, \underset{\sim}{F})$ is called a quotient fuzzy space.

If $\underset{\sim}{F} = F_{\underset{\sim}{R}}$ then $(X/\sim, F_{\underset{\sim}{R}})$ is called the fuzzy ordered quotient fuzzy topological space.

3.2.6 <u>Result</u>

Let (X,F_R) be a fuzzy ordered fuzzy topological space and $(X/\sim, F_{\underset{\sim}{R}})$ be a fuzzy ordered quotient fuzzy topological space. Then $q: (X,F_R) \longrightarrow (X/\sim, F_{\underset{\sim}{R}})$ is a quotient map.

<u>Proof</u>:

Let δ be a fuzzy set in $X/\sim$ and $q^{-1}(\delta)$ be open in (X,F_R). Then $q^{-1}(\delta) = \cup\mu_i$, where μ_i's are basic open sets in (X,F_R).

Now $\delta = q(\cup\mu_i) = \cup q(\mu_i)$

$q(\mu_i)$ is open in $(X/\sim, F_{\underset{\sim}{R}})$, since $q^{-1}(q(\mu_i)) = \mu_i$ is open in (X,F_R).

3.2.7 Result

Let $\mathcal{L}(F_R)$ and $\mathcal{L}(F_{\underset{\sim}{R}})$ be the associated topologies of F_R and $F_{\underset{\sim}{R}}$, then

$$q: (X, \mathcal{L}(F_R)) \longrightarrow (X/\sim, \mathcal{L}(F_{\underset{\sim}{R}})) \text{ is a quotient map.}$$

Proof:

Let A be any subset of $X/\sim$ such that

$q^{-1}(A)$ is open in $(X, \mathcal{L}(F_R))$.

Then, $q^{-1}(A) = \bigcup f_i^{-1}(\alpha_i, 1]$

Now, $A = q(\bigcup f_i^{-1}(\alpha_i, 1]) = \bigcup q(f_i^{-1}(\alpha_i, 1])$ is open in $(X/\sim\ , \mathcal{L}(F_{\underset{\sim}{R}}))$ since $q^{-1}q(f_i^{-1}(\alpha, 1]) = f_i^{-1}(\alpha, 1]$ is open in $(X, \mathcal{L}(F_R))$.

Hence q is a quotient map.

3.3 Union and intersection of fuzzy orders

In [Cha] union of two perfect antisymmetric orders R_1 and R_2 has been found to be perfectly antisymmetric if and only if for every $x, y \in X$, $x \neq y$ either $R_1(x,y) = R_2(y,x) = 0$ or $R_1(x,y) \neq R_2(y,x)$.

Also, it has been observed that if R_1 and R_2 are perfect antisymmetries then $R_1 \cup R_2$ is also a perfect antisymmetry if and only if

$$R_i(x,y) > 0 \implies R_j(y,x) = 0; \quad i,j = 1,2.$$

In the following result union of strong fuzzy orders is proved to be a strong fuzzy order if

$$R_i(x,y) \wedge R_j(y,x) = 0; \quad \begin{array}{l} i = 1,2,\ldots,n \\ j = 1,2,\ldots,n, \ i \neq j \text{ and} \\ \qquad\qquad\qquad x \neq y \end{array}$$

3.3.1 Result

Let R_i, $i \in \Lambda$ be a strong fuzzy order defined on X. Then,

$\cup R_i$ is a strong fuzzy order if

$$R_i(x,y) \wedge R_j(y,x) = 0 \qquad \begin{array}{l} i \in \Lambda \\ j \in \Lambda \end{array}, \ i \neq j, \ x \neq y$$

Proof:

i) $\cup R_i(x,x) = \bigvee_i R_i(x,x) = 0$ i.e., irreflexive

ii)
$$\begin{aligned} \cup R_i(x,y) > 0 &\implies \bigvee_i R_i(x,y) > 0 \\ &\implies \bigvee_i R_i(y,x) = 0 \\ &\implies \cup R_i(y,x) = 0 \end{aligned}$$

i.e., $\cup R_i$ is perfectly antisymmetric.

iii) $\bigcup R_i(x,z) \; = \; \bigvee_i R_i(x,z)$

$$\geqslant \bigvee_i \left[\bigvee_y (R_i(x,y) \wedge R_i(y,z)) \right]$$

$$= \bigvee_y \left[\bigcup_i R_i(x,y) \wedge \bigcup_i R_i(y,z) \right] \text{ by hypothesis}$$

i.e., $\bigcup R_i(x,z) \; \geqslant \; \bigvee_y \left[\bigcup R_i(x,y) \wedge \bigcup R_i(y,z) \right]$

$\therefore$ $\bigcup R_i$ is transitive and hence $\bigcup R_i$ is a strong fuzzy order.

3.3.2 Remark

Intersection of $\bigcap R_i$; $i \in \Lambda$ of strong fuzzy orders R_i, $i \in \Lambda$ on X is defined by $\bigcap R_i(x,y) = \bigwedge_{i \in \Lambda} R_i(x,y)$.

3.3.3 Remark

Intersection of fuzzy orders is a subspace of $\prod R_\alpha$ which has been introduced in (3.1.1).

Also, the intersection of two fuzzy orders need not be a fuzzy order if they are not strong. For example, see (3.1.5).

3.4 Fuzzy Ordered Subspaces

We consider a crisp subset of a foset and analyse the induced fuzzy order in it.

3.4.1 <u>Result</u>

Let (X,R) be a foset and Y be a subset of X. Define R_Y on Y as $R_Y(y_1,y_2) = R(y_1\ y_2)$ for every $y_1,y_2 \in Y$. Then R_Y is a fuzzy order on Y.

<u>Proof</u>: Obvious

3.4.2 <u>Result</u>

Let $Y \subset X$. Let (X,F_R) and (Y,F_{R_Y}) be the fuzzy ordered fuzzy topological spaces defined by R and R_Y on X and Y respectively. Then, $F_{R_Y} = F_R \wedge Y$.

