S.p.27. BABY, B.V. – Some Non-linear Problems in the Theoretical Physics – 1985 – Dr. K. Babu Joseph

An immense variety of problems in theoretical physics are of the non-linear type. Non-linear partial differential equations (NPDE) have almost become the rule rather than an exception in diverse branches of physics such as fluid mechanics, field theory, particle physics, statistical physics and optics, and the construction of exact solutions of these equations constitutes one of the most vigorous activities in theoretical physics today. The thesis entitled "Some Non-linear Problems in Theoretical Physics" addresses various aspects of this problem at the classical level. For obtaining exact solutions we have used mathematical tools like the bilinear operator method, base equation technique and similarity method with emphasis on its group theoretical aspects.

A new era in theoretical physics was ushered in by the discovery of a non-linear transformation called inverse scattering transform (IST) and the collisional stability of a particular solitary wave solution, called soliton, of a class of NPDEs by Gardner, Greene, Kruskal and Miura [1] and Zabusky and Kruskal [2]. Further rapid development added a number of non-linear field theoretical models such as Korteweg-de Vries (Kdv), two-dimensional sine-Gordon (sG), non-linear Schrodinger, Thirring model etc. The solitary wave solutions of an integrable system are often called solitons [3] which are either topological or nontopological, depending on the nonvanishing or vanishing of the topological charge. Equations such as KdV, sG etc. belong to this class that is characterisd by the existence of an infinite number of conserved quantities. Backlund transformation (BT) constitutes one of the oldest approaches to the solution of NPDEs. The method of prolongation structures has been introduced recently to support the studies using IST and BT.

The bilinear operator method pioneered by Horota [4] is closely associated to the numerical method of Pade approximants. We have applied this method to develop single solitary wave solutions of the Double sinh-Grodon (DshG) equation in (1 + 1) dimensions. The DshG system is a newly introduced system and bears close resemblance to the Louville and the Toda models. For massive and massless $\varnothing^4$ equations this method yields some previously known solutions.

Non-abelian gauge theories of the Yang-Mills (YM) type are of great interest in contemporary field theory and particle physics, especially in the context of unifield models of fundamental interactions. By using some suitable ansatz one can reduce the SU(2) YM or YM-Higgs theory to non-linear diffrential equations (NDE) (the massless $\varnothing^4$ equation, or one-dimensional Liouville equation) or to a set of coupled NPDEs. Educlidean space solutions of the massless $\varnothing^4$equations lead to the celebrated instanton and merons of SU (2) pure YM theory. Monopole solutions of YM-Higgs system can be obtained from the one dimensional Liouville equation or a pair of coupled NPDEs.

We have used the singular solutions of the massless $\varnothing^4$ model to generate are interpreted as localized fluctuations involving no flux transport. It is conjectured that these objects having infinite action and infinite topological charge, closely resemble the merons and may play a tunnelling role. The idea of employing the soluytions of a know differential equation to construct a solution of a given NPDE was developed by Pinney, Reid anmd Burt [5] who called it the base equation technique. Using this approach we developed multisolitary wave solutions of the DshG system in arbitrary dimensions, which collapse to a single solitary wave in (1 + 1) dimensions.

We have developed a generalisation of the base equation method and called it the composite mapping method, wherein a sequence of maps is applied to several members of the non-linear Klein-Gordon (NKG) family to produce new solutions. In a broad scenario like this where one deals with a whole class of NDEs rather than a specific one, besides yielding new solutions, this procedure can expose certain 'family relationships' between different equations which we later confirmed through the similarity group method. Starting from the classical $\varnothing^4$ equation we have generated through sequential maps, arbitrary dimensional solutions of Liouville, double sine-Gordon (DsG), massive and massless $\varnothing^6$ equations of the NKG family by imposing simple constraints at each stage. While all other known solutions of the DsG collapse to a single wave in (1 + 1) dimensions, our solutions behave differently. Since all the four distinct solutions obtained by us can be simultaneously constructed for given values of a parameter, it will be possible to study their interactions. In this context we have also highlighted the appearance of one set of solutions and the disappearance of another set at a critical point.

The Lie point transformation theory has emerged as a most outstanding attempt to study continuous symmetry, particular solutions and dimensional reduction of NPDE S [6]. When a second order NPDE is invariant under these transformations, known as similarity transformations, it is possible to reduce the number of independent variables by one, and find similarity solutions of the equation [7]. In general the similarity transformations form an extended group, the similarity group, which upon a suitable redefinition of the generators, leads to the Poincare group in the case of Klein-Gorddon (KG) equations. This suggests a three-fold classification of solutions of two dimensional KG equations into translation invariant, heperbolic rotation (boost) invariant and similarity invariant types. Similarity invariance denotes invariance under the full ssimilarity group. Such a description emphasises the behaviour of the solutions rather than that of the equations. Most of the known solitary wave solutions are of the translation invariant type. We have produced rotationally invariant solutions of several NKG equations. The group theoretical meaning of the base equation technique has also been examined. We have found that the similarity groups of the original equation and the constraint equation are identical in all the cases studied in the literature.

It has conjectured that the existence of the Painleve property (PP) (ie., the absence of movable critical points) is a signal to the original equations integrability. We have shown that the translation-invariant sector of the SG equation does not possess PP whereas the rotation invariant sector does possess PP. This may restrict the integrability property of the SG system in some sense.

The SU (2) Yang-Mills field interacting with a Higgs scalar triplet is known to give monopole solutions through the 't Hooft ansatz. Prasad and Sommerfield developed spherically symmetric solutions to this system in a special limit in the static case. Aftérwards several attempts based on guesswork were made to obtain time-dependent exact solutions. We have carried out a similarity group analysis of the coupled system of equations representing the SU (2) YM-Higgs model equations and shown that the equations reduce to the one-dimensional form under the full similarity group or under one of its subgroups. This approach gives two new exact time-dependent solutions along with some previously known solutions.

The material reported in the present thesis has been published in the form of the following papers:

1. Solitary wave solutions in double sinh-Gordon system, J.Phys., *A16*, 2685 (1983).
2. Fluctuations in SU (2) Yang-Mills theory, Pramana, *22*, 111 (1984).
3. Composite mapping method for generations of kinks and solitons in the Klein-Gordon family, Phys. Rev. A., *29*, 2899 (1984).
4. Symmetry classification of solutions of two dimensional non-linear Klein-Gordon equations, J. Math. Phys (submitted).
5. Classical SU (2) Yang-Mills-Higgs system: Time-dependent solutions by similarity method, J. Math. Phys. (submitted).