DISCRETE GEOMETRY AND ITS APPLICATIONS

STUDIES ON CONVEXITY
IN
SOME DISCRETE STRUCTURES

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CERTIFICATE

This is to certify that the work reported in this thesis entitled "STUDIES ON CONVEXITY IN SOME DISCRETE STRUCTURES" that is being submitted by Shri. Manoj Changat, for the award of the Degree of Doctor of Philosophy to Cochin University of Science and Technology, Cochin 682 022, is based on the bona fide research work carried out by him under my supervision and guidance in the Department of Mathematics and Statistics, Cochin University of Science and Technology. The results embodied in this thesis have not been included in any other thesis submitted previously for the award of any degree or diploma.

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CONTENTS

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Chapter-1

INTRODUCTION

1.1 HELLY'S THEOREM AND AXIOMATIC CONVEXITY

The applicability and the intuitive appeal of the notion of convexity have led to a wide range of notions of " Generalized Convexity ". For several of them, theorems related to Helly's, were either a motive or a by-product of the investigation. Helly's theorem, which was first published by Johan Radon in 1921 and later in 1925 by Helly himself states that "each family of convex sets in R^d, which is finite or whose members are compact, has a nonempty intersection, provided each subfamily of at most d+1 sets has nonempty intersection." The formulation of Helly's theorem can be found in the famous paper of L. Danzer, B. Grunbaum and V. Klee [6], called "Helly's theorem and its Relatives". Restricting Helly's theorem to finite families of convex sets, it is clear that the theorem is formulated completely in terms of convex sets, their intersections and the dimension d of the underlying space.

A convex set can be defined as the intersection of large basic convex sets (For example, half spaces in vector spaces) or by the property of being closed with respect to a certain family of finitary operators (For example, n-ary operators of the form

$$(x_1, x_2, \ldots, x_n) \longmapsto \sum_{i=1}^{n} \lambda_i x_i \text{ in } R^d, \text{ where}$$

the λ_i's are non-negative and sum to 1). This remark leads to the following definition.

A set X, together with a collection $\mathcal{C}$ of distinguished subsets of X, called convex sets, forms a convexity space or aligned space, if the following axioms are satisfied:

C_1: $\emptyset \in \mathcal{C}$, $X \in \mathcal{C}$

C_2: $\mathcal{C}$ is closed under arbitrary intersections

C_3: $\mathcal{C}$ is closed for the unions of totally ordered subcollections. $\mathcal{C}$ is called an alignment or convexity on X. The convex hull of a set S in X (the smallest convex set containing S) is defined as $\operatorname{conv}(S) = \bigcap \{A \in \mathcal{C} \mid S \subseteq A\}$. Those families of sets which satisfy C_1 and C_2 are known as Moore families or closure systems. The axioms C_1 and C_2 were first used by F.W. Levi [31] in 1951 and later on by Eckhoff [10], Jamison [24], Kay and Wamble [29] and Sierksma [43]. The term "alignment" is due to Jamison [24]. Hammer [21] has shown that for Moore families the axiom C_3

is equivalent to the " domain finiteness" condition which states that for each $S \subseteq X$, $conv(S) = \bigcup \{conv(T) \mid T \subseteq S, |T| < \infty\}$. ($|T|$ denotes the cardinality of T). Alternative terminologies for convexity spaces are "algebraic closure systems" ([5]) and "domain finite convexity spaces" ([10], [21], [29], [43]-[45]). As mentioned earlier, the axiomatization of convexity is motivated by the fact that most combinatorial properties of ordinary convex sets in R^d like Helly, Radon and Caratheodory theorems can be studied in the general context of convexity spaces.

1. Helly property

A convexity space $(X, \mathcal{C})$ has the Helly property H_k, if a finite family of convex sets of X has an empty intersection, then this family contains at most k members with an empty intersection. The Helly number of $(X, \mathcal{C})$ is the smallest integer k, such that H_k holds. Helly's theorem states that the Helly number for the ordinary convexity in R^d is d+1. For further examples, see Danzer [6], Jamison [25] and Sierksma [45].

2. Partition property

Closely related to Helly's theorem is the classical theorem of Radon published in 1921. The theorem states that each set of d+2 or more points in R^d can be expressed as the union of two disjoint sets, whose convex hulls have a common point. See Danzer et al. ([6]). Radon's theorem was generalized in 1966 by H.Tverberg ([50]). Instead of 2-partitions, he has investigated arbitrary m partitions. The theorem states that each set S in R^d with $|S| \geq (m-1)(d+1)+1$ can be partitioned into m pairwise disjoint sets with intersecting convex hulls.

Thus, we have, that the convexity space $(X, \mathcal{C})$ has Partition property $P_{k,n}$, if $(P_i)_{i \in I}$ is a family of $n=|I|$ points, there exists a partition of I into k parts $I_1, I_2, \ldots, I_k$ such that

$$\bigcap_{1 \leq j \leq k} \mathrm{conv}(\{P_i \mid i \in I_j\}) \neq \emptyset.$$

Tverberg's theorem states that the ordinary convexity in R^d has property $(P_{k,(k-1)(d+1)+1})$ and for k=2, we get Radon's theorem. An important problem related to Radon partitions posed by Eckhoff in analogy with Tverberg's theorem is the following:

Eckhoff's conjecture

Suppose an aligned space $(X,\mathcal{C})$ has Radon number r. Does the partition inequality $P_m \leqslant (m-1)(r-1)+1$ always hold? Jamison [27] has shown that the partition conjecture holds for order convexities, tree-like convexities etc.

3. Caratheodory property

The classical theorem of Caratheodory, states that, when $A \subseteq R^d$, each point of conv A is a convex combination of d+1 or fewer points of A. The theorem of Caratheodory was published in 1907. See Danzer [6]. A convexity space $(X,\mathcal{C})$ has the Caratheodory property C_k, if $x \in \mathrm{conv}(A)$, then $x \in \mathrm{conv}(F)$, for some $F \subseteq A$, with $|F| \leqslant k$, for any $A \subseteq X$. The Caratheodory number of $(X,\mathcal{C})$ is the smallest number such that C_k holds. Ordinary convexity in R^d has Caratheodory number d+1 (Caratheodory theorem).

1.2. INTERVAL CONVEXITIES

An interval I on a set X is a mapping $I: X \times X \longrightarrow 2^X$. The I-closed subsets of X are subsets $C \subseteq X$ such that $I(x,y) \subseteq C$, for every $x,y \in C$. The collection $\mathcal{C}_I$ of I-closed subsets satisfies the axioms C_1, C_2, C_3 of convexity spaces. The axiom C_3 is a consequence of the finitary property of convex hulls and the fact that, for a subset A of X, $\text{conv}(A) = \bigcup_{k \in N} I^k(A)$, where $I^k(A)$ is defined as $I^o(A) = A$ and $I^{k+1}(A) = I(I^k(A) \times I^k(A))$. The function I is called an interval-function of the convexity space $(X, \mathcal{C}_I)$. Convexity spaces admitting an interval function are named Interval Convexity Spaces, see Calder([4]). Most of the usual convexities are interval convexities. For example, ordinary convexity in R^d, metric convexity (d-convexity) in metric spaces, order convexity in partially ordered sets and geodesic convexity and minimal path convexity in graphs.

Metric convexity

The concept of convexity in metric spaces was introduced by Menger. It is the interval convexity generated by the metric interval $d\text{-}[x,y] = \{ z \in X \mid d(x,z)+d(z,y)=d(x,y) \}$, for points x,y in the metric space (X,d). For various geometric developments involving Menger's and other closely related notions of metric convexity, see Blumenthal ([2]) and Buseman ([3]).

Order convexity

The usual order convexity in a partially ordered set $(P, \leqslant)$ is the interval convexity generated by the usual order interval $[x,y] = \{z \in P \mid x \leqslant z \leqslant y \text{ or } y \leqslant z \leqslant x\}$, for points $x,y \in P$. Order convexity generated by the order interval function has been studied by Franklin ([17]) in 1962. See also Jamison-Waldner ([27]), Jamison ([25)].

1.3 GRAPH CONVEXITIES AND CONVEX GEOMETRIES

Convexity in Graphs

The first explicit use of convexity in graphs has been made perhaps by Feldman and Hogassen. Most of their results deal with geodesic convexity. A more general point of view appeared in Sekanina ([42]) in 1975 and Mülder ([33]) in 1980. A systematic approach arises in Farber-Jamison ([15]).

A graph convexity (Düchet) is a pair $(G, \mathscr{C})$ formed with a connected graph G with vertex set V and a convexity $\mathscr{C}$ on V such that $(V, \mathscr{C})$ is a convexity space, satisfying the additional axiom,

GC: Every convex subset of V induces a connected subgraph. See ([9]).

In the study of convexity in graphs, two types of convexity have played a prominent role, namely the " minimal path convexity or monophonical convexity and geodesic convexity or d-convexity".

Minimal path convexity

The minimal path convexity in a connected graph G is the interval convexity in V(G), generated by the minimal path interval m-[x,y], where m-[x,y] is the set of all vertices of all chordless pathsfrom the vertex x to the vertex y in G, and a chord of a path in G is an edge joining two nonconsecutive vertices in the path. See Jamison ([25]) and Düchet ([9]).

Geodesic convexity

Let d-[x,y] denote the set of all vertices of all shortest paths between the vertices x and y in G. The convexity generated by the interval function d-[x,y] is called the geodesic convexity or distance convexity in G. The d-convexity is the metric convexity associated with the usual distance function d(x,y) in G.

Early researches on d-convexity in graphs were motivated by an important problem posed by Ore in 1962, which is the following: " Characterize the geodetic graphs: that is, graphs in which every pair of vertices is joined by a unique shortest path ".

Graphs with only the trivial geodesic subgraphs have been called distance convex simple graphs by Hebbare and others. See Hebbare ([23]), Batten ([1]). Unlike m-convexity, the geodesic convexity is very general and has been intensively studied since 1981. See Jamison ([25]), Soltan ([49]) and Farber ([13]).

Convex geometries

Convex geometries were introduced independently by Edelman and Jamison in 1980. They are finite convexity spaces in which the finite Krein-Milman property holds. That is every convex set is the convex hull of its extreme points. There are numerous equivalent ways of defining a convex geometry. See Edelman-Jamison ([12]).

We have the following characterizations of graphs.

(i) The m-convexity in a graph G is a convex geometry if and only if G is chordal. A chordal graph is one in which every cycle of length at least four has a chord.

(ii) The geodesic convexity in G is a convex geometry if and only if G is a disjoint union of Ptolemaic graphs. G is a Ptolemaic graph, if for every four vertices x,y,z,w in G, the Ptolemaic inequality $d(x,y)\ d(z,y) \leqslant d(x,z)\ d(y,w) + d(x,w)\ d(z,y)$ holds. See Farber-Jamison ([14], [15]). Major references on the abstract theory of convexity are Jamison ([24], [25]), Sierksma ([45]) and Soltan ([46]). A recent survey of various convexities in discrete structures is in Düchet ([8]).

1.4. DIGITAL AND COMPUTATIONAL CONVEXITIES

The growing field of computer science has also seen the emergence of studies dealing with convexity. This began in the early 1960's, when Freeman ([56]) investigated the representation of straight line segments on a digital grid and Bilanski ([54]), gave an algorithm for determining the vertices of a convex polyhedron. Convexity can be discussed in computer science from the following view:
(1) Digital Geometry, and (2) Computational Geometry.

1. Digital geometry

To generalize convexity and related notions such as straight line segments to the geometry of digital grids, and analyse their properties, in this framework.

Convexity in the two dimensional digital images has been studied by several authors in particular Kim ([57]), Kim and Rosenfeld ([58]) and Ronse ([59]). In contrast with Euclidean images, several non equivalent definitions can be given for digital images. The rectangular grid of two dimensions can be viewed as the set Z^2, where Z is the set of integers, so that pixels can be represented by integer co-ordinates. The basic notions of k-adjacency, k-connected paths, k-connectedness (k=4 or 8) in the geometry of rectangular digital grids can be realized in Z^2 with the integer valued metrics (graph metrics), denoted as d_1 (for k=4) and d_2 (for k=8), defined as $d_1(x,y) = |x_1-y_1| + |x_2-y_2|$ and $d_2(x,y)= \max(|x_1-y_1|,|x_2-y_2|)$, for $x = (x_1,x_2)$ and $y = (y_1,y_2)$ in Z^2.

We can view a k-connected path in the rectangular grid as a path in the graph metric space (Z^2,d), where d is d_1 or d_2, according as k=4 or k=8 respectively. Thus the distance geometry in Z^2, generated by the integer valued metrics d_1 and d_2 is closely related to the geometry of the digital rectangular grid of two-dimension. This is the motivation of our study of the d_1-convexity and d_2-convexity in the integer lattice.

2. Computational Geometry

One wants to evaluate the computational complexity of various operations related to convex sets, and to find optimal computer algorithms for them. Important problems are the determination of the convex hull, that of vertices, faces, volume or diameter of convex bodies, intersection of convex polyhedra, extremal distance between convex polyhedra and maximal convex subsets of non-convex sets.

The first computational question relating to convexity is the design of algorithms, for finding the convex hull of a set of points. The digital convex hull is dealt with in Yau ([62]). A related problem is the determination of the computational complexity of the construction of the convex hull of a set of points. A bibliography on digital and computational convexity is seen in ([61]). See Preperata-Shamos ([38]), for recent developments in computational geometry.

1.5 PRELIMINARIES

Let $(X, \mathcal{C})$ be any convexity space. That is, $\mathcal{C}$ is a collection of subsets of the set X, such that (i) $\emptyset, X \in \mathcal{C}$, (ii) $\mathcal{C}$ is closed under arbitrary intersections,

(iii) $\mathcal{C}$ is closed for the unions of totally ordered subcollections. $\mathcal{C}$ is called an alignment or convexity on X. The convex hull of a set A in X is defined as $\operatorname{conv}(A) = \bigcap \{B \in \mathcal{C} \mid A \subseteq B\}$.

Definition 1.5.1.

The Caratheodory number of a convexity space $(X, \mathcal{C})$ is defined as the smallest nonnegative integer 'c', such that

$\operatorname{conv}(A) = \bigcup \{\operatorname{conv}(B) \mid B \subseteq A \text{ and } |B| \leqslant c\}$, for all $A \subseteq X$.

Definition 1.5.2.

The Helly number h of $(X, \mathcal{C})$ is defined to be the infimum of all nonnegative integers k, such that the intersection of any finite collection of convex sets is nonempty, provided the intersection of each subcollection of at most k elements is nonempty. Or equivalently,

Definition 1.5.3.

A convexity space $(X, \mathcal{C})$ has the Helly number h, if h is the smallest nonnegative integer such that $A \subseteq X$ and $|A| = h+1 \Longrightarrow \bigcap \{\operatorname{conv}(A \setminus a) \mid a \in A\} \neq \emptyset$, for all $A \subseteq X$, where $A \setminus a$ denotes $A \setminus \{a\}$.

Definition 1.5.4.

A convexity space $(X, \mathcal{C})$ has the Radon number r, if r is the infimum of all positive integers k, with the property that, each set A in X with $|A| \geqslant k$, admits a partition $A = A_1 \cup A_2$ with $A_1 \cap A_2 = \emptyset$ and such that $\mathrm{conv}(A_1) \cap \mathrm{conv}(A_2) \neq \emptyset$. Such a partition is called a Radon partition of A.

Definition 1.5.5.

The generalized Radon number or Tverberg type Radon number P_m of a convexity space $(X, \mathcal{C})$ is defined as the infimum of all positive integers k, with the property that, each set A in X with $|A| \geqslant k$ admits an m-partition $A = A_1 \cup \cdots \cup A_m$, into pairwise disjoint sets A_i such that $\mathrm{conv}(A_1) \cap \mathrm{conv}(A_2) \cap \cdots \cap \mathrm{conv}(A_m) \neq \emptyset$. Such an m-partition of A is called a Radon m-partition of A. We need the theorem of Levi.

Theorem 1.5.6. (Levi)

Let $(X, \mathcal{C})$ be a convexity space. If the Radon number r of $(X, \mathcal{C})$ exists, then the Helly number h exists, and $h \leqslant r-1$.

Theorem 1.5.7. (Eckhoff and Jamison)

Let $(X, \mathcal{C})$ be a convexity space with Caratheodory number c and Helly number h. Then the Radon number r of $(X, \mathcal{C})$ exists, and $r \leqslant c(h-1)+2$.

Definition 1.5.8.

Let $(X, \mathcal{C})$ be a convexity space. A subset B of X is said to be (convexly) independent if $b \notin \mathrm{conv}(B \setminus b)$, for each $b \in B$.

Definition 1.5.9.

The rank of a convexity space $(X, \mathcal{C})$ is defined as the supremum of the cardinalities of the independent sets. It is noted that the rank of a convexity space $(X, \mathcal{C})$ is an upper bound for both the Helly number h and the Caratheodory number c.

$N = \{1,2,3, \ldots\}$ is the set of natural numbers and Z denotes the set of integers. The graph theoretic terminology used in this thesis are as in Harary ([22]). We use induction in some of the proofs.

1.6. AN OVERVIEW OF THE MAIN RESULTS OF THIS THESIS

A rather active area in modern convexity theory is concerned with the computation of several "invariants" in general convexities. This thesis contributes mainly to this in some interval convexities, where the underlying set is a discrete set. The "invariants" that we discuss in this thesis are the Caratheodory, Helly, Radon and Tverberg type Radon numbers.

In chapter 2, we consider Z^n as a model. Metric convexity (d-convexity), with respect to the integer valued metrics d_1, d_2, d_3 are defined. For $x=(x_1,\ldots,x_n)$, $y=(y_1,\ldots,y_n) \in Z^n$, the metrics d_1, d_2 and d_3 are defined respectively as $d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i|$, $d_2(x,y) = \max_{1 \leqslant i \leqslant n} |x_i - y_i|$ and $d_3(x,y)$= the number of co-ordinates in which x and y differ. The order convexity is defined with respect to the partial order $x \leqslant y$ if and only if $x_i \leqslant y_i$ for all i. It is shown that every d_1-convex set is both order convex and d_3-convex. Also it is obtained that there is no finite Helly and Radon numbers for the order convexity and d_3-convexity. The d_1-convex sets has Caratheodory number 'n' and Helly number 2. Using Jamison-Eckhoff theorem, it is shown that the Radon number 'r' of the d_1-convexity attains the bound n+2, for n=2 and n=3. For d_2-convexity, the rank is found to be 2^n, and the Helly number equals the rank. The Radon number for d_2-convexity is found to be 2^n+1 and the Caratheodey number is 2^{n-1}. Tverberg type Radon number is also obtained for d_2-convexity. For d_3-convexity the Caratheodory number is n.

In chapter 3, we extend the definitions of order convexity and d-convexity in Z^n to the infinite dimensional sequential space Z^∞. The d-intervals are defined using the d-intervals in the finite dimensional submodules of Z^∞. The analogous results of Caratheodory, Helly and Radon type numbers are obtained for these convexities in Z^∞.

Chapter 4 deals with the geodesic convexity in the finite geometric structure known as " Generalized Polygons", considering it as a bipartite graph Γ. The geodesic convexity in Γ is not exactly a convex geometry but finite Krein-Milman property holds for every proper d-convex subset of Γ. It is shown that a d-convex subset K of Γ has the Krein-Milman property if and only if diam(K) $<$ n. Various center concepts, such as center, centroid and distance centre in Γ are studied. Finally, the Helly, Radon and Caratheodory type theorems for the geodesic convexity are obtained. It is shown that the m-convexity in Γ is the trivial convexity, consisting of the null set $\emptyset$ and whole vertex set V of Γ. In the last section of this chapter (4.5), we discuss an interesting result, which holds for any finite connected bipartite graph G. We order the vertices of G called the "the canonical ordering of G", as given by Mülder, and show that the "geodesic alignment" on G is the join of order alignments, with respect to all possible canonical orderings of G.