<u>Proof</u>:

Let $\mu_{a_Y} \in F_{R_Y}$

Then $\mu_{a_Y}(y_1) = R_Y(a,y_1)$

$= R\ (a,y_1)$

$= \mu_a(y_1)$

$= \mu_a(y_1) \wedge Y(y_1)$ since $y_1 \in Y$, $Y(y_1)=1$

$\in F_R \wedge Y$

Conversely,

Let $\mu_a \in F_R \wedge Y$

then $\mu_a(x) = R(a,x) \wedge Y(x)$

$= R_Y(a,x)$ if $x \in Y$

if $x \notin Y$, $\mu_a(x) = 0 = R_Y(a,x) \wedge Y(x)$

$\therefore\ F_{R_Y} = F_R \wedge Y.$

3.4.3 <u>Result</u>

Let $Y \subset X$ and let $(X, \ell(F_R))$ and $(Y, \ell(F_{R_Y}))$ be the associated topologies of F_R and F_{R_Y} on X and Y. Then

$$\ell(F_{R_Y}) = \ell(F_R) \cap Y.$$

<u>Proof</u>

Let $\mu_{a_Y}^{-1}(\alpha,1] \in \ell(F_{R_Y})$

$$\begin{aligned}
\mu_{a_Y}^{-1}(\alpha,1] &= \{y \in Y \mid \mu_{a_Y}(y) > \alpha\} \\
&= \{y \in Y \mid R_Y(a,y) > \alpha\} \\
&= \{y \in Y \mid R(a,y) > \alpha\} \\
&= \{y \in Y \mid \mu_a(y) > \alpha\} \\
&= \mu_a^{-1}(\alpha,1] \cap Y
\end{aligned}$$

CHAPTER 4

SOME SEPARATION PROPERTIES OF THE FUZZY ORDERED FUZZY TOPOLOGICAL SPACE

4.0 Introduction

Several definitions of various separation axioms have been proposed and studied by many authors, from different perspectives, [RO_3], [Sinh], [Sri-L]. Among these separation properties, fuzzy Hausdorff separation axiom has been studied most. Many characterizations are formulated in terms of fuzzy points. However, fuzzy sets are also used for describing separation axioms as well [Kat]. In this chapter, we recall a few such definitions and study some separation properties of the fuzzy ordered fuzzy topological space.

4.1. Fuzzy T_1 spaces

4.1.1 Definition [Kat]

A fuzzy topological space (X,F) is fuzzy T_1 if for any two distinct points $x,y \in X$, there exist fuzzy neighbourhoods β and δ in F such that

$$\beta(y) = 0, \qquad \beta(x) > 0$$

$$\delta(x) = 0 \quad \text{and} \quad \delta(y) > 0.$$

In the following result we prove that the fuzzy ordered fuzzy topological space is fuzzy T_1.

4.1.2 <u>Result</u>

The fuzzy ordered fuzzy topological space (X,F_R) is fuzzy T_1.

<u>Proof</u>:

Let $x \neq y$ be two points of X.

$$\text{Define } \beta = ({}_y\mu) \vee (1 - {}_y\mu - \chi_y)$$

$$\delta = ({}_x\mu) \vee (1 - {}_x\mu - \chi_x)$$

then β and δ are in F_R and

$$\beta(x) = {}_y\mu(x) \vee 1 - {}_y\mu(x) - \chi_y(x) > 0$$

$$\beta(y) = {}_y\mu(y) \vee 1 - {}_y\mu(y) - \chi_y(y) = 0$$

$$\delta(x) = {}_x\mu(x) \vee 1 - {}_x\mu(x) - \chi_x(x) = 0$$

$$\delta(y) = {}_x\mu(y) \vee 1 - {}_x\mu(y) - \chi_x(y) > 0$$

Hence (X,F_R) is fuzzy T_1.

4.2 Fuzzy Hausdorff Spaces

We recall the definition of fuzzy Hausdorfness as

4.2.1 Definition [Kat]

A fuzzy topological space (X,F) is fuzzy Hausdorff if for any two distinct points $x,y \in X$, there exist fuzzy neighbourhoods β and δ in F such that

$$\beta(x) > 0, \quad \delta(y) > 0 \text{ and } \beta \wedge \delta = 0.$$

4.2.2 Result

The fuzzy ordered fuzzy topological space (X,F_R), when R is a strong fuzzy order on X, is fuzzy Hausdorff.

Proof:

Let (X,F_R) be a strong fuzzy ordered fuzzy topological space. By [2.1.2], R induces a crisp total order $\prec$ on X by $x \prec y$ iff $R(x,y) > R(y,x)$.

Case-1:

There exists an $z \in X$ such that $x \prec z \prec y$. Define $\beta = {}_z\mu$ and $\delta = \mu_z$, then $\beta,\delta \in F_R$ and

$$\beta(x) = {}_z\mu(x) = R(x,z) > R(z,x) = 0,$$ since $x \prec z$ and R is perfectly antisymmetric.

i.e., $\beta(x) > 0$.

Similarly $\delta(y) = \mu_z(y) = R(z,y) > R(y,z) = 0$

Since $z \prec y$ and R is perfectly antisymmetric

i.e., $\delta(y) > 0$

Also $\beta \wedge \delta = 0$

For; if there exists $a \in X$ such that $(\beta\wedge\delta)(a) \neq 0$;

then, ${}_z\mu(a) \wedge \mu_z(a) \neq 0$

But ${}_z\mu(a)$ and $\mu_z(a)$ cannot be non zero simultaneously as R is perfectly antisymmetric.

Therefore $\beta\wedge\delta = 0$.

Hence (X,F_R) is fuzzy Hausdorff.

<u>Case-2</u>

There does not exist any z in X such that $x \prec z \prec y$.

Here we take $\beta = {}_y\mu$ and $\delta = \mu_x$.

Then,

$$\beta(x) = {}_y\mu(x) = R(x,y) > R(y,x)$$

$\therefore$ $\beta(x) > 0$.

Similarly, $\delta(y) > 0$.

Also, $\beta \wedge \delta = 0$. i.e., $\beta(t) \wedge \delta(t) = 0$ for every $t \in X$.

