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SOME NEW INTEGRAL GRAPHS

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The eigenvalue of a graph is the eigenvalue of its adjacency matrix. A graph G is integral if all of its eigenvalues are integers. In this paper some new classes of integral graphs are constructed.

1. INTRODUCTION

Let G be a graph with $|V(G)| = n$ and adjacency matrix A. The eigenvalues of A are called the eigenvalues of G and form the spectrum of G denoted by $spec\,(G)$ in Cvetković [**2**]. The graph G is integral if $spec\,(G)$ consists of only integers.

In Balińska [**1**] constructions and properties of integral graphs are discussed in detail. The graphs K_p and $K_{p,p}$ are examples of integral graphs. Some recent work on these lines pertaining to the class of trees is found in Wang [**4**]. Moreover, several graph operations such as Cartesian product, Strong sum and Product on integral graphs can be used for constructing infinite families of integral graphs, Balińska [**1**].

In this paper we provide some new constructions to obtain integral graphs.

All graph theoretic terminology is from Cvetković [**2**].

2. MAIN THEOREMS

The characteristic polynomial of G, $|\lambda I - A|$ is denoted by $P(G)$. A graph G, is said to be rooted at u if u is a specified vertex of G. We use the following lemmas in this paper.

Lemma 1 (Schwenk [**3**]). *Let G and H be graphs rooted at u and v respectively.*

1. *Let F be the graph obtained by making u and v adjacent. Then*

$$P(F) = P(G)P(H) - P(G-u)P(H-v).$$

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2. *Let F' be the graph obtained by identifying u and v. Then*

$$P(F') = P(G)P(H-v) + P(G-u)P(H) - xP(G-u)P(H-v).$$

Lemma 2 (Cvetković [2]). *Let M, N, P and Q be matrices with M invertible. Let $S = \begin{bmatrix} M & N \\ P & Q \end{bmatrix}$. Then $\det S = |M|\,|Q - PM^{-1}N|$.*

Lemma 3 (Cvetković [2]). *Let G be an r-regular connected graph on p vertices with $r = \lambda_1, \lambda_2, \ldots, \lambda_m$ as the distinct eigenvalues. Then there exists a polynomial $Q(x)$ such that $Q\{A(G)\} = J$, where J is the all one square matrix of order p and $Q(x)$ is given by $Q(x) = p \times \dfrac{(x-\lambda_2)(x-\lambda_3)\cdots(x-\lambda_m)}{(r-\lambda_2)(r-\lambda_3)\cdots(r-\lambda_m)}$, so that $Q(r) = p$ and $Q(\lambda_i) = 0$, for all $\lambda_i \neq r$.*

Definition 1 (Cvetković [2]). *Let G be an r_1-regular graph on p_1 vertices and H, an r_2-regular graph on p_2 vertices. Then the complete product of G and H, denoted by $G\nabla H$ is obtained by joining every vertex of G to every vertex of H.*

Note 1. *The characteristic polynomial of $G\nabla H$ is given by*

$$P(G\nabla H) = \frac{P(G)P(H)}{(x-r_1)(x-r_2)}\left(x^2 - (r_1+r_2)\,x + r_1r_2 - p_1p_2\right).$$

Notation 1. *Let $k *_G H$ denote the graph obtained by joining roots in k copies of H to all vertices of G. This graph can be obtained by first forming the complete product $G\nabla\overline{K_k}$ and then successively identifying the vertices in $\overline{K_k}$ one by one with roots in the k copies of H.*

Let F_k^t, $t \leq k$ denote the graph obtained by identifying roots of t copies of H with t vertices of $\overline{K_k}$ in $G\nabla\overline{K_k}$. Then $F_k^0 = G\nabla\overline{K_k}$ and $F_k^k = k *_G H$.

