SOME PROBLEMS IN TOPOLOGY AND ANALYSIS

BOX PRODUCTS IN FUZZY TOPOLOGICAL SPACES AND RELATED TOPICS

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CERTIFICATE

*This is to certify that the thesis entitled "**BOX PRODUCTS IN FUZZY TOPOLOGICAL SPACES AND RELATED TOPICS**" is an authentic record of research carried out by **Smt. Susha D** under my supervision and guidance in the Department of Mathematics, Cochin University of Science and Technology for the PhD degree of the Cochin University of Science and Technology and no part of it has previously formed the basis for the award of any other degree or diploma in any other university.*

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CHAPTER - 1
INTRODUCTION

1.1 BOX PRODUCTS

The product of a family of topological spaces was considered first by H. Tietze in 1923. He described the topology as the product set with a base chosen as all products of open sets in the individual spaces. This topology is now known as box topology. But Tietze's definition was not followed much because it was found to be deficient in getting nice theorems like "product of compact spaces is compact". A new topology introduced by Tychonov became more popular and much work was done using this product topology.

Later in 1977, M.E. Rudin found that box topologies are useful at least as counter examples. So box products also became an area of study in topology.

Mathematicians like Williams [WIL], Van Douwen [VA], Miller [MI], Kato [KAT] and Roitman[ROI] have done a detailed study in this area. A complete survey regarding the covering and separation properties were given by Van Douwen [VA].

A comprehensive survey of results on box products available in Williams [WIL] including his own contributions also. It is proved that complete uniform space is the richest structure preserved by box products. He also introduced paracompactness in box products by making use of uniformity.

Thus the notion of 'Box Products' is well known in topology. In some sense perhaps the generalization of finite products of spaces to infinite products leads to the box product rather than Tychonov product.

1.2 Fuzzy Set Theory

Decision –making is all pervasive in human activities. But under uncertainty it is as old as mankind. Just like most of the real world systems in which human perception and intuitive judgement play important roles, the conventional approaches to the analysis of large scale systems were ineffective in dealing with systems that are complex and mathematically ill defined. Thus an answer to capture the concept of imprecision in a way that would differentiate imprecision from uncertainty, the very simple idea put forward by the American Cyberneticist L.A. Zadeh [ZA] in 1965 as the generalization of the concept of the characteristic function of a set to allow for immediate grades of membership was the genesis of the concept of a fuzzy set.

In Mathematics, a subset A of X can be equivalently represented by its characteristic function – a mapping χ_A from the universe X of discourse (region of consideration ie, a larger set) containing A to the two element set {0,1}. That is to say x belongs to A if and only if χ_A (x) = 1. But in the "fuzzy" case the "belonging to" to relation χ_A (x) between x and A is no longer "either 0 or otherwise 1", but it has a membership degree belonging to [0,1] instead of {0,1}, or more generally, to a lattice L, because all membership degrees in mathematical view form an ordered structure, a lattice. A mapping from X to a lattice L is called a generalized characteristic function and it describes the fuzziness of the set in general. A fuzzy set on a universe X is simply a function from X to I or to a lattice L. Zadeh took the closed unit interval [0,1] as the membership set. Later J.A. Goguen [GO] suggested that a complete and distributive lattice would be a minimum structure for the membership set.

Thus the fuzzy set theory extended the basic mathematical concept of a set. Owing to the fact that set theory is the corner stone of modern Mathematics, a new and more general framework of Mathematics was established. Fuzzy Mathematics is just a kind of Mathematics developed in this frame work. Hence in a certain sense, fuzzy Mathematics is the kind of mathematical theory which contains wider content than the classical theory. Also it has found numerous applications in different fields such as Linguistics, Robotics, Pattern Recognition, Expert Systems, Military Control, Artificial Intelligence, Psychology, Taxonomy and Economics.

1.3 Fuzzy Topology

The theory of general topology is based on the set operations of unions, intersections and complementation. Fuzzy sets were assumed to have a set theoretic behaviour almost identical to that of ordinary sets. It is therefore natural to extend the concept of point set topology to fuzzy sets resulting in a theory of fuzzy topology. Using fuzzy sets introduced by Zadeh, C.L. Chang [CH] defined fuzzy topological space in 1968 for the first time. In 1976 Lowen $[LO]_1$ suggested a variant of this definition. Since then an extensive work on fuzzy topological space has been carried out by many researchers.

Many Mathematicians while developing fuzzy topology have used different lattices for the membership sets like (1) Completely distributive lattice with 0 and 1 by T.E. Gantner, R.C. Steinlage and R.H. Warren [G;S;W] (2) Complete and completely distributive lattice equipped with order reversing involution by Bruce Hutton and Ivan Reilly [H;R] (3) Complete and completely distributive non atomic Boolean Algebra by Mira Sarkar [SA] (4) Complete Chain by Robert Bernard [BE] and F. Conard [CO] (5) Complete Brouwerian lattice with its dual also Brouwerian by Ulrich Hohle $[HO]_1$, (6) Complete and distributive lattice by S.E. Rodabaugh [ROD] (7) Complete Boolean Algebra by Ulrich Hohle $[HO]_2$.

Fuzzy topology is just a kind of topology developed on fuzzy sets and in his very first paper Chang [CH] gives a strong basement for the development of fuzzy topology in the [0,1] membership value framework. Compactness and its different versions are always important concepts in topology. In fuzzy topology, after the initial work of straight description of ordinary compactness in the pattern of covers of a whole space, many authors tried to establish various reasonable notions of compactness with consideration of various levels in terms of fuzzy open sets and obtained many important results. Since the level structures or in other words stratification of fuzzy open sets is involved, compactness in fuzzy topological spaces is one of the most complicated problems in this field. Many kinds of fuzzy compactness using different tools were raised, and each of them has its own advantages and shortcomings. In $[LO]_2$ Lowen gives a comparative study of different compactness notions introduced by himself. Chang, T.E. Gantner, R.C. Steinlage, R.H. Warren etc and all the value domains used in these notions are [0,1].

Gantner and others [G; S; W] used the concept of shading families to study compactness and related topics in fuzzy topology. The shading families are a very natural generalization of coverings. Using these concepts Malghan and Benchalli defined point finite and locally finite families of fuzzy sets and introduced the concept of fuzzy paracompact and fuzzy α-paracompact spaces.

We take the definition of fuzzy topology in the line of Chang with membership set as the closed unit interval [0,1]

1.4 About this thesis

The main purpose of our study is to extend the concept of box products to fuzzy box products and to obtain some results regarding them. Owing to the fact that box products have plenty of applications in uniform and covering properties, we have made an attempt to explore some inter relations of fuzzy uniform properties and fuzzy covering properties in fuzzy box products. Even though our main focus is on fuzzy box products, some brief sketches regarding hereditarily fuzzy normal spaces and fuzzy nabla product is also provided.

1.5 Summary of the thesis

The thesis is divided into six chapters.

The general preliminary definitions and results which are used in the succeeding chapters are given in the next section of this chapter. Due references are given wherever necessary. Some of the preliminary results which are relevant to each chapter are given at the beginning of the corresponding chapter itself.

In the second chapter we introduce the notion of fuzzy box product and investigate some properties including separation properties, local compactness, connectedness etc. The main results obtained include characterization of fuzzy Hausdroffness and fuzzy regularity of box· products of fuzzy topological spaces.

The concept of fuzzy uniformity was introduced in the literature by many authors. Here we are interested in the fuzzy uniform structure $\mathscr{U}$ in the sense of Lowen [LO]3. In this chapter fuzzy uniform fuzzy topological space, compatible fuzzy uniform base, fuzzy uniform fuzzy box product, fuzzy topologically complete spaces etc are defined and some results related to them are obtained. We also investigate the completeness of fuzzy uniformities in fuzzy box products. Here we have proved that a fuzzy box product of spaces is fuzzy topologically complete if each co-ordinate space is fuzzy topologically complete.

The notion of shading family was introduced in the literature by T.E. Gantner and others in [G;S;W] during the investigation of compactness in fuzzy topological spaces. The shading families are a very natural generalization of coverings. Approach to fuzzy α-paracompactness using the notion of shading families was introduced by S.R. Malghan and S.S. Benchalli in $[M; B]_1$. In this chapter we introduce and study fuzzy α-paracompactness in fuzzy box products. For this we make use of the concept of fuzzy entourages in fuzzy uniform spaces. Here we give a characterization of fuzzy α-paracompactness through fuzzy entourages. We also prove that the fuzzy box product of a family of fuzzy α-paracompact spaces is fuzzy topologically complete.

Fuzzy box product of hereditarily fuzzy normal spaces is considered in chapter V. The main result obtained is that if a fuzzy box product of spaces is hereditarily fuzzy normal, then every countable subset of it is fuzzy closed.

Chapter VI deals with the notion of fuzzy nabla product of spaces which is a quotient of fuzzy box product. Here we study the relation connecting fuzzy box product and fuzzy nabla product. We also discuss fuzzy uniform fuzzy nabla product. Even though there is a close association between fuzzy box product and fuzzy nabla product, the latter one is meaningful only for countable products.

1.6 Basic Definitions

The following definitions are adapted from [ZA] and [CH].

1.6.1 Definition [ZA] Let X be a set. A fuzzy set U in X is characterized by a membership function $x \rightarrow U(x)$ from X to the unit interval $I = [0,1]$.

Let U and V be fuzzy sets in X. Then

$$U = V \Leftrightarrow U(x) = V(x) \text{ for all } x \in X$$

$$U \leq V \Leftrightarrow U(x) \leq V(x) \text{ for all } x \in X$$

$$W = U \vee V \Leftrightarrow W(x) = \text{Max}\,\{U(x), V(x)\} \text{ for all } x \in X$$

$$W = U \wedge V \Leftrightarrow W(x) = \text{Min}\,\{U(x), V(x)\} \text{ for all } x \in X$$

Complement of U, $U' = S \Leftrightarrow S(x) = 1 - U(x)$ for all $x \in X$

More generally, for a family of fuzzy sets $\mathscr{U} = \{U_i : i \in I\}$, the union $W = \bigcup_{i \in I} U_i$ and the intersection $Y = \bigcap_{i \in I} U_i$ are defined by

$$W(x) = \sup_{i \in I} \{ U_i(x) \} \quad x \in X \text{ and}$$

$$Y(x) = \inf_{i \in I} \{ U_i(x) \} \quad x \in X.$$

The symbol 0 and 1 will be used to denote the empty fuzzy set ($U(x) = 0$ for all $x \in X$) and the full set X ($U(x) = 1$ for all $x \in X$) respectively.

1.6.2 Definition [CH] A fuzzy topology on X is a family T of fuzzy sets in X which satisfies the following conditions.

i) $0, 1 \in T$

ii) If $U, V \in T$ then $U \wedge V \in T$

iii) If $U_i \in T$ for each $i \in I$ then $\bigcup_{i \in I} U_i \in T$.

Then pair (X, T) is then called a fuzzy topological space or fts for short. Every member of T is called a T-fuzzy open set (or simply a fuzzy open set). A fuzzy set is called T-closed (fuzzy closed or simply f-closed) if and only if its complement is T-open.

1.6.3 Definition [CH] Let U be a fuzzy set in a fuzzy topological space (X, T). The largest fuzzy open set contained in U is called the interior of U and is denoted by int U or U^0

ie, $U^0 = \vee\{K : K \in T, K \leq U\}$

The smallest fuzzy closed set containing U is called the closure of U, denoted as cl(U) or $\overline{U}$

ie, $\overline{U} = \wedge\{K : K' \in T \text{ and } U \leq K\}$

1.6.4 Definition [CH] Let θ be a function from X to Y. Let f be a fuzzy set in Y. Then the inverse of f, written as θ^{-1} (f) is the fuzzy set in X whose membership function is given by

$\theta^{-1}(f)(x) = f(\theta(x))$ for all $x \in X$.

On the other hand, let g be a fuzzy set in X. Then the image of g written as θ(g) is the fuzzy set in Y whose membership function is given by

$$\theta(g)(y) = \begin{cases} \sup g(z) / z \in \theta^{-1}(y) & \text{if } \theta^{-1}(y) \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

1.6.5 Definition [CH] A function θ from a fuzzy topological space (X,T) to a fuzzy topological space (Y, U) is fuzzy continuous if the inverse image of each U-open fuzzy set is T-open.

1.6.6 Definition [CH] A function θ from a fuzzy topological space (X,T) to a fuzzy topological space (Y, U) is fuzzy open (resp. fuzzy closed) if it maps every open (resp. closed) fuzzy set in (X,T) onto an open (resp. closed) fuzzy set in (Y, U).