In chapter 5, our discussion is mainly in the discrete plane Z^2. Using the concept of hemispaces, it is shown that an intersection convex set A of Z^2 (an intersection convex set of Z^n is defined by Doignon as the intersection of a convex set in R^n with Z^n) is d_1-convex if and only if the supporting lines of A are parallel to the co-ordinate axes and A is d_2-convex if and only if the supporting lines of A have slope $\pm$ 1. Finally, a computational problem is dealt with. An algorithm for computing the d_2-convex hull of a finite set of points in Z^2 is given and also the complexity of the algorithm is computed.

Chapter-2

ORDER AND METRIC CONVEXITIES IN Z^n *

2.1 INTRODUCTION

We consider the n-dimensional integer lattice $Z^n = \{(m_1, m_2, \ldots, m_n) \mid m_i \in Z\}$. In this chapter we discuss the order convexity and metric convexity with respect to three integer valued metrics d_1, d_2 and d_3. The theory discussed here may work well in any discrete set, isometric to Z^n. In particular $H^n = \{(q^{m_1}x_1, q^{m_2}x_2, \ldots, q^{m_n}x_n) \mid m_i \in Z\}$, $q \in (0,1)$ is fixed and $(x_1, x_2, \ldots, x_n)$ is fixed in R^n. In [52], Vijayakumar has defined D-convex sets for the discrete plane $\{(q^{m_1}x_1, q^{m_2}x_2) \mid m_1, m_2 \in Z\}$, $q \in (0,1)$ is fixed, and studied concepts like the D-convex hull and D-convex domain. The d_1-convex sets that we define are generalizations of D-convex sets. If $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n) \in Z^n$, then $d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i|$, $d_2(x,y) = \max_{1 \leqslant i \leqslant n} |x_i - y_i|$ and $d_3(x,y) =$ the number of co-ordinates in which x and y differ, are three integer valued metrics in Z^n. A partial order relation '$\leqslant$' in Z^n

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* Some of the results in Sections 2.2 and 2.3 are to appear in Compo. Math.

can be defined as $(x_1,x_2,\ldots,x_n) \leqslant (y_1,y_2,\ldots,y_n)$ if and only if $x_i \leqslant y_i$, for all $i=1,2,\ldots,n$. We note that d_1-convex sets are boxes with sides parallel to the co-ordinate axes. The box alignment has been studied by many authors. See Eckhoff ([11]), Jamison and Waldner ([27]), Sierksma ([44]), Reay ([39]).

Definition 2.1.1.

A point $z \in Z^n$ is said to be order between $x,y \in Z^n$ if $x \leqslant z \leqslant y$ or $y \leqslant z \leqslant x$. The set of all points order between x and y is denoted by $[x,y]$. Conventionally $[x,y] = \emptyset$, if x and y are not comparable.

Definition 2.1.2.

A point $z \in Z^n$ is said to be metrically between $x,y \in Z^n$, if $d(x,z) + d(z,y) = d(x,y)$, where 'd' is a metric in Z^n. The set of all metrically between points of x and y is denoted by d-$[x,y]$ and is called the metric interval or d-interval determined by x and y.

Definition 2.1.3.

$A \subseteq Z^n$ is said to be order convex, if $[x,y] \subseteq A$, for each pair of points x and $y \in A$.

Definition 2.1.4.

$A \subseteq Z^n$ is said to be metrically convex or d-convex, if the metric interval d-$[x,y] \subseteq A$, for each pair of points $x,y \in A$.

Definition 2.1.5.

The order (metric) convex hull of a set A is the intersection of all order (metric) convex sets containing A. The order (metric) convex hull of a set A is denoted by order conv(A) (d-conv(A)) and **is order convex (metric convex).**

2.2. ORDER CONVEXITY AND d_1-CONVEXITY

Lemma 2.2.1.

For any two points $x,y \in Z^n$,

$$d_1\text{-}[x,y] = \Big\{ z = (z_1,z_2,\ldots,z_n) \in Z^n \,|\, z_i \text{ is order between } x_i \text{ and } y_i \text{ for every } i = 1,\ldots,n \Big\}.$$

Proof:

$$z \in d_1\text{-}[x,y] \Longleftrightarrow d_1(x,z) + d_1(z,y) = d_1(x,y)$$

$$\Longleftrightarrow \sum_{i=1}^{n} |x_i - z_i| + \sum_{i=1}^{n} |z_i - y_i| = \sum_{i=1}^{n} |x_i - y_i|$$

$$\Longleftrightarrow |x_i - z_i| + |z_i - y_i| = |x_i - y_i|, \quad \text{for every } i = 1,\ldots,n.$$

for if not, there exists $j \in \{1,\ldots,n\}$ such that

$$|x_j-z_j| + |z_j-y_j| > |x_j-y_j|$$

and thus

$$\sum_{i=1}^{n}|x_i-z_i| + \sum_{i=1}^{n}|z_i-y_i| = \sum_{i\neq j}|x_i-z_i| + \sum_{i\neq j}|z_i-y_i|$$

$$+ \sum_{j}|x_j-z_j| + \sum_{j}|z_j-y_j|$$

$$> \sum_{i=1}^{n}|x_i-y_i|, \text{ since}$$

$$|x_j-z_j|+|z_j-y_j| > |x_j-y_j|,$$

which is a contradiction. Therefore

$$|x_i-z_i| + |z_i-y_i| = |x_i-y_i|, \text{ for every } i=1,\ldots,n$$

$\Longleftrightarrow z_i$ is order-between x_i and y_i , for every $i=1,\ldots,n$

and hence the lemma.

Lemma 2.2.2.

If $x \leqslant y$, then $[x,y] = d_1\text{-}[x,y]$.

Proof follows from Lemma 2.2.1.

Note 2.2.3.

It follows from lemma 2.2.2 that every d_1-convex set is order convex. But the converse is not true, for example

$A = \{(1,0),\ (0,1)\} \subseteq Z^2$ is trivially order convex, but not d_1-convex.

Lemma 2.2.4.

If $A \subseteq Z^n$ is finite, then

$$d_1\text{-conv}(A) = d_1\text{-}[u,v], \quad \text{where } u = \inf A \text{ and } v = \sup A.$$

Proof:

We have $u \leqslant a \leqslant v$, for all $a \in A$.
Therefore $A \subseteq [u,v] = d_1\text{-}[u,v]$, by lemma 2.2.2. Also $d_1\text{-conv}(A) \subseteq d_1\text{-}[u,v]$, since $d_1\text{-}[u,v]$ is d_1-convex.
Since A is finite both u and v belong to $d_1\text{-conv}(A)$.
Therefore $d_1\text{-conv}(A) = d_1\text{-}[u,v]$.

In [17], Franklin has proved that the Caratheodory number for order convexity in any poset is 2 .

We have

Theorem 2.2.5.

The Caratheodory number for d_1-convexity in Z^n is n, if $n \geqslant 2$.

Proof:

We have for any $A \subseteq Z^n$,

$$d_1\text{-conv}(A) = \bigcup \{d_1\text{-conv}(B) \mid B \subseteq A \text{ and } |B| < \infty\}.$$

By lemma 2.2.4, if $|B| < \infty$, then d_1-conv(B) = d_1-[u,v], where u = inf B and v = sup B. Also if $|B| < \infty$, u is the infimum of at most n elements of B and v is the supremum of at most n elements of B.

Let $u = \inf \{a_1,\ldots,a_n \mid a_i \in B\}$ and
$v = \sup \{b_1,\ldots,b_n \mid b_i \in B\}$

Note that a_i and b_i need not be distinct for all i=1,...,n. Therefore, we have d_1-conv(B) = d_1-conv $\{a_1,\ldots,a_n, b_1,\ldots,b_n\}$. We shall now show that any point $z \in d_1$-conv(B) belongs to the d_1-convex hull of at most n points among $a_1,\ldots,a_n$, $b_1,\ldots,b_n$. Let $z = (z_1,z_2,\ldots,z_n)$. We select the n points $a_1',\ldots, a_n'$, among $a_1,a_2,\ldots,a_n$, $b_1,\ldots,b_n$ as follows. a_i' is chosen such that the i^{th} co-ordinate of a_i' is at most z_i, for all i = 1,...,n.

If the j^{th} co-ordinate of a_i' is at most z_j, i=1,...,n, $i \neq j$, then we delete a_j' and replace it with one among $a_1,\ldots,a_n$, $b_1,\ldots,b_n$, whose j^{th} co-ordinate is greater than or equal to z_j. The points $a_1',\ldots,a_n'$ selected in this way satisfies the inequality $u' \leqslant z \leqslant v'$, where $u' = \inf\{a_1',\ldots,a_n'\}$ and $v' = \sup\{a_1',\ldots,a_n'\}$ and $z \in d_1$-[u',v'] = d_1-conv $\{a_1',\ldots,a_n'\}$ and hence the theorem. We note that this theorem can be obtained as a particular case of a product theorem due to Sierksma ([44]).

We shall now prove the Helly-type theorem for d_1-convexity. We begin with a lemma.

Lemma 2.2.6.

If $\mathcal{F} = \{B_1, B_2, B_3\}$ is a family of three nonempty d_1-convex sets in Z^n, such that any two members of $\mathcal{F}$ have nonempty intersection, then $\bigcap_{i=1}^{3} B_i \neq \emptyset$.

Proof:

Let $x \in B_1 \cap B_2$, $y \in B_2 \cap B_3$ and $z \in B_3 \cap B_1$. If one of x,y,z belongs to the d_1-convex hull of the remaining two, then we are done. If not, then there are three different cases.

Case (i):

One of x,y,z, say x is comparable with y and z, and y and z are not comparable. Take $x \preccurlyeq y$ and $x \preccurlyeq z$.

Case (ii):

Only one pair say x and y are comparable. Take $x \preccurlyeq y$.

Case (iii):

None of x,y,z is comparable with each other.
(case (iii) happens only if $n > 2$.)

We will show that in all these cases, there exists a point p, which belongs to all the three d_1-intervals d_1-$[x,z]$, d_1-$[y,z]$ and d_1-$[z,x]$.

Let $x = (x_1,\ldots,x_n)$, $y = (y_1,y_2,\ldots,y_n)$ and $z = (z_1,\ldots,z_n)$.

Case (i):

We have $x_i \leqslant y_i$ and $x_i \leqslant z_i$ for all $i=1,\ldots,n$, and there exists j such that $z_j < y_j$ and $y_i \leqslant z_i$ for $i \neq j$. Thus we have

$$x_i \leqslant y_i \leqslant z_i \qquad i \neq j$$
$$x_j \leqslant z_j \leqslant y_j$$

Then $p = (y_1,\ldots,y_i,\ \ldots,\ z_j,\ldots,y_n)$.

Case (ii):

Here $x_i \leqslant y_i$ for all $i = 1,\ldots,n$ and there exists at least two co-ordinates, with subscripts j and k such that

$$z_j < x_j < y_j$$
$$x_k < z_k < y_k$$

and $$x_i \leqslant y_i \leqslant z_i, \quad i \neq j \neq k.$$

Then $p = (y_1,\ldots,x_j,\ldots,z_k,\ldots,y_i,\ldots,y_n)$.

Case (iii):

Here there exists i, j, k, ℓ such that

$x_i \leqslant y_i \leqslant z_i \quad i \neq j \neq k \neq \ell$

$x_j < z_j < y_j$

$y_k < x_k < z_k$

$z_\ell < y_\ell < x_\ell$

Here $p = (y_1, \ldots, y_i, z_j, \ldots, x_k, \ldots, y_\ell, \ldots, y_n)$

Thus in all these three cases $p \in \bigcap_{i=1}^{3} B_i$, and hence the lemma.

Theorem 2.2.7.

The Helly number h for the d_1-convexity in Z^n is 2.

Proof:

We use induction to prove the theorem. Let $\mathcal{F} = \{B_1, \ldots, B_k\}$ be a family of k nonempty d_1-convex sets in Z^n, with $k \geqslant 2$ and every two members of $\mathcal{F}$ has nonempty intersection. For k=2, conclusion trivially holds and for k=3 it follows from lemma 2.2.6.

Assuming the result for k=m, to prove for k=m+1, let $\mathcal{F} = \{B_1, \ldots, B_{m+1}\}$.

Define $B_i{}' = B_i \cap B_{m+1}$, for $i=1,\dots,m$. Then $B_i{}' \neq \emptyset$ and $\{B_1{}',\dots,B_m{}'\}$ is a family of m nonempty d_1-convex sets satisfying the induction hypothesis, by lemma 2.2.6. Therefore $\bigcap_{i=1}^{m} B_i{}' \neq \emptyset$. That is $\bigcap_{i=1}^{m+1} B_i \neq \emptyset$, which completes the proof by induction. We note that the Helly number h for the d_1-convexity in Z^n can also be obtained from the following facts. See [30], [44], [47].

The d_1-convexity in Z is the same as the order convexity in Z with respect to the usual order and the Helly number for the usual order convexity in Z is 2. The d_1-convexity in Z^n is the product convexity of n copies of d_1-convexity in Z, and the Helly number of a product convexity is the maximum of the Helly numbers of the factors. Hence h=2, for the d_1-convexity in Z^n.

Note 2.2.8.

For the d_1-convexity in Z^n, we have $r \leq n+2$, using Theorem 1.5.7. We will show that r attains the bound n+2, for n=2 and n=3.

Theorem 2.2.9.

The Radon number r for the d_1-convexity in Z^n is 4 if n=2 and is 5, if n=3.

Proof:

We will show that there are sets with cardinality 3 and 4, which have no Radon partition when n=2 and n=3 respectively. Consider the sets

$A = \{a=(a_1,a_2),\ b=(b_1,b_2),\ c=(c_1,c_2)\} \subseteq Z^2$, where

$a_1 < b_1 < c_1$ and $a_2 < c_2 < b_2$ and

$B = \{a=(a_1,a_2,a_3),\ b=(b_1,b_2,b_3),\ c=(c_1,c_2,c_3),\ d=(d_1,d_2,d_3)\} \subseteq Z^3$

where $a_1 < b_1 < c_1 < d_1$, $a_2 < c_2 < d_2 < b_2$ and $a_3 < d_3 < b_3 < c_3$

Now the sets A and B have cardinalities 3 and 4 respectively and it can be shown that they have no Radon partitions. Therefore for the d_1-convexity in Z^n, r=4 and r=5, when n=2 and n=3 respectively.

The following example illustrates that the family of order convex sets in $Z^n (n \geq 2)$ has an infinite Helly and Radon number.

Example 2.2.10.

Suppose that there exists finite Helly number 'h' and Radon number 'r' for the order convexity in Z^n. Consider the set $A = \{(x,y,0,\ldots,0),\ (x-1,y+1,0,\ldots,0),\ldots,(x-h,y+h,0,\ldots,0)\} \subseteq Z^n$. Then $|A| = h+1$ and A is trivially order convex. Now consider subsets of A defined as

$$A_o = A \setminus (x,y,0,\ldots,0),\ A_1 = A \setminus (x-1,y+1,0,\ldots,0),\ldots$$
$$A_h = A \setminus (x-h,y+h,0,\ldots,0).$$

Then $\{A_o,A_1,\ldots,A_h\}$ is a family of h+1 order convex sets, such that every h members of the family have nonempty intersection, but $\bigcap_{i=o}^{h} A_i = \emptyset$, which is a contradiction to the assumption that h is the Helly number.

Since $h \leqslant r-1$, by theorem 1.5.6, for any convexity it follows that the order convexity in Z^n has no finite Radon number also.

2.3. d_2-CONVEXITY

In this section, we discuss d_2-convexity in Z^n where d_2 is the metric defined by

$$d_2(x,y) = \max_{1 \leqslant i \leqslant n} |x_i - y_i|, \text{ for } x=(x_1,\ldots,x_n), \; y=(y_1,\ldots,y_n) \in Z^n$$

In the following discussion, by independent sets, we mean d_2-convexly independent sets.

Lemma 2.3.1.

Let $A \subseteq Z^n$ be a set with $r=2^n$ independent points. Let $\pi_j \colon Z^n \longrightarrow Z$, denote the projection to the j^{th} factor. Then for each $x \in A$ and $j=1,\ldots,n$, there is a point $y \in A$ with $d_2(x,y) = | \pi_j(x) - \pi_j(y) |$.

Proof:

We prove the lemma by induction on the dimension n of Z^n.

For n=1, it is trivially true.

For n=2, let $A = \{x_1, x_2, x_3, x_4\}$ be a set of $2^2=4$ independent points in Z^2. Required to show that, for each $x_i \in A$ and j=1,2, there is a point $x_k \in A$ with

$$d_2(x_i, x_k) = |\pi_j(x_i) - \pi_j(x_k)|$$

Suppose not, that is, for at least one $x_i \in A$, say x_1,

$$d_2(x_1, x_k) = |\pi_1(x_1) - \pi_1(x_k)| \text{ or } |\pi_2(x_1) - \pi_2(x_k)|,$$
$$\text{for all } x_k \in A.$$

Let $m = \min_{x_k \in A \setminus x_1} \{ d_2(x_1, x_k) \}$ and

$$A' = \{ x_k \in A \mid d_2(x_1, x_k) = m \} .$$

Then $A' \neq \emptyset$ and $x_k \in d_2\text{-conv}(A \setminus x_1)$, for every $x_k \in A'$, which is a contradiction to the assumption that A is an independent set, hence the lemma for n=2. Now assume the result for n-1. Let $A = \{x_1, \ldots, x_r\}$ $r = 2^n$ be an independent set in Z^n. For each $x_i \in A$, any (n-1) dimensional projection A' of A containing x_i, contains 2^{n-1} independent points. So by induction assumption, for each $j = j_1, j_2, \ldots, j_{n-1}$, there is a point

$$x_k \in A' \text{ with } d_2(x_i, x_k) = |\pi_j(x_i) - \pi_j(x_k)|,$$
$$\text{where } j_1, j_2, \ldots, j_{n-1} \in \{1, \ldots, n\}.$$

Consider another (n-1) dimensional projection B' of A, containing x_i and again by inductive assumption, there is a point $x_k' \in B'$, such that

$$d_2(x_i,x_k') = |\pi_j(x_i)-\pi_j(x_k')| \text{ for each } j=j_2,\ldots,j_n$$
$$\text{where } j_2,j_3,\ldots,j_n \in \{1,\ldots,n\}.$$

Therefore, for each $x_i \in A$, and $j=1,\ldots,n$, there is a point $x_k \in A$, with $d_2(x_i,x_k) = |\pi_j(x_i)-\pi_j(x_k)|$ and hence the lemma for all n.

Theorem 2.3.2.

Rank of the d_2-convexity in Z^n is 2^n.

Proof:

We prove that every set with cardinality 2^n+1 is dependent. Let $B = \{x_1,x_2,\ldots,x_{r+1}\}$, $r = 2^n$ be any subset of Z^n. Let $A = \{x_1,\ldots,x_r\}$ be any subset of B, containing 2^n independent points. If $x_{r+1} \in d_2\text{-conv}(A)$, then we are done. If not,

let $m = \inf \{d_2(x_i,x_{r+1}) \mid x_i \in A\}$.

Define $C = \{x_j \in A \mid d_2(x_j,x_{r+1}) = m\}$. Then $C \neq \emptyset$ and for $x_j \in C$, let $d_2(x_j,x_{r+1})$ be the difference between the

k^{th} co-ordinates $(1 \leqslant k \leqslant n)$. By lemma 2.3.1, there exists a point $x_p \in A$ such that $d_2(x_p, x_j)$ is also the difference between the k^{th} co-ordinates. Since $x_{r+1} \notin d_2$-conv(A) and $d_2(x_j, x_{r+1})$ is the minimum, $d_2(x_p, x_{r+1})$ is also the difference between the k^{th} co-ordinates. Therefore we have,

$$d_2(x_p, x_{r+1}) = d_2(x_p, x_j) + d_2(x_j, x_{r+1})$$

That is,

$$x_j \in d_2\text{-}[x_p, x_{r+1}]$$

Therefore

$x_j \in d_2$-conv$(B \setminus x_j)$, which completes the proof.

