STUDIES ON MAGNETIC MONOPOLE SOLUTIONS OF NON-ABELIAN GAUGE THEORIES AND RELATED PROBLEMS

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CERTIFICATE

Certified that the work reported in the present thesis is based on the bonafide work done by Shri C. M. Ajithkumar, under my guidance in the Department of Physics, Cochin University of Science and Technology, and has not been included in any other thesis submitted previously for the award of any degree.

Cochin - 22
May 03, 1986.

K. Babu Joseph
Supervising teacher

DECLARATION

Certified that the work presented in this thesis is based on the original work done by me under the guidance of Prof. K. Babu Joseph in the Department of Physics, Cochin University of Science and Technology, and has not been included in any other thesis submitted previously for the award of any degree.

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PREFACE

This thesis reports a study of some aspects of classical solutions of non-abelian gauge theories with emphasis on magnetic monopole solutions. This work has been carried out by the author in the Department of Physics, Cochin University of Science and Technology, during 1981-86.

This thesis contains six chapters. The first chapter gives a brief account of the gauge theory formalism, giving importance to the SU(2) gauge theory which is used throughout in this work. The first chapter gives an introduction to various classical solutions of the theory. Some topological aspects, which are necessary for a better understanding of the subject, are also included. In Chapter 2 we report a simple derivation of the relationship between the topological index of the gauge fields of a dyon and its magnetic charge. A systematic derivation of various monopole solutions from the field equations is given in Chapter 3. A pair of new complex finite action solutions of SU(2) gauge theory and its properties are discussed in Chapter 4. Euclidean solutions related to monopoles and dyons and their properties are presented in Chapter 5. In the final chapter we report our study of the bound states of dyons with fermions and bosons.

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SYNOPSIS

In 1931 Dirac studied the motion of an electron in the field of a magnetic monopole and found that the quantization of electric charge can be explained by postulating the mere existence of a magnetic monopole. Electric charge quantization is actually observed in nature, and no other explanation for this deep phenomenon is known. Since 1974 there has been a resurgence of interest in magnetic monopole due to the work of 't Hooft and Polyakov who independently observed that monopoles can exist as finite energy topologically stable solutions to certain spontaneously broken gauge theories. In spite of a lack of supporting experimental evidence, there are compelling reasons based on unified theories of fundamental interactions, for believing that magnetic monopoles exist in nature. The thesis, "Studies on Magnetic Monopole Solutions of Non-abelian Gauge Theories and Related Problems", reports a systematic investigation of classical solutions of non-abelian gauge theories with special emphasis on magnetic monopoles and dyons which possess both electric and magnetic charges. The formation of bound states of a dyon with fermions and bosons is also studied in detail.

The thesis opens with an account of a new derivation of a relationship between the magnetic charge of a dyon and the topology of the gauge fields associated with it. Although this formula has been reported earlier in the literature, the present method has two distinct advantages. In the first place, it does not depend either on the mechanism of symmetry breaking or on the nature of the residual symmetry group. Secondly, the results can be generalised to finite temperature monopoles.

We have carried out a systematic search for solutions of the nonlinear radial field equations of the 't Hooft-Polyakov monopole theory by using the direct method of Hirota. In the literature this method has hitherto been applied to various scalar theories in two space-time dimensions. We have applied this method to the second order nonlinear differential equations corresponding to a vanishing Higgs self-interaction and reproduced all the previously known solutions which satisfy the first order Bogomolny equations.

A pair of exact, complex conjugate solutions is constructed for SU(2) gauge theory in the Prasad-Sommerfield limit. The Euclidean actions corresponding to these solutions are found to be finite and complex. Such solutions are important for the evaluation of the partition function in finite temperature theories. These flat-space solutions are transformed to de Sitter space by a standard procedure.

We have constructed new time-dependent solutions to pure SU(2) gauge theory which are related to monopoles and dyons. A set of solutions, which approaches the Wu-Yang monopole as Euclidean time $t \longrightarrow \pm \infty$, is derived. These are non-self-dual real solutions in Euclidean space with zero topological index and infinite action. The divergent action is due to the singular behaviour of the solutions at the origin. Another set of solutions, which approaches a singular dyon configuration is also obtained. These are solutions to self-daulity equations and possess severe singularities. The action is infinite. In both cases, owing to the infiniteness of action, semiclassical approximation cannot be directly applied to extract more physical content. However, their Euclidean time development is interesting.

In the final section of the thesis we carry out a study of the bound states of spin 1/2 and spin zero particles in the background field of a point dyon for isospinor and isovector representations. Energy levels and eigenfunctions for all angular momenta are obtained for isospinor fermions, isospinor bosons and isovector bosons. For isovector fermions bound state spectrum for lowest angular momentum is determined. The method of separation of angular and radial parts is achieved by using spherical harmonics. This method is compared with an alternative method of separation using monopole harmonics, and the two methods are shown to be equivalent. A relation connecting mono pole harmonics and spherical harmonics is also derived. The study of the bound states of monopoles and dyons is important for their possible experimental detection.

The material reported in the thesis has been published in the form of the following papers :

1. Relation between magnetic and charge and the topology of dyon fields, J. Phys.G:Nucl.Phys. **8**(1982) 887.
2. Systematic derivation of Prasad-Sommerfield solution. Presented at the VI High Energy Physics Symposium, Mysore, December 6-11, 1982.
3. Complex SU(2) Yang-Mills-Higgs configurations with finite, complex Euclidean actions, J.Phys.G:Nucl.Phys. **9**(1983) 1469.
4. New Euclidean solutions of SU(2) gauge theory, Phys.Rev. D**30**(1984) 2247.
5. Bound states of non-abelian dyons with fermions and bosons, Ann.Phys. (In press).

CONTENTS

CHAPTER - 1

INTRODUCTION

1.1 Gauge theories

The well known gauge invariance property of electromagnetism may be expressed in terms of gauge transformations that are elements of a U(1) group. It is generally believed that the four fundamental interactions can be described by gauge theories [1-5]. The unified theory of a weak and electromagnetic interaction proposed by Weinberg, Salam and Glashow [6] is a gauge theory with the gauge group SU(2)xU(1). Grand unifying schemes, which incorporate strong, weak and electromagnetic interactions into a single theory, are also based on gauge theories. It is believed that strong interaction can be described by a gauge theory with SU(3)-colour as the gauge group.

Let us consider, for instance, the Lagrangian of a complex scalar field

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \qquad (1.1)$$

which is invariant under the transformation :

$$\phi(x) \longrightarrow \phi'(x) = e^{-i\theta} \phi(x) \qquad (1.2)$$

where θ is an aribitrary real parameter. The transformation (1.2) is called a global gauge transformation or a gauge transformation of the first kind. The symmetry of $\mathcal{L}$ with respect to (1.2) leads to the conservation of electric charge.

If θ is made space-time dependent,

$$\phi(x) \longrightarrow \phi'(x) = e^{-i\theta(x)} \phi(x) , \qquad (1.3)$$

the Lagrangian (1.1) is not invariant. This is because the derivative $\partial_\mu \phi$ does not transform like the field ϕ:

$$\partial_\mu \phi(x) \longrightarrow (\partial_\mu \phi(x))' = e^{-i\theta(x)} \partial_\mu \phi(x) - i(\partial_\mu \theta(x)) e^{-i\theta(x)} \phi(x) . \tag{1.4}$$

There is a standard way to restore the invariance. This is done by introducing a vector field $A_\mu(x)$ called the gauge field into the theory and replacing all derivatives by covariant derivatives :

$$D_\mu \phi(x) = (\partial_\mu - ie A_\mu(x)) \phi(x) , \tag{1.5}$$

where e is a real parameter which determines the coupling strength of gauge-scalar field interaction. Its transformation is obtained by imposing the condition that covariant derivative transforms like the fields, that is,

$$D_\mu \phi(x) \longrightarrow (D_\mu \phi(x))' = e^{-i\theta(x)} D_\mu \phi(x) . \tag{1.6}$$

From this we get

$$A_\mu(x) \longrightarrow A'_\mu(x) = A_\mu(x) - \frac{1}{e} \partial_\mu \theta(x) . \tag{1.7}$$

The transformations (1.3) and (1.7) are called local gauge transformations or gauge transformations of the second kind or simply, gauge transformations. The Lagrangian in which all derivatives replaced by covariant derivatives,

$$\begin{aligned} \mathcal{L} &= (D_\mu \phi)^* (D^\mu \phi) - m^2 \phi^* \phi \\ &= (\partial_\mu + ie A_\mu) \phi^* (\partial^\mu - ie A^\mu) \phi - m^2 \phi^* \phi , \end{aligned} \tag{1.8}$$

is invariant under the local gauge transformations (1.3) and (1.7). The Lagrangian of the Dirac field,

$$\mathcal{L} = \bar{\Psi} (i\gamma^\mu \partial_\mu - m) \Psi , \tag{1.9}$$

can be made gauge invariant by the above procedure. We get

$$\begin{aligned} \mathcal{L}' &= \bar{\Psi}\,(i\gamma^\mu D_\mu - m)\,\Psi \\ &= \bar{\Psi}\,(i\gamma^\mu(\partial_\mu - ieA_\mu) - m)\,\Psi \\ &= \mathcal{L} + e\,\bar{\Psi}\,\gamma^\mu A_\mu \Psi\,. \end{aligned} \tag{1.10}$$

Lagrangians (1.8) and (1.10) are nothing but Lagrangians of quantum electrodynamics without the kinetic energy term of the electromagnetic field. We obtained this by imposing local gauge symmetry on the theory. The gauge field $A_\mu(x)$ is nothing but the electromagnetic field. Here the advantage is that the interaction Lagrangian (for fermions the second term on the last line of (1.10)) is uniquely obtained from the symmetry principle relating to gauge invariance. To complete the Lagrangian one should add the kinetic energy term for the gauge field to (1.8) and (1.10). Since we have identified the gauge field with the electromagnetic field we can take

$$\mathcal{L}_{kin} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{1.11}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,. \tag{1.12}$$

One readily verifies that (1.11) and (1.12) are invariant under the local gauge transformations (1.7). However, a mass term for the gauge field, $-M^2 A_\mu A^\mu$, is not invariant under the local gauge transformation. Thus gauge invariance implies masslessness of the photon.

The transformations (1.3) and (1.7) form an abelian group U(1). The idea of local gauge transformations was extended to the non-abelian group SU(2) by Yang and Mills in 1954 [7] and later generalised to arbitrary non-abelian groups by Utiyama [8]. In this case a set of gauge fields, whose number is equal to that of the generators of the group, should be added into the theory. In what follows we consider the example of the SU(2) group and let the scalar fields belong to the n dimensional representation of SU(2), namely,

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \end{pmatrix} . \tag{1.13}$$

The globally symmetric Lagrangian is

$$\mathcal{L} = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - m^2 \phi^\dagger \phi . \tag{1.14}$$

A global transformation is defined by

$$\phi(x) \longrightarrow \phi'(x) = e^{-i\theta^a T^a} \phi(x) , \tag{1.15}$$

where the T^a are the 3 generators of the SU(2) group in the n dimensional representation satisfying the Lie algebra,

$$[T^a, T^b] = i\epsilon_{abc} T^c . \tag{1.16}$$

θ^a are three arbitrary real parameters. If we make θ^a space-time dependent,

$$\begin{aligned} \phi(x) \longrightarrow \phi'(x) &= e^{-i\theta^a(x) T^a} \phi(x) \\ &= U(\theta(x)) \phi(x) , \end{aligned} \tag{1.17}$$

we obtain a local gauge transformation belonging to the group SU(2). The Lagrangian (1.14) is no longer invariant under the local gauge transformations (1.17). As in the abelian case, (1.14) can be made gauge invariant by replacing ordinary derivatives by covariant derivatives defined by

$$D_\mu \phi_n = (\partial_\mu \delta_{nm} - ig A_\mu^a T_{nm}^a)\, \phi_m \quad , \tag{1.18}$$

where the A_μ^a are 3 vector fields called SU(2) gauge fields and g is the coupling constant. As in the earlier case, the transformation of the gauge fields can be obtained by requiring covariant derivatives to transform like the fields. That is

$$D_\mu \phi(x) \longrightarrow (D_\mu \phi(x))' = U(\theta(x))\, D_\mu \phi(x) \quad . \tag{1.19}$$

Solving (1.19) we get

$$\begin{aligned} A_\mu^a T^a \longrightarrow A_\mu^{'a} T^a = {} & U(\theta(x))\, A_\mu^a T^a\, U^{-1}(\theta(x)) \\ & - \frac{i}{g} (\partial_\mu U(\theta(x)))\, U^{-1}(\theta(x)) \quad . \end{aligned} \tag{1.20}$$

The SU(2) gauge invariant Lagrangian is

$$\mathcal{L}' = (D_\mu \phi)^\dagger (D^\mu \phi) - m^2 \phi^\dagger \phi \quad . \tag{1.21}$$

To complete the Lagrangian one should add the kinetic energy term for the gauge fields. The simplest gauge invariant form of kinetic energy is

$$\mathcal{L}_{kin} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} \quad , \tag{1.22}$$

where

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\, \epsilon_{abc} A_\mu^b A_\nu^c \quad . \tag{1.23}$$

As in the abelian case, a mass term of the form $M^2_{ab} A^a_\mu A^{\mu b}$ is not gauge invariant. Hence non-abelian gauge fields are massless.

Gauge theories are very interesting because the interaction is uniquely fixed by the symmetry of the theory. Further, they are believed to be the only theories of vector mesons that are renormalisable.

1.2 Spontaneous symmetry breaking

If the states of a system do not respect the symmetry of the Lagrangian or Hamiltonian, the symmetry is said to be spontaneously broken. Ferromagnetism is an example, from solid state physics, of this phenomenon. The Hamiltonian of a ferromagnet is rotationally symmetric. A ferromagnet above its Curie temperature, possesses this symmetry because the individual dipoles are randomly oriented. All directions are equally important in this case. However, below the Curie temperature the dipoles are aligned in some particular direction which violates the rotational invariance. So in this case the ferromagnetic ground state violates the rotational symmetry of the Hamiltonian. The rotational symmetry in this case is said to be spontaneously broken.

The idea of spontaneous symmetry breaking can be studied in field theory. When degenerate vacuum states exist spontaneous symmetry breaking occurs. We shall demonstrate this with a Lagrangian which is an extension of (1.1) :

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - V(\phi^*\phi) , \tag{1.24}$$

where

$$V(\phi^*\phi) = m^2 \phi^*\phi + \lambda (\phi^*\phi)^2 . \tag{1.25}$$

The last term in (1.25) is added to to obtain degenerate minima. The free field situation corresponds to $\lambda = 0$ and $m^2 > 0$. The case $\lambda < 0$ is unphysical because the potential would not be bounded. For $\lambda > 0$ and $m^2 > 0$ the minimum of $V(\phi^*\phi)$ corresponds to

$$\phi = 0 \tag{1.26}$$

In this case the ground state is non-degenerate. For $\lambda > 0$ and $m^2 < 0$, the minimum of $V(\phi^*\phi)$ is at

$$|\phi| = (-m^2/2\lambda)^{1/2} = v/\sqrt{2} \quad , \tag{1.27}$$

which corresponds to the points on a circle in the complex ϕ plane, having radius $(-m^2/2\lambda)^{1/2}$. All the points on the circle are equally good ground states. Choosing any one of them breaks the global symmetry (1.2) of the Lagrangian (1.24) spontaneously. In quantum field theory (1.27) is replaced by the equation

$$|\langle 0|\phi|0\rangle| = (-m^2/2\lambda)^{1/2} = v/\sqrt{2} \quad , \tag{1.27}$$

which means that the vacuum expectation value of ϕ is nonzero.

What is the consequence of spontaneous symmetry breaking? According to a theorem due to Goldstone [9], spontaneous breaking of a continuous global symmetry implies the existence of a massless spin zero particle. The massless particle is called a Goldstone boson. To demonstrate this, let us choose the vacuum on the positive real axis of the complex ϕ plane and redefine fields with respect to the vacuum :

$$\phi(x) = \frac{1}{\sqrt{2}}\left(v + \psi(x) + i\,\chi(x) \right) \tag{1.28}$$

where ψ and χ are two real fields having zero vacuum expactation value.

Substituting (1.28) in (1.24) we get

$$\mathcal{L}(\psi,\chi) = \frac{1}{2}(\partial_\mu\psi)(\partial^\mu\psi) + \frac{1}{2}(\partial_\mu\chi)(\partial_\mu\chi) - \frac{1}{2}(-2m^2)\psi^2 - \lambda v\psi(\psi^2+\chi^2) - \frac{\lambda}{4}(\psi^2+\chi^2)^2 \tag{1.29}$$

There is no mass term for the χ field while the bare mass of the ψ field is $\sqrt{-2m^2}$. The χ field corresponds to the Goldstone boson.

In ferromagnets the Goldstone boson corresponds to spin waves. One has to note that a spontaneously broken symmetry is still a symmetry of the Lagrangian. However, it is not manifested by its ground state.

In general, when the number of group generators is more than one, one can have a number of Goldstone bosons. In fact, the number of Goldstone bosons is equal to the number of broken generators. Goldstone bosons exist only when the broken symmetry is global. When a local gauge symmetry is spontaneously broken two spectacular things happen: Some of the gauge fields become massive and all Goldstone bosons disappear. This phenomenon was studied by several authors [10] and is known as Higgs mechanism. The scalar fields effecting the spontaneous symmetry breaking are called Higgs fields.

To demonstrate the Higgs mechanism explicitly, we consider the locally gauge invariant version of (1.24)

$$\mathcal{L}' = (D_\mu\phi)^*(D^\mu\phi) - m^2\phi^*\phi - \lambda(\phi^*\phi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} . \tag{1.30}$$

For $m^2 < 0$ and $\lambda > 0$, the local gauge symmetry is spontaneously broken. As in the earlier case, we take the ground state on the positive real axis. Here instead of the previous parametrisation it is advantageous to use the following one :

$$\phi(x) = e^{i\xi(x)/v}(v + \eta(x))/\sqrt{2}$$
$$= \frac{1}{\sqrt{2}}(v + \eta(x) + i\xi(x) + \text{quadratic and higher order terms}) . \tag{1.31}$$

Substituting this in (1.30) we get

$$\mathcal{L}'(A_\mu, \eta, \xi) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) + \frac{1}{2}(\partial_\mu\xi)(\partial^\mu\xi)$$
$$+ \frac{1}{2}e^2v^2A_\mu A^\mu - ev A_\mu \partial^\mu \xi \tag{1.32}$$
$$- \frac{1}{2}(-2m^2)\eta^2 + \text{cubic and higher order terms.}$$

From this we conclude that the mass of the η field is $\sqrt{-2m^2}$. The coupling of A_μ and ξ in the quadratic term prevents us from directly predicting the full particle spectrum. However, it is possible to eliminate the ξ field altogether from the Lagrangian by the local gauge transformation

$$\phi(x) \longrightarrow \phi'(x) = e^{-i\xi(x)/v}\phi(x) = (v + \eta(x))/\sqrt{2}$$
$$A_\mu(x) \longrightarrow A'_\mu(x) = A_\mu(x) - \frac{1}{ev}\partial_\mu\xi(x) . \tag{1.33}$$

This gauge is called the unitary gauge. Substituting this reparametrisation with gauge transformation, (1.33), in (1.30) we obtain

$$\mathcal{L}'(A'_\mu, \eta) = -\frac{1}{4}F'_{\mu\nu}F^{\mu\nu\prime} + \frac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) + \frac{1}{2}e^2v^2A'_\mu A^{\mu\prime} \tag{1.34}$$
$$+ \frac{1}{2}e^2\eta(2v + \eta)A'^2_\mu - \frac{1}{2}(-2m^2)\eta^2 - \lambda v\eta^3 - \frac{\lambda}{4}\eta^4 ,$$

where

$$F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu .$$

The Goldstone boson field $\xi(x)$ is not present in (1.34). The particle spectrum can be read off from the quadratic terms. There is a scalar η meson with mass $\sqrt{-2m^2}$ and a massive vector meson with mass $e v$. The degree of freedom corresponding to the Goldstone boson appears in the form of the longitudinal degree of freedom of the massive gauge particle.