For, if there exists an $a \in X$ such that

$\beta(a)$ and $\delta(a) > 0$; then,

$R(a,y) > 0$ and $R(x,a) > 0$.

implies $R(y,a) = 0$ and $R(a,x) = 0$.

i.e., $R(x,a) > R(a,x)$ and $R(a,y) > R(y,x)$.

i.e., $x \prec a \prec y$ which is a contradiction

Hence $\beta \wedge \delta = 0$.

Therefore (X, F_R), R being a strong fuzzy order is fuzzy Hausdorff.

4.2.3 <u>Note</u>

In fact if R is not a strong fuzzy order then (X, F_R) need not be fuzzy Hausdorff.

4.2.4 <u>Example</u>

Let $X = \mathbb{N} \cup \{0, -1\}$ and let

$R : X \times X \longrightarrow [0,1]$ be defined by

$R(n,o) = 1$ $\quad R(o,n) = 0$ for every $n=2m+1$, $m=0,1,2,\ldots$

$R(o,2m) = 1$ $\quad R(2m,o) = 0$

$R(2m, 2n+1) = 0 \quad R(2n+1,2m) = \frac{1}{2}$ for every $m=1,2,\ldots$

$R(2m+1,2n+1) = 0 \quad R(2n+1,2m+1) = \frac{1}{2}$ for every $n<m$.

$R(2n,2m) = 0, \quad R(2m,2n) = \frac{1}{2}$ for every $n<m$

$R(0,0) = R(-1,-1) = 0$

$R(m,m) = 0$ for every m.

$R(n,-1) = 1 \quad R(-1,n) = 0$ for every $n=2m+1$, $m=0,1,\ldots,$

$R(2m,-1) = 0 \quad R(-1,2m) = 1$ for every m

$R(0,-1) = \frac{1}{4} \quad R(-1,0) = \frac{1}{2}$

Then R is clearly a fuzzy order which is not strong. We claim that 0 and -1 cannot be separated. For; if we take any subbasic open sets α and β with $\alpha(0) > 0$ and $\beta(-1)>0$ in F_R, then they cannot be disjoint. For example, let μ_a and μ_b be the subbasic open sets in F_R with $\mu_a(-1) > 0$ and $\mu_b(0) > 0$.

Then, $\mu_a(-1) > 0 \quad \mu_b(0) > 0$

i.e., $\mu_a(-1) = R(a,-1) > 0$ and $\mu_b(0) = R(b,0) > 0$

i.e., $a = 2m$ or 0 and $b = -1$.

Here, $(\mu_{2m} \wedge \mu_{-1})(x) \neq 0$ and $(\mu_o \wedge \mu_{-1})(x) \neq 0$ for every x.

Similarly it can be verified that any pair of subbasic open sets containing -1 and 0 respectively, will not be disjoint. Hence (X,F_R) is not fuzzy Hausdorff.

In the definition [4.1] of fuzzy Hausdorffness if we relax the condition $\beta \wedge \delta = 0$ and put $\beta \wedge \delta < 1$ then we call (X,F) as weak fuzzy Hausdorff.

4.2.5 Definition

A fts (X,F) is weak fuzzy Hausdorff if for any two distinct points $x,y \in X$, there exist fuzzy neighbourhoods β and δ in F such that

$$\beta(x) > 0, \quad \delta(y) > 0 \text{ and } \beta \wedge \delta < 1.$$

In the following result we find that (X,F_R) is weak fuzzy Hausdorff, R being any fuzzy order.

4.2.6 Result

(X,F_R) is weak fuzzy Hausdorff, R being any fuzzy order defined on X.

Proof:

Let $x \neq y$ be two points of X, and let R induce $\prec$ on X as in [2.1.1].

Case-1

There exists $z \in X$ such that $x \prec z \prec y$.

Define $\beta = {}_z\mu$ and $\delta = \mu_z$,

then β and δ belong to F_R.

Also, $\beta(x) = {}_z\mu(x) = R(x,z) > 0$ as $x \prec z$

$\delta(y) = \mu_z(y) = R(z,y) > 0$ as $z \prec y$

and $\beta \wedge \delta < 1$

For, if $(\beta \wedge \delta)(x) = 1$, then $\beta(x) \wedge \delta(x) = 1$

i.e., $R(x,z) = R(z,x) = 1$.

But R is antisymmetric. So $R(x,z) \neq R(z,x)$.

Hence (X, F_R) is weak fuzzy Hausdorff.

Case-2

There does not exist any $z \in X$ such that $x \prec z \prec y$

Take $\beta = 1 - \mu_y - \chi_y$ and $\delta = \mu_x$.

Then, $\beta(x) = 1 - \mu_y(x) - \chi_y(x)$

$= 1 - R(y,x) - 0 > 0$, since if $R(y,x)$ were 1 then $R(y,x) > R(x,y)$ and this would lead to $y \prec x$.

Also $\delta(y) = \mu_x(y) = R(x,y) > 0$.

Now $(\beta \wedge \delta) < 1$

For, if not,

$$(1 - \mu_y(t) - \chi_y(t)) \wedge \mu_x(t) = 1$$

implies $1 - \mu_y(t) - \chi_y(t) = 1$ and $\mu_x(t) = 1$.

i.e., $\mu_y(t) = 0$ and $\mu_x(t) = 1$

i.e., $R(y,t) = 0$ and $R(x,t) = 1$.

Now $R(t,x) = 0$ and $R(t,y) > 0$.

imply that $R(x,t) > R(t,x)$ and $R(t,y) > R(y,t)$

i.e., $x \prec t \prec y$

which is a contradiction to the fact that there does not exist any z such that $x \prec z \prec y$. Therefore, (X,F_R) is weak fuzzy Hausdorff.

CHAPTER 5

FUZZY ORDER PRESERVING MAPS*

5.0 Introduction

This chapter deals with fuzzy order preserving maps and strict fuzzy order preserving maps, which we define, from a fuzzy ordered fuzzy topological space (X,F_R) to another fuzzy ordered fuzzy topological space (Y,F_S). It is proved that a bijective strict fuzzy order preserving map from (X,F_R) to (Y,F_S) is a fuzzy homeomorphism. However, a counter example is provided to show that a fuzzy order preserving map (not strict) is not necessarily fuzzy continuous, even if it is bijective.