1.6.7 Definition [CH] Let T be a fuzzy topology. A subfamily B of T is a base for T if every member of T can be expressed as the join of some members of B.

1.6.8 Definition [CH] Let T be a fuzzy topology. A subfamily S of T is a subbase for T if the family of finite meets of S form a base for T.

1.6.9 Definition [CH] Let (X,T) be a fuzzy topological space. A family $\mathscr{U}$ of fuzzy sets is a cover of a fuzzy set V if and only if

$$V \leq \{U: U \in \mathscr{U}\}$$

It is an open cover if and only if each member of $\mathscr{U}$ is an open fuzzy set. A subcover of $\mathscr{U}$ is a subfamily, which is also a cover.

For the elementary definitions and results in topology reference may be made to [BO], [WI] and [JO].

For the theory of box products to [WIL], [VA], [ROI] and [KAT].

CHAPTER - 2

ON FUZZY BOX PRODUCTS*

2.1 Introduction

In 1923, Tietze considered topology on a set product of infinitely many spaces which is now known as box topology. He described this topology as the product set with the base chosen as all products of open sets in the individual spaces.

Later in 1977, M.E. Rudin observed that box products are useful at least as counter examples. Thus the notion of box products is well-known in topology. In some sense perhaps the generalization of finite products of spaces to infinite products leads to box product rather than Tychonov product.

In this chapter we are doing the fuzzy analogue of the concept "Box Products".

In the second section of this chapter we give the necessary preliminary ideas. In the next section, we introduce the concept of fuzzy box product and investigate properties like separation properties, local compactness and connectedness.

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* Some Results mentioned in this Chapter are accepted for publication in the paper titled *On Fuzzy Box Products* in the Journal of Tripura Mathematical Society Vol V (2004)

Some Results of this Chapter are published in the paper titled Some *Separation Properties of Fuzzy Box Products* in the Proceedings of the National Seminar on Graph Theory and Fuzzy Mathematics.

2.2 Preliminaries

The following definitions are adapted from [CH], $[WO]_3$ and $[M; B]_1$

2.2.1 Definition A fuzzy topological space (X, T) is Hausdroff if $x, y \in X$ with $x \neq y$ imply that there exists U and V in T with $U(x) = 1 = V(y)$ and $U \wedge V = 0$.

2.2.2 Definition A fuzzy topological space (X, T) is regular if for each

$x \in X$ and a $g \in T$ with $g(x) = 1$, there exists $h \in T$ with $h(x) = 1$ and $h \leq \bar{h} \leq g$,where $\bar{h}$ is the closure of h.

2.2.3 Definition A fuzzy topological space (X, T) is locally compact if for every point $x \in X$ there exists a member $U \in T$ with $U(x) = 1$ and U is compact (U is compact means each fuzzy open cover of U has a finite subcover).

2.2.4 Definition A fuzzy topological space (X, T) is said to be separable if there exists (countable) sequence of points $\{p_i\}$, i = 1, 2, ... such that for every member U of T and $U \neq \varnothing$ there exists a p_i such that

$p_i \in U$ (i.e., $U(p_i) = 1$).

2.2.5 Definition A fuzzy topological space (X,T) is said to be first countable at a point if there exists a countable local base.

2.2.6 Definition A fuzzy topological space (X, T) is connected if there do not exist fuzzy open sets $U \neq 0$ and $V \neq 0$ of X with $U \vee V = 1$ and $U \wedge V = 0$

2.2.7 Definition A fuzzy topological space (X,T) is locally connected at a point x if for any fuzzy open set U of x with $U(x) = 1$, there exists a connected fuzzy open set V of x with $V(x) = 1$ such that $V \leq U$.

2.2.8 Definition A fuzzy topological space (X, T) is perfect if each fuzzy closed set in X is a fuzzy G_δ - set.

The following definitions and theorems are from $[P;Y]_1$

2.2.9 Definition A fuzzy point x_α is a fuzzy set defined by

$$x_\alpha(y) = \alpha \text{ if } y = x$$

$$= 0 \text{ otherwise. Where } 0<\alpha \leq 1$$

Here x is called its support and α its value.

Now $x_\alpha \in A$ if $A(x) \geq \alpha$

$x_\alpha \notin A$ if $A(x) < \alpha$

Every fuzzy set A can be expressed as the union of all fuzzy points which belongs to A.

2.2.10 Definition A fuzzy set A in a fuzzy topological space (X,T) is called a neighbourhood fuzzy point x_α if and only if there exists $B \in T$ such that $x_\alpha \in B \leq A$.

2.2.11 Definition A fuzzy point x_α is said to be quasi-coincident with A, denoted by x_α qA if and only if $\alpha > A'(x)$,

That is, $\alpha + A(x) > 1$

2.2.12 Definition A fuzzy set A is said to be quasi-coincident with B denoted by AqB if and only if there exists $x \in X$ such that $A(x) > B'(x)$ or $A(x)+B(x) > 1$.

We say that A and B are quasi-coincident at x, both A(x) and B(x) are not zero and hence A and B intersect at x. That is, $(A \wedge B)(x) \neq 0$

2.2.13 Definition A fuzzy set A in fuzzy topological space (X,T) is called

Q-neighbourhood of x_α if and only if there exists $B \in T$ such that $x_\alpha qB \leq A$

2.2.14 Theorem A fuzzy point $e = x_\alpha \in \overline{A}$ if and only if each

Q-neighbourhood of e is quasi-coincident with A.

2.2.15 Definition A fuzzy point e is called an adherence point of a fuzzy set A if and only if every Q-neighbourhood of e quasi-coincident with A.

Note A fuzzy point e = $x_\alpha \in \overline{A}$ if and only if it is an adherence point of A.

$\overline{A}$ is the union of all adherence point of A.

2.2.16 Definition A fuzzy point e is called an accumulation point of a fuzzy set A if and only if e is an adherence point of A and every

Q-neighbourhood of e and A are quasi-coincident at some point different from supp (e), whenever e $\in$ A.

The union of all the accumulation points of A is called the derived set of A, denoted by A^d

Note that $A^d \subset \overline{A}$.

2.2.17 Theorem $\overline{A} = A \cap A^d$, for any fuzzy set A in (X, T)

2.2.18 Theorem A fuzzy set A is closed if and only if A contains all the accumulation points of A.

The following definition of 'Box Products' is in [WIL]

2.2.19 Definition Let $\{X_i : i \in I\}$ be a family of topological spaces. The box topology on the product set $X = \Pi_i X_i$ is the topology generated by the base of all open boxes of the form $U = \Pi_i U_i$ where U_i is open in X_i for each $i \in I$.

2.3 Fuzzy Box Products

In this section we introduce the concept of fuzzy box product and investigate some elementary properties.

2.3.1 Definition Let X_i be a fuzzy topological space for each $i \in I$. Then the fuzzy box product is the set $\prod_i X_i$ with the fuzzy topology whose basis consists of all fuzzy open boxes of the form $U = \prod_i U_i$, where U_i is fuzzy open in X_i for each $i \in I$.

$$\text{ie, } U(x) = \prod_i U_i(x)$$

$$= \inf_{i \in I} U_i(x_i) \quad \text{for each} \quad x = (x_i)_{i \in I} \text{ in } \prod_i X_i$$

This space is represented by using $\boxed{f}$ or $\boxed{f}_i X_i$. If each X_i is identical to fixed set X, then the fuzzy box product is denoted as $\boxed{f}^{I} X$.

2.3.2 Theorem Let $\{X_i : i \in I\}$ be finite or infinite family of fuzzy topological spaces. Then the following hold.

i) $\prod_j : \boxed{f}_i X_i \rightarrow X_j$ is fuzzy continuous and fuzzy open for each $j \in I$.

i) $F = \prod_i F_i$ is a fuzzy closed (open) subset of $\boxed{f}_i X_i$ if and only of for each $i \in I$, F_i is a fuzzy closed (open) subset of X_i.

iii) $X = \boxed{f}_i X_i$ is fuzzy Hausdroff if and only of for each $i \in I$, X_i is fuzzy Hausdroff.

iv) $X = \boxed{f}_i X_i$ is fuzzy regular if and only of for each $i \in I$, each X_i is fuzzy regular.

Proof:

Proof of (i) follows from the definition

(ii) Assume that F_i is a fuzzy closed subset of X_i for each $i \in I$.

Let $X = \boxed{f}_i X_i$

We know that the projection map $\prod_i : X \rightarrow X_i$ is fuzzy continuous for each $i \in I$. Since F_i is fuzzy closed in X_i, $\prod_i^{-1}(F_i)$ is fuzzy closed in X for each $i \in I$.

Now $F = \wedge\{\prod_i^{-1}(F_i) : i \in I\}$

Therefore F is fuzzy closed in X.

Conversely assume that $F = \prod_i F_i$ is fuzzy closed in $X = \boxed{f}_i X_i$.

Claim F_i is fuzzy closed in X_i for each $i \in I$

Let $j \in I$ be arbitrary.

We prove that F_j is fuzzy closed.

Let $(z_j)_\alpha$ be any accumulation point of F_j in X_j.

Consider z_α where $\prod_j(z_\alpha) = (z_j)_\alpha$

Now $\prod_i(z_\alpha)$ is an element of F_i for $i \neq j$.

Let G be a fuzzy open set containing z_α (ie, $G(z) \geq \alpha$)

Then $\prod_j(G)$ is fuzzy open (since every projection map is fuzzy open) and it contain $(z_j)_\alpha$, since $G_j(z_j) \geq \inf G_i(z_i) = G(z) \geq \alpha$.

Since $(z_j)_\alpha$ is an accumulation point of F_j in X_j every Q-neighbourhood G_j of $(z_j)_\alpha$ and F_j are quasi-coincident at some point say $(x_j)_\alpha$ of F_j different from $(z_j)_\alpha$.

Therefore $\prod_i(G)$ must contain a point $(x_j)_\alpha$ of F_j different from $(z_j)_\alpha$.

Hence G contains the point x_α.

That is, $G(x) \geq \alpha$ and

$$\prod_i(x_\alpha) = \prod_i(z_\alpha) \quad \text{for } i \neq j.$$

Also $\prod_j(x_\alpha) = (x_j)_\alpha$

So we get $x_\alpha \in F$.

Since x_α and z_α differ in the jth co-ordinate we have $x_\alpha \neq z_\alpha$.

Therefore every Q-neighbourhood G containing z_α contains a point of F different from z_α.

Hence z_α is an accumulation point of F.

Since F is fuzzy closed in X,

$$z_\alpha \in F \Rightarrow \prod_j(z_\alpha) \in \prod_j(F)$$

That is $F(z) \geq \alpha \Rightarrow F_j(z_j) \geq \alpha$

Therefore F_j contains all its accumulation points. Hence F_j is fuzzy closed.

Since j was arbitrary, F_i is fuzzy closed for each $i \in I$.

Hence the theorem. .

iii) Assume that X_i is fuzzy Hausdroff for each $i \in I$.

Let $x = (x_i)_{i \in I}$ and $y = (y_i)_{i \in I}$ be two distinct points of $X = \boxed{f}_i X_i$

Since $x \neq y$, there is some index j such that $x_j \neq y_j$, where x_j & $y_j \in X_j$.

Since X_j is fuzzy Hausdroff there exist fuzzy open sets G_j and H_j of x_j and y_j in X_j with $G_j(x_j) = 1$, $H_j(y_j) = 1$ and $G_j \wedge H_j = 0$.

Since $\Pi_j : \boxed{f}_i X_i \rightarrow X_j$ is fuzzy continuous,

$\Pi_j^{-1}(G_j)$ and $\Pi_j^{-1}(H_j)$ are fuzzy open sets in $X = \boxed{f}_i X_i$ containing x & y.

That is,
$$\Pi_j^{-1}(G_j)(x) = G_j((\Pi_j(x)) = G_j(x_j) = 1$$

$$\Pi_j^{-1}(H_j)(y) = H_j(\Pi_j(y)) = H_j(y_j) = 1$$

Also $\Pi_j^{-1}(G_j \wedge H_j) = 0$ since $G_j \wedge H_j = 0$

That is, $\Pi_j^{-1}(G_j) \wedge \Pi_j^{-1}(H_j) = 0$

Therefore $X = \boxed{f}_i X_i$ is fuzzy Hausdroff.

Conversely assume that $X = \boxed{f}_i X_i$ is fuzzy Hausdroff. Claim X_i is fuzzy Hausdroff for each $i \in I$.

We prove that X_j is fuzzy Hausdroff for an arbitrary $j \in I$.

Let x_j & y_j be two distinct points of X_j.