As a non-abelian example of spontaneous symmetry breaking and Higgs mechanism we consider SU(2) gauge theory with a Higgs triplet defined by the Lagrangian,

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + \frac{1}{2} D_\mu \phi^a D^\mu \phi^a - \frac{m^2}{2} \phi^a \phi^a - \frac{\lambda}{4} (\phi^a \phi^a)^2 , \tag{1.35}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g \epsilon_{abc} A^b_\mu A^c_\nu$$

$$D_\mu \phi^a = \partial_\mu \phi^a + g \epsilon_{abc} A^b_\mu \phi^c$$

$$(a = 1, 2, 3) .$$

For $m^2 < 0$ and $\lambda > 0$ the minimum of the potential corresponds to

$$\phi_a \phi_a = \phi_1^2 + \phi_2^2 + \phi_3^2 = -m^2/\lambda = v^2 \tag{1.36}$$

which represents a two dimensional surface of a sphere in the internal space with a radius $\sqrt{-m^2/\lambda}$. Let us choose the vacuum on the third axis (ϕ_3 axis) so that the vacuum value of ϕ is given by

$$\langle \phi \rangle_0 = \langle 0 | \phi | 0 \rangle = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} . \tag{1.37}$$

The Lagrangian (1.35) is invariant under SU(2) gauge transformations. The infinitesimal transformation for the scalar field is

$$\phi_i(x) \longrightarrow \phi_i'(x) = (\delta_{ik} - i\,\theta^j(x)\,T^j_{ik})\,\phi_k(x) \tag{1.38}$$

Hence

$$\delta\phi_i(x) = \epsilon_{ijk}\,\theta^j(x)\,\phi_k(x) \tag{1.39}$$

where we have substituted $i\epsilon_{ijk}$ for T^j_{ik} because the scalar field is in the adjoint representation.

One can readily see that the vacuum is no longer invariant under T_1 and T_2, but T^3 remains a good symmetry ($\delta\phi_i$ in this case is zero). As in the abelian case we re-express the fields with respect to the vacuum :

$$\begin{aligned}\phi(x) &= \exp\left\{\frac{i}{v}\left(\xi_1(x)\,T^1 + \xi_2(x)\,T^2\right)\right\}\begin{pmatrix}0\\0\\v+\eta(x)\end{pmatrix}\\ &= \langle\phi\rangle_0 + \begin{pmatrix}\xi_2(x)\\ \xi_1(x)\\ \eta(x)\end{pmatrix} + \text{higher order terms}.\end{aligned} \tag{1.40}$$

In a global theory we expect two Goldstone bosons corresponding to the fields $\xi_1(x)$ and $\xi_2(x)$. Since we started with a local theory, (1.35), they do not exist. They are eliminated by the gauge transformation

$$\begin{aligned}\phi(x) &\longrightarrow \phi'(x) = U(\xi)\,\phi(x) = \begin{pmatrix}0\\0\\v+\eta(x)\end{pmatrix}\\ A^a_\mu T^a &\longrightarrow A'^a_\mu T^a = U(\xi)\,A^a_\mu T^a\,U^{-1}(\xi) - \frac{i}{g}\left(\partial_\mu U(\xi)\right)U^{-1}(\xi)\end{aligned} \tag{1.41}$$

where

$$U(\xi) = \exp\left\{-\frac{i}{v}\left(\xi_1(x)T^1 + \xi_2(x)T^2\right)\right\} . \tag{1.42}$$

when the Lagrangian (1.35) is rewritten using (1.41), it will be independent of ξ_1 and ξ_2. Further there will be a gauge field mass term,

$$-\frac{1}{2}g^2 v^2\left(A^1_\mu A^{\mu 1} + A^2_\mu A^{\mu 2}\right) \tag{1.43}$$

while A^3_μ will remain massless. This is because T^3 is still a good symmetry of the vacuum. There will be one massive scalar particle, with mass $\sqrt{-2m^2}$ corresponding to the η field.

In general, for arbitrary gauge groups, the number of gauge fields acquiring mass is equal to the number of 'would be' Goldstone bosons. Further, the number of the Goldstone bosons is equal to the number of broken generators.

1.3 Topological considerations

First we consider how maps can be classified into equivalence classes. Two maps $f(x)$ and $g(x)$ from a topological space X to Y,

$$\begin{aligned} f(x) &: X \longrightarrow Y \\ g(x) &: X \longrightarrow Y \end{aligned} \tag{1.44}$$

are said to be equivalent or homotopic if there exists a continuous function

$$h(x,\alpha) : X \otimes [0,1] \longrightarrow Y, \tag{1.45}$$

with $\alpha \in [0,1]$, such that

$$
\begin{aligned}
h(x,0) &= f(x) \\
h(x,1) &= g(x) \quad .
\end{aligned}
\tag{1.46}
$$

We can say that $f(x)$ is continuously deformable into $g(x)$. The continuous function, $h(x,\alpha)$, effecting this deformation is called homotopy. Eqn. (1.46) is an equivalence relation. Accordingly, maps can be classified into equivalence classes called homotopic classes. In (1.46) $f(x)$ and $g(x)$ are said to be homotopic to each other. Each homotopic class contains maps which are homotopic (continuously deformable) to each other.

This concept is important in the study of classical solutions of field theories. Classical solutions played an important role in understanding many features, which were never unveiled in the usual perturbation scheme, of the structure of non-abelian gauge theories. These theories also possess solutions with energy density confined to a small region in space, which can be interpreted as particles. These are coherent excitations of the basic fields and a consistent quantum theory exists for them.

For obtaining physically relevent classical solutions, some condition like the finiteness of energy or action is imposed. This condition often defines a map between non-trivial topological spaces. Such maps fall into different homotopic classes. Often, these classes are labelled by a number called the winding number.

Let us elucidate the concept of winding number and homotopic classification by considering a simple map $S^1 \rightarrow S^1$ [5]

$$
f(\theta) = e^{i(n\theta + a)} \tag{1.47}
$$

where θ corresponds to the points on a unit circle with θ and $\theta + 2\pi$ identified. $f(\theta)$s are unimodular complex numbers and the space formed by them is equivalent to a unit circle. Hence (1.47) is a map from one dimensional sphere to one dimensional sphere, represented by $S^1 \rightarrow S^1$.

For fixed n, all maps for different a are homotopic because we can construct a homotopy

$$F(\theta, \alpha) = e^{i(n\theta + (1-\alpha)\theta_0 + \alpha\theta_1)} . \tag{1.48}$$

We have then

$$\begin{aligned} F(\theta, 0) &= f(\theta) = e^{i(n\theta + \theta_0)} \\ F(\theta, 1) &= g(\theta) = e^{i(n\theta + \theta_1)} . \end{aligned} \tag{1.49}$$

$f(\theta)$ and $g(\theta)$ are homotopic. However, maps with different n are not homotopic and belongs to different topological classes. In (1.47) when the domain of the map S^1 is covered once, its image space S^1 is covered n times. The number n is called the winding number or Pontryagin index. The winding number of the map (1.47) can be written as

$$n = \int_0^{2\pi} \frac{d\theta}{2\pi} \left[-\frac{i}{f(\theta)} \frac{df(\theta)}{d\theta} \right] . \tag{1.50}$$

The winding number of magnetic monopole solution originates from the finite energy condition on the Higgs field ϕ_a :

$$\phi^2 \longrightarrow m^2/\sqrt{\lambda} \quad \text{as} \quad r \longrightarrow \infty \tag{1.51}$$

where

$$\phi^2 = \phi_a \phi_a = \phi_1^2 + \phi_2^2 + \phi_3^2$$

and

$$r^2 = x_1^2 + x_2^2 + x_3^2 .$$

We will explain how the boundary condition (1.51) is obtained in the next section. The condition (1.51) defines a mapping from the two dimensional surface of a three dimensional sphere having infinite radius, S^2_∞ , into the surface of the sphere, having radius $m/\sqrt{\lambda}$, in the space spanned by ϕ_1 , ϕ_2 and ϕ_3 ,

$$\phi_a : \quad S^2_\infty \longrightarrow S^2_{m/\sqrt{\lambda}} . \tag{1.52}$$

To satisfy (1.51) ϕ_a should be of the form

$$\phi_a \longrightarrow \frac{m}{\sqrt{\lambda}} n_a(\hat{r}) \quad \text{as} \quad r \longrightarrow \infty \tag{1.53}$$

with $n_a n_a = 1$. So one can alternatively define a map

$$n_a(\hat{r}) : \quad S^2_\infty \longrightarrow S^2_I \tag{1.54}$$

where S^2_I is the unit sphere in the ϕ space. We know that the map (1.52) (or (1.54)) can be classified into homotopic classes. As in the example considered earlier a winding number characterises each map. These homotopic classes form a group called the second homotopy group denoted by $\Pi_2(S^2)$ (The n^{th} homotopy group $\Pi_n(X)$ is the set of equivalence classes of maps from S^n into a topological space X. The homotopy group of the map (1.47) is the first homotopy group $\Pi_1(S^1)$). From the theory of homotopy groups we have

$$\Pi_2(S^2) = Z \tag{1.55}$$

where the elements of Z are integers. Thus the winding number n is an integer.

Since all finite energy solutions satisfy (1.51) we can classify them with respect to their winding numbers. For a fixed winding number n, a solution having the lowest energy will be the stable one. Since the time development of a system induces only continuous change, a solution cannot decay into another one with a different winding number and having a lower energy. For magnetic monopoles the magnetic charge is related to the winding number [12]. Sometimes the winding number is referred to as the topological charge The conservation of magnetic charge follows from the conservation of winding number. So the conservation of magnetic charge is of topological origin. It does not follow from a symmetry via Noether's theorem [12].

In the case of instanton solutions [13] the winding number is related to the number of instantons. In this case the finite action condition,

$$A_\mu(x) \longrightarrow -\frac{i}{g}\left(\partial_\mu U(x)\right) U^{-1}(x) \quad \text{as} \quad x \longrightarrow \infty , \tag{1.56}$$

where

$$x = \sqrt{x_0^2 + |\vec{x}|^2} ,$$

in Euclidean space, defines a map

$$U(x) : \quad S^3_\infty \longrightarrow S^3 \tag{1.57}$$

for SU(2) gauge potentials $A_\mu(x)$. We will discuss how the condition (1.56) is arrived at in Section 1.5. In (1.56) $U(x)$ takes values in SU(2) and SU(2) group space is topologically equivalent to S^3. This is because any matrix of SU(2) can be parametrised in the form

$$U = u_0 + i\vec{u}\cdot\vec{\tau} , \tag{1.58}$$

where the τ^i are Pauli matrices and the us are real. The condition $U^\dagger U = UU^\dagger = 1$ implies $u_0^2 + |\vec{u}|^2 = 1$. Hence SU(2) group space is isomorphic to S^3.

The homotopy group, the elements of which are equivalence classes, corresponding to the map (1.57) is

$$\Pi_3(S^3) = Z . \tag{1.59}$$

1.4 Magnetic monopoles

Dirac in 1931 [14] proposed that the mere existence of a single magnetic monopole could explain the quantisation of electric charge observed in nature. Further, the Maxwell equations will be symmetric under the exchange of electric and magnetic fields. This symmetry, eventhough present in vacuum, is broken by an electric current [15]. By considering the interaction of a charged particle with a magnetic monopole, Dirac showed that the magnetic charge Q_m should satisfy the condition

$$q\, Q_m = \frac{n}{2} \tag{1.60}$$

where n is an integer and q is the electric charge of the particle. This means that the electric charge carried by any particle is an integral multiple of the basic unit [15].

Since this thesis is mainly concerned with monopoles [4, 5, 11, 15-21] in non-abelian gauge theories we do not discuss Dirac monopoles in detail. But we shall say something about the string singularity, which is absent in non-abelian gauge theories, present in the vector potential of the Dirac monopole. The vector potential is related to the magnetic field $\vec{B}$ through the relation

$$\vec{B} = \nabla \times \vec{A} \tag{1.61}$$

and the magnetic charge is given by

$$Q_m = \frac{1}{4\pi} \int_S \vec{B} \cdot d\vec{S}, \tag{1.62}$$

where S is a closed surface enclosing the magnetic monopole. Now if (1.61) is true everywhere Q_m will be automatically zero. Since S is an arbitrary surface we see that (1.61) should fail atleast at one point on every surface enclosing the magnetic monopole. For this it is sufficient to assume that $\vec{A}$ is singular on a line joining the origin to infinity. This singularity of the vector potential is called the Dirac string. In polar coordinates the vector potential of a magnetic monopole of charge Q_m is

$$\vec{A} = \frac{Q_m}{r} \left(\frac{1 - \cos\theta}{\sin\theta} \right) \hat{\phi} \tag{1.63}$$

with the Dirac string on the negative z-axis.

Ever since 1974, there has been a resurgence of interest in magnetic monopoles due to the theoretical discovery of them as classical finite energy solutions of non-abelian gauge theories, by 't Hooft [22] and Polyakov [23]. In their formulation the basic fields are regular everywhere in a suitable gauge and the energy density of the solution is concentrated in a finite region of space so that it has a particle interpretation. No Dirac string is present eventhough it can be created by a singular gauge transformation [11]. 't Hooft and Polyakov obtained the monopole solution in the Georgi-Glashow model [24] where the gauge group is SU(2). In this model the SU(2) symmetry is broken down to U(1) by a triplet of Higgs fields (see Section 1.2). Eventhough this model is ruled out experimentally by the discovery of neutral current phenomena, it is the simplest example of a non-abelian gauge theory having monopole solutions. Further the SU(2) solution can be embedded in larger gauge groups.

The Lagrangian density of the model is

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + \frac{1}{2} D_\mu \phi^a D^\mu \phi^a - V(\phi) , \qquad (1.64\ a)$$

where

$$\begin{aligned} F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\,\epsilon_{abc} A^b_\mu A^c_\nu \\ D_\mu \phi_a &= \partial_\mu \phi_a + g\,\epsilon_{abc} A^b_\mu \phi_c \\ V(\phi) &= \frac{\lambda}{4} (\phi^2 - m^2/\lambda)^2 , \end{aligned} \qquad (1.64\ b)$$

which is the same as that considered in Section 1.2 but for the difference of an additional constant to make the minimum value of $V(\phi)$ zero.

The equations of motion are

$$\begin{aligned} D_\nu F^{\mu\nu a} &= -g\,\epsilon_{abc} (D^\mu \phi_b)\, \phi_c \\ D_\mu D^\mu \phi_a &= (m^2 - \lambda \phi^2)\, \phi_a . \end{aligned} \qquad (1.65)$$

For $m^2 > 0$ the minimum of $V(\phi)$ corresponds to the value of ϕ given by the relation

$$\phi^2 = \frac{m^2}{\lambda} . \qquad (1.66)$$

The total energy of any static solution is given by the integral

$$
\begin{aligned}
E &= \int \mathcal{H}\, d^3x = -\int \mathcal{L}\, d^3x \\
&= \int d^3x \left[\tfrac{1}{4} F_{ij}^a F_{ij}^a + \tfrac{1}{2} D_i\phi^a D_i\phi^a + V(\phi) \right. \\
&\qquad \left. + \tfrac{1}{2} F_{0i}^a F_{0i}^a + \tfrac{1}{2} D_0\phi^a D_0\phi^a \right] .
\end{aligned}
\tag{1.67}
$$

From (1.67) it is clear that all finite energy solutions assume the ground state configuration (1.66) as $r = \sqrt{x_1^2 + x_2^2 + x_3^2} \longrightarrow \infty$, that is ,

$$\phi^2 \longrightarrow m^2/\lambda \quad \text{as} \quad r \longrightarrow \infty . \tag{1.68}$$

As explained in the previous section, this finite energy condition is a map from $S_\infty^2 \longrightarrow S^2$:

$$\phi_a : \; S_\infty^2 \longrightarrow S^2 , \tag{1.69}$$

where

$$
\begin{aligned}
S_\infty^2 &= \left\{ x_1, x_2, x_3 : \sqrt{x_1^2 + x_2^2 + x_3^2} = \infty \right\} \\
S^2 &= \left\{ \phi_1, \phi_2, \phi_3 : \phi_1^2 + \phi_2^2 + \phi_3^2 = \frac{m^2}{\lambda} \right\} .
\end{aligned}
\tag{1.70}
$$

In Section 1.3 we have seen that such maps fall into homotopic classes labelled by a winding number. Accordingly, finite energy solutions are classified on the basis of the winding number. An ansatz with winding number **1** is [22, 23]

$$
\begin{aligned}
A_0^a &= 0 \\
A_i^a &= \frac{1}{g} \epsilon_{ain}\, r_n \frac{1 - K(r)}{r^2} \\
\phi_a &= \frac{1}{g} r_a \frac{H(r)}{r^2} ,
\end{aligned}
\tag{1.71}
$$

where $r_n = x_n$ and r is the radial variable. This static spherically symmetric ansatz converts the equations of motion (1.65) to two coupled nonlinear ordinary differential equations :

$$\begin{aligned} r^2 K'' &= K(K^2 - 1 + H^2) \\ r^2 H'' &= H(2K^2 - m^2 r^2 + \frac{\lambda}{g^2} H^2) , \end{aligned} \tag{1.72}$$

where K'' denotes the second derivative with respect to the argument. The energy integral, when expressed in terms of the ansatz functions, becomes

$$\begin{aligned} E = \frac{4\pi}{g^2} \int_0^\infty dr \Big\{ (K')^2 + \frac{(rH' - H)^2}{2r^2} + \frac{(K^2-1)^2}{2r^2} \\ + \frac{K^2 H^2}{r^2} + \frac{\lambda r^2}{4g^2}\left(\frac{H^2}{r^2} - \frac{g^2 m^2}{\lambda}\right)^2 \Big\} . \end{aligned} \tag{1.73}$$

For finiteness of the above integral the ansatz functions should satisfy the conditions

$$H \longrightarrow 0 , \quad K \longrightarrow 1 \quad \text{as} \quad r \longrightarrow 0 \tag{1.74 a}$$

$$H \longrightarrow \frac{gm}{\sqrt{\lambda}} r , \quad K \longrightarrow 0 \quad \text{as} \quad r \longrightarrow \infty . \tag{1.74 b}$$

The condition on H in (1.74 b) evidently results in (1.68) as expected. From (1.74 b), (1.71) and (1.53), one readily verifies the winding number to be 1 . This can be seen as follows. Comparing these three equations we find $n_a(\hat{r}) = \hat{r}_a$. This is a one-to-one mapping $S^2_\infty \longrightarrow S^2_I$ and when S^2_∞ is covered once, S^2_I is also covered once, and hence the result.