Also, Y^X– the set of all maps from a set X to Y, when Y is a strong fuzzy ordered set is made a strong fuzzy ordered set. Besides, certain homomorphisms between the monoids $\hat{Y}$, $\widehat{\widehat{(X,Y)}}$ and between $\hat{X} \times \hat{Y}$, $\widehat{\widehat{(X,Y)}}$ are studied, where $\hat{Z}$ denotes the set of all fuzzy order preserving maps from Z to Z.

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* Many of the results given in this chapter appeared in The Journal of Fuzzy Mathematics, Los Angeles, Vol.1, No.2 (1993) pp.359-365.

5.1 Fuzzy order preserving maps

The definition of order preserving map in the crisp sense is recalled first.

5.1.1 Definition

A map f: $(X,<_1) \longrightarrow (Y,<_2)$ is order preserving iff $x_1 <_1 x_2$ implies $f(x_1) <_2 f(x_2)$, for every $x_1,x_2 \in X$.

5.1.2 Note

Every surjective order preserving map is continuous in the order topology.

Proof

Let $f:X \longrightarrow Y$ is surjective and order preserving and $x \in X$ and (y_1,y_2) is an open set containing f(x). Then choose x_1 from $f^{-1}(y_1)$ and x_2 from $f^{-1}(y_2)$. Clearly $f(x_1,x_2) \subset (y_1,y_2)$ and $x \in (x_1,x_2)$.

However, the converse is not true. Also, if $f:(X, <_1) \longrightarrow (Y,<_2)$ is a bijective continuous map, then f is a homeomorphism. In particular, if f is a bijective strict order preserving map then f is a homeomorphism.

Proof

Let $y = f(x)$ and let (a,b) be an open interval containing x. Put $c = f(a)$ and $d = f(b)$. Then we have $c < y < d$. Similarly if $m \in (c,d)$ then $n = f^{-1}(m) \in (a,b)$. Hence (c,d) is an open set which contains y and its image $f^{-1}((c,d))$ is a subset of (a,b). Therefore f^{-1} is continuous at y.

Hence f is a homeomorphism.

Let us analyse the corresponding results in the fuzzy setting. First we define fuzzy order preserving maps in two ways.

5.1.3 Definition

Let (X,F_R) and (Y,F_S) be two fuzzy ordered fuzzy topological spaces. A map $\alpha : X \longrightarrow Y$ is fuzzy order preserving iff

$$R(x_1,x_2) > R(x_2,x_1)$$

$$\Longrightarrow S(\alpha(x_1), \alpha(x_2)) \geq S(\alpha(x_2),\alpha(x_1))$$

and strict fuzzy order preserving iff

$$R(x_1,x_2) = S(\alpha(x_1),\alpha(x_2)), \ x_1,x_2 \in X.$$

5.1.4 Result

A strict fuzzy order preserving surjective map is fuzzy continuous with respect to the fuzzy ordered fuzzy topologies.

Proof:

Let α: $(X,R) \longrightarrow (Y,S)$ be a strict fuzzy order preserving surjective map.

Let δ_{y_o} be a subbasic open set in (Y,F_S)

$$\alpha^{-1}(\delta_{y_o}) = \delta_{y_o} \circ \alpha$$

$$(\delta_{y_o} \circ \alpha)(x) = \delta_{y_o}(\alpha(x)) = S(y_o, \alpha(x))$$

$$= R(x_o,x), \text{ where } y_o = \alpha(x_o)$$

$$= \mu_{x_o}(x), \text{ where } \mu_{x_o} \text{ is a subbasic}$$

open set in F_R.

i.e., $\alpha^{-1}(\delta_{y_o})$ is open.

Similarly, for the other subbasic open sets.

Therefore, α is fuzzy continuous.

5.1.5 Note

A bijective fuzzy order preserving map (not strict) need not be fuzzy continuous.

We have the following counter example.

5.1.6 Example

Let $X = \mathbb{N}$ and $Y = 2\mathbb{N}$

Let the fuzzy orders R and S be defined on X and Y respectively as follows:

$$R : (a,a) \longmapsto 0$$
$$(a,b) \longmapsto \tfrac{1}{2} \text{ if } a \mid b;\ a \neq b$$
$$(a,b) \longmapsto \tfrac{1}{4} \text{ if } a \nmid b$$

$$S : (a,a) \longmapsto 0$$
$$(a,b) \longmapsto \tfrac{1}{2} \text{ if } 0 < (b-a) \leqslant 2 \text{ i.e., } b-a = 2$$
$$(a,b) \longmapsto 1 \text{ if } (b-a) > 2$$
$$(a,b) \longmapsto 0 \text{ if } (b-a) < 0$$

Let $\alpha : (X, F_R) \longrightarrow (Y, F_S)$ be defined by

$\alpha(x) = 2x$ for every $x \in X$.

We claim that α is a bijective fuzzy order preserving (not strict) map but not fuzzy continuous.

<u>Proof</u>:

Obviously α is bijective.

Now, for $x_1, x_2 \in X$, if $R(x_1,x_2) > R(x_2,x_1)$

then $x_1 | x_2$; $x_1 \neq x_2$.

Thus $2x_2 - 2x_1 > 2$ and $S(2x_1,2x_2) \geqslant S(2x_2,2x_1)$

i.e., $S(\alpha(x_1),\alpha(x_2)) \geqslant S(\alpha(x_2), \alpha(x_1))$

i.e., α is fuzzy order preserving.

In fact, α is not strict.

For, $R(2,4) = \frac{1}{2}$

$S(\alpha(2), \alpha(4)) = S(4,8) = 1$

i.e., $R(2,4) \neq S(\alpha(2), \alpha(4))$.

Here, α is not fuzzy continuous.