Choose x & y in X such that

$x = (x_i)$, $y = (y_i)$ where $x_i = y_i = a_i$ (say) for $j \neq i$.

Here a_i is a chosen fixed point in X_i for each i&j.

Since X is fuzzy Hausdroff there exists fuzzy open sets G and H in X with $G(x) = 1$, $H(y) = 1$ and $G \wedge H = 0$.

Then there exists basic fuzzy open sets $U = \Pi_i U_i$ & $V = \Pi_i V_i$ such that $x \in U \leq G$, $y \in V \leq H$ and $U \wedge V = 0$.

Now U_j and V_j are fuzzy open sets in X_j with $U_j(x_j) = 1$, $V_j(y_j) = 1$

and $U_j \wedge V_j = 0$.

Thus X_j is fuzzy Hausdroff.

Before proving (iv) of theorem 2.3.2, we prove a lemma.

2.3.3 Lemma : Let $\{X_i : i \in I\}$ be a family of spaces, let U_i be a fuzzy subset of X_i for each $i \in I$, then $\overline{\prod_i U_i} = \prod_i \overline{U_i}$ for each $i \in I$.

Proof:

If U_i is fuzzy closed in X_i for each $i \in I$ then $\prod_i U_i$ is fuzzy closed in $\boxed{f}_i X_i$ by (ii) of theorem 3.2.

We have

$U_i \leq \overline{U_i}$ for each $i \in I$

$\prod_i U_i \leq \prod_i \overline{U_i}$

$\overline{\prod_i U_i} \leq \overline{\prod_i \overline{U_i}} = \prod_i \overline{U_i}$ by (ii) of theorem 3.2. .... (1)

Suppose $U_i = 0$ for some i, then

$\overline{\prod_i U_i} = \prod \overline{U_i} = 0$ and so we are done.

Assume that $U_i \neq 0$ for all i.

Let x_α be any element of $\prod_i \overline{U_i}$.

Therefore $(x_i)_\alpha \in \overline{U_i}$ for each $i \in I$.

That is every Q-neighbourhood of $(x_i)_\alpha$ is quasi-coincident with U_i..

Let $G = \prod_i G_i$ be a fuzzy open box containing x_α, where G_i is fuzzy open in X_i for each $i \in I$.

We have $G(x) \geq \alpha$

That is $\inf_{i \in I} G_i(x_i) \geq \alpha$

Therefore $G_i(x_i) \geq \alpha$

Since $(x_i)_\alpha \in \overline{U_i}$ for every $i \in I$, every Q-neighbourhood G_i of $(x_i)_\alpha$ with $G_i(x_i) \geq \alpha$ is quasi coincident with U_i.

Therefore every Q-neighbourhood G of x_α with $G(x) \geq \alpha$ is quasi coincident with $\prod_i U_i$.

Hence $x_\alpha \in \overline{\prod_i U_i}$

Thus we get

$\prod_i \overline{U_i} \leq \overline{\prod_i U_i}$ .... (2).

So from (1) & (2) $\overline{\prod_i U_i} = \prod_i \overline{U_i}$

H.ence the lemma.

iv) Assume that each X_i is fuzzy regular.

Let $x = (x_i)_{i \in I}$ be any point in X and $G = \prod_i G_i$ be a fuzzy open set in X with $G(x) = 1$, where G_i is fuzzy open in X_i for each $i \in I$.

Then there exists a basic fuzzy open set V in X with $V(x) = 1$ and $V \leq G$. Let $V = \prod_i V_i$ where V_i is fuzzy open in X_i for each $i \in I$.

Since each X_i is fuzzy regular, for each $x_i \in X_i$ and $V_i \in T_i$ with $V_i(x_i) = 1$, there exists $U_i \in T_i$ with $U_i(x_i) = 1$ and $U_i \leq \overline{U_i} \leq V_i$.

Let $U = \prod_i U_i$. Since each U_i is fuzzy open in X_i, U is also fuzzy open in X.

$$\text{Therefore } U(x) = \prod_i U_i(x)$$

$$= \inf_{i \in I} U_i(x_i) = 1$$

$$\text{But } \overline{U} = \overline{\prod_i U_i} = \prod_i \overline{U_i} \text{ by above lemma.}$$

Since $\overline{U_i} \leq V_i$ for every $i \in I$.

We have $\overline{U} = \prod_i \overline{U_i} \leq \prod_i V_i = V$

Thus for every point $x \in X$ and $V \in T$ with $V(x) = 1$, there exists $U \in T$ with $U(x) = 1$ and $U \leq \overline{U} \leq V$.

Therefore X is fuzzy regular.

Conversely assume that X is fuzzy regular. Claim X_i is fuzzy regular. We prove that X_j is fuzzy regular for an arbitrary $j \in I$.

Let $x_j \in X_j$

Let G_j be a fuzzy open set in X_j containing x_j. So that $G_j(x_j) = 1$.

Choose a point $x = (x_i)_{i \in I}$ in X where $x_j = a_j$ for $j \neq i$.

Since G_j is fuzzy open set in X_j for each $j \in I$, $G = \prod_j G_j$ is fuzzy open in X and G(x) = 1.

Since X is fuzzy regular, for each $x \in X$ and a $G \in T$ with G(x) = 1 there exists a basic fuzzy open set $V = \prod_i V_i$ with V(x) = 1 and $V \leq \overline{V} \leq G$.

If follows that $V_j(x_j) = 1$

Again $\prod_j \overline{V_j} = \overline{\prod_j V_j} = \overline{V}$

Thus $\prod_j \overline{V_j} = \overline{V} \leq G = \prod_j G_j$

That is, $\overline{V_j} \leq G_j$

Thus to each $x_j \in X_j$ and each fuzzy open set G_j in X_j with $G_j(x_j) = 1$ there exists fuzzy open set V_j in X_j with $V_j(x_j) = 1$ and $V_j \leq \overline{V_j} \leq G_j$.

Therefore X_j is fuzzy regular.

Hence the theorem.

2.3.4 Theorem Let $\{X_i : i \in I\}$ be an infinite family of fuzzy topological spaces. Then the fuzzy box product $\boxed{f}_i X_i$ is not any of the following:

(i) Locally compact, (ii) Separable, (iii) connected or locally connected (iv) first countable (v) Perfect.

Proof:

Let $\boxed{f}$ denote the fuzzy box product $\boxed{f}_i X_i$. Choose a point $p \in \boxed{f}$

That is, $p_i \in X_i$ for each $i \in I$.

Let $\prod_i G_i$ be a fuzzy open box which contains p

So that $\prod_i G_i(p) = 1$ for each $i \in I$

That is, $\inf_{i \in I} G_i(p_i) = 1$ for each $i \in I$

i.e., $G_i(p_i) = 1$ for each $i \in I$.

Choose another point $x_i \neq p_i$ in G_i, for each $i \in I$.

Since each X_i is fuzzy regular, for each $p_i \in X_i$ and $G_i \in T_i$ with $G_i(p_i) = 1$, there exists $H_i \in T_i$ with $H_i(p_i) = 1$ and $H_i \leq \overline{H_i} \leq G_i$.

Choose a set as follows.

$\{y \in \boxed{f}$ such that $y_i \in \{p_i, x_i\}$ for all $i \in I\}$ .... (1).

This is an uncountable closed set.

Now take the product as

$\mathscr{H} = \{\prod_i K_i : K_i \in \{H_i, G_i - \overline{H_i}\}$ for all $i \in I\}$

Here $\mathscr{H}$ is uncountable and pairwise disjoint.

Thus it is a fuzzy open covering of the uncountable closed set as defined in (1).

Therefore $\boxed{f}_i X_i$ is not locally compact.

(ii) $\boxed{f}_i X_i$ is not separable

$\boxed{f}_i X_i$ is separable if there exists countable sequence of points $\{p_j\}$ $j = 1, 2, \ldots$ such that for every member $G = \prod_i G_i$ of T and $G \neq \phi$, there exists a p_j such that $G_j(p_j) = 1$.

But from the above result, we have seen that

$\mathscr{H} = \{\prod_i K_i;\ K_i \in \{H_i, G_i - \overline{H_i}\}$ for all $i \in I\}$ is uncountable and pairwise disjoint. Therefore the above condition does not hold for $\mathscr{H}$.

Thus $\boxed{f}_i X_i$ is not separable.

(iii) $\boxed{f}_i X_i$ is not connected or locally connected.

Let $G = \prod_i G_i$ be a fuzzy open box which contains p.

That is, $G(p) = 1$

So $\inf_{i \in I} G_i(p_i) = 1$

$G_i(p_i) = 1$

So for each $i \in I$, there is a family $\{G_{i,n} : n \in \omega\}$ of fuzzy open sets of X_i with $G_{i,n}(p_i) = 1$

Also there exist another family $\{G_{i,n+1} : n \in \omega\}$ of fuzzy open sets of X_i with $G_{i,n+1}(p_i) = 1$ and $G_{i,n+1} \leq \overline{G_{i,n+1}} \leq G_{i,n}$ for all $n \in \omega$ & for each $i \in I$

This is possible since each X_i is fuzzy regular.

Let $\phi:\omega \rightarrow I$ be an injection.

Define a fuzzy closed box E_k^m, for k, m $\in \omega$ by

$$E_k^m(i) = \begin{cases} \overline{G_{i,n+k+1}} & \text{if } \phi(n) = i \;\&\; m < n \\ \overline{G_{i,1}} & \text{otherwise} \end{cases}$$

Let $E_k = \bigvee_{m \in \omega} E_k^m$ and

$$E = \bigwedge_{k \in \omega} E_k$$

Then we have $p \in E \leq G$.

Now $\boxed{f}$ - E is a fuzzy open set, since it is the union of $\boxed{f} - \prod_i \overline{G_{i,1}}$ with all

boxes of the shape $(X_j - \overline{G_{i,n+1}}) \times \prod_{i \neq j} X_i$ with $j \in \phi(\omega)$.

Next we prove that E is fuzzy open.

For, let $x \in E$

Define a fuzzy open box U by

$$U_i = \begin{cases} \wedge \{G_{i,n+h+1} : k \leq n, G_{i,n+k+l}(x_i) = 1\} \text{if } i = \phi(n) \;\&\; G_{i,2}(x_i) = 1 \\ G_{i,1} \text{ otherwise} \end{cases}$$

Since U_i is fuzzy open for each $i \in I$.

U is also fuzzy open and $U(x) = 1$.

So for given $k \in \omega$, $x \in E_{k+2}$.

Hence there exist $m(k) \in \omega$ such that $\phi(n) = i$ and $m(k) \leq n \Rightarrow x_i \in G_{i,n+k+3}$

Thus we have $U \leq E_k^{m(k)}$ for all $k \in \omega$ and $U \leq E$.

So E is both fuzzy open and fuzzy closed.

Hence $\boxed{f}_i X_i$ is not connected or locally connected.

(iv) We have $\prod_i G_{i,n} \leq \prod_i G_i = G$ for each $i \in I$ and $n \in \omega$

Consider $\{\prod_i G_{i,n} : n \in \omega\}$ and E as defined in the above result (iii).

Clearly $p \in E \leq G$.

Also $E \leq \prod_i G_{i,n}$ for all $n \in \omega$.

Thus every point in X can not have a countable local base. Therefore X is not first countable.

(v) $\boxed{f}_i X_i$ is not fuzzy perfect

It is sufficient to prove the result for the case $I = \omega$.

Let $G_i = X_i - \{p_i\}$ for all $i \in I$

$\boxed{f}_i X_i$ is fuzzy perfect if each fuzzy closed set is a fuzzy G_δ - set or if each fuzzy open set is a fuzzy F_σ - set.

Thus for proving our claim, we have to construct a fuzzy open set $G = \prod_i G_i$ which is not a fuzzy F_σ - set.

Suppose F_n is fuzzy closed in $\boxed{f}$ for all $n \in \omega$ and

$F_n \leq F_{n+1} \leq G$ for all $n \in \omega$.

Since $\boxed{f}$ is fuzzy regular there exists a fuzzy open set $G_{i,0}$ of each p_i with $G_{i,0}(p_i) = 1$ such that $F_0 \wedge \prod_i G_{i,0} = 0$.