Now let us see why the solution (1.71), with the properties (1.74), is called a magnetic monopole. There is spontaneous symmetry breaking in this theory because the minimum of $V(\phi)$ corresponds to values of ϕ_a on S^2 defined in (1.70). Choosing any one out of these degenerate minima breaks the symmetry spontaneously. However, any arbitrarily chosen vacuum is still invariant under SO(2) symmetry. This can be qualitatively deduced as follows. In Section 1.2 we have chosen the vacuum value of ϕ_a pointing along the third direction (ϕ_3 axis) in the internal ϕ space. This is invariant with respect to rotation about the third axis, the relevant symmetry group being SO(2). This argument directly generalises to an arbitrarily chosen vacuum. Since SO(2) = U(1) we can say that the U(1) symmetry survives and the gauge field corresponding to this symmetry is long ranged. One can identify this U(1) gauge field with electromagnetic field. There is however no unique way to identify this U(1) gauge field throughout the space [11, 15]. 't Hooft proposed the gauge invariant definition [22]

$$F_{\mu\nu} = \frac{1}{\phi} F^a_{\mu\nu} \phi_a - \frac{1}{g\phi^3} \epsilon_{abc} \phi_a (D_\mu \phi_b)(D_\nu \phi_c) \tag{1.75}$$

for electromagnetic field tensor. This can be rewritten as [12]

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{1}{g} \epsilon_{abc} \hat{\phi}_a (\partial_\mu \hat{\phi}_b)(\partial_\nu \hat{\phi}_c) , \tag{1.76}$$

where $\hat{\phi}_a = \phi_a / \phi$ and $A_\mu = A^a_\mu \hat{\phi}_a$. From (1.71) we find for $r > 0$, $A_\mu = 0$ and $\hat{\phi}_a = \hat{r}_a$. Substituting this in (1.76) and separating the electric and magnetic fields, we find,

$$\begin{aligned} E_i &= F_{0i} = 0 \\ B_i &= \frac{1}{2} \epsilon_{ijk} F_{jk} = \frac{1}{g} \frac{\hat{r}_i}{r^2} \end{aligned} \tag{1.77}$$

This is the static electromagnetic field of a magnetic monopole with magnetic charge $1/g$ situated at the origin.

According to the 't Hooft definition (1.75), $F_{\mu\nu}$ is singular at the origin [11, 15] eventhough all fields of a non-abelian monopole are regular everywhere. Another equivalent definition is [25]

$$F_{\mu\nu} = F^{a}_{\mu\nu} \hat{\phi}_a \tag{1.78}$$

This is nonsingular everywhere. Both the definitions coincide in the $r \to \infty$ limit and the magnetic field becomes indistinguishable from that of a Dirac monopole.

The relation between the magnetic charge and the winding number of the map (1.69) has been established by Arafune, Freund and Goebel [12]. They have shown that the magnetic charge Q_m is given by

$$Q_m = n/g \tag{1.79}$$

where n is the winding number. They also observed that the conservation of magnetic charge is of topological origin and it does not follow from a symmetry of the theory.

Let us come to the solution part of the subject. In order to find the detailed structure and the classical mass, which is equal to the energy E, the static field equations (1.72) should be solved. Due to the complexity of the coupled nonlinear differential equations, an exact analytic solution has not been obtained so far. A unique [27] solution satisfying the boundary conditions (1.74) has been shown to exist for the system (1.72), and it should be expressible in terms of real analytic functions of r [28]. Only its numerical form is known so far. The value of the integral can be written as

$$E = \frac{4\pi}{g^2} M_W f(\lambda/g^2) \tag{1.80}$$

where M_W is the vector boson mass $gm/\sqrt{\lambda}$. The value of f obtained by various workers is given below [15].

$$f(0) = 1 \quad (\text{Prasad and Sommerfield [29]})$$

$$f(0.1) = 1.1 \quad (\text{'t Hooft [22]})$$

$$f(0.5) = 1.42 \quad (\text{Julia and Zee [30]})$$

$$f(10) = 1.44 \quad (\text{'t Hooft [22]}) .$$

Further it has been shown [31, 32, 33] that f is a slowly varying function of λ/g^2 . For $\lambda/g^2 \to \infty$ the value of f is 1.787 [31, 32].

Eventhough an exact solution has not been obtained for general m and λ , a solution has been obtained [29] in the limit of vanishing Higgs potential. In this limit, called Prasad - Sommerfield (PS) limit [29], $m \to 0$ and $\lambda \to 0$ with m^2/λ finite.

The spontaneous symmetry breaking survives because a classical solution can have its Higgs field assuming non-zero value $m/\sqrt{\lambda}$ as $r \to \infty$. So a nontrivial map $S^2 \to S^2$ is possible which gives a topologically stable solution. The solution to (1.72) in the PS limit is [29]

$$K(r) = \beta r/\sinh \beta r \quad , \quad H(r) = \beta r \coth \beta r - 1 , \tag{1.81}$$

where $\beta = M_W = gm/\sqrt{\lambda}$, and this was originally obtained by trial and error. In Chapter 3 we give a systematic derivation of the PS solution (1.81).

It has been shown that the PS solution satisfied the lower bound of energy in its homotopic class [26] . Solutions, in any topological sector satisfying the lower bound of energy (see page 16) obey a first order equation called the Bogomolny equation [26],

$$F_{ij}^{a} = \epsilon_{ijk} D_k \phi_a \quad . \tag{1.82}$$

All solutions of Bogomolny equation are solutions of the second order field equations (1.65) with m and λ zero. But the converse is not true.

It can be shown that K and H approach the boudnary condition (1.74 b) in the following way [15] :

$$\begin{aligned} &\text{As} \quad r \to \infty \\ &\qquad K(r) \longrightarrow O(e^{-M_w r}) \\ &\qquad H(r) \longrightarrow \frac{gm}{\sqrt{\lambda}} r + O(e^{-\mu r}) \ , \end{aligned} \tag{1.83}$$

where μ is the mass of the massive Higgs particle $\sqrt{2}\, m$. This means that the 't Hooft-Polyakov monopole has a definite size determined by the Compton wavelength of massive fields. The massive fields exist inside this core and outside they vanish exponentially leaving a field configuration exactly similar to that of the Dirac monopole.

In the PS limit there exist several solutions, albeit of infinite energy such as the following [34, 35]:

$$K = 0 \ , \quad H = 1 \tag{1.84a}$$

$$K = \frac{\beta r}{1+\beta r} \ , \quad H = \frac{1}{1+\beta r} \tag{1.84b}$$

$$K = \frac{\beta r}{\sinh(\beta r + \alpha)} \ , \quad H = \beta r \coth(\beta r + \alpha) - 1 \tag{1.84c}$$

$$K = \frac{\alpha \beta r e^{\beta r}}{\alpha^2 e^{2\beta r} - 1} \ , \quad H = \beta r \frac{\alpha^2 e^{2\beta r} + 1}{\alpha^2 e^{2\beta r} - 1} - 1 \ . \tag{1.84d}$$

All these solutions obey the first order Bogomolny equations and can be obtained by integrating them [34, 35]. In general these solutions are singular at $r = 0$ and the energy is infinite. They are called point monopoles and their significance is not clear at present. The finite energy PS solution is a special case of (1.84c) and (1.84d).

Another point solution is given by

$$K = 0 \quad , \quad H = 0 \tag{1.85}$$

In this case the Higgs field is zero everywhere and the theory consists only of SU(2) gauge fields. All the three gauge fields are long ranged. In this case it has been shown [16, 36] that no classical finite energy solution exists. The gauge fields corresponding to (1.85) are

$$A_0^a = 0 \quad , \quad A_i^a = \frac{1}{g}\,\epsilon_{ain}\,\frac{r_n}{r^2} \quad . \tag{1.86}$$

This field configuration was obtained by Wu and Yang [37] five years before the work of 't Hooft and Polyakov. This is singular at the origin and the energy is infinite. One may think that the singularity of Wu-Yang monopole is smoothed by the scalar field to yield the regular 't Hooft-Polyakov monopole.

The possibility of non-self-dual solutions* (solutions which do not satisfy the Bogomolny lower bound (1.82)), having energy greater than the PS solution, in the PS limit was investigated by Frampton [38]. He found that the PS solution is unique and there are no other finite energy solutions. However, according to Kerner [39] a family of finite energy solutions exists for (1.72) in the PS limit and only one out of it (which is nothing but the PS solution), having the lowest energy, satisfies the Bogomolny equation. Recently we

---

* Bogomolny equation (1.82) is equivalent to static self-duality equations (see page 31) [11] .

came to know that this result is wrong and Frampton's result is correct [40].

The 't Hooft-Polyakov monopole possesses only magnetic charge and does not carry electric charge. Julia and Zee [30] showed that this is because A_0^a is put equal to zero. If A_0^a is non-zero, the electric field does not vanish. They showed that a finite energy solution is still possible with non-zero A_0^a. Particles carrying both electric and magnetic charges are called dyons [41]. Julia and Zee proposed the ansatz

$$A_0^a = \frac{1}{g} r_a \frac{J(r)}{r^2} , \quad A_i^a = \frac{1}{g} \epsilon_{ain} \frac{r_n}{r^2} (1 - K(r)) ,$$
$$\phi_a = \frac{1}{g} r_a \frac{H(r)}{r^2} . \tag{1.87}$$

The equations motion now become

$$\begin{aligned} r^2 J'' &= J(2K^2) \\ r^2 H'' &= H(2K^2 - m^2 r^2 + \frac{\lambda}{g^2} H^2) \\ r^2 K'' &= K(K^2 - 1 + H^2 - J^2) . \end{aligned} \tag{1.88}$$

The energy integral will have contributions from A_0^a :

$$E = \frac{4\pi}{g^2} \int_0^\infty dr \left\{ (K')^2 + \frac{1}{2} \frac{(rH' - H)^2}{r^2} + \frac{1}{2} \frac{(rJ' - J)^2}{r^2} \right.$$
$$\left. + \frac{(K^2 - 1)^2}{2r^2} + \frac{K^2(H^2 - J^2)}{r^2} + \frac{\lambda r^2}{4g^2} \left(\frac{H^2}{r^2} - \frac{g^2 m^2}{\lambda} \right)^2 \right\} . \tag{1.89}$$

A finite energy solution is shown to exist with the ansatz functions having the following boundary behaviour : As $r \longrightarrow \infty$

$$\begin{aligned} K(r) &\longrightarrow O(\exp[-\sqrt{M_w^2 - M^2}\, r]) \\ J(r) &\longrightarrow Mr + b + O(1/r) \\ H(r) &\longrightarrow \frac{gm}{\sqrt{\lambda}} r + O(e^{-\mu r}) , \end{aligned} \tag{1.90}$$

where M and b are two parameters [11]. From the boundary behaviour of $K(r)$ we conclude that the values of M are restricted,

$$|M| < M_W . \tag{1.91}$$

for finite energy. However there is no resgtriction on the continuous parameter b. It is related to the electric charge of the dyon [11]

$$Q = \frac{4\pi}{g} b \tag{1.92}$$

and this is not quantised at the classical level. In the quantum theory Q is quantised [42, 43].

Exact dyon solutions were obtained in the PS limit [29] :

$$K(r) = \frac{\beta r}{\sinh \beta r} , \quad J(r) = \sinh \eta \, (\beta r \coth \beta r - 1) \tag{1.93}$$

$$H(r) = \cosh \eta \, (\beta r \coth \beta r - 1)$$

where η is an arbitrary parameter. The electric charge of the PS dyon is

$$Q = - \frac{4\pi}{g} \sinh \eta . \tag{1.94}$$

Magnetic monopoles are present in almost all grand unified theories and together with standard hot big bang cosmology, they predict an unacceptably large density of monopoles in the universe [44]. Eventhough the existence of monopoles has not been confirmed by experiment [45], it is possible to predict an upper limit on their abundance. This imposes severe constraints on cosmological models [46]. Monopoles and dyons in non-abelian gauge theories lead to several other interesting physical phenomena. These include solitons with fractional fermion number [47], spin isospin mixing [48], fermion number breaking in the presence of monopoles [49] etc.

1.5 Euclidean solutions

Classical solutions of gauge theories in Euclidean space-time have played an important role in unravelling their structure. Three types of Euclidean solution have been discovered. They are the instanton, meron and elliptic solutions. In this section we shall give a brief account of instanton and meron solutions. (The Euclidean solutions are reviewed in Ref. 11).

Instantons are finite action solutions in Euclidean space having zero energy. In gauge theory it was first obtained by Belavin *et. al.* [1 3]. As in the case of monopoles, they can be classified by a winding number that arises from the condition of finiteness of the Euclidean action,

$$
\begin{aligned}
S &= \int \mathcal{L}\, d^4x = \int \frac{F^a_{\mu\nu} F^{\mu\nu a}}{4} d^4x \\
&= \int tr\left(\frac{F_{\mu\nu} F_{\mu\nu}}{2} \right) d^4x \quad ,
\end{aligned}
\tag{1.95}
$$

where we have used the matrix notation :

$$
\begin{aligned}
F_{\mu\nu} &= \frac{1}{2} F^a_{\mu\nu} \tau^a \\
A_\mu &= \frac{1}{2} A^a_\mu \tau^a \quad .
\end{aligned}
\tag{1.96}
$$

For finite action we require

$$
F_{\mu\nu} \longrightarrow 0 \quad as \quad x \longrightarrow \infty \quad , \tag{1.97}
$$

where

$$
x = \sqrt{x_0^2 + |\vec{x}|^2} \quad .
$$

This does not necessarily mean $A_\mu(x) \longrightarrow 0$ as $x \rightarrow \infty$. We can have a non trivial A_μ :

$$A_\mu(x) \longrightarrow -\frac{i}{g}\left(\partial_\mu U(x)\right) U^{-1}(x) \tag{1.98}$$

where $U(x)$ is an arbitrary gauge transformation function of SU(2).

The condition (1.98) defines a map $S^3 \rightarrow S^3$ as explained in Section 1.3. The winding number of this map can be expressed as [13]

$$n = \frac{g^2}{16\pi^2}\int d^4x \; tr\left(F_{\mu\nu}\tilde{F}_{\mu\nu}\right) . \tag{1.99}$$

There is a lower bound for the action (1.95) in each homotopic class as in the case of monopoles. To obtain this we start with the inequality

$$\int tr\left(F_{\mu\nu} \pm \tilde{F}_{\mu\nu}\right)^2 \geq 0 .$$

Using the relation

$$\left(F_{\mu\nu} \pm \tilde{F}_{\mu\nu}\right)^2 = 2\left(F_{\mu\nu}F_{\mu\nu} \pm F_{\mu\nu}\tilde{F}_{\mu\nu}\right) ,$$

this can be rewritten as

$$\int tr\left(F_{\mu\nu}F_{\mu\nu}\right) d^4x = \left| \int tr\left(F_{\mu\nu}\tilde{F}_{\mu\nu}\right) d^4x \right| .$$

Hence we have the action

$$S \geq \frac{8\pi^2}{g^2}|n| . \tag{1.100}$$

The equality (lower bound) is achieved if

$$F_{\mu\nu} = \pm \tilde{F}_{\mu\nu} = \pm \frac{1}{2}\epsilon_{\mu\nu\rho\lambda}F_{\rho\lambda} . \tag{1.101}$$

For positive (negative) sign, (1.101) is called self-daulity (anti-self-duality) equation. Gauge fields satisfying the lower bound are called self-dual fields.

The instanton solution discovered by Belavin *et.al.* [13] is self-dual and has a winding number 1. Its explicit form is

$$A_\mu(x) = -\frac{i}{g}\frac{x^2}{x^2+\lambda^2}(\partial_\mu U(x))\,U^{-1}(x) \tag{1.102}$$

where

$$U(x) = (x_0 + i\,\vec{x}\cdot\vec{\tau})/x$$

Here $\vec{\tau}$ are Pauli matrices and λ is some scale parameter which determines the size of the instanton (The action density $\frac{1}{2}\,tr\,F_{\mu\nu}F_{\mu\nu}$ is concentrated in a finite volume in Euclidean space). For $x \gg \lambda$ we have

$$A_\mu(x) = -\frac{i}{g}(\partial_\mu U(x))\,U^{-1}(x)\,. \tag{1.103}$$

Instantons cannot be interpreted as particles because they are imaginary time solutions. It has been shown [50, 51] that they correspond to tunneling between vacuum states having different topological numbers. There exist inequivalent vacua in a gauge theory labelled by a winding number. $A_\mu = 0$ corresponds to only one of them having zero winding number. Let us see how different vacua arise in a gauge theory. Choosing the gauge

$$A_0 = 0 \tag{1.104}$$

we see that the vacuum solution $(F_{\mu\nu} = 0)$ is given by

$$A_\mu(x) = -\frac{i}{g}(\partial_\mu U(\vec{x}))\,U^{-1}(\vec{x})\,, \tag{1.105}$$

where $U(\vec{x})$ is a time-independent gauge transformation of SU(2). Further, assume the condition [50, 52]

$$U(\vec{x}) \longrightarrow I \quad as \quad r \longrightarrow \infty \tag{1.106}$$

on the function $U(\vec{x})$. Then it can be seen that $U(\vec{x})$ can be classified into equivalence classes. $U(\vec{x})$ defines a map from 3-dimensional space with all points at infinity identified into SU(2) group space. 3-space with all points at infinity identified is equivalent to S^3. So $U(\vec{x})$ defines the map

$$U(\vec{x}) \; : \quad S^3 \longrightarrow S^3 , \tag{1.107}$$

which, we know, can be classified into homotopic classes labelled by an integer winding number. Vacuum with winding number zero is given by

$$U(\vec{x}) = U_0(\vec{x}) = I \tag{1.108}$$

which gives $A_\mu = 0$. A gauge function $U(\vec{x})$ which is not continuously deformable to $U_0(\vec{x})$ and having winding number 1 is [50]

$$U_1(\vec{x}) = \frac{r^2 - \lambda^2}{r^2 + \lambda^2} - 2i\lambda \frac{\vec{x}\cdot\vec{\tau}}{r^2 + \lambda^2} . \tag{1.109}$$

A guage function belonging to the n^{th} homotopic class (winding number n) is

$$U_n(\vec{x}) = (U_1(\vec{x}))^n \tag{1.110}$$

Now consider the instanton solution. In the $A_0 = 0$ gauge it has the following behaviour

$$\begin{aligned} A_\mu(x) &\longrightarrow -\frac{i}{g}\,(\partial_\mu U_1(\vec{x}))\, U_1^{-1}(\vec{x}) \quad as \quad x_0 \longrightarrow -\infty \\ A_\mu(x) &\longrightarrow 0 \quad as \quad x_0 \longrightarrow +\infty . \end{aligned} \tag{1.111}$$

So the $n=1$ and $n=0$ vacua are connected via an imaginary time solution, the instanton. Instantons with different winding numbers and arbitrary location have been obtained [53] and any two vacua are connected by the appropriate instanton solutions. Imaginary time solutions interpolating between distinct vacua corresponds to tunneling between the vacua.

Due to this tunneling the inequivalent vacua of a gauge theory are non-degenerate. It has been shown that the states, which are superpositions of the different vacua, can be parameterised by an angle θ and the energy depends on θ $(0 < \theta < 2\pi)$. There exists an energy band $E(\theta)$ in the theory [50, 51]. The study of instantons and vacuum tunneling lead to the solution of the U(1) problem [54].

There are also certain other types of interesting Euclidean solutions of SU(2) gauge theory. These are called meron solutions. Since they are imaginary time solutions they also correspond to some kind of tunneling. But this tunneling process is not well understood. The main reason for this is that a multimeron configuration representing an arbitrary number of merons located at arbitrary points in Euclidean space-time is not obtained so far [11]. Unlike in the case of instantons, meron solutions are singular and the Euclidean action is infinite. Further, they are non-self-dual and have a half integral topological charge.

The following are some meron configurations in the $A_0^a = 0$ gauge [11]:

1) One-meron solution

$$A_0^a = 0 \ , \ A_i^a = \frac{1}{g} \epsilon_{ain} \frac{r_n}{r^2} \left(1 - \frac{x_0}{\sqrt{x^2}}\right) . \tag{1.112}$$

The solution is singular at $x=0$ and the topological charge is concentrated at the origin.

