SOME PROBLEMS RELATED TO TOPOLOGY
AND
THEORY OF SEMIGROUPS

STUDIES ON SEMIGROUP COMPACTIFICATIONS OF TOPOLOGICAL SEMIGROUPS

THESIS SUBMITTED FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

By

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MAY 1990

CERTIFICATE

Certified that the work reported in this thesis is based on the bonafide work done by Smt.K.S.Kripalini, under my guidance in the Department of Mathematics and Statistics, Cochin University of Science and Technology, and has not been included in any other thesis submitted previously for the award of any degree.

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INTRODUCTION

1. HISTORICAL SURVEY

(a) Origin

The study of topological semigroups as an independent subject was possibly started in the fifties. This area was highlighted by A.D.Wallace in 1953 in his address to the American Mathematical Society as " What topological spaces admit a continuous associative multiplication with unit? " . As noted by Wallace, the answers to these questions involved more algebra and topology than was the case for compact groups, where there is a representation theory due to the presence of Haar measure. During these thirty seven years, the subject has developed in many directions and the literature is so vast that number of papers in the subject runs to several hundreds. Some of the main early contributors are K.H. Hofmann and P.S. Mostert (1966) [HO-M_2], A.B. Paalman-De-Miranda (1970) [P], A.C.Shershin (1979) [SH], K. Numakura [NU_1], E. Hewitt [HE], W.M.Faucett [F], R.P. Hunter [HUN_2] and R.J. Koch [KO_1].

(b) Main directions of development

(i) Structure theory

By definition, a topological semigroup S is a Hausdorff space with continuous associative multiplication $(x,y) \longmapsto xy : S \times S \longrightarrow S$. Topological semigroups which are compact will be called compact semigroups [HO-M_2]. The theory of compact semigroups is a major area and a good account of the standard results in this area are available in the book "The Theory of Topological Semigroups" by J.H. Carruth, J.A. Hildebrant and R.J. Koch (1983) [C-H-K_1]. One of the historically primary observations about compact semigroups concerns with the existence of an idempotent and it leads to the structure theorem for the minimal ideal. This observation appears independently in papers by K. Iwasawa (1948) [I], K. Numakura (1952) [NU_1], A.D.Wallace (1953) [WA_1] and R. Ellis (1957b) [E_2]. The developments in internal structure theory of compact semigroups started with the work of A.D.Wallace (1955) [WA_3]. A first systematic treatment of monothetic compact semigroups (the compact subsemigroups generated by one element) was perhaps given by K. Numakura (1952) [NU_1] in which he derived most of the results from the fact that minimal ideal

$$\cap\{x^n, x^{n+1}, \ldots \}^* = M [\Gamma(x)]$$ is a group.

A thorough description of the structure of monothetic compact semigroup is available in E. Hewitt (1956)[HE]. Some other contributions on compact monothetic semigroups are due to J. Los and S. Schawarz (1958)[LO-S], E. Hewitt and K.A. Ross (1963) [HE-R] and to R.J.Koch (1957a) [KO_1].

In the theory of compact semigroups, the concept of schützenberger group was developed by Schützenberger in 1957 [S], its topologization was given by Wallace[WA_3] and further developments were made by Anderson and Hunter (1962a) [A-H], J.H. Carruth, J.A. Hildebrant and R.J. Koch (1983) [$C-H-K_1$].

A.M. Gleason (1950a) [GL], P.S.Mostert and A.L. Shields (1957) [MO-S], J.H. Carruth and J.D.Lawson (1970b) [C-L] have improved the theory of compact semigroups by introducing the existence of one parameter semigroups in compact semigroups.

K.H. Hofmann and P.S. Mostert (1966) [$HO-M_2$] introduced 'atoms' of compact connected monoids called solenoidal semigroups to develop compact semigroup theory.

A more systematic treatment of a class of compact semigroups, including both monothetic and solenoidal compact semigroup may be found in Hofmann (1976) [HO].

Contributions of P.S. Mostert and A.L. Shields (1957) [MO-S] and W.M. Faucett (1955) [F] are mainly on interval semigroups, i.e., I-semigroups. For the theory of compact semigroups the structure of I-semigroups was determined by P.S. Mostert and A.L. Shields (1957) [MO-S]. In particular, H.Cochen and I.S. Krule (1959) [CO-K] discussed closed congruences on I-Semigroups. The existence of idempotent I-semigroups in compact connected semigroups under suitable conditions was proved by R.J. Koch around 1957 and its existence in certain special cases has been demonstrated by R.J. Koch (1959) [KO_2], R.P. Hunter (1960, 1961b) [HUN_1, HUN_2] and A.L. Hudson (1961a) [HU].

W.M. Faucett (1955a) [F], introduced the concept of irreducibility in the study of semigroups, where he discusses semigroups irreducibly connected between two idempotents. R.P. Hunter formulated the concept of irreducibility. K.H. Hofmann and P.S. Mostert (1964a) [$HO\text{-}M_1$] developed various characterizations of I-semigroups in terms of the concept of irreducibility.

K. Numakura (1957) [NU_2], R.P. Hunter (1961b) [HUN_2] , K.H. Hofmann and P.S. Mostert (1966)[HO-M_2], and J.H. Carruth, J.A. Hildebrant and R.J. Koch (1983) [C-H-K_1] used projective limits for the investigation of compact semigroups.

A.D. Wallace (1953c) [WA_2] introduced the geometric structure of compact semigroups and A.L.Hudson and P.S. Mostert (1963) [HU-M] studied its very important applications in compact semigroups.

D.R.Brown and M. Friedberg (1968) [BR-F_1]and J.A. Hildebrant (1968) [HI] have improved the theory of compact semigroups by introducing compact divisible semigroups.

(ii) <u>Applications to functional analysis</u>

Unlike the topological theory, the semitopological theory, seems to lean strongly towards functional analysis. Applications of the theory of topological semigroups in certain branches of functional analysis called for a distinguished subclass of topological semigroups, namely, semigroups which are compact Hausdorff spaces with the multiplication being continuous in each variable separately. Such semigroups are called semitopological semigroups.

Application to functional analysis start from K. Deleeuw's and I. Glicksberg's work on compact semitopological semigroups. His foundations of the theory of almost periodic and weakly almost periodic functions [D-G_1] based on a general theory of semigroups of operators on topological vector spaces, where the semigroups are compact in the strong operator topology or in the weak operator topology. In this study he used general methods concerning topological vector spaces and compactness criteria for function spaces. Developments in this direction constitute a rich theory.

J.F. Berglund and K.H. Hofmann (1967) [BE-H], J.F. Berglund (1970) [BE] and J.F. Berglund and P. Milnes (1976) [BE-M] applied semigroup theory to operator semigroups and thus developed the theory of almost periodic functions in the spirit of K. Deleeuw and I. Glicksberg [D-G_1].

Other main contributions in this direction are due to L.M.Anderson and R.P.Hunter (1969) [A-H_1], M.Friedberg (1981) [FR_2], W.G.Rosen (1956) [RO] and to J.S. Pym (1965) [PY_2].

Also, K. Deleeuw and I. Glicksberg [$D-G_1$], J.F. Berglund and K.H. Hofmann (1967) [BE-H], J.F. Berglund, H.D. Junghenn and P. Milnes (1978) [BE-J-M] and A.L.T. Paterson (1978) [PA] studied means for bounded functions and developed amenable semigroups.

(iii) Applications to Algebra

H. Cochen and H.S. Collins (1959) [CO-C] developed affine semigroups. R. Ellis (1957) [E_1] established results on locally compact transformation groups. J.F. Berglund and K.H.Hofmann (1967) [BE-H] formulated Ellis's results and discussed some fixed point theorems for semigroups of continuous affine transformations of compact convex sets.

(iv) Semigroup compactifications

In the theory of topological semigroups, another branch is the study of semigroup compactifications. This area of research started from the information about compact semigroups from which the Bohr compactifications were constructed [D-G]. The theory of semigroup compactification of topological semigroups is still in the stage of infancy. However, there are developments in particular cases. For example,

(a) Bohr (almost periodic) compactification of topological semigroups has been studied by K.Deleeuw and I. Glicksberg (1961) [D-G], J.F. Berglund and K.H. Hofmann (1967) [BE-H] and L.W. Anderson and R.P. Hunter (1969) [$A-H_1$].

(b) J.F. Berglund, H.D. Junghenn and P. Milnes (1978) [BE-J-M] developed the theory of almost periodic and weakly almost periodic compactifications of semitopological semigroups.

By semigroup compactification, they mean a compact right topological semigroup which contains a dense continuous homomorphic image of a given semitopological semigroup. Possible techniques developed for semigroup compactification are (i) by the use of operator theory a technique employed by K. Deleeuw and I. Glicksberg, (ii) based on the adjoint functor of category theory, and (iii) based on the Gelfand-Naimark theory of commutative C*-algebras.

Other main contributions in this direction are due to M. Friedberg and J.W. Steep (1973) [FR-S],

P. Holm (1964) [HOL], H.D. Junghenn and R.D. Pandian (1984) [JU-P], P. Milnes (1973) [MI_1] and to J.S.Pym (1963) [PY_1].

(c) J.H. Carruth, J.A. Hildebrant and R.J.Koch(1983) [C-H-K_1] indicated other types of compactifications for a given topclogical semigroup S such as the group compactification, one-point compactification, etc.

In topological spaces, the theory of compactification is well-developed starting with the work of A. Tychonoff (1930) [TY]. E. Čeh (1937) [CE] and M.H. Stone (1948) [ST] who independently defined the maximal Hausdorff compactification βX, now known as Stone-Čech compactification of X, and stated its fundamental properties. In the theory of topological semigroups J.W. Baker and R.J. Butcher (1976) [B-B] and H.M.Umoh (1985) [U] studied the Stone-Čech compactification of a topological semigroup.

In topology, contributions of R.E. Chandler(1976) [CH], M.C. Rayburn(1973) [R_2], H. Tamano (1960) [T] and R.C. Walker (1974) [W] are mainly on the theory of Hausdorff compactifications. Also if X is a locally compact space, all Hausdorff compactifications of X are obtained by considering Hausdorff quotients of βX-X [CH].

In this thesis, we have attempted to present our studies in this direction based on the Bohr compactification of a topological semigroup.

2. SUMMARY OF MAIN RESULTS ESTABLISHED IN THIS THESIS

The main part of Chapter-1 is devoted to (i) define semigroup compactifications of a topological semigroup, and (ii) prove that semigroup compactifications of a topological semigroup S are precisely the quotients of the Bohr compactification of S under closed congruences.

We define semigroup compactification of a topological semigroup S as a compact semigroup which contains a dense continuous homomorphic image of S. The contrast with the usual notion of compactification of a topological space may be noted that- it contains a continuous not necessarily a homeomorphic image. Section-1 of this chapter contains some background material from (1) The algebraic theory of semigroups [HOW] and (2) The Theory of topological semigroups [C-H-K_1] needed in later chapters also.

In 1941, Lubben [LU] observed some properties of K(X), the family of compactifications of a completely regular space X and proved that K(X) is a complete lattice if and only if X is locally compact. Also, K.D. Magill Jr. (1968) [M_2]. M.C. Rayburn (1969) [R_1] and T.Thrivikraman (1972) [TH] have improved the theory of Lattice of Hausdorff compactifications. K.D.Magill Jr. (1968) [M_2] proved that if X and Y are locally compact Hausdorff spaces, then their lattices of compactifications K(X) and K(Y) are isomorphic if and only if $\beta X-X$ and $\beta Y-Y$ are homeomorphic. In the second chapter, we conduct a study in this direction.

It is known that the family of congruences on a semigroup is a complete lattice [HOW]. In Section 2.1 of this thesis, we study the properties of $K_1(S)$- family of all semigroup compactifications of S, each obtained by a closed congruence on (β,B) and show that $(K_1(S), \geqslant)$ is a complete lattice, under the ordering $(\alpha,A) \geqslant (\gamma,C)$ if there exists a continuous surmorphism $f: A \longrightarrow C$ such that $f\alpha = \gamma$. In Section 2.2, we prove that for topological semigroups S_1 and S_2 if (β_1,B_1) and (β_2,B_2) are topologically isomorphic then the lattices $K_1(S_1)$ and $K_1(S_2)$ are isomorphic and observe that converse need not be true. This may be compared with the theorem of K.D. Magill Jr. [M_2] on compactifications. In Section 2.3, we give certain possible generalizations. Also using the category language, as a special case, we establish that the association of (B,I) with $K_1(S)$ where S is a topological semigroup, B its Bohr compactification and I a closed ideal of B, is a contravariant functor into the category of all complete lattices with suitable morphisms.

In the third chapter, we describe some more results on the lattice $K_1(S)$ of semigroup compactifications of a topological semigroup S.

In 1961, A.H. Clifford and G.B.Preston [CL-P_1] considered the concept of ideals on semigroups. It is known that if S is a semigroup and 'ω' is an ideal of S, then $(\omega x \omega) \cup \Delta$ is a congruence on S [HOW]. In Section 3.1, we introduce weak ideals, joint weak ideals and complementary joint ideals of a semigroup and discuss congruences determined by them. Also we prove that

(i) A topological semigroup S with Bohr compactification (β,B) has a semigroup compactification (α, A) determined by 'n' disjoint closed proper weak ideals (ideals) of B, at least one of which is non-singleton only if S has a semigroup compactification strictly bigger than (α,A).

(ii) A topological semigroup S with Bohr compactification (β,B) has an n-point compactification (α,A) determined by 'n' non-empty subsets of B does not imply that it has an (n-1)-point compactification, nor does it imply that there is a semigroup compactification strictly bigger than (α,A) different from (β,B).

(iii) A topological semigroup S with (β,B) has an n-point compactification (α,A) determined by 'n' non-empty weak ideals (ideals) of B implies that S has a semigroup compactification strictly bigger than (α,A), but it does not imply that S has an (n-1)-point compactification.

In Section 3.2, we describe the dual atoms and atoms of $K_1(S)$, when B is finite.

In 1961, K. Deleeuw and I. Glicksberg [D-G] observed that the product of Bohr compactification of $\{S_\alpha\}_{\alpha \in A}$, collection of abelian topological monoids is the Bohr compactification of $P\{S_\alpha\}_{\alpha \in A}$. In Section 4.1 of this thesis, we discuss semigroup compactification of the product $P\{A_\alpha\}_{\alpha \in A}$, where A_α is a semigroup compactification of S_α, for each $\alpha \in A$. Also, we consider the family of topological semigroups $\{S_\alpha\}_{\alpha \in A}$ with Bohr compactifications $\{B_\alpha\}_{\alpha \in A}$ and tne lattices of semigroup compactifications $\{K_1(S_\alpha)\}_{\alpha \in A}$ and show that $P\{K_1(S_\alpha)\}_{\alpha \in A} \subset K_1(P\{S_\alpha\}_{\alpha \in A})$ is a complete lattice. In Section 4.2, we discuss semigroup compactifications, Bohr compactification and lattice of semigroup compactifications of the limit of a projective system of topological semigroups.

In the category of Tychnoff spaces, the subcategory of compact space is epireflective. Also, in the category of all Hausdorff spaces, epimorphisms are dense continuous maps and extremal monomorphisms are closed embeddings. In the category of topological semigroups, the compact semigroups form an epireflective subcategory. However, the other results mentioned above do not hold in the category of topological semigroups. We investigate this problem in Chapter-5 and give the possible results in these directions.

For example, we prove that

(1) If the images are ideals, the epimorphisms in the category TS of all topological semigroups are morphisms with dense range.

(2) If the images are ideals, the weak extremal monomorphisms in the category TS are the closed embeddings and epireflective subcategories are closed under weak extremal subobjects.

We do not claim that the study made in this thesis is complete in all respects- rather, there are various problems connected with the work here, worth investigating, as has been pointed at relevant places in the thesis.

Chapter 1

SEMIGROUP COMPACTIFICATIONS

Introduction

A considerable body of information about the structure of topological semigroups is now available, and is given in books and monographs by K.H. Hofmann and P.S. Mostert (1966) [HO-M_2]. A.B. Paalman-de Miranda (1970) [P] and J.H. Carruth, J.A. Hildebrant and R.J. Koch (1983) [C-H-K_1].

In topological spaces, the notion of a compactification was considered for the first time by A. Tychnoff (1930) [TY]. In 1937, E. Čech [CE] and M.H. Stone [ST] independently defined the maximal compactification βX and stated its fundamental properties. But in topological semigroups, the theory of semigroup compactification is still in the stage of infancy. However, there are results in special types of compactifications. Also the theory of semitopological semigroups develops in this direction. For example, in [BE-J-M] J.F. Berglund, H.D. Junghenn and P. Milnes develops the theory of compact right topological semigroups and in particular of semigroup compactifications of semitopological semigroups. In 1961,

K. Deleeuw and I. Glicksberg [D-G] constructed almost periodic and weakly almost periodic compactifications of any semitopological semigroup. Bohr [almost periodic] compactification of topological semigroups has been studied by K. Deleeuw and I. Glicksberg in 1961 [D-G], J.F. Berglund and K.H. Hofmann in 1967 [BE-H] and Anderson and Hunter in 1969 [A-H_1]. In [C-H-K_1] J.H.Carruth, J.A. Hildebrant and R.J. Koch have discussed the problems that arise when we work with the Bohr compactification in contrast with Stone-Čech compactification of topological spaces. They have also discussed the concepts of group compactification, one-point compactification, etc. for a given topological semigroup.

In this chapter in Section 1.2, we introduce another type of compactification for a given topological semigroup named as "Semigroup Compactification" and discuss some results relating them to the Bohr compactification. Section 1.1 contains some background material needed in later chapters also.

1.1 Preliminaries

Semigroups 1.1.1 A semigroup is a non-empty set S together with an associative multiplication $(x,y) \longmapsto xy$ from S x S into S. If S has a Hausdorff topology such that

$(x,y) \longmapsto xy$ is continuous with the product topology on $S \times S$, then S is called a topological semigroup. If S is a compact topological semigroup then S is called a compact semigroup [C-H-K_1].

Examples 1.1.2

(a) Let S be a topological space. Define multiplication in S by $xy = x$ ($xy = y$) for every x,y in S. Then S is a topological semigroup, called the left zero (right zero) semigroup.

(b) Let S be a topological space. Let $z \in S$ be fixed. Define multiplication in S by $xy = z$ for every x,y in S. Then S is called a zero semigroup which is a topological semigroup with zero 'z'.

(c) Let $I_u = [0,1]$ with usual topology and usual multiplication. Then I_u is a compact abelian semigroup.

(d) Let $I_m = [\frac{1}{2},1]$ with the usual topology and multiplication $(x,y) \longmapsto \min\{x,y\}$. Then I_m is a compact semigroup.

Definition 1.1.3.

A non-empty subset T of a topological semigroup S is called a subsemigroup of S if $TT \subset T$, a left ideal of S if $ST \subset T$, a right ideal if $TS \subset T$ and an ideal if $TS \cup ST \subset T$.

If T is a subsemigroup of S, T itself is a topological semigroup under the restriction of multiplication on S to T x T and the closure $\overline{T}$ of T is also a subsemigroup of S [C-H-K_1].

In 1976, J.M. Howie [HOW] and in 1961 A.H. Clifford and G.B. Preston [CL-P_1] considered the concept of congruence on semigroups.

Definition 1.1.4

Let S be a semigroup. A relation R on S is said to be left (right) compatible (with the operation on S) if $(x,y) \in R \Longrightarrow (ax,ay) \in R$ [$(xa,ya) \in R$] $\forall x,y,a \in S$ and compatible if R is both left and right compatible.

Definition 1.1.5

A compatible equivalence on a semigroup S is called a congruence [HOW].

Proposition 1.1.6

(a) An equivalence R on a semigroup S is a congruence if and only if $(a,b) \in R$ and $(c,d) \in R \Longrightarrow (ac,bd) \in R$.

(b) the intersection of any collection of congruences on a semigroup S is a congruence on S.

(c) $S \times S$ is a congruence on S. [HOW]

Definition 1.1.7

If S is a semigroup and I is an ideal of S then the semigroup $S/(I \times I) \cup \Delta$ is called the Rees quotient semigroup of S mod I and is denoted as S/I [C-H-K_1].

Definition 1.1.8

If R is an equivalence (congruence) on a topological space (semigroup) S, then R is called a closed equivalence (congruence) if R is a closed subset of $S \times S$ [C-H-K_1].