Since p_i is not an isolated point of X_i for each $i \in I$, there exists $x_0 \in \boxed{f}$ such that

$$(x_0)_0 \in G_{0,0} - \{p_0\} \text{ and}$$

$$(x_0)_i = p_i \quad \text{for } i > 0$$

So by induction on $n \in \omega$, we can construct fuzzy open boxes $\prod_i G_{i,n}$ and $x_n \in \boxed{f}$ subject to the following restrictions.

i) $\prod_i G_{i,n}$ is a fuzzy neighbourhood of x_{n-1} and $\prod_i G_{i,n} \wedge F_n = 0$

ii) $\overline{G_{i,n}} \leq G_{i,n-1}$ for all $i,n \in \omega$.

iii) $(x_n)_i \in \begin{cases} G_{i,n} - \{p_i\} \text{ if } n = i \\ (x_{n-1})_i \text{ otherwise for all } i,n \in \omega \end{cases}$

If $x_i = (x_i)_i$ for all $i \in \omega$, then

$x \in (\wedge \prod_i G_{i,n}) - (\vee_n F_n)$

Hence $\boxed{f}_i X_i$ is not fuzzy perfect.

Hence the theorem

CHAPTER - 3

FUZZY UNIFORM FUZZY BOX PRODUCTS*

3.1 Introduction

The concept of fuzzy uniformity has been defined by many authors in more or less similar terms. Here we are interested in the fuzzy uniform structure $\mathscr{U}$ in the sense of Lowen $[LO]_3$.

In the second section of the chapter we give the necessary preliminary ideas like fuzzy filter, fuzzy uniform space, compatible fuzzy uniform base, fuzzy uniform fuzzy topological space etc.

In the third section we introduce the concept fuzzy uniform fuzzy box product.

In the fourth section we investigate the completeness property of fuzzy uniformities in fuzzy box products. Also we introduce the notion of fuzzy topologically complete spaces and prove the main theorem that for a family of fuzzy topologically complete spaces, their fuzzy box product is also fuzzy topologically complete.

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* Some results of this chapter were communicated to the Journal of Fuzzy Mathematics.

Some results of this chapter were presented in the Annual Conference of Kerala Mathematical Association at Payyanur, January 2004.

3.2 Preliminaries

3.2.1 Definition **[KA]₁** A fuzzy filter on X is a family $\mathscr{U}$ of non empty fuzzy subsets of $X \times X$ which satisfies the following conditions.

i) If $U, V \in \mathscr{U}$ then $U \wedge V \in \mathscr{U}$

ii) If $U \in \mathscr{U}$ and $U \leq V$ then $V \in \mathscr{U}$.

3.2.2 Definition **[P;A]** A family $\mathscr{U}$ of non empty fuzzy subsets of X x X is called a fuzzy filter base if it satisfies the condition:

if $U_1, U_2 \in \mathscr{U}$ then there exists $U_3 \in \mathscr{U}$ such that $U_3 \leq U_1 \wedge U_2$.

Let U be a fuzzy subset of X x X, U is symmetric if $U = U^{-1}$

If X is a set, the diagonal of X x X is denoted by D(X).

That is, $D(X) = \{ (x,x) : x \in X \}$.

The following definitions are from [LO]₃

3.2.3 Definition A fuzzy uniformity on a set X is a fuzzy filter $\mathscr{U}$ on. $X \times X$ which satisfies the following conditions.

(U1) For all $U \in \mathscr{U}$, $D(X) \subset U$

(U2) For all $U \in \mathscr{U}$, $U^{-1} \in \mathscr{U}$ where $U^{-1}(x,y) = U(y,x)$

(U3) For all $U \in \mathscr{U}$, there exists $V \in \mathscr{U}$ such that $V \circ V \leq U$ where

$$V \circ V(x,y) = \sup_{z \in X} \{V(x,z) \wedge V(z,y)\} \text{ for all } (x,y) \in X \times X$$

The pair $(X, \mathscr{U})$ is called a fuzzy uniform space.

3.2.4 Note

(i) (U1) is equivalent to saying that $U(x,x) = 1$ for all $U \in \mathscr{U}$

(ii) If $V \in I^{X \times X}$ and $n \in N$, we denote by V^n the fuzzy set $V^n = V \circ V^{n-1}$ inductively defined from $V^2 = V \circ V$. Clearly then (U3) is equivalent to saying that for all $n \in N$ and for all $U \in \mathscr{U}$ there exists $V \in \mathscr{U}$ such that $V^n \leq U$.

3.2.5 Definition If V is a fuzzy subset of X x X and A is a fuzzy subset of X, then the section of V over A is the fuzzy subset of X, defined by

$V\langle A\rangle(x) = \sup_{y \in X} (A(y) \wedge V(y,x))$ for all $x \in X$.

If $A = \{x\}$ we write $V\langle x\rangle$ for $V\langle A\rangle$ and call it the x-section of V.

3.2.6 Definition A fuzzy uniform base is a fuzzy filter base $\mathscr{U}$ on XxX which satisfies (U1),(U2) and (U3).

3.2.7 Definition Let $\mathscr{U}$ be a fuzzy uniform base. Then the fuzzy topology $T_{\mathscr{U}}$ induced by the fuzzy uniform base $\mathscr{U}$ is called fuzzy uniform fuzzy topology, is given by

$T_{\mathscr{U}} = \{G \in I^X$ / If $x \in X$ is such that $G(x) = 1$ then there exists $U \in \mathscr{U}$ such that $U\langle x\rangle \leq G\}$

where $U\langle x\rangle(y) = U(x, y)$ for all $y \in X$.

3.2.8 Definition If (X, T) is a fuzzy topological space and $\mathscr{U}$ is a fuzzy uniform base such that $T_{\mathscr{U}}$ = T then we say that (X, T) fuzzy uniformizable or that the fuzzy uniform base $\mathscr{U}$ on X is compatible with (X,T). Then (X, $T_{\mathscr{U}}$) is called a fuzzy uniform fuzzy topological space.

3.2.9 Definition The neighbourhoods of the diagonal in the fuzzy product. topology on X x X with respect to the fuzzy uniform topology on X are called fuzzy entourages.

3.3 Fuzzy uniformities in fuzzy box products

3.3.1 Definition. Let $\mathscr{U}_i$ be a compatible fuzzy uniform base on X_i for all $i \in I$. Let U_i be a fuzzy subset of X_i x X_i for all $i \in I$.

That is, $U_1 \times U_2 \times U_3 \ldots \leq (X_1 \times X_1) \times (X_2 \times X_2) \times \ldots\ldots\ldots\ldots$

$$\leq (X_1 \times X_2 \times \ldots\ldots) \times (X_1 \times X_2 \times \ldots\ldots\ldots)$$

$$= \Pi_i X_i \times \Pi_i X_i$$

$$= (\Pi_i X_i)^2$$

Let $x = (x_i)_{i \in I}$ and $y = (y_i)_{i \in I}$ be elements of $(\Pi_i X_i)^2$

Then

$$\boxed{f}_i U_i (x,y) = \inf_{i \in I} U_i (x_i, y_i)$$

for each $x = (x_i)_{i \in I}$ and $y = (y_i))_{i \in I}$ in $(\Pi_i X_i)^2$.

And for each $j \in I$,

$$(\boxed{f}_i U_i)(j) = U_j$$

So $\boxed{f}_i \mathscr{U}_i = \sup_{i \in \omega} \{ \boxed{f}_i U_i \}$

Here $\boxed{f}_i \mathscr{U}_i$ is the fuzzy uniform fuzzy box product of $\{ \mathscr{U}_i : i \in I \}$

3.3.2 Theorem Let $\mathscr{U}_i$ be a compatible fuzzy uniform base on X_i, for each $i \in I$. Then $\mathscr{U} = \boxed{f}_i \mathscr{U}_i$ is a compatible fuzzy uniform base on $X = \boxed{f}_i X_i$.

Proof:

Given that $\mathscr{U}_i$ is a compatible fuzzy uniform base on X_i, for each $i \in I$. This means that $\mathscr{U}_i$ is a fuzzy filter base on X_i x X_i which satisfies the conditions (U1),(U2) and (U3) of definition 3.2.3 and $T_{\mathscr{U}_i} = T_i$ for all $i \in I$. Now we can verify that

$\mathscr{U} = \boxed{f}_i \mathscr{U}_i$ is a compatible fuzzy uniform base on $X = \boxed{f}_i X_i$.

(U1) Let $U = \boxed{f}_i U_i \in \mathscr{U} = \boxed{f}_i \mathscr{U}_i$ where $U_i \in \mathscr{U}_i$ for all $i \in I$.

$$\begin{aligned} U(x,x) &= \boxed{f}_i U_i(x, x) \\ &= \inf_{i \in I} U_i(x_i, x_i) \\ &= 1 \qquad \text{for all } U \in \mathscr{U} \end{aligned}$$

(U2) Let $U = \boxed{f}_i U_i$ and $\mathscr{U} = \boxed{f}_i \mathscr{U}_i$ where $U_i \in \mathscr{U}_i$ for all $i \in I$.

$$U(x,y) = \boxed{f}_i U_i (x, y) \in \mathscr{U} \text{ for all } U \in \mathscr{U}.$$

$$= \inf_{i \in I} U_i (x_i, y_i)$$

$$= \inf_{i \in I} U_i^{-1} (y_i, x_i)$$

$$= (\inf_{i \in I} U_i (y_i, x_i))^{-1}$$

$$= [\boxed{f}_i U_i (y, x)]^{-1}$$

$$= [U (y, x)]^{-1}$$

$$= U^{-1} (x, y) \in \mathscr{U} .$$

That is, for all $U \in \mathscr{U}$, $U^{-1} \in \mathscr{U}$.

(U3) Let $U = \boxed{f}_i U_i \in \mathscr{U}$ and $V = \boxed{f}_i V_i \in \mathscr{U}$ where $U_i, V_i \in \mathscr{U}_i$ for all $i \in I$

Consider

$$(V \circ V)(x,y) = \sup_{z \in X} \{V(x,z) \wedge V(z,y)\}$$

$$= \sup_{z \in X} \{\boxed{f}_i V_i (x, z) \wedge \boxed{f}_i V_i (z, y)\}$$

$$= \sup_{z \in X} \{\inf_{i \in I} V_i (x_i, z_i) \wedge \inf_{i \in I} V_i (z_i, y_i)\}$$

$$\leq \inf_{i \in I} \{\sup_{i \in I} (V_i (x_i, z_i) \wedge V_i (z_i, y_i))\}$$

$\leq \inf_{i \in I} U_i\ (x_i, y_i)$

$\leq \boxed{f}_i U_i\ (x, y) = U\,(x, y)$

That is, $V \circ V \leq U$ for all $U \in \mathscr{U}$

Therefore $U = \boxed{f}_i U_i$ satisfies the conditions (U1),(U2) and (U3).

Hence $\mathscr{U} = \boxed{f}_i \mathscr{U}_i$ is a fuzzy uniform base on $X = \boxed{f}_i X_i$

Next we prove that $\mathscr{U}$ is compatible.

That is, to prove that $T_{\mathscr{U}} = T$.

Since $\mathscr{U}_i$ is compatible, we have $T_{\mathscr{U}_i} = T_i$ for all $i \in I$.

where $T_{\mathscr{U}_i} = \{G_i \in I^{X_i}$ / If $x_i \in X_i$ is such that $G_i\,(x_i) = 1$ then

there exists $U_i \in \mathscr{U}_i$ s.t. $U_i\,{<}x_i{>} \leq G_i\}$ for all $i \in I$.

We have $X = \boxed{f}_i X_i$ & $U = \boxed{f}_i U_i$

Now $U{<}x{>}(y) = U(x,y)$ for all $y \in X$

That is, $U\,(x,y) = \boxed{f}_i U_i\,(x,y)$

$= \inf_{i \in I} U_i\ (x_i, y_i)$

$\leq G_i\,(y_i)$ for all $i \in I$.

$\leq G\,(y)$

Therefore $U{<}x{>} \leq G$ and

$T_{\mathscr{U}} = \{G \in I^X$ / If $x \in X$ is such that $G(x) = 1$ then there exists $U \in \mathscr{U}$

such that $U{<}x{>} \leq G\}$

Thus $T_{\mathscr{U}} = T$ holds.

Therefore $\mathscr{U} = \boxed{f}_i \mathscr{U}_i$ is compatible and it is a fuzzy uniform base on $X = \boxed{f}_i X_i$.

Hence the theorem.

3.4 Fuzzy Topologically Complete Spaces

The following concepts are available in literature.

3.4.1 Definition Let $\mathscr{U}$ be a compatible fuzzy uniform base on (X,T). A fuzzy filter $\mathscr{F}$ in a fuzzy uniform space (X, $\mathscr{U}$) is said to be Cauchy if $U \in \mathscr{U} \Rightarrow$ there exists $x \in X$ with $U<x> \in \mathscr{F}$.