Corollary 2.3.3.

Let $S \subseteq Z^n$ be finite with $|S| \geqslant 2^n$. Then there exists an independent subset A of S with $|A| \leqslant 2^n$, such that d_2-conv(S) = d_2-conv(A).

We note that if $A = \{x_1, \ldots, x_r\}, r \leqslant 2^n$ is a set of r independent points in Z^n, then for any point $x \in d_2$-conv(A), there is an (n-1)-dimensional submodule of Z^n, containing x. In Z^{n-1}, there are at most 2^{n-1} independent points of A, the d_2-convex hull of which contains x. Thus any point $x \in d_2\text{-conv}(A)$, belongs to the d_2 convex hull of a subset of A, containing at most 2^{n-1} points of A.

Therefore,

$$d_2\text{-conv}(A) = \bigcup \left\{ d_2 \text{ conv}(T) \,\middle|\, T \subseteq A \text{ and } |T| \leqslant 2^{n-1} \right\}.$$

In fact this bound is sharp. For example, consider the subset

$$A = \left\{ (x_1, x_2, \ldots, x_n) \in Z^n \,|\, x_i=0 \text{ or } x_i=2 \text{ for all } i=1,\ldots,n \right\}.$$

Define the subsets A_j and A_j' of A with cardinality 2^{n-1} as

$$A_j = \left\{ (x_1, \ldots, x_j, \ldots, x_n) \in A \,|\, x_j = 0 \right\} \text{ and}$$
$$A_j' = \left\{ (x_1, \ldots, x_j', \ldots, x_n) \in A \,|\, x_j' = 2 \right\}, \text{ for } j=1,\ldots,n.$$

Then we have

$$d_2\text{-conv}(A) = \bigcup \left\{ d_2 \text{ conv}(B) \,|\, B=A_j \text{ or } A_j' \right\}$$

Now consider the point $z = (4,1,1,\ldots,1) \in d_2\text{-conv}(A)$.

Then $z \in d_2 \text{ conv}(A_1')$ and it can be verified easily that z cannot lie in the d_2-convex hull of a subset of A of cardinality less than 2^{n-1}. Thus we have

Theorem 2.3.4.

The Caratheodory number for the family of d_2-convex sets in Z^n is 2^{n-1}.

Theorem 2.3.5.

The Helly number h for the d_2-convexity in Z^n is 2^n.

Proof:

The method of proof is by induction.
Let $\mathcal{F} = \{B_1, B_2, \ldots, B_r\}$, $r \geqslant 2^n$ be a family r d_2-convex sets such that each 2^n members of $\mathcal{F}$ has nonemtpy intersection. When $r = 2^n+1$, then there exists $x_1, \ldots, x_r$, such that

$$x_i \in \bigcap_{\substack{j=1 \\ j \neq i}}^{r} B_j.$$ Now $A = \{x_1, \ldots, x_r\}$ is a set

consisting of 2^n+1 distinct points in Z^n, which by Theorem 2.3.2 is dependent.
Therefore $x_i \in d_2\text{-conv}\{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_r\}$, for some i
Clearly $x_i \in B_i$, for all $i = 1, \ldots, r$, completing the proof for $r = 2^n+1$.

Now assuming the result for $r = 2^n+m$, consider $r=2^n+m+1$.
Define

$$B_i' = B_i \cap B_r \neq \emptyset, \text{ for } i = 1, \ldots, r-1.$$

Now $\{B_1', B_2', \ldots, B_{r-1}'\}$ is a family of 2^n+m nonempty d_2 convex sets, satisfying the conditions of the theorem.

Therefore by inductive assumption $\bigcap_{i=1}^{r-1} B_i' \neq \emptyset$.

That is $\bigcap_{i=1}^{r} B_i \neq \emptyset$ and that completes the proof by induction.

Theorem 2.3.6.

The Radon number r for the d_2 convexity in Z^n is 2^n+1.

Proof:

We have $r \leqslant 2^n+1$, since by theorem 2.3.2, any 2^n+1 points in Z^n is dependent and therefore any set $S \subseteq Z^n$ with $|S| \geqslant 2^n+1$ has a partition into two disjoint sets S_1 and S_2, whose d_2-convex hulls contain at least one common point. Therefore $r \leqslant 2^n+1$. To prove that $r = 2^n+1$, we will show that there exists a set A with $|A| = 2^n$, which has no Radon partition. Consider the set
$A = \left\{(x_1,\dots,x_n) \in Z^n \mid x_i = 0 \text{ or } 1, \text{ for all } i=1,\dots,n\right\}$.
Then $|A| = 2^n$, and every subset of A is d_2-convex, and hence A has no Radon partition. Therefore $r = 2^n+1$.

Theorem 2.3.7.

The Tverberg type generalized Radon number P_m for the d_2-convexity in Z^n is $(m-1)\, 2^n+1$.

Proof:

We will show that every subset S of Z^n, with $|S| = (m-1)\ 2^n+1$ have a Radon m-partition, and there exists a subset B with $|B| = (m-1)2^n$, having no Radon m-partition. Let $S \subseteq Z^n$ be such that $|S| = (m-1)2^n+1$. Choose $F_1 \subseteq S$ with $|F_1| \leqslant 2^n$ and $d_2\text{-conv}(F_1)=d_2\text{-conv}(S)$, which is possible, since rank of the d_2-convexity is 2^n. Again choose $F_2 \subseteq S \setminus F_1$ with $|F_2| \leqslant 2^n$ and $d_2\text{-conv}(F_2) = d_2\text{-conv}(S \setminus F_1) \subseteq d_2\text{-conv}(F_1)$. Proceeding in this way, we get a partition of S into (m-1) sets F_i with $|F_i| \leqslant 2^n$, for each i, and there remains at least one point $z \in d_2\ \text{conv}(F_{m-1}) \subseteq d_2\ \text{conv}(F_{m-2}) \subseteq \cdots \cdots \subseteq d_2\text{-conv}(F_1) = d_2\ \text{conv}(S)$. Thus we get a Radon m-partition, for any subset S of Z^n with $|S| = 2^n(m-1)+1$.

Now consider the subsets B_i of Z^n for i=0,...,m-2 defined as

$$B_o = \{(x_1,\ldots,x_n) \in Z^n \mid x_i=0 \text{ or } x_i=2m-3 \text{ for all } i\}$$
$$B_1 = \{(x_1,\ldots,x_n) \in Z^n \mid x_i=1 \text{ or } x_i=2m-2 \text{ for all } i\}$$
$$\cdot\cdot \qquad \cdot\cdot \qquad \cdot\cdot \qquad \cdot\cdot \qquad \cdot\cdot$$
$$B_{m-2} = \{(x_1,\ldots,x_n) \in Z^n \mid x_i=m-2 \text{ or } x_i=m-1 \text{ for all } i\}$$

Then $\{B_o, B_1,\ldots,B_{m-2}\}$ are m-1 disjoint sets with $|B_i|=2^n$, for each i=0,..,m-2 and the set $B=B_o \cup \ \ldots \ \cup B_{m-2}$ has cardinality $(m-1)2^n$ and has no Radon m-partition. Hence $P_m = (m-1)2^n+1$.

2.4 d_3-CONVEXITY

In this section, we discuss d_3-convexity in Z^n, where d_3 is the metric defined by

$d_3(x,y)$ = Number of co-ordinates in which x and y differ, where $x = (x_1,\ldots,x_n)$ and $y = (y_1,\ldots,y_n)$. We have $d_3(x,y) \leqslant n$ for every $x,y \in Z^n$. Suppose that $z=(z_1,\ldots,z_n)$ belongs to the d_3-interval d_3-$[x,y]$. That is $d_3(x,z) + d_3(z,y) = d_3(x,y)$. We note that $z \in d_3$-$[x,y]$ if and only if $z_i = x_i$ or $z_i = y_i$, for all $i=1,\ldots,n$.
We have

Lemma 2.4.1.

$$d_3\text{-}[x,y] \subseteq d_1\text{-}[x,y] \text{ for all } x,y \in Z^n.$$

Proof:

$$z \in d_3\text{-}[x,y] \Longleftrightarrow z_i=x_i \text{ or } z_i = y_i \text{ for every } i=1,\ldots,n.$$

Suppose $z \in d_3$-$[x,y]$.

$$\text{Now } d_1(x,z)+d_1(z,y) = \sum_{i=1}^{n} |x_i-z_i| + \sum_{i=1}^{n} |z_i-y_i|$$

$$= \sum_{i} |x_i-y_i| + \sum_{j\neq i} |x_j-y_j|,$$

since $z_i=x_i$ or $z_i=y_i$ for all $i=1,\ldots,n$.

$$= \sum_{i=1}^{n} |x_i - y_i| = d_1(x,y)$$

$$\Longrightarrow z \in d_1\text{-}[x,y]$$

and hence the lemma.

Theorem 2.4.2.

$d_3\text{-}[x,y] = d_1\text{-}[x,y]$ if and only if

$d_3(x,y) = d_1(x,y)$.

Proof:

Suppose $d_3(x,y) = d_1(x,y)$ (1)

We have $d_3(x,y) \leqslant n$ for all $x,y \in Z^n$.

Therefore (1) gives $d_1(x,y) \leqslant n$

i.e., $\sum_{i=1}^{n} |x_i - y_i| \leqslant n$

$\Longleftrightarrow |x_i - y_i| \leqslant 1$, for all $i=1,\ldots,n$

Now $z \in d_1\text{-}[x,y] \Longleftrightarrow d_1(x,z)+d_1(z,y) = d_1(x,y)$

$$\Longleftrightarrow \sum_{i=1}^{n} |x_i - z_i| + \sum_{i=1}^{n} |z_i - y_i| = \sum_{i=1}^{n} |x_i - y_i|$$

$\Longleftrightarrow x_i = z_i$ or $z_i = y_i$ for all $i=1,\ldots,n$,

since $|x_i - y_i| \leqslant 1$ for all $i=1,\ldots,n$.

$\Longleftrightarrow z \in d_3\text{-}[x,y]$

$\Longrightarrow d_1\text{-}[x,y] \subseteq d_3\text{-}[x,y]$

Therefore d_1-$[x,y]$ = d_3-$[x,y]$, by lemma 2.4.1.

Conversely suppose d_1-$[x,y]$ = d_3-$[x,y]$.

Therefore $z \in d_1\text{-}[x,y] \Longleftrightarrow z \in d_3\text{-}[x,y] \Longleftrightarrow z_i=x_i$ or $z_i=y_i$,

for all $i = 1,\ldots,n$.

That is $z \in d_1\text{-}[x,y] \Longleftrightarrow \sum_{i=1}^{n}|x_i-z_i| + \sum_{i=1}^{n}|z_i-y_i|$

$\Longleftrightarrow z_i=x_i$ or $z_i=y_i$ for all $i=1,\ldots,n$.

This is possible only if $|x_i-y_i| \leqslant 1$ for all $i=1,\ldots,n$.

If not, for at least one j, $|x_j-y_j| > 1$, then there exists

z_j such that $x_j < z_j < y_j$ or

$y_j < z_j < x_j$, so that $z_j \neq x_j$ or $z_j \neq y_j$.

Therefore

$$d_1(x,y) = \sum_{i=1}^{n} |x_i-y_i| =$$

$$= \sum_j |x_j-y_j| + \sum_k |x_k-y_k|, \text{ where}$$

$$|x_j-y_j|=0 \text{ and } |x_k-y_k|=1$$

$$\text{for } 1\leqslant j<n,\ 1\leqslant k\leqslant n$$

$$= \sum_k |x_k-y_k|$$

$= m < n$, if there are m co-ordinates for which $x_k \neq y_k$.

= The number of co-ordinates in which x and y differ

$= d_3(x,y)$, hence the theorem.

Theorem 2.4.3.

If $A \subseteq Z^n$ is d_1-convex, then A is d_3-convex.

Proof:

Follows from lemma 2.4.1.

Lemma 2.4.4.

For any $A \subseteq Z^n$,

$$d_3\text{-conv}(A) = \{z=(z_1,\ldots,z_i,\ldots,z_n) \in Z^n \mid z_i=a_i, \text{ for some } a=(a_1,\ldots,a_i,\ldots,a_n) \in A \text{ for all } i\}$$

Proof:

Let $B = \{z=(z_1,\ldots, z_i,\ldots,z_n) \in Z^n \mid z_i=a_i \text{ for some } a=(a_1,\ldots,a_i,\ldots,a_n) \in A \text{ for all } i\}$

To prove that B is d_3-convex, let $z,w \in B$,

Now $y \in d_3\text{-}[z,w] \iff y_i=z_i$ or $y_i=w_i$ for all $i=1,\ldots,n$

$y_i=a_i$ or $y_i=b_i$, for some

$a=(a_1,\ldots,a_i,\ldots,a_n)$ and

$b=(b_1,\ldots,b_i,\ldots,b_n) \in A$ for all $i=1,\ldots,n$.

Therefore we have

$$y = (y_1,\dots,y_i,\dots,y_n), \text{ where}$$

$$y_i = a_i \text{ for some } a=(a_1,\dots,a_i,\dots,a_n) \in A$$

$$\text{for all } i=1,\dots,n.$$

$\Rightarrow y \in B.$

$\Rightarrow B$ is d_3-convex.

Since $A \subseteq B$, we have $d_3\text{-conv}(A) \subseteq B$ (1)

Now let $z \in B$, then $z=(z_1,\dots,z_i,\dots,z_n)$, where $z_i=a_i$, for some $a=(a_1,\dots,a_i,\dots,a_n) \in A$ for all $i=1,\dots,n$.

Thus there are at most n points, say $C_1,C_2,\dots,C_n$ in A, such that z_i, the i^{th} co-ordinate of z is equal to the i^{th} co-ordinate of C_i, for all $i=1,\dots,n$.

Therefore

$$z \in d_3\text{-conv}\{C_1,\dots,C_n\} \subseteq d_3\text{-conv}(A)$$

$$\Rightarrow B \subseteq d_3\text{-conv}(A)$$

Therefore $d_3\text{-conv}(A) = B$ by (1), and hence the lemma.

From lemma 2.4.4. we have

Theorem 2.4.5.

The Caratheodory number for d_3-convexity in Z^n is 'n'.

Note 2.4.6.

It may be noted that there is no finite Helly and Radon number for the d_3-convexity in Z^n. Suppose, if possible, that there exists finite Helly number h, for the d_3-convexity in Z^n. Now consider the set

$$A = \{(a_1,a_2,0,\ldots,0),\ (a_1+1,\ a_2+1,\ 0,\ldots),\ \ldots,\ (a_1+h,\ a_2+h,\ 0,\ldots,0)\} \subseteq Z^n.$$

It is clear that $a \notin d_3\text{-conv}(A \setminus a)$, for all $a \in A$.

Consider the h+1 member family of d_3-convex sets defined as $\mathcal{F} = \{d_3\text{-conv}(A \setminus a) \mid a \in A\}$. Every h members of $\mathcal{F}$ has nonempty intersection, but $\bigcap \mathcal{F} = \emptyset$, which is a contradiction to the fact that h is the Helly number. Therefore, there is no finite Helly number h, for the d_3-convexity in Z^n. Since $h \leqslant r-1$ by 1.5.6, for any convexity, there is no finite Radon number for the d_3-convexity in Z^n.

Chapter-3

ORDER AND METRIC CONVEXITIES IN Z^∞

In this chapter, we extend the definitions of order convexity and d-convexity in Z^n to the infinite dimensional sequential space Z^∞. Being interval convexities, these are all domain finite convexities, having no finite Caratheodory number, with the exception of order convexity. Convexity spaces having finite Caratheodory number is known as domain bounded convexities. Therefore these convexities are domain finite, but not domain bounded. See Hammer ([21)], Sierksma ([45]), Kay and Womble ([29]).

π_ν, π_μ denote the projection from Z^∞ to Z^n and Z^m respectively.

3.1. ORDER CONVEXITY

We consider the infinite dimensional sequential space $Z^\infty = \left\{(m_1, m_2, \ldots) \mid m_i \in Z\right\}$, where Z is the set of integers.

For $x = (x_1, x_2, \ldots)$, $y=(y_1, y_2, \ldots) \in Z^\infty$, the relation $x \leqslant y$ if and only if $x_i \leqslant y_i$ for every i, is a partial order in Z^∞.

Definition 3.1.1.

A point $z \in Z^\infty$ is said to be order-between two points $x,y \in Z^\infty$, if $x \leqslant z \leqslant y$ or $y \leqslant z \leqslant x$. The usual order interval $[x,y]$ is the set of all points order-between x and y. Note that $[x,y] = \emptyset$, if x and y are not comparable.

Definition 3.1.2.

$A \subseteq Z^\infty$ is said to be order convex, if $[x,y] \subseteq A$ for every $x,y \in A$. This is a weak definition of convexity so that even the finite dimensional projections of order convex sets need not be order convex in the corresponding finite dimensional submodule.

For, Example 3.1.3:

The two element set

$A = \{x=(1,0,1,0,1,0,\dots),\ y = (0,1,0,1,0,1,\dots)\}$ is trivially order convex, but the 2-dimensional projection $\pi_2(A)$ on say the first and third co-ordinates, defined by

$$\begin{aligned} x &\longrightarrow (1,1) \\ y &\longrightarrow (0,0) \end{aligned}$$ is not order convex in Z^2, since

$$[(1,1),\ (0,0)] = \{(0,0),\ (1,0),\ (0,1),\ (1,1)\} \not\subseteq \pi_2(A).$$

Hence we modify the definition of the interval $[x,y]$ as follows.

Definition 3.1.4.

For $x = (x_1, x_2, \ldots)$, $y=(y_1, y_2, \ldots) \in Z^\infty$, define $\langle x,y\rangle$ as $\left\{z=(z_1, z_2, \ldots) \in Z^\infty \mid z_i \text{ lies between } x_i \text{ and } y_i \text{ for all } i\right\}$.

Thus we have a stronger definition of order convexity.

Definition 3.1.5.

$A \subseteq Z^\infty$ is said to be strongly order convex, if $\langle x,y\rangle \subseteq A$, for every $x,y \in A$. As the definition indicates, A is strongly order convex, implies that A is order convex, for $[x,y] = \langle x,y\rangle$, if $x \leqslant y$. Example 3.1.3 itself shows that the converse is not true.

3.2. d-CONVEXITY

We extend the definition of integer valued metrics in Z^n that we have considered in Chapter 1, to Z^∞ as follows.

For $x = (x_i)_{i=1}^\infty$, $y=(y_i)_{i=1}^\infty \in Z^\infty$, define the extended metrics

$$d_1(x,y) = \sum_{i=1}^\infty |x_i - y_i|, \text{ if the sum is finite.}$$

That is if and only if all except a finite number of x_i's and y_i's are zero, or $x_i=y_i$ for all except a finite number of i's.

and $d_1(x,y) = \infty$, otherwise.

$$d_2(x,y) = \max_{1 \leqslant i \leqslant \infty} |x_i - y_i| \text{ if and only if the}$$

sequence $\{z_n\}$, where $z_n = |x_n - y_n|$ is bounded

and $d_2(x,y) = \infty$, otherwise.

and $d_3(x,y) =$ the number of co-ordinates in which x and y differ, if $x_i = y_i$, for all except a finite number of i's.

and $d_3(x,y) = \infty$, otherwise.

Note that these extended metrics are integer valued, when they are finite.