Definition 1.1.9

Let X,Y be spaces and $f:X \longrightarrow Y$ a function which is surjective, then f is said to be a quotient map if W being open (closed) in Y is equivalent to $f^{-1}(W)$ being open (closed) in X.

Definition 1.1.10

A semigroup S is said to be left (right) cancellative provided $x,y,z \in S$ and $xy = xz \Longrightarrow y = z$ [$yx=zx \Longrightarrow y=z$]. If S is both left and right cancellative, then S is said to be cancellative.

Next theorem is an algebraic hypothesis on a compact semigroup which implies that it must be a group [C-H-K_1].

Theorem 1.1.11

Let S be a compact cancellative semigroup. Then S is a group [C-H-K_1].

Definition 1.1.12

If S and T are semigroups, a function $\emptyset$: S $\longrightarrow$ T is called a homomorphism if $\emptyset(xy) = \emptyset(x). \emptyset(y)$ for each $x,y \in S$. If $\emptyset$ is surjective, then $\emptyset$ is called a surmorphism. If $\emptyset$ is also injective then $\emptyset$ is called an algebraic isomorphism and S and T are said to be algebraically isomorphic [C-H-K_1].

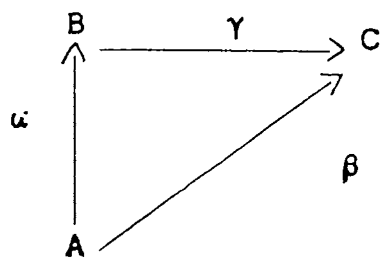
If S and T are topological semigroups and $\emptyset$:S $\longrightarrow$ T is both an algebraic isomorphism and a homeomorphism, then $\emptyset$ is called a topological isomorphism and S and T are said to be topologically isomorphic [C-H-K_1].

If $\emptyset$: S $\longrightarrow$ T is a homomorphism, then $\emptyset$ preserves subsemigroups and subgroups. In the case that $\emptyset$ is a surmorphism then $\emptyset$ preserves ideals and minimal ideals of all three types (left, right, two-sided) and $\emptyset^{-1}$ preserves subsemigroups, (left, right) ideals.

If S and T are semigroups and $\emptyset : S \longrightarrow T$ is a homomorphism, then $K(\emptyset)$ is a relation defined as $\{(x,y) \in S \times S : \emptyset(x) = \emptyset(y)\}$

Theorem 1.1.13 Induced Homomorphism theorem

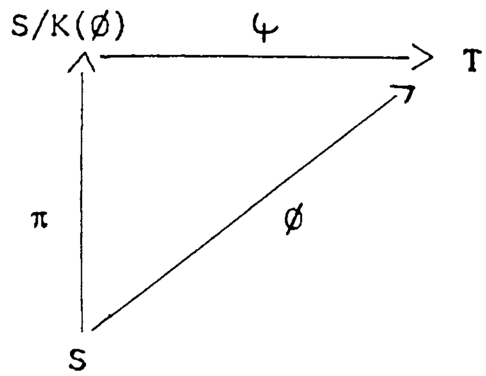
Let A,B and C be (topological) semigroups, $\alpha: A \longrightarrow B$ a (quotient) surmorphism, and $\beta: A \longrightarrow C$ a (continuous) homomorphism such that $K(\alpha) \subset K(\beta)$. Then there exists a unique (continuous) homomorphism $\gamma : B \longrightarrow C$ such that the diagram commutes [C-H-K_1]



Theorem 1.1.14 First Isomorphism theorem

Let S and T be semigroups and let $\emptyset : S \longrightarrow T$ be a surmorphism. Then $K(\emptyset)$ is a congruence on S and there exists a unique algebraic isomorphism $\psi : S/K(\emptyset) \longrightarrow T$ such that the diagram commutes.

Moreover, if S and T are topological semigroups and $\phi : S \longrightarrow T$ is a continuous surmorphism, then $K(\phi)$ is a closed congruence on S and the following are equivalent.



(a) ψ^{-1} is continuous

(b) ψ is a topological isomorphism, and

(c) ϕ is quotient

Finally, if these equivalent statements hold, then $S/K(\phi)$ is a topological semigroup [C-H-K_1].

If S is a topological semigroup and R is a closed congruence on S, then S/R with the induced operation and the quotient topology need not be a topological semigroup. This situation has been studied by J.H. Carruth, J.A. Hildebrant and R.J. Koch (1983) [C-H-K_1] and some conditions under which S/R is a topological semigroup have been established. This result was established for compact semigroups by Wallace (1955) [WA_3] and extended to locally compact σ-compact semigroups by Lawson and Madison (1971) [LA-M].

Lemma 1.1.15

Let S be a topological semigroup and let R be a closed congruence on S such that $P \times P : S \times S \longrightarrow S/R \times S/R$ is a quotient map. Then S/R is a topological semigroup [C-H-K_1].

Theorem 1.1.16.

Let S be a compact semigroup and let R be a closed congruence on S. Then S/R is a compact semigroup.

Let $\{S_i\}_{i \in I}$ be a collection of (topological) semigroups. Then co-ordinatewise multiplication on $P\{S_i\}_{i \in I}$ is given by $(fg)(j) = f(j)\, g(j)$, the latter product being taken in S_j for each $j \in I$ [C-H-K_1].

Theorem 1.1.17

Let $\{S_i\}_{i \in I}$ be a collection of (topological) semigroups and $S = P\{S_i\}_{i \in I}$. Then S with coordinatewise multiplication is a (topological) semigroup and each projection $P_j : S \longrightarrow S_j$ is a (continuous open) surmorphism.

The concepts of projective (inverse) limits of topological semigroups are developed in [C-H-K_1] and some results on compact semigroups are studied by Hofmann and Mostert (1966) [HO-M_2], Numakura (1957) [NU_2], J.H. Carruth, J.A. Hildebrant and R.J. Koch (1983) [C-H-K_1].

Definition 1.1.18

A projective system of (topological) semigroups is a triple $((D, \leqslant), \{S_\alpha\}_{\alpha \in D}, \{\phi_\alpha^\beta\}_{\alpha \leqslant \beta})$ where

(a) $(D, \leqslant)$ is a directed set

(b) $\{S_\alpha\}_{\alpha \in D}$ is a family of (topological) semigroups indexed by D, and

(c) $\{\phi_\alpha^\beta\}_{\alpha \leqslant \beta}$ is a family of functions indexed by $\leqslant$ such that

(i) $\phi_\alpha^\beta : S_\beta \longrightarrow S_\alpha$ is a (continuous) homomorphism for each $(\alpha,\beta) \in \leqslant$

(ii) $\phi_\alpha^\alpha = 1_{S_\alpha}$ identity map on S_α, for each $\alpha \in D$, and

(iii) $\phi_\alpha^\beta \circ \phi_\beta^\gamma = \phi_\alpha^\gamma$ for all $\alpha \leqslant \beta \leqslant \gamma$ in D. This projective system is denoted by $\{S_\alpha, \phi_\alpha^\beta\}_{\alpha \in D}$. Each ϕ_α^β is called a bonding map and $\{S_\alpha, \phi_\alpha^\beta\}_{\alpha \in D}$ is said to be strict if each bonding map is surjective [C-H-K_1].

Definition 1.1.19

If $S = \{x \in P\{S_\alpha\}_{\alpha \in D} : \phi_\alpha^\beta(x(\beta)) = x(\alpha)$ for all $\alpha \leqslant \beta$ in $D\}$

is non-empty, then S is called the projective limit of $\{S_\alpha, \phi_\alpha^\beta\}_{\alpha \in D}$ and is denoted by

$$S = \varprojlim \{S_\alpha, \phi_\alpha^\beta\}_{\alpha \in D} \text{ or } S = \varprojlim S_\alpha$$

If $\{S_\alpha, \phi_\alpha^\beta\}_{\alpha \in D}$ is a strict projective system, then S is called the strict projective limit of $\{S_\alpha, \phi_\alpha^\beta\}_{\alpha \in D}$ [C-H-K_1].

Theorem 1.1.20

Let $\{S_\alpha, \phi_\alpha^\beta\}_{\alpha \in D}$ be a projective system of (topological) semigroups such that $S = \varprojlim S_\alpha$ exists. Then S is a (closed) subsemigroup of $P_{\alpha \in D}\{S_\alpha\}$[HO-$M_1$].

Some results on compact semigroups that we would require are

Theorem 1.1.21

Let $\{S_\alpha, \phi_\alpha^\beta\}_{\alpha \in D}$ be a projective system of compact semigroups. Them $\varprojlim S_\alpha$ is a compact semigroup [C-H-K_1].

Theorem 1.1.22.

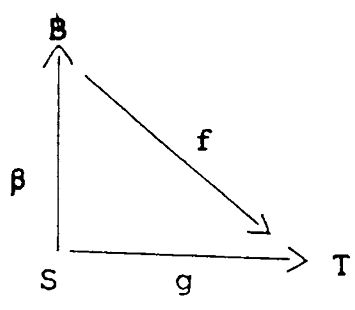
Let $\{S_\alpha, \phi^\beta_\alpha\}_{\alpha \in D}$ be a projective system of compact semigroups and let $S = \varprojlim S_\alpha$. Then $P_\alpha|_S : S \longrightarrow S_\alpha$ is surjective for each $\alpha \in D$, where P_α is the projection map [C-H-K_1].

Note.

Associated with each topological semigroup S, there is a compact semigroup called the Bohr compactification of S which is universal over the compact semigroups containing dense continuous homomorphic images of S. The existence and uniqueness of Bohr compactification can be proved [C-H-K_1].

Definition 1.1.23 Bohr Compactification

If S is a topological semigroup, then Bohr compactification of S is a pair (β, B) such that B is a compact semigroup, $\beta : S \longrightarrow B$ is a continuous homomorphism with $B = \overline{\beta(S)}$ and if $g : S \longrightarrow T$ is a continuous homomorphism of S into a compact semigroup T, then there exists a unique continuous homomorphism $f.B \longrightarrow T$ such that the diagram commutes



For each topological semigroup S, there exists a Bohr compactification which is unique upto topological isomorphism [C-H-K_1].

In 1961, K. Deleuuw and I. Glicksberg [D-G] developed product theorem on Bohr compactifications.

The Product Theorem 1.1.24

Let $\{S_\alpha : \alpha \in A\}$ be a collection of abelian topological monoids, (β_α, B_α) the Bohr compactifications of S_α for each $\alpha \in A$ and $\beta : \underset{\alpha \in A}{P} \{S_\alpha\} \longrightarrow \underset{\alpha \in A}{P} \{B_\alpha\}$ the function defined by $\beta(x)(\delta) = \beta_\delta \; P_\delta(x)$, where $P_\delta : \underset{\alpha \in A}{P} \{S_\alpha\} \longrightarrow S_\delta$ is projection for each $\delta \in A$.

Then $(\beta, \; P\{B_\alpha\}_{\alpha \in A})$ is the Bohr compactification of $\underset{\alpha \in A}{P} \{S_\alpha\}$.

Remark.

This result is true even in non-abelian case.

1.2. Semigroup Compactification

Here we introduce our definition of semigroup compactification.

Definition 1.2.1.

A semigroup compactification of a topological semigroup S is an ordered pair (g,T) where T is a compact semigroup and $g:S \longrightarrow T$ is a dense continuous homomorphism of S into T. (Here g is dense means g(S) is dense in T).

Examples 1.2.2.

(1). Let N be the multiplicative semigroup of +ve integers with the discrete topology.

$$T = \{\frac{1}{n} : n \in N\} \cup \{0\} \text{ is a closed subsemigroup}$$

of $I_u = [0,1]$ with usual topology and usual multiplication. If $\emptyset: N \longrightarrow T$ is defined by $\emptyset(n) = \frac{1}{n}$ for all $n \in N$. Then $(\emptyset,T)$ is a semigroup compactification of N.

(2). Let (R, +) be the additive (semi) group of real numbers with the usual topology. Let T be the circle group with usual multiplication and usual topology. If $\emptyset : R \longrightarrow T$ is defined by $\emptyset(x) = \exp(2\pi i x)$. Then $(\emptyset,T)$ is a semigroup compactification of R.

(3). Bohr compactification (β,B) of a topological semigroup S is a semigroup compactification.

Wallace, A.D. has shown that if B is a compact semigroup and R is a closed congruence on B, then the quotient space B/R is a compact semigroup [1.1.16]. We prove below that semigroup compactifications of a topological semigroup S are precisely the quotients of the Bohr compactification of S under closed congruences.

Result 1.2.3.

Let S be a topological semigroup with Bohr compactification (β,B). If (α,A) is any semigroup compactification of S, then

(a) there exists a continuous surmorphism (ie. surjective homomorphism) $\Theta:B \longrightarrow A$ such that $\Theta\beta = \alpha$

(b) and the equivalence defined by Θ on B is a closed congruence.

(c) (α,A) is the quotient of (B,β) with respect to the congruence in (b).

Proof

(a) from the definition of (β,B) there exists a continuous homomorphism $\Theta: B \longrightarrow A$ such that $\Theta\beta = \alpha$.

Again Θ is surjective, for,

$$A = \overline{\alpha(S)} \quad (\because \alpha \text{ is dense in } A)$$

$$= \overline{\Theta\beta(S)} = \Theta\, \overline{\beta(S)} \quad (\because \Theta \text{ is a closed map being from a compact space to a } T_2 \text{ space})$$

$$= \Theta(B)$$

We have $\Theta: B \longrightarrow A$ is a continuous surmorphism such that $\Theta\beta = \alpha$.

(b) Let R be relation defined on B by Θ

$R = \{(x,y) \in B \times B : \Theta(x) = \Theta(y)\}$ is clearly an equivalence.

$R = (\Theta \times \Theta)^{-1} (\Delta_A)$ is closed since Δ_A diagonal in $A \times A$ of Hausdorff space A is closed.

R is a congruence, for,

$(a,b) \in R \Longrightarrow \Theta(a) = \Theta(b)$

$(c,d) \in R \Longrightarrow \Theta(c) = \Theta(d)$

$\therefore \quad \Theta(ac) = \Theta(a).\ \Theta(c) = \Theta(b).\ \Theta(d) = \Theta(bd)$

$\therefore \quad (ac, bd) \in R.$

Hence R is a closed congruence.

(c) We have Θ: B ——> A is a quotient map, since it is a closed continuous surmorphism.

Result 1.2.4.

Let S be a topological semigroup with Bohr compactification (β, B). If R is a closed congruence on B, then there exists a semigroup compactification (α, A) of S so that the congruence defined by this compactification is R.

Proof.

Let R be a closed congruence on B. Define $\Theta: B \longrightarrow B/R$ the natural map. Then $A = B/R$, with the quotient topology and multiplication induced by Θ is a compact semigroup [1.1.16].

Define $\alpha: S \longrightarrow A$ such that $\alpha = \Theta\beta$. Clearly α is a well-defined continuous homomorphism.

Also α is dense, for,

$$\overline{\alpha(S)} = \overline{\Theta\beta(S)} = \Theta\,\overline{\beta(S)} \qquad (\because\ \Theta \text{ is closed})$$

$$= \Theta(B) = A \qquad (\because\ \Theta \text{ is surjective})$$

Thus we have $\alpha: S \longrightarrow A$ is a dense continuous homomorphism.

$\therefore$ (α,A) is a semigroup compactification of S and the congruence defined by (α,A) is that defined by Θ which is R.

Remark 1.2.5.

Thus we have proved that if (β,B) is a Bohr compactification of S and R is any closed congruence on B, then the quotient space B/R is a semigroup compactification of S and conversely any semigroup compactification (α,A) of S is topologically isomorphic to B/R for some closed congruence R on B.

Chapter-2

LATTICES OF SEMIGROUP COMPACTIFICATIONS

Introduction

In 1941, Lubben [LU] observed two general results concerned with properties of K(X), the family of compactifications of a completely regular space X. The results are:

(i) K(X) is a complete upper semilattice.

(ii) K(X) is a complete lattice if and only if X is locally compact.

It is known that the family of congruences on a semigroup is a complete lattice [HOW]. Here we consider $K_1(S)$- the family of semigroup compactifications of S. We define a pre-order in $K_1(S)$ and show that the equivalence classes form a complete lattice. This is the content of Section 2.1.

Kenneth D. Magill Jr. [M_2] has shown that if X and Y are locally compact Hausdorff spaces, then their lattices of compactifications K(X) and K(Y) are isomorphic if and only if $\beta X-X$ and $\beta Y-Y$ are homeomorphic. But this theorem does not hold in general for completely regular Hausdorff spaces [TH].

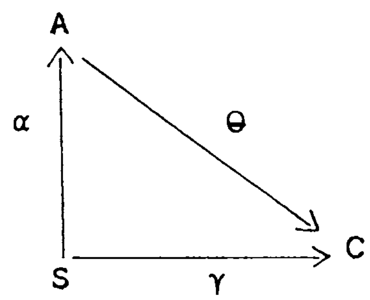
In Section 2.2 we prove that for topological semigroups S_1 and S_2 if (β_1, B_1) and (β_2, B_2) are topologically isomorphic then the lattices $K_1(S_1)$ and $K_1(S_2)$ are isomorphic. We construct examples to show that the converse of this theorem is false.

In Section 2.3 we study certain possible generalizations of this result.

2.1 Some Properties of $K_1(S)$

Definition 2.1.1

Two semigroup compactifications (α,A) and (γ,C) of a topological semigroup S are regarded as being equivalent if there exists a topological isomorphism $\Theta : A \longrightarrow C$ such that the following diagram commutes i.e., $\Theta\alpha = \gamma$.



Remark 2.1.2.

We have a pre-order $\geqslant$ in the class of all semigroup compactifications of S if we define $(\alpha,A) \geqslant (\gamma,C)$ whenever there is a continuous surmorphism $f: A \longrightarrow C$ such that $f\alpha = \gamma$.

Lemma 2.1.3

Two semigroup compactifications (α,A), (γ,C) are equivalent if and only if $(\alpha,A) \geqslant (\gamma,C)$ and $(\gamma,C) \geqslant (\alpha,A)$

Proof

Assume that (α,A) and (γ,C) are equivalent. Then by definition (2.1.1) there exists a topological isomorphism $\theta : A \longrightarrow C$ such that $\theta\alpha = \gamma$. Now $\theta : A \longrightarrow C$ is a continuous surmorphism such that $\theta\alpha = \gamma$.

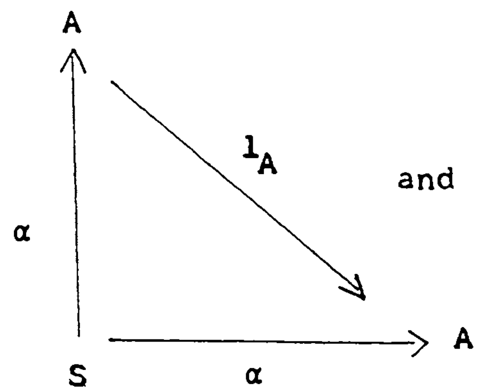
$$\therefore \quad (\alpha,A) \geqslant (\gamma,C)$$

Again $\alpha = \theta^{-1}\gamma$

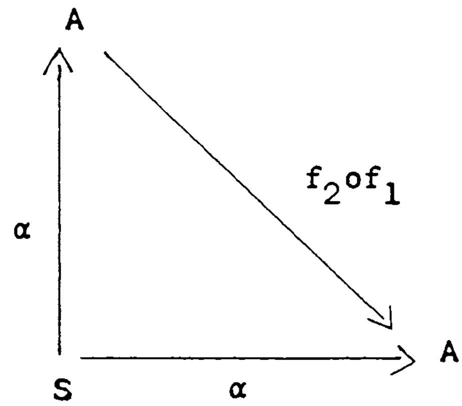
$\therefore$ $\theta^{-1} : C \longrightarrow A$ is a continuous surmorphism such that $\theta^{-1}\gamma = \alpha$.

$$\therefore \quad (\gamma,C) \geqslant (\alpha,A)$$

Conversely, if $(\alpha,A) \geqslant (\gamma,C)$ and $(\gamma,C) \geqslant (\alpha,A)$ then there exists a continuous surmorphism $f_1: A \longrightarrow C$ such that $f_1\alpha = \gamma$ and a continuous surmorphism $f_2: C \longrightarrow A$ such that $f_2\gamma = \alpha$.



and



Then $f_2 \circ f_1$: A ——> A is a continuous surmorphism such that $f_2 \circ f_1 \circ \alpha = f_2 \circ \gamma = \alpha$. $f_2 \circ f_1$ is unique, since α is dense in A and A is a Hausdorff space.