Notation 2. *$H_k = k \bullet H$, denote the graph obtained by identifying the root v in k copies of H.*

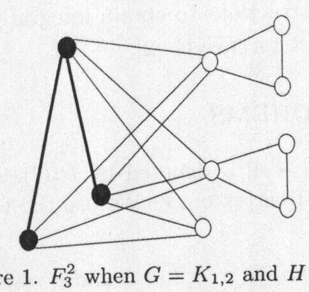

Figure 1. F_3^2 when $G = K_{1,2}$ and $H = K_3$

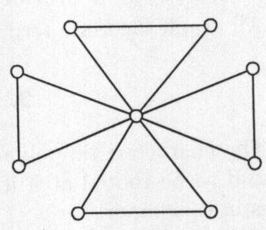

Figure 2. H_4 when $H = K_3$

Theorem 1. *Let G be an m-regular graph on p vertices and H be rooted at v. Then with the notations as described above*

$$P\left(F_k^t\right) = \frac{P(G)}{(x-m)}\, x^{k-(t+1)} \left(P(H)\right)^{t-1} \Big(P(H)\big(x(x-m) - p(k-t)\big) - tpxP(H-v)\Big).$$

Proof. We shall prove the Theorem by mathematical induction on t.

When $t = 0$, $F_k^0 = G\nabla \overline{K_k}$ and in this case

$$P\left(F_k^0\right) = \frac{P(G)}{(x-m)}\, x^{k-1}\left(x(x-m) - pk\right),$$

which is true from Note 1.

Now assume that the Theorem is true when $t = r < k$. Thus

$$P(F_k^r) = \frac{P(G)}{(x-m)}\, x^{k-(r+1)}\left(P(H)\right)^{r-1}\Big(P(H)\big(x(x-m) - p(k-r)\big) - rpxP(H-v)\Big).$$

Now F_k^{r+1} is the graph obtained from F_k^r by identifying the $(r+1)^{th}$ vertex of $\overline{K_k}$ in $G\nabla \overline{K_k}$ with the root v in the $(r+1)^{th}$ copy of H. Now by Lemma 1 and by the induction hypothesis

$$\begin{aligned}
P(F_k^{r+1}) &= P(F_k^r)P(H-v) + P(F_{k-1}^r)P(H) - xP(F_{k-1}^r)P(H-v)\\
&= \frac{P(G)}{(x-m)}\, x^{k-(r+1)}\left(P(H)\right)^{r-1}\Big(P(H)\big(x(x-m)\\
&\qquad -p(k-r)\big) - rpxP(H-v)\Big)P(H-v)\\
&\quad + \frac{P(G)}{(x-m)}\, x^{k-1-(r+1)}\left(P(H)\right)^{r-1}\Big(P(H)\big(x(x-m)\\
&\qquad -p(k-1-r)\big) - rpxP(H-v)\Big)P(H)\\
&\quad -xP(H-v)\,\frac{P(G)}{(x-m)}\, x^{k-(r+2)}\left(P(H)\right)^{r-1}\Big(P(H)\big(x(x-m)\\
&\qquad -p(k-1-r)\big) - rpxP(H-v)\Big)\\
&= \frac{P(G)}{(x-m)}\, x^{k-(r+2)}\left(P(H)\right)^{r}\Big(P(H)\big(x(x-m)\\
&\qquad -p(k-(r+1))\big) - (r+1)pxP(H-v)\Big).
\end{aligned}$$

Thus the Theorem is true for $t = r+1$. Hence by mathematical induction the Theorem follows. □

Corollary 1.

$$P(k *_G H) = P(F_k^k) = \frac{P(G)}{(x-m)}\left(P(H)\right)^{k-1}\left(P(H)(x-m) - pkP(H-v)\right).$$

Theorem 2. *The characteristic polynomial of H_k is given by*

$$P(H_k) = \left(P(H-v)\right)^{k-1}\left(kP(H) - (k-1)xP(H-v)\right).$$

Proof. We shall prove the Theorem by mathematical induction on k and by Lemma 1. The Theorem is trivially true when $k = 1$. Assume that the result is true for $t < k$. Thus

$$P(H_t) = \big(P(H-v)\big)^{t-1}\big(tP(H) - (t-1)xP(H-v)\big).$$

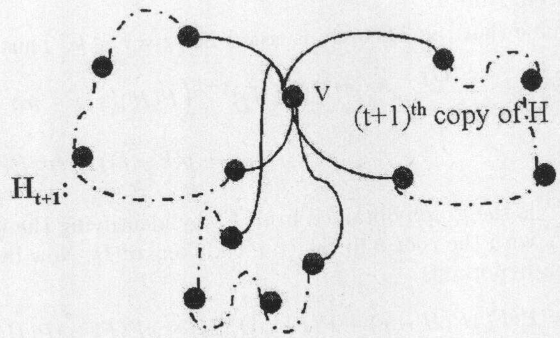

Figure 3.