For, δ_2 is a subbasic open set in (Y,F_S)

$\delta_2(x) = S(2,x)$

$\alpha^{-1}(\delta_2) = \delta_2 \circ \alpha$

$(\delta_2 \circ \alpha)(x) = \delta_2(\alpha(x))$

$= S(2,2x) = 0$ if $x=1$

$= \frac{1}{2}$ if $1 < x \leq 2$, i.e., $x=2$

$= 1$ if $x > 2$

which is not in F_R.

Therefore, α is not fuzzy continuous.

5.1.7 <u>Result</u>

Let (X,F_R) and (Y,F_S) be two fuzzy ordered fuzzy topological spaces and $\alpha: (X,R) \longrightarrow (Y,S)$ be a bijective strict order preserving map, then α is a fuzzy homeomorphism.

<u>Proof</u>:

By [5.1.4] α is fuzzy continuous.

We will prove that α^{-1} is also fuzzy continuous.

Let μ_{x_o} be a subbasic open set in (X,F_R)

$$(\alpha^{-1})^{-1}\mu_{x_o} = \mu_{x_o} \circ \alpha^{-1}.$$

$$(\mu_{x_o} \circ \alpha^{-1})y = \mu_{x_o}(\alpha^{-1}(y))$$

$$= R(x_o,x) \text{ where } x = \alpha^{-1}(y)$$

But $R(x_o,x)$ $= S(\alpha(x_o), \alpha(x)) = S(y_o,y)$

$= \delta_{y_o}(y)$ where δ_{y_o} is a subbasic open set in F_S.

Similarly for the other subbasic open sets. Therefore, α^{-1} is fuzzy continuous and α is a fuzzy homeomorphism.

5.2 <u>Fuzzy order preserving self maps</u>

In this section fuzzy order preserving self maps are analysed.

The following result envisages the fact that Y^X- the set of all maps from X to Y in which Y is a strong fuzzy ordered set can be strongly fuzzy ordered by virtue of the strong fuzzy order defined on Y.

5.2.1 <u>Result</u>

Let X be any set and (Y,S) be a strong fuzzy ordered set. Let Y^X be the set of all maps from X to Y. Define θ: $Y^X \times Y^X \longrightarrow [0,1]$ as $\theta(f,g) = \bigwedge_x [S(f(x), g(x))]$ for every $f,g \in Y^X$. Then θ is a strong fuzzy order.

<u>Proof</u>:

i) θ is irreflexive as,

$$\theta(f,f) = \bigwedge_x [S(f(x), f(x))] = \bigwedge_x 0 = 0 \text{ for every } f \in Y^X.$$

ii) θ is perfectly antisymmetric.

Let $\Theta(f,g) > 0$,

then $\bigwedge_x [S(f(x),g(x))] > 0$ for every $x \in X$

i.e., $S(f(x),g(x)) > 0$ for every $x \in X$

i.e., $S(g(x),f(x)) = 0$ for every x, as S is perfectly antisymmetric.

i.e., $\bigwedge_x S(g(x),f(x)) = 0$ for every $x \in X$

i.e., $\Theta(g,f) = 0$

Therefore Θ is perfectly antisymmetric.

iii) Θ is transitive

$$S(f(x),g(x)) \geqslant \bigvee_h [S(f(x),h(x)) \wedge S(h(x),g(x))]$$

for every x

implies $\bigwedge_x [S(f(x),g(x))] \geqslant \bigvee_h \{\bigwedge_t S(f(t),h(t))$

$\wedge \bigwedge_s S(h(s),g(s))\}$

as $S(f(x),g(x)) \geqslant \bigvee_h S(f(x),h(x)) \wedge S(h(x),g(x))$

for every x

and $S(f(x),h(x)) \geqslant \bigwedge_t S(f(t),h(t))$

$S(h(x),g(x)) \geqslant \bigwedge_s S(h(s),g(s))$

then,

$$S(f(x),g(x)) \wedge S(h(x),g(x))$$

$$\geqslant \bigwedge_t S(f(t),h(t)) \wedge \bigwedge_s S(h(s),g(s))$$

Therefore,

$$\bigvee_h [S(f(x),h(x)) \wedge S(h(x),g(x))]$$

$$\geqslant \bigvee_h [\bigwedge_t S(f(t),h(t)) \wedge \bigwedge_s S(h(s),g(s))]$$

Therefore,

$$S(f(x),g(x)) \geqslant \bigvee_h [\bigwedge_t S(f(t),h(t)) \wedge \bigwedge_s S(h(s),g(s))]$$

Thus,

$$\theta(f,g) \geqslant \bigvee_h [\theta(f,h) \wedge \theta(h,g)]$$

Therefore θ is transitive and θ is a strong fuzzy order.

5.2.2 <u>Notation</u>

When X is also a strict fuzzy ordered set, then $\widehat{(X,Y)}$ denotes the set of all strict fuzzy order preserving maps from X to Y.

$\widehat{(Y,Y)}$ is denoted by $\widehat{Y}$.

Clearly $\widehat{(X,Y)} \subset Y^X$.

The next result provides a monoid homomorphism between $\widehat{Y}$ and $\widehat{\widehat{(X,Y)}}$ where (X,R) and (Y,S) are two strong fuzzy ordered sets. Recall that a monoid is a nonempty set endowed with an associative binary operation with a unit element, and a monoid homomorphism is a homomorphism between monoids.

5.2.3 <u>Result</u>

Let (X,R) and (Y,S) be two strong fuzzy ordered sets. Given any $\beta \in \widehat{Y}$ and a strict fuzzy order preserving map $f: X \longrightarrow Y$, then $f^* = \beta \circ f \in \widehat{(X,Y)}$. Also the mapping $f \longmapsto f^*$ is order preserving, i.e., $(f \longmapsto f^*) \in \widehat{\widehat{(X,Y)}}$ and $\beta \longmapsto (f \longmapsto f^*)$ is a monoid homomorphism between $\widehat{Y}$ and $\widehat{\widehat{(X,Y)}}$. Conversely, given a map $f:X \longrightarrow Y$ there exists $\beta \in \widehat{Y}$ such that * is order preserving.