2) Two-meron solution

$$A_0^a = 0 \ , \ A_i^a = \frac{1}{g} \epsilon_{ain} \frac{r_n}{r^2} \left(1 + \frac{r^2 + (x_0 - a)(x_0 - b)}{\sqrt{(x-a)^2 (x-b)^2}} \right) , \tag{1.113}$$

where a and b are two constant four-vectors. The topological charges are concentrated at the two singular points $x = a$ and $x = b$.

3) Meron-antimeron solution

$$A_0^a = 0 \ , \ A_i^a = \frac{1}{g} \epsilon_{ain} \frac{r_n}{r^2} \left(1 - \frac{x^2 - a^2}{\sqrt{(x+a)(x-a)}} \right) . \tag{1.114}$$

This describes an antimeron at a and a meron at $-a$.

An interesting property of the meron is its Euclidean time development. For the one-meron solution (1.112)

$$\begin{aligned} A_i^a &\longrightarrow 0 \quad as \quad x_0 \longrightarrow +\infty \\ A_i^a &\longrightarrow \frac{1}{g} \epsilon_{ain} \frac{2 r_n}{r^2} = -\frac{i}{g} (\partial_i \omega) \, \omega^{-1} \quad as \quad x_0 \longrightarrow -\infty , \end{aligned} \tag{1.115}$$

where $\omega = \omega^{-1} = \hat{r} \cdot \vec{\sigma}$ [11]. So a meron solution starts from the trivial vacuum configuration and ends in a configuration which is not trivially zero. However, both vacua are in the same homotopic class and both have zero winding number. When $x_0 = 0$ the field configuration becomes that of the Wu-Yang monopole (1.86).

The significance of meron solutions is not clear at present. It has been argued that they may lead to colour confinement [55].

CHAPTER - 2

RELATION BETWEEN MAGNETIC CHARGE AND TOPOLOGY OF DYON FIELDS

2.1 Introduction

It is a well known fact that the conservation of magnetic charge in Yang-Mills-Higgs system is of topological character and that the magnetic charge is related to the topology of Higgs fields [12]. However in the case of dyons, which possess electric as well as magnetic charge, the gauge fields themselves may belong to non-trivial topological classes [56, 57]. In this case, the magnetic charge is related to the topological charge of the gauge field configuration. This situation is very desirable because it gives a description of monopoles even in the absence of Higgs scalars, that is, in theories where the symmetry breaking is achieved dynamically.

Christ and Jackiw [56] showed, without mentioning Higgs scalars, that the magnetic pole strength is related to a topological index. Pak and Percacci rederived their result using topological methods [57]. They proved the topological equivalence of gauge and Higgs fields in the dyon sector for any simply connected group broken down to an abelian subgroup.

In this chapter we present our investigation on this subject. We use an altogether different and simpler approach to establish the connection between the magnetic charge and the topological index. For this we use an alternative definition of electromagnetic field in the spontaneously broken theory [11]. The result is demonstrated when the residual symmetry is U(1) and then generalised to the case where the residual symmetry group is arbitrary.

2.2 Relation between magnetic charge and topological charge

In a Yang-Mills theory based on an arbitrary Lie group spontaneously broken down to a U(1) subgroup or any one parameter subgroup, it is possible to define an electromagnetic field tensor using a unit vector n_a is the group space [11]. This unit vector is obtained from the boundary behaviour of the time component of the gauge field :

$$A_0^a \longrightarrow M\, n_a(\theta,\phi) \quad \text{as} \quad r \longrightarrow \infty . \tag{2.1}$$

with $n_a n_a = 1$ and M is a parameter determined by the asymptotic form of the solution (see Section 1.4). The electromagnetic field tensor is defined to be

$$F_{\mu\nu} = F_{\mu\nu}^{a}\, n_a(\theta,\phi) \tag{2.2}$$

From the antisymmetry of the dual tensor $\tilde{F}^{a}_{\mu\nu}$ we get the current conservation

$$\partial_\mu J^\mu = 0 \ , \tag{2.3}$$

where

$$J^\mu = \partial_\nu \left(\tilde{F}^{\mu\nu a} n_a \right) . \tag{2.4}$$

(2.3) implies the conserved magnetic charge

$$\begin{aligned} Q_m &= \frac{1}{4\pi} \int \partial_i \left(\tilde{F}^{0ia} n_a \right) d^3x \\ &= \frac{1}{4\pi} \int \vec{\nabla} \cdot \left(\vec{B}^a n^a \right) d^3x , \end{aligned} \tag{2.5}$$

where we have used the notation *

$$B_i^a = \frac{1}{2} \epsilon_{ijk} F^{jka} = \tilde{F}_{0i}^a$$
$$E_i^a = F_{0i}^a \quad . \tag{2.6}$$

Using Gauss's theorem we have

$$Q_m = \frac{1}{4\pi} \int_S \left(\tilde{F}^{0ia} n_a \right) dS_i \tag{2.7}$$

where S is the surface at spatial infinity.

Since A_0^a must be nonvanishing, if the above definition of magnetic charge is to be useful, then it should be applicable only to dyons. We know that dyons are static finite energy solutions. However, unlike pure monopoles, dyons are not strictly static in that if we go to a gauge where $A_0^a = 0$ by means of a time-dependent gauge transformation, the spatial components become time-dependent. Further, they will be periodic upto a gauge transformation [57]:

$$\underset{\sim}{A}_i (r, t + T) = G^{-1}(r) \underset{\sim}{A}_i(r,t) G(r) + \frac{i}{g} G^{-1}(r) \partial_i G(r) \tag{2.8}$$

where $\underset{\sim}{A}_i = A_i^a T^a$ and the T^a are gauge group generators in the adjoint representation. The period T is given by* [57]

$$T = \frac{2\pi}{Mg} \quad . \tag{2.9}$$

For finite energy we must have the condition

$$\underset{\sim}{F}_{\mu\nu} \longrightarrow 0 \quad \text{as} \quad r \longrightarrow \infty \quad . \tag{2.10}$$

---

* The factor g is absent in Ref.57. This is an error which we have corrected in this work.

This gives

$$\underset{\sim}{A}_\mu \longrightarrow -\frac{i}{g}(\partial_\mu U)\, U^{-1} \quad \text{as} \quad r \longrightarrow \infty \,, \tag{2.11}$$

where U takes values in the gauge group. Now U defines the map

$$U : \; S^1 \otimes S^2_\infty \longrightarrow G \tag{2.12}$$

where G is the group space. The winding number of the map being given by [57]

$$\nu = \int_{S^1 \otimes S^2_\infty} \frac{dS_\mu}{24\pi^2}\, \epsilon^{\mu\nu\rho\lambda}\, tr\,[\, U^{-1}(\partial_\nu U) \cdot U^{-1}(\partial_\rho U) \cdot U^{-1}\partial_\lambda U \,]$$

$$= \frac{g^2}{16\pi^2} \int_0^T dt \int d^3x \;\; tr(\underset{\sim}{F}_{\mu\nu}\, \underset{\sim}{\tilde{F}}^{\mu\nu})$$

$$= \frac{g^2}{32\pi^2} \int_0^T dt \int d^3x \;\; F^a_{\mu\nu}\, \tilde{F}^{\mu\nu a} \quad . \tag{2.13}$$

To establish the connection between magnetic charge defined in (2.7) and winding number we note that since the surface integral there is to be evaluated on a surface at infinity, n_a can be replaced by A^a_0/M using (2.1). Hence

$$Q_m = \frac{1}{4\pi M} \int \tilde{F}^{0ia}\, A^a_0\, dS_i$$

$$= \frac{1}{4\pi M} \int \partial_i(\tilde{F}^{0ia}\, A^a_0)\, d^3x \quad . \tag{2.14}$$

This can be rewritten as (See Appendix 2.A).

$$Q_m = \frac{1}{4\pi M} \int \tilde{F}^{0ic}\left(-g f_{abc} A^b_i A^a_0 + \partial_i A^c_0\right) , \tag{2.15}$$

where f_{abc} are the structure constants of the gauge group.

From the definition of the field strength it is obvious that Q_m can be expressed as

$$Q_m = -\frac{1}{4\pi M}\int (\tilde{F}^{0ia} F^a_{0i})\, d^3x - \frac{1}{4\pi M}\int (\tilde{F}^{0ia}\, \partial_0 A^a_i)\, d^3x$$

$$= -\frac{1}{16\pi M}\int (\tilde{F}^{\mu\nu a} F^a_{\mu\nu})\, d^3x$$

$$- \frac{1}{4\pi M}\int (\vec{B}^a \cdot \frac{\partial \vec{A}^a}{\partial t})\, d^3x \; . \qquad (2.16)$$

Since Q_m is a conserved quantity, integration of (2.16) with respect to t yields

$$T\, Q_m = -\frac{1}{16\pi M}\int_0^T dt \int d^3x\, (\tilde{F}^a_{\mu\nu} F^{\mu\nu a})$$

$$- \frac{1}{4\pi M}\int_0^T dt \int d^3x\, (\vec{B}^a \cdot \frac{\partial \vec{A}^a}{\partial t}) \, . \qquad (2.17)$$

Multiplying throughout by $\frac{g^2 M}{2\pi}$ we get

$$T\, Q_m \frac{g^2 M}{2\pi} = -\frac{g^2}{32\pi^2}\int_0^T dt \int d^3x\, (\tilde{F}^a_{\mu\nu} F^{\mu\nu a})$$

$$- \frac{g^2}{8\pi^2}\int_0^T dt \int d^3x\, (\vec{B}^a \cdot \frac{\partial \vec{A}^a}{\partial t}) \, . \qquad (2.18)$$

For static fields the second term on the right hand side of (2.18) vanishes. The remaining integral is nothing but the topological index (winding number) (2.13). Substituting for T from (2.9) we find, for static fields,

$$Q_m = -\frac{\nu}{g} \; . \qquad (2.19)$$

In the general case where the gauge potential have a periodic time dependence, it will be convenient to work in the gauge $A_0^a = 0$. Using the relation [56]

$$
-\frac{1}{8\pi^2}\int_0^T dt \int d^3x \left(\vec{B}^a \cdot \frac{\partial \vec{A}}{\partial t} \right) - \frac{1}{8\pi^2}\int_0^T dt \int d^3x \left(\vec{B}_a \cdot \vec{D} A_0^a \right)
$$

$$
= -\frac{1}{32\pi^2}\int_0^T dt \int d^3x \left(\tilde{F}^a_{\mu\nu} F^{\mu\nu a} \right) , \tag{2.20}
$$

where now T is the period of the gauge potentials, (2.18) can be rewritten as

$$
Q_m \frac{T M g^2}{8\pi^2} = -2\nu \tag{2.21}
$$

where ν is the topological index (2.13).

2.3 Generalisation

The procedure described above in Section 2.2 can be generalised to the case where the unbroken symmetry group is non-abelian. If p generators remain unbroken we will have p independent vectors in the internal space corresponding to each of which we can have a conserved magnetic charge,

$$
Q_{m(\alpha)} = \frac{1}{4\pi}\int \partial_i \left(\tilde{F}^{0ia} n_{a(\alpha)} \right) d^3x , \tag{2.22}
$$

$$
\alpha = 1, 2, \dots p ,
$$

where $n_{a(\alpha)}$ is defined through the boundary condition

$$
A_0^a \xrightarrow[r\to\infty]{} \sum_{\alpha}^{p} M_{(\alpha)} n_{a(\alpha)} . \tag{2.23}
$$

Multiplying (2.22) by $M_{(\alpha)}$ and summing over α yields

$$
\sum_{\alpha} Q_{m(\alpha)} M_{(\alpha)} = \frac{1}{4\pi}\int \partial_i \left(\tilde{F}^{0ia} A_0^a \right) d^3x , \tag{2.24}
$$

where we have replaced $\sum_{\alpha} M_{(\alpha)} n_{a\,(\alpha)}$ by A_0^a in the integral, by the argument given previously (following Eq. 2.13). The right hand side of (2.24) contains the same integral as in (2.14) and following the same line of reasoning as developed earlier, we can identify it as the topological index of the dyon configuration.

Finally we observe that the above established connection between magnetic charge and topological index can readily be applied to finite temperature monopoles also since these are just the periodic solutions in Euclidean space [58].

2.A Appendix

In this section we shall prove

$$\partial_i (\tilde{F}^{0ia} A_0^a) = \tilde{F}^{0ic} (-g f_{abc} A_i^b A_0^a + \partial_i A_0^c) . \tag{2.25}$$

We have the Bianchi identity

$$D_\nu \tilde{F}^{\mu\nu a} = 0 . \tag{2.26}$$

For $\mu = 0$ this equation becomes

$$\partial_i \tilde{F}^{0ia} + g f_{abc} A_i^b \tilde{F}^{0ic} = 0 \tag{2.27}$$

Now

$$\begin{aligned} \partial_i (\tilde{F}^{0ia} A_0^a) &= (\partial_i \tilde{F}^{0ia}) A_0^a + \tilde{F}^{0ia} (\partial_i A_0^a) \\ &= -g f_{abc} A_i^b \tilde{F}^{0ic} A_0^a + \tilde{F}^{0ia} \partial_i A_0^a \\ &= \tilde{F}^{0ic} (-g f_{abc} A_i^b A_0^a + \partial_i A_0^c) , \end{aligned} \tag{2.28}$$

which is the required result.

CHAPTER - 3

SYSTEMATIC DERIVATION OF MONOPOLE SOLUTIONS

3.1 Introduction

In this chapter we report a systematic study on the second order nonlinear differential equations of the 't Hooft-Polyakov theory. We have analysed these equations in the PS limit. The method we have used is the direct method of Hirota [59]. This method has been very useful in constructing solutions to (1+1) dimensional scalar field theories. A first application of the method to a coupled system of nonlinear differential equations was recently made by Hirota and Satsuma [60]. Here we apply this method to find solutions of the effectively one dimensional coupled field equations of the monopole theory.

All the exact monopole solutions, regular or point singular, reported in the literature were either obtained by guess work [29] or by integrating the first order Bogmolny equation [26, 34, 35]. However, all solutions of the second order field equations may not satisfy the Bogomolny equation*. All the solutions we have obtained using the Hirota method satisfied the Bogomolny equation and were reported earlier in the literature. Eventhough there are no new solutions, ours is a systematic method to generate all the monopole solutions from the second order field equations in the PS limit.

In Section 3.2 we introduce the Hirota method and in Section 3.3 this method is applied to the 't Hooft-Polyakov equations in the PS limit and the solutions

* After the completion of this work we learned [40] that there are no such regular solutions for the case of the spherically symmetric ansatz (1.71) which is used in our work. In this connection see also papers by Frampton [38] and Kerner [39].

are obtained as a ratio of infinite series which depend upon a number of parameters. By adjusting the parameters the series and the solution are expressed in terms of elementary functions. This is discussed in Section 3.4.

3.2 Direct method of Hirota

Hirota developed a direct method [59] of finding exact solutions of a number of nonlinear differential equations. In this method the dependent variable is expressed as the ratio of two dependent variables g and f. When this ratio is substituted in the original differential equation we get an equation with two dependent variables. The derivatives of functions can always be combined and expressed in terms of bilinear derivatives, the n^{th} order bilinear derivative being defined as

$$D_x^n A(x,t)\cdot B(x,t) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n A(x,t)\, B(x',t)\Big|_{x=x'} . \tag{3.1}$$

The nonlinear equation which contains bilinear derivatives of g and f is split into two coupled nonlinear equations. The functions g and f are then expanded as power series in a parameter ϵ , as in perturbation theory. The individual functions in the power series are evaluated by successively integrating the differential equations that follow from equating the same powers of ϵ on both sides of the split nonlinear equations. Solutions can be obtained either by terminating the series by some technique or by summing the infinite power series.

3.3 Hirota's method for 't Hooft-Polyakov equations

In the PS limit the equations of motion of the SU(2) gauge theory (1.72) become

$$r^2 K'' = K(K^2 - 1 + H^2) \tag{3.2a}$$

$$r^2 H'' = 2HK^2 . \tag{3.2b}$$

We make a dependent variable transformation,

$$K(r) = \frac{A(r)}{B(r)} \quad , \quad H(r) = \frac{C(r)}{B(r)} \quad , \tag{3.3}$$

which modifies (3.2) into

$$r^2 (B\, D^2 A\cdot B \; - \; A\, D^2 B\cdot B) \;=\; A\,(A^2 - B^2 + C^2) \tag{3.4a}$$

$$r^2 (B\, D^2 C\cdot B \; - \; C\, D^2 B\cdot B) \;=\; 2 C A^2 \tag{3.4b}$$

where $D^2 A\cdot B$ and $D^2 B\cdot B$ are second order bilinear operators :

$$D^2 A\cdot B = \left(\frac{d}{dr} - \frac{d}{dr'}\right)^2 A(r)\, B(r') \Big|_{r=r'}$$

$$= A''B - 2A'B' + AB'' \; . \tag{3.5}$$

We split (3.4b) using an arbitrary function $\eta(r)$ to get

$$r^2 D^2 B\cdot B + \eta\, B^2 = -2A^2 \tag{3.6a}$$

$$r^2 D^2 C\cdot B + \eta\, CB = 0 \tag{3.6b}$$

One readily verifies that solutions to (3.6) are solutions of (3.4b). One may consider alternative splitting patterns as well. However the present splitting procedure is advantageous because it reduces the degree of nonlinearity from three to two. (3.4a) now becomes

$$r^2 B\, D^2 A\cdot B = A\,(C^2 + (\eta + 1) B^2 - A^2) \; . \tag{3.6c}$$

In Hirota's method the functions A, B and C are expanded as perturbation series. A consistent expansion of this kind is

$$\begin{aligned} A(r) &= \epsilon A_1(r) + \epsilon^2 A_2(r) + \cdots\cdots \\ B(r) &= 1 + \epsilon B_1(r) + \epsilon^2 B_2(r) + \cdots\cdots \\ C(r) &= 1 + \epsilon C_1(r) + \epsilon^2 C_2(r) + \cdots\cdots \quad , \end{aligned} \tag{3.7}$$

where ϵ is a parameter. Substituting (3.7) in (3.6) and comparing the zeroth power of ϵ on both sides we see that $\eta(\lambda)$ should be zero for consistency. Hence (3.6) can be rewritten as

$$\lambda^2 D^2 B\cdot B = -2A^2 \tag{3.8a}$$

$$D^2 C\cdot B = 0 \tag{3.8b}$$

$$\lambda^2 B D^2 A\cdot B = A\,(C^2 - B^2 - A^2)\ . \tag{3.8c}$$

The functions $A_1, B_1, C_1;\ A_2, B_2, C_2;\ \cdots\cdots$ are obtained by integrating successively the linear equations which follow by substituting (3.7) in (3.8) and comparing coefficients of $\epsilon, \epsilon^2, \cdots\cdots$ respectively. For example, the first two sets of equations are

$\epsilon \Rightarrow$

$$\frac{d^2 B_1}{d\lambda^2} = 0\ , \quad \frac{d^2 C_1}{d\lambda^2} = 0\ , \quad \frac{d^2 A_1}{d\lambda^2} = 0 \tag{3.9a}$$

$\epsilon^2 \Rightarrow$

$$\begin{aligned} \lambda^2\,(2D^2 B_2\cdot 1 + D^2 B_1\cdot B_1) &= -2A_1^2 \\ D^2 C_2\cdot 1 + D^2 C_1\cdot B_1 + D^2 1\cdot B_2 &= 0 \\ \lambda^2(D^2 A_2\cdot 1 + D^2 A_1\cdot B_1 + B_1 D^2 A_1\cdot 1) &= 2A_1(C_1 - B_1)\ . \end{aligned} \tag{3.9b}$$

We have choosen

$$\begin{aligned} A_1 &= a\lambda \\ B_1 &= b\lambda + d \\ C_1 &= c\lambda + d\ , \end{aligned} \tag{3.10}$$

as solutions of (3.9a) to insure the simplicity of the successive integrations. General solutions to (3.9a) lead to logarithmic functions in the second order, and hence the successive calculations become formidable.