Now

$$\begin{aligned} P(H_{t+1}) &= P(H)\big(P(H-v)\big)^t + P(H-v)P(H_t) - xP(H-v)\big)\big(P(H-v)\big)^t \\ &= P(H)\big(P(H-v)\big)^t + P(H-v\Big(\big(P(H-v)\big)^{t-1}\big(tP(H) \\ &\qquad -(t-1)xP(H-v)\big)\Big) - x\big(P(H-v)\big)^{t+1} \\ &= \big(P(H-v)\big)^t\big((t+1)P(H) - txP(H-v)\big). \end{aligned}$$

Hence the Theorem is true for $t+1$ and by mathematical induction Theorem follows. □

3. SOME NEW INTEGRAL GRAPHS

In this section we shall give some new constructions of integral graphs.

Construction 1. *Let G be any $m-$ regular integral graph and H be K_{m+2}. Then by Theorem 1, the graph $k *_G H$ is integral if and only if the roots of $(x-m-1)(x+1) - pk = 0$ are integers. That happens if and only if $(m+2)^2 + 4pk$ is a perfect square. Thus for $k = \frac{h^2-(m+2)^2}{4p}$, $h > m+2$, we get an infinite family of integral graphs.*

EXAMPLE 1.

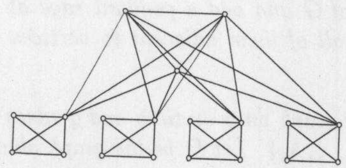

Figure 4. $G = K_3,\ m = 2,$
$H = K_4,\ k = 4$

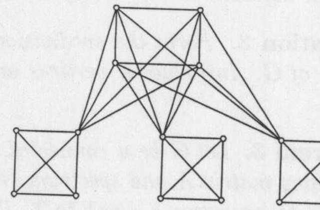

Figure 5. $G = C_4,\ m = 2,$
$H = K_4,\ k = 3$

Construction 2. *Let $G = K_{m,n}$ with any vertex in the n vertex set as root. Then by Theorem 2, G_k is integral if and only if both $m(n-1)$ and $m(n-1)+mk$ are perfect squares. Now $m = t$; $n = t+1$; $k = 3t$ is a feasible solution.*

Construction 3. *Let $G = K_4 - e$ rooted at any of the two non adjacent vertices. Then by Theorem 2, G_k is integral if and only if $8k+9$ is a perfect square. Then for integer k of the form $k = \frac{t^2-9}{8}$, $t \geqslant 4$, we get an infinite family of integral graphs.*

EXAMPLE 2.

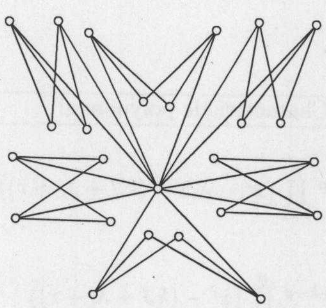

Figure 6. $G = K_{2,3},\ k = 6$
$H = K_4,\ k = 4$

EXAMPLE 3.

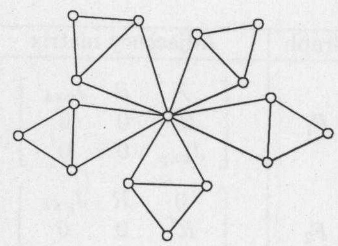

Figure 7. $G = K_4 - e,\ k = 5$
$H = K_4,\ k = 3$

4. SOME OPERATIONS ON GRAPHS

In this section we define some operations on a regular graph and thus provide some infinite families of integral graphs. Let G be a connected $r-$ regular graph on p vertices and q edges whose adjacency matrix is A and spectrum $\{\lambda_1 = r,\ \lambda_2,\ \lambda_3, \ldots, \lambda_p\}$.