<u>Proof</u>:

Let $f,g \in \widehat{(X,Y)}$

$$\begin{aligned}
\theta(f^*,g^*) &= \bigwedge_y [S(f^*(y),\ g^*(y))] \\
&= \bigwedge_y [S((\beta\circ f)(y)\ (\beta\circ g)(y))] \\
&= \bigwedge_y [S(f(y),g(y))], \text{ since } \beta \text{ is strict order preserving.} \\
&= \theta(f,g).
\end{aligned}$$

Therefore,

$$\Theta(*f, *g) = \Theta(f^*,g^*) = \Theta(f,g)$$

Converse is trivial.

We generalise the result as follows.

5.2.4 <u>Result</u>

Let (X,R) and (Y,S) be two fuzzy ordered sets. Let $(\alpha,\beta) \in \hat{X} \times \hat{Y}$ and α be surjective. Then $f \longmapsto f^* = \beta \circ f \circ \alpha$ is order preserving. Consequently, $(\alpha,\beta) \longmapsto (f \longmapsto f^*)$ is a monoid homomorphism from $\hat{X} \times \hat{Y}$ to $\hat{\hat{(X,Y)}}$.

<u>Proof</u>

Let $f,g \in \widehat{(X,Y)}$

$$\begin{aligned}
\Theta(*f,*g) &= \Theta\ (f^*,g^*) \\
&= \bigwedge_y S((\beta \circ f \circ \alpha)(y),\ (\beta \circ f \circ \alpha)(y)) \\
&= \bigwedge_y S(f(\alpha(y),g(\alpha(y))) \\
&= \bigwedge_x S(f(x),g(x)) \\
&= \Theta(f,g)
\end{aligned}$$

$\therefore$ Θ is order preserving.

5.2.5 <u>Note</u>

$f \longrightarrow f^*$ is an isomorphism if

i) β is left invertible, and

ii) f is onto.

CHAPTER 6

FUZZY ORDER COMPLETENESS

6.0 Introduction

This chapter deals with different characterizations of fuzzy order completeness of a foset. Definitions of gap, cut, least upper bound, greatest lower bound are recalled and their fuzzy analogues are proposed. Finally different versions of fuzzy order completeness are obtained.

6.1 Fuzzy gap

Let us recall the definition of a cut in the crisp case.

6.1.1 Definition [Ru]

Let Q be any nonempty ordered set. A cut is any set $\alpha \subset Q$ with following properties.

(i) α is nonempty, $\alpha \neq Q$

(ii) If $p \in \alpha$, $q \in Q$ and $p<q$ then $q \in \alpha$

(iii) If $p \in \alpha$, then $p<r$ for some $r \in \alpha$ (α has no largest element)

Now we propose the fuzzy analogue of the crisp cut.

6.1.2 Definition

Let (X,R) be a foset. Any fuzzy set σ in X is called an R-fuzzy cut or simply a cut if

(i) $0 \underset{\neq}{<} \sigma \underset{\neq}{<} 1$

(ii) $\sigma(x) > 0$ and $R(y,x) > R(x,y)$ imply $\sigma(y) > 0$

(iii) $\sigma(x) > 0$ there exists $y \in X$ such that $R(x,y) > R(y,x)$ and $\sigma(y) > 0$.

The upper bound of a fuzzy set is defined as follows.

6.1.3 Definition

Let σ be a fuzzy set in (X,R). An element a in X is an upper bound for σ if either $\sigma(a) = 0$ or $R(x,a) > R(a,x)$ for every $x \in X$. Such that $\sigma(x) > 0$.

$a \in X$ is a least upper bound (l.u.b) if a is an upper bound and $a \leqslant b$ for every upper bound b.

Lower bound and greatest lower bound (g.l.b) are defined analogously.

6.1.4 <u>Definition</u>

If an R-fuzzy cut has no l.u.b, then it is called an R-fuzzy gap or fuzzy gap in (X,R).

The following result follows immediately.

6.1.5 <u>Result</u>

σ is a fuzzy cut in (X,R) iff σ_o, the support of σ is a cut in $(X,<)$ where $<$ is the crisp order induced as in [2.1.1].

<u>Proof</u>

Assume that σ is a fuzzy cut in (X,R). Then σ_o, the support of σ, given by $\sigma_o = \{x \in X | \sigma(x) > 0\}$ is a cut in $(X,<)$ as

(i) $\sigma_o \neq \emptyset$, and $\sigma_o \neq X$. Since $0 \lneq \sigma \lneq 1$

(ii) If $x \in \sigma_o$, $y \in X$ and $y<x$, then $R(x,y) < R(y,x)$ and $\sigma(x) > 0$ which implies $\sigma(y) > 0$ by condition (ii) in Definition 6.1.2.

(iii) If $x \in \sigma_o$, then $\sigma(x) > 0$ implies that there exists $y \in X$ such that $R(x,y) > R(y,x)$ and $\sigma(y) > 0$ by condition (iii) in Definition 6.1.2, which implies that there exists $y \in \sigma_o$ such that $x<y$.

$\therefore$ σ_o is a cut in $(X, <)$.

Conversely, it can be easily proved that if

σ_o be a cut in $(X,<)$,

then σ is a fuzzy cut in (X,R).

Hence the result.

6.2 Fuzzy Order Completeness

As in the case of crisp sets, we define fuzzy order completeness as follows.

6.2.1 Definition

A foset (X,R) is fuzzy order complete iff every non empty fuzzy set in X with an upper bound has a l.u.b in X. (This may be called the l.u.b property of (X,R)). The g.l.b property can also be defined analogously. The following result is immediate.

6.2.2 Result

A foset (X,R) is fuzzy order complete iff (X,R) has no fuzzy gap.

Proof:

Obvious.

6.2.3 Note

A foset (X,R) is Dedekind complete iff no fuzzy cut is a fuzzy gap.

Now we have the following result.

6.2.4 Result

A foset (X,R) is Dedekind complete iff the induced crisp order set $(X,\prec)$ is Dedekind complete.