3.4.2 Definition A fuzzy filter is convergent if it contains a fuzzy neighbourhood base at some point.

3.4.3 Definition A fuzzy uniform space (X, $\mathscr{U}$) is said to be complete if every Cauchy fuzzy filter converges.

3.4.4 Definition A fuzzy topological space (X,T) is said to be fuzzy topologically complete if there exists a fuzzy uniformity $\mathscr{U}$ for X such that (X, $\mathscr{U}$) is complete and $T_{\mathscr{U}} = T$.

3.4.5 Theorem If X_i is fuzzy topologically complete for each $i \in I$, then $\boxed{f}_i X_i$ is fuzzy topologically complete.

Proof:

Since X_i is fuzzy topologically complete, it possess a compatible complete fuzzy uniform base $\mathscr{U}_i$, for each $i \in I$. But we proved (in theorem 3.3.2) that $\mathscr{U} = \boxed{f}_i \mathscr{U}_i$ is a compatible fuzzy uniform base on $X = \boxed{f}_i X_i$. So it is enough to prove that $\mathscr{U} = \boxed{f}_i \mathscr{U}_i$ is complete.

Suppose $\mathscr{F}$ is a $\boxed{f}_i \mathscr{U}_i$ Cauchy fuzzy filter on $\boxed{f}_i X_i$.

Define

$$\mathscr{F}_i = \{ F \subseteq X_i : \Pi_i^{-1}(F) \in \mathscr{F} \}$$

That is, $\Pi_i^{-1}(F)(x) = F(\Pi_i(x)) = F(x_i) \in \mathscr{F}_i$, for all $i \in I$.

Now $\mathscr{F}_i$ is a $\mathscr{U}_i$ - Cauchy fuzzy filter on X_i for each $i \in I$.

For,

$U_i \in \mathscr{U}_i \Rightarrow$ there exists $x_i \in X_i$ with $U_i\langle x_i\rangle \in \mathscr{F}_i$

where $U_i \langle x_i \rangle (y_i) = U_i(x_i, y_i)$ for each $i \in I$.

Since $\mathscr{U}_i$ is complete, every Cauchy fuzzy filter converges.

Let $x \in \Pi_i X_i$

Assume that $\mathscr{F}_i$ converges to x_i for all $i \in I$. By definition, for $U = \boxed{f}_i U_i \in \boxed{f}_i \mathscr{U}_i$ there is a symmetric $V = \boxed{f}_i V_i \in \boxed{f}_i \mathscr{U}_i$ such that $VoVoV \leq U$

That is, $(V \circ V \circ V)(p, q) \leq U(p, q)$

where $(V \circ V \circ V)(p, q) = \sup_{r,s \in v} \{V(p,r) \wedge V(r,s) \wedge V(s,q)\}$

Since $\mathscr{F}$ is a $\boxed{f}_i\, \mathscr{U}_i$ Cauchy fuzzy filter there exists $y \in \prod_i X_i$

such that $V\langle y \rangle \in \mathscr{F}$, where $V\langle y \rangle(x) = V(y,x)$ for all $x \in X$

$$
\begin{aligned}
\text{Therefore } V(y,x) &= \boxed{f}_i\, V_i(y, x) \\
&= \inf_{i \in I} V_i(y_i, x_i)
\end{aligned}
$$

where $x = (x_i)i \in I$ & $y = (y_i)i \in I$ in $(\prod i\, Xi)^2$

$\in \mathscr{F}$.

Also, for $U = \boxed{f}_i\, U_i \in \boxed{f}_i\, \mathscr{U}_i$, there exists $x \in \prod_i X_i$

such that $U\langle x \rangle \in \mathscr{F}$, where $U(x, y) = \inf_{i \in I} U_i(x_i, y_i)$

Thus we get, $V\langle y \rangle \leq U\langle x \rangle$

Therefore $\mathscr{F}$ converges to x.

Thus $\boxed{f}_i\, \mathscr{U}_i$ is complete.

Hence the theorem.

CHAPTER - 4

FUZZY α-PARACOMPACTNESS IN FUZZY BOX PRODUCTS*

4.1 Introduction

The notion of shading family was introduced in the literature by T.E. Gantner and others in [G;S;W] during the investigation of compactness in fuzzy topological spaces. The shading families are a very natural generalization of coverings. An approach to fuzzy α-paracompactness using the notion of shading families was introduced by S.R. Malghan and S.S. Benchalli in $[M;B]_1$.

The second section of this chapter describes the necessary definitions and results of shading families.

In the third section, we introduce and study the notion of fuzzy α-paracompactness in fuzzy box products. Here we give a characterization of fuzzy α-paracompactness through fuzzy entourages.

In the last section we introduce fuzzy α-paracompact fuzzy topologically complete spaces. Here we have the main theorem that for a family of fuzzy α-paracompact spaces, their fuzzy box product is fuzzy topologically complete.

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* Some results of this Chapter were communicated to the Journal of Fuzzy Mathematics.

4.2 Shading families

The following definitions and results are from [M;B]$_1$

4.2.1 Definition Let (X,T) be a fuzzy topological space and $\alpha \in [0,1)$. A collection $\mathscr{U}$ of fuzzy sets is called an α- shading of X if for each $x \in X$ there exists $g \in \mathscr{U}$ with $g(x) > \alpha$. A subcollection of an α - shading of X which is also an α – shading is called an α – subshading of X.

4.2.2 Definition Let X be a set . Let $\mathscr{U}$ and $\mathscr{V}$ be any two collections of fuzzy subsets of X.. Then $\mathscr{U}$ is a refinement of $\mathscr{V}$ ($\mathscr{U} < \mathscr{V}$) if for each $g \in \mathscr{U}$ there is an $h \in \mathscr{V}$ such that $g \leq h$.

If $\mathscr{U}$, $\mathscr{V}$, $\mathscr{W}$ are collections such that $\mathscr{U} < \mathscr{V}$ and $\mathscr{U} < \mathscr{W}$ then $\mathscr{U}$ is called a common refinement of $\mathscr{V}$ and $\mathscr{W}$.

4.2.3 Definition A family $\{a_s : s \in S\}$ of fuzzy sets in a fuzzy topological space (X, T) is said to be locally finite if for each $x \in X$ there exists a fuzzy open set g with $g(x) = 1$ such that $a_s \leq 1\text{-}g$ holds for all but atmost finitely many $s \in S$.

4.2.4 Definition A family $\{a_s : s \in S\}$ of fuzzy sets in a fuzzy topological space (X, T) is said to be σ-locally finite if it is the union of countably many locally finite sets.

4.2.5 Theorem Let $\{a_s\}$ and $\{b_t\}$ be two α–shadings of a fuzzy topological space (X, T), where $\alpha \in [0,1)$. Then

i) $\{a_s \wedge b_t\}$ is an α – shading of X which refines both $\{a_s\}$ and $\{b_t\}$..
Further if both $\{a_s\}$and $\{b_t\}$are locally finite so is $\{a_s \wedge b_t\}$.

ii) Any common refinement of $\{a_s\}$ and $\{b_t\}$ is also a refinement of $\{a_s \wedge b_t\}$.

4.2.6 Theorem Let $\{a_s : s \in S\}$ be a locally finite family of fuzzy sets in a fuzzy topological space (X,T) then

i) $\{\overline{a_s} : s \in S\}$ is also locally finite.

ii) for each $S' \subset S$, $\vee\{\overline{a_s} : s \in S'\}$is a fuzzy closed set.

4.3 A Characterization of fuzzy α-paracompactness

4.3.1 Definition A fuzzy topological space (X, T) is said to be α-paracompact if each α - shading of X by fuzzy open sets has a locally finite α–shading refinement by fuzzy open sets.

We quote the following theorem from [SU]

4.3.2 Theorem For a fuzzy regular space the following are equivalent

1) X is. α-paracompact.

2) Every α-shading of X by fuzzy open sets has a σ- locally finite α - shading refinement by fuzzy open sets.

3) Every α - shading of X by fuzzy open sets has a locally finite α - shading refinement by fuzzy open sets.

4) Every α - shading of X by fuzzy open sets has a locally finite α - shading refinement by fuzzy closed sets.

We prove the following theorem.

4.3.3 Theorem For a fuzzy regular space X, X is α-paracompact if and only if (*) every. α - shading $\mathscr{U}$ of X by fuzzy open sets is refined by a fuzzy entourage D.

Remark: We say that D refines $\mathscr{U}$ for some $D \subset X x X$ if

$\mathscr{S} = \{D\langle x\rangle : x \in X\}$ refines $\mathscr{U}$. In particular, this gives a refinement by fuzzy entourages.

Proof of the above theorem

We first prove that (4) in theorem (4.3.2) implies (*)

Let $\mathscr{U}$ be an α – shading of X by fuzzy open sets. So for each $x \in X$ there exists $U_\beta \in \mathscr{U}$ such that $U_\beta(x) > \alpha$.

Let $\mathscr{V} = \{V_\beta : \beta \in \wedge\}$ be a locally finite α – shading refinement by closed sets. For each $\beta \in \wedge$ and $V_\beta < U_\beta,$

let $W_\beta = (U_\beta \times U_\beta) \cup (V'_\beta \times V'_\beta)$

Now W_β is a fuzzy open neighbourhood of the diagonal in $X \times X$.

Let $V = \inf \{W_\beta : \beta \in \wedge\}$

So $V<x> \leq W_\beta<x>$ for each $x \in X$.

Therefore $\{V<x> : x \in X\}$ is a refinement of $\mathscr{U}$.

Next we prove that V is a fuzzy neighbourhood of the diagonal.

For each point of the diagonal we choose a fuzzy open set g of x with

$g(x) = 1$ and $V_\beta \leq 1-g$ holds for all but atmost finitely many $\beta \in \wedge$.

If $g \wedge V_\beta = 0$ then $g \leq 1 - V_\beta$

That is $g \times g \leq W_\beta$

But $V = \inf \{W_\beta : \beta \in \wedge\}$.

This means that V is a fuzzy neighbourhood of the diagonal.

Before proving (*) implies (1) of theorem 4.3.2, we prove a lemma

4.3.4 Lemma Let X be a fuzzy topological space such that each α - shading of X by fuzzy open sets is refined by a fuzzy entourage and let $\mathscr{A} = \{a_s : s \in S\}$ be a locally finite family of fuzzy subsets of X. Then there is a neighbourhood V of the diagonal in $X \times X$ such that the family of all sets $V<a_s>$ for $s \in S$ is locally finite.

Proof

Let $\mathscr{U}$ be an α – shading of X by fuzzy open sets. That is, for each $x \in X$ there exists a fuzzy open set $U_\beta \in \mathscr{U}$ such that $U_\beta(x) > \alpha$.

Since $\{a_s : s \in S\}$ is locally finite, for each $x \in X$ there exists a fuzzy open set g with $g(x) = 1$ and $a_s \leq 1 - g$ holds for all but atmost finitely many $s \in S$.

Let U be neighbourhood of the diagonal such that $\{U<x>: x \in X\}$ refines $\mathscr{U}$. Then there exists a symmetric neighbourhood V of the diagonal such that

$$V \circ V \leq U, \text{where } V = V^{-1}.$$

If $V \circ V<x> \wedge a_s = 0$ then $V<x> \wedge V<a_s> = 0$

For,

If $(y_\alpha)_{\alpha > 0} \in V<x> \wedge V<a_s>$ then $y_\alpha \in V<x>$ and $y_\alpha \in V<a_s>$ where $\alpha > 0$.

That is, $V(x, y) = \alpha$ and $V<a_s>(y) = \alpha$ where $\alpha > 0$.

Now $V<a_s>(y) = \sup_{z \in X} (a_s(z) \wedge V(z,y)) = \alpha$

Therefore given $\varepsilon > 0$, there exists $z \in X$ such that

$$a_s(z) \wedge V(z,y) > \alpha - \varepsilon$$

That is, $a_s(z) > \alpha - \varepsilon$ and $V(z,y) > \alpha - \varepsilon$.

So $V \circ V(x,z) = \sup_{y \in X} \{V(x,y) \wedge V(y,z)\} > \alpha - \varepsilon$

$\therefore V \circ V(x, z) \wedge a_s(z) > \alpha - \varepsilon$

Which is a contradiction.

Therefore the family of all sets $V\langle a_s \rangle$ for $s \in S$ is locally finite.

Hence the lemma.

We prove (*) implies (1) of theorem 4.3.2.

Let $\mathcal{U}$ be an α – shading of X by fuzzy open sets.

Therefore for each $x \in X$ there exists $U_\beta \in \mathcal{U}$ such that $U_\beta(x) > \alpha$.

By (*) there exists a fuzzy neighbourhood V of the diagonal which refines $\mathcal{U}$.