In view of the commuting diagrams, and the uniqueness of $f_2 \circ f_1$, we see that $f_2 \circ f_1 = 1_A$, and similarly $f_1 \circ f_2 = 1_C$. We conclude that f_1 is a topological isomorphism.

$\therefore$ (α,A) and (γ,C) are equivalent.

Notation

$K_1(S)$ denotes the set of equivalence classes of semigroup compactifications of S.

Lemma 2.1.4.

$K_1(S)$, under the ordering '$\geq$' described in (2.1.2) is partially ordered set.

Theorem 2.1.5.

Let S be a topological semigroup with family of semigroup compactifications $K_1(S)$. Then $(K_1(S), \geqslant)$ is an upper complete semilattice.

Proof.

By lemma 2.1.4, we have $(K_1(S), \geqslant)$ is a partially ordered set. For the required proof, let $\{\alpha_i S\}_{i \in I}$ be a subset of $K_1(S)$. We must show that this set has a least upper bound with respect to ' $\geqslant$ '.

Define $\alpha : S \longrightarrow \underset{i \in I}{P} \{\alpha_i S\}$ by $(\alpha(x))_i = \alpha_i(x)$

Since each α_i is a continuous homomorphism it follows α is also one such. $\underset{i \in I}{P} \{\alpha_i S\}$ is a compact semigroup under co-ordinatewise multiplication and product topology [C-H-K_1].

Let $\alpha S = \overline{\alpha(S)}$; then it is a closed subsemigroup of the compact semigroup $\underset{i \in I}{P} \{\alpha_i S\}$ and so is a compact semigroup.

$\therefore$ α: $S \longrightarrow \alpha S$ is a dense continuous homomorphism. Thus $(\alpha, \alpha S)$ is a semigroup compactification of S.

For each $i \in I$ let $P_i : \alpha S \longrightarrow \alpha_i S$ be the restriction to αS of the projection map.

For each $i \in I$,

$$(P_i \circ \alpha)(x) = (\alpha(x))_i = \alpha_i(x)$$

so that

$$P_i\alpha = \alpha_i \text{ and thus } (\alpha,\alpha S) \geqslant (\alpha_i,\alpha_i S) \text{ for all } i \in I$$

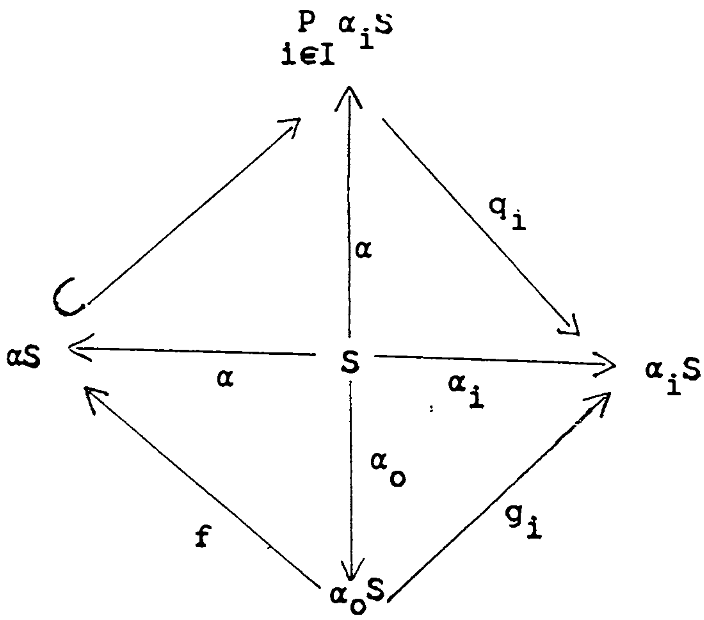
Suppose $(\alpha_o,\alpha_o S) \geqslant (\alpha_i,\alpha_i S)$ for all $i \in I$

where

$$g_i : \alpha_o S \longrightarrow \alpha_i S \text{ such that } g_i\alpha_c = \alpha_i.$$

Define

$$f: \alpha_o S \longrightarrow \underset{i \in I}{P}\{\alpha_i S\} \text{by } (f(y))_i = g_i(y)$$



Then $q_i \circ f = g_i$ so that f is a continuous homomorphism and also,

$$f(\alpha_o(x))_i = g_i(\alpha_o(x)) = \alpha_i(x) = (\alpha(x))_i$$

We conclude that $f\alpha_o = \alpha$

$$\text{ie. } (f\alpha_o)(S) = \alpha(S)$$

Now f is dense for,

$$f(\alpha_o S) \supset f(\alpha_o(S)) = (f\alpha_o)(S) = \alpha(S)$$

$$\therefore \quad f(\alpha_o S) \supset \alpha(S)$$

and so

$$\overline{f(\alpha_o S)} \supset \overline{\alpha(S)} = \alpha S$$

$$\therefore \quad \overline{f(\alpha_o S)} = \alpha S$$

and f is a surmorphism.

For,

$$\alpha S = \overline{f(\alpha_o S)}$$

$$= f(\overline{\alpha_o S}) = f(\alpha_o S) \qquad (\because \text{ f is a closed map})$$

$\therefore$ f is a continuous surmorphism from

$\alpha_o S \longrightarrow \alpha S$ such that $f\alpha_o = \alpha$.

Thus $(\alpha_o, \alpha_o S) \succ (\alpha, \alpha S)$

$\therefore$ $(\alpha, \alpha S)$ is the least upper bound of

$$\{(\alpha_i, \alpha_i S)\}_{i \in I}$$

Corollary 2.1.6 $K_1(S)$ has the largest element (β,B)

If X is a locally compact space, it is known that all compactifications of X are obtained by 'adjoining' Hausdorff quotients of $\beta X - X$ to X [CH]. Here we consider $K_1(S)$, the family of semigroup compactifications of S each obtained by a closed congruence on Bohr compactification (β,B).

Note 2.1.7

We say a congruence R_1 refines a congruence R_2 and write $R_1 \geqslant R_2$ if $R_1 \subseteq R_2$.

Lemma 2.1.8

Let (α,A), $(\gamma,C) \in K_1(S)$. Then $(\alpha,A) \geqslant (\gamma,C)$ if and only if R_1 refines R_2, where R_1 and R_2 are the closed congruences corresponding to (α,A) and (γ,C) respectively.

Proof.

Let S be a topological semigroup with Bohr compactification (β,B). Let R_1 and R_2 be closed congruences on B such that $R_1 \geqslant R_2$. Consider $T_1 : B \longrightarrow B/R_1$ and $T_2 : B \longrightarrow B/R_2$ the natural maps where $B/R_1 \simeq (\alpha,A)$ and $B/R_2 \simeq (\gamma,C)$. Then by induced homomorphism theorem [1.1.13] there exists a continuous surmorphism $\Theta : A \longrightarrow C$ such that $\Theta\alpha = \gamma$. Hence $(\alpha,A) \geqslant (\gamma,C)$. On the other hand, now suppose that $(\alpha,A) \geqslant (\gamma,C)$. Then by definition there exists a continuous surmorphism $\Theta:A \longrightarrow C$ such that $\Theta\alpha = \gamma$. Again let (β,B) be the Bohr compactification of S, then (α,A) and (γ,C) are the quotient spaces of B, it follows that $f_1:B \longrightarrow A$ is a continuous dense homomorphism such that $f_1\beta = \alpha$ and $f_2: B \longrightarrow C$ is a continuous dense homomorphism such that $f_2\beta = \gamma$.

Now let R_1 and R_2 be closed congruences on B defined by f_1 and f_2 respectively. Then (α,A) topologically isomorphic to B/R_1 and (γ,C) topologically isomorphic to B/R_2, and given $\Theta: A \longrightarrow C$ such that $\Theta\alpha = \gamma$

ie. $\Theta f_1\beta = \gamma$

$\Theta f_1\beta = f_2\beta$

ie. $\Theta f_1 = f_2$

ie. $R_1 \geqslant R_2$

Theorem 2.1.9

Let S be a topological semigroup with Bohr compactification (β,B). Then $K_1(S)$ is a complete lattice. Moreover if B is cancellative, then $K_1(S)$ is a modular lattice.

Proof

$K_1(S)$ is an upper complete semilattice follows from theorem (2.1.5). For the required proof only to show that $K_1(S)$ has a smallest element.

Since B is a compact semigroup $R_o = B \times B$ is the largest closed congruence on B. Then $B/R_o = \{0\}$ determines a semigroup compactification $(q_\beta; \{0\}) \in K_1(S)$. Again $(q_\beta, \{0\})$ is the smallest compactification denoted by $(\alpha_o, \{0\})$.

If $(\alpha_1, A_1) \in K_1(S)$, then $(\alpha_1, A_1) \geqslant (\alpha_o, \{0\})$, since the constant map $\emptyset$ from A_1 to $\{0\}$ is a continuous surmorphism such that $\emptyset\alpha_1 = \alpha_o$. Thus $K_1(S)$ has a smallest element.

Again if B is cancellative, using [theorem 1.10 (C-H-K_1)], B is a group. Also it is known that the lattice of congruences on a group is modular [HOW]. $K_1(S)$ is a modular lattice, for,

if $(\alpha_1, A_1) \geqslant (\alpha_3, A_3)$, i.e. $R_1 \subseteq R_3$

Then $R_1 \subset R_1 \cup R_2 = R_1 \circ R_2$ and $R_1 \subseteq R_3$

given $R_1 \subseteq (R_1 \cup R_2) \cap R_3$

ie. the corresponding compactification satisfies

$$(\alpha_1, A_1) \geqslant ((\alpha_1, A_1) \wedge (\alpha_2, A_2)) \vee (\alpha_3, A_3) \qquad (1)$$

Also $R_2 \cap R_3 \subseteq R_2 \subseteq R_1 \cup R_2$ and $R_2 \cap R_3 \subseteq R_3$

ie. $R_2 \cap R_3 \subseteq (R_1 \cup R_2) \cap R_3$

ie. the corresponding compactification

$$(\alpha_2,A_2)\vee(\alpha_3,A_3) \geqslant [(\alpha_1,A_1)\wedge(\alpha_2,A_2)]\vee(\alpha_3,A_3) \quad (2)$$

from (1) and (2) we have

$$(\alpha_1,A_1)\wedge[(\alpha_2,A_2)\vee(\alpha_3,A_3)]$$

$$\geqslant [(\alpha_1,A_1)\wedge(\alpha_2,A_2)]\vee(\alpha_3,A_3)$$

Hence the result.

2.2 Functorial relation between B and $K_1(S)$

By a lattice isomorphism from a lattice L_1 to a lattice L_2, we mean a bijection f from L_1 to L_2 which preserves meet and join.

Theorem 2.2.1.

Let S_1 and S_2 be two topological semigroups with Bohr compactification (β_1,B_1) and (β_2,B_2) respectively. If (β_1,B_1) topologically isomorphic to (β_2,B_2), then their lattices of semigroup compactifications $K_1(S_1)$ and $K_1(S_2)$ are isomorphic.

Proof

Assume that (β_1,B_1) and (β_2,B_2) are topologically isomorphic. ie. there exists a topological isomorphism $f: B_1 \longrightarrow B_2$ such that $f\beta_1 = \beta_2$.

Let $(\alpha_1, A_1) \in K_1(S_1)$, then $\alpha_1 : S \longrightarrow A_1$ is a dense continuous homomorphism, where A_1 is a compact semigroup. By the definition of Bohr compactification there exists a continuous homomorphism $\Theta_1 : B_1 \longrightarrow A_1$ such that $\Theta_1 \beta_1 = \alpha_1$ and Θ_1 induces a congruences on B_1 say R_1.

Define $(x,y) \in R_2 \Longleftrightarrow (f^{-1}(x), f^{-1}(y)) \in R_1$

Then $R_2 = \{(x,y) \in B_2 \times B_2 : (f^{-1}(x), f^{-1}(y)) \in R_1\}$

$= (\Theta_1 f^{-1} \times \Theta_1 f^{-1})^{-1} \triangle_{A_1}$ which is a closed subset of A_1.

Clearly R_2 is an equivalence.

Again R_2 is a congruence; for,

if $(a,b) \in R_2$, $(c,d) \in R_2$

$\Longrightarrow (f^{-1}(a), f^{-1}(b)) \in R_1$, $(f^{-1}(c), f^{-1}(d)) \in R_1$

$\Longrightarrow (f^{-1}(a).f^{-1}(c), f^{-1}(b).f^{-1}(d)) \in R_1$

($\because$ R_1 is a congruence)

$\Longrightarrow (f^{-1}(ac), f^{-1}(bd)) \in R_1$

($\because$ f^{-1} is a homomorphism)

$\Longrightarrow (ac, bd) \in R_2$.

Define $\Theta_2 : B_2 \longrightarrow B_2/R_2 = A_2$

Then A_2 is a compact semigroup.

Define $\alpha_2 : S_2 \longrightarrow A_2$ such that $\alpha_2 = \theta_2 \beta_2$.

Then α_2 is a dense continuous homomorphism.

$\therefore (\alpha_2, A_2) \in K_1(S_2)$

$\therefore$ corresponding to each $(\alpha_1, A_1) \in K_1(S_1)$ there exists $(\alpha_2, A_2) \in K_1(S_2)$.

Conversely, if $(\alpha_2, A_2) \in K_1(S_2)$ there exists $(\alpha_1, A_1) \in K_1(S_1)$.

Define $\phi: (\alpha_1, A_1) \longmapsto (\alpha_2, A_2)$

Clearly ϕ is one-one and onto.

Again $K_1(S_1)$ isomorphic to $K_1(S_2)$

ie. ϕ preserves order in both directions

for, $(\alpha_1, A_1) \geqslant (\alpha_1', A_1')$ in $K_1(S_1)$

$\Longleftrightarrow R_1 \subseteq R_1'$

$\Longleftrightarrow R_2 \subseteq R_2'$

$\Longleftrightarrow (\alpha_2, A_2) \geqslant (\alpha_2', A_2')$

$\Longleftrightarrow \phi(\alpha_1, A_1) \geqslant \phi(\alpha_1', A_1')$

and also ϕ preserves the meet and join.

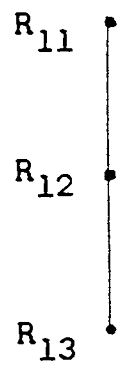
Hence the result.

Note

But in contrast with Magill's theorem on compactification [M_2], the converse of this theorem is false. We construct examples to show that (β_1, B_1) and (β_2, B_2) are not topologically isomorphic while $K_1(S_1)$ and $K_1(S_2)$ are lattice isomorphic.

Example 2.2.2

1. Let S_1 be a topological semigroup with Bohr compactification $(\beta_1, C \cup \{1\})$, where $C \cup \{1\}$ is a compact semigroup by adjoining an identity 1 to C discretely, where $C = \{z \in \mathbb{C} : |z| \leq \frac{1}{2}\}$ is a compact subsemigroup of complex plane $\mathbb{C}$ with complex multiplication and usual topology. (Note that S_1 can be $C \cup \{1\}$ itself). Set of all closed congruences on $C \cup \{1\}$ are $R_{11} = \triangle$, $R_{12} = C \times C \cup \triangle$, $R_{13} = B_1 \times B_1$.



Let S_2 be a topological semigroup with Bohr compactification

$$(\beta_2, \{ \overset{x}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \ \overset{y}{\begin{bmatrix} 0 & 1/2 \\ 0 & 1 \end{bmatrix}} \ \overset{z}{\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}} \}) = (\beta_2, B_2), \text{ where}$$

B_2 is a compact semigroup with discrete topology and multiplication defined by usual matrix multiplication (Note that S_2 can be B_2 itself).

ie,

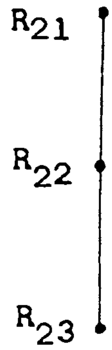
Set of all closed congruences on B_2 are

$R_{21} = \Delta$

$R_{22} = \{y,z\} \times \{y,z\} \cup \Delta$

$R_{23} = B_2 \times B_2$

	x	y	z
x	x	y	z
y	y	y	y
z	z	z	z



Then $K_1(S_1)$, family of all semigroup compactifications determined by R_{11}, R_{12} and R_{13} are lattice isomorphic to $K_1(S_2)$ family of all semigroup compactifications determined by R_{21}, R_{22} and R_{23}. But B_1 and B_2 are not topologically isomorphic.

2. Let S_1 be a topological semigroup with Bohr compactification (β_1, B_1)

β_1, $\{o, e, f, g, x, y\}$

where

$$o = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad e = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad f = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$g = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad y = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with discrete topology and usual matrix multiplication.

ie.

Set of all closed congruences on B_1 are

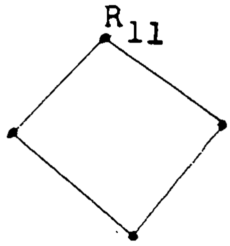
$R_{11} = \triangle$

$R_{12} = \{f,g\} \times \{f,g\} \cup \triangle$

$R_{13} = \{o,e,g,x,y\} \times \{o,e,g,x,y\} \cup \triangle$

$R_{14} = B \times B$

	o	e	f	g	x	y
o	o	o	o	o	o	o
e	o	e	o	o	x	o
f	o	o	f	g	o	y
g	o	o	g	g	o	y
x	o	o	x	x	o	e
y	o	y	o	o	g	o



Let S_2 be a topological semigroup with $(\beta_2, B_2) = (\beta_2, \{\frac{1}{2}, \frac{1}{3}, 1\})$ with discrete topology and multiplication defined as

$$xy = \max\left(\frac{1}{2}, xy\right)$$

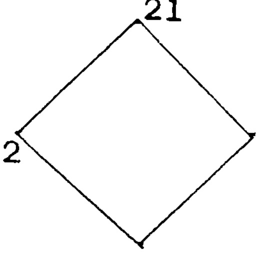
Set of all closed congruence on B_2 are

$R_{21} = \triangle$

$R_{22} = \{\frac{1}{2}, \frac{1}{3}\} \times \{\frac{1}{2}, \frac{1}{3}\} \cup \triangle$

$R_{23} = \{\frac{1}{2}, 1\} \times \{\frac{1}{2}, 1\} \cup \triangle$

$R_{24} = B_2 \times B_2$



Then $K_1(S_1)$ and $K_1(S_2)$ are lattice isomorphic but B_1 and B_2 are not topologically isomorphic.

2.3 Generalizations

We give certain possible generalizations of theorem (2.2.1) and some particular cases are given as corollaries.

Theorem 2.3.1.

Let S and S' be topological semigroups with Bohr compactification (β, B) and (β', B') respectively. If h is a continuous homomorphism from a closed ideal I contained in B into B', then there is a lattice isomorphism $\emptyset$ from an interval in $K_1(S')$ onto an interval in $K_1(S)$.

Proof

Let $h : I \longrightarrow B'$ be a continuous homomorphism. We have h(I), a compact subsemigroup of B'. Let (α_o, A_o) and (α_o', A_o') be the semigroup compactifications corresponding to R_o and R_o' respectively, where R_o the largest closed congruence in B' containing $h(I) \times h(I)$ and R_o' the smallest closed congruence in B' which restricted to h(I) is $\Delta h(I)$.

Consider the interval $[(\alpha_o,A_o), (\alpha_o',A_o')]$ in $K_1(S')$. Then $h^{-1}(R_o) = \{(x,y) \in I \times I : (h(x), h(y)) \in R_o\}$ is a closed congruence in I, since R_o is a closed congruence containing $h(I) \times h(I)$. Similarly

$$h^{-1}(R_o') = \{(x,y) \in I x I : (h(x), h(y) \in R_o\}$$

is a closed congruence in I.

Define $R_1 = h^{-1}(R_o) \cup \Delta_B$ and

$R_1' = h^{-1}(R_o') \cup \Delta_B$, both

are closed equivalence relations in B.

Again both are congruences.

for,

Consider the case if $(a,b) \in h^{-1}(R_o)$ and c=d in B, then $(ac,bd) \in I x I$ and $(h(ac), h(bc)) \in R_o$ follows from the fact that R_o is a closed congruence containing $h(I) \times h(I)$ and I is an ideal. Thus R_1 is a closed congruence in B. Similarly, we can prove that R_1' is a closed congruence in B.