Proof:

If (X,R) is Dedekind complete, then no fuzzy cut is a fuzzy gap.

Let α be a cut in the induced ordered set $(X,\prec)$. Then, α is a fuzzy cut.

Therefore, α is not a fuzzy gap.

i.e., α is not a gap.

Hence $(X,\prec)$ is Dedekind complete.

Conversely, let $(X, \prec)$ be Dedekind complete.

Then to prove that (X,R) is Dedekind complete.

Suppose σ is a cut in (X,R), then σ_o– the support of σ, is a cut in $(X,\prec)$ by (6.1.5).

By l.u.b property of Dedekind Completeness in the crisp case, σ_o has a l.u.b.

Obviously this l.u.b is a l.u.b for σ, (in the fuzzy sense).

Hence (X,R) is Dedekind complete.

6.2.5 Remark

We have the following characterizations of completeness of an ordered set in the classical case [Co-E].

An ordered set $(X, <)$

i) is Dedekind complete iff no cut is a gap.

ii) has l.u.b property iff every bounded above subset has a l.u.b.

iii) has g.l.b property iff every bounded below subset has a g.l.b.

iv) is Cauchy complete iff every Cauchy sequence converges in X.

v) Cantor complete iff for each $n \in \mathbb{N}$, X_n is a closed interval and $X_{n+1} \subset X_n$, then $\bigcap_{n \in \mathbb{N}} X_n \neq \emptyset$.

In an ordered field the above characterization are equivalent. A similar study in the fuzzy setting may be interesting.

CHAPTER 7

GENERALISED FUZZY ORDERED FUZZY TOPOLOGICAL SPACES

7.0 Introduction

In the earlier chapters, the fuzzy orders under consideration were defined on crisp sets. This chapter is an attempt to extend the definition of fuzzy order defined on a crisp set to fuzzy order defined on a fuzzy set, which may be considered as a generalization of the notion of fuzzy order. Also, it is observed that the generalised fuzzy ordered fuzzy topological space defined by means of this generalised fuzzy order is the counter part in the fuzzy setting of the generalised ordered spaces [Lu] in the classical sense.

7.1 Fuzzy order defined on a fuzzy subset

A fuzzy order defined on a fuzzy set is defined by imposing one more condition than those of a fuzzy order defined on a crisp set as given below.

7.1.1 Definition

Let X be a non empty set and μ be a fuzzy subset of X. Let $R_\mu: X \times X \longrightarrow [0,1]$ be a fuzzy binary relation satisfying

i) $R_\mu(x,x) = 0$ for every $x \in X$ -- (irreflexivity)

ii) For $x \neq y$, $R_\mu(x,y) \neq R_\mu(y,x)$ whenever $\mu(x)$,
$\mu(y) > 0$ -- (antisymmetry)

iii) $R_\mu(x,z) \geq \bigvee_y [R_\mu(x,y) \wedge R_\mu(y,z)$ -- (transitivity),

and

iv) $R_\mu(x,y) \leq \mu(x) \wedge \mu(y)$; $x,y,z \in X$.

Then R_μ is called a fuzzy order defined on a fuzzy set μ.

7.1.2 <u>Note</u>

If R_μ satisfies the stronger condition

$R_\mu(x,y) > 0 \Longrightarrow R_\mu(y,x) = 0$ for $x \neq y$ -- (perfect antisymmetry) instead of (ii), in (7.1.1), then R_μ is called a strong fuzzy order defined on the fuzzy set μ.

Clearly every strong fuzzy order defined on μ is also a fuzzy order defined on μ.

7.1.3 <u>Remark</u>

When $\mu=1$, then R_μ is the same as the fuzzy order R defined on X. Therefore, all the results which hold good for R are applicable to R_μ with $\mu=1$.

Note that a strong fuzzy order on X induces, in the natural way, a strong fuzzy order on any fuzzy subset of X.

This is given in the following result.

7.1.4 Result

Let (X,R) be a strong fuzzy ordered set and μ be a fuzzy set in X.

Define $R_\mu(x,y) = R(x,y) \wedge [\mu(x) \wedge \mu(y)]; x,y \in X$.

Then R_μ is a strong fuzzy order defined on μ.

Proof:

i) $R_\mu(x,x) = R(x,x) \wedge [\mu(x) \wedge \mu(x)] = 0$

i.e., R_μ is irreflexive.

ii) $R_\mu(x,y) > 0 \implies R(x,y) \wedge [\mu(x) \wedge \mu(y)] > 0$

$\implies R(x,y) > 0$

$\implies R(y,x) = 0$ for $x \neq y$

$\implies R_\mu(y,x) = 0, \quad x,y \in X$ and $x \neq y$.

i.e., R_μ is perfectly antisymmetric.

(iii) Now,

$$R(x,z) \geq \bigvee_y [R(x,y) \wedge R(y,z)]$$

$$\Rightarrow R(x,z) \wedge [\mu(x) \wedge \mu(z)]$$

$$\geq \bigvee_y [R(x,y) \wedge R(y,z)] \; [\mu(x) \wedge \mu(z)]$$

$$\geq \bigvee_y [\{R(x,y) \wedge (\mu(x) \wedge \mu(y))\} \wedge \{R(y,z) \wedge (\mu(y) \wedge \mu(z))\}]$$

Therefore, $R_\mu(x,z) \geq \bigvee_y [R_\mu(x,y) \wedge R_\mu(y,z)]$

i.e., R_μ is transitive.

(iv) $R_\mu(x,y) = R(x,y) \wedge [\mu(x) \wedge \mu(z)]$

$$\leq \mu(x) \wedge \mu(y).$$

Hence R_μ is a strong fuzzy order defined on the fuzzy set μ in X.

7.1.5 Note

For each μ, R_μ is a fuzzy order defined on X and conversely, every fuzzy order on X is an R_μ with $\mu=1$. Therefore, in a sense, R_μ is a generalization of a fuzzy order defined on X.

Next, we note that a fuzzy order defined on μ also determines a crisp order (which is total) but only on the support μ_o of μ. Recall that the support of μ is $\mu_o = \{x \in X \mid \mu(x) > 0\}$.

