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The Edge C_4 Graph of a Graph

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Abstract. The edge C_4 graph $E_4(G)$ of a graph G has all the edges of G as its vertices, two vertices in $E_4(G)$ are adjacent if their corresponding edges in G are either incident or are opposite edges of some C_4. In this paper, characterizations for $E_4(G)$ being connected, complete, bipartite, tree etc are given. We have also proved that $E_4(G)$ has no forbidden subgraph characterization. Some dynamical behaviour such as convergence, mortality and touching number are also studied.

1 Introduction

The study of graph dynamics has been receiving wide attention since Ore's work on the line graph operator $L(G)$, see [5,6]. Of concern in this paper is the notion of edge C_4 graph $E_4(G)$ of a graph G. The vertices of $E_4(G)$ are in one one correspondence with the edges in G and two vertices in $E_4(G)$ are adjacent if their corresponding edges in G either intersect or are opposite edges of some C_4 in G. So two vertices are adjacent vertices in $E_4(G)$ if the union of the corresponding edges in G induces any one of the graphs P_3, C_3, C_4, $K_4 - \{e\}$, K_4. This edge C_4 graph is the edge graph mentioned in [6].

Clearly the edge C_4 graph coincides with the line graph for any acyclic graph. But they differ in many properties. As a case, for a connected graph G, $E_4(G) = G$ if and only if $G = C_n$, $n \neq 4$. Then we say that C_n, $n \neq 4$ is fixed under E_4. Also Beineke has proved in [2] that the line graph has nine forbidden subgraphs. In this paper we see that $E_4(G)$ has no forbidden subgraphs.

In [1], Bandelt and others proved that a bipartite graph is dismantlable if and only if its edge C_4 graph is dismantlable and a bipartite graph is neighbourhood-Helly if and only if its edge C_4 graph is neighbourhood-Helly.

All the graphs considered here are finite, undirected and simple. We denote by P_n (respectively C_n), a path (respectively cycle) on n vertices. The graph obtained by deleting any edge of K_n is denoted by $K_n - \{e\}$. A triangle with a pendant edge attached to any one of its vertices is called a 'paw'. A graph G is H-free if G does not contain H as an induced subgraph. P_4-free graphs are called cographs [3, 4]. A graph H is a forbidden subgraph for a property P of graphs if no graph having property P contains an induced subgraph isomorphic to H. The cross product $G_1 \times G_2$ of two graphs G_1 and G_2 is a simple graph with $V(G_1) \times V(G_2)$ as its vertex set and two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G_1 \times G_2$ if and only if either $u_1 = u_2$ and v_1 is adjacent to v_2 in G_2, or u_1 is adjacent to u_2 in G_1

and $v_1 = v_2$. For all basic concepts and notations not mentioned in this paper we refer [7].

In this paper, characterizations for the edge C_4 graph of a graph G being connected, complete, bipartite etc are obtained. We have also proved that the edge C_4 graph has no forbidden subgraph characterization. The dynamical behaviour such as convergence, periodicity, mortality, transition number and touching number of $E_4(G)$ are also studied.

2 The Edge C_4 Graph of a Graph

Consider the edge C_4 graph $E_4(G)$ of a graph G . If $a_1 - a_2$ is an edge in G, the corresponding vertex in $E_4(G)$ is denoted by a_1a_2.

Theorem 1. *$E_4(G)$ is connected if and only if exactly one component of G contains edges.*

Theorem 2. *For a connected graph G, $E_4(G)$ is complete if and only if G is a complete multipartite graph.*

Proof. Let G be a connected graph such that $E_4(G)$ is complete. We shall first show that G is a cograph and is paw-free. If G contains an induced P_4 then the first and the third edges in P_4 correspond to two non adjacent vertices in $E_4(G)$, and $E_4(G)$ is not complete. Thus G must be a cograph. Further if G contains a paw as an induced subgraph then the pendant edge and the edge in the triangle of the paw to which the pendant edge is not adjacent correspond to non adjacent vertices in $E_4(G)$. Hence G is also paw-free. Claim: G is a complete multipartite graph. If not, $\overline{G}$ is not a union of complete graphs. Then $\overline{G}$ contains an induced P_3. But, since G is a connected cograph, $\overline{G}$ is disconnected. Hence $\overline{G}$ is a disconnected graph containing an induced P_3 and so G has a paw, giving a contradiction. This proves the claim.

Conversely suppose that G is a complete multipartite graph. Let e_1 and e_2 be any two edges in G. If they are not adjacent, then since G is complete multipartite, they are opposite edges of some C_4 in G. Hence $E_4(G)$ is a complete graph.