Let (α_1,A_1) and (α_1', A_1') be the compactifications of S corresponding to R_1 and R_1' respectively and if $(\alpha,A) \in [(\alpha_o,A_o), (\alpha_o',A_o')]$ in $K_1(S')$,

where $A = B'/R$, R is a closed congruence on B'.

Then

$$(\alpha_o, A_o) \leqslant (\alpha, A) \leqslant (\alpha_o', A_o')$$

$\therefore$ $R_o \supseteq R \supseteq R_o'$

$\therefore$ $h^{-1}(R_o) \supseteq h^{-1}(R) \supseteq h^{-1}(R_o')$

$\therefore$ $h^{-1}(R_o) \cup \Delta_B \supseteq h^{-1}(R) \cup \Delta_B \supseteq h^{-1}(R_o') \cup \Delta_B$

$\therefore$ $(\alpha_1, A_1) \leqslant (\alpha', A') \leqslant (\alpha_1', A_1')$

Thus we have

$(\alpha', A') \in K_1(S_1)$ and

$(\alpha', A') \in [(\alpha_1, A_1), (\alpha_1', A_1')]$

$\therefore$ each $(\alpha, A) \in [(\alpha_o, A_o), (\alpha_o', A_o')]$ in $K_1(S')$ determines a $(\alpha', A') \in [(\alpha_1, A_1), (\alpha_1', A_1')]$ in $K_1(S)$.

Then the mapping

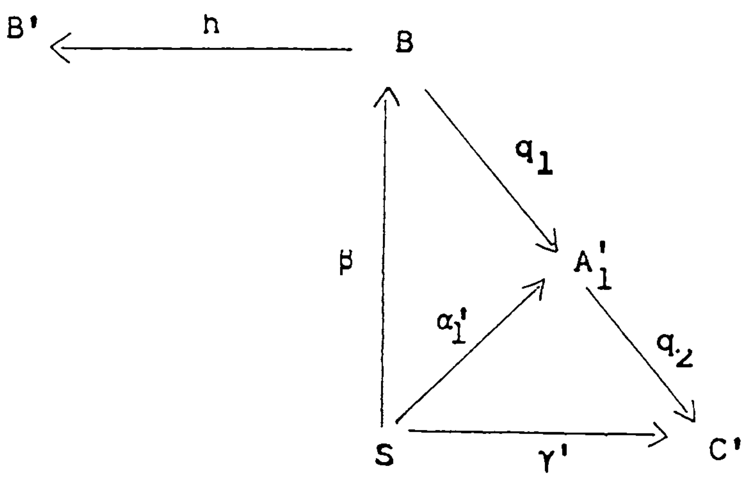
$$\emptyset: [(\alpha_o, A_o), (\alpha_o', A_o')] \longmapsto [(\alpha_1, A_1), (\alpha_1', A_1')]$$

is onto, one-one and order preserving.

$\emptyset$ is onto. for, if $(\gamma', C') \in [(\alpha_1, A_1), (\alpha_1', A_1')]$

ie. $(\alpha_1, A_1) \leqslant (\gamma', C') \leqslant (\alpha_1', A_1')$

$\therefore$ (γ', C') is the quotient space of (α_1', A_1')



Define a closed congruence R_1 on B such that for all x, x' $\in$ B

$$x \, R_1 x' \Longleftrightarrow q_2 q_1(x) = q_2 q_1(x')$$

where $q_1 : B \longrightarrow A_1'$

$q_2 : A_1' \longrightarrow C'$ are quotient maps.

Define a closed congruence

R_o on B' as

$xR_oy \Longleftrightarrow$ there exist $t \in h^{-1}(x)$, $t' \in h^{-1}(y)$
such that $t \, R_1 \, t'$.

This is well defined. for,

if there exists $t_o \in h^{-1}(x)$, $t_o' \in h^{-1}(y)$ such that
$t_o \, R_1 \, t_o'$

Then

$$t \, R_1 \, t' \text{ for all } t \in h^{-1}(x), \; t' \in h^{-1}(y).$$

for, it is given that

$$t_o \, R_1 \, t_o'. \quad \text{ie.} \quad q_2 \, q_1(t_o) = q_2 q_1(t_o')$$

$$q_2 q_1(t) = q_2(q_1(t_o)) \quad (\because \; t_o \in h^{-1}(x)$$
$$t \in h^{-1}(x)$$
$$h(t) = h(t_o)$$
$$\therefore \; q_1(t) = q_1(t_o))$$

$$= q_2(q_1(t_o'))$$

$$= q_2(q_1(t')) \quad (\because \; t_o' \in h^{-1}(y)$$
$$t' \in h^{-1}(y)$$
$$h(t_o') = h(t')$$
$$\therefore \; q_1(t_o') = q_1(t'))$$

$$= q_2 \, q_1(t')$$

$\therefore$ $t \, R_1 \, t'$ for all $t \in h^{-1}(x)$, $t' \in h^{-1}(y)$.

Thus R_o is a closed congruence in B'. Then R_o corresponds to semigroup compactification say (γ, C) in $K_1(S')$ such that

$$\phi(\gamma, C) = (\gamma', C')$$

$\emptyset$ is one-one. for if

$$(\gamma_2, C_2) = (\gamma_2', C_2') \in [(\alpha_o, A_o), (\alpha_o', A_o')]$$

ie. $R_2 = R_2'$ where $C_1 = B'/R_2$,

$C_2 == B'/R_2'$

ie. $h^{-1}(R_2) = h^{-1}(R_2')$

ie. $h^{-1}(R_2) \cup \triangle_B = h^{-1}(R_2') \cup \triangle_B$

ie. $(\gamma_3, C_3) = (\gamma_3', C_3') \in [(\alpha_1, A_1), (\alpha_1', A_1')]$

ie. $\emptyset(\gamma_2, C_2) = \emptyset(\gamma_2', C_2')$

$\emptyset$ preserves order in both ways .

For, $(\gamma_2, C_2) \geqslant (\gamma_2', C_2') \in [(\alpha_o, A_o), (\alpha_o', A_o')]$

$$\Longleftrightarrow R_2 \subseteq R_2'$$
$$\Longleftrightarrow h^{-1}(R_2) \subseteq h^{-1}(R_2')$$
$$\Longleftrightarrow h^{-1}(R_2) \cup \triangle_B \subseteq h^{-1}(R_2') \cup \triangle_B$$
$$\Longleftrightarrow (\gamma_3, C_3) \geqslant (\gamma_3', C_3')$$
$$\Longleftrightarrow \emptyset(\gamma_2, C_2) \geqslant \emptyset(\gamma_2', C_2')$$

Corollary-1. 2.3.2

Suppose h is a continuous surmorphism from B onto B'. Then $K_1(S')$ is lattice isomorphic to the ideal generated by (α_1', A_1') in $K_1(S)$.

Proof

Here we denote (α_1', A_1') as the semigroup compactification corresponding to $h^{-1}(\triangle)$, where $\triangle$ is a diagonal in B'. Then the image of each element in $K_1(S')$ is contained in ideal generated by (α_1', A_1').

Corollary-2. 2.3.3.

Suppose h is a continuous surmorphism from a closed ideal I in B onto B'. Then $K_1(S')$ is lattice isomorphic to the ideal generated by (α_1', A_1') in $K_1(S)$.

Proof

Here (α_1', A_1') is the semigroup compactification corresponding to the closed congruence

$$R_1' = \{(x,y) \in I \times I : h(x) = h(y)\} \cup \{(x,x)/x \in B\} \text{ on } B.$$

From the above theorem (2.3.1) and corollaries, we have the following theorem in categorical language.

Theorem 2.3.4

Let CS be the category whose objects are ordered pairs (B,I) where I is a closed ideal of a compact semigroup B. If (B_1,I_1) and (B_2,I_2) are two such pairs, by

morphism $f : (B_1,I_1) \longrightarrow (B_2,I_2)$, we mean continuous homomorphism $f:I_1$ into B_2 such that $f(I_1) = I_2$. Let L be the category of all complete lattices, a morphism from L to L' being a lattice isomorphism from an interval in L onto an interval in L'. Let F associate to each (B,I) in CS, the corresponding $K_1(S)$ in L, where B is the Bohr compactification of topological semigroup S.
Then F is a contravariant functor from CS to L.

Chapter-3

SOME MORE RESULTS ON THE LATTICE $K_1(S)$

Introduction

In this chapter, we discuss some special types of congruences on a topological semigroup S and related results about the lattice $K_1(S)$ of semigroup compactifications of a topological semigroup S, also some results about atoms and dual atoms of $K_1(S)$ are obtained. These arose as a result of our attempt (though not successful) to obtain at least some partial converse of the theorem-for topological semigroups S_1 and S_2 if (β_1,B_1) and (β_2,B_2) are topologically isomorphic then the lattices $K_1(S_1)$ and $K_1(S_2)$ are isomorphic.

In Section 3.1, we prove that

(i) A topological semigroup S with Bohr compactification (β,B) has a semigroup compactification (α,A) determined by 'n' disjoint closed proper weak ideals (ideals) of B, at least one of which is non-singleton only if S has a semigroup compactification strictly bigger than (α,A).

(ii) A topological semigroup S with Bohr compactification (β,B) has an n-point compactification determined by 'n' non-empty subsets of B does not imply that it has an (n-1)-point compactification, nor does it imply that there is a semigroup compactification strictly bigger than (α,A) different from (β,B).

(iii) If a topological semigroup S with (β,B) has an n-point compactification (α,A) determined by 'n' non-empty weak ideals (ideals) of B, then there exists semigroup compactification strictly bigger than (α,A), but it does not imply that S has an (n-1)-point compactification.

In Section 3.2 we describe the dual atoms and atoms of $K_1(S)$, when B is finite.

3.1 Special types of congruences

In this section we introduce weak ideals, joint weak ideals and complementary joint ideals of a semigroup S and describe special types of congruences on S.

Let S be a semigroup and ω an ideal of S, then $(\omega x \omega) \cup \Delta$ is a congruence on S [HOW]. But converse need not be true. i.e., if R is a congruence of the form $(\omega x \omega) \cup \Delta$, then ω need not be an ideal of S.

Example 3.1.1.

1) $Z_4 = \{0,1,2,3\}$ with multiplication modulo 4 is a semigroup.

Here $\{1,3\} \times \{1,3\} \cup \Delta$ is a congruence on Z_4, but $\{1,3\}$ is not an ideal of Z_4.

2) Let $T = \{o,e,f,g,x,y\}$ with usual matrix multiplication where

$$o = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad e = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$f = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad g = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad y = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Here $\{f,g\} \times \{f,g\} \cup \Delta$ is a congruence, but $\{f,g\}$ is not an ideal.

	o	e	f	g	x	y
o	0	0	0	0	0	0
e	0	e	0	0	x	0
f	0	0	f	g	0	y
g	0	0	g	g	0	y
x	0	0	x	x	0	e
y	0	y	0	0	g	0

In this situation we introduce the following definitions.

Definition 3.1.2.

A non-empty subset ω of a semigroup S is said to be a

(i) weak right ideal of S

if either $ax, bx \in \omega$ or $ax = bx$ for all $a,b \in \omega$ and for all $x \in S$.

(ii) weak left ideal of S

if either $xa, xb \in \omega$ or $xa=xb$ for all $a,b \in \omega$ and for all $x \in S$.

(iii) weak ideal of S, if it is both weak right and weak left ideal of S

i.e., if either $ax,bx \in \omega$ or $ax=bx$ and either $xa, xb \in \omega$ or $xa=xb$ for all $a,b \in \omega$ and for all $x \in S$

Result 3.1.3

A topological semigroup S has a non-trivial closed congruence of the form $\omega \times \omega \cup \Delta$ if and only if ω is a closed non-singleton proper weak ideal of S.

Proof

Assume that S has a non-trivial closed congruence of the form $\omega \times \omega \cup \Delta = R$ (say)

i.e., $\Delta \subsetneq R \subsetneq S \times S$

and for all $(a,b) \in R$ both $a,b \in \omega$ or $a = b$.

Since R is non-trivial, there exist at least one (a,b) such that $a \neq b \in \omega$.

i.e., ω is a non-singleton proper subset of S.

ω is a weak ideal. For,

since R is a congruence,

both (ax,bx), $(xa,xb) \in R$ for all $a,b \in \omega$ and for all $x \in S$.

i.e., either $ax, bx \in \omega$ or $ax=bx$

and either $xa, xb \in \omega$ or $xa = xb$

for all $a,b \in \omega$ and for all $x \in S$.

i.e, ω is a weak ideal.

Again ω is closed; for,

let (x_α) be a net in ω, $(x_\alpha) \longrightarrow x \in S$.

Since ω is non-singleton, let $y \ (\neq x) \in \omega$.

$\therefore$ (x_α, y) is a net in R, which is closed.

$\therefore$ the limit (x,y) of (x_α, y) belongs to R.

i.e., $(x,y) \in \omega \times \omega$ ($\because$ $x \neq y$)

$\therefore$ $x \in \omega$.

On the other hand, consider ω as a closed non-singleton proper weak ideal of S, then clearly $R = \omega \times \omega \cup \Delta$ is closed, since Δ is closed in $S \times S$ and ω is closed in S.

Clearly R is an equivalence.

Again R is compatible. For,

since ω is a weak ideal both $(ax,bx), (xa,xb) \in R$ for all $a,b \in \omega$ and for all $x \in S$.

Clearly R is non-trivial, since ω is a non-singleton proper subset of S.

Hence the result.

Remark 3.1.4.

If B is the Bohr compactification of a topological semigroup S, then $B/(\omega \times \omega) \cup \Delta$ is called the semigroup compactification of S determined by ω. Thus S has a semigroup compactification defined by ω if and only if ω is a closed non-singleton proper weak ideal of B.

Definition 3.1.5.

Two non-empty disjoint subsets ω_1 and ω_2 of a semigroup S are said to be

(i) joint weak right ideals if
either $ax, bx \in \omega_1$ or $ax, bx \in \omega_2$ or $ax = bx$
for all $a,b \in \omega_1$ or $a,b \in \omega_2$ and for all $x \in S$.

(ii) joint weak left ideals if
either $xa, xb \in \omega_1$ or $xa, xb \in \omega_2$ or $xa = xb$
for all $a,b \in \omega_1$ or $a,b \in \omega_2$ and for all $x \in S$.

(iii) joint weak ideals if they are both joint weak right and joint weak left ideals of S.

i.e., either $ax, bx \in \omega_1$ or $ax, bx \in \omega_2$ or $ax = bx$
and either $xa, xb \in \omega_1$ or $xa, xb \in \omega_2$ or $xa = xb$
for all $a,b \in \omega_1$ or $a,b \in \omega_2$, and for all $x \in S$.

Result 3.1.6

A topological semigroup S has a non-trivial closed congruence of the form $\omega_1 \times \omega_1 \uplus \omega_2 \times \omega_2 \cup \Delta$ ($\uplus$ indicates the sets whose union is taken, are disjoint) if and only if ω_1 and ω_2 are disjoint closed proper joint weak ideals, at least one of which is non-singleton.

Proof

Suppose that S has a non-trivial closed congruence of the form

$\omega_1 \times \omega_1 \cup \omega_2 \times \omega_2 \cup \Delta = R$ (say)

$\Delta \subsetneq R \subsetneq S \times S$, since R is non-trivial

and for all $(a,b) \in R$, both $a,b \in \omega_1$ or both $a,b \in \omega_2$ or $a=b$.

Clearly ω_1, ω_2 are disjoint proper subsets of S and at least one of them is non-singleton.

ω_1, ω_2 are joint weak ideals. For,

suppose first that $a,b \in \omega_1$.

Then both (ax,bx), $(xa,xb) \in R$ for all $x \in S$

($\because$ R is compatible)

i.e., either $ax, bx \in \omega_1$ or $ax, bx \in \omega_2$ or $ax = bx$ and either $xa, xb \in \omega_1$ or $xa, xb \in \omega_2$ or $xa = xb$ for all $x \in S$.

i.e., ω_1, ω_2 are joint weak ideals.

Similarly if for all $a,b \in \omega_2$ and for $a=b$. Also they are closed. If ω_1 is a singleton, then clearly it is closed. If not, we proceed as follows.

Let (x_α) be a net in ω_1, $(x_\alpha) \longrightarrow x \in S$. Since ω_1 is non-singleton, let $y(\neq x) \in \omega_1$, (x_α, y) be a net in $\omega_1 \times \omega_1$.

$\therefore$ (x_α, y) be a net in $\omega_1 \times \omega_1 \cup \omega_2 \times \omega_2 \cup \Delta = R$, which is closed.

$\therefore$ the limit (x,y) of (x_α, y) belongs to R

$\therefore$ $(x,y) \in \omega_1 \times \omega_1$.

i.e., both $x,y \in \omega_1$. $\therefore$ $x \in \omega_1$. Thus ω_1 is closed.

Similarly ω_2 is closed.

Hence the result.

On the other hand, if ω_1, ω_2 are disjoint closed proper joint weak ideals of S, at least one of which is non-singleton, then $\omega_1 \times \omega_1 \cup \omega_2 \times \omega_2 \cup \Delta = R$ is closed, since ω_1, ω_2 are closed in S and Δ is closed in $S \times S$.

R is clearly reflexive and symmetric.

R is transitive. For,

$(a,b) \in R$ and $(b,c) \in R$

imply either both $a,b \in \omega_1$ or both $a,b \in \omega_2$ or $a=b$ and either both $b,c \in \omega_1$ or both $b,c \in \omega_2$ or $b=c$.

Since ω_1,ω_2 are disjoint, the following cases are not possible.

(1) $a,b \in \omega_1$ and $b,c \in \omega_2$

(2) $a,b \in \omega_2$ and $b,c \in \omega_1$

all other cases imply $(a,c) \in R$

R is compatible. For,

since ω_1,ω_2 are joint weak ideals

either both $ax,bx \in \omega_1$ or both $ax,bx \in \omega_2$ or $ax=bx$

and either both $xa,xb \in \omega_1$ or both $xa,xb \in \omega_2$ or $xa=xb$

for all $(a,b) \in R$ and for all $x \in S$

i.e., both (ax,bx) and $(xa,xb) \in R$

Also R is non-trivial since at least one of ω_1,ω_2 is non-singleton.

Hence the result.

Remark 3.1.7.

A topological semigroup S with Bohr compactification (β,B) has a semigroup compactification "determined" by $\{\omega_1,\omega_2\}$ if and only if ω_1,ω_2 are disjoint closed proper joint weak ideals of B, at least one of which is non-singleton.

Definition 3.1.8.

A finite disjoint family $\{\omega_1, \omega_2, .., \omega_n\}$ of S is said to be joint weak ideals if either both $ax, bx \in \omega_1$ or both $ax, bx \in \omega_2$ or ... or both $ax, bx \in \omega_n$ or $ax = bx$ and either both $xa, xb \in \omega_1$ or both $xa, xb \in \omega_2$ or ... both $xa, xb \in \omega_n$ or $xa = xb$, for all $a, b \in \omega_1$ or in ω_2 or in ω_3 or ... in ω_n and for all $x \in S$.

By a similar argument as to that in result (3.1.6) we obtain the following.

Result 3.1.9.

A topological semigroup S has a non-trivial closed congruence of the form $\bigcup_{i=1}^{n} \omega_i \times \omega_i \cup \Delta \Longleftrightarrow \omega_i$'s are disjoint closed proper joint weak ideals of S, at least one of which is non-singleton.

Definition 3.1.10.

Two non-empty subsets ω_1 and ω_2 of a semigroup S are said to be

(i) joint right ideals, if either $ax, bx \in \omega_1$ or $ax, bx \in \omega_2$ for all $a, b \in \omega_1$ or $a, b \in \omega_2$ and for all $x \in S$.

(ii) joint left ideals, if either $xa, xb \in \omega_1$ or $xa, xb \in \omega_2$ for all $a, b \in \omega_1$ or $a, b \in \omega_2$ and for all $x \in S$.

(iii) joint ideals, if both joint right and joint left ideals.

i.e., either $ax, bx \in \omega_1$ or $ax, bx \in \omega_2$ and either $xa, xb \in \omega_1$ or $xa, xb \in \omega_2$ for all $a, b \in \omega_1$ or $a, b \in \omega_2$ and for all $x \in S$.

Definition 3.1.11 Complementary joint ideals.