With an initial set of solutions in the form (3.10) we obtain the following:

$$
\begin{aligned}
A_1 &= a r \\
A_2 &= a c r^2 \\
A_3 &= -d a c r^2 + a c^2 r^3/2! \\
A_4 &= d^2 a c r^2 - 2d\, a c^2 r^3/2! + a c^3 r^4/3! \\
A_5 &= -d^3 a c r^2 + 3d^2 a c^2 r^3/2! - 3d\, a c^3 r^4/3! + a c^4 r^5/5! \\
&\vdots
\end{aligned}
\tag{3.11a}
$$

$$
\begin{aligned}
B_1 &= b r + d \\
B_2 &= E^2 r^2/2! \\
B_3 &= -d E^2 r^2/2! + (E^2 b - 2a^2 c) r^3/3! \\
B_4 &= d^2 E^2 r^2/2! - 2d (E^2 b - 2a^2 c) r^3/3! + (E^4 - 4a^2 c^2) r^4/4! \\
B_5 &= -d^3 E^2 r^2/2! + 3d^2 (E^2 b - 2a^2 c) r^3/3! - 3d (E^4 - 4a^2 c^2) r^4/4! \\
&\qquad + \left(b E^4 + 4a^2 c\, (cb - E^2) - 8a^2 c^3\right) r^5/5! \\
&\vdots
\end{aligned}
\tag{3.11b}
$$

$$
\begin{aligned}
C_1 &= c r + d \\
C_2 &= (2bc - E^2) r^2/2! \\
C_3 &= -d (2bc - E^2) r^2/2! + \left([c - 2b] E^2 + 2c\, [b^2 + a^2]\right) r^3/3! \\
C_4 &= d^2 (2bc - E^2) r^2/2! - 2d \left([c - 2b] E^2 + 2c\, [b^2 + d^2]\right) r^3/3! \\
&\qquad + \left(4a^2 c^2 + 4bc E^2 - 3E^4\right) r^4/4! \\
&\vdots
\end{aligned}
\tag{3.11c}
$$

where $E = \sqrt{b^2 - a^2}$.

3.4 Solutions

Equation (3.11) together with (3.7) and (3.3) yield a solution to (3.2) labelled by five parameters a, b, c, d, and ϵ. But the series corresponding to (3.11) are too complicated to permit summation for arbitrary values of the parameters. However, by adjusting the parameters, one can make summable series out of (3.11) which can be expressed in terms of elementary functions. Before discussing this aspect, let us consider some simple cases of (3.11)

Setting $a = b = c = d = 0$ in (3.11), we find

$$K = 0 \;, \quad H = 1 \;. \tag{3.12}$$

Likewise, the choice $a = b$, $c = 0$ yields

$$K = \frac{\eta r}{1 + \eta r} \;, \quad H = \frac{1}{1 + \eta r} \tag{3.13}$$

where $\eta = a\epsilon/(1+d)$ is an arbitrary constant. Solutions (3.12) and (3.13) are precisely the point monopole solutions (1.84a) and (1.84b) discussed in Section 1.4.

For $a \neq b$, $c = 0$ and $|d\epsilon| < 1$, (3.11b) and (3.11c) lead to series for $B(r)$ and $C(r)$ which possess a representative term while the series (3.11a) terminates. Using binominal theorem, the series for $B(r)$ becomes

$$\begin{aligned} B(r) &= 1 + d\epsilon + b(\epsilon r) + \frac{E^2}{1+d\epsilon}\frac{(\epsilon r)^2}{2!} + \frac{bE^2}{(1+d\epsilon)^2}\frac{(\epsilon r)^3}{3!} \\ &\quad + \frac{E^4}{(1+d\epsilon)^3}\frac{(\epsilon r)^4}{4!} + \frac{bE^4}{(1+d\epsilon)^4}\frac{(\epsilon r)^5}{5!} + \cdots\cdots \\ &= \frac{\epsilon}{\eta}\left(b \sinh \eta r + E \cosh \eta r\right) , \end{aligned} \tag{3.14}$$

where

$$\eta = \frac{E\epsilon}{1+d\epsilon} = \frac{\sqrt{b^2-a^2}\ \epsilon}{1+d\epsilon} . \tag{3.15}$$

Due to the large arbitrariness* of d and ϵ we can take η as an arbitrary constant independent of a and b. After a straightforward calculation, the series for $C(\lambda)$ becomes

$$C(\lambda) = \frac{\epsilon}{\eta}\left[(E - b\eta\lambda)\cosh\eta\lambda + (b - E\eta\lambda)\sinh\eta\lambda\right] . \tag{3.16}$$

Transforming back to the original dependent variables we find

$$K(\lambda) = \frac{p\eta\lambda e^{\eta\lambda}}{p^2 e^{2\eta\lambda} - 1} \tag{3.17a}$$

$$H(\lambda) = -\eta\lambda \frac{p^2 e^{2\eta\lambda} + 1}{p^2 e^{2\eta\lambda} - 1} + 1 \tag{3.17b}$$

where $p = a/(b-\sqrt{b^2-a^2})$ is an arbitrary constant. This coincides** with the general point monopole solution (1.84d) obtained by Ju [35]. The Protogenov solution (1.84c) is a special case of (3.17) (with $p = e^{\alpha}$). By setting $p = 1$ one obtains the regular PS solution (1.81). We were not able to sum $B(\lambda)$ and $C(\lambda)$ series (3.11b) and (3.11c) for nonzero c. However the $A(\lambda)$ series (3.11a) can be summed in this case because it possesses a representative term. The function $A(\lambda)$ obtained after summation can be substituted in (3.8a) to obtain an uncoupled nonlinear equation in B. This can further be reduced to the one dimensional Liouville equation. From the known solutions of this equation $B(\lambda)$ can be obtained. From a knowledge of $A(\lambda)$ and $B(\lambda)$, $K(\lambda)$ can be evaluated. Then $H(\lambda)$ can be constructed by direct substitution of $K(\lambda)$ in (3.2a). However, this procedure does not give any new result. This we prove in Appendix 3.A.

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* The only condition on d and ϵ is $|d\epsilon| < 1$.

** Comparing (3.17) with (1.84d) one notices a sign change. This does not matter because (3.2) is symmetric under the operations $K \to -K$ and $H \to -H$ (either together or separately).

3.A Appendix

From (3.11a) the $(n+1)^{th}$ term of $A(r)$ series is given by

$$A_{n+1} = ar(-1)^n d^n \sum_{k=0}^{n-1} (-1)^{k+1} \frac{(cr/d)^{k+1}}{(k+1)!} \binom{n-1}{k} . \quad (3.18)$$

From this

$$A_{n+2} = ar(-1)^n d^{n+1} \sum_{k=0}^{n} (-1)^k \frac{(cr/d)^{k+1}}{(k+1)!} \frac{n!}{k!(n-k)!}$$

$$= ar(-1)^n \frac{d^{n+1}}{n+1} \left(\frac{cr}{d}\right) \sum_{k=0}^{n} (-1)^k \frac{(cr/d)^k}{k!} \binom{n+1}{k+1}$$

$$= ar(-1)^n \frac{d^{n+1}}{n+1} \left(\frac{cr}{d}\right) \sum (-1)^k \frac{(cr/d)^k}{k!} \binom{n+1}{n-k}$$

$$= ar(-1)^n \frac{d^{n+1}}{n+1} \left(\frac{cr}{d}\right) L_n^1(cr/d) , \quad (3.19)$$

where $L_n^1(x)$ is the associated Laguerre Polynomial [61]. $A(r)$ now becomes

$$A(r) = \epsilon ar \left[1 + \left(\frac{cr}{d}\right) \sum_{n=0}^{\infty} (-1)^n \frac{(\epsilon d)^{n+1}}{n+1} L_n^1(cr/d) \right] . (3.20)$$

Using the relation [62]

$$\sum_{n=0}^{\infty} a_n d^n = \int_0^{\infty} e^{-t} \sum_{n=0}^{\infty} \frac{a_n t^n d^n}{n!} dt , \quad (3.21)$$

$A(r)$ can be rewritten as

$$A(r) = \epsilon ar \left[1 + \epsilon cr \int_0^{\infty} dt\, e^{-t} \sum_{n=0}^{\infty} (-1)^n \frac{(td\epsilon)^n L_n^1(cr/d)}{(n+1)!} \right] (3.22)$$

After performing the summation [61] we get

$$A(r) = \epsilon ar \left[1 + \epsilon cr \int_0^{\infty} dt (-crt\epsilon)^{-1/2} e^{-(1+d\epsilon)t} J_1(2\sqrt{-crt\epsilon}) \right] . (3.23)$$

which upon integration [61] yields , for $c < 0$,

$$A(r) = \epsilon a r e^{\epsilon c r/(1+d\epsilon)} \tag{3.24}$$

Substituting this in (3.8a), an uncoupled nonlinear equation for B is obtained :

$$D^2 B \cdot B = -2(\epsilon a)^2 \exp[2\epsilon c r/(1+d\epsilon)] \,. \tag{3.25}$$

By putting

$$B(r) = \epsilon a \exp\left[\frac{\epsilon c r}{1+d\epsilon} - f(r)\right] \tag{3.26}$$

we get the one dimensional Liouville equation,

$$f'' = e^{2f} \,. \tag{3.27}$$

Three distinct solutions of this equation are [35]

$$f = -\ln(r+\beta) \tag{3.28a}$$

$$f = -\ln\left(\frac{\sin(\alpha(r+\beta))}{\alpha}\right) \tag{3.28b}$$

$$f = \alpha r + \ln|2\alpha\eta|^{1/2}| - \ln(1-\eta e^{2\alpha r}) \,. \tag{3.28c}$$

Using (3.26), (3.24) and (3.3), $K(r)$ can be calculated for each solution (3.28). In each case $H(r)$ can be constructed by direct substitution of $K(r)$ in (3.2a). The solutions which follow from (3.28a), (3.28b) and (3.28c) can be reduced to the form (1.84b), (1.84c) and (1.84d) respectively.

CHAPTER - 4

COMPLEX SU(2) YANG-MILLS-HIGGS CONFIGUARATIONS WITH FINITE COMPLEX EUCLIDEAN ACTION

4.1 Introduction

Finite action solutions in Euclidean space have played an important role in understanding the structure of gauge theories (See Section 1.5). One usually considers real guage potentials. The importance of complex Euclidean solutions in functinal integral calculations has been pointed out recently by several authors [63-65]. Richard and Rouet [63] observed that, within the complex saddle point method, a complex solution can represent a superposition of instantons and anti-instantons. Moreover, as shown by Lapedes and Mottola [64], the inclusion of complex classical paths in the evaluation of the partition function would yield a better semiclassical approximation compared with the dilute gas approximation [51] which is not generally valid at finite temperatures.

Boutaleb-Joutei *et.al.* [66] obtained a pair of complex conjugate solutions of SU(2) gauge theory with finite, complex Euclidean action. In this chapter we present a similar pair of complex conjugate solutions to the time dependent field equations of the 't Hooft-Polyakov monopole theory in the PS limit. The time-dependent ansatz for the gauge and Higgs fields has already been constructed by Mecklenburg and O'Brein [67]. But the solution constructed by them has infinite energy in Minkowski space or when continued to Euclidean space, has infinite action. To our knowledge, no finite action solution (real or complex) has been reported so far for the SU(2) gauge theory (1.64). Such a solution was obtained earlier [68] in the de Sitter space without the ϕ^4 term in the Lagrangian. This we discuss in Section 4.3.

4.2 The pair of solutions and their actions

We have used the time-dependent extension of the 't Hooft-Polyakov ansatz [67] :

$$A_0^a = 0 \quad , \quad A_i^a = \frac{1}{g} \epsilon_{ain} r_n \frac{1 - K(r,t)}{r^2}$$
$$\phi_a = \frac{1}{g} r_a \frac{H(r)}{r^2} \tag{4.1}$$

This reduces the Euclidean space field equations of the SU(2) gauge theory to

$$r^2 \left(\frac{\partial^2 K}{\partial r^2} + \frac{\partial^2 K}{\partial t^2} \right) = K (K^2 - 1 + H^2) \tag{4.2a}$$

$$r^2 \left(\frac{\partial^2 H}{\partial r^2} + \frac{\partial^2 H}{\partial t^2} \right) = 2 H K^2 . \tag{4.2b}$$

in the PS limit. For fields corresponding to K and H to be regular at the origin $(r = 0)$, one should insist

$$K \longrightarrow 1 \ , \ H \longrightarrow 0 \quad \text{as} \quad r \longrightarrow 0 . \tag{4.3}$$

The classical action in terms of the ansatz functions is

$$\begin{aligned} S &= \int \mathcal{L} \, d^4x \\ &= \frac{4\pi}{g^2} \int_{-\infty}^{\infty} dt \int_0^{\infty} dr \left\{ \left(\frac{\partial K}{\partial r}\right)^2 + \left(\frac{\partial K}{\partial t}\right)^2 + \frac{1}{2}\left[\left(\frac{\partial H}{\partial r}\right)^2 + \left(\frac{\partial H}{\partial t}\right)^2 \right] \right. \\ &\quad \left. + \frac{1}{2r^2}(K^2 - 1)^2 + \frac{K^2 H^2}{r^2} - \frac{H}{r}\frac{\partial H}{\partial r} + \frac{H^2}{2r} \right\} . \end{aligned} \tag{4.4}$$

We observe that the substitution,

$$K = 1 - H \tag{4.5}$$

is not only consistent with (4.3), but turns (4.2a) and (4.2b) into a single nonlinear equation :

$$r^2\left(\frac{\partial^2 K}{\partial r^2}+\frac{\partial^2 K}{\partial t^2}\right) = 2K^2(K-1)\,. \tag{4.6}$$

Inspired by the work of Boutaleb-Joutei *et.al.* [66], we try a solution of the form

$$K(r,t) = \frac{4ar^2+(1+r^2+t^2)^2}{4br^2+(1+r^2+t^2)^2} \tag{4.7}$$

which is found to solve (4.6) for the choice :

$$\begin{aligned} a &= -\tfrac{1}{3}(1\pm i\sqrt{8}) \\ b &= -\tfrac{1}{9}(7\pm 2i\sqrt{8})\,. \end{aligned} \tag{4.8}$$

The solution evidently satisfies (4.3) and results in a field configuration which is regular everywhere. We now show that the resulting action is finite (and complex). First we note that the contributions to the action from the last two terms of (4.4) cancel each other for our solution. This can easily be seen by a partial integration. Consider the radial integral of the last but one term

$$\begin{aligned} \int_0^\infty dr\left(-\frac{H}{r}\frac{\partial H}{\partial r}\right) &= \int_0^\infty dr\left(-\frac{1}{2r}\frac{\partial}{\partial r}(H^2)\right) \\ &= -\frac{1}{2}\frac{H^2}{r}\Big|_0^\infty - \int_0^\infty dr\,\frac{H^2}{2r^2} \\ &= -\int_0^\infty dr\,(H^2/2r^2)\,, \end{aligned}$$

which cancels with the last term of (4.4). The remaining action integral, using (4.5), becomes

$$S = \frac{4\pi}{g^2}\int_{-\infty}^{\infty} dt\int_0^\infty dr\left\{\frac{3}{2}\left[\left(\frac{\partial K}{\partial r}\right)^2+\left(\frac{\partial K}{\partial t}\right)^2\right] + \frac{1}{2}(K^2-1)^2/r^2 \right. \tag{4.9}$$

The integrations can be made simpler by a co-ordinate transformation $(r,t) \longrightarrow (\rho,\tau)$ defined through the relations [66],

$$\tan\tau = \frac{2t}{1-r^2-t^2} \qquad (0 \le \tau \le 2\pi)$$
$$\rho = \frac{2r}{1+r^2+t^2} \qquad (0 \le \rho \le 1) . \tag{4.10}$$

The action integral, when expressed in terms of the (ρ,τ) co-ordinate system, becomes (see appendix 4.A)

$$S = \frac{4\pi}{g^2}(a-b)^2 \int_0^{2\pi} d\tau \int_0^1 d\rho \frac{\rho^2}{(b\rho^2+1)^4} \Big[10 + 2(2a+b-3)\rho^2 + \frac{1}{2}(3a^2+b^2+2ab)\rho^4 \Big] , \tag{4.11}$$

where we have substituted for K from (4.7). The τ integration is trivial. After performing the ρ integration using standard formulae [61], and substituting the value of a and b from (4.8), we find

$$S \approx -\frac{8\pi^2}{g^2}(6.545 \pm 9.89 i) , \tag{4.12}$$

for the corresponding signs in (4.8).

The topology of the gauge field configuration is trivial and is essentially a property of the ansatz (4.1) and regularity of K. $F^a_{\mu\nu}\tilde{F}^a_{\mu\nu}$ is zero for this ansatz for non-singular K.

4.3 A solution in the de Sitter space

A similar solution can be constructed for an SU(2) gauge theory in the de Sitter space. Consider the Lagrangian density [68]

$$\mathcal{L} = |g|^{1/2}\left(-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} - \frac{1}{2}D_\mu\phi^a D^\mu\phi^a - \frac{\Lambda}{6}\phi^2\right) . \tag{4.13}$$

The de Sitter line element is given by

$$ds^2 = -\left(1 - \tfrac{\Lambda}{3} r^2\right) dt^2 + \left(1 - \tfrac{\Lambda}{3} r^2\right)^{-1} dr^2 + r^2 d\Omega , \tag{4.14}$$

with $0 \leq r \leq \sqrt{\frac{3}{\Lambda}}$ and $-\infty < t < \infty$. In Ref.68 equations of motion similar to (4.2) are obtained by a co-ordinate transformation $(r,t) \longrightarrow (\eta, \xi)$ given in two steps :

$$\left(\tfrac{\Lambda}{3}\right)^{1/2} r = \tanh \rho \qquad (0 \leq \rho < \infty)$$

$$\left(\tfrac{1}{3}\right)^{1/2} t = \tau \qquad (-\infty < \tau < \infty)$$

and

$$\xi_{\pm} = \tanh \tau_{\pm} ,$$

where

$$\xi_{\pm} = \xi \pm \eta \quad \text{and} \quad \tau_{\pm} = \tau \pm \rho$$

with $0 \leq \eta \leq 1$ and $-1 \leq \xi \leq 1$. The real time equations of motion now reduce to

$$\begin{aligned} \eta^2 \left(\frac{\partial^2 K}{\partial \eta^2} - \frac{\partial^2 K}{\partial \xi^2} \right) &= K\left(K^2 - 1 + H^2\right) \\ \eta^2 \left(\frac{\partial^2 H}{\partial \eta^2} - \frac{\partial^2 H}{\partial \xi^2} \right) &= 2 H K^2 . \end{aligned} \tag{4.15}$$

A pair of complex solutions is evidently

$$K = 1 - H = \frac{4a\eta^2 + \left(1 + \eta^2 - \xi^2\right)^2}{4b\eta^2 + \left(1 + \eta^2 - \xi^2\right)^2} , \tag{4.16}$$

with the same constants a and b being given by (4.8). To get the solution corresponding to the Euclidean signature one merely changes $\xi^2 \longrightarrow -\xi^2$ in (4.16).