7.1.6 Result

Every R_μ defined on the fuzzy subset μ of X determines a crisp order $<_\mu$ on the support of μ_o of μ as

$$x <_\mu y \text{ iff } R_\mu(x,y) > R_\mu(y,x), \quad x,y \in \mu_o.$$

Proof:

Similar to the proof of [2.1.1].

Using R_μ-the fuzzy order defined on μ, μ-fuzzy ordered fuzzy topological space can be defined as follows.

7.1.7 Definition

Let R_μ be a fuzzy order defined on the fuzzy set μ in X and μ_o be the support of μ. Then,

$$\{\mu_a', {}_a\mu', (1-{}_a\mu'-\chi_a), (1-\mu_a'-\chi_a) \mid a \in \mu_o\}^*$$

where,

$$\mu_a'(x) = R_\mu(a,x)$$

$${}_a\mu'(x) = R_\mu(x,a), \text{ for every } x \in \mu_o$$

will form a subbasis for a fuzzy topology on μ_o, provided μ_o has neither the largest element nor the smallest element (in the induced order). This fuzzy topology is called the μ-fuzzy ordered fuzzy topology induced by R_μ on μ_o and is denoted by F_{R_μ}. The pair (X, F_{R_μ}) is called the μ-fuzzy ordered fuzzy topological space.

If μ_o has the largest element ℓ we include $P_\ell \vee {}_\ell\mu'$ and $(1-\mu'_\ell)$ where $a \neq \ell$ to the elements of * and if μ_o has the smallest element s, $P_s \vee \mu_s'$ and $(1-{}_s\mu')$ where $a \neq s$ are included to the elements of * .

7.1.8 Note

All the results obtained for F_R in the earlier chapters could be carried over to the F_{R_μ} as well.

7.2 Generalised fuzzy ordered fuzzy topological space

David J. Lutzer [Lu] and P.R. Meyer-R.G.Wilson [Me] have studied generalised ordered spaces in detail. We recall the relevant definitions of a generalised ordered space and conclude that the μ-fuzzy ordered fuzzy topological space (X, F_{R_μ}) for a special type of R_μ is the fuzzy analogue of the generalised ordered space in the crisp sense. We call it the generalised fuzzy ordered fuzzy topological space.

7.2.1 Definition

Let $T_<$ be the order topology defined on a linearly ordered set $(X,<)$. Take a subspace Y of X and consider the relative topology T_Y on Y and the order $<_Y$ obtained by restricting $<$ to Y. Then $(Y, T_{<_Y})$ is called a Generalised Ordered Space (GO space for short).

7.2.2 Remark

$T_<$ need not be the order topology of $<_Y$, $T_{<_Y}$. But it is true that $T_{<_Y} \subset T_Y$. There is another equivalent way to obtain GO spaces: start with a linearly ordered set $(Y,<)$ and equip Y with any topology which contains $T_<$ and has a base of open sets each of which is order convex. (A set S in Y is called order convex if $x \in S$ for every point x lying between two points of S).

E. Čech has called a topological space (X,T) a GO space if there is a total ordering for X such that each point x has a T-neighbourhood base which consists of order intervals and which contains all order-open intervals containing x.

7.2.3 Note

GO spaces are also called suborderable because it is precisely these topologies which can arise as subspaces of ordered spaces.

Historically it is the linearly ordered spaces which have been given the most attention, but recently GO spaces are getting more attention.

Let us now consider the result [7.1.4] once again.

Here the fuzzy order defined on the fuzzy set μ in the fuzzy ordered set (X,R) is given by

$$R_\mu(x,y) = R(x,y) \wedge [\mu(x) \wedge \mu(y)]$$

The following result holds good for this particular R_μ.

7.2.3 <u>Result</u>

Let (X,R) be a foset and μ be a fuzzy set in X. Let R_μ be a fuzzy order defined on μ. Let $<$ and $<_\mu$ be the crisp orders induced by R and R_μ respectively on the support μ_o of μ as given in (2.1.2). Then $<$ contains $<_\mu$ and the inclusion can be strict.

<u>Proof</u>

Let $x <_\mu y$, $x,y \in \mu_o$

then $\quad R_\mu(x,y) > R_\mu(y,x)$

$$\Rightarrow R(x,y) > R(y,x)$$

$$\Rightarrow x < y$$

The converse need not be true as

$R(x,y) > R(y,x)$ need not imply that

$R_\mu(x,y) > R_\mu(y,x)$, for example

7.2.4 <u>Example</u>

$$X = \{a,b\}$$

$$\mu : \begin{array}{l} a \longmapsto 0.2 \\ b \longmapsto 0.2 \end{array}$$

$$R : \begin{array}{l} (a,b) \longmapsto 0.2 \\ (a,a) \longmapsto 0 \\ (b,b) \longmapsto 0 \\ (b,a) \longmapsto 0.4 \end{array}$$

Here $R(b,a) > R(a,b)$

Let $R_\mu(x,y) = R(x,y) \wedge [\mu(x) \wedge \mu(y)]$

then $R_\mu(b,a) = R(b,a) \wedge [\mu(b) \wedge \mu(a)] = 0.2$

$R_\mu(a,b) = R(a,b) \wedge [\mu(a) \wedge \mu(a)] = 0.2$

We now define the generalised fuzzy ordered fuzzy topology as the fuzzy topology determined by this particular R_μ.

7.2.5 Definition

Let (X,R) be a fuzzy ordered set and μ be a fuzzy set in X. Let the fuzzy order R_μ defined on μ be given as follows.

$$R_\mu(x,y) = R(x,y) \wedge [\mu(x) \wedge \mu(y)]$$

Then the μ-fuzzy ordered fuzzy topology defined in terms of this R_μ as described in (7.1.7) is called the generalised fuzzy ordered fuzzy topology of X. The pair (X, F_{R_μ}) is called the generalised fuzzy ordered topological space.

Detailed investigation of the generalised fuzzy ordered fuzzy topological space (X, F_{R_μ}) analogous to the GO space is yet to be done.

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