Two joint ideals ω_1 and ω_2 of a semigroup S are said to be complementary if they are disjoint and $\omega_1 \cup \omega_2 = S$.

Result 3.1.12.

A topological semigroup S has a non-trivial closed congruence of the form $\omega_1 \times \omega_1 \Psi \omega_2 \times \omega_2$ if and only if ω_1 and ω_2 are disjoint closed proper complementary joint ideals of S, at least one of which is non-singleton.

Suppose S has a non trivial closed congruence of the form $\omega_1 \times \omega_1 \Psi \omega_2 \times \omega_2 = R$ (say).

Then,

1) ω_1, ω_2 are proper subsets of S, at least one of which is non-singleton.

2) ω_1, ω_2 are complementary joint ideals

for,

for all $a \in S$ $(a,a) \in \omega_1 \times \omega_1 \Psi \omega_2 \times \omega_2 = R$

($\because$ R is a congruence)

i.e., either $a \in \omega_1$ or in ω_2, for all $a \in S$

i.e., $S = \omega_1 \cup \omega_2$.

Clearly ω_1, ω_2 are disjoint (R being an equivalence)

Again ω_1, ω_2 are joint ideals for,

for all $a, b \in \omega_1$ or $a, b \in \omega_2$ and for all $x \in S$
both (ax, bx), $(xa, xb) \in \omega_1 \times \omega_1 \Psi \omega_2 \times \omega_2$

($\because$ R is compatible)

i.e., either both $ax, bx \in \omega_1$ or $ax, bx \in \omega_2$
and either both $xa, xb \in \omega_1$ or $xa, xb \in \omega_2$

$\Longrightarrow \omega_1, \omega_2$ are joint ideals.

ω_1, ω_2 are closed. For, consider ω_1.

If ω_1 is a singleton, then clear. If ω_1 is not a singleton, we proceed as follows.

Let (x_α) be a net in ω_1, $(x_\alpha) \longrightarrow x \in S$.
Since ω_1 is non-singleton, there exists $y \in \omega_1$, $y \notin \omega_2$.

$\therefore$ (x_α, y) be a net in $\omega_1 \times \omega_1 \Psi \omega_2 \times \omega_2$,
which is closed.

$\therefore$ the limit (x,y) of (x_α, y) belongs to $\omega_1 \times \omega_1 \Psi \omega_2 \times \omega_2$.

i.e., both $(x,y) \in \omega_1 \times \omega_1 (\because y \in \omega_1)$.

$\therefore$ $x \in \omega_1$. Thus ω_1 is closed.

Similarly ω_2 is closed.

Hence the result.

On the other hand, if ω_1, ω_2 are disjoint closed proper complementary joint ideals of S, at least one of which is non-singleton, then $R = \omega_1 \times \omega_1 \Psi \omega_2 \times \omega_2$ is a closed non-trivial subset of S x S.

R is an equivalence for,

for all $a \in S$, either $a \in \omega_1$ or in ω_2

$(\because \omega_1 \cup \omega_2 = B,\ \omega_1 \cap \omega_2 = \emptyset)$

$\therefore (a,a) \in \omega_1 \times \omega_1 \Psi \omega_2 \times \omega_2$

i.e., $\Delta \subset \omega_1 \times \omega_1 \Psi \omega_2 \times \omega_2$

i.e., R is reflexive.

Clearly R is symmetric. Also R is transitive for,

let $(a,b), (b,c) \in R$.

i.e., either both $a,b \in \omega_1$ or both $a,b \in \omega_2$

and either both $b,c \in \omega_1$ or both $b,c \in \omega_2$.

Since ω_1, ω_2 are disjoint, the possible cases are

$a,b \in \omega_1$, $b,c \in \omega_1$

and $a,b \in \omega_2$, $b,c \in \omega_2$

$\therefore$ $(a,c) \in \omega_1 \times \omega_1 \cup \omega_2 \times \omega_2$

Again,

R is compatable.

For,

since ω_1, ω_2 are joint ideals, by the definition, we have either

$ax, bx \in \omega_1$ or $ax, bx \in \omega_2$

and either

$xa, xb \in \omega_1$ or $xa, xb \in \omega_2$

for all $a,b \in \omega_1$ or $a,b \in \omega_2$

and for all $x \in S$.

i.e., $(ax,bx), (xa,xb) \in \omega_1 \times \omega_1 \cup \omega_2 \times \omega_2$.

Hence the result.

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Remark 3.1.13.

A topological semigroup S with Bohr compactification (β, B) has a semigroup compactification determined by a non-trivial closed congruence of the form $\omega_1 x \omega_1 \uplus \omega_2 \times \omega_2$ if and only if ω_1 and ω_2 are disjoint closed proper complementary joint ideals of B, at least one of which is non-singleton.

Theorem 3.1.14.

A topological semigroup S has a non-trivial closed congruence of the form $\biguplus_{i=1}^{n} \omega_i \times \omega_i$ if and only if ω_i's are disjoint closed proper complementary joint ideals of S (i.e., $\omega_i \cap \omega_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^{n} \omega_i = S$), at least one of which is non-singleton.

As a result we get

Theorem 3.1.15.

A topological semigroup S has an n-point compactification if and only if its Bohr compactification has n disjoint closed proper complementary joint ideals, at least one of which is non-singleton.

Remark 3.1.16.

A semigroup S has a set $\{\omega_i\}_{i=1}^{n}$ of finite number of joint weak ideals does not imply that any of the ω_i's is a weak ideal, nor does it imply that a proper subset of $\{\omega_i\}_{i=1}^{n}$ forms joint weak ideals.

Example.

$Z_8 = \{0,1,2,3,4,5,6,7\}$ with multiplication modulo 8 is a semigroup

(1) $\{\omega_1, \omega_2\} = \{\{1,5\}, \{3,7\}\}$ is a set of joint weak ideals, but neither $\{1,5\}$ nor $\{3,7\}$ is a weak ideal of Z_8.

(2) $\{\omega_1, \omega_2, \omega_3\} = \{\{1,7\}, \{2,6\}, \{3,5\}\}$ is a set of joint weak ideals, but $\{\{1,7\}, \{3,5\}\}$ is not a set of joint weak ideals. Also $\{1,7\}$, $\{3,5\}$ are not weak ideals.

Result 3.1.17

A semigroup S has a congruence of the form $(\bigcup_{i=1}^{n} \omega_i \times \omega_i) \cup \Delta$ does not imply

(1) $(\omega_i \times \omega_i) \cup \Delta$ is a congruence on S for any $i=1,2,\ldots,n$.

(2) $(\bigcup_{j \in A} \omega_j \times \omega_j) \cup \Delta$ is a congruence on S for some proper subset A of $\{1,2,\ldots,n\}$.

Theorem 3.1.18.

A semigroup S has a congruence of the form $(\bigcup_{i=1}^{n} \omega_i \times \omega_i) \cup \triangle$, with ω_i's weak ideals (ideals), then S has a congruence of the form $(\bigcup_{j \in A} \omega_j \times \omega_j) \cup \triangle$, where A is any proper subset of $\{1,2,\ldots n\}$, contained in $(\bigcup_{i=1}^{n} \omega_i \times \omega_i) \cup \triangle$.

Proof.

Given $(\bigcup_{i=1}^{n} \omega_i \times \omega_i) \cup \triangle$ is a congruence, with ω_i's weak ideals (ideals).

i.e., ω_i's are disjoint closed proper weak ideals (ideals)

$\therefore$ $(\omega_i \times \omega_i) \cup \triangle$ is a congruence for any $i=1,2,\ldots,n$

Consider $\{\omega_j\}_{j \in A}$, where A is any proper subset of $\{1,2,\ldots,n\}$.

Then $(\omega_j \times \omega_j) \cup \triangle$ is a congruence for each $j \in A$.

i.e., $(\bigcup_{j \in A} \omega_j \times \omega_j) \cup \triangle$ is a congruence contained in

$$(\bigcup_{i=1}^{n} \omega_i \times \omega_i) \cup \triangle, \text{ since } \{\omega_j\}_{j \in A} \subset \{\omega_i\}_{i=1}^{n}$$

Hence the result.

Theorem 3.1.19

Let S be a topological semigroup with closed congruence $(\bigcup_{i=1}^{n} \omega_i \times \omega_i) \cup \Delta$, where ω_i's are either ideals or weak ideals. Then S has a closed congruence of the form $(\bigcup_{j \in A} \omega_j \times \omega_j) \cup \Delta$ contained in $(\bigcup_{i=1}^{n} \omega_i \times \omega_i) \cup \Delta$, where A is any proper subset of $\{1,2,\ldots, n\}$.

Proof.

Given $(\bigcup_{i=1}^{n} \omega_i \times \omega_i) \cup \Delta$ is a closed congruence with ω_i's are weak ideals (ideals).

i.e., ω_i's are closed disjoint proper weak ideals, at least one of which is non-singleton [3.1.9]

i.e., each $(\omega_i \times \omega_i) \cup \Delta$ is a closed congruence

i.e., $R = (\bigcup_{j \in A} \omega_j \times \omega_j) \cup \Delta$, where A is any proper subset of $\{1,2,\ldots,n\}$ is a congruence contained in $(\bigcup_{i=1}^{n} \omega_i \times \omega_i) \cup \Delta$ [3.1.18].

Also $R = (\bigcup_{j \in A} \omega_j \times \omega_j) \cup \Delta$ is closed, since ω_i's are closed in S and Δ is closed in S x S.

We obtained the following theorem about the lattice $K_1(S)$ of a given topological semigroup S with Bohr compactification (β,B).

Theorem 3.1.20.

Let S be a topological semigroup with Bohr compactification (β,B). If S has a semigroup compactification (α,A) determined by 'n' disjoint closed proper weak ideals (ideals) $\{\omega_i\}_{i=1}^{n}$ of B, at least one of which is non-singleton, then there is a semigroup compactification in $K_1(S)$ strictly bigger than (α,A).

Proof.

Since $\{\omega_i\}_{i=1}^{n}$ are disjoint closed proper weak ideals (ideals) of B, at least one of which is non-singleton, B has a non-trivial closed congruence of the form

$R = \left(\bigcup_{i=1}^{n} \omega_i \times \omega_i\right) \cup \Delta$. Let (α,A) denote the semigroup compactification determined by R.

i.e., $(\beta,B) > (\alpha,A) \in K_1(S)$

Again since ω_i's are weak ideals (ideals) for each $i \in \{1,2,...,n\}$, by theorem (3.1.19), B has a closed congruence of the form

$$R' = \left(\bigcup_{j \in I} \omega_j \times \omega_j\right) \cup \Delta,$$ where I is any proper subset of $\{1,2,\ldots,n\}$ and R' is contained in R.

Let (α_1, A_1) denotes semigroup compactification determined by R' and $(\beta,B) > (\alpha_1,A_1) > (\alpha,A)$.

Hence the result.

Remark 3.1.21.

Theorem (3.1.20) need not be true, if (α,A) is determined by $\{\omega_i\}_{i=1}^{n}$ closed disjoint proper weak ideals, at least one of which is non-singleton.

For example,

Let S be a topological semigroup with (β,B), where

$B = Z_8 = \{0,1,2,3,4,5,6,7\}$ with discrete topology and multiplication modulo 8.

(1) Here $\{\omega_1,\omega_2,\omega_3\} = \{\{1,7\}, \{2,6\}, \{3,5\}\}$ set of joint weak ideals and Z_8 has a congruence

$$\{1,7\} \times \{1,7\} \cup \{2,6\} \times \{2,6\} \cup \{3,5\} \times \{3,5\} \cup \Delta$$

but $\{1,7\} \times \{1,7\} \cup$ and $\{3,5\} \times \{3,5\} \cup \Delta$ are not congruences.

(2) $\{\{1,5\}, \{3,7\}\}$ set of joint weak ideals and $\{1,5\} \times \{1,5\} \cup \{3,7\} \times \{3,7\} \cup \triangle$ is a congruence but $\{1,5\} \times \{1,5\} \cup \triangle$ and $\{3,7\} \times \{3,7\} \cup \triangle$ are not congruences.

Remark 3.1.22.

A semigroup S has a set $\{\omega_i\}_{i=1}^{n}$ of finite number of complementary joint ideals does not imply that any of the ω_i's is a weak ideal, nor does it imply that a proper subset of $\{\omega_i\}_{i=1}^{n}$ forms joint weak ideals, or complementary joint ideals.

Example.

(1) Let S = {e,a,f,b} with multiplication defined below is a semigroup

	e	a	f	b
e	e	a	f	b
a	a	e	b	f
f	f	b	f	b
b	b	f	b	f

Here $\{\omega_1, \omega_2\} = \{\{e,f\}, \{a,b\}\}$ is a set of complementary joint ideals, but neither {e,f} nor {a,b} is a weak ideal.

(2) $Z_6 = \{0,1,2,3,4,5\}$ with multiplication modulo 6 is a semigroup.

Here $\{\omega_1, \omega_2, \omega_3\} = \{\{2,5\}, \{1,4\}, \{0,3\}\}$ is a set of complementary joint ideals but $\{\{2,5\}, \{1,4\}\}$ are not sets of joint weak ideals, nor complementary joint ideals.

From the above remark we have the following result.

Result 3.1.23.

A semigroup S has a congruence of the form $\bigcup_{i=1}^{n} \omega_i \times \omega_i$ does not imply

(1) $(\bigcup_{j \in A} \omega_j \times \omega_j) \cup \Delta$ is a congruence on S, where A is any proper subset of $\{1,2,...,n\}$.

(2) $\bigcup_{j \in A} \omega_j \times \omega_j$ is a congruence on S, where A is any proper subset of $\{1,2,...,n\}$.

Theorem 3.1.24.

A semigroup S has a non-trivial congruence of the form $\bigcup_{i=1}^{n} \omega_i \times \omega_i$ with ω_i's weak ideals (ideals) then S has a congruence of the form $(\bigcup_{j \in A} \omega_j \times \omega_j) \cup \Delta$, where A is any proper subset of $\{1,2,...,n\}$. But $\bigcup_{j \in A} \omega_j \times \omega_j$ is not a congruence.

Proof

Given ($\bigcup_{i=1}^{n} \omega_i \times \omega_i$) is a non-trivial congruence with ω_i's weak ideals (ideals).

i.e., ω_i's are disjoint proper weak ideals for each $i = 1,\ldots,n$, at least one of which is non-singleton.

$\therefore$ $(\omega_i \times \omega_i) \cup \Delta$ is a congruence for each $i = 1,\ldots,n$.

i.e., $(\omega_j \times \omega_j) \cup \Delta$ is a congruence for each $j \in A$, where A is any proper subset of $1,\ldots,n$

$\therefore$ $(\bigcup_{j \in A} \omega_j \times \omega_j) \cup \Delta$ is a congruence contained in $\bigcup_{i=1}^{n} \omega_i \times \omega_i$.

But $\bigcup_{j \in A} \omega_j \times \omega_j$ is not a congruence, since it is not reflexive.

Theorem 3.1.25

Let S be a topological semigroup with non-trivial closed congruence $\bigcup_{i=1}^{n} \omega_i \times \omega_i$, where ω_i's are weak ideals (ideals), then S has a closed congruence of the form $(\bigcup_{j \in A} \omega_j \times \omega_j) \cup \Delta$ contained in $\bigcup_{i=1}^{n} \omega_i \times \omega_i$, where A is any proper subset of $\{1,\ldots,n\}$

Proof

This is immediate from (3.1.12) and (3.1.24).

Result 3.1.26.

A topological semigroup S with (β,B) has an n-point compactification does not imply that it has an (n-1)-point compactification, nor does it imply that there is a semigroup compactification strictly bigger than (α,A) and different from (β,B).

For example,

Let S be a topological semigroup with (β,B) where $B = \{e,a,f,b\}$ with discrete topology and multiplication defined below

	e	a	f	b
e	e	a	f	b
a	a	e	b	f
f	f	b	f	b
b	b	f	b	f

Closed congruences on B are

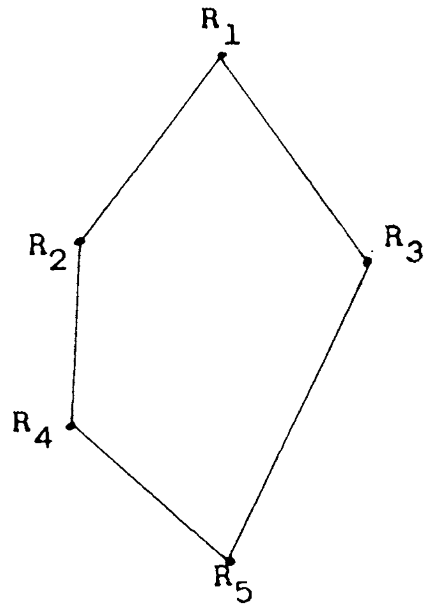
$R_1 = \Delta$

$R_2 = \{f,b\} \times \{f,b\} \cup \Delta$

$R_3 = \{e,f\} \times \{e,f\} \cup \{a,b\} \times \{a,b\}$

$R_4 = \{e,a\} \times \{e,a\} \cup \{f,b\} \times \{f,b\}$

$R_5 = B \times B$



R_3 determines a 2-point compactification say (α,A), where $\{\{e,f\}, \{a,b\}\}$ is a set of closed proper disjoint non-singleton complementary joint ideals but $\{e,f\} \times \{e,f\}$ and $\{a,b\} \times \{a,b\}$ are not closed congruences on B.

i.e., two point compactification doesnot imply the existence of one-point compactification.

Also,

$\{e,f\} \times \{e,f\} \cup \Delta$ and $\{a,b\} \times \{a,b\} \cup \Delta$ are not closed congruences contained in R_3. So 2-point compactification does not imply there exist a semigroup compactification strictly bigger than (α,A) and different from (β,B).

Next theorem shows that if a topological semigroup S with (β,B) has an n-point compactification (α,A) determined by 'n' weak ideals (ideals) of B, then there exists semigroup compactification strictly bigger than (α,A). And in this case also it doesnot imply that S has an (n-1)-point compactification.

Theorem 3.1.27.

A topological semigroup S with (β,B) has an n-point compactification (α,A) determined by 'n' weak ideals (ideals) of B, then there exists semigroup compactification strictly bigger than (α,A). And in this case also it does not imply that S has an (n-1)-point compactification.

Proof.

Since (α,A) is an n-point compactification of S, (α,A) is determined by a non-trivial closed congruence of the form $\bigcup_{i=1}^{n} \omega_i \times \omega_i$, where ω_i's closed proper complementary joint ideals of B, at least one of which is non-singleton. Also given that ω_i's are weak ideals (ideals).

i.e., ω_i's are closed disjoint proper weak ideals (ideals) of B, at least one of which is non-singleton.

By theorem (3.1.19) B has a closed congruence of the form $\bigcup_{j \in A} (\omega_j \times \omega_j) \cup \Delta$, where A is any proper subset of $\{1,2, \ldots, n\}$.

Also it determines a semigroup compactification (α_1,A_1) such that $(\beta,B) \geqslant (\alpha_1,A_1) > (\alpha,A)$.

i.e., there is a semigroup compactification strictly bigger than (α,A).

But it does not imply that S has an (n-1)-point compactification.

For example,

Let S be a topological semigroup with (β, B), where $B = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, 1\}$ with discrete topology and multiplication defined by $xy = \max\{\frac{1}{2}, xy\}$

Here S has a 2-point compactification determined by

$$R = \{\tfrac{1}{2}, 1\} \times \{\tfrac{1}{2}, 1\} \cup \{\tfrac{1}{3}, \tfrac{1}{4}\} \times \{\tfrac{1}{3}, \tfrac{1}{4}\}$$

where,

$\{\{\frac{1}{2}, 1\}, \{\frac{1}{3}, \frac{1}{4}\}\}$ is a set of disjoint proper closed non-singleton complementary joint ideals.

Also $\{\frac{1}{2}, 1\}$ and $\frac{1}{3}, \frac{1}{4}\}$ are weak ideals.

But $\{\frac{1}{2}, 1\} \times \{\frac{1}{2}, 1\}$ and $\{\frac{1}{3}, \frac{1}{4}\} \times \{\frac{1}{3}, \frac{1}{4}\}$ are not congruences.

$\therefore$ 2-point compactifications determined by weak ideals does not imply existence of one-point compactification.