4.4 Discussion

The Euclidean flat space solutions herein obtained have properties similar to those of the 'bounce' solutions which have been used to calculate the false vacuum decay rate [69]. Like the bounce solutions, they assume the vacuum configuration,

$$A_\mu^a = 0 \quad , \quad \phi_a = 0 \; , \tag{4.17}$$

as $t \longrightarrow \pm\infty$ and $r \longrightarrow \infty$. Detailed implications of our solution in the quantum theory and finite-temperature calculations will be clear only if this path is included in the complex saddle-point method.

4.A Appendix

To write the action integral in terms of the co-ordinates (ρ, τ) defined in (4.11) we note that

$$d\tau \, d\rho = \frac{\partial(\rho,\tau)}{\partial(r,t)} \, dr \, dt = \frac{4}{1+r^2+t^2} \, dr \, dt \; , \tag{4.18}$$

where $\frac{\partial(\rho,\tau)}{\partial(r,t)} = \frac{4}{1+r^2+t^2}$ is the Jacobian of the (r,t) to (ρ,τ) transformation. Since $K(r,t)$ is a function of ρ only,

$$K(r,t) = K(\rho) = \frac{a\rho^2+1}{b\rho^2+1} \; , \tag{4.19}$$

we can write

$$\left(\frac{\partial K}{\partial t}\right)^2 + \left(\frac{\partial K}{\partial r}\right)^2 = \left(\frac{dK}{d\rho}\right)^2 \left[\left(\frac{\partial \rho}{\partial t}\right)^2 + \left(\frac{\partial \rho}{\partial r}\right)^2 \right] \; . \tag{4.20}$$

We have

$$\left(\frac{\partial\rho}{\partial t}\right)^2 + \left(\frac{\partial\rho}{\partial r}\right)^2 \;=\; \frac{4}{1+r^2+t^2}\,(1-\rho^2)\;. \tag{4.21}$$

Using (4.18), (4.20) and (4.21), the action integral (4.10) can be written as

$$S = \frac{4\pi}{g^2}\int_0^{2\pi} d\tau \int_0^1 d\rho\,\Big\{\ \tfrac{3}{2}(1-\rho^2)\left(\frac{dK}{d\rho}\right)^2 + \frac{1}{2\rho^2}(K^2-1)^2 + \frac{1}{\rho^2}K^2(K-1)\Big\}\;. \tag{4.22}$$

Substituting for K from (4.19) we get (4.11).

CHAPTER - 5

NEW EUCLIDEAN SOLUTIONS OF SU(2) GAUGE THEORY

5.1 Introduction

In the last chapter we have described a complex finite action solution of the SU(2) Yang-Mills-Higgs theory and herein we shall discuss the construction of a Euclidean solution of pure SU(2) gauge theory (without Higgs fields). In a pure SU(2) gauge theory the instanton and meron configurations represent finite and infinite action solutions, respectively. Instantons are self-dual solutions but merons are not.

We have constructed a Euclidean non-self-dual solution by making use of a solution recently discovered by Arodz [70] in Minkowski space. Its action is infinite but its Euclidean time evolution is very much different from that of a meron; the solution becomes a Wu-Yang monopole asymptotically. Similar self-dual solutions are also constructed which, in the asymptotic limit, become point dyons.

5.2 Solution of Arodz

Arodz [70] used the ansatz ,

$$\begin{aligned} A_0^a &= 0 \\ A_i^a &= \frac{1}{g}\epsilon_{ain}\frac{r_n}{r^2}\left(1 - K(r,t)\right) \end{aligned} \tag{5.1}$$

to find the time dependent solutions of pure SU(2) gauge theory (1.95). This ansatz reduces the equations of motion,

$$\mathcal{D}_\mu F^{\mu\nu a} = 0 \ , \tag{5.2}$$

to

$$r^2\left(\frac{\partial^2 K}{\partial r^2} - \frac{\partial^2 K}{\partial t^2}\right) + K - K^3 = 0 . \tag{5.3}$$

In Ref. 70 this is further reduced by an independent variable transformation,

$$\tau = \frac{t - t_0}{r} - 1 , \tag{5.4}$$

to an ordinary differential equation :

$$(2+\tau)\,\tau\,\frac{d^2K}{d\tau^2} + 2\,(1+\tau)\,\frac{dK}{d\tau} + K - K^3 = 0 \quad . \tag{5.5}$$

The fields were considered as evolving from an initial time $t = t_0$ to $t \to \infty$. The domain of τ variable was thus fixed as

$$-1 \leq \tau < \infty \quad . \tag{5.6}$$

A family of regular solution in this domain was obtained with the following properties :

$$\begin{aligned} K &\longrightarrow 0 \quad \text{as} \quad \tau \longrightarrow \infty \\ 0 < |K| &< 1 \quad \text{as} \quad \tau \longrightarrow -1 \ . \end{aligned} \tag{5.7}$$

5.3 A new solution in Euclidean space

We observe that (5.5) can be obtained from (5.3) by a more general transformation* :

$$\tau = \left(A + Bt + C(r^2 - t^2)\right)/r \quad . \tag{5.8a}$$

---

* The derivation of the Euclidean version of this transformation was given by Ray [71]. If we put $x = 1 + \tau$ (5.5) becomes

$$(x^2 - 1)\,\frac{d^2K}{dx^2} + 2x\,\frac{dK}{dx} + K - K^3 = 0 \ .$$

Comparing this with the corresponding equation in Ray's paper we get the condition, (5.8b), on the constants.

The constants A, B and C are such that

$$B^2 + 4AC = 1 \; . \tag{5.8b}$$

Transformation (5.8), however, sets the domain of the variable τ to be

$$-\infty < \tau < \infty \; . \tag{5.9}$$

This change is not in anyway advantageous because no regular solution exists for $\tau < 1$ [10]. However, the situation is different if we consider the Euclidean version of (5.3) obtained by the substitution $t \to -it$:

$$r^2\left(\frac{\partial^2 K}{\partial r^2} + \frac{\partial^2 K}{\partial t^2}\right) + K - K^3 = 0 \; . \tag{5.10}$$

In this case the transformation

$$\tau = \left(A + Bt + C(r^2+t^2)\right)/r \quad , \tag{5.11a}$$

$$B^2 - 4AC = -1 \quad , \tag{5.11b}$$

reduces (5.10) to

$$(2+\tau)\,\tau\,\frac{d^2K}{d\tau^2} + 2\,(1+\tau)\,\frac{dK}{d\tau} + K - K^3 = 0 \; .$$

This is nothing but (5.5) obtained earlier. The linear dependence of t in (5.11a) can be removed since (5.10) is invariant under time translations, $t \to t + \beta$. If we take $\beta = -B/2C$, then

$$\tau = \frac{1 + 4C^2(r^2+t^2)}{4Cr} - 1$$

Further (5.10) is invariant under scale transformations $r \to \lambda r$, $t \to \lambda t$. Choosing $\lambda = 1/2C$ we find,

$$\tau = \frac{1 + r^2 + t^2}{2r} - 1 . \tag{5.12}$$

The domain of the τ variable (for $-\infty < t < \infty$, $0 \leq r < \infty$) is

$$0 \leq \tau < \infty . \tag{5.13}$$

The only difference compared with (5.5) is the change in the domain of τ from (5.6) to (5.13). Hence to construct a solution in Euclidean space, we need take only a section (by excluding the domain $-1 \leq \tau < 0$) of the solution of Arodz.

5.4 Properties of the solution

The solution discussed in the previous section is not self-dual. One readily verifies that self-dual solutions within the ansatz (5.1) are trivial solutions ± 1 of (5.3) and all other solutions, including $K = 0$, are non-self-dual*. The $K = 0$ solution is nothing but the Wu-Yang [37] monopole solution.

As this is a Euclidean solution, the important thing to consider would be the evaluation of the action. The action of the solution is infinite. This can be concluded from the nature of the solution in the following way. We have the action

$$S = \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a}$$

$$= \frac{4\pi}{g^2} \int_{-\infty}^{\infty} dt \int_0^{\infty} dr \left\{ \left(\frac{\partial K}{\partial r}\right)^2 + \left(\frac{\partial K}{\partial t}\right)^2 + (K^2 - 1)^2/2r^2 \right\} . \tag{5.14}$$

---

* In this case the self-duality equation becomes

$$\frac{\partial K}{\partial r} = \frac{\partial K}{\partial t} = 0 \quad \text{and} \quad K^2 - 1 = 0 .$$

This can be rewritten as

$$S = \frac{4\pi}{g^2}\int_0^{2\pi} d\tau' \int_0^1 d\rho \left\{ (1-\rho^2)\left(\frac{dK}{d\rho}\right)^2 + \frac{1}{2\rho^2}(K^2-1)^2 \right\} , \tag{5.15}$$

where (see Appendix 4.A)

$$\begin{aligned} \rho &= \frac{1}{1+\tau} = \frac{2r}{1+r^2+t^2} \\ \tau' &= \tan^{-1}\left(\frac{2t}{1-r^2-t^2}\right) . \end{aligned} \tag{5.16}$$

In obtaining (5.15) we have used the fact that $K(r,t)$ depends on τ, and hence on ρ only. After the τ' integration, which is trivial because the integrand is independent of τ', we get

$$S = \frac{8\pi^2}{g^2}\int_0^1 d\rho \left\{ (1-\rho^2)\left(\frac{dK}{d\rho}\right)^2 + \frac{1}{2\rho^2}(K^2-1) \right\} . \tag{5.17}$$

The integral of the first term is evidently convergent. However, the second term is singular at $\rho = 0$ and the integral is finite only if $K \longrightarrow 1$ as $\rho \longrightarrow 0$. The limit $\rho \longrightarrow 0$ corresponds to the limit $\tau \longrightarrow \infty$, and in Ref. 70 it was shown that the limiting value of regular solutions is zero as $\tau \longrightarrow \infty$. Hence the Euclidean counterpart of the Arodz solution given through (5.12), is an infinite action solution. This is, however, not an unexpected result; all the known non-self-dual solutions are of infinite action. However, as has been shown by Boutaleb-Joutei, Chakrabarti, and Comtet [66], complex solutions can be found for (5.10) with finite complex action.

Comparing the above obtained solution with other known solutions in the existing literature [11], we find that it is neither an instanton nor a meron. Eventhough merons are infinite action non-self-dual Euclidean solutions, they carry half unit of topological charge. In the present case, since $K(r,t)$ is regular, the topological charge density is zero everywhere and the topological charge of the solution is zero. Furthermore, the Euclidean time evolution of this solution is quite different from that of the meron. A meron solution starts from a vacuum configuration (at $t = -\infty$) and evolves through a Wu-Yang monopole configuration (at $t = 0$), and finally ends up in a vacuum configuration (at $t = +\infty$) . In contrast, the present solution assumes the form of a Wu-Yang monopole configuration as $t \longrightarrow \pm\infty$ ($\tau \to \infty$) because $K(\tau) \to 0$ at these limits. In this regard the present solution looks similar to the bounce solutions [69] of scalar field theories which start from a vacuum at $t = -\infty$ and return to the same vacuum at $t = +\infty$. However, because of infiniteness of action, the implications of the newly obtained solution in the quantised theory can be understood only by going beyond semiclassical approximations.

5.5 A solution related to dyons

Finally we note that it is possible to obtain self-dual solutions possessing properties analogous to those of the solution presented above. Considering a special case of Witten's ansatz [72],

$$\begin{aligned} g A_0^a &= -\frac{r_a}{r^2}\left(1 - K(r,t)\right) \\ g A_i^a &= \epsilon_{ain}\frac{r_n}{r^2}\left(1 - K(r,t)\right) \mp \delta_{ia}\frac{K(r,t)}{r} , \end{aligned} \tag{5.18}$$

it may be verified that the self-duality equation in Euclidean space (1.101) can be satisfied if

$$r \frac{\partial K}{\partial r} = K(1-K)$$
$$r \frac{\partial K}{\partial t} = \mp K^2 . \tag{5.19}$$

The solution to (5.19) is given by

$$K(r,t) = \frac{r}{r \pm t + \beta} , \tag{5.20}$$

where β is an arbitrary constant. It may be noted that the solution corresponding to (5.20) may also be obtained within the ϕ^4 ansatz [11]. The solutions have singularities at $r = 0$ as well as on a hypersurface, $r \pm t + \beta = 1$ As Euclidean time $t \longrightarrow \pm \infty$, the gauge potentials become that of a point dyon configuration [73].

(5.21)

CHAPTER - 6

BOUND STATES OF NON-ABELIAN DYONS WITH FERMIONS AND BOSONS

6.1 Introduction

The study of bound states of magnetic monopoles with fermions and bosons has a long history. Dirac, in his seminal paper on monopoles [14], showed that with the usual boundary condition of quantum mechanics, there exist no bound states of monopoles with electrons. The conclusion of Dirac, reinforced by other workers [74],had to be abandoned in the seventies following the theoretical observation of several unusual properties of the charge-pole system. It was shown thast if the boundary conditions are chosen to ensure the self-adjointness of the Hamiltonian operator [75, 76], there can exist a spectrum of bound states with the lowest angular momentum value. There are several subtle problems in the charge-pole system which call for careful treatment and the question of the bound state formation between Dirac monopoles and charged particles is far from closed. It may be mentioned that the study of such bound states is also important in the context of experimental searches for monopoles [77].

The quantum mechanics of fermions and bosons in the background of non-abelian monopoles and dyons has been investigated by several workers. The bound state spectrum of a fermion in the background of a Wu-Yang monopole [37] and a dyon [73] of pure gauge theory was determined by Dereli, Swank and Swank [78]. They showed that while Wu-Yang monopoles have no bound states with fermions, dyons can have such bound states. For a 't Hooft-Polyakov monopole, Jackiw and Rebbi [47] demonstrated the existence of non-degenerate zero energy bound states of monopoles with isospinor or isovector fermions. These solutions, incidentally, imply a doublet of solitions with fractional fermion number. A general analysis of the Dirac equation or Klein-Gordon equation in the background of the 't Hooft-Polyakov monopole is not possible at the moment because the regular monopole solution has not been cast in a closed form. In the PS limit where closed expression is available for the monopole solution, scattering solutions for the lowest partial wave were recently constructed by Marciano and Muzinich [79]. Bound states have not been obtained in Ref.79, probably because the Higgs-Fermi coupling is neglected in this work. Most of the studies have been done in the point limit of a 't Hooft-Polyakov monopole by allowing the size of the monopole core to tend to zero. In this limit the system is essentially abelian, and with special boundary conditions, there exist bound states in the lowest angular momentum channel [76]. Callias [76] has argued that for a regular monopole a finite number of bound states will exist. In the asymptotic (point) limit of the PS monopole it has been shown by Cox and Yildiz [80] that the bound states can occur for all values of J. It is the additional $-1/r$ term present in the asymptotic Higgs field which is responsible for the bound states. Cox and Yildiz [80], however, have determined only the energy eigenvalues and did not construct the eigenfunctions. For PS dyon solutions in the point limit also, there exist an infinite number of bound states with all J values [81].

In addition to the regular monopole solution, there exist point singular monopoles and dyons [35] in the PS limit. In a recent work, Din and Roy [82] showed that an isospinor fermion in the background of a singular non-abelian monopole has a well defined Hamiltonian with ordinary boundary conditions imposed on the wavefunctions at the origin. Monopole-fermion bound states were shown to occur for all $\mathcal{J}$ values.

In this chapter we study the quantum mechanics of spin 1/2 and spinless particles in the background of a point dyon potential. The form of the background dyon potential chosen is such that the asymptotic PS dyon solution arises as a special case. The background may be interpreted as due either to a point singular dyon or to a regular dyon solution with the size of the core neglected. In addition to the isospinor fermions and bosons which have already been discussed in the literature [81, 83] we have also studied isovector fermions and bosons. Exact bound state solutions to the Jackiw-Rebbi equations for all $\mathcal{J}$, are obtained for isovector bosons and isospinor bosons and fermions. For isovector fermions a bound state has been obtained for the lowest angular momentum. Furthermore, we have shown that no bound state having $I_3 = 0$ exists for this system.

As mentioned above, part of our work concerning isospinor fermions and bosons overlaps that of Tang [81, 83] who considered the same problem with the asymptotic PS dyon as background. However, our method of solution is different. Tang introduced a singular string in the gauge potential by using a singular gauge transformation. The resulting abelianised equations are separated into radial and angular parts with the help of monopole harmonics [84]. We on the other hand, follow the method of Jackiw and Rebbi [47] who used spherical harmonics to separate radial and angular parts. The equivalence of the two procedures is demonstrated by utilizing a relationship between monopole harmonics and spherical harmonics which we have deduced.

In Section 6.2 we review briefly the classical SU(2) gauge theory and obtain the point singular background dyon potential. In Section 6.3 we obtain bound state solutions to the relevant Dirac and Klein-Gordon equations. We discuss various results in Section 6.4. The method of solving the radial equations of Section 6.3 is given in the Appendix 6.A. The proof of a relationship between spherical harmonics and monopole harmonics that we have deduced is given in the Appendix 6.B.

6.2 The background dyon potential

In Section 1.4 we have seen that the equations of motion following from the Lagrangian density,

$$\mathcal{L} = -\frac{1}{4} F^{a}_{\mu\nu} F^{\mu\nu a} + \frac{1}{2} D_{\mu}\phi^{a} D^{\mu}\phi^{a} - V(\phi) , \tag{6.1}$$

can be reduced to

$$\begin{aligned} r^2 J'' &= 2 J K^2 \\ r^2 H'' &= H\left(2K - m^2 r^2 + \frac{\lambda}{g^2} H^2\right) \\ r^2 K'' &= K\left(K^2 - 1 + H^2 - J^2\right) , \end{aligned} \tag{6.2}$$

using the spherically symmetric 't Hooft-Polyakov-Julia-Zee ansatz [22, 23, 30]

$$\begin{aligned} A_0^a &= \frac{1}{g} r_a J(r)/r^2 \\ A_i^a &= \frac{1}{g} \epsilon_{aim} r_m A(r) \\ \phi_a &= \frac{1}{g} r_a H(r)/r^2 , \end{aligned} \tag{6.3}$$

where $K(r) = 1 - r A(r)$.

We consider a particular solution of (6.2)

$$\begin{aligned} K(r) &= 0 \qquad \left(A(r) = \frac{1}{r} \right) \\ H(r) &= ar + b \\ J(r) &= cr + d , \end{aligned} \tag{6.4}$$

where $a, b, c,$ and d are arbitrary constants, as the background potential. Unlike the PS solution (1.81), this solution is singular at $r = 0$ and the classical energy is infinite. But at large distances this solution mimics the behaviour of a regular solution. In fact at large distances $r \gg 1/(gm/\sqrt{\lambda})$, this solution coincides with the PS dyon solution (1.93) with the identification

$$\begin{aligned} a &= \beta \cosh \eta \qquad & b &= - \cosh \eta \\ c &= \beta \sinh \eta \qquad & d &= - \sinh \eta . \end{aligned} \tag{6.5}$$

So if the particles do not penetrate the dyon core, then (6.4) with (6.5) will be a good approximation to the regular solution. We can also consider (6.4) as a point dyon [35] solution. The relevance of such a solution is unclear at present, mainly because their classical energy is infinite. To our knowledge the quantum field theory of such objects has not been worked out so far. The fact that the singularity is at the origin seems to be a favourable point since one encounters a similar situation in the case of electron. The electric charge of this field configuration is given by

$$Q = - \frac{4\pi d}{g} . \tag{6.6}$$

To include fermions we add to (6.1) the fermionic Lagrangian

$$\mathcal{L}_\psi = \bar{\Psi}_n (i \not{D} - M) \Psi_n - g G \bar{\Psi}_n T^a_{nm} \phi_a \Psi_m \tag{6.7}$$

where

$$D_\mu \Psi_n = \partial_\mu \Psi_n - i g T^a_{nm} A^a_\mu \Psi_m \quad .$$

and the T^a are SU(2) generators satisfying

$$\begin{aligned} [T^a, T^b] &= i \epsilon_{abc} T^c . \\ T^a_{nm} &= \tau^a_{nm} \quad \text{for } I = \tfrac{1}{2} \text{ representation} \\ &= i \epsilon_{nam} \quad \text{for } I = 1 \text{ representation} . \end{aligned} \tag{6.8}$$

We will consider fermions in the above two representations moving in the background potential (6.4).