That is $\{V\langle x \rangle : x \in X\}$ refines $\mathcal{U}$.

That is $V\langle x \rangle \leq U_\beta$ where $U_\beta \in \mathcal{U}$.

Let $\{a_s : s \in S\}$ be a locally finite family of fuzzy subsets of X. Then by above lemma there exists a neighbourhood V of the diagonal in XxX such that $\{V\langle a_s \rangle : s \in S\}$ is locally finite,.

where $V\langle a_s \rangle(y) = \sup_{x \in X} (a_s(x) \wedge V(x,y))$ for all $y \in X$

So for each $s \in S$, choose a fuzzy open set $U_\beta \in \mathcal{U}$ such that $a_s \leq U_\beta$.

Let $W_\beta = U_\beta \wedge V\langle a_s \rangle$.

Therefore W_β is a locally finite α – shading refinement of $\mathscr{U}$.

Hence X is fuzzy α-paracompact.

4.4. Fuzzy α-paracompact fuzzy topologically complete spaces

We first prove a lemma.

4.4.1 Lemma If X is a fuzzy α-paracompact space, then the fuzzy filter of entourages of X is a complete fuzzy uniformity compatible with X.

Proof

Let $\mathscr{N}$ be the fuzzy filter of entourages of X. We prove that $\mathscr{N}$ is a fuzzy uniformity.

Let $D \in \mathscr{N}$. For each $x \in X$ choose an α – shading U_x of x by fuzzy open sets with $U_x(x) > \alpha$ and $U_x \times U_x \leq D$. By theorem 4.3.3, every α – shading of X by fuzzy open sets is refined by a fuzzy entourage.

That is, there exists $E \in \mathscr{N}$ which refines $\mathscr{U} = \{U_{x:}\, x \in X\}$.

Let $D = E \wedge E^{-1}$

So we have.

i) $D(x, x) = 1$

ii) $D \in \mathscr{N} \Rightarrow D^{-1} \in \mathscr{N}$

iii) Let $(x, y) \in E$ and $(y, z) \in E$

Consider $E \circ E\,(x, z) = \sup_{y \in X} \{E(x,y) \wedge E(y,z)\} \leq D(x,z)$.

That is, for $D \in \mathcal{N}$, there exists $E \in \mathcal{N}$ such that $E \circ E \leq D$.

Now $D \in \mathcal{N}$ is a fuzzy open subset of $X \times X$ and $D\langle x\rangle$ is a fuzzy open subset of X for every point $x \in X$.

Again, if given a fuzzy open set G of y in X with $G(y) = 1$ for all $y \in X$, then there exists $F \in \mathcal{N}$ such that $F\langle y\rangle \leq G$,

Where $F = (G \times G) \cup ((X - \{y\}) \times (X - \{y\}))$.

That is $F\langle y\rangle(x) \leq G(x)$ for all $x \in X$.

Therefore $T_{\mathcal{N}} = \{G \in I^X$ / If $y \in X$ is such that $G(y) = 1$ then there exists $F \in \mathcal{N}$ such that $F\langle y\rangle \leq G\}$.

Thus $\mathcal{N}$ is compatible with X.

Claim $\mathcal{N}$ is complete.

We have to prove that every $\mathcal{N}$ - Cauchy fuzzy filter is convergent.

It is enough to prove that a non – convergent fuzzy filter is not $\mathcal{N}$ - Cauchy.

Suppose $\mathcal{F}$ is a non-convergent fuzzy filter on X. Then for each $y \in X$ there exists an α – shading U_y of y with $U_y(y) > \alpha$ and $U_y \notin \mathcal{F}$.

But by theorem 4.3.3, every α - shading of X by fuzzy open sets is refined by fuzzy entourages.

That is there exists $D \in \mathscr{N}$ which refines $\mathscr{U} = \{U_x : x \in X\}$.

But this is not possible by our above argument. Therefore $\mathscr{F}$ is not $\mathscr{N}$-Cauchy.

Thus $\mathscr{N}$ is complete. Hence the theorem.

4.4.2 Corollary Each fuzzy α-paracompact space is fuzzy topologically complete.

4.4.3 Theorem Suppose that $\{X_i : i \in I\}$ be a family of fuzzy α-paracompact spaces. Then $\boxed{f}_i X_i$ is fuzzy topologically complete.

Proof

Proof follows from lemma 4.4.1 and theorem 3.4.5.

CHAPTER - 5

HEREDITARILY FUZZY NORMAL SPACES*

5.1 Introduction

In this chapter we introduce the concept hereditarily fuzzy normal spaces.

Katetov in 1948 proved the following theorem in the crisp case "If X x Y is hereditarily normal, then either X is perfectly normal or every countable subset of Y is closed and discrete.

Here we obtain the fuzzy analogue of the above theorem. We also prove that the above result holds for fuzzy box product of hereditarily fuzzy normal spaces. So we have the main theorem that if a fuzzy box product of spaces is hereditarily fuzzy normal then every countable subset of it is fuzzy closed.

5.2 Preliminaries

5.2.1 Definition[A;P] Let A be a fuzzy set in a fuzzy topological space (X,T). Let x_α be any fuzzy point in X with support x (where $0< \alpha \leq 1$). Then x_α is a fuzzy accumulation point of A if every fuzzy open set B containing x_α contains a fuzzy point of A with support different from x.

* Some results of this chapter were communicated to Indian Journal of Mathematics, Allahabad Mathematical Society.

Let A be a fuzzy set in a fuzzy topological space (X,T). Let p_n, $n = 1,2, \ldots$ be a sequence of fuzzy points in a fuzzy topological space (X,T) with support x_n, $n = 1, 2, \ldots\ldots$ Then p is a fuzzy accumulation point of A if for every member A of T such that $p \in A$, there exists a number m such that $p_n \in A$ for all $n \geq m$.

A fuzzy subset A of a fuzzy topological space (X,T) is said to be discrete if it has no fuzzy accumulation point in X.

5.2.2 Definition [M;B]$_1$ A fuzzy topological space (X,T) is called fuzzy normal if for any two fuzzy closed sets C and D in X such that $C \leq 1\text{-}D$, there exists two fuzzy open sets U and V such that $C \leq U$, $D \leq V$ and $U \leq 1\text{-}V$.

Associated with a given fuzzy topological space (X,T) and an ordinary subset F of X, (F,T_F) is called a fuzzy subspace of (X,T) where $T_F = \{F \cap A \,/\, A \in T\}$.

A fuzzy subspace (F,T_F) of a fuzzy topological space (X,T) is called a fuzzy open (fuzzy closed) subspace if and only if the basis set F is T-fuzzy open (T-fuzzy closed).

A property P in a fuzzy topological space (X,T) is said to be hereditary if it is satisfied by each subset of X.

5.2.3 Definition A fuzzy topological space (X, T) is perfect if each fuzzy closed set is a fuzzy G_δ - set.

5.2.4 Definition A fuzzy topological space (X, T) is perfectly fuzzy normal if it is perfect and fuzzy normal.

Note : All spaces under consideration are assumed to be fuzzy T_1.

5.3 Hereditarily fuzzy normal spaces

Here we prove the fuzzy analogue of Katetov's theorem which is used in the main theorem.

5.3.1 Theorem Suppose X and Y are fuzzy topological spaces. If $X \times Y$ is hereditarily fuzzy normal then either X is perfectly fuzzy normal or every fuzzy subset of Y whose support is countable is fuzzy closed and discrete.

Proof

Suppose F is a fuzzy closed subset of X which is not a G_δ set.

Let D be a fuzzy subset of Y whose support is $\{d_n / n < \omega\}$ (which is countable). Consider the sequence $(d_n)_\alpha$ of fuzzy points in Y with accumulation point $\{y_\alpha\}$ for given $\alpha \in (0,1]$.

Assume that $y_\alpha \notin D$

That is, $D(y) < \alpha$

We prove that the fuzzy open set

$G = (X \times Y) \setminus (F \times \{y_\alpha\})$ of $X \times Y$ is not fuzzy normal.

Assume if possible that G is fuzzy normal

Consider two fuzzy subsets $K = F \times (Y - \{y_\alpha\})$

$L = (X \setminus F) \times \{y_\alpha\}$

Here $K \leq 1 - L$ and they are fuzzy closed subsets of G.

Suppose that there exists fuzzy open sets U and V of $X \times Y$ such that

$K \leq U, L \leq V$ and $U \leq 1-V$

Let $U_n(x) = U(x, d_n), \forall x \in X$ and $\forall n < \omega$.

We have $F(x) \leq \inf_{n<\omega} U_n(x)$ ....(1)

If $x \in \inf_{n<\omega} U_n$ then $x \in U_n$ for each n.

Therefore $(x, y_\alpha) \in \overline{U}$

That is, $x \in F$

Therefore $\inf_{n<\omega} U_n \leq F$ ....(2)

From (1) and (2) we have $F = \inf_{n<\omega} U_n$

Thus we get F as a fuzzy G_δ set which is a contradiction to our first assumption. This is due to the assumption that G is fuzzy normal. Therefore G is not fuzzy normal. Thus X is not perfectly fuzzy normal.

Hence every fuzzy subset of Y whose support is countable is fuzzy closed and discrete.

5.3.2 Theorem Suppose $\{X_i : i \in \omega\}$ is a family of fuzzy topological spaces such that $\boxed{f}_i X_i$ is hereditarily fuzzy normal. Then every countable subset of $\boxed{f}_{i<\omega} X_i$ is fuzzy closed.

Proof

Fix $j \in \omega$

Set $\boxed{f}' = \boxed{f}_{i \neq j} X_i$

Let $\boxed{f}_{i<\omega} X_i = \boxed{f}' \times X_j$

Applying the theorem 5.3.1 to $\boxed{f}_{i<\omega} X_i$

Then

$\boxed{f}'$ is not fuzzy perfect due to the theorem 2.3.4.

Thus every fuzzy subset of X_j whose support is countable is fuzzy closed and discrete. Hence every countable subset of $\boxed{f}_{i<\omega} X_i$ is fuzzy closed. Hence the theorem.

CHAPTER - 6

FUZZY NABLA PRODUCT*

6.1 Introduction

It is known that in the crisp situation the nabla product is the quotient of box product [cf. WIL]. In the second chapter we have already introduced fuzzy box product and its properties. In an analogous way we consider a certain quotient of fuzzy box product call it the fuzzy nabla product.

Here we introduce the concept fuzzy nabla product in the third section and investigate the relation connecting fuzzy box product and fuzzy nabla product.

In the third section of chapter 3, we have already defined the concept of fuzzy uniform fuzzy box product by making use of fuzzy uniformity of Lowen. Similar to this concept here we introduce fuzzy uniform fuzzy nabla product in the fourth section of this chapter.

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* Some results of this chapter were communicated to Iranian Journal of Fuzzy Systems.

6.2 Preliminaries

Here we reproduce the definition of fuzzy box product and fuzzy uniform fuzzy box product for countable products.

6.2.1 Definition Let X_i be a fuzzy topological space for each $i \in \omega$, where ω is the countably infinite set of all natural numbers. Then the fuzzy box product is the set $\prod_i X_i$ with the fuzzy topology whose basis consists of all fuzzy open boxes of the form

$$U = \prod_i U_i \text{ where } U_i \text{ is fuzzy open in } X_i \text{ for each } i \in \omega.$$

That is, $U(x) = \prod_i U_i\,(x)$

$$= \inf_{i \in \omega} U_i\,(x_i) \text{ for each } x = (x_i)_{i \in \omega} \text{ in } \prod_i X_i.$$

This space is denoted by $\boxed{f}_{i \in \omega} X_i$

6.2.2 Definition Let $\mathscr{U}_i$ be a compatible fuzzy uniform base on X_i for each $i \in \omega$. Let U_i be a fuzzy subset of X_i x X_i for each $i \in \omega$.

Then

$$\boxed{f}_i U_i\,(x,y) = \inf_{i \in \omega} U_i\,(x_i,y_i)$$

for each $x = (x_i)_{i \in \omega}$ and $y = (y_i)_{i \in \omega}$ in $(\prod_i X_i)^2$.

And for each $j \in \omega$,

$$(\boxed{f}_i U_i)\,(j) = U_j$$

So $\boxed{f}_i \mathscr{U}_i = \sup_{i \in \omega} \{\boxed{f}_i U_i\}$

Here $\boxed{f}_i \mathcal{U}_i$ is the fuzzy uniform fuzzy box product of $\{\mathcal{U}_i : i \in \omega\}$

6.3 Fuzzy nabla product

Here we consider in an analogous way a certain quotient of fuzzy box product and call it the fuzzy nabla product.