3.2 Some results about atoms and dual atoms of $K_1(S)$

In this section, we describe the dual atoms and atoms of $K_1(S)$, family of all semigroup compactifications of a topological semigroup S with (β,B), where B is finite.

An element $(\alpha,A) \in K_1(S)$ is a dual atom of $K_1(S)$ provided $(\alpha,A) < (\beta,B)$ and there does not exist $(\alpha_1,A_1) \in K_1(S)$ for which $(\alpha,A) < (\alpha_1,A_1) < (\beta,B)$.

An element $(\alpha_o,A_o) \in K_1(S)$ is an atom of $K_1(S)$ provided $(\alpha_o,A_o) > (\alpha, \{o\})$, where $(\alpha, \{o\})$ is the smallest semigroup compactification of S and there does not exist $(\alpha_1,A_1) \in K_1(S)$ for which $(\alpha_o,A_o) > (\alpha_1,A_1) > (\alpha, \{o\})$

Theorem 3.2.1.

Let S be a topological semigroup with Bohr compactification (β,B), where B is finite, and ω^* be the collection of all weak ideals, joint weak ideals, complementary joint ideals of B. If there exists a closed non-singleton proper weak ideal ω minimal (maximal) in ω^*, then (α,A) the semigroup compactification determined by ω is a dual atom (atom) of $K_1(S)$.

Proof.

Let $|B| = n$, where n is finite and ω^* be the collection of all weak ideals, joint weak ideals, complementary joint ideals of B.

(a) Let 'ω' be a closed non-singleton proper weak ideal of B minimal in ω^*.

i.e., there exists no weak ideal, no joint weak ideals, no complementary joint ideals properly contained in ω and $(\omega x \omega) \cup \triangle$ is a non-trivial closed congruence on B.

i.e., $\triangle \subsetneq (\omega x \omega) \cup \triangle$, and there exists no non-trivial closed congruence properly contained in $(\omega x \omega) \cup \triangle$. If not, let R' be a non-trivial closed congruence properly contained in $(\omega x \omega) \cup \triangle$.

i.e., $R' \subsetneq (\omega x \omega) \cup \triangle \subset BxB$

Since R' is a non-trivial closed congruence, R' is **determined** by at least one non-singleton subset A (say) of B; if not, let $|A| = 1$, R' determined by A is $\triangle$, this is not possible since $R' \neq \triangle$. Then the possible cases of R' are the following:

<u>Case-1</u>

R' is determined by a subset A of B with $1 < |A| < n$

If $|A| = 2$, i.e., $A = \{a,b\}$ (say)

Then $R' = \{a,b\} \times \{a,b\} \cup \triangle \subset (\omega x \omega) \cup \triangle$

Since $a \neq b$, $\{a,b\} \subset \omega$

and since R' is a congruence, for all $a,b \in A$

$ax, bx \in A$ or $ax = bx$

and $xa, xb \in A$ or $xa = xb$ for all $x \in B$.

i.e., $A = \{a,b\}$ is a weak ideal, also we have $\{a,b\} \subset \omega$, which is a contradiction.

Similarly we have a contradiction if R' is determined by any non-empty subset A of B, with $1 < |A| < n$.

Case-2

If R' is determined by two non-singleton subsets say $A_1 = \{a,b\}$, $A_2 = \{c,d\}$

i.e., $\{a,b\} \times \{a,b\} \cup \{c,d\} \times \{c,d\} \cup \triangle$ is a closed congruence contained in $\omega \times \omega \cup \triangle$

Since $a \neq b$, $c \neq d$, $\{a,b,c,d\} \subset \omega$

and since R' is a congruence, for all $x \in B$

and for all $a,b \in \{a,b\}$ or in $\{c,d\}$

$ax,bx \in \{a,b\}$ or $ax,bx \in \{c,d\}$ or $ax = bx$

and $xa,xb \in \{a,b\}$ or $xa,xb \in \{c,d\}$ or $xa = xb$

i.e. $\{\{a,b\}, \{c,d\}\}$ is a set of joint weak ideals contained in ω, which is a contradiction.

Similarly, we have a contraidction if R' is determined by any collection of subsets of B, at least one of which is non-singleton.

Case-3

If R' is determined by any two non-singleton subsets A_1, A_2 of B such that $A_1 \cup A_2 = B$.

Let $A_1 = \{a,b\}$, $A_2 = \{c,d\}$

$R' = \{a,b\} \times \{a,b\} \cup \{c,d\} \times \{c,d\}$ is a closed congruence and

$$\{a,b\} \times \{a,b\} \cup \{c,d\} \times \{c,d\} \subset \omega \times \omega \cup \triangle$$

Since $a \neq b$, $c \neq d$, $\{a,b,c,d\} \subset \omega$

Since R' is a congruence for all $x \in B$ and for all $a,b \in A_1$ or in A_2

$ax, bx \in A_1$ or $ax, bx \in A_2$

and $xa, xb \in A_1$ or $xa, xb \in A_2$

i.e., $\{A_1, A_2\}$ is a set of complementary joint ideals contained in ω, which is a contradiction.

Similarly we have a contradiction, if R' is determined by any disjoint collection of subsets B, whose union is B, at least one of which is non-singleton.

Thus in all these possible cases, there exists no non-trivial closed congruence properly contained in $(\omega \times \omega) \cup \Delta$.

$\therefore$ (α, A) the semigroup compactification determined by $(\omega \times \omega) \cup \Delta$ is a dual atom of $K_1(S)$.

(b) Let ω be a closed non-singleton proper weak ideal of B maximal in ω^*.

$\therefore$ $\Delta \subset (\omega \times \omega) \cup \Delta \subset B \times B$ is a closed congruence on B and there exists no proper closed congruence properly contains $(\omega \times \omega) \cup \Delta$.

If not, let R' be a closed congruence properly contains $(\omega x \omega) \cup \Delta$

i.e., $\Delta \subsetneq (\omega \times \omega) \cup \Delta \subsetneq R' \subsetneq B \times B$

Since R' is non-trivial, the possible cases of R' are same as that in (a) and we have a contradiction

(1) if R' is determined by any non-singleton subset A of B with $1 < |A| < n$.

(2) if R' is determined by any disjoint collection of subsets of B atleast one of which is non-singleton.

∴ there exists no proper closed congruence properly contains $(\omega \times \omega) \cup \triangle$

∴ (α,A) the semigroup compactification corresponding to $(\omega \times \omega) \cup \triangle$ is an atom of $K_1(S)$.

By similar argument we have the following.

Remark-1

If ω is a set of closed joint weak ideals of B at least one of which is non-singleton minimal (maximal) in ω^*, then (α,A) the semigroup compactification corresponding to $(\omega \times \omega) \cup \triangle$ is a dual atom (atom) of $K_1(S)$.

Remark-2

If ω is a set of closed complementary joint ideals of B at least one of which is non-singleton minimal (maximal) in ω^*, then (α,A) the semigroup compactification corresponding to $(\omega \times \omega) \cup \triangle$ is a dual atom (atom) of $K_1(S)$.

Chapter-4

SEMIGROUP COMPACTIFICATION OF PRODUCTS AND PROJECTIVE LIMITS

Introduction

Let $\{S_\alpha\}_{\alpha \in A}$ be a family of topological semigroups with semigroup compactifications $\{A_\alpha\}_{\alpha \in A}$. We discuss in this chapter about the corresponding semigroup compactification of $P\{S_\alpha\}_{\alpha \in A}$. In 1961, K. Deleeuw and I. Glicksberg [D-G] observed that the product of Bohr compactifications of a collection of abelian topological monoids is the Bohr compactification of their product. They showed by an example that the identity is not necessary. This work was further extended and supplemented in [BE]. Here the distinction between the Bohr compactification and its topological analogue the Stone-Čech compactification βX is more pronounced, since βX does not generally have the product property even for a finite number of factors. A necessary and sufficient condition for the equality $\beta \underset{\alpha \in A}{P} X_\alpha = \underset{\alpha \in A}{P} \beta X_\alpha$ was given by I.Glicksberg in 1959 [GLI].

In this chapter, in Section 4.1, we prove that if $\{S_\alpha\}_{\alpha \in A}$ is any family of topological semigroups with semigroup compactifications $\{A_\alpha\}_{\alpha \in A}$, then $\underset{\alpha \in A}{P} \{A_\alpha\}$ is a semigroup compactification of $\underset{\alpha \in A}{P} \{S_\alpha\}$. Also we consider the family of topological semigroups $\{S_\alpha\}_{\alpha \in A}$ with Bohr compactifications $\{B_\alpha\}_{\alpha \in A}$ and the lattices of semigroup compactifications $\{K_1(S_\alpha)\}_{\alpha \in A}$. Then we show that $\underset{\alpha \in A}{P} \{K_1(S_\alpha)\} \subset K_1(\underset{\alpha \in A}{P} \{S_\alpha\})$ is a complete lattice.

In Section 4.2, we discuss semigroup compactifications, Bohr compactification and lattice of semigroup compactifications of the limit of a projective system of topological semigroups.

4.1 Semigroup Compactification of Products

Theorem 4.1.1.

Let $\{S_\alpha\}_{\alpha \in A}$ be a collection of topological semigroups with semigroup compactification $(\gamma_\alpha, A_\alpha)$ for each $\alpha \in A$. Define $\gamma : P\{S_\alpha\}_{\alpha \in A} \longrightarrow P\{A_\alpha\}_{\alpha \in A}$ by $\gamma(x)_k = \gamma_k P_k(x)$, where $P_k : P\{S_\alpha\}_{\alpha \in A} \longrightarrow S_k$ is projection for each $k \in A$. Then $(\gamma, P\{A_\alpha\}_{\alpha \in A})$ is a semigroup compactification of $P\{S_\alpha\}_{\alpha \in A}$.

Proof

Let $S = P\{S_\alpha\}_{\alpha \in A}$, $A^* = P\{A_\alpha\}_{\alpha \in A}$

Define $\gamma : S \longrightarrow A^*$ by $\gamma(x)_k = \gamma_k P_k(x)$. It is known that A*, being the cartesian product of a family of compact semigroups with co-ordinatewise multiplication and Tychnoff topology, is a compact semigroup [C-H-K_1]. A straight forward argument shows that γ is a continuous homomorphism. Again γ is dense for,

Let $y = (y_k)_{k \in A}$ be any element of A*, where $y_k \in A_k$, $k \in A$. To show that A* has a net in $\gamma(S)$ converging to y.

Since $\overline{\gamma_k(S_k)} = A_k$ and $y_k \in A_k$ for each $k \in A$, A_k has a net in $\gamma_k(S_k)$ converging to y_k, $k \in A$.

i.e., for $i \in I_k$, there exist $(x_k^i) \in S_k$ such that $\gamma_k(x_k^i)_{i \in I_k} \longrightarrow y_k$, for each $k \in A$, where $(I_k, <_k)$ is a directed set for each $k \in A$. Then $(P\{I_k : k \in A\}, \leqslant)$ is a product directed set by defining $i \leqslant j \Longleftrightarrow i_k \leqslant j_k$ (i.e., $i(k) \leqslant j(k)$) for each $k \in A$. Also we have $A \times P\{I_k : k \in A\}$ is a directed set by defining

$$(k,i) \leqslant (k',j) \Longleftrightarrow k \leqslant k' \text{ and } i_\gamma \leqslant j_\gamma \text{ for every } \gamma \in A.$$

and $(\gamma_k (x_k^i)_i)_{k \in A}$ is a net in $\gamma(S)$

such that $(\gamma_k(x_k^i)_i)_k \longrightarrow (y_k)_k = y$

$\therefore$ $\gamma(S)$ contains a net converging to 'y'.

i.e., $\overline{\gamma(S)} = A^*$

$\therefore$ $(\gamma, A^*) = (\gamma, P\{A_\alpha\}_{\alpha \in A})$ is a semigroup compactification of $\underset{\alpha \in A}{P}\{S_\alpha\}$.

Next we consider the quotients of Bohr compactifications and prove the following theorem.

Theorem 4.1.2.

Let $\{S_\alpha\}_{\alpha \in A}$ be a collection of topological monoids with Bohr compactification $(\beta_\alpha, B_\alpha)_{\alpha \in A}$. Then $(\gamma, P\{A_\alpha\}_{\alpha \in A})$ is a semigroup compactification of $P\{S_\alpha\}_{\alpha \in A}$, where $A_\alpha = B_\alpha/R_\alpha$ for each $\alpha \in A$. And $P\{B_\alpha\}_{\alpha \in A}/R$ is topologically isomorphic to $P\{A_\alpha\}_{\alpha \in A}$, where

$$R = \{((a_\alpha)_{\alpha \in A}, (b_\alpha)_{\alpha \in A}) \in (P\{B_\alpha\}_{\alpha \in A} \times P\{B_\alpha\}_{\alpha \in A}) : (a_\alpha, b_\alpha) \in R_\alpha \quad \text{for each } \alpha \in A\}.$$

Moreover, any semigroup compactification of $P\{S_\alpha\}_{\alpha\in A}$ is a quotient space of $P\{B_\alpha\}_{\alpha\in A}$.

Proof.

Let $S = P\{S_\alpha\}_{\alpha\in A}$,

$B = P\{B_\alpha\}_{\alpha\in A}$ and

$A^* = P\{A_\alpha\}_{\alpha\in A}$

Define $\gamma : S \longrightarrow A^*$ by $\gamma(x)_k = \gamma_k P_k(x)$, where P_k is projection and $\gamma_k = \emptyset_k \beta_k$, where $\emptyset_k : B_k \longrightarrow A_k$ and $\beta_k : S_k \longrightarrow B_k$ for each $k \in A$.

Then by theorem (4.1.1), we have (γ, A^*) is a semigroup compactification of S. Using product theorem on Bohr compactification [D-G] (β, B) is the Bohr compactification of S, where $\beta : S \longrightarrow B$ defined by $\beta(x)_k = \beta_k P_k(x)$, where P_k is projection. By the definition of Bohr compactification there exists a continuous homomorphism $h : B \longrightarrow A^*$ such that $h\beta = \gamma$. Moreover, h is a quotient map and h determines a closed congruence

$$R = \{((a_\alpha)_{\alpha\in A}, (b_\alpha)_{\alpha\in A}) \in B \times B : h((a_\alpha)_{\alpha\in A}) = h((b_\alpha)_{\alpha\in A})\}$$

i.e., $\{((a_\alpha)_{\alpha\in A}, (b_\alpha)_{\alpha\in A}) \in B \times B : (h(a_\alpha))_\alpha = (h(b_\alpha))_\alpha$

for each $\alpha \in A\}$

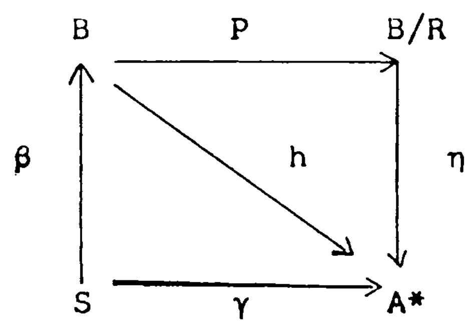
i.e., $\{((a_\alpha)_{\alpha\in A}, (b_\alpha)_{\alpha\in A}) \in B \times B : h_\alpha(a_\alpha) = h_\alpha(b_\alpha)$

for each $\alpha \in A\}$

i.e., $R = \{((a_\alpha)_{\alpha\in A}, (b_\alpha)_{\alpha\in A}) \in B \times B : (a_\alpha, b_\alpha) \in R_\alpha$

for each $\alpha \in A\}$

Define $P : B \longrightarrow B/R$, the natural map, then $(P\beta, B/R)$ determines a semigroup compactification of S [1.2.4]. Then by induced homomorphism theorem [1.1.13] and first isomorphism theorem [1.1.14], there exists a topological isomorphism $\eta : B/R \longrightarrow A^*$ such that the diagram commutes



i.e., $\underset{\alpha\in A}{P}\{B_\alpha\} / R \simeq \underset{\alpha\in A}{P}\{B_\alpha/A_\alpha\}$

Moreover, if (α,C) is any semigroup compactification of S, then it is the quotient space of B follows from [1.2.3].

Theorem 4.1.3.

Let $\{S_\alpha\}_{\alpha \in A}$ be a family of topological semigroups with lattices of semigroup compactifications $\{K_1(S_\alpha)\}$, for each $\alpha \in A$. Then

$$\underset{\alpha \in A}{P}\{K_1(S_\alpha)\} \subset K_1(\underset{\alpha \in A}{P}\{S_\alpha\})$$ is a lattice of semigroup compactifications of $\underset{\alpha \in A}{P}\{S_\alpha\}$.

<u>Proof</u>

Define $\underset{\alpha \in A}{P}\{K_1(S_\alpha)\} = \{(a_\alpha^k)_{\alpha, k \in A} : P_\alpha((a_\alpha^k)) = a_\alpha^k\}$

where P_α, projection to the α^{th} factor and

$$a_\alpha^k \in K_1(S_\alpha) = \{a_\alpha^k\}_{\alpha, k \in A} \text{ for each } \alpha \in A.$$

Since $\{S_\alpha\}_{\alpha \in A}$ is the family of topological semigroups with families of semigroup compactification

$$\{\{a_\alpha^k\}_{\alpha, k \in A}\} = \{K_1(S_\alpha)\}$$

where,

$\{a_\alpha^k\}_{\alpha, k \in A}$ is the family of semigroup compactifications of S_α, for each $\alpha \in A$. Then $(a_\alpha^k)_{\alpha, k \in A}$ is a semigroup compactification of $\underset{\alpha \in A}{P}\{S_\alpha\}$ for each α, $k \in A$, by theorem i.e., $\{(a_\alpha^k)_{\alpha, k \in A}\}_{\alpha \in A}$ is the family of semigroup compactifications of $\underset{\alpha \in A}{P} \{S_\alpha\}$

$$\therefore \quad \underset{\alpha \in A}{P} \{K_1(S_\alpha)\} \subset K_1(\underset{\alpha \in A}{P} \{S_\alpha\})$$

Moreover $\underset{\alpha \in A}{P} \{K_1(S_\alpha)\}$ is a partially ordered set by defining an order $(a_\alpha^s)_{\alpha \in A} \leqslant (a_\alpha^t)_{\alpha \in A} \Longleftrightarrow a_\alpha^s \leqslant a_\alpha^t$,

for each $\alpha \in A$, $s, t \in A$.

Also $(\underset{\alpha \in A}{P} \{K_1(S_\alpha)\}, \leqslant)$ is a complete lattice with join and meet defined by

$$(a_\alpha^s)_{\alpha, s \in A} \wedge (a_\alpha^t)_{\alpha, t \in A} = (a_\alpha^s \wedge a_\alpha^t)_{\alpha \in A}, \; s, t \in A$$

and $$(a_\alpha^s)_{\alpha, s \in A} \vee (a_\alpha^t)_{\alpha, t \in A} = (a_\alpha^s \vee a_\alpha^t)_{\alpha \in A}, \; s, t \in A$$

for each $\alpha \in A$.

Note.

If $\{S_\alpha\}_{\alpha \in A}$ be a collection of topological monoids with Bohr compactifications $\{\beta_\alpha, B_\alpha\}_{\alpha \in A}$ and lattices of semigroup compactifications $\{K_1(S_\alpha)\}_{\alpha \in A}$, determined by quotients of B_α for each $\alpha \in A$ and

$$(\underset{\alpha \in A}{P} B_\alpha) / R \simeq \underset{\alpha \in A}{P} (B_\alpha / R_\alpha)$$

Then $\underset{\alpha \in A}{P} \{K_1(S_\alpha)\} = K_1(\underset{\alpha \in A}{P} \{S_\alpha\})$ is a complete lattice.

4.2 Semigroup compactification of Projective Limits

In this section, we consider the projective system of semigroup compactifications of a topological semigroup S and show that projective limit itself is a semigroup compactification of S.

Theorem 4.2.1.

Let $\{(\eta_\alpha, A_\alpha) : \phi_\alpha^\beta\}_{\alpha \leqslant \beta \in D}$ be a projective system of semigroup compactifications of a topological semigroup S, where $\eta_\alpha = \phi_\alpha^\beta \eta_\beta$ for every pair $\alpha \leqslant \beta$ in a directed set D. Then $\varprojlim (\eta_\alpha, A_\alpha)$, itself is a semigroup compactification of S.

Proof.