For bosons instead of (6.7), we consider

$$\mathcal{L}_U = D_\mu U^*_n D^\mu U_n - M^2 |U|^2 - g G U^*_n T^a_{nm} \phi^a U_n - \tfrac{1}{2} h^2 \phi^2 |U|^2 . \tag{6.9}$$

6.3 Solutions

i) Isospinor fermions

In this case the Dirac equation is given by

$$i \gamma^\mu D_\mu \Psi_n - \tfrac{1}{2} g G \tau^a_{nm} \phi^a \Psi_m = M \Psi_m \quad . \tag{6.10}$$

Putting $\Psi_n(x) = e^{-iEt}\,\psi_n(\vec{x})$ and substituting the gauge and Higgs field ansatz, we obtain the equation for $\psi(x)$ as

$$\Big\{ \vec{\alpha}\cdot\Big[\vec{p} - \frac{A(r)}{2}(\hat{r}\times\vec{\tau})\Big] + \frac{J(r)}{2r}(\vec{\tau}\cdot\hat{r}) - \frac{G\,H(r)}{2r}\beta(\vec{\tau}\cdot\hat{r})\Big\}\psi(\vec{x}) = (E-\beta M)\,\psi(\vec{x}) \ , \tag{6.11}$$

where $\vec{\alpha}$ and β are the Dirac matrices

$$\vec{\alpha} = \gamma^0\vec{\gamma} \ , \quad \beta = \gamma^0$$

We now proceed exactly as in Ref.47 to separate the radial and angular parts. But we use a different representation of Dirac matrices,

$$\vec{\alpha} = \begin{pmatrix} 0 & i\vec{\sigma} \\ -i\vec{\sigma} & 0 \end{pmatrix} \ , \quad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{6.12}$$

It will be seen later that this choice is advantageous when we come to the solution of the radial equations.

Splitting the wavefunction into upper and lower components,

$$\psi_n(\vec{x}) = \begin{pmatrix} \chi^+(\vec{x}) \\ \chi^-(\vec{x}) \end{pmatrix} \ , \tag{6.13}$$

we find that the Dirac equation becomes

$$\Big\{ \vec{\sigma}_{ij}\cdot\Big[\vec{\nabla}\,\delta_{nm} - \tfrac{1}{2}A(r)(\hat{r}\times\tau_{nm})\Big] \pm \delta_{ij}\Big[\frac{G\,H(r)}{2r}(\hat{r}\cdot\tau_{nm}) - M\,\delta_{nm}\Big]\Big\}\chi^{\pm}_{jm} = \pm\Big\{\frac{J(r)}{2r}(\vec{\tau}_{nm}\cdot\hat{r}) - E\,\delta_{nm}\Big\}\chi^{\mp}_{im} . \tag{6.14}$$

Here the first index on $\chi^{\pm}$ refers to the spin part and the second one to the isospin part. $\chi^{\pm}_{in}$ is then expressed in terms of two scalar and vector functions,

$$\chi^{\pm}_{in}(\vec{x}) = \left(g^{\pm}(\vec{x})\, \delta_{im} + \vec{g}^{\pm}(\vec{x})\cdot\vec{\sigma}_{im} \right) \tau^{2}_{mn} \,. \tag{6.15}$$

The scalar and vector functions are now expressible in terms of ordinary and vector spherical harmonics respectively,

$$\begin{aligned} g^{\pm}(\vec{x}) &= G^{\pm}_{J}(r)\, Y^{M}_{J}(\Omega) \\ g^{\pm}_{a}(\vec{x}) &= P^{\pm}_{J}(r)\, \hat{r}_{a}\, Y^{M}_{J}(\Omega) + B^{\pm}_{J}(r)\, \frac{1}{j}\, r\, \partial_{a} Y^{M}_{J}(\Omega) \\ &\quad + C^{\pm}_{J}(r)\, \frac{1}{ij}\, \epsilon_{abc}\, r_{b}\, \partial_{c}\, Y^{M}_{J}(\Omega) \,. \end{aligned} \tag{6.16}$$

Here $j = \sqrt{J(J+1)}$ and J is the total angular momentum. Total angular momentum is obtained by combining orbital and spin angular momenta and isospin. In this case it takes values $0, 1, 2, \ldots\ldots$. $B_0 = C_0 = 0$ by definition.

Substituting (6.15) and (6.16) in (6.14) we get eight radial equations :

$$\left(\frac{d}{dr} + \frac{1}{r} \mp \frac{G\,H(r)}{2r}\right) P^{\pm}_{J} - \frac{j}{r} B^{\pm}_{J} \mp M G^{\pm}_{J} = \mp \left(\frac{J(r)}{2r} P^{\mp}_{J} + E\, G^{\mp}_{J} \right) \qquad (\text{all } J)$$

$$\left(\frac{d}{dr} + \frac{1}{r} \mp \frac{G\,H(r)}{2r}\right) G^{\pm}_{J} - \frac{j}{r} C^{\pm}_{J} \mp M P^{\pm}_{J} = \mp \left(\frac{J(r)}{2r} G^{\mp}_{J} + E\, P^{\mp}_{J} \right) \qquad (\text{all } J)$$

(6.17)

$$\left(\frac{d}{dr} + \frac{1}{r} \pm \frac{G\,H(r)}{2r}\right) B^{\pm}_{J} - \frac{j}{r} P^{\pm}_{J} \pm M C^{\pm}_{J} = \pm \left(\frac{J(r)}{2r} B^{\mp}_{J} + E\, C^{\mp}_{J} \right) \qquad (J > 0)$$

$$\left(\frac{d}{dr} + \frac{1}{r} \pm \frac{G\,H(r)}{2r}\right) C^{\pm}_{J} - \frac{j}{r} G^{\pm}_{J} \pm M B^{\pm}_{J} = \pm \left(\frac{J(r)}{2r} C^{\mp}_{J} + E\, B^{\mp}_{J} \right) \qquad (J > 0).$$

where we have substituted $A(r)$ from (6.4). These equations, unlike the corresponding equations in Ref. 47 have, a $\pm$ sign before E and $J(r)$. This is because of our choice of the representation for Dirac matrices. The advantage of this choice is that the equations can now be transformed into a set of four independent coupled pairs of first order differential equations. Each of them can be decoupled and the resulting second order differential equations can be solved exactly. In contrast, the decoupling of equations in Ref. 47 gives a fourth order differential equation [82]. We now discuss $J=0$ and $J>0$ solutions separately.

$J=0$ *solutions*

Setting

$$
\begin{aligned}
P_0^{\pm} + G_0^{\pm} &= R^{\pm} \\
P_0^{\pm} - G_0^{\pm} &= S^{\pm}
\end{aligned}
\tag{6.18}
$$

the $J=0$ radial equations become

$$\left[\frac{d}{dr} + \frac{1}{r} \mp \left(\frac{B}{r} + m_+\right)\right] R^{\pm} = \mp \left(\frac{D}{r} + \epsilon_+\right) R^{\mp} \tag{6.19a}$$

$$\left[\frac{d}{dr} + \frac{1}{r} \mp \left(\frac{B}{r} + m_-\right)\right] S^{\pm} = \mp \left(\frac{D}{r} + \epsilon_-\right) S^{\mp}, \tag{6.19b}$$

where we have substituted $J(r)$ and $H(r)$ from (6.4). Also $B = Gb/2$, $D = d/2$, $m_{\pm} = (aG/2) \pm M$ and $\epsilon_{\pm} = \frac{c}{2} \pm E$. Here we need solve only for the first set. The solution of the second set can be obtained by suitable replacements.

For solving (6.19a), consider one more dependent variable transformation,

$$X^{\pm} = R^{+} \pm R^{-} . \tag{6.20}$$

(6.19a) now becomes

$$\left(\frac{d}{dr} + \frac{1}{r}\right) X^{\pm} = \left(\frac{B \pm D}{r} + m \pm \epsilon\right) X^{\mp} , \tag{6.21}$$

where we have suppressed the symbol over m and ϵ . This equation is exactly the same as that of the hydrogen atom problem in Dirac theory [85] if the B/r term is absent, and it can be solved by a similar technique (see Appendix 6.A). For $\epsilon < m$ we get discrete bound states and for $\epsilon > m$ we get continuum states. Here we give only the bound state solutions. The continuous spectrum can be obtained by suitable replacements [85].

The solution for $\epsilon < m$ is given by

$$X^{\pm} = \sqrt{m \pm \epsilon} \; e^{-\rho/2} \rho^{\gamma - 1} \left(Q_2 \pm Q_1 \right) , \tag{6.22}$$

where

$$\rho = 2\lambda r \quad , \quad \lambda = \sqrt{\epsilon^2 - m^2} \quad , \quad \gamma = \sqrt{B^2 - D^2} \; ,$$

$$Q_1 = {}_1F_1\left(\gamma + \frac{Bm - D\epsilon}{\lambda} \, , \, 2\gamma + 1 \, , \, \rho\right) , \tag{6.23}$$

$$Q_2 = \frac{\gamma + (Bm - D\epsilon)/\lambda}{(Dm - B\epsilon)/\lambda} \; {}_1F_1\left(1 + \gamma + \frac{Bm - D\epsilon}{\lambda} \, , \, 2\gamma + 1 \, , \, \rho\right) ,$$

and ${}_1F_1(a, b, \rho)$ are Kummer functions. For (6.22) to vanish at the origin, we should set $B^2 > D^2$. The corresponding solutions to (6.19) are

$$R^{\pm} = \alpha\, e^{-\rho_+/2}\, \rho_+^{\gamma-1} \Big\{ \sqrt{m_+ + \epsilon_+}\, \big(Q_2^+ + Q_1^+ \big) \pm \sqrt{m_+ - \epsilon_+}\, \big(Q_2^+ - Q_1^+ \big) \Big\} \tag{6.24a}$$

$$S^{\pm} = \beta\, e^{-\rho_-/2}\, \rho_-^{\gamma-1} \Big\{ \sqrt{m_- + \epsilon_-}\, \big(Q_2^- + Q_1^- \big) \pm \sqrt{m_- - \epsilon_-}\, \big(Q_2^- - Q_1^- \big) \Big\} , \tag{6.24b}$$

where

$$\rho_\pm = 2\lambda_\pm r \quad , \quad \lambda_\pm = \sqrt{\epsilon_\pm^2 - m_\pm^2} \quad , \quad \gamma = \sqrt{B^2 - D^2}$$

$$\begin{aligned} Q_1^{\pm} &= {}_1F_1\Big(\gamma + \frac{Bm_\pm - D\epsilon_\pm}{\lambda_\pm} \, , \, 2\gamma + 1 \, , \, \rho_\pm \Big) \\ Q_2^{\pm} &= \frac{\gamma + (Bm_\pm - D\epsilon_\pm)/\lambda_\pm}{(Dm_\pm - B\epsilon_\pm)/\lambda_\pm}\; {}_1F_1\Big(1 + \gamma + \frac{Bm_\pm - D\epsilon_\pm}{\lambda_\pm} \, , \, 2\gamma + 1 \, , \, \rho_\pm \Big) \end{aligned} \tag{6.25}$$

and α and β are arbitrary constants (fixed by normalisation). The normalisation condition after the angular integration becomes

$$\begin{aligned} 1 &= 2 \sum_{\pm} \int_0^\infty r^2 dr \, \big(P_0^{\pm *} P_0^{\pm} + G_0^{\pm *} G_0^{\pm} \big) \\ &= \sum_{\pm} \int_0^\infty r^2 dr \, \big(R^{\pm *} R^{\pm} + S^{\pm *} S^{\pm} \big) . \end{aligned} \tag{6.26}$$

For the convergence of this integral the Kummer functions should reduce to polynomials. From this requirement we obtain the eigenvalue spectrum. The conditions are

$$\gamma + \frac{Bm_+ - D\epsilon_+}{\lambda_+} = -n_1 \tag{6.27a}$$

$$\gamma + \frac{Bm_- - D\epsilon_-}{\lambda_-} = -n_2 , \tag{6.27b}$$

where n_1 and n_2 are positive integers. Zero is not possible because in this case $\gamma + (Bm - D\epsilon)/\lambda = (Dm - B\epsilon)/\lambda$ and Q_2 remains divergent*. Solving for ϵ_+ and ϵ_- we get

$$\frac{\epsilon_+}{m_+} = \frac{BD/(n_1+\gamma)^2 \pm \sqrt{1-(B^2-D^2)/(n_1+\gamma)^2}}{1 + D^2/(n_1+\gamma)^2} \tag{6.28a}$$

$$\frac{\epsilon_-}{m_-} = \frac{BD/(n_2+\gamma)^2 \pm \sqrt{1-(B^2-D^2)/(n_2+\gamma)^2}}{1 + D^2/(n_2+\gamma)^2} \tag{6.28b}$$

Inspection of Eq. (6.28a,b) shows that it is not possible to satisfy these two equations simultaneously for the same value of energy. So we can take either (6.28a) and set $\beta = 0$ $(P_0^{\pm} = G_0^{\pm})$ or we take (6.28b) and set $\alpha = 0$ $(P_0^{\pm} = -G_0^{\pm})$. The corresponding energy values in terms of the original parameters are

$$E_{n_1} = -\frac{c}{2} + \frac{M + aG/2}{1 + d^2/4(n_1+\gamma)^2}\left[\frac{bdG}{4(n_1+\gamma)^2} \pm \sqrt{1 - \frac{b^2G^2 - d^2}{4(n_1+\gamma)^2}}\,\right] \tag{6.29a}$$

$$E_{n_2} = \frac{c}{2} + \frac{M - aG/2}{1 + d^2/4(n_2+\gamma)^2}\left[\frac{bdG}{4(n_2+\gamma)^2} \pm \sqrt{1 - \frac{b^2G^2 - d^2}{4(n_2+\gamma)^2}}\,\right]. \tag{6.29b}$$

We get different solutions for each sign in (6.29) because $\rho_{\pm}$ depends on energy.

---

* If n_1 and n_2 are zero we get $Q_2^{\pm} = \beta\ {}_1F_1(1, 2\gamma+1, \rho_{\pm})$, which does not reduce to a polynomial.

$\mathcal{J} > 0$ Solutions

As remarked earlier, in this case the eight coupled differential equations can be transformed to a system of four independent coupled pairs of first order differential equations. This is achieved by defining eight new functions in the following way :

$$
\begin{aligned}
X_J^{\pm} &= P_J^{\pm} + G_J^{\pm} + B_J^{\mp} + C_J^{\mp} \\
Y_J^{\pm} &= P_J^{\pm} + G_J^{\pm} - B_J^{\mp} - C_J^{\mp} \\
Z_J^{\pm} &= P_J^{\pm} - G_J^{\pm} + B_J^{\mp} - C_J^{\mp} \\
W_J^{\pm} &= P_J^{\pm} - G_J^{\pm} - B_J^{\mp} - C_J^{\mp} \,.
\end{aligned}
\tag{6.30}
$$

The radial equations take the form

$$
\begin{aligned}
\mathcal{D}_{m_+}^{\pm} X_J^{\pm} &= \left[\frac{j}{r} \mp \left(\frac{D}{r} + \epsilon_+ \right) \right] X_J^{\mp} \\
\mathcal{D}_{m_+}^{\pm} Y_J^{\pm} &= \left[-\frac{j}{r} \mp \left(\frac{D}{r} + \epsilon_+ \right) \right] Y_J^{\mp} \\
\mathcal{D}_{m_-}^{\pm} Z_J^{\pm} &= \left[\frac{j}{r} \mp \left(\frac{D}{r} + \epsilon_- \right) \right] Z_J^{\mp} \\
\mathcal{D}_{m_-}^{\pm} W_J^{\pm} &= \left[-\frac{j}{r} \mp \left(\frac{D}{r} + \epsilon_- \right) \right] W_J^{\mp} \,,
\end{aligned}
\tag{6.31}
$$

where $\mathcal{D}_m^{\pm} = \frac{d}{dr} + \frac{1}{r} \mp \left(\frac{B}{r} + m \right)$ and $\epsilon_{\pm}$ and $m_{\pm}$ are the same as before.

These equations can be solved as in the previous case. (See also Appendix 6.A). We need solve only the first equation. Solutions to the remaining three can be obtained by suitable replacements. We give only the final results (for $\epsilon < m$) :

$$X_J^{\pm} = \alpha\, e^{-\rho_+/2}\, \rho_+^{\gamma-1} \left[\sqrt{m_+ + \epsilon_+}\left(Q_2^+(j) + Q_1^+\right) \pm \sqrt{m_+ - \epsilon_+}\left(Q_2^+(j) - Q_1^+\right)\right]$$

$$Y_J^{\pm} = \beta\, e^{-\rho_+/2}\, \rho_+^{\gamma-1} \left[\sqrt{m_+ + \epsilon_+}\left(Q_2^+(-j) + Q_1^+\right) \pm \sqrt{m_+ - \epsilon_+}\left(Q_2^+(-j) - Q_1^+\right]\right.$$

$$Z_J^{\pm} = \eta\, e^{-\rho_-/2}\, \rho_-^{\gamma-1} \left[\sqrt{m_- + \epsilon_-}\left(Q_2^-(j) + Q_1^-\right) \pm \sqrt{m_- - \epsilon_-}\left(Q_2^-(j) - Q_1^-\right)\right]$$

$$W_J^{\pm} = \delta\, e^{-\rho_-/2}\, \rho_-^{\gamma-1} \left[\sqrt{m_- + \epsilon_-}\left(Q_2^-(-j) + Q_1^-\right) \pm \sqrt{m_- - \epsilon_-}\left(Q_2^-(-j) - Q_1^-\right)\right] \tag{6.32}$$

where

$$\gamma = \sqrt{B^2 + j^2 - D^2}$$

$$Q_1^{\pm} = {}_1F_1\left(\gamma + \frac{Bm_\pm - D\epsilon_\pm}{\lambda_\pm},\ 2\gamma + 1,\ \rho_\pm\right) \tag{6.33}$$

$$Q_2^{\pm}(j) = \frac{\gamma + (Bm_\pm - D\epsilon_\pm)/\lambda_\pm}{-j + (Dm_\pm - B\epsilon_\pm)/\lambda_\pm}\ {}_1F_1\left(1 + \gamma + \frac{Bm_\pm - D\epsilon_\pm}{\lambda_\pm},\ 2\gamma + 1,\ \rho_\pm\right)$$

and $\rho_\pm$ and $\lambda_\pm$ are given by the same expressions as before. $Q_2^{\pm}(-j)$ is obtained by changing the sign of j in $Q_2^{\pm}(j)$. As in the earlier case the requirement of normalisability of wave functions leads to the conditions

$$\gamma + \frac{Bm_+ - D\epsilon_+}{\lambda_+} = -n_1 \tag{6.34a}$$

$$\gamma + \frac{Bm_- - D\epsilon_-}{\lambda_-} = -n_2\ . \tag{6.34b}$$

Here n_1 and n_2 can be zero contrary to the previous case. Depending on the sign of $Dm - B\epsilon$, one normalisable solution exists. Suppose $(Dm - B\epsilon) > 0$. In this case $Q_2(j) = 0$ and $Q_2(-j)$ is nonzero and divergent. If $(Dm - B\epsilon) < 0$, $Q_2(-j) = 0$ and $Q_2(j)$ is nonzero and divergent. Since ϵ is given by a quadratic equation it can have two roots. For $n_1 = 0$ and $n_2 = 0$, $\epsilon_\pm$ is given by

$$\epsilon_+ = m_+ \frac{BD/j^2 \pm \sqrt{1 + (B^2 - D^2)/j^2}}{1 + B^2/j^2} \tag{6.35a}$$

$$\epsilon_- = m_- \frac{BD/j^2 \pm \sqrt{1 + (B^2 - D^2)/j^2}}{1 + B^2/j^2} \tag{6.35b}$$

Hence for both (m_+, ϵ_+) and (m_-, ϵ_-) we obtain

$$Dm - B\epsilon = m \frac{D \mp \sqrt{1 + (B^2 - D^2)/j^2}}{1 + B^2/j^2} \tag{6.36}$$

Consider the case $D > 0$. Then if $|D| \geq (1 + (B^2 - D^2)/2)^{1/2}$ we find $(Dm - B\epsilon)$ always positive*. So solutions containing $Q_2(-j)$ should be discarded (by setting $\beta = \delta = 0$). Similarly for $D < 0$ we should discard solutions containing $Q_2(j)$ $(\alpha = \eta = 0)$. If $|D| < (1 + (B^2 - D^2)/j^2_{max})^{1/2}$, where $j^2_{max} = J_{max}(J_{max} + 1)$ is the highest value satisfying this condition (all lower angular momenta will evidently satisfy this), we should again consider two cases. For energy obtained by taking the upper sign in (6.35) we discard solutions containing $Q_2(j)$. For the lower sign the solution containing $Q_2(-j)$ should be discarded. This is the same for both positive and negative values of D and for all angular momentum $J \leq J_{max}$.