We define, for any extended metric 'd', the d-interval d-[x,y] as follows:

Definition 3.2.1.

For $x = (x_i)_{i=1}^{\infty}$, $y = (y_i)_{i=1}^{\infty} \in Z^{\infty}$, d-[x,y] is defined as $\{z \in Z^{\infty} | \pi_{\nu}(z) \in d^n\text{-}[\pi_{\nu}(x), \pi_{\nu}(y)]$ for all $n \in N$, where π_{ν} denotes the projection to the first n co-ordinates, and $d^n\text{-}[\pi_{\nu}(x), \pi_{\nu}(y)]$ denotes the d-interval in the corresponding submodule Z^n. Note that, when $d(x,y) < \infty$,

$$d\text{-}[x,y] = \{z \in Z^{\infty} | d(x,z) + d(z,y) = d(x,y)\}.$$

Definition 3.2.2.

$A \subseteq Z^\infty$ is said to be d-convex, if the d-interval d-$[x,y] \subseteq A$ for all $x,y \in A$. We will show that the "d-convexity in Z^∞" is stronger than the d-convexity in all the finite dimensional projections. We need a lemma from Z^n, namely

Lemma 3.2.3.

In Z^n, the projection of a d-convex set to any lower dimensional submodule $Z^m (m<n)$ is d-convex in Z^m. The proof follows easily,

since $\pi_\mu(d\text{-}[x,y]) = d\text{-}[\pi_\mu(x), \pi_\mu(y)]$.

Now we have

Theorem 3.2.4.

If A is a d-convex subset of Z^∞, then every finite dimensional projection of A is d-convex, in the corresponding finite dimensional submodule of Z^∞.

Proof:

Suppose that A is a d-convex subset of Z^∞. Assume that m is the largest integer, for which the projection $\pi_\mu(A)$ is not d-convex in the corresponding submodule Z^m. That is, $\pi_\nu(A)$ is d-convex, for every $n>m$. There are two cases.

Case (i):

When $\pi_\mu = \pi_\mu{}^*$, where $\pi_\mu{}^*$ denotes the projection to the first m-cordinates. Consider $\pi_{\mu+1}{}^*$. Now fix some $a \in A$ and consider $\pi_{m+1}(a)$, the projection to the $m+1^{th}$ co-ordinate. Let $A_{\mu+1}$ be the subset of $\pi_{\mu+1}{}^*(A)$, consisting of points with $m+1^{th}$ co-ordinate $= \pi_{m+1}(a)$. By assumption $A_{\mu+1}$ is d-convex and $\pi_\mu(A_{\mu+1}) = \pi_\mu(A)$, and the assumption that $\pi_\mu(A)$ is not d-convex, contradicts lemma 3.2.3, and hence the theorem for case (i).

Case (ii):

When $\pi_\mu \neq \pi_\mu{}^*$. Let n be a natural number greater than m, such that Z^m is a submodule of Z^n, where $Z^n = \pi_\nu{}^*(Z^\infty)$. Now using the same argument to that of case (i), we get a contradiction to lemma 3.2.3, and hence the theorem for case(ii).

The following example shows that the converse of theorem 3.2.4 need not be true.

Example 3.2.5.

Let $a = (a_i)_{i=1}^\infty$, $b = (b_i)_{i=1}^\infty$ be two members of Z^∞, having all entries nonzero and distinct in all co-ordinates.

Let $A = \{z=(z_1,\ldots,z_n,\ z_{n+1},0,0,\ldots) \in Z^\infty|$

$(z_1,\ldots,z_n) \in d^n\text{-}[\pi_\nu(a),\ \pi_\nu(b)]$ and

$z_{n+1}=a_{n+1}$ or $z_{n+1} = b_{n+1},\ z_j=0,$ for $j>n+1\}$

for $n \in N$ varies.

It is clear that every n-dimensional projection of A is d-convex (since projection to the first n- co-ordinates is d-convex) in the corresponding submodule Z^n, but A is not d-convex in Z^∞,

For if $x,y \in A$, then $x = (x_1,x_2,\ldots,x_n,\ x_{n+1},0,\ \ldots\quad)$

and $y = (y_1,y_2,\ldots,y_m,y_{m+1},0,\ \ldots\quad)$, where

$(x_1,x_2,\ldots,x_n) \in d^n\text{-}[\pi_\nu(a),\pi_\nu(b)]$ and $x_{n+1}=a_{n+1}$ or b_{n+1}

$x_j=0$, for $j > n+1$

and

$(y_1,y_2,\ldots,y_m) \in d^m\text{-}[\pi_\mu(a),\ \pi_\mu(b)]$ and $y_{m+1}=a_{m+1}$ or b_{m+1}

and $y_j=0$, for all $j > m+1$.

Assume that $m > n$.

Then

$$d\text{-}[x,y] = \{z \in Z^\infty|\pi_\nu(z) \in d^n\text{-}[\pi_\nu(x),\pi_\nu(y)] \text{ for all } n$$

$$= \{z \in Z^\infty|\pi_{\mu+1}(z) \in d^{m+1}\text{-}[\pi_{\mu+1}(x),\pi_{m+1}(y)],$$

$$\pi_j(z) = 0, \text{ for } j > m+1\}.$$

Now there exist an $z \in d-[x,y]$, such that $z \notin A$, for, if $z = (z_1,\ldots,z_{m+1}, z_{m+2}, 0,0,\ldots)$, where $a_{m+2}<z_{m+2}<b_{m+2}$ or $b_{m+2}<z_{m+2}<a_{m+2}$, then $z \in d-[x,y]$, but $z \notin A$. Since $z \in A$ is of the form $z = (z_1,\ldots,z_{m+1}, z_{m+2}, 0, 0, \ldots)$, where $(z_1,\ldots,z_{m+1}) \in d^{m+1}-[\pi_{\mu+1}(a),\pi_{\mu+1}(b)]$, $z_{m+2}=a_{m+2}$ or $z_{m+2} = b_{m+2}$ and $z_j = 0$, for all $j > m+2$.

Now we will show that d_1-convexity and strong order convexity are equivalent.

Theorem 3.2.6.

$A \subseteq Z^\infty$ is strongly order convex, if and only if A is d_1-convex.

Proof:

A is strongly order convex implies that for any pair of points $x,y \in A$, $\langle x,y\rangle \subseteq A$.

Let $z \in \langle x,y\rangle \Longleftrightarrow x_i\leqslant z_i\leqslant y_i$ or $y_i\leqslant z_i\leqslant x_i$ for all i

$$\Longleftrightarrow |x_i-z_i| + |z_i-y_i| = |x_i-y_i| \text{ for all } i$$

$$\Longleftrightarrow \sum_{i=1}^{n}|x_i-z_i| + \sum_{i=1}^{n}|z_i-y_i| = \sum_{i=1}^{n}|x_i-y_i|$$

for all $n \in N$, by lemma 2.2.1.

$$\Longleftrightarrow d_1\ (\pi_{\nu}(x),\ \pi_{\nu}(z)) + d_1\ (\pi_{\nu}(z),\ \pi_{\nu}(y))$$

$$= d_1\ (\pi_{\nu}(x),\ \pi_{\nu}(y)) \text{ for all } n \in N$$

where π_{ν} denotes the projection to the first n co-ordinates.

$$\Longleftrightarrow \pi_{\nu}(z) \in d_1{}^n\text{-}[\pi_{\nu}(x),\ \pi_{\nu}(y)] \text{ for all } n \in N$$

$\Longleftrightarrow z \in d_1\text{-}[x,y]$, and hence the theorem.

3.3. INVARIANTS OF d-CONVEXITY

For the order convexity in Z^{∞}, the Caratheodory number is 2 ([17]), and there is no finite Helly and Radon numbers, since there is no finite Helly and Radon numbers, for the order convexity in Z^n.
We have

Theorem 3.3.1.

Helly number for the d_1-convexity in Z^{∞} is 2 .

Proof:

Let $\mathcal{F} = \{A_1, A_2, \ldots, A_m\}$ be a finite family of order convex sets in Z^{∞}, with pairwise nonempty intersection. To show that $\bigcap_{j=1}^{m} A_j \neq \emptyset$.

By theorem 3.2.4, every finite dimensional projection of members of $\mathcal{F}$ is nonempty, d_1-convex and pairwise intersecting in the corresponding finite dimensional submodule of Z^{∞}. Therefore by Helly's theorem in the finite dimensions (2.2.7), we have $\bigcap_{j=1}^{m} \pi_{\nu}(A_j) \neq \emptyset$,

for any n-dimensional projection $\pi_{\nu}: Z^{\infty} \longrightarrow Z^{n}$. Let

$\pi_i(A_j)$ denotes the projection to the i^{th} co-ordinate for all i and $j=1,\ldots,m$.

Since $\bigcap_{j=1}^{m} \pi_i(A_j) \neq \emptyset$, let $x_i \in \bigcap_{j=1}^{m} \pi_i(A_j)$, for all i

Now $x = (x_i)_{i=1}^{\infty} \in A_j$ for all i

i.e., $x \in \bigcap_{j=1}^{m} A_j$, hence the theorem.

Note 3.3.2.

The example given by Kay and Womble [29] itself shows that there is no finite Radon number for the d_1-convexity in Z^{∞}. Also note that there is no finite Caratheodory number for the d_1-convexity in Z^{∞}, since the Caratheodory number for the d_1-convexity in Z^{n} is n (2.2.5).

In chapter 1, it is shown that the Caratheodory, Helly and Radon numbers for the d_2-convexity in Z^{n} is 2^{n-1},

2^n and 2^n+1 respectively (2.3.4, 2.3.5 and 2.3.6 respectively). Also the Caratheodory number for the d_3-convexity in Z^n is n (2.4.5), and there is no finite Helly and Radon numbers for d_3-convexity in Z^n (2.4.6). Hence we have

Theorem 3.3.3.

There exists no finite Caratheodory, Helly and Radon numbers for the d_2-convexity in Z^∞, and

Theorem 3.3.4.

There exists no finite Caratheodory, Helly and Radon numbers for the d_3-convexity in Z^∞.

Chapter-4

CONVEXITY IN GENERALIZED POLYGONS

4.1. INTRODUCTION

In this chapter, we study the convexity in the finite geometric structure, known as " Generalized Polygons", considering it as a bipartite graph, denoted by Γ . It is observed that the m-convexity in Γ is the trivial convexity, consisting of the whole vertex set of Γ and $\emptyset$. But the geodesic convexity (d-convexity) in Γ has close similarity with a convex geometry ([12]). We believe that the generalized polygons can be characterized using the geodesic convexity in certain bipartite graphs. For details about Generalized Polygons, see ([16], [37], [40]).

Definition 4.1.1.

A finite incidence structure is a triple S:(P,L,I) in which P and L are nonempty disjoint finite sets of objects, called points and lines respectively and I is a symmetric point-line incidence relation.

Definition 4.1.2.

A path from an element x to an element y in $P \cup L$

is a sequence $(x_i)_{i=o}^{r}$ such that $x_o=x$, x_{i-1} I x_i, for $i=1,2,\ldots,r$ and $x_r=y$. 'r' is called the length of the path. A finite incidence structure $S = (P,L,I)$ is said to be connected if every two elements in $P \cup L$ can be joined by a path. Note that if $S = (P,L,I)$ is a finite connected incidence structure, then S is a finite metric space with $d(x,y)$ = length of the shortest path from x to y.

Definition 4.1.3.

A finite connected incidence structure $S=(P,L,I)$ is called a generalized n-gon, for some positive integer n, if the following are satisfied:

(i) $d(x,y) \leqslant n$, for all $x,y \in P \cup L$

(ii) If $d(x,y) < n$, then there is a unique path between x and y.

(iii) For each $x \in P \cup L$, there is a $y \in P \cup L$, such that $d(x,y) = n$.

If S is a generalized n-gon, then S is said to have order (s,t) $(s \geqslant 1,\ t \geqslant 1)$, if there are exactly $t+1$ lines incident with each point and $s+1$ points incident with each line. A generalized polygon is a generalized n-gon, for some integer n. When $s=t=1$, we get ordinary polygons. We have a famous non-existence theorem of a generalized n-gon due to Feit-Higman. See ([16] and [40]).

Theorem 4.1.4 (Feit-Higman)

Apart from the ordinary polygons, with $s=t=1$, a generalized n-gon can exist, only if $n \in \{2,3,4,6,8,12\}$.

Now S can be considered as a bipartite graph with vertex set $V = P \cup L$ and two vertices adjacent in the graph if and only if they are incident in the n-gon. We denote by Γ, the bipartite graph corresponding to a generalized n-gon of order (s,t).

The theory of convexity has a natural role in graph theory. See Farber ([13]), Farber and Jamison ([14],[15]), Düchet ([8],[9]) for the notions of geodesic convexity (d-convexity) and minimal path convexity (m-convexity) in a finite connected graph. Let G be any graph with vertex set V. A chord of a path in G is an edge joining two non consecutive vertices in the path.

Definition 4.1.5:

A set $K \subseteq V$ is said to be d-convex (m-convex) if for any pair of vertices $x,y \in K$, all vertices on all shortest (chordless) paths from x to y also lie in K.

That is $K \subseteq V$ is d-convex, if $d\text{-}[x,y] \subseteq K$, for every pair of vertices $x,y \in K$, where

$$\begin{aligned} d\text{-}[x,y] &= \{z \in V \mid z \text{ lies in a shortest } x\text{-}y \text{ path}\} \\ &= \{z \in V \mid d(x,z) + d(z,y) = d(x,y)\} \end{aligned}$$

and $K \subseteq V$ is m-convex, if $m\text{-}[x,y] \subseteq K$, for every $x,y \in K$, where $m\text{-}[x,y] = \{z \in V \mid z \text{ lies in a chordless path from } x \text{ to } y\}$

Definition 4.1.6.

If $K \subseteq V$ is convex (d-convex or m-convex), a vertex $v \in K$ is said to be an extreme point of K, if $K \setminus v$ is again convex. EX(K) denotes the set of all extreme points of K and K is said to have the Krein-Milman property if $K = \mathrm{conv}(EX(K))$.

For any vertex $v \in V$, $N_j(v)$ denotes the neighbourhood of radius j about v. That is $N_j(v) = \{z \in V \mid d(z,v) \leqslant j\}$, for some integer j. For $S \subseteq V$, the diameter of S, denoted by diam(S) is $\mathrm{Sup}\,\{d(x,y) \mid x,y \in S\}$. The radius of S with respect to V is $\inf\,\{r : S \subseteq N_r(x), \text{ for some } x \in V\}$. It is noted that $\mathrm{diam}(V) = \mathrm{radius}(V) = n = \mathrm{diam}(\Gamma)$. We need two lemmas, the proofs of which are seen in [16].

Lemma 4.1.7.

Let Γ be the bipartite graph corresponding to a generalized n-gon of order (s,t), with vertex set $V:P\cup L$. If $x \in P$ and $d(x,y) = n$, then there are exactly t+1 distinct paths of length 'n' from x to y. Similarly if $x \in L$ and $d(x,y) = n$, there are exactly s+1 distinct paths of length n from x to y.

Lemma 4.1.8.

Let Γ be as in lemma 4.1.7. Then n is odd implies that s=t.

4.2 GEODESIC CONVEXITY

We have

Lemma 4.2.1.

In Γ, if $d(x,y)=n$, then d-$[x,y]$, contains every neighbour of x as well as every neighbour of y.

Proof:

When n is even, then both x and y either belong to P or belong to L. That is, x and y are of the same type and when n is odd then s=t, by Lemma 4.1.8. Let $x \in P$, then x is adjacent with t+1 distinct vertices in L.

Let $x_1, x_2, \ldots, x_{t+1}$ be the t+1 distinct vertices in L adjacent with x. We have $d(x_i, y) = n-1$, for all $i = 1, \ldots, t+1$, for if $d(x_i, y) < n-1$, for some i, then there is a path of length less than n from x to y, which is a contradiction. If $d(x_i, y) = n$, for some i, then $d(x, y) = n$ and $d(x_i, y) = n$ implies that x and x_i are vertices of the same type, which is also a contradiction, since x and x_i are adjacent. Therefore all the t+1 neighbours of x belong to d-[x,y]. Similarly if $x \in L$, all the s+1 neighbours of x belong to d-[x,y]. By the same argument, we can show that all the neighbours of y also belong to d-[x,y].

Note 4.2.2.

d-[x,y] is not d-convex always. For example, let one of s,t is greater than one, say t and $d(x,y)=n$. Then d-[x,y] is not d-convex, for there are t+1 or s+1 distinct shortest paths from x to y, according as $x \in P$ or $x \in L$. Suppose $x \in L$ and let

$$\begin{array}{llllll}
x, & x_{11}, & x_{12}, & \ldots & , x_{1n-1}, & y \\
x, & x_{21}, & x_{22}, & \ldots & , x_{2n-1}, & y \\
\ldots & \ldots & \ldots & \ldots & \ldots & \\
x, & x_{s+11}, & x_{s+12}, & \ldots & , x_{s+1\,n-1}, & y
\end{array}$$

be the s+1 distinct shortest paths of length n from x to y. We can easily see that $d(x_{i1},x_{jn-1}) = n$, for $i \neq j$, $i,j \in \{1,\ldots,s+1\}$. Now $x_{i1} \in P$ and therefore there are t+1 distinct shortest paths of length n from x_{i1} to x_{jn-1} of which only 2 paths namely $x_{i1},x_{i2},\ldots,x_{in-1}y\ x_{jn-1}$ and $x_{i1}\ x\ x_{j},\ldots,x_{jn-1}$ belong to d-[x,y]. Thus d-[x,y] is not d-convex. Note that in a generalized n-gon Γ, when $d(x,y) = n$, the d-interval d-[x,y] contains, for each neighbour x_{i1} of x, a neighbour x_{jn-1} of y with $d(x_{i1}, x_{jn-1}) = n$. Now we have

Theorem 4.2.3.

If K is a d-convex subset of $V(\Gamma)$, with $\mathrm{diam}(K) = n$, then $K = V$.

Proof:

Since $\mathrm{diam}(K) = n$, there are vertices $x,y \in K$ with $d(x,y) = n$. Let z be any vertex of Γ. Since Γ is connected, there is a path from x to z. Let $x,x_1,x_2,\ldots,x_r=z$ be the shortest path from x to z. Since $d(x,y) = n$ and K is d-convex, by lemma 4.2.1, every neighbour of x belongs to d-[x,y] and hence belongs to K. Therefore $x_1 \in K$.

Also as in Note 4.2.2, d-$[x,y]$ contains a vertex y_1 with $d(x_1,y_1) = n$. Therefore again by Lemma 4.2.1, $x_2 \in K$. Again there exists a neighbour y_2 of y in d-$[x_1,y_1]$ with $d(x_2,y_2) = n$, and by Lemma 4.2.1, $x_3 \in K$. Applying Lemma 4.2.1, successively, we get $z \in K$. Hence $K = V$.

Corollary 4.2.4.

If $K \subseteq V$ is a proper d-convex subset of Γ, then $\operatorname{diam}(K) < n$.

Corollary 4.2.5.

If $K \subseteq V$ is a proper d-convex subset of Γ, then K is a subtree of Γ **and hence K has** the Krein-Milman property.

Proof:

By Corollary 4.2.4, if $K \subseteq V$ is a proper d-convex subset, then $\operatorname{diam}(K) < n$ and therefore, for every pair x,y of vertices of K, $d(x,y) < n$, and by the defining condition (ii) (4.1.3) of a generalized n-gon, there is a unique path from x to y, which is contained in K. In other words, the subgraph induced by K of Γ contains no cycle and hence is a tree. Clearly K has the Krein-Milman property and all the end vertices of K (vertices of K having degree one in the subgraph induced by K) are the extreme points of K.

Now we have a theorem of a general nature.

Theorem 4.2.6.