By the definition of semigroup compactification. $\eta_\alpha : S \longrightarrow A_\alpha$ is a dense continuous homomorphism for each $\alpha \in D$ and A_α is a compact semigroup. Moreover, each bonding map ϕ_α^β is surjective, for, when $\alpha \leqslant \beta$

$$A_\alpha = \overline{\eta_\alpha(S)} = \overline{\phi_\alpha^\beta \, \eta_\beta(S)}$$

$$= \phi_\alpha^\beta \, (\overline{\eta_\beta(S)})$$ ($\because$ ϕ_α^β is continuous and closed, being a continuous map from compact semigroup to Hausdorff space)

$$= \phi_\alpha^\beta(A_\beta)$$

Hence the system $\{ A_\alpha, \phi_\alpha^\beta \}_{\alpha \leqslant \beta}$ is a strict projective system of compact semigroups. Then $A^* = \varprojlim \{ A_\alpha, \phi_\alpha^\beta \}$ exists and is a compact semigroup [C-H-K_1].

So it is enough to show that there exists a dense continuous homomorphism from S into A*.

Define,

$$\eta : S \longrightarrow P\{A_\alpha\}_{\alpha \in A} \text{ by } \eta(x)(\alpha) = \eta_\alpha(x)$$

for each $\alpha \in D$, $x \in S$.

then η actually maps S into A*.

for,

if $x \in S$ and $\alpha \leqslant \beta \in D$

$$\eta(x)(\alpha) = \eta_\alpha(x) = \phi_\alpha^\beta \, \eta_\beta(x)$$

$$= \phi_\alpha^\beta(\eta(x)(\beta))$$

Then $\eta(x) \in A^*$.

Since each η_α is a continuous homomorphism, so is η.

Claim. $\overline{\eta(S)} = A^*$

For this, we show that each non-empty basic open set in A^* contains points of $\eta(S)$. Since the system is strict projective, the restricted map $\rho_\alpha = P_\alpha|_{A^*} : A^* \longrightarrow A_\alpha$ is surjective for each $\alpha \in D$.

Given $\overline{\eta_\alpha(S)} = A_\alpha$ for every $\alpha \in D$

Let U be an open set in A_α containing points of $\eta_\alpha(S)$.

i.e., $\rho_\alpha^{-1}(U)$ contains points of $\rho_\alpha^{-1}\ \eta_\alpha(S)$

i.e., $\rho_\alpha^{-1}(U)$ contains points of $\eta(S)$ for every $\alpha \in D$.

$\therefore$ each non-empty basic open set in A^* contains points of $\eta(S)$, since $\{\rho_\alpha^{-1}(U) \ / \text{ all } \alpha, \text{ all open } U \subset A_\alpha\}$ forms a basis for A^* [EN]

i.e., $\overline{\eta(S)} = A^*$

$\therefore$ (η, A^*) is a semigroup compactification of S.

Theorem 4.2.2.

Let $\{S_\alpha, \phi_\alpha^\beta\}_{\alpha \leqslant \beta \in D}$ be a projective system of topological semigroups with projective system of semigroup compactification $\{(\eta_\alpha, A_\alpha),\ e_\alpha^\beta\}_{\alpha \leqslant \beta \in D}$, where

$e_\alpha^\beta\ \eta_\beta = \eta_\alpha\ \phi_\alpha^\beta$ for every pair $\alpha \leqslant \beta \in D$ such that

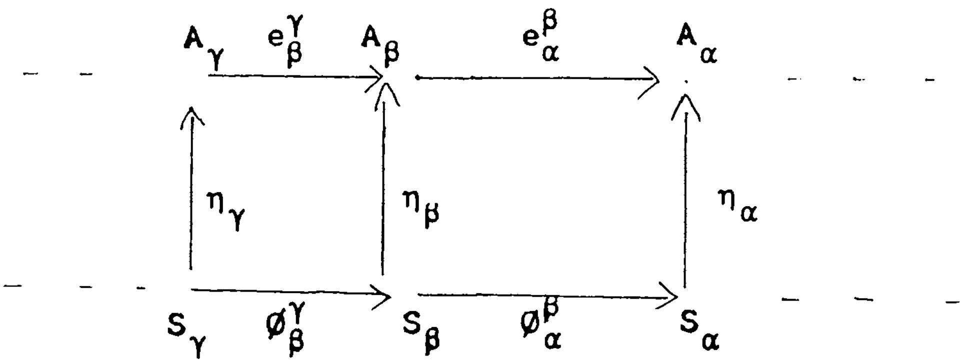
$S^* = \varprojlim S_\alpha$ exists and $\lambda_\alpha : P_\alpha|_{S^*} : S^* \longrightarrow S_\alpha$ is surjective for each $\alpha \in D$, where P_α is projection.

Then $\varprojlim (\eta_\alpha, A_\alpha) = A^*$ is a semigroup compactification of S^*.

Proof.

Since $\{(\eta_\alpha, A_\alpha),\ e_\alpha^\beta\}_{\alpha \leqslant \beta \in D}$ is a projective system of compact semigroups,

$A^* = \varprojlim A_\alpha$ exists and is a compact semigroup [C-H-K_1].



Define $\eta : P\{S_\alpha\}_{\alpha \in A} \longrightarrow P\{A_\alpha\}_{\alpha \in A}$ by $(\eta(x))(\alpha) = \eta_\alpha(x(\alpha))$.

Then η actually maps S^* into A^*.

i.e., if x $P\{S_\alpha\}_{\alpha \in A}$ is in S^*, then $\eta(x) \in A^*$.

For,

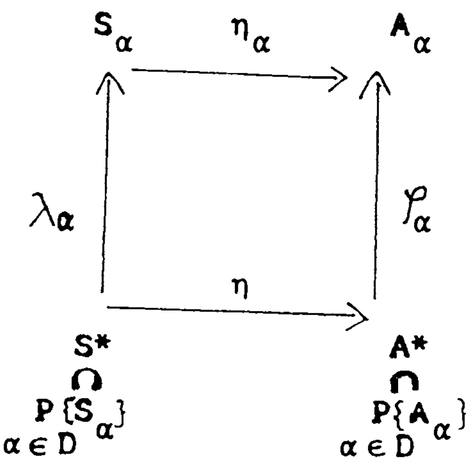
since $x \in S^*$, when $\alpha \leqslant \beta$, $x(\alpha) = \phi_\alpha^\beta(x(\beta))$

$$(\eta(x))(\alpha) = \eta_\alpha(x(\alpha)) = \eta_\alpha\, \phi_\alpha^\beta(x(\beta))$$

$$= e_\alpha^\beta\, \eta_\beta(x(\beta))$$

$$= e_\alpha^\beta(\eta(x)(\beta))$$

$\therefore \quad \eta(x) \in A^*$



Again $\eta : S^* \dashrightarrow A^*$ is a continuous homomorphism such that the diagram is commutative.

for,

$$\varphi_\alpha\, \eta(x) = (\eta(x))(\alpha)$$

$$= \eta_\alpha(x(\alpha))$$

$$= \eta_\alpha(\lambda_\alpha(x))$$

$$= \eta_\alpha\, \lambda_\alpha(x) \quad (\because \lambda_\alpha \text{ is surjective})$$

This is true for all $x \in S^*$.

$\therefore \quad \varphi_\alpha \eta = \eta_\alpha\, \lambda_\alpha .$

and $\quad \varphi_\alpha = P_\alpha|_{A^*} : A^* \longrightarrow A_\alpha$ is surjective

for,

$$A_\alpha = \overline{\eta_\alpha(S_\alpha)} \quad (\because A_\alpha \text{ is the semigroup compactification of } S_\alpha)$$

$$= \overline{\eta_\alpha \lambda_\alpha(S^*)} (\because \lambda_\alpha \text{ is surjective})$$

$$= \overline{\rho_\alpha \eta(S^*)} \quad (\because \rho_\alpha \eta = \eta_\alpha \lambda_\alpha)$$

$$= \rho_\alpha \overline{\eta(S^*)} \quad (\because \rho_\alpha \text{ is a closed map})$$

$$\subset \rho_\alpha(A^*) \quad (\because \rho_\alpha : A^* \longrightarrow A_\alpha \text{ and}$$

$$\eta(S^*) \subset A^*$$

i.e. $\overline{\eta(S^*)} \subset A^*$

$$\rho_\alpha \overline{\eta(S^*)} \subset \rho_\alpha(A^*))$$

also

$$\rho_\alpha(A^*) \subset A_\alpha$$

$\therefore \quad \rho_\alpha(A^*) = A_\alpha$ for each $\alpha \in D$

Claim.

$$\overline{\eta(S^*)} = A^*$$

Since $\overline{\eta_\alpha \lambda_\alpha(S^*)} = \overline{\eta_\alpha(S_\alpha)} = A_\alpha$ for each $\alpha \in D$, each non-empty basic open set in A_α contains points of $\eta_\alpha \lambda_\alpha(S^*)$

Let U be an open set in A_α

U contains points of $\eta_\alpha \lambda_\alpha(S^*)$

i.e., $\varphi_\alpha^{-1}(U)$ contains points of $\varphi_\alpha^{-1} \eta_\alpha \lambda_\alpha(S^*)$

i.e., each non-empty basic open set in A* contains points of $\eta(S^*)$

(since $\varphi_\alpha^{-1}(U)$/ for all α, all open $U \subset B_\alpha$ forms a basis for A*)

$\therefore \quad \overline{\eta(S^*)} = A^*$

$\therefore$ (η, A^*) is a semigroup compactification of S*.

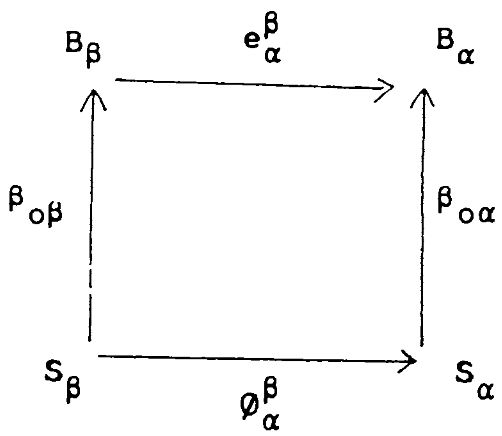
Specialise to Bohr compactification, we have the following theorem.

Theorem 4.2.3.

Let $\{S_\alpha, \phi_\alpha^\beta\}_{\alpha \leq \beta \in D}$ be a projective system of topological semigroups with Bohr compactifications $\{(\beta_{o\alpha}, B_\alpha)\}_{\alpha \in D}$ such that $S^* = \varprojlim S_\alpha$ exists and $\lambda_\alpha = P_\alpha|_{S^*} : S^* \longrightarrow S_\alpha$ is surjective for each $\alpha \in D$, where P_α is projection. Then $\varprojlim \{B_\alpha\}$ is a Bohr compactification of $\varprojlim \{S_\alpha\} = S^*$

Proof.

Since $(\beta_{p\alpha}, \beta_\alpha)$ is a Bohr compactification of S_α and $\beta_{o\alpha} \phi_\alpha^\beta : S_\beta \longrightarrow B_\alpha$ is a continuous homomorphism for each $\alpha \leqslant \beta \in D$, there exists a unique continuous homomorphism $e_\alpha^\beta : B_\beta \longrightarrow B_\alpha$ such that the diagram commutes [1.1.23].



i.e., $e_\alpha^\beta \beta_{o\beta} = \beta_{o\alpha} \phi_\alpha^\beta$ for all $\alpha \leqslant \beta \in D$ and satisfies

(i) $e_\alpha^\alpha = 1_{B\alpha}$, identity function on $B\alpha$

(ii) $e_\alpha^\beta \circ e_\beta^\gamma = e_\alpha^\gamma$ for all $\alpha \leqslant \beta \leqslant \gamma$

Thus we have $\{B_\alpha, e_\alpha^\beta\}$ as a projective system of compact semigroups. Then $B^* = \varprojlim B_\alpha$ exist is a compact semigroup.

Define $\beta : \underset{\alpha \in D}{P} \{S_\alpha\} \longrightarrow \underset{\alpha \in D}{P} \{B_\alpha\}$ by

$$\beta(x)(\alpha) = \beta_{o\alpha}(x(\alpha))$$

Then β actually maps S^* into B^* is a dense continuous homomorphism.

$\therefore$ $\beta : S^* \longrightarrow B^*$ is a dense continuous homomorphism. (Proof is same as that in theorem 4.2.2).

To complete the proof if $g: S^* \longrightarrow T$ is a continuous homomorphism of S^* into a compact semigroup T. We need to exhibit a continuous homomorphism $f : B^* \longrightarrow T$ such that the diagram commutes.

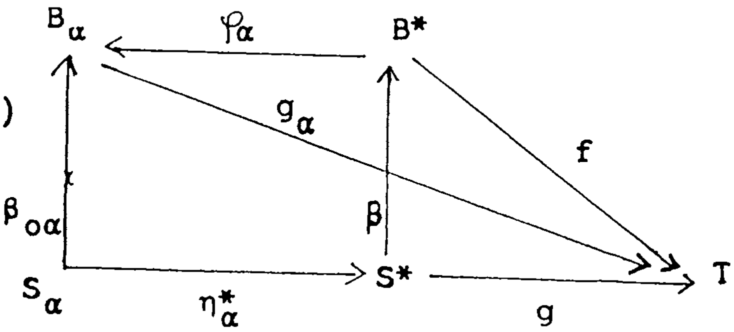
Define

$$\eta^*_\alpha : S_\alpha \longrightarrow S^* \text{ so that}$$

$\lambda_\alpha \circ \eta^*_\alpha(x(\alpha)) = x(\alpha)$

$\lambda_k \circ \eta^*_\alpha(x(\alpha)) = \emptyset^\alpha_k x(\alpha)$

for all $k \leq \alpha \in D$.



Then $\eta^*_\alpha : S_\alpha \longrightarrow S^* \subset P_{\alpha \in D}\{S_\alpha\}$ is a continuous homomorphism, since it is composite with P_α is a continuous homomorphism.

Then $g \circ \eta^*_\alpha : S_\alpha \longrightarrow T$ is a continuous homomorphism and since $(\beta_{o\alpha}, B_\alpha)$ is a Bohr compactification, there exists a unique continuous homomorphism $g_\alpha : B_\alpha \longrightarrow T$ such that the diagram commutes for each $\alpha \in D$.

i.e., $g_\alpha \beta_{o\alpha} = g \eta^*_\alpha$ for each $\alpha \in D$.

Then define $f: B^* \longrightarrow T$ by $f = g_\alpha \varphi_\alpha$, for each $\alpha \in D$ and is a continuous homomorphism such that $f\beta = g$

for,

$$f\beta(x) = g_\alpha \, \rho_\alpha \beta(x)$$
$$= g_\alpha \, \beta_{o\alpha} \, \lambda_\alpha(x)$$
$$= g_\alpha \beta_{o\alpha} x(\alpha)$$
$$= g \, \eta^*_\alpha x(\alpha) = g(x) \text{ for all } x \in S^*.$$

Also f is unique, since $\beta(S^*)$ is dense in B^* and $f\beta = g$.

$\therefore$ (β, B^*) is a Bohr compactification of S^*.

Theorem 4.2.4.

Let $\{(K_1(S_\lambda), \geq), \phi_\lambda^k\}_{\lambda \leq k \in D}$ be a projective system of lattices of semigroup compactifications of $\{S_\lambda\}_{\lambda \in D}$ with ϕ_λ^k's as lattice isomorphism. Then $K_1(S^*) = \varprojlim \{K_1(S_\lambda)\}$ is a complete sub-lattice of $\underset{\lambda \in D}{P} \{K_1(S_\lambda)\}$

$$K_1(S^*) = \{(A_\lambda^\gamma) \in \underset{\lambda \in D}{P} K_1(S_\lambda) : P_\lambda(A_\lambda^\gamma) = \phi_\lambda^k(P_k(A_\lambda^\gamma)) \text{ for all } \lambda \leq k \in D\}$$

and $K_1(S^*) \neq \emptyset$. Since $\{\phi_\lambda^k\}_{\lambda \leq k \in D}$ is an isomorphism there exist $(A_\lambda^\gamma) \in \underset{\lambda \in D}{P} \{K_1(S_\lambda)\}$ such that

$$A_\lambda^\gamma = \phi_\lambda^k(A_k^\gamma), \lambda \leq k \in D.$$

$\therefore$ $K_1(S^*)$ is a subset of $\underset{\lambda \in D}{P} \{K_1(S_\lambda)\}$

Again $K_1(S^*)$ is a partially ordered set by defining an order

$$(A_\lambda^s)_{\lambda, s \in D} \leqslant (A_\lambda^t)_{\lambda, t \in D} \Longleftrightarrow A_\lambda^s \leqslant A_\lambda^t ,$$

$$\text{where } A_\lambda^\gamma = P_\lambda((A_\lambda^\gamma))$$

for each $\lambda \in D$, $\gamma \leqslant t \in D$.

If (A_λ^s), $(A_\lambda^t) \in K_1(S^*)$, then both $(A_\lambda^s) \wedge (A_\lambda^t)$, $(A_\lambda^s) \vee (A_\lambda^t) \in K_1(S^*)$;

for, since (A_λ^s), $(A_\lambda^t) \in K_1(S^*)$

When $\lambda \leqslant k \in D$, $A_\lambda^s = \phi_\lambda^k (A_k^s)_{s \in D}$

$$A_\lambda^t = \phi_\lambda^k (A_k^t)_{t \in D}$$

Since $\underset{\lambda \in D}{P} \{K_1(S_\lambda)\}$ is a complete lattice

$$(A_\lambda^s) \wedge (A_\lambda^t) \text{ and } (A_\lambda^s) \vee (A_\lambda^t) \in \underset{\lambda \in D}{P} \{K_1(S_\lambda)\}$$

and when $\lambda \leqslant k$

we have

$$\phi_\lambda^k (P_k(A_\lambda^s) \vee (A_\lambda^t))$$

$$= \phi_\lambda^k P_k(A_\lambda^s) \vee \phi_\lambda^k P_k(A_\lambda^t)$$

$$= P_\lambda (A_\lambda^s) \vee P_\lambda (A_\lambda^t)$$

$$= P_\lambda ((A_\lambda^s) \vee (A_\lambda^t)) \quad \text{for each } \lambda \in D \text{ and } s,t \in D.$$

Then $(A_\lambda^s) \vee (A_\lambda^t) \in K_1(S^*)$ for every $\lambda \in D$.

Similarly,

$$(A_\lambda^s)_{\lambda, s \in D} \wedge (A_\lambda^t)_{\lambda, t \in D} \in K_1(S^*)$$

$\therefore$ $K_1(S^*)$ is a sublattice of $\underset{\lambda \in D}{P} \{K_1(S_\lambda)\}$

Similarly we can prove that $\vee$ and $\wedge$ exist in $K_1(S^*)$ for every non-empty subset of $K_1(S^*)$.

$\therefore$ $K_1(S^*)$ is a complete sublattice of $\underset{\lambda \in D}{P} \{K_1(S_\lambda)\}$

Chapter-5

ON THE CATEGORY TS OF ALL TOPOLOGICAL SEMIGROUPS

Introduction

J.H. Carruth, J.A. Hildebrant and R.J. Koch [C-H-K_2] interpret several categorical concepts in various categories of topological semigroups like category of compact semilattices and category of compact Lawson semilattices. In 1973, Crawley [CR] made an extensive study in this direction.

In topology, in the category of Hausdorff spaces, the epimorphisms are the mappings with dense range [W]. But in the category of topological semigroups, every epimorphism need not be of this form [C-H-K_2]. In 1973, Herrlich and Strecker [H-S] showed that group epimorphisms are surjective. In 1975, Hofmann and Mislove [HO-M] established that discrete-semilattice epimorphisms are surjective. In 1966, Husain [HUS] proved that in the category of Topological abelian groups (locally compact abelian groups) each epimorphism is dense. The compact abelian group epimorphisms are surjective follows from the result of Section 5 of chapter 1 [C-H-K_1] and showed that

Abelian group, Topological abelian group, Locally compact abelian group epimorphisms are dense. In 1966, Hofmann and Mostert [HO-M_1] gave an example to show that compact semigroup epimorphisms are not necessarily surjective. In 1975, Lamatrin [L] showed that epimorphisms in the category of Hausdorff [abelian] K-groups need not be dense. However, question remains unanswered in various other categories of topological semigroups.