6.3.1 Definition Let $\{X_i : i \in \omega\}$ be a family of fuzzy topological spaces. Define an equivalence relation $\sim$ on $X = \prod_i X_i$ by $x \sim y$ if and only if

$\{i : x_i \neq y_i\}$ is finite .

Consider the set $X = \prod_i X_i$ with the fuzzy box topology F. We consider the largest fuzzy topology on $X/\sim$ such that the projection map

X onto $X/\sim$ is fuzzy continuous. That is, consider the family $\tilde{F}$ of fuzzy sets

in $X/\sim$ defined by $\tilde{F} = \{B / q^{-1}(B) \in F\}$;

where $q: X \to X/\sim$ is the projection map. Then $\tilde{F}$ is a fuzzy topology called the quotient topology on $X/\sim$ and is the largest fuzzy topology on $X/\sim$ such that the projection map q from X to $X/\sim$ is fuzzy continuous. This quotient fuzzy topological spaces $(X/\sim, \tilde{F})$ is called the fuzzy nabla product of the given family.

Notation This fuzzy nabla product is denoted by $\boxed{f}_i X_i/\sim$ or $\nabla^{f}_{i \in \omega} X_i$.

If $X_i = X$ for all $i \in \omega$ then $\nabla_f{}_i X_i$ is denoted as $\nabla_f{}^{\omega} X$.

6.3.2 Note:

Let $A = \prod_i X_i$.

$$\text{Denote } \nabla_f{}_{i\in\omega} A_i = q(\textstyle\prod_i A_i) = q(A) = \tilde{A}$$

$$\begin{aligned} \text{Then } \nabla_f{}_{i\in\omega} A_i\,[x] &= \tilde{A}\,[x] \\ &= \sup\{A(z) \;/\; q(z) = [x]\} \\ &= \sup\{\textstyle\prod_i A_i \;/\; q(z) = [x]\} \\ &= \sup\{\inf_{i\in\omega} A_i(z_i) \;/\; q(z_i) = [x_i]\} \end{aligned}$$

Therefore $\{\nabla_f{}_{i\in\omega} A_i$:where A_i is fuzzy open in X_i for each $i \in \omega\}$ is a base for the fuzzy nabla product.

6.4 Fuzzy Uniform Fuzzy Nabla Product

6.4.1 Definition Let $\mathcal{U}_i$ be a compatible fuzzy uniform base on X_i for each $i \in \omega$. Let U be a fuzzy subset of $X \times X$. Define

$$\begin{aligned} q(U)([x],[y]) &= \tilde{U}([x],[y]) \\ &= \sup\{U(a,b) \;/\; q(a,b) = ([x],[y])\} \\ &= \sup\{\inf_{i\in\omega} U_i(a_i,b_i) \;/ q(a_i,b_i) = ([x_i],[y_i])\} \end{aligned}$$

$$\text{Therefore } \nabla_f{}_{i\in\omega} U_i([x],[y] = \sup\{\inf_{i\in\omega} U_i(a_i,b_i) \;/ q(a_i,b_i) = ([x_i],[y_i])\}$$

$$\text{So } \nabla_f{}_i \mathcal{U}_i = \sup_{i\in\omega}\{\nabla_f{}_i U_i\}$$

6.4.2 Theorem Suppose that $\mathcal{U}_i$ is a compatible fuzzy uniform base on X_i for each $i \in \omega$. Then $\tilde{\mathcal{U}} = \nabla\!\!\!\!f_i\, \mathcal{U}_i$ is a fuzzy uniform base compatible with $\nabla\!\!\!\!f_i X_i$.

Proof

$$\text{(U1)}\quad \text{Let } U = \boxed{f}_i U_i \in \mathcal{U} = \boxed{f}\, i\, \mathcal{U}_i$$

$$\tilde{U} = \nabla\!\!\!\!f_i U_i \in \tilde{\mathcal{U}} = \nabla\!\!\!\!f\, i\, \mathcal{U}_i \quad \text{for each } i \in \omega$$

$$q(U)([x],[x]) = \tilde{U}([x],[x])$$

$$= 1 \quad \text{for all } \tilde{U} \in \tilde{\mathcal{U}}$$

$$\text{(U2)}\quad \text{Let } \tilde{U} = \nabla\!\!\!\!f_i U_i \in \nabla\!\!\!\!f_i\, \mathcal{U}_i \text{ where } U_i \in \mathcal{U}_i \text{ for all } i \in \omega.$$

$$\tilde{U}([x],[y]) = \nabla\!\!\!\!f_{i \in \omega} U_i([x],[y])$$

$$= \sup \{ \inf_{i \in \omega} U_i (a_i,b_i) \,/\, q(a_i,b_i) = ([x_i],[y_i]) \}$$

$$= \sup \{ \inf_{i \in \omega} U_i^{-1}(b_i ,a_i)/q(b_i,a_i) = ([y_i],[x_i]) \}$$

$$= \{ \sup \{ \inf_{i \in \omega} U_i (b_i ,a_i)/q(b_i,a_i) = ([y_i],[x_i]) \} \}^{-1}$$

$$= \{ \nabla\!\!\!\!f_i U_i([y],[x]) \}^{-1}$$

$$= \{ \tilde{U}([y],[x]) \}^{-1}$$

$$= \tilde{U}^{-1}([y],[x]) \in \mathcal{U} \quad \text{for all } \tilde{U} \in \tilde{\mathcal{U}}$$

$$\text{(U3)}\quad \text{Let } \tilde{U} = \nabla\!\!\!\!f_i U_i \in \tilde{\mathcal{U}} = \nabla\!\!\!\!f_i\, \mathcal{U}_i$$

$$\tilde{V} = \nabla\!\!\!\!f_i V_i \in \nabla\!\!\!\!f_i\, \mathcal{U}_i \text{ where } U_i, V_i \in \mathcal{U}_i \text{ for all } i \in \omega.$$

$$(\tilde{V} \circ \tilde{V})([x],[y]) = \sup_{[z]\in X/\sim} (\tilde{V}([x],[z]) \wedge \tilde{V}([z],[y]))$$

$$= \sup_{[z]\in X/\sim} (\nabla\!\!\!/_f{}_i V_i([x],[z]) \wedge \nabla\!\!\!/_f{}_i V_i([z],[y]))$$

$$= \sup_{[c_i]\in X_i/\sim} \{\sup(\inf_{i\in\omega} V_i(a_i,c_i) / q(a_i,c_i) = ([x_i],[z_i]))$$

$$\wedge \sup(\inf_{i\in\omega} V_i(c_i,b_i)/q(c_i,b_i) = ([z_i],[y_i]))\}$$

$$\leq \sup_{q(a_i,b_i)=([x_i],[y_i])} \inf_{i\in\omega} \{ \sup_{[c_i]\in X_i/\sim} V_i(a_i,c_i) \wedge V_i(c_i,b_i)\}$$

$$\leq \sup_{q(a_i,b_i)=([x_i],[y_i])} \inf_{i\in\omega} (V_i \circ V_i)(a_i,b_i)$$

$$\leq \nabla\!\!\!/_f{}_i U_i([x],[y])$$

$$\leq \tilde{U}([x],[y])$$

Therefore $\tilde{V} \circ \tilde{V} \leq \tilde{U}$ for all $\tilde{U} \in \tilde{\mathscr{U}}$

Therefore $\tilde{\mathscr{U}} = \nabla\!\!\!/_f\, i\, \mathscr{U}_i$ is a fuzzy uniform base on $\nabla\!\!\!/_f{}_i X_i$.

Next we prove that $\tilde{\mathscr{U}}$ is compatible.

That is to prove that $\tilde{F}_{\tilde{\mathscr{U}}} = \tilde{F}$

Since $\mathscr{U}_i$ is compatible, $\tilde{\mathscr{U}}_i$ is also compatible.

That is, $\tilde{F}_{\tilde{\mathscr{U}}_i} = \tilde{F}_i$ for all $i\in\omega$

Where $\tilde{F}_{\tilde{\mathscr{U}}_i} = \{\tilde{G}_i \in I^{X_i/\sim}$ / If $[x_i] \in X_i/\sim$ is such that $\tilde{G}_i[x_i] = 1$

then there exists $\tilde{U}_i \in \tilde{\mathscr{U}}_i$ such that $\tilde{U}_i\langle[x_i]\rangle \leq \tilde{G}_i\}$

where $\tilde{U}_i\langle[x_i]\rangle[y_i] = \tilde{U}_i([x_i],[y_i])$ for all $i\in\omega$.

We have $\tilde{X} = \nabla\!\!\!\!/_{f}\, {}_i X_i$ and $\tilde{U} = \nabla\!\!\!\!/_{f}\, {}_i U_i$

$$\tilde{U}([x],[y]) = \nabla\!\!\!\!/_{f}\, {}_i U_i([x],[y])$$

$$= \sup \{ \inf_{i\in\omega} U_i(a_i,b_i) / q(a_i,b_i) = ([x_i],[y_i]) \} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

But

$$\tilde{U}_i([x_i],[y_i]) = q(U_i)([x_i],[y_i])$$

$$= \sup \{ U_i(a_i,b_i) / q(a_i,b_i) = ([x_i],[y_i]) \}$$

So from (1) $\tilde{U}([x],[y]) = \sup \{ \inf_{i\in\omega} U_i(a_i,b_i) / q(a_i,b_i) = ([x_i],[y_i]) \}$

$$\leq \inf_{i\in\omega} \{ \operatorname{Sup} U_i(a_i, b_i) / q(a_i,b_i) = ([x_i],[y_i]) \}$$

$$\leq \inf_{i\in\omega} \tilde{U}_i([x_i],[y_i])$$

$$\leq \inf_{i\in\omega} \tilde{G}_i[y_i] \text{ for all } i \in \omega.$$

$$\leq \tilde{G}[y]$$

Thus $\tilde{U}[x] \leq \tilde{G}$

Therefore $\tilde{F}_{\tilde{\mathscr{U}}} = \{ \tilde{G} \in I^{X/\sim} / \text{If } [x] \in X/\sim \text{ is such that } \tilde{G}[x] = 1$

then there exists $\tilde{U} \in \tilde{\mathscr{U}}$ such that $\tilde{U}([x]) \leq \tilde{G} \}$

Thus $\tilde{F}_{\tilde{\mathscr{U}}} = \tilde{F}$ holds.

Therefore $\tilde{\mathscr{U}} = \nabla\!\!\!\!/_{f}\, {}_i \mathscr{U}_i$ is compatible and it is a fuzzy uniform base on

$\tilde{X} = \nabla\!\!\!\!/_{f}\, {}_i X_i$

Hence the theorem.

BIBLIOGRAPHY

[A;M]$_1$ Artico, G; Moresco, R. *Fuzzy Proximities and Totally Bounded Fuzzy Uniformities*, J. Math.Anal.Appl.Vol.99, No.2, (1984).

[A;M]$_2$ Artico, G; Moresco, R. *Fuzzy Proximities Compatible with Lowen Fuzzy Uniformities*, Fuzzy sets and Systems 21 (1987) 85-98.

[A;M]$_3$ Artico, G; Moresco, R. *Fuzzy Uniformities induced by Fuzzy Proximities*, Fuzzy Sets and Systems 31 (1989)111-121.

[A;M]$_4$ Artico,G; Moresco, R. *Uniform Properties of Normal and Compact Fuzzy Topological Spaces*, J.Math.Anal.Appl.Vol 132, No. 2, June 1988.

[A;P] Arya, S.P; Purushottam Jha, *Countable Fuzzy Paracompactness, Fuzzy Meta Compactness and the Extension Theorem for Point Finite Fuzzy Covers*, J.Ind.Math.Soc.Vol.63, No.1-4 (1997) 53-61.

[BA]$_1$ Bayoumi,F. *The α-levels of a Fuzzy Uniform Structure and of a Fuzzy Proximity*, Fuzzy sets and Systems 116 (2000) 421-428.

[BA]$_2$ Bayoumi,F. On initial and final Fuzzy Uniform Structures, Fuzzy Sets and Systems133 (2003) 299-319.

[B;D] Benchalli, S.S; Desai, S.C. *Perfect maps and Paracompactness in Fuzzy Topological Spaces*, Proce. National Seminar at Karnataka University, Dharward (1996), 198-200.

[BE] Bernard, R. *Fuzzy Topological Spaces and Their Representation*,J.Math.Anal.Appl.81 (1981), 378-390.