Let G be any connected bipartite graph in which every vertex has degree at least two. Let K be a d-convex subset of G. If K has the Krein-Milman property, then diam(K) < diam(G).

Proof:

Suppose diam(K) = diam(G) = n. Since K has the Krein-Milman property, it is the d-convex hull of its extreme points. Now there exists two extreme points x,y of K, which are diametrically opposite vertices of K. We have that x is an extreme point of the d-convex subset K, if and only if x is a simplicial vertex in the subgraph induced by K. A vertex x of K is called a simplicial vertex, if the neighbourhood $(N_1(x))$ of x induces a complete subgraph in the subgraph induced by K. Since G is a bipartite graph, x is a simplicial vertex implies that $N_1(x)$ consists of a single vertex in the subgraph induced by K.

Since each vertex has degree at least 2, x is adjacent with a vertex $z \notin K$. Now $d(x,y) = n$ and $d(x,z) = 1$ implies that $d(y,z) = n$, for if $d(y,z) < n$, then $z \in K$, a contradiction. Therefore we have $d(x,y) = n$, $d(z,y) = n$ and $d(x,z) = 1$. In a bipartite graph, this is not possible, because x and z are vertices belonging to distinct partition classes of vertices. Therefore our assumption is wrong, and hence $\mathrm{diam}(K) < \mathrm{diam}(G)$.

Now we have

Theorem 4.2.7.

If Γ is the bipartite graph corresponding to a generalized n-gon and K is a d-convex subset of Γ. Then K has the Krein-Milman property if and only if $\mathrm{diam}(K) < n$.

Proof:

The necessity part follows from Theorem 4.2.6, and sufficiency follows from Corollary 4.2.5, since $\mathrm{diam}(K) < n$ implies that K is a proper d-convex subset and by Corollary 4.2.5, K has the Krein-Milman property.

Theorem 4.2.8.

In Γ, $N_j(v)$ is d-convex, for $j \leqslant [\frac{n-1}{2}]$, for all $v \in V$.

Proof:

For any two vertices $x, y \in N_j(v)$, we have $d(x,y) \leqslant n-1$. Therefore, there is a unique path from x to y in Γ, which is contained in $N_j(v)$.

Next we show that the m-convexity in Γ is the trivial convexity.

Theorem 4.2.9.

The m-convexity in Γ is the trivial convexity consisting of the null set $\emptyset$ and the whole vertex set V.

Proof:

For any vertex $x \in V$, by the defining condition (iii), of a generalized n-gon (condition (iii) of 4.1.3), there exists a vertex $y \in V$ with $d(x,y) = n$, and by Lemma 4.1.7, there exists t+1 or s+1 distinct shortest paths from x to y according as $x \in P$ or $x \in L$, and hence there are $\binom{t+1}{2}$ or $\binom{s+1}{2}$ distinct cycles containing both x and y. Note that all the cycles are chordless, for if there is a chord in one of the cycles, then there will be a path of length less than n from x to y. Hence $m\text{-}[x,x]$ or $m\text{-}[y,y]$ contains both x and y. We have by Theorem 4.2.3 $d\text{-conv}\{x,y\} = V$, since $d(x,y) = n$, and d-conv $\{x,y\} \subseteq$ m-conv $\{x,y\}$, since every shortest path is a chordless path. Therefore m-conv $\{x,y\} = V$. Since $y \in$ m-$[x,x]$, m-conv $\{x\}$ = m-conv $\{x,y\}$

$= V$.

Thus the m-convexity in Γ is the trivial convexity consisting only the null set $\emptyset$ and the whole vertex set V of Γ.

4.3. CENTRALITY

Nieminen has studied various center concepts in connection with the geodesic convexity in connected graphs. See([34], [35], [36]). In this section, we discuss the center, centroid and distance center of Γ, in connection with the geodesic convexity in Γ. We need the following definitions. See ([22], [34], [35], [36]). Let G be any finite connected undirected graph, without loops and multiple edges.

Definition 4.3.1.

The eccentricity $e(v)$ of a vertex $v \in V(G)$ is $e(v) = \max\{d(u,v) \mid u \in V\}$. The center C_e of G is the set consisting of vertices of G having minimum eccentricity. If $K \subseteq V(G)$, then the center of K with respect to V, denoted as center (K) is the center of the subgraph of G induced by K.

Definition 4.3.2.

For any vertex $v \in V$, a copoint of v is a maximal convex subset of $V \setminus v$, denoted by C_v. That is C_v is a convex subset of $V \setminus v$ having maximum cardinality.

Definition 4.3.3.

The centroid C of G is $\{v \in V \mid |C_v| = m\}$, where $m = \inf \{ |C_v| \mid v \in V \}$. That is, the centroid of G consists of vertices v with the property that, their copoints C_v has minimum cardinality.

Definition 4.3.4.

The distance d(v) of a vertex v in V(G) is the sum $d(v) = \sum_{u \in V} d(u,v)$. The distance center C_d of G, also called the median of G consists of vertices of G, having minimum distance.

If Γ is the bipartite graph corresponding to a generalized n-gon of order (s,t), then we have the following theorems. (In this section by a convex subset of V, we mean a d-convex subset of V.)

Theorem 4.3.5.

The center of a d-convex subset K of Γ is d-convex.

Proof:

If $K = V(\Gamma)$, the whole vertex set, then we have center(V) = V, since $e(v) = n$, for all $v \in V(\Gamma)$ and therefore the vertices of minimum eccentricity consists of all the vertices of Γ, and V is d-convex. If $K \subseteq V$, then by Corollary 4.2.5, the subgraph induced by K is a subtree of Γ, and for a tree, the center consists of either one vertex or two adjacent vertices (see Harary ([22]), and hence d-convex. Hence the theorem.

Theorem 4.3.6.

The centroid C of Γ is the whole vertex set $V(\Gamma)$.

Proof:

For any vertex $v \in V(\Gamma)$, let C_v be a copoint of v. That is C_v is a maximal d-convex subset of Γ not containing v. Therefore C_v is such that diam $(C_v) = n-1$ and diam $(C_v \cup \{v\}) = n$. We claim that C_v is a maximal proper d-convex subset of Γ. If K is a proper d-convex subset of Γ, with $C_v \subsetneq K \subsetneq V$, then $v \in K$ and hence diam(K) = n, and by theorem 4.2.3, K=V and hence C_v is a maximal proper d-convex subset of V. Therefore, C_v is a copoint of every vertex of $V \setminus C_v$. Now $V \setminus C_v$ contains vertices in P as well as vertices in L. Therefore C_v is a copoint of a vertex in P as well as a vertex in L. Since every vertex in P is identical and every vertex in L is identical, the copoint of every vertex in Γ has got the same cardinality. Therefore the centroid C of Γ is the whole vertex set V.

Theorem 4.3.7.

If C_d is the distance center of Γ of order (s,t) with vertex set $V = P \cup L$, then (i) $C_d = P$ if and only if $s<t$, (ii) $C_d = L$ if and only if $t<s$, and (iii) $C_d = P \cup L = V$ if and only if $s=t$.

Proof:

To find the distance center C_d of Γ, we shall find out the vertices in Γ having minimum distance. Since every vertex in P is identical, every vertex in P has got the same distance. Similarly every vertex in L has got the same distance.

For a vertex $p \in P$, we have

$$d(p) = 1(t+1) + 2(t+1)s + 3(t+1)st + \ldots + (n-1)(t+1)\, s^{[\frac{n}{2}]-1}\, t^{[\frac{n}{2}]-1}$$

$$+ n(|V| - ((t+1)+(t+1)s+(t+1)st + (t+1)st + \ldots + (t+1)s^{[\frac{n}{2}]-1}\, t^{[\frac{n}{2}]-1}))$$

$$= n|V| - ((n-1)(t+1)+(n-2)(t+1)s + (n-3)(t+1)st + \ldots + 1(t+1)s^{[\frac{n}{2}]-1}\, t^{[\frac{n}{2}]-1})$$

Similarly, for a vertex $\ell \in L$, we have,

$$d(\ell) = n|V| - ((n-1)(s+1) + (n-2)(s+1)t + (n-3)(s+1)ts + \ldots + 1(s+1)t^{[\frac{n}{2}]-1}\, s^{[\frac{n}{2}]-1})$$

Therefore we have

$d(\ell) < d(p)$ if and only if

$$(n-1)(s+1)+(n-2)(s+1)t+ \ldots + 1(s+1)t^{[\frac{n}{2}]-1}\, s^{[\frac{n}{2}]-1}$$

$$> (n-1)(t+1)+(n-2)(t+1)s + \ldots + (t+1)s^{[\frac{n}{2}]-1}\, t^{[\frac{n}{2}]-1}.$$

That is, if and only if

$$(n-1)(s-t)+(n-2)(t-s)+(n-3)ts(s-t)+(n-4)ts(t-s)+ \dots +$$

$$(s-t)s^{[\frac{n}{2}]-1}\ t^{[\frac{n}{2}]-1} > 0.$$

That is $d(\ell) < d(p)$ if and only if

$$(s-t)+(s-t)ts + (s-1)t^2s^2 + \dots + (s-t)t^{[\frac{n}{2}]-1}\ s^{[\frac{n}{2}]-1} > 0.$$

That is if and only if

$$(s-t)\ (1+ts+t^2s^2+ \dots + t^{[\frac{n}{2}]-1}\ s^{[\frac{n}{2}]-1}) > 0$$

That is $d(\ell) < d(p)$ if and only if $(s-t) > 0$,

Since $1+ts+t^2s^2+ \dots + t^{[\frac{n}{2}]-1}\ s^{[\frac{n}{2}]-1} > 0$,

since $s \geqslant 1$ and $t \geqslant 1$ always.

Therefore we have

(i) $d(\ell) < d(p)$ if and only if $t < s$

(ii) $d(p) < d(\ell)$ if and only if $s < t$

(iii) $d(p) = d(\ell)$ if and only if $s = t$.

Therefore the vertices of Γ having minimum distance belong to L if and only if $t<s$, belong to P if and only if $s<t$, and all vertices of Γ have the same distance if and only if $s=t$. Hence we have

$C_d = L$ if and only if $t < s$

$C_d = P$ if and only if $s < t$

and $C_d = P \cup L = V$ if and only if $s=t$.

Also we have,

Corollary 4.3.8.

The distance center C_d of Γ order (s,t) is d-convex if and only if $s=t$.

Proof:

The distance center C_d is d-convex implies that $C_d=V$ by Theorem 4.3.7, since P and L are not d-convex subsets of Γ and again by Theorem 4.3.7 $C_d=V$ if and only if $s=t$.

4.4. INVARIANTS OF GEODESIC CONVEXITY.

In this section, we shall compute the invariants of the geodesic convexity in Γ, like the Helly, Caratheodory, Radon and Generalized Radon type numbers.

Theorem 4.4.1.

The Helly number h for the geodesic convexity in Γ is 2 if n=2 and is 3 if $n \geqslant 3$.

We prove the theorem using a lemma.

Lemma 4.4.2.

If $A \subseteq V(\Gamma)$ is an independent set with $|A| \geqslant 3$, and d-conv(A) $\neq V$, then there exists a vertex $v \in V$ $v \notin A$ such that

$$v \in \bigcap \left\{ \text{d-conv}(A \setminus a) \mid a \in A \right\}.$$

Proof:

Let $A \subseteq V$ be an independent set with $|A| = 3$. A is said to be independent, if $a \notin$ d-conv$(A \setminus a)$, for any $a \in A$. Let $A = \{v_1, v_2, v_3\}$ and let diam(A) = r. Suppose $d(v_1, v_2) = r$. Clearly $r \leqslant n-1$, since d-conv(A) $\neq V$.

Suppose $\quad \bigcap \left\{ \text{d-conv}(A \setminus v_i) \mid v_i \in A \right\} = \emptyset \qquad (1)$

Since d-conv(A) is a proper d-convex subset of Γ, d-$[v_1, v_2]$, d-$[v_2, v_3]$ and d-$[v_3, v_1]$ are all proper d-convex subsets of Γ.

Let $d(v_2,v_3) = r_1$ and $d(v_3,v_1) = r_2$. Clearly $r_1 \leqslant r$ and $r_2 \leqslant r$.

(1) gives that $d\text{-}[v_1,v_2] \cap d\text{-}[v_2,v_3] \cap d\text{-}[v_3,v_1] = \emptyset$.

Let $v_1,y_1,y_2,\ldots,y_r = v_2$, $v_2,z_1,z_2,\ldots,z_{r_1} = v_3$ and $v_3,w_1,w_2,\ldots,w_{r_2} = v_1$ be the unique paths from v_1 to v_2, v_2 to v_3 and v_3 to v_1 respectively. Now $v_1,y_1,\ldots,y_r = v_2$, $z_1,z_2,\ldots,z_r = v_3,w_1,\ldots,w_{r_2} = v_1$ is a cycle in d-conv(A), which is a contradiction, since d-conv(A) is a subtree of Γ, being a proper d-convex subset. Hence the lemma.

Proof of Theorem 4.4.1.

We use the definition given by Sierksma of the Helly number h (1.5.3). If n=2, consider a subset A of $V(\Gamma)$ with $|A| = 3$. Clearly, $\mathrm{diam}(A) = \mathrm{diam}(\Gamma) = 2$, and hence A contains 2 vertices a_1,a_2 with $d(a_1,a_2) = 2$ and hence $d\text{-conv}(A) = d\text{-conv}(\{a_1,a_2\}) = V$ by Theorem 4.2.3, and thus $\bigcap \{d\text{-conv}(A \setminus a) \mid a \in A\} \neq \emptyset$ hence h=2, if n=2.

If $n \geqslant 3$, consider $A \subseteq V(\Gamma)$ with $|A|=4$ and let A be an independent set, for if A is dependent then we are done.

Now there exists a subset $B \subseteq A$ with $|B| = 3$ and d-conv$(B) \neq V$, for if not, then A is dependent. Therefore by Lemma 4.4.2, there is a vertex $v \in V$, $v \notin B$ such that $v \in \bigcap \{\text{d-conv}(B \setminus b) \mid b \in B\}$. Clearly $v \in \bigcap \{\text{d-conv}(A \setminus a) \mid a \in A\}$. Thus $h \leq 3$. To show that $h = 3$, consider a subset S of $V(\Gamma)$ with $|S| = 3$ and d-conv$(S) = V$. Let $S = \{v_1, v_2, v_3\}$ be such that $d(v_1,v_2) = n-1$, $d(v_2,v_3) = n-1$ and $d(v_3,v_1) = 2$ (we can always find such a set of 3 vertices in Γ). Thus d-$[v_1,v_2]$, d-$[v_2,v_3]$ and d-$[v_3,v_1]$ are all proper d-convex subsets of $V(\Gamma)$ and they are all pairwise intersecting but d-$[v_2,v_3] \cap$ d-$[v_3,v_1] \cap$ d-$[v_1,v_2] = \emptyset$. Therefore $h > 2$. Therefore $h = 3$, if $n > 2$.

Before we look into the Caratheodory number, we need a lemma.

Lemma 4.4.3.

Let $A = \{a_1, a_2, \ldots, a_r\} \subseteq V(\Gamma)$ be an independent set with $|A| \geq 4$ and diam$(A) \neq n$, then d-conv$(A) \neq V$.

Proof:

We prove the lemma using induction on the cardinality of A.

When $|A| = 4$, let $A = \{a_1, a_2, a_3, a_4\}$.

Since A is independent, d-conv($A \setminus a_i$) is a proper d-convex subset of V, for all $a_i \in A$. Consider d-conv($A \setminus a_4$). d-conv($A \setminus a_4$) being a proper d-convex subset of V has the Krein-Milman property by Corollary 4.2.5, and

$$\text{EX}(\text{d-conv}(A \setminus a_4)) = \{a_1, a_2, a_3\}, \text{ since } \{a_1, a_2, a_3\}$$

is independent.

We have by Lemma 4.4.2. there exists a vertex $v \in V$, $v \notin \{a_1, a_2, a_3\}$ such that

$$v \in \text{d-}[a_1, a_2] \cap \text{d-}[a_2, a_3] \cap \text{d-}[a_3, a_1]$$

Now consider d-conv(A).

We claim that $v \in$ d-$[a_4, a_i]$, for all i=1,2,3, for if $v \notin$ d-$[a_4, a_i]$, for some a_i, say a_1, then d-$[v_1, a_1] \cup$ d-$[a_1, a_4] \cup$ d-$[a_4, v_1]$ is a cycle in d-conv $\{a_1, a_2, a_4\}$ and hence

$$\text{d-conv}\{a_1, a_2, a_4\} = V, \text{ implies that } a_3 \in \text{d-conv}\{a_1, a_2, a_4\}$$

a contradiction to the assumption that $\{a_1, a_2, a_3, a_4\}$ is independent. Therefore $v \in$ d-$[a_4, a_i]$ for all i=1,2,3.

Thus we have

for any $x \in$ d-conv $\{a_1, a_2, a_3\}$

$$d\text{-}[x, a_4] \subseteq d\text{-}[a_i, a_4] \subseteq d\text{-conv}(A), \text{ for some } a_i \in \{a_1, a_2, a_3\},$$

since d-$[a_i, a_4]$ is a proper d-convex subset of V. Therefore any point $x \in$ d-conv $\{a_1, a_2, a_3\}$ can be joined by a unique path to the vertex a_4, since diam(A) $\neq$ n and thus every two vertices in d-conv(A) can be joined by a unique shortest path. Hence d-conv(A) does not contain a cycle and d-conv(A) $\neq$ V.

Induction:

If $A = \{a_1, \ldots, a_m\} \subseteq V$ is an independent set with $|A| \geqslant 4$ and diam(A) $\neq$ n, then d-conv(A) $\neq$ V.

To prove for A with $|A| = m+1$

Let $A = \{a_1, \ldots, a_m, a_{m+1}\} \subseteq V$ be an independent set with diam(A) $\neq$ n.

To prove that d-conv(A) $\neq$ V.

Suppose d-conv(A) = V.

We have d-conv $\{a_1, a_2, \ldots, a_m\} \neq V$, d-conv $\{a_1, \ldots, a_m\}$ has the Krein-Milman property by Corollary 4.2.5 and $EX(\text{d-conv}\{a_1, \ldots, a_m\}) = \{a_1, \ldots, a_m\}$, since $\{a_1, \ldots, a_m\}$ is independent.

Therefore d-conv(A) = d-conv $\{a_1,\ldots,a_{m+1}\} = V$, only if there exists at least two $a_i, a_j \in \{a_1,\ldots,a_m\}$, such that $d\text{-}[a_{m+1},a_i] \cap d\text{-}[a_{m+1},a_j] = \emptyset$.

Therefore d-conv $\{a_i,a_j,a_{m+1}\} = V$, which shows that $\{a_1,\ldots,a_{m+1}\}$ is dependent, a contradiction. Therefore d-conv(A) $\neq V$, and the lemma follows by induction.

Corollary 4.4.4.

Let $A = \{a_1,\ldots,a_r\} \subseteq V$ be an independent set and diam(A) $\neq n$. If d-conv(A) = V, then $|A| = 3$.

Proof:

By Lemma 4.4.3. we have $|A| \leqslant 3$. That is $|A| = 2$ or $|A| = 3$. Since diam $A \neq n$, $|A| \neq 2$, for then d-conv(A) is a proper d-interval and cannot be V.

Therefore $|A| = 3$.

Now we have

Theorem 4.4.5.

The Caratheodory number c, for the geodesic convexity in Γ is 2, if n=2 and is 3 if $n \geqslant 3$.

Proof:

For n=2, the proof follows easily. For $n \geq 3$, there are two cases.

Case (i):

Let $A \subseteq V$ be any subset with $|A| \geq 3$ and d-conv(A) $\neq$ V. For case (i), by Corollary 4.2.5. d-conv(A) is the d-convex hull of its extreme points.