Operation 1. *Corresponding to each edge of G introduce a vertex and make it adjacent to vertices incident with it. Now introduce k isolated vertices and make all of them adjacent to vertices of G only.*

Operation 2. *Form the subdivision graph of G. Introduce k vertices and make all of them adjacent to vertices of G only.*

Operation 3. *Form the subdivision graph of G and add a pendant edge at each vertex of G. Introduce k vertices and make all of them adjacent to vertices of G only.*

Theorem 3. *Let G be a connected r-regular graph on p vertices and q edges with adjacency matrix A and spectrum $\{r, \lambda_2, \lambda_3, \ldots, \lambda_p\}$. Let F_i be the graph obtained from G by operation i, i = 1 to 3. Then*

$$spec(F_1) = \begin{pmatrix} 0 & \frac{r\pm\sqrt{r^2+4(pk+2r)}}{2} & \frac{\lambda_2\pm\sqrt{\lambda_2^2+4(\lambda_2+r)}}{2} & \cdots & \frac{\lambda_p\pm\sqrt{\lambda_p^2+4(\lambda_p+r)}}{2} \\ k+q-p & \text{each once} & \text{each once} & \cdots & \text{each once} \end{pmatrix}$$

$$spec(F_2) = \begin{pmatrix} 0 & \pm\sqrt{(pk+2r)} & \pm\sqrt{(\lambda_2+r)} & \cdots & \pm\sqrt{(\lambda_p+r)} \\ k+q-p & \text{each once} & \text{each once} & \cdots & \text{each once} \end{pmatrix}$$

$$spec(F_3) = \begin{pmatrix} 0 & \pm\sqrt{(pk+2r+1)} & \pm\sqrt{(\lambda_2+r+1)} & \cdots & \pm\sqrt{(\lambda_p+r+1)} \\ k+q & \text{each once} & \text{each once} & \cdots & \text{each once} \end{pmatrix}$$

Proof. The proof follows from the Table 1 which gives the adjacency matrix and characteristic polynomial of F_i, $i = 1, 2, 3$.

Table 1

Graph	Adjacency matrix	Characteristic polynomial
F_1	$\begin{bmatrix} A & R & J_{p\times k} \\ R^T & 0 & 0 \\ J_{k\times p} & 0 & 0 \end{bmatrix}$	$x^{q+k-p} \prod_{i=1}^{p} (x^2 - \lambda_i x - (kJ + \lambda_i + r))$
F_2	$\begin{bmatrix} 0 & R & J_{p\times k} \\ R^T & 0 & 0 \\ J_{k\times p} & 0 & 0 \end{bmatrix}$	$x^{q+k-p} \prod_{i=1}^{p} (x^2 - (kJ + \lambda_i + r))$
F_3	$\begin{bmatrix} 0 & R & I & J_{p\times k} \\ R^T & 0 & 0 & 0 \\ I & 0 & 0 & 0 \\ J_{k\times p} & 0 & 0 & 0 \end{bmatrix}$	$x^{q+k} \prod_{i=1}^{p} (x^2 - (kJ + \lambda_i + r + 1))$

where R is the incidence matrix with $RR^T = A+rI$ and J is the the all one matrix as in Lemma 3. □

EXAMPLES:

1. $G = K_{p,p}$. F_1 is integral if and only if $p = t^2$, and $k = 2\ell^2 \pm \ell t - 1$, $\ell \geqslant t$, $t \geq 1$.

2. $G = K_{p,p}$. F_2 is integral if and only if $p = t^2$, and $k = 2h^2 - 1,\ t \geq 1,\ h \geq 1$.
3. $G = K_p$. F_3 is integral when $p = t^2$, and $k = t^2h^2 \pm 2h - 2,\ t \geq 1, h \geq 1$.
4. $G = K_{p,p}$. F_3 is integral when $p = t^2 - 1$, and $k = 2(t^2 - 1)h^2 \pm 2h - 1, t \geq 1, h \geq 1$.

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