[BO] Bourbaki, N. *General Topology*, Addison-Wesley Publishing Company (1996)

[BU] Bruke, D.K. *Covering Properties* in the Hand Book of Set theoretic Topology edited by J.E. Vaughen and K. Kunen, Elsevier Science Pub. BV (1984), 349-422

[BU]$_1$ Bulbul, A. *Some results on Paracompactness of Fuzzy Topological Spaces*, Journal of Karandeniz Technical University, National Math. Symposium issue Vol.11 (1989) 113-117.

[B;W] Bulbul, A ; Warner, M.W. *On the Goodness of Some Types of Fuzzy Paracompactness*, Fuzzy Sets and Systems 55 (1993) 187-191.

[CH] Chang, C.L. *Fuzzy Topological Spaces*, J.Math.Anal.Appl.24 (1968) 182-190.

[CO] Conard, F. *Fuzzy Topological Concepts*, J. Math. Anal. Appl. (1980) 432-440.

[C;J] Chun-hai Yu ; Ji-linag MA. *On Fuzzy Topological Groups*, Fuzzy Sets and Systems 23 (1987) 281-287.

[DE]$_1$ Dexue Zhang, *On the relationship between several Basic Categories in Fuzzy Topology,* Quaestiones Mathematicae 25 (2002), 289-301.

[DE]$_2$ Dexue Zhang, *Stratifiied Hutton Uniform Spaces*, Fuzzy Sets and Systems131 (2002) 337-346.

[D;P] Dubois, D; Prade, H. *Fuzzy Sets and Systems Theory and Applications*, Academic Press New York (1980).

[DU] Dugundji, F. *Topology.* Allyn & Bacon (1966).

[E;G]$_1$ Eklund, P; Gahler, W. *Basic Notions for Fuzzy Topology 1*, Fuzzy Sets and Systems 26 (1988) 333-356

[E;G]$_2$ Eklund, P; Gahler, W. *Basic Notions for Fuzzy Topology 11*, Fuzzy Sets and Systems 27, 333-356 (1988)

[FO] Foster, D.H. *Fuzzy Topological Groups*, J. Math. Anal. Appl. 67, 549-564 (1979)

[GA] Gaal, S.A. *Point Set Topology*, Academic Press Publications (1964).

[G;K;M] Ghanim, M.H ; Kerre E.E ; Mashhour, A.S. *Separation Axioms, Subspaces and Sums in Fuzzy Topology*, J.Math.Anal.Appl.102, (1984)189-202.

[GO] Goguen, J.A. *L-fuzzy sets*, J.Math.Anal.Appl.18 (1967) 145-174

[G;S]$_1$ Ganguly, S ; Saha, S. *On Separation Axiom and T_1 Fuzzy continuity*, Fuzzy Sets and Systems 16, 265-275 (1985).

[G;S]$_2$ Ganguly, S ; Saha, S. *A note on Compactness in Fuzzy Setting*, Fuzzy Sets and Systems, 54 (1990) 117-124.

[G;S;W] Gantner, T.E ; Steinlage, R.C ; Warren, R.H. *Compactness in Fuzzy Topological Spaces*, J. Math. Anal. appl. 62 (1978) 547-562.

[GU] Guo-Jan, Wang. *A new Fuzzy Compactness defined by Fuzzy Nets*, J.Math.Anal.Appl.94,(1983) 1-23.

[G;Y] Gui Shi, Fu ; Yu Zheng, Chong. *Paracompactness in L-fuzzy topological spaces*, Fuzzy Sets and Systems129 (2002) 29-37.

[HO]$_1$ Hohle, U. *Fuzzy Topologies on Algebric Structures*, J. Math. Anal. appl. 108 (1985) 115-150.

[HO]$_2$ Hohle, U. *Fuzzy Topologies and Topological Spaces Objects in a TOPOS*, Fuzzy Sets and Systems 19 (1986) 299-304.

[H;R] Hutton, B ; Reilly, I. *Separable Axioms in Fuzzy Topological Spaces*, Fuzzy Sets and Systems 3 (1980) 93-104.

[HU] Hutton, B. *Normality in Fuzzy Topological Spaces*, J. Math. Anal. appl. 50 (1975) 74-79.

[JA] James, I.M. *Topological and Uniform Spaces*, Springer-Verlag Pubs.(1987)

[JO] Joshy, K.D. Introduction to General Topology, wiley Eastern (1983).

[J;C] Ji Liang MA; Chun Hai Yu. *On the direct product of Fuzzy Topological Groups*, Fuzzy Sets and Systems 17 (1985) 91-97.

[J;S] Jha, P.K.; Singh, B.T. *Some local properties of Fuzzy Topology and Fuzzy Topological Groups*, Bull. Cal. Math. Soc. 87, 441-448 (1995).

[KA]$_1$ Katsaras, A.K. *Fuzzy Proximity Spaces*, J. Math. Anal. appl. 68 (1979) 100-110.

[KA]$_2$ Katsaras, A.K. *On Fuzzy Proximity Spaces*, J. Math. Anal. Appl. 75 (1980) 571-583.

[KAT] Kato, A. Real *Compactness of Box Products*, Defence Acad. Japan, 19, No-1, 1-4.

[KATS] Katsaras, A.K. *On Fuzzy Uniform Spaces*, J. Math. Anal.. Appl. 101, 97-113 (1984).

[KAU] Kaufmann, A. *Introduction to the Theory of Fuzzy subsets*, Academic Press. Inc New York (1975).

[KE] Kelley, J.L. *General Topology*, Van Nostrand Princeton, New Jersey (1955).

[K;F] Klir, G; Folger, T.A. *Fuzzy Sets Uncertainty and Information*, Prentice Hall of India (1993).

[K;N] Kerre, E.E; Noug, A.A. *Operations on the class of Fuzzy sets on a Universe Endowed with a Fuzzy Topology*, J. Math. Anal. Appl. 180, 325-341 (1993).

$[LO]_1$ Lowen, R. *Fuzzy Topological Spaces and Fuzzy Compactness*, J. Math. Anal. Appl. 56 (1976) 621-633.

$[LO]_2$ Lowen, R. *A Comparison of Different Compactness notions in Fuzzy Topological Spaces*, J.Math.Anal.Appl.64 (1978) 446-454.

$[LO]_3$ Lowen, R. *Fuzzy Uniform Spaces*, J.Math.Anal.Appl.82, 370-385 (1981).

$[LO]_4$ Lowen, R. *Initial and Final Fuzzy Topologies and The Fuzzy Tychnoff Theorem*, J.Math.Anal.appl.58, (1977) 11-21

$[LO]_5$ Lowen, R. *Fuzzy Neighbourhood spaces*, Fuzzy Sets and Systems 7 (1982) 165-189.

[L;P] Lou, S.P; Pan, S.H. *Fuzzy Structure*, J.Math.Anal.Appl.76, 631-642 (1980).

[L;Z] Li Sanjiang; Zhang Dexue. *A note on Super Uniform Spaces*, Chinese Journal of Contemporary Mathematics Vol.22, No.2, 2001.

[MA] Mao-Kang, Luo. *Paracompactness in Fuzzy Topological Spaces*, J. Math. Anal. Appl. 130 (1988) 55-97.

[M;B]$_1$ Malghan, S.R ; Benchalli,S.S. *On Fuzzy topological Spaces*, Glasnik Mathematiki 16 (36) (1981), 313-325.

[M;B]$_2$ Malghan, S.R ; Benchalli, S.S. *Open maps, Closed maps and Local compactness in Fuzzy Topological Spaces*, J.Math.Anal.Appl.99 (1984) 338-349.

[MI] Miller, A.W. *On Box Products*, Topology and its Applications, 14 (1982) 313-317.

[MU] Munkres, R.J. *Topology A First Course*, Prentice-Hall of India (1994).

[M;G] Mukherjee; Ghosh. *Some Stronger forms of Fuzzy Continuous mappings on Fuzzy Topological Spaces*, Fuzzy Sets and Systems 38 (1990) 375-387.

[M;N] Morita, K ; Nagata, J. *Topics in General Topology*, North Holland Publications (1989).

[M;Y] MA Ji Liang; Yu Chun Hai, *Fuzzy Topological Groups*, Fuzzy Sets and Systems 12 (1984) 289-299.

[NA] Nanda, S. *On Fuzzy Topological Spaces*, Fuzzy Sets and Systems 19 (1986) 193-197.

[NAG] Nagata, J. *Modern General Topology*, North Holland Pub (1974).

[P;A] De Prade Vicente, M.A; Aranguren, M.S. *Fuzzy Filters*, J. Math. Anal. Appl. 129 (1988) 560-568.

[PR] Przymusinski, T. *Product of Normal spaces* in the Hand Book of Set Theoretic Topology, Elsevier Science Publishers B.V.,(1984) 781-826.

[P;Y]$_1$ Pao-Ming, Pu ; Ying- Ming, Liu. *Fuzzy Topology.I. Neighbourhood Structure of a Fuzzy Point and Moore-Smith Convergence*, J.Math.Anal.Appl.76, 571-599 (1980).

[PY]$_2$ Pao-Ming, Pu ; Ying- Ming, Liu. *Fuzzy Topology.II. Product and Quotient spaces*, J.Math.Anal.Appl.77 (1980) 20-37.

[RO] Rosa, M.V. *Separation Axioms in Fuzzy Topology Fuzzy Convexity Spaces*, The Journal of Fuzzy Mathematics Vol.2, No.3, 1994.

[ROD] Rodabaugh, S.E. *A Categorical Accommodation of Different Notions in Fuzzy Topology*, Fuzzy Sets and Systems 9 (1983) 241-265.

[ROI]$_1$ Roitman, J. *More Paracompact box Products*, Proceedings of the American Mathematical society, Vol.74,No.1(1979).

[ROI]$_2$ Roitman, J. *Paracompact Box Products in Forcing Extensions*, Fundamenta Mathematicae (1979).

[ROS] Rosenfield, A. *Fuzzy Sets, Fuzzy numbers and Fuzzy Mathematics*, The Mathematical Intelligence Vol.4, No.3, 1982,139-143.

[SA]$_1$ Sarkar, M. *On Fuzzy Topological Spaces*, J. Math. Anal. appl. 79 (1981) 384-394.

[SA]$_2$ Sarkar, M. *On L-fuzzy Topological Spaces*, J. Math. Anal. Appl. 84 (1981) 431-442.

[S;A] Srivastava, A.K; Ali, D.M. *A comparison of some FT2 concepts*, Fuzzy Sets and Systems 23, (1987) 289-294.

[S;S] Steen, L.A ; Seebach, A. *Counter Examples in Topology*, Springer Verlag (1970).

[SU] Sunil Jacob John, Ph.D Thesis *On fuzzy topological Games and Related Topics*, Cochin University of Sci.& Tech. (2000).

[VA] Van Douwen, E.K. *Covering and Separation Properties of Box Products*, Surveys in General Topology (Academic Press, NewYork) 55-130 (1980).

[WI] Willard, S. *General topology*, Addison Wesley Pub.company (1970).

[WIL] Williams, S.W. *Box products* in the Hand Book of Set theoretic Topology edited by J.E. Vaughen and K. Kunen, Elsevier Science Pub. BV (1984), 169-200.

[WO]$_1$ Wong, C.K. *Covering Properties in Fuzzy Topological Spaces*, J. Math. Anal. Appl. 43 (1973), 697-704.

[WO]$_2$ Wong C.K. *Fuzzy Topology. Product and Quotient Theorems*, J.Math.Anal.Appl.45 (1973) 512-521.

[WO]$_3$ Wong, C.K. *Fuzzy Points and Local Properties of fuzzy Topology*, J.Math.Anal.Appl.46, 316-328 (1974).

[WO]$_4$ Wong, C.K. *Fuzzy Topology* in Fuzzy Sets and Their Applications to Cognitive and Decision Process, Academic Press, New York (1975).

[W;L] Wuyts, P; Lowen, R. *On Separation Axioms in Fuzzy Topological Spaces, Fuzzy Neighbourhood Spaces and Fuzzy Uniform spaces*, J. Math. Anal. Appl. 93, 27-41 (1983).

[Y;F] Yager, R.R; Filev, P.D. *Essentials of Fuzzy Modelling and Control*, John Wiley & Sons Pub.(1994)

[Y;M] Ying- Ming, Liu; Mao-Kang, Luo. *Fuzzy Topology,* Advances in Fuzzy Systems, Application and Theory Vol 9, World Scientific (1997).

[YI] Yixiang Chen. *Paracompactness on L-fuzzy topological spaces*, Fuzzy Sets and Systems 53 (1993) 335-342.

[ZA] Zadeh, L.A. *Fuzzy Sets*, Information and Control 8 (1965) 338-353.

[ZI] Zimmerman, H.J. *Fuzzy Set Theory and its Applications*, Kluwer Academic (1996).