Clearly we have

$$\begin{aligned} \text{d-conv}(A) &= \bigcup \left\{ \text{d-}[v_i, v_j] \mid v_i, v_j \in EX(\text{d-conv}(A)) \right\} \\ &= \bigcup \left\{ \text{d-}[v_i, v_j] \mid v_i, v_j \in A \right\}, \quad \text{since} \end{aligned}$$

every extreme point of d-conv(A) is a member of A. Hence for case (i), the d-convex hull of any subset A of V can be expressed as the union of d-convex hulls of two point subsets of A.

Case (ii):

Let $A \subseteq V$ be any subset with d-conv(A) = V and diam(A) $\neq$ n.

If diam(A) = n, then by Theorem 4.2.3,

$\text{d-conv}(A) = V = \text{d-conv}\{a_1, a_2\}$, where $a_1, a_2 \in A$ with $d(a_1, a_2) = n$, and hence we are done.

For case (ii), by Corollary 4.4.4, there exists an independent subset A' of A with $|A'| = 3$ such that d-conv(A') = d-conv(A) = V and hence the theorem.

Theorem 4.4.6.

The Radon number r, for the geodesic convexity in Γ is 3 if n=2, and is 4 if $n \geqslant 3$.

Proof:

The proof follows from Theorem 4.4.1, Lemma 4.4.2 and Theorem 4.4.5.

Note 4.4.7.

Note that diam(A) = diam (d-conv(A)) for all $A \subseteq V(\Gamma)$, except for subsets A, which are of the type of case (ii) of Theorem 4.4.5 and that it is shown in the proof of Theorem 4.4.5 that the minimum cardinality of such exceptional subsets is 3.

Next we shall find a bound of the generalized Radon number P_m for the geodesic convexity in Γ. We need a theorem of Jamison. See Jamison-Waldner ([27]).

If P is a point in an aligned space X, then a copoint at P is a maximal convex subset of $X \setminus P$. We shall say that X satisfies the copoint intersection property CIP (m,k), if for each P in X, among any m distinct copoints at P, there are k with empty intersection. We state the theorem as

Remark 4.4.8. (Jamison-Waldner):

Suppose that an aligned space X satisfies CIP (3,2), and has finite Helly number h. Then the partition number P_m satisfy

(i) $P_m \leqslant 2m$ if $h=2$

(ii) $P_m = (m-1)h+1$ if $h \geqslant 3$

In [27], it is proved that if G is a graph theoretic tree with vertex set V, then the geodesic alignment on V(G) satisfies CIP(3,2). Since the geodesic alignment on a tree has helly number 2, $P_m \leqslant 2m$, for the geodesic alignment on a graph theoretic tree.

Now we have

Theorem 4.4.9.

The generalized Radon number P_m, for the geodesic alignment in Γ satisfies the inequality $P_m \leqslant 3m-2$.

Proof:

The proof is a successive application of Remark 4.4.8. and Caratheodory's theorem (4.4.5). Let $A \subseteq V(\Gamma)$ be a subset with $|A| \geqslant 3m-2$. We will show that A has a Radon m-partition. If d-conv(A) $\neq$ V, then the subgraph induced by d-conv(A) is a subtree of Γ by Theorem 4.2.3, and hence A has a Radon m-partition by Remark 4.4.8 , since $3m-2 \geqslant 2m$ if $m \geqslant 2$. Now if d-conv(A) = V, then by Theorem 4.4.5, there exist a subset A_1 of A with $|A_1| = 3$ and d-conv(A_1) = d-conv(A)=V. Now $B_1 = A \setminus A_1$ has cardinality $\geqslant 3m-5$. If d-conv(B_1) $\neq$ V, then B_1 has a Radon (m-1) partition if $m \geqslant 3$, by Remark 4.4.8, since $3m-5 \geqslant 2(m-1)$ if $m \geqslant 3$, and hence A has a Radon m-partition. If d-conv(B_1) = V, then by Theorem 4.4.5, there exists a subset A_2 of B_1 with $|A_2|=3$ and d-conv(A_2) = d-conv(B_1) = V. Now $B_2 = B_1 \setminus A_2$ has cardinality $\geqslant 3m-8$. If d-conv(B_2) $\neq$ V, then by the same argument, B_2 has a Radon (m-2)-partition, if $m \geqslant 4$, by Remark 4.4.8, and B_1 has a Radon (m-1) partition, and hence A has a Radon m-partition. If d-conv(B_2)=V, then again by Theorem 4.4.5, there exist $A_3 \subseteq B_2$ with $|A_3| = 3$ and d-conv(A_3) = d-conv(B_2) = V. This procedure continues, until we get a Radon (m-2) partition of A with each partition class containing 3 members and the d-convex hull of each partition class being V. Thus we are left with a subset B of A of cardinality greater than or equal to 4 and by Radon's theorem (4.4.6) B has a Radon partition and hence A has a Radon m-partition. Thus $P_m \leqslant 3m-2$.

Remark 4.4.10.

If Γ is a generalized 3-gon, then it can be shown that $P_m = 3m-2$.

Consider a generalized 3-gon of order s. Take s sufficiently larger than m. (There exists generalized 3-gons of order $s = p^{\alpha}$, p prime, $\alpha \in N$, being finite projective planes). Consider (m-1) distinct vertices $\ell_1, \ell_2, \ldots, \ell_{m-1}$ of L and define a subset A of P with cardinality 3(m-1) as follows.

$A = A_1 \cup \cdot \ldots\ldots \cup A_{m-1}$, where A_i's are disjoint subsets of P each with cardinality 3 and such that ℓ_i is the common vertex adjacent to all the three vertices of A_i, for i=1,..., m-1. Then $|A| = 3(m-1)$ and it can be shown easily that A has no Radon m-partition. If A has a Radon m-partition, then there exist at least two partition classes each with cardinality 2, say $B_i = \{P_i, P_i'\}$ and $B_j = \{P_j, P_j'\}$.

Now,

$$\text{d-conv}(B_i) = \{P_i,\ P_i',\ \ell\} \text{ and}$$
$$\text{d-conv}(B_j) = \{P_j,\ P_j',\ \ell'\}\ , \text{ where } \ell \text{ and } \ell'$$

are the unique vertices of L adjacent with P_i and P_i' and P_j and P_j' respectively. Thus $\text{d-conv}(B_i) \cap \text{d-conv}(B_j) = \emptyset$. Therefore A cannot have a Radon m-partition, and thus $P_m = 3m-2$, for the geodesic convexity in Γ, if Γ is a generalized 3-gon of order $s > m$. A generalized polygon of order (s,t) is called **thick**, if $s>1$ and $t>1$. We believe that if Γ is a thick generalized polygon, then $P_m = 3m-2$. This is one of the problems, which we have attempted, but could not answer completely.

4.5. ORDER AND GEODESIC ALIGNMENTS OF A CONNECTED BIPARTITE GRAPH

In this section, we will show that the geodesic alignment on the vertex set V of any finite connected bipartite graph G is the join of order alignments with respect to all possible canonical orderings on V.

Let G be any finite connected bipartite graph with vertex set V. We can order V with respect to any vertex u, which give rise to a graded poset on V. A graded poset V is a poset V with a height function $h:V \longrightarrow Z$, such that

(i) if $u \leqslant v$, then $h(u) \leqslant h(v)$

(ii) if v covers u, then $h(v) = h(u)+1$, the integer $h(v)$ is called the height of v.

Let u be any vertex of G. For $i=0,1,\ldots \operatorname{diam}(G)-1$, we direct the edges between $N_i(u)$ and $N_{i+1}(u)$ from $N_{i+1}(u)$ to $N_i(u)$, where $N_i(u)$ denotes the i^{th} level of u. That is, $N_i(u) = \{v \in V \mid d(u,v) = i\}$.

Define $v \leqslant_u w$, whenever there is a directed path from w to v. With this ordering on V with respect to the vertex u, gives a graded poset $(V \leqslant_u)$, of which G is the digraph. The height function is $h_u(v)=d(u,v)$ for $v \in V$. That is $h_u(v) = i$, for any $v \in N_i(u)$. Since G is connected, we have $u \leqslant_u v$, for all $v \in V$, and so u is the universal lower bound of this poset. This kind of ordering on the vertex set V of a finite connected bipartite graph G has been considered by Mülder in [33]. The ordering so constructed on V is called the canonical ordering of G with respect to the vertex u.

We have a theorem of Mülder, we state it as

Remark 4.5.1. (Mülder)

A graph G is connected and bipartite if and only if G is the digraph of a finite graded poset with a universal lower bound.

Let E denote any canonical ordering of G, and let D_E denote the order alignment on V with respect to E, where, as usual $K \subseteq V$ is said to be order convex, if $[x,y] = \{z \in V \mid x \leqslant z \leqslant y \text{ or } y \leqslant z \leqslant x\} \subseteq K$, for every $x, y \in K$.

Definition 4.5.2.

For any set X, if $(L_i)_{i \in I}$ is a collection of alignments on X, then the smallest alignment R on X, containing all L_i's is called the join of L_i's in the lattice of all alignments on X, denoted by $R = \bigvee_{i \in I} L_i$.
It is shown by Jamison in [25] that if $R = \bigvee_{i \in I} L_i$, then $R(A) = \bigcap_{i \in I} L_i(A)$ for all finite subsets A of X.
If this is true for all subsets of X, then $R = \bigvee_{i \in I} L_i$ is called the strong join of L_i's.

Now we have

Theorem 4.5.3.

If $\mathcal{L}$ denotes the geodesic alignment on the vertex set V of a finite connected bipartite graph G, then $\mathcal{L} = \bigvee \{D_E \mid E \in$ Canonical orderings of $G\}$, where D_E denotes the order alignment on V with respect to the canonical ordering E.

Proof:

Suppose $K \in \mathcal{L}$. Now every d-convex subset of V induces a connected subgraph of G. Therefore the subgraph induced by K of G is connected and bipartite, since G is bipartite. Therefore by Remark 4.5.1., there exists an ordering on K which gives a graded poset on K with a universal lower bound, say u. Since $u \in V$, there is a canonical ordering E on V, with u as the universal lower bound. Clearly K is a sub poset of the graded poset $(V, \leqslant_u)$ and hence $K \in D_E$.
Therefore,

$$K \in \bigvee D_E, \text{ thus } \mathcal{L} \subseteq \bigvee D_E.$$

Conversely suppose

$$K \in \bigvee \{D_E \mid E \in \text{canonical orderings of } G\}.$$

That is $K \in D_E$, for all canonical orderings E of G.
Let $K = \{u_1, \ldots, u_n\}$. Then $K \in D_{E_{u_i}}$, for all $i=1,\ldots,n$,
where $D_{E_{u_i}}$ denotes the canonical ordering on V, with u_i
as the universal lower bound.

Now consider any two $u_i, u_j \in K$ and $u \in d\text{-}[u_i, u_j]$.
We have $u_i \leqslant_{u_i} u \leqslant_{u_i} u_j$.

That is $u \in [u_i, u_j]$, the order interval with
respect to the canonical ordering E_{u_i} on V.

and $[u_i, u_j] \subseteq K$, since $K \in D_{E_{u_i}}$.

Therefore $d\text{-}[u_i, u_j] \subseteq K$, for every $u_i, u_j \in K$ and hence $K \in \mathcal{L}$

Thus $\mathcal{L} = \bigvee \{ D_E \mid E \in \text{canonical ordering of } G \}$
and hence the theorem.

Remark 4.5.4.

Several graphs are characterized using the convexity
structure on the vertex sets. For example chordal graphs,
Ptolemaic graphs, block graphs, bridged graphs etc. See
([12], [13], [14], [15]).

We believe that the following statement 'P' will hold, although we do not have a complete proof; it trivially holds for $n=2$ and for $s=t=1$.

P: A connected bipartite graph Γ with $\mathrm{diam}(\Gamma)=n$ and vertex set $V = P \cup L$, in which every vertex of P has degree $t+1$ and every vertex of L has degree $s+1$ $(s \geqslant 1,\ t \geqslant 1)$ and having the property that every proper d-convex subset of V has the Krein-Milman property, is the bipartite graph corresponding to a generalized n-gon of order (s,t).

Chapter-5

INTERSECTION CONVEX SETS AND d-CONVEX SETS IN Z^2

5.1. INTRODUCTION

In this chapter, we will show that the d-convex sets (d_1-convex or d_2-convex) in Z^n are intersection convex sets defined by Doignon, with special supporting half lattices. We discuss mainly the case with n=2, and finally an algorithm for computing the d_2-convex hull of a finite planar set in the discrete plane Z^2 is given, and the time complexity of the algorithm is computed. We consider Z^n as a crystallographic lattice defined by Doignon. We need some preliminary definitions and theorems. See Doignon [7].

Definition 5.1.1.

A lattice in R^n is the set $\left\{ \sum_{i=1}^{n} n_i a_i \mid n_i \in Z \right\}$ where $\{a_1, \ldots, a_n\}$ is any basis of R^n and R is the set of real numbers. When we take the standard basis $\{(0,\ldots,i,0,\ldots,0) \mid i=1, \text{ for } i=1,2,\ldots,n\}$, we get the lattice Z^n.

Definition 5.1.2.

The lattice lines, lattice half lines and lattice line segments are defined as the intersections of Z^n with the lines, half lines and line segments of R^n respectively, intersecting Z^n in at least two points. When n=2, the slope of the lattice line ℓ is defined as the slope of the line in R^2, whose intersection with Z^2, will give the line ℓ .

Definition 5.1.3.

A subset of Z^n is called intersection convex, if it is the intersection of Z^n with an ordinary convex subset of R^n.

Definition 5.1.4.

A half space of R^n is any subset of R^n, whose intersection with every line of R^n is either the empty set, a half line or a line.

Definition 5.1.5.

The intersections of Z^n with the half spaces of R^n are called half lattices. Note that every half lattice of Z^n is intersection convex, since every half space of R^n is ordinary convex.

Definition 5.1.6.

A half lattice H is said to support the intersection convex set C of Z^n, if it is minimal among all the half lattices containing C. Points in $H \cap C$ are said to be the contact points of C with H, and H is said to support C at each contact point.

Definition 5.1.7.

For any intersection convex set C of Z^n, the boundary of C is defined as the set $bd(C) = Z^n \cap fr(conv(C))$ where conv(C) denotes the ordinary convex hull of C in R^n and fr(conv(C)) denotes the boundary of conv(C) in R^n. We call the boundary of a supporting half lattice H of an intersection convex set C, the supporting line of C at the contact points of C with H.

Theorem 5.1.8.

Any intersection convex set of Z^n is the intersection of half lattices of Z^n and conversely.

The theorem states that every intersection convex set is an intersection of supporting half lattices.

We need a few more definitions in the discrete plane Z^2.

Definition 5.1.9.

Two points $x=(x_1,x_2)$ and $y=(y_1,y_2)$ of Z^2 are said to lie in a horizontal set or in a vertical set, if $y_1=y_2$ or $x_1=x_2$ respectively. We call a lattice line ℓ of Z^2 an axial line, if it is a horizontal set or a vertical set.

Definition 5.1.10.

A lattice line ℓ of Z^2 is called a diagonal line, if the slope of ℓ is +1 or -1.

Definition 5.1.11.

A half lattice H of Z^2 is said to be an axial half lattice or a diagonal half lattice, if the boundary of H is an axial line or a diagonal line respectively.

5.2. INTERSECTION CONVEX SETS AND d-CONVEX SETS

Theorem 5.2.1.

If A is either a d_1-convex set or a d_2-convex set of Z^n, then A is an intersection convex set of Z^n.

Proof:

Proof follows easily since A can be expressed as

$A = \operatorname{conv}(A) \cap Z^n$, where conv(A) denotes the ordinary convex hull of A in R^n.

Note 5.2.2.

Note that a d_3-convex subset of Z^n need not be intersection convex. For example, the set $A = \{(0,0),\ (2,0),\ (0,2),\ (2,2)\} \subseteq Z^2$ is d_3-convex in Z^2, but A is not intersection convex, since any ordinary convex subset containing A contains the points (1,0), (0,1), (1,1), which does not belong to A.

Lemma 5.2.3.

A lattice line ℓ of Z^2 is d_1-convex if and only if ℓ is axial.

Proof:

ℓ is an axial line implies that ℓ is d_1-convex. Conversely suppose ℓ is a lattice line which is d_1-convex, but not axial. Then there exist two points $x = (x_1,x_2)$, $y = (y_1,y_2)$ in ℓ with $x_1 \neq y_1$ and $x_2 \neq y_2$. Clearly the points $z_1 = (x_1,y_2)$ and $z_2 = (y_1,x_2) \in d_1$-$[x,y]$, but $z_1\ z_2 \notin \ell$, and hence the lemma.

Lemma 5.2.4.

A lattice line ℓ of Z^2 is d_2-convex, if and only if ℓ is diagonal.

Proof:

ℓ is a diagonal line implies that ℓ is d_2-convex. Conversely suppose that ℓ is a lattice line, which is d_2-convex, but not diagonal. Then there exist points $x=(x_1,x_2)$, $y=(y_1,y_2) \in \ell$ with $|x_1-y_1| < |x_2-y_2|$ and $d_2(x,y) > 2$.

Consider the point $z = (z_1,z_2)$, where $z_i=x_i \pm 1$. ($z_i=x_i+1$ if $x_i<y_i$ and $z_i=x_i-1$ if $x_i>y_i$ for $i=1,2$). Clearly $z \in d_2$-$[x,y]$ and x and z lie in a diagonal line. Since ℓ is not a diagonal line $z \notin \ell$, a contradiction, and hence the lemma.

Theorem 5.2.5.

A half lattice H of Z^2 is d_1-convex (respectively d_2-convex) if and only if H is an axial half lattice (respectively a diagonal half lattice).

Proof:

H is d_1-convex (respectively d_2-convex) if and only if the boundary of H is d_1-convex (respectively d_2-convex). Therefore the theorem follows by Lemma 5.2.3 and Lemma 5.2.4. Now we have the main theorem.

Theorem 5.2.6.

Every d_1-convex set (respectively d_2-convex set) of Z^2 is the intersection of axial half lattices (respectively diagonal half lattices) and conversely.

Proof:

Intersection of axial half lattices (respectively diagonal half lattices) in Z^2 is d_1-convex (respectively d_2-convex), since axial half lattices and diagonal half lattices are d_1-convex and d_2-convex respectively by Theorem 5.2.5. Conversely, every d_1-convex set and d_2-convex set in Z^2 is intersection convex by Theorem 5.2.1, and using Theorem 5.1.8, they are the intersections of supporting half lattices. Now the boundaries of the supporting half lattices must be d_1-convex or d_2-convex. Therefore the supporting half lattices must be axial respectively diagonal, if the set is d_1-convex, respectively d_2-convex. Hence the theorem.

Note 5.2.7.

A hemispace of a convexity space is a convex subset with a convex complement. In this situation, we observe that the axial half lattices and diagonal half lattices are the hemispaces of the d_1-convexity and d_2-convexity in Z^2 respectively, since they have convex complements.

5.3. AN ALGORITHM FOR DETERMINING THE d_2-CONVEX HULL OF A FINITE PLANAR SET IN Z^2.

Computation of the convex hull of a finite set of points particularly in the plane is an interesting problem in computational geometry. Preparata and Shamos ([38]) have described various convex hull algorithms in the plane and in higher dimensional spaces, its time complexity, and other related computational problems. In this section we describe an algorithm, which determines the d_2-convex hull of a finite set of points S in the discrete plane Z^2. The computation of the d_1-convex hull of S in Z^2 is very trivial, since it is the smallest rectangle in Z^2, containing S, with sides parallel to the co-ordinate axes.