Theorem 3. *There is no forbidden subgraph characterization for $E_4(G)$.*

Proof. We shall prove that given any graph G, we can find a graph H such that G is an induced subgraph of $E_4(H)$. For any graph G, let $H = G \times K_2$. Then in $E_4(H)$, all the vertices of the form uu' where u is a vertex in G and u' is the corresponding vertex in the copy of G used in the construction of $G \times K_2$ will induce G. For, if u and v are any two adjacent vertices in G, uu' and vv' correspond to adjacent vertices in $E_4(H)$ as $uu'v'v$ forms a C_4 in H. If u and v are any two non adjacent vertices in G then uu' and vv' are non adjacent vertices in $E_4(H)$. □

Theorem 4. *For a connected graph G, $E_4(G)$ is bipartite if and only if G is either a path or an even cycle of length greater than five.*

Corollary 1. *For a connected graph G, $E_4(G)$ is a tree if and only if G is a path.*

3 Dynamical Properties

We shall first list some graph dynamical terminologies from [6].
Let G be any graph. The n^{th} iterated graph is iteratively defined as $E_4^2(G) = E_4(E_4(G))$ and $E_4^n(G) = E_4(E_4^{n-1}(G))$ for $n > 2$. We say that G is convergent

under E_4 if $\{E_4^n(G),\ n \in N\}$ is finite. If G is not convergent under E_4, then G is divergent under E_4. A graph G is periodic if there is some natural number n with $G = E_4^n(G)$. The smallest such number is called the period of G. The transition number $t(x)$ of a convergent graph G is defined as zero if G is periodic and as the smallest number n such that $E_4^n(G)$ is periodic. A graph G is mortal if for some $n \in N$, $E_4^n(G) = \phi$, the empty graph. The touching number of a cycle is the cardinality of the set of all edges having exactly one of its vertices on the cycle. For every integer $n \geq 3$, the n-touching number $t_n(G)$ of a graph G is the supremum of all touching numbers of C_n, provided G contains some C_n. If not, $t_n(G)$ is undefined.

Theorem 5. *The graphs P_n, $K_{1,3}$, $C_n(n \neq 4)$ are the only E_4 convergent graphs.*

Proof. If G contains a vertex of degree > 3, then $E_4(G)$ contains K_4. In the subsequent iterations the clique size goes on increasing and hence G diverges. So, for convergent graphs $\Delta(G) \leq 3$. If G is a tree which is neither P_n nor $K_{1,3}$, then K_4 is contained at least in the third iterated graph and hence G cannot converge. The paths P_n converge to ϕ and $K_{1,3}$ converges to the triangle. □

Consider the graphs which are not trees. If G is not a cycle, then G contains a cycle with a pendant edge as a subgraph (need not be induced). Then K_4 is a subgraph at least in the second iteration and hence in the subsequent iterations the clique size will go on increasing and hence cannot converge. All cycles except C_4 are fixed under E_4 and C_4 is not convergent.

Corollary 2. *For $E_4(G)$, the only periodic graphs are the cycles $C_n, n \neq 4$ and they have period one.*

Proof. Proof is clear from the remark in [6] that a graph G is convergent if and only if G is either periodic or there is some positive integer n with $E_4^n(G)$ periodic. □

Corollary 3. *The transition number $t(K_{1,3}) = 1$ and $t(C_n)$, $n \neq 4 = 0$.*

Corollary 4. *For $E_4(G)$, the paths are the only mortal graphs.*

Proof. Among the convergent graphs, cycles other than C_4 are fixed and $K_{1,3}$ converges to K_3. The paths are the only graphs converging to ϕ. □

In the following theorem, we consider only the graphs G for which the touching number $t_n(G)$ is defined.

Theorem 6. *For any graph G, $t_n(E_4(G)) \geq 2\ t_n(G)$. Further if G contains C_4 as a subgraph where either an edge or two consecutive edges of C_4 are the edges of the C_n which determines the touching number then $t_n(E_4(G) > 2t_n(G)$.*

Proof. Let the n-cycle in G be $x_1, x_2, \ldots, x_n, x_1$. Then x_1x_2, x_2x_3, $\ldots$, x_nx_1 is an n-cycle in $E_4(G)$. If yx_i is a touching edge in G, then $E_4(G)$ contains two touching edges say $yx_i - x_ix_{i+1}$ and $yx_i - x_{i-1}x_i$.

Let $C_4 = a_1a_2a_3a_4$ be a subgraph of G. Further, suppose that the edge a_3a_4 is a touching edge in G . Then a_1a_4, a_2a_3, a_1a_2 are touching edges in $E_4(G)$. Hence $t_n(E_4(G)) > 2\ t_n(G)$. The proof is similar for the case when any two consecutive edges of C_4 are the edges on the C_n mentioned above. □

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