In this chapter, in Section 5.2, we discuss epimorphisms in the category of all topological semi-groups. In Section 5.3, we define weak extremal monomorphism and prove that if the images are ideals the weak extremal monomorphisms in the category of all topological semigroups are the closed embeddings.

5.1 Preliminaries

In the theory of (topological) semigroups morphisms are simply (continuous) homomorphisms except that in the monoid categories morphisms are required to be identity-preserving. The rule of composition in each category is ordinary composition of functions. Isomorphisms are precisely the topological isomorphisms [C -H-K_2].

Definition 5.1.1.

A morphism e: A ——> B is an epimorphism if for every pair of morphisms the equality foe = goe implies that f = g.

$$A \xrightarrow{e} B \underset{g}{\overset{f}{\rightrightarrows}} C \qquad [W]$$

Definition 5.1.2.

A morphism f is a monomorphism if for every pair of morphisms the equality

fog = foh implies that f = g

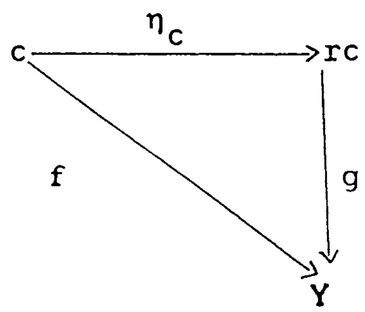
$$A \xleftarrow{f} B \underset{h}{\overset{g}{\leftleftarrows}} C \qquad [W]$$

Note:

Monomorphisms in the category of all topological semigroups are precisely injective homomorphisms [C-H-K_2].

Definition 5.1.3.

A functor r from a category C* to a subcategory R* of C* is a <u>reflective functor</u> if there is a unique morphism η_c : c ——> rc and if every morphism from c to any object Y of R* factors uniquely through rc via η_c so that the following diagram commutes [W].



If r: C* ——> R* is a reflective functor, the subcategory R* is called a reflective subcategory. The object rc is called the reflection of c in R*. [W]

Definition 5.1.4

A reflective functor r is said to be <u>epi-reflective</u> if the morphism

η_X : X ——> rX is an epimorphism. [W]

5.2 <u>Epimorphisms in the category TS</u>

The epimorphisms in the category of Hausdorff spaces are the mappings with dense range. [W]

In the case of topological semigroups also, continuous homomorphisms with dense range are epimorphisms. But the converse need not be true. For example, let S be a semigroup of non-negative integers under addition with discrete topology and let $\emptyset : S \longrightarrow Z$ be the inclusion homomorphism. Then $\emptyset$ is not dense in Z but $\emptyset$ is an epimorphism. [C-H-K_2].

In this situation we study when will the converse hold. As a result, we have the following propositions.

Proposition 5.2.1.

Let $f:X \longrightarrow Y$ be a continuous homomorphism such that $\langle f(X) \rangle$ is dense in Y. If g,h agree on $\langle f(X) \rangle$, then g=h (where $\langle f(X) \rangle = f(X) \cup Yf(X) \cup f(X)Y \cup Yf(X)Y$, the ideal generated by f(X)).

Proof.

Let $f:X \longrightarrow Y$ be a continuous homomorphism such that $\overline{\langle f(X) \rangle} = Y$ and g,h agree on $\langle f(X) \rangle$.

Claim.

f is an epimorphism.

For this show that $g(x)=h(x)$ for all $x \in Y$

$$X \xrightarrow{f} Y \underset{h}{\overset{g}{\rightrightarrows}} Z$$

If not, assume that $g(x) \neq h(x)$ for at least one $x \in Y \setminus \langle f(X) \rangle$. Since Z is a Hausdorff space, there exists disjoint open sets say U and U' containing g(x) and h(x) respectively, $U \cap U' = \emptyset$. Choose a neighbourhood V of x such that $g(V) \subset U$, $h(V) \subset U'$. This is possible, since g,h are continuous. Since $x \in Y = \overline{\langle f(X) \rangle}$, V intersects $\langle f(X) \rangle$ in some point say 'y' other than x.

Then $g(y) \in U$, $h(y) \in U'$

but $g(y) = h(y)$ ($\because y \in \langle f(X) \rangle$)

i.e., $U \cap U' \neq \emptyset$

This contradicts the fact that U and U' are disjoint.

$\therefore$ $g(x) = h(x)$ for all $x \in Y$

$\therefore$ f is an epimorphism.

Proposition 5.2.2 (converse of 5.2.1)

Suppose $f:X \longrightarrow Y$ be an epimorphism then $\langle f(X) \rangle$ is dense in Y.

Proof

For this assume that $\overline{\langle f(X) \rangle} \neq Y$, then we will show that f cannot be an epimorphism by constructing a topological semigroup Z and two continuous homomorphisms L_1 and L_2 from Y into Z which agree on f(X), but which are not equal.

for,

let $Y_1 = Y \times \{1\}$, $Y_2 = Y \times \{2\}$

Y_1 and Y_2 are topological semigroups with product topology and multiplication defined by $(x,i)\ (y,i) = (xy,i)$, for each $i=1,2$. Let $h_i: Y \longrightarrow Y_i$ be defined by $h_i(y) = (y,i)$, for each $i = 1,2$. Then h_i, $i = 1,2$, are topological isomorphisms.

$$Y_1 \cup Y_2 = \{(x,i) : (x,i) \in Y_1 \text{ or } Y_2 \text{ for each } i = 1,2\}$$

The disjoint topological sum $Y_1 \cup Y_2$ is a topological semigroup with multiplication defined by

$$(x,i)\ (y,j) = (xy, \min\{i,j\})$$

Multiplication is well-defined.

For,

$$\text{if } (x,i) = (x',j) \text{ then } x = x',\ i = j$$

and if

$$(y,j) = (y',i) \text{ then } y = y',\ i = j$$

$$\therefore \quad xy = x'y',\ \ i = j$$

$$\begin{aligned} \text{i.e.,}\ (x,i)(y,j) &= (xy, \min\{i,j\}) \\ &= (x'y', \min\{i,j\}) \\ &= (x',j)\ (y',i) \end{aligned}$$

Clearly multiplication is associative and continuous.

Let $i_1 : Y_1 \longrightarrow Y_1 \cup Y_2$

$i_2 : Y_2 \longrightarrow Y_1 \cup Y_2$

be the inclusion maps.

$i_1 o h_1 < \overline{f(X)} > \cup\, i_2 o h_2 < \overline{f(X)} >$ is the set of copies of $< \overline{f(X)} >$ contained in $Y_1 \cup Y_2$. Let Z be the image of the quotient map 'q' obtained byidentifying $i_1 o h_1(y) = i_1(y,1)$ and $i_2 o h_2(y) = i_2(y,2)$ if $y \in < \overline{f(X)} >$

Define $q(x,i)\, q(y,j) = q(xy, \min\{i,j\})$.

This multiplication is well defined. For,

if $q(x,i) = q(x',j)$ then either $x = x'$ and $i = j$ or $i \neq j$ and $x = x' \in < \overline{f(X)} >$ and if $q(y,j) = q(y',i)$ then either $y = y'$ and $i = j$ or $i \neq j$ and $y = y' \in < \overline{f(X)} >$.

Then there are four cases.

1. $i = j$ and $x = x'$

   $i = j$ and $y = y'$

i.e., $xy = x'y'$, $i = j$

i.e., $q(xy,i) = q(x'y',j) = q(xy,j)$

$= q(x'y',i)$

$\therefore$ $q(xy, \min\{i,j\}) = q(x'y', \min\{i,j\})$

i.e., $q(x,i)\, q(y,j) = q(x',j)\, q(y',i)$

2. $i \neq j$ and $x = x' \in \langle \overline{f(X)} \rangle$
   $i \neq j$ and $y = y' \in \langle \overline{f(X)} \rangle$

Then $xy = x'y' \in \langle \overline{f(X)} \rangle$ ($\because$ $\langle \overline{f(X)} \rangle$ is a subsemigroup)

i.e., $q(xy,i) = q(x'y',j)$ ($\because$ $xy = x'y' \in \langle \overline{f(X)} \rangle$)

$= q(xy,j) = q(x'y',i)$

$\therefore$ $q(xy, \min\{i,j\}) = q(x'y', \min\{i,j\})$

3. $i = j$ and $x = x'$
   $i \neq j$ and $y = y' \in \langle \overline{f(X)} \rangle$

Then $xy = x'y' \in \langle \overline{f(X)} \rangle$ ($\because \langle \overline{f(X)} \rangle$ is an ideal)

i.e., $q(xy,i) = q(x'y',i) = q(x'y',j) = q(xy,j)$

i.e., $q(xy,\min\{i,j\}) = q(x'y', \min\{i,j\})$

$\therefore$ $q(x,i)\, q(y,j) = q(x',j)\, q(y',i)$

4. $i \neq j$ and $x = x' \in \langle \overline{f(X)} \rangle$
   $i = j$ and $y = y'$

Then $xy = x'y' \in \langle \overline{f(X)} \rangle$

and similarly we have

$$q(x,i)\, q(y,j) \;=\; q(x',j)\, q(y',i)$$

Clearly multiplication is associative. Thus Z is a semigroup with multiplication continuous and q is a homomorphism.

$\therefore$ $L_1 = qoi_1oh_1$ and $L_2 = qoi_2oh_2$

are continuous homomorphisms from Y into Z.

Now if x is a point of X, then the maps i_1oh_1 and i_2oh_2 split the point f(x) into two and is joined again by

$$q : Y_1 \cup Y_2 \longrightarrow Z$$

Thus we see that

$$((qoi_1oh_1)of)(x) = ((qoi_2oh_2)of)(x)$$

Hence

$$(qoi_1oh_1)of = (qoi_2oh_2)\, of$$

However, any point lying outside of $< \overline{f(X)} >$ in Y is split by i_1oh_1 and i_2oh_2, but is not joined again by q.

Hence

$$qoi_1oh_1 \neq qoi_2oh_2$$

This would show that f cannot be an epimorphism if we show that Z is a topological semigroup. So it remains to show that the quotient space Z is Hausdorff.

Let p and r be two distinct points of Z. We have to find two disjoint open sets containing p and r respectively. Then we have six cases.

Case-1.

$$p, r \in qoi_1oh_1(Y \setminus \langle \overline{f(X)} \rangle\ ) .$$

Since $\langle\ \overline{f(X)}\ \rangle$ is closed, $Y \setminus \langle \overline{f(X)} \rangle$ is open

$$(qoi_1oh_1)^{-1}(p),\ (qoi_1oh_1)^{-1}(r) \in Y \setminus\ \langle \overline{f(X)} \rangle,$$

there exists open sets U_p and U_r in $Y \setminus\ \langle\ \overline{f(X)}\ \rangle$ such that

$$(qoi_1oh_1)^{-1}(p) \in U_p \subset Y \setminus \langle \overline{f(X)} \rangle$$

$$(qoi_1oh_1)^{-1}(r) \in U_r \subset Y \setminus \langle \overline{f(X)} \rangle$$

Again since Y is Hausdorff we get disjoint neighbourhoods V_p and V_r of $(qoi_1oh_1)^{-1}(p)$ and $(qoi_1oh_1)^{-1}(r)$ respectively. Thus the required neighbourhoods are

$$qoi_1oh_1(U_p) \cap qoi_1oh_1(V_p) \quad \text{and}$$

$$qoi_1oh_1(U_r) \cap qoi_1oh_1(V_r)$$

<u>Case-2</u>

$p, r \in qoi_2oh_2(Y \setminus < \overline{f(X)} >)$. This is the same as Case-1 with suffix changed.

<u>Case-3</u>

$p \in qoi_1oh_1\ (Y \setminus < \overline{f(X)} >)$ and

$r \in qoi_2oh_2\ (Y \setminus < \overline{f(X)} >)$

Here the two given sets $qoi_1oh_1\ (Y \setminus < \overline{f(X)} >)$, $qoi_2oh_2\ (Y \setminus < \overline{f(X)} >)$ containing the points are already disjoint.

<u>Case-4</u>

$p \in qoi_1oh_1(Y \setminus < \overline{f(X)} >)$ and

$r = qoi_1oh_1(y)$ for some $y \in < \overline{f(X)} >$

Since $Y \setminus < \overline{f(X)} >$ is open, there exists open set U such that $(qoi_1oh_1)^{-1}(p) \subset U \subset Y \setminus < \overline{f(X)} >$.
Since Y is Hausdorff there exists disjoint open sets U and V with $y \in V$,

$p \in qoi_1oh_1(U)$ and

$r \in q[i_1oh_1(V) \cup i_2oh_2(V)]$

are disjoint and open.

Case-5

If $p \in qoi_2oh_2\,(Y \setminus \langle \overline{f(X)} \rangle)$ and

$r = qoi_1oh_1(y)$ for some $y \in \langle \overline{f(X)} \rangle$

same as that of case-4 with suffix changed.

Case-6

$p = qoi_1oh_1(x)$ and $r = qoi_1oh_1(y)$

where, $x \neq y \in \langle \overline{f(X)} \rangle$. Since Y is Hausdorff there exists disjoint neighbourhoods U and V such that $x \in U$, $y \in V$. Then disjoint neighbourhoods of p and r in Z is given by

$$q\,[i_1oh_1(U) \cup i_2oh_2(U)]$$

and

$$q\,[i_1oh_1(V) \cup i_2oh_2(V)]$$

Hence Z is Hausdorff

$\therefore$ Z is a topological semigroup.

Notation.

TS- denotes the category of all topological semigroups.

From propositions (5.2.1) and (5.2.2) we obtain the following proposition as a particular case.

Proposition 5.2.3

If the images are ideals, the epimorphisms in the category TS are morphisms with dense range.

Proof.

Let $f:X \longrightarrow Y$ be a continuous homomorphism with $f(X)$ an ideal and $\overline{f(X)} = Y$, then f is an epimorphism (proof is same as that of (5.2.1), since $f(X) = \langle f(X) \rangle$ an ideal).

Conversely, let $f : X \longrightarrow Y$ be an epimorphism with $f(X)$ an ideal, then $\overline{f(X)} = Y$ (proof is same as that of (5.2.2), since $f(X) = \langle f(X) \rangle$ an ideal).

Note.

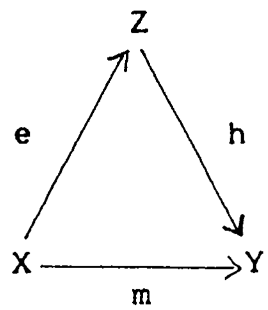
Proofs of proposition (5.2.1) and (5.2.2) are on the same lines as those of the corresponding results in the category of all Hausdorff spaces given by R.C. Walker [W].

5.3 Weak extremal monomorphisms in the Category TS

When a mapping is factored through the closure of its image, the second factor is a closed embedding. These maps also have a categorical characterization in the category of all Hausdorff spaces [W].

Definition 5.3.1

A monomorphism m is an extremal monomorphism if whenever m can be factored as illustrated



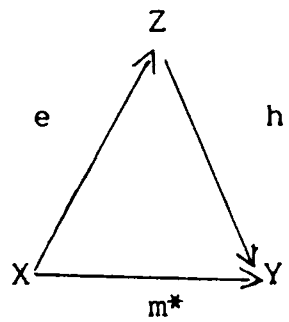
so that e is an epimorphism, then e is an isomorphism. In the diagram, the object X is said to be an extremal subobject of Y. [W]

It is known that the extremal monomorphisms in the category of Hausdorff spaces are the closed embeddings and thus, the extremal subobjects are the closed subspaces. [W]

Next we define weak extremal monomorphism in the category TS of all topological semigroups.

Definition 5.3.2.

A monomorphism m* is a <u>weak extremal monomorphism</u> if whenever m* can be factored as illustrated so that e(X)is an ideal and e is an epimorphism, then e is a topological isomorphism.



The object X is said to be a weak extremal subobject of Y. We will show that if the images are ideals, the weak extremal monomorphisms in the category TS are the closed embeddings.

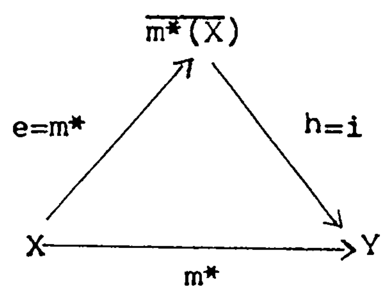
Proposition 5.3.3.

If the images are ideals, the weak extremal monomorphisms in the category TS are the closed embeddings.

Proof

We first show that a weak extremal monomorphism $m^* : X \longrightarrow Y$ with $m^*[X]$ is an ideal is a closed embedding.

We can factor $m^*: X \longrightarrow Y$ through the closure of its image. Since e(X) is an ideal, and the range of e is dense, e is an epimorphism (5.2.1).



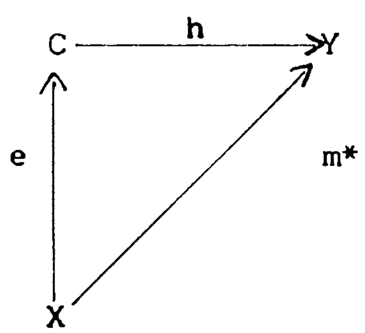
Then e must be a topological isomorphism, since m*
is a weak extremal monomorphism.

i.e., $X \xrightarrow[m^*]{} \overline{m^*(X)}$ is a topological isomorphism,

where $\overline{m^*(X)}$ is a closed ideal of Y.

$\therefore$ m* is a closed embedding.

On the other hand, let m* : X $\longrightarrow$ Y be a closed embedding. Assume that m* = hoe is a factorization of m*, where e is an epimorphism and e(X) an ideal .

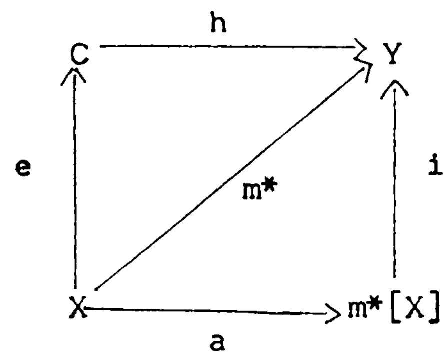


Claim: 1) m* : X $\longrightarrow$ Y is a monomorphism

2) e is a topological isomorphism

Clearly m* is a monomorphism, because it is one-one homomorphism. Thus it remains to show that 'e' is a topological isomorphism.

We can also factor m* through its image thus obtaining the diagram.



We will show that the epimorphism e is a topological isomorphism by obtaining a left inverse for e. Since e(X) an ideal, and e is an epimorphism e has dense range. i.e., $\overline{e(X)} = C$ [5.2.2] and m*[X] is closed in Y (given).

$$h[C] = h[\ \overline{e(X)}\ ] \subset \overline{h[e(X)]} = \overline{m^*[X]} = m^*[X]$$

$\therefore$ h(C) is contained in m*[X].

Thus if we define h' = C ——> m*[X] by h'(x) = h(x), we have that

$$h = ioh'$$

But then we also have

$$ioh'oe = hoe = ioa,$$

where i is a monomorphism (since i is a one-one homomorphism)

$\therefore$ h'oe = a

Since a is a topological isomorphism we have

$$1_X = (a^{-1} o h') o e$$

Thus e is an epimorphism with a left inverse and is therefore a topological isomorphism.

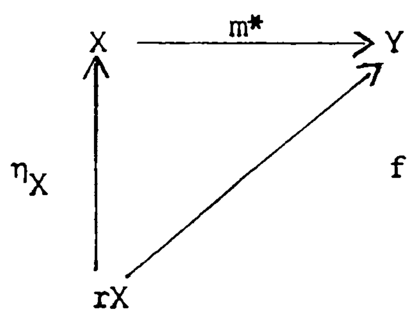
Proposition 5.3.4

If the images are ideals, epireflective subcategories are closed under weak extremal subobjects.

Proof.

Let R* be an epi-reflective subcategory of C*. Let Y belong to R* and let m* : X ——> Y be a weak extremal monomorphism.

Since η_X is an epi-reflective functor, there exists f:rX ——> Y such that $f\eta_X$ = m*, where η_X is an epimorphism [5.1.4] and $\eta_X(X)$ is an ideal, then η_X is a topological isomorphism (since m* is a weak extremal monomorphism).



i.e. weak extremal subobject X ∈ R*

<u>Note</u>

Proofs of propositions (5.3.3) and (5.3.4) are on the same lines as those of the corresponding results in the category of all Hausdorff spaces given by R.C.Walker[W].

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