Our algorithm is based on the fact that the d_2-conv(S) contains the intersection convex hull of S in Z^2 and the supporting half lattices of d_2-conv(S) are diagonal half lattices (Theorem 5.2.6). We first compute the intersection convex hull of S in Z^2, using the Graham Scan, described in [18], and add the necessary points to get the d_2-conv(S). The algorithm works in not more than $\left(\frac{n \log n}{\log 2}\right)$ + an + b operations where a and b are positive constants and n is the cardinality of S. The algorithm we give determines which points of S are the end vertices of d_2-conv(S), which of course define d_2-conv(S). The algorithm proceeds in four steps.

Let $S = \{X_1, X_2, \ldots, X_n\}$ be the given set in Z^2.

<u>Step-1</u>:

Find the intersection convex hull of S using Graham Scan ([18]), and obtain a list of points S' ordered by polar angle, which determines the extreme points of the intersection convex hull of S. It has been proved in [18], that step-1 takes not more than $\frac{n \log n}{\log 2} + cn$ operations. Let $S' = \{X_1, \ldots, X_r\}$, clearly $r \leqslant n$.

Step-2:

In this step, we remove the d_2-convexly dependent points in S', because we have d_2-conv(S)=d_2-conv(S')=d_2-conv(T), where T is an independent subset of S'. Start with 3 consecutive points in S', say X_i, X_{i+1},X_{i+2}. There are two possibilities.

(i) $d_2(X_i,X_{i+1}) + d_2(X_{i+1},X_{i+2}) = d_2(X_i,X_{i+2})$

Then we delete X_{i+1} from S', since it cannot be an independent point of S', and return to the beginning of Step-2, with the points X_i, X_{i+1}, X_{i+2} replaced by X_{i-1},X_i, X_{i+2} (where indices are reduced modulo r).

(ii) $d_2(X_i,X_{i+1})+d_2(X_{i+1},X_{i+2}) > d_2(X_i,X_{i+2})$. Then return to the beginning of Step-2 with the points X_i,X_{i+1}, X_{i+2} replaced by X_{i+1},X_{i+2},X_{i+3}. Note that each application of Step-2 either reduces by one the number of possible dependent points of S' or increases by one the current total number of points of S' considered. By arguing similarly as in Step-5 of Graham Scan, with less than 2r iterations of Step-2, we must be left with a subset T of d_2-convexly independent points of S'. The cardinality of T is at most four because the rank of the d_2-convexity (cardinality of the maximal independent set) in Z^2 is four. Let T = $\{X_i, X_{i+1},\dots,X_{i+t}\}$, where $t \leq 4$.

Step-3:

Add new points to obtain d_2-conv(S). Since the Caratheodory number for the d_2-convexity in Z^2 is 2, every point in d_2-conv(S) belongs to the d_2-convex hull of a subset of T of cardinality two. So we add points corresponding to two consecutive points in T. Start with two consecutive points X_i, X_{i+1} in T. Let ℓ be the lattice line joining X_i and X_{i+1}. There are two possibilities.

I. If the slope of ℓ is +1 or -1, then no point is to be added, because in this case ℓ is the boundary of a supporting diagonal half lattice of d_2-conv(S). Return to the beginning of Step-3 with the points X_i, X_{i+1} replaced by X_{i+1}, X_{i+2}.

II. If the slope of ℓ is different from +1 and -1, then proceed as follows:

Let $X_i = (x_i, y_i)$ and $X_{i+1} = (x_{i+1}, y_{i+1})$.

Let $x_{i+1} = x_i+k$ and $y_{i+1} = y_i+h$.

(i) If $k > h$ and $k-h$ is even, then add two points

$$\left(x_i + \left\lfloor\frac{k-h}{2}\right\rfloor,\ y_i - \left\lceil\frac{k-h}{2}\right\rceil\right),\ \left(x_{i+1} - \left\lfloor\frac{k-h}{2}\right\rfloor,\ y_{i+1} + \left\lceil\frac{k-h}{2}\right\rceil\right)$$

(ii) If $k > h$ and $k-h$ is odd, then

(a) if $|k| > |h|$, add four points

$$(x_i+ \lfloor\tfrac{k-h}{2}],\ y_i-\lfloor\tfrac{k-h}{2}]),\ (x_i+ \lfloor\tfrac{k-h}{2}]+1,\ y_i-[\tfrac{k-h}{2}])$$

$$(x_{i+1}-[\tfrac{k-h}{2}],\ y_{i+1}+[\tfrac{k-h}{2}]),\ (x_{i+1}-([\tfrac{k-h}{2}]+1), y_{i+1}+[\tfrac{k-h}{2}])$$

(b) if $|k| < |h|$, then add 4 points

$$(x_i+[\tfrac{k-h}{2}],\ y_i-[\tfrac{k-h}{2}]),\quad (x_i+[\tfrac{k-h}{2}],\ y_i-([\tfrac{k-h}{2}]+1)),$$

$$(x_{i+1}-[\tfrac{k-h}{2}],\ y_{i+1}+[\tfrac{k-h}{2}]),\ (x_{i+1}-[\tfrac{k-h}{2}], y_{i+1}+[\tfrac{k-h}{2}]+1)$$

(iii) If $h > k$ and $h-k$ is even, then add two points

$$(x_i-[\tfrac{h-k}{2}],\ y_i+[\tfrac{h-k}{2}]),\ (x_{i+1}+[\tfrac{h-k}{2}],\ y_{i+1}-[\tfrac{h-k}{2}])$$

(iv) If $h>k$ and $h-k$ is odd, then

(c) if $|h| > |k|$, add four points

$$(x_i-[\tfrac{h-k}{2}],\ y_i+[\tfrac{h-k}{2}]),\quad (x_{i+1}+[\tfrac{h-k}{2}],\ y_{i+1}-[\tfrac{h-k}{2}])$$

$$(x_i-[\tfrac{h-k}{2}],\ y_i+[\tfrac{h-k}{2}]+1), (x_{i+1}+[\tfrac{h-k}{2}],\ y_{i+1}-([\tfrac{h-k}{2}]+1))$$

(d) if $|h| < |k|$, then add four points,

$$(x_i-[\frac{h-k}{2}],\ y_i+[\frac{h-k}{2}]),\ (x_i-([\frac{h-k}{2}]+1), y_i+[\frac{h-k}{2}]),$$

$$(x_{i+1}+[\frac{h-k}{2}],\ y_{i+1}-[\frac{h-k}{2}]), (x_{i+1}+[\frac{h-k}{2}]+1, y_i-[\frac{h-k}{2}])$$

Return to the beginning of Step-3 with X_i, X_{i+1} replaced by X_{i+1}, X_{i+2}. Since there are at most four points in T, Step-3 requires the addition of at most sixteen new points and hence requires less than a constant number say c_1 operations. Let T' be the new set obtained. Clearly $|T'| \leqslant 20$.

Step-4:

Since the new set T' may contain interior points d_2-conv(S), we need to find the intersection convex hull of T' to obtain d_2-conv(S). So Step-4 is to determine the intersection convex hull of T' as in Step-1. By the end of Step-4, we are left with the end vertices of d_2-conv(S).

Since T' contain at most twenty points, step-4 requires at most $\frac{20 \log 20}{\log 2} + c.20 = c_2$ operations.

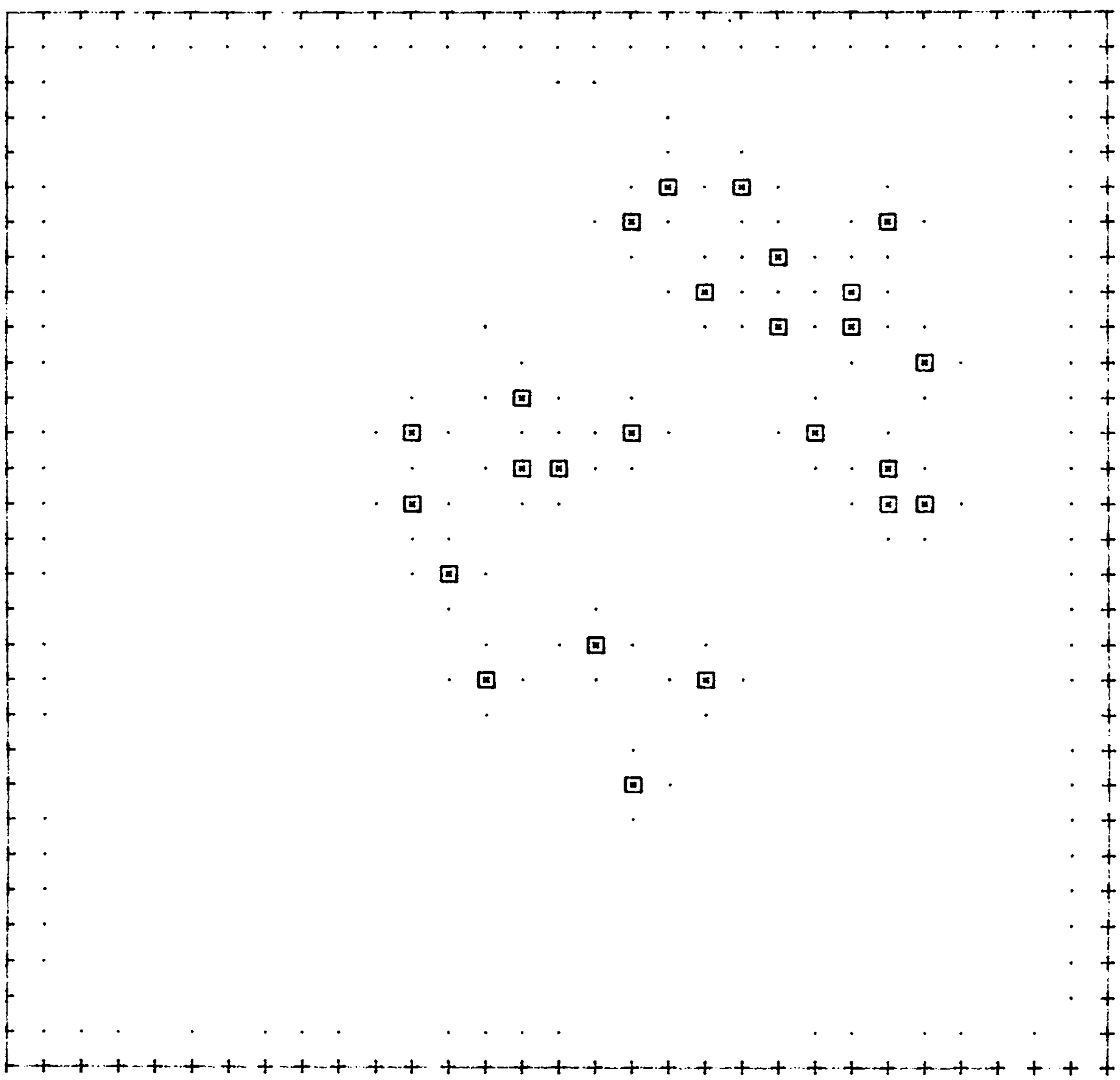
Therefore the time complexity of the algorithm is

$$\frac{n \log n}{\log 2} + cn + 2r + c_1+c_2, \text{ where } r \leqslant n$$

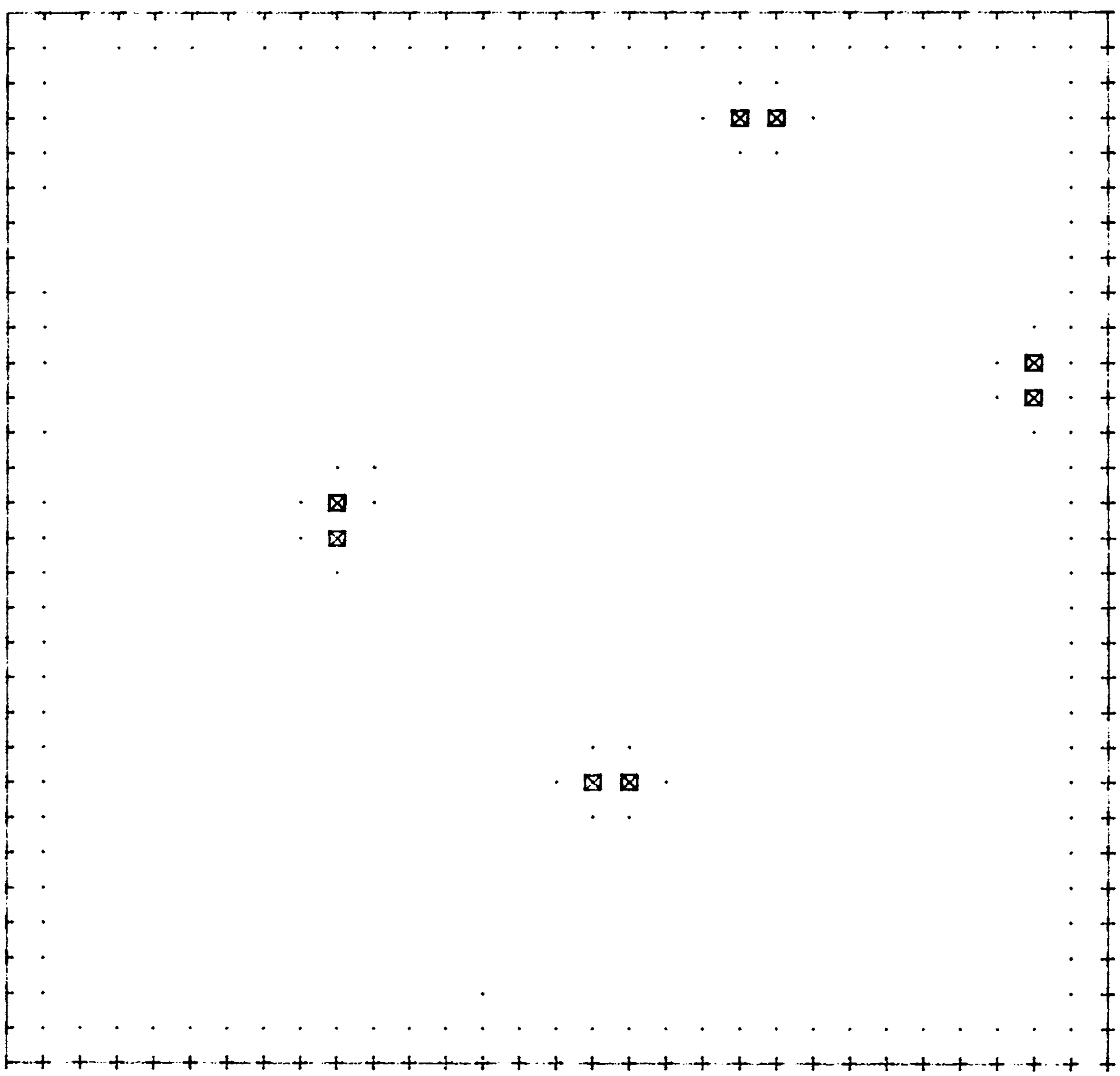
$$= \frac{n \log n}{20} + an + b, \text{ where a and b are positive constants.}$$

Computer implementation of this algorithm makes it quite feasible to consider examples with large n. Say n = 50000. We give some examples to illustrate this algorithm in the following pages. $\boxed{\times}$ represents the data point and $\boxtimes$ represents the end vertex of the d_2-convex hull.

A



d_2-conv(A)



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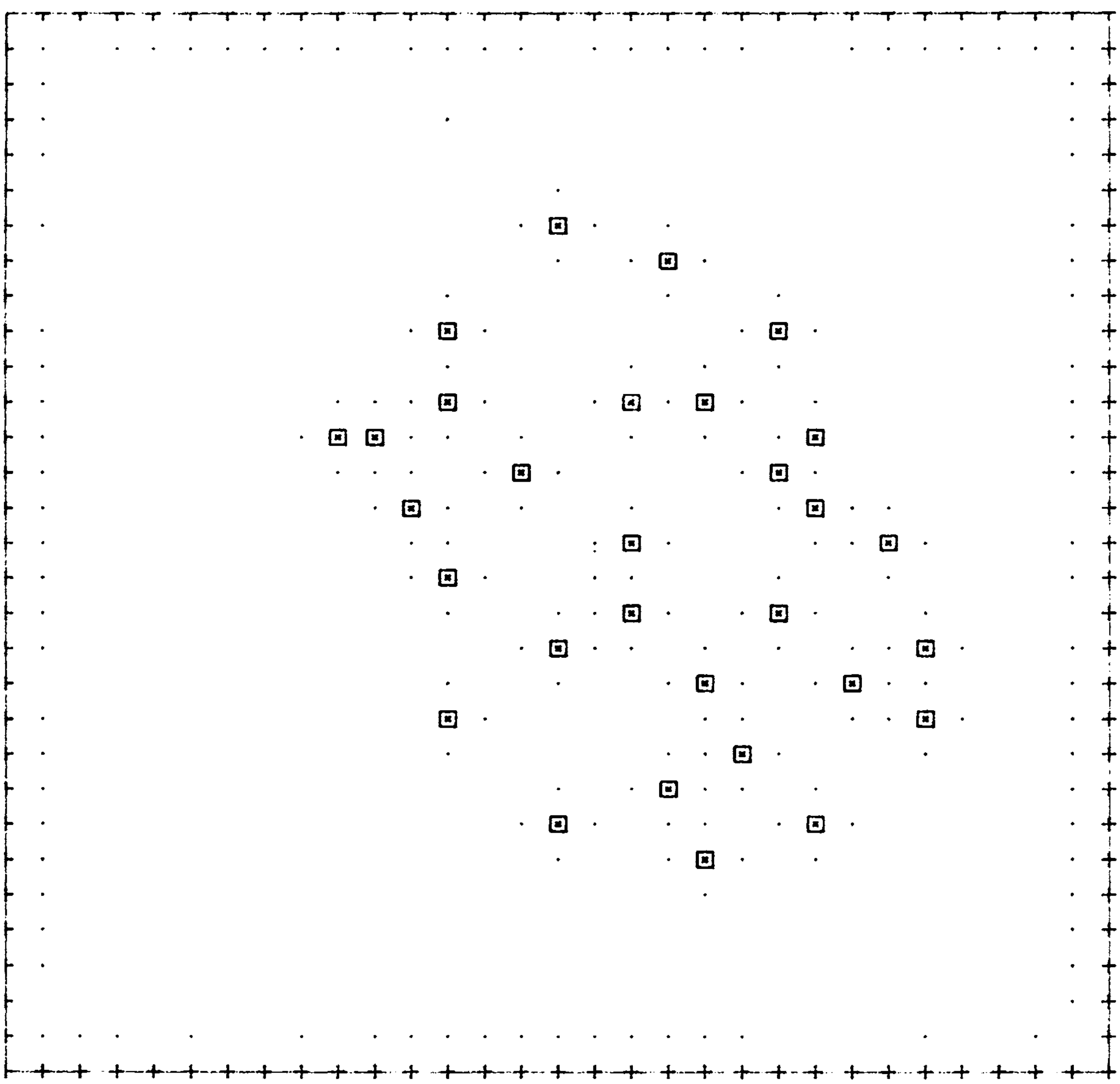

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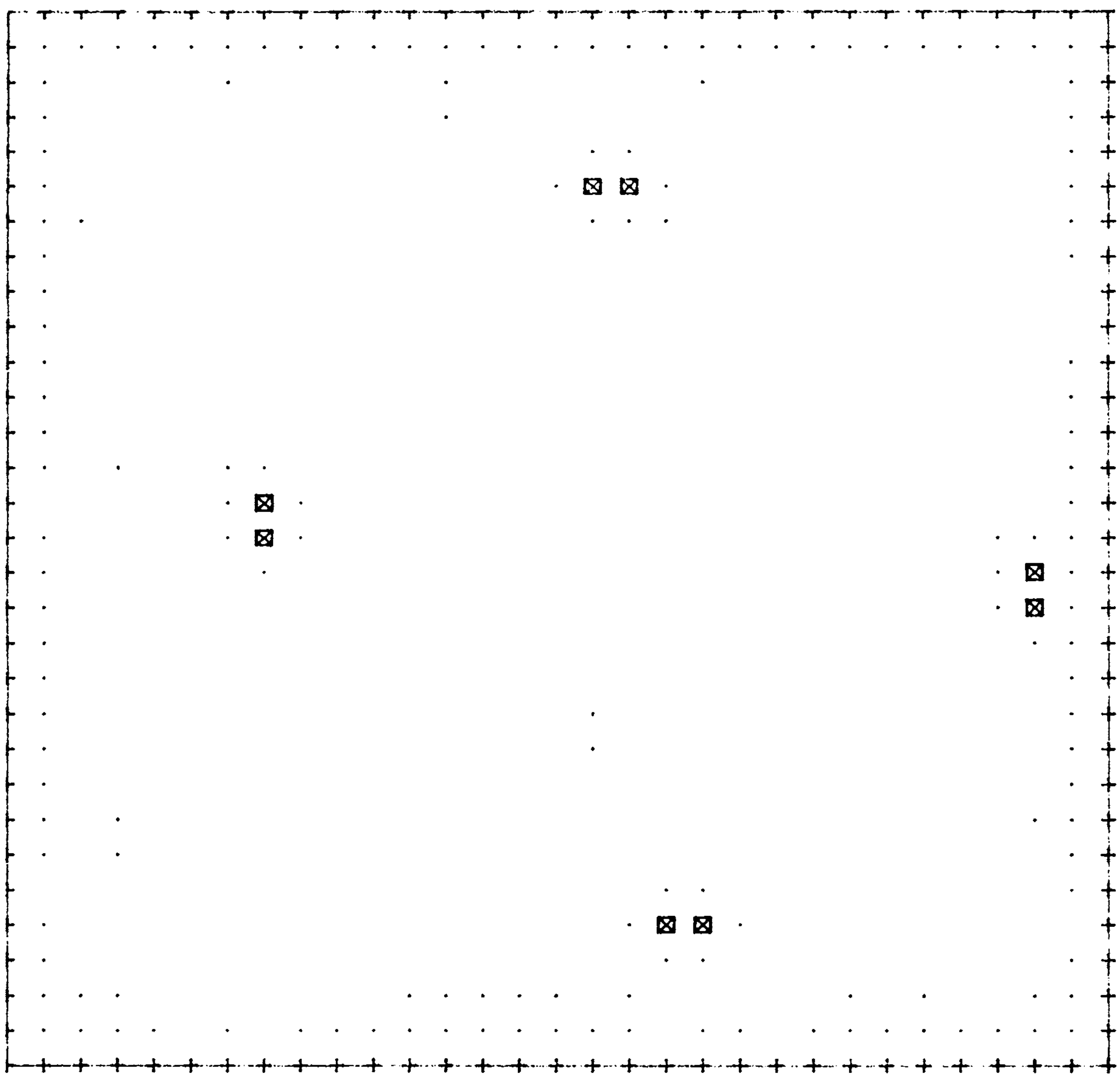

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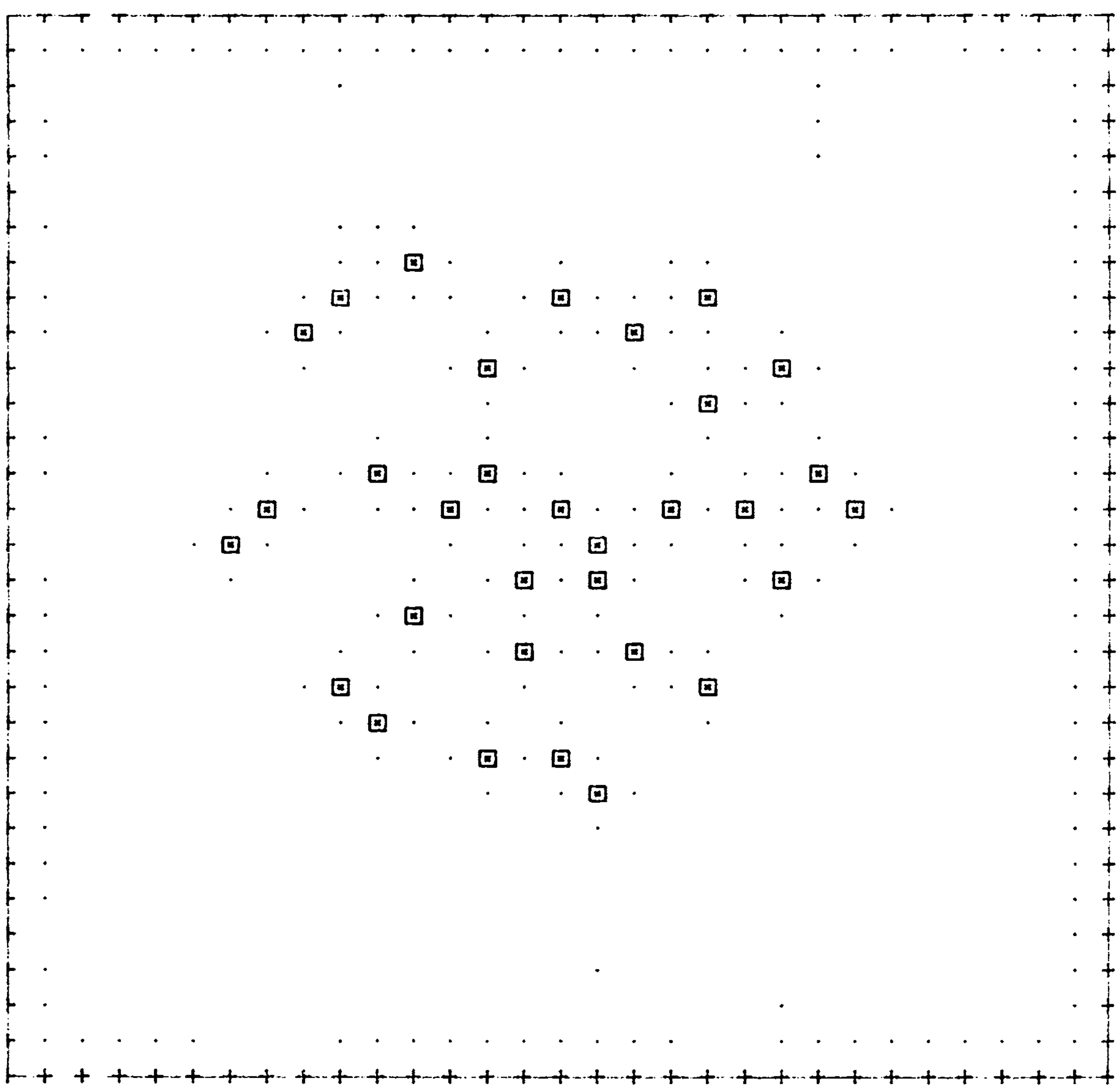

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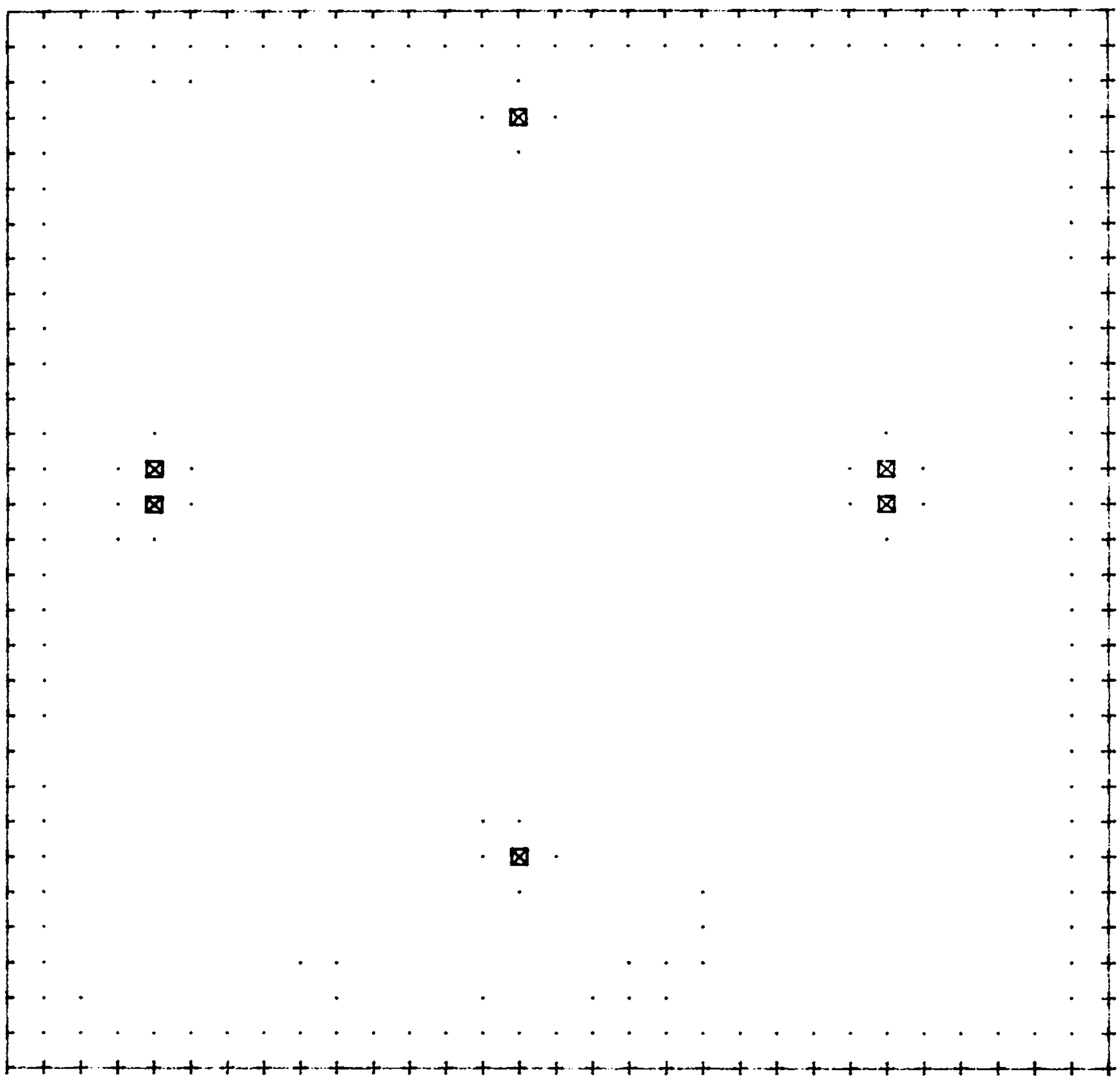

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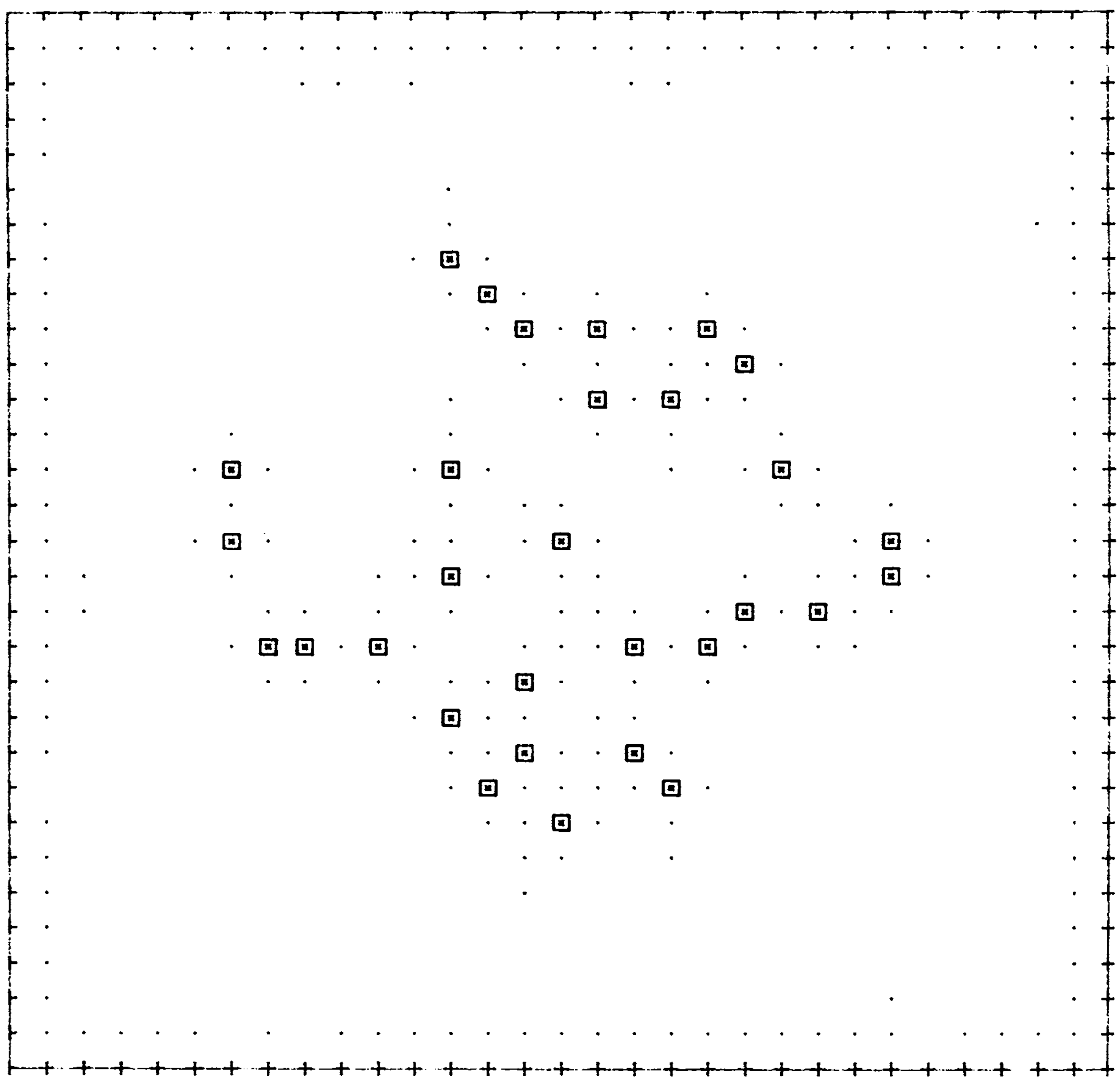
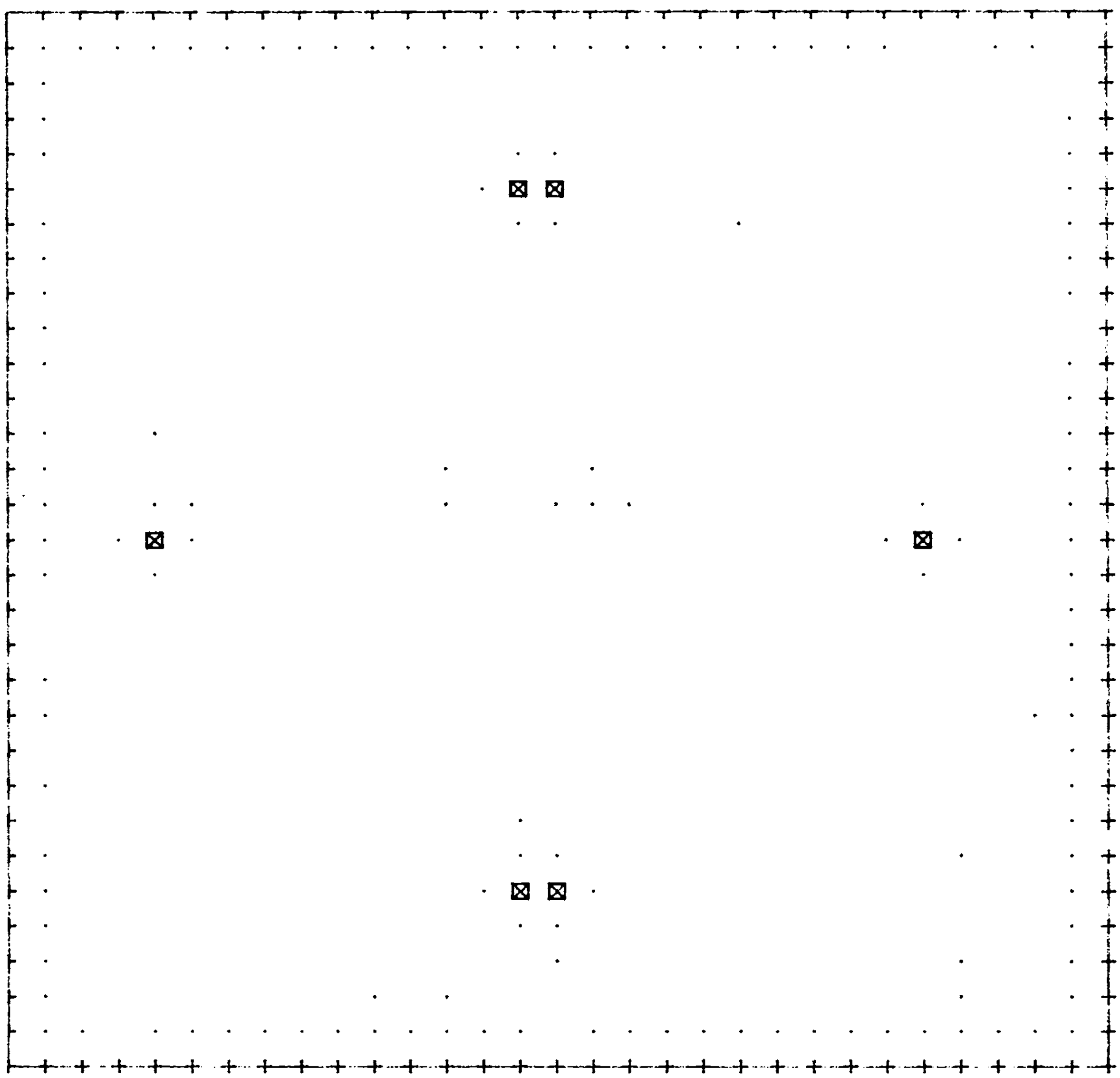


d_2-conv(A)



CONCLUDING REMARKS

Kay and Chartrand ([28]) defined a metric space (M,d) as a graph metric space, if there exists a connected graph G, whose vertex set can be put in one to one correspondence with the points of M in such a way that the distance between every two points of M is equal to the distance between the corresponding vertices of G. In that case the metric spaces (Z^2,d_1), (Z^2,d_2), (Z^2,d_3) are graph metric spaces.

In this thesis, we have studied mainly the d-convexity in these graph metric spaces. But we can easily show that the minimal path convexity (m-convexity) in all these graphs (graphs corresponding to the graph metric spaces (Z^2,d_1), (Z^2,d_2) and (Z^2,d_3)) is the trivial convexity consisting of the whole vertex set and $\emptyset$. We proved that the m-convexity in the bipartite graph Γ corresponding to a generalized n-gon is the trivial convexity. Thus in all the situations, where the d-convexity have been discussed the m-convexity is found to be the trivial convexity.

Hebbare ([23]) called the graphs, which have only the trivial d-convex sets as distance convex simple graphs. Similarly, we call the graphs, which have only the trivial m-convex sets, the m-convex simple graphs. Thus all the graphs which we have discussed in this thesis are m-convex simple graphs.

Another thing that we want to mention is about the Eckhoff's conjecture. For all the convexities that we have discussed in this thesis and for which the Tverberg type Radon number P_m or its bound has been computed, the Eckhoff's conjecture hold.

That is, $P_m \leqslant (m-1)(r-1)+1$, where r is the Radon number.

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SECTION-B : Those which were consulted during the investigation.

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