---

* Lowest nonzero value of $j^2 = J(J+1) = 2$. If this value satisfies the above condition all higher angular momenta will automatically satisfy the same.

For integers $n_1, n_2 > 0$ the energy levels are the same as given in (6.29a) and (6.29b) except for the expression for γ and when $j=0$ we get exactly (6.29). Here again we should consider two separate cases since (6.29a) and (6.29b) cannot both be satisfied simultaneously for the same value of energy. Corresponding to each sign in (6.29a), we get two levels: $\alpha \neq 0$, $\beta = \eta = \delta = 0$ and $\alpha = 0$, $\beta \neq 0$, $\eta = \delta = 0$. In either case we get $P_J^\pm = G_J^\pm$ and $C_J^\pm = B_J^\pm$. Similarly for each sign in (6.29b) we get two solutions $\alpha = \beta = 0$, $\eta \neq 0$, $\delta = 0$ and $\alpha = \beta = 0$, $\eta = 0$, $\delta \neq 0$. In this case we get solutions satisfying $P_J^\pm = -G_J^\pm$ and $C_J^\pm = -B_J^\pm$. Further discussion of these results will be given in Section 6.4.

ii) Isovector fermions

In order to facilitate solution we use a different set of Dirac matrices:

$$\vec{\alpha} = \begin{pmatrix} 0 & i\vec{\sigma} \\ -i\vec{\sigma} & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.37}$$

With this representation the Dirac equation can be reduced to the form

$$\begin{aligned}&[\vec{\sigma}\cdot\vec{\nabla}\,\delta_{nm} - A(r)(\hat{r}_n\sigma_m - \sigma_n\hat{r}_m)]\,\chi_m^\pm \\ &= -\Big[\frac{G\,H(r)}{r}\,i\epsilon_{nam}\hat{r}_a + M\,\delta_{nm} \pm \Big(\frac{J(r)}{r}\,i\epsilon_{nam} r_a \\ &\qquad + E\,\delta_{nm}\Big)\chi_m^\mp .\end{aligned} \tag{6.38}$$

Separation of radial and angular parts is achieved by the use of vector spinor harmonics :

$$\begin{aligned}\chi_m^\pm &= F_{1+}^\pm(r)\,\hat{r}_m\,\mathcal{Y}_{JM}(\Omega) + F_{2+}^\pm(r)\,r\,\partial_m\,\mathcal{Y}_{JM}(\Omega) + F_{3-}^\pm(r)\,L_m\,\mathcal{Y}_{JM}(\Omega) \\ &\quad + F_{1-}^\pm(r)\,\hat{r}_m\,\mathcal{Y}'_{JM}(\Omega) + F_{2-}^\pm(r)\,r\,\partial_m\,\mathcal{Y}'_{JM} + F_{3+}(r)\,L_m\,\mathcal{Y}'_{JM}(\Omega),\end{aligned} \tag{6.39}$$

where $\mathcal{Y}_{JM}$ and $\mathcal{Y}'_{JM}$ are spinor harmonics :

$$\mathcal{Y}_{JM}(\Omega) = \begin{pmatrix} \sqrt{\frac{J+M}{2J}} \; Y^{M-1/2}_{J-1/2} \\ \sqrt{\frac{J-M}{2J}} \; Y^{M+1/2}_{J-1/2} \end{pmatrix}$$

$$\mathcal{Y}'_{JM}(\Omega) = (\vec{\sigma}\cdot\hat{r})\, \mathcal{Y}_{JM}(\Omega) = \begin{pmatrix} \sqrt{\frac{J-M+1}{2J+2}} \; Y^{M-1/2}_{J+1/2} \\ -\sqrt{\frac{J+M+1}{2J+2}} \; Y^{M+1/2}_{J+1/2} \end{pmatrix} \qquad (6.40)$$

and $L_m = -i\,\epsilon_{mij}\, r_i\, \partial_j$ is the angular momentum operator. $F^{\pm}_{2+} = F^{\pm}_{3-} = 0$ for $J = 1/2$ by definition. The radial equations are

$$\left(\frac{d}{dr} + \frac{j+1}{r}\right) F^{\pm}_{1-} = -(M \pm E)\, F^{\mp}_{1+} \qquad (\text{all } J) \qquad (6.41a)$$

$$\left(\frac{d}{dr} - \frac{j-1}{r}\right) F^{\pm}_{1+} = -(M \pm E)\, F^{\mp}_{1-} \qquad (\text{all } J) \qquad (6.41b)$$

$$(j-1)\left(\frac{d}{dr} - \frac{j-1}{r}\right) F^{\pm}_{3-} + \left(\frac{d}{dr} + \frac{1}{r}\right) F^{\pm}_{2+} = j\left(\frac{G H(r) \pm J(r)}{r} F^{\mp}_{2-} - (M \pm E)\, F^{\mp}_{3+}\right) \qquad (\text{all } J) \qquad (6.42a)$$

$$(j-1)\left(\frac{d}{dr} - \frac{j-1}{r}\right) F^{\pm}_{2+} + \left(\frac{d}{dr} + \frac{1}{r}\right) F^{\pm}_{3+} = j\left(\frac{G H(r) \pm J(r)}{r} F^{\mp}_{3+} - (M \pm E)\, F^{\mp}_{2-}\right) \qquad (\text{all } J) \qquad (6.42b)$$

$$(j+1)\left(\frac{d}{dr} + \frac{j+1}{r}\right) F^{\pm}_{2-} - \left(\frac{d}{dr} + \frac{1}{r}\right) F^{\pm}_{3-} = j\left(\frac{G H(r) \pm J(r)}{r} F^{\mp}_{3-} - (M \pm E)\, F^{\mp}_{2+}\right) \qquad (J > 1/2) \qquad (6.43a)$$

$$(j+1)\left(\frac{d}{dr} + \frac{j+1}{r}\right) F^{\pm}_{3+} - \left(\frac{d}{dr} + \frac{1}{r}\right) F^{\pm}_{2+} = j\left(\frac{G H(r) \pm J(r)}{r} F^{\mp}_{2+} - (M \pm E)\, F^{\mp}_{3-}\right) \qquad (J > 1/2)\,, \qquad (6.43b)$$

where $j = J + 1/2$. Here $F^{\pm}_{1+}(r)$ and $F^{\pm}_{1-}(r)$ do not depend on either the Higgs field or the time component of gauge field. We will later see that $F_{1+}^{\pm}(r)$ and $F_{1-}^{\pm}(r)$ correspond to solutions having the third component of isospin $I_3 = 0$.

$J = 1/2$ solutions

Solutions to (6.41) are readily obtained by directly decoupling them. There are no normalisable solutions corresponding to bound states $(E < M)$; only a continuous spectrum exists.

Equations (6.42), however, possess bound state solutions. Energy levels are similar to those of zero angular momentum isospinor fermions. To see this, define

$$\begin{aligned} X^{\pm} &= F_{2-}^{\pm} + F_{3+}^{\pm} \\ Y^{\pm} &= F_{2-}^{\pm} - F_{3+}^{\pm} \, . \end{aligned} \tag{6.44}$$

The four mutually coupled equations now change to two independent coupled equations

$$\left(\frac{d}{dr} + \frac{1}{r}\right) X^{\pm} = \left(\frac{B \pm D}{r} + m_{-} \pm \epsilon_{-}\right) X^{\mp} \tag{6.45a}$$

$$\left(\frac{d}{dr} + \frac{1}{r}\right) Y^{\pm} = \left(\frac{B \pm D}{r} + m_{+} \pm \epsilon_{+}\right) Y^{\mp} \tag{6.45b}$$

where we have substituted for $H(r)$ and $J(r)$ and $B = Gb$, $D = d$, $m_{\pm} = Ga \pm M$ and $\epsilon_{\pm} = c \pm E$. These equations are similar to (6.21) studied earlier. The solutions to (6.45a) and (6.45b) are not normalisable simultaneously. So the two solutions are $X^{\pm} \neq 0, Y^{\pm} = 0$ and $X^{\pm} = 0$, $Y^{\pm} \neq 0$. The corresponding energy levels are given by

$$E_{n_1} = c \pm \frac{M - aG}{1 + d^2/(n_1+\gamma)^2}\left[\frac{bdG}{(n_1+\gamma)^2} \pm \sqrt{1 - \frac{b^2G^2 - d^2}{(n_1+\gamma)^2}}\,\right] \tag{6.46a}$$

$$E_{n_2} = -c + \frac{M + aG}{1 + d^2/(n_2+\gamma)^2}\left[\frac{bdG}{(n_2+\gamma)^2} \pm \sqrt{1 - \frac{b^2G^2 - d^2}{(n_2+\gamma)^2}}\right] \qquad (6.46b)$$

respectively.

$J > 1/2$ solutions

As in the $J = 1/2$ case there are no normalisable state solutions for equations (6.41); only continuum solutions exist. We have not been able to solve the remaining equations for $J > 1/2$ since decoupling them leads to atleast fourth order differential equations.

iii) Isospinor bosons

In this case the Klein-Gordon equation

$$D_\mu D^\mu U(x) = -\left(M^2 + g^2h^2\phi^2 + gG\frac{\tau^a}{2}\phi^a\right)U(x) \qquad (6.47)$$

can be simplified to

$$\left[\nabla^2 - \frac{A(r)}{r}(\vec{L}\cdot\vec{\tau}) + \left(\frac{G\,H(r)}{2r} - \frac{E\,J(r)}{r}\right)(\vec{\tau}\cdot\hat{r}) - \frac{A^2(r)}{2}\right.$$
$$\left. + \frac{J^2(r)}{4r^2} - \left(\frac{h\,H(r)}{r}\right)^2 + E^2 - M^2\right]U(\vec{r}) = 0, \qquad (6.48)$$

where $U(\vec{x}) = e^{iEt}\,U(x)$. The angular part can be separated using spinor harmonics :

$$U(\vec{r}) = F_+(r)\,\mathcal{Y}_{JM}(\Omega) + F_-(r)\,\mathcal{Y}'_{JM}(\Omega). \qquad (6.49)$$

$F_+(r)$ and $F_-(r)$ obey the radial equations,

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + \frac{\frac{1}{4} - \gamma^2}{r^2} + \frac{\beta}{r} + \epsilon^2 - m^2\right]\begin{pmatrix}F_+\\ F_-\end{pmatrix}$$
$$= \left[Ec - \frac{Ga}{2} + \frac{Ed - Gb/2}{r}\right]\begin{pmatrix}F_-\\ F_+\end{pmatrix}, \qquad (6.50)$$

where

$$\begin{aligned}
\gamma^2 &= J(J+1) + b^2 h^2 - \frac{d^2}{4} \\
\beta &= \frac{cd}{2} - 2abh^2 \\
\epsilon^2 &= E^2 + c^2/4 \\
m^2 &= M^2 + h^2 a^2 .
\end{aligned} \tag{6.51}$$

Here we need not consider lowest angular momentum $(J = 1/2)$ and higher angular momenta $(J > 1/2)$ separately, since both equations are valid for all J.

To solve (6.50) we define two new functions

$$X^{\pm} = \frac{F_+ + F_-}{r} . \tag{6.52}$$

The radial equations become

$$\frac{d^2 X^{\pm}}{dr^2} + \left[\frac{\frac{1}{4} - \gamma^2}{r^2} + \frac{\beta_\pm}{r} + \epsilon_\pm^2 - m_\pm^2 \right] X^{\pm} = 0 \tag{6.53}$$

where

$$\begin{aligned}
\epsilon_\pm^2 &= (E \mp c/2)^2 \\
m_\pm^2 &= m^2 \pm Ga/2 \\
\beta_\pm &= \beta \pm \left(\frac{Gb}{2} - Ed \right) .
\end{aligned} \tag{6.54}$$

The solution is given by

$$X^{\pm} = e^{-\rho_\pm/2} \, \rho_\pm^{\gamma + 1/2} \, {}_1F_1 \left(\gamma + 1/2 - \beta_\pm / 2\lambda_\pm \, , \, 2\gamma + 1 \, , \, \rho_\pm \right) , \tag{6.55}$$

where

$$\rho_\pm = 2\lambda_\pm r$$
$$\lambda_\pm = (m_\pm^2 - \epsilon_\pm^2)^{1/2} . \qquad (6.56)$$

For (6.55) to vanish at the origin we require $b^2 h^2 > d^2/4$. Also for normalisability, the Kummer functions should reduce to polynomials. From this we arrive at the condition

$$\gamma + \tfrac{1}{2} - \beta_\pm / 2\lambda_\pm = -n \qquad (6.57)$$

where $n = 0, 1, 2 \cdots$. The energy levels are explicitly given by

$$E_n^+ = \frac{c}{2} + \left\{ \tfrac{1}{2} d (Gb - 4h^2 ab) \pm \left[(-G^2 b^2 - 16\, a^2 b^2 h^4 + 8 h^2 G a b^2) \right.\right.$$
$$\left.\left. \times (4 n_\gamma^2)^{-1} + (M^2 + h^2 a^2 + \frac{Ga}{2})(1 + d^2/n_\gamma^2) \right]^{1/2} \right\} \left(1 + \frac{d^2}{n_\gamma^2}\right)^{-1} \qquad (6.58a)$$

for the upper sign in (6.57) and

$$E_n^- = -\frac{c}{2} + \left\{ \tfrac{1}{2} d (Gb + 4h^2 ab) \pm \left[(-G^2 b^2 - 16 a^2 b^2 h^4 - 8 h^2 G a b^2) \right.\right.$$
$$\left.\left. \times (4 n_\gamma^2)^{-1} + (M^2 + h^2 a^2 - \frac{Ga}{2})(1 + \frac{d^2}{n_\gamma^2}) \right]^{1/2} \right\} \left(1 + \frac{d^2}{n_\gamma^2}\right)^{-1} \qquad (6.58b)$$

for the lower sign in (6.57) and $n_\gamma = 4(n + \gamma + 1/2)$. As in the previous cases (6.58a) and (6.58b) cannot be satisfied simultaneously. So we take one solution as $X^+ \neq 0$, $X^- = 0$, the corresponding energy levels are given by (6.58a). Similarly eigenfunctions with $X^+ = 0$ and $X^- \neq 0$ will have energy values (6.58b). Since (6.57) can be satisfied only with a positive $\beta_\pm$ the $\pm$ sign appearing before the square root in (6.58) should be chosen to satisfy this.

iv) Isovector bosons

The static Klein-Gordon equation in this case can be written in the form

$$\nabla^2 U_n + E^2 U_n + 2A(r)\left[\hat{r}_n \partial_i U_i - \hat{r}_k \partial_n U_k\right] + 2iE\,\epsilon_{nbc}\hat{r}_b \frac{J(r)}{r} U_c$$
$$- A^2(r)\left[U_n + \hat{r}_n \hat{r}_k U_k\right] + \left(\frac{J(r)}{r}\right)^2 \left[U_n - \hat{r}_n \hat{r}_b U_b\right]$$
$$= \left[M + \left(\frac{hH(r)}{r}\right)^2\right] U_n + \frac{GH(r)}{r}\, i\epsilon_{nam}\hat{r}_a U_m . \tag{6.59}$$

The angular part can be separated using vector spherical harmonics ,

$$U_n = X_J(r)\,\hat{r}_n Y_J^M(\Omega) + Y_J(r)\frac{1}{j} r\,\partial_n Y_J^M(\Omega) + \frac{Z_J}{ij}\epsilon_{nab} r_b \partial_c Y_J^M(\Omega),$$

where $Y_J^M(\Omega)$ are ordinary spherical harmonics and $j = \sqrt{J(J+1)}$. J is the total angular momentum and it takes values $0, 1, 2 \cdots\cdots$. Y_0 and Z_0 are zero by definition. The radial equations are

$$X_J'' + \frac{2}{r}X_J' - \frac{j^2}{r^2}X_J + \left[E^2 - M^2 - \left(hH(r)/r\right)^2\right]X_J = 0 \qquad (\text{all } J) \tag{6.60a}$$

$$Y_J'' + \frac{2}{r}Y_J' + \frac{1-j^2}{r^2}Y_J + \left[E^2 + \left(\frac{J(r)}{r}\right)^2 - M^2 - \left(hH(r)/r\right)^2\right]Y_J$$
$$= \left[\frac{2E\,J(r) - GH(r)}{r}\right] Z_J \qquad (J>0) \tag{6.60b}$$

$$Z_J'' + \frac{2}{r}Z_J' + \frac{1-j^2}{r^2}Z_J + \left[E^2 + \left(\frac{J(r)}{r}\right)^2 - M^2 - \left(hH(r)/r\right)^2\right]Z_J$$
$$= \left[\frac{2E\,J(r) - GH(r)}{r}\right] Y_J \qquad (J>0), \tag{6.60c}$$

where we have substituted for $A(r)$.

We note that (6.60a) is independent of the electric degree of freedom of the monopole. We will later prove that $X_J(r)$ corresponds to the third component of isospin zero. Contrary to the case of isospinor fermions there is an interaction with the Higgs field. This is due to the addition of a fourth power boson term (last term in (6.9)) in the Lagrangian. Note that X_J does not couple to the linear term in the Higgs field. Since (6.60a) is the only equation for the lowest partial wave $(J = 0)$ we conclude that the electric charge of the dyon has no effect on the lowest angular momentum boson. The lowest angular momentum bound state corresponds to a singlet with $I_3 = 0$.

Solutions to (6.60a) are easily obtained in terms of Kummer functions. Substituting for $H(r)$ we get

$$X_J = \alpha\, e^{-\rho/2} \rho^{\gamma + 3/2}\ {}_1F_1\left(\gamma + \tfrac{1}{2} + \frac{ab h^2}{\lambda},\ 2\gamma + 1,\ \rho\right), \tag{6.61}$$

where

$$\begin{aligned} \rho &= 2\lambda r \\ \lambda &= \left(M^2 + h^2 a^2 - E^2\right)^{1/2} \\ \gamma &= \left(\tfrac{1}{4} + h^2 b^2 + j^2\right)^{1/2} \end{aligned} \tag{6.62}$$

and α is an arbitrary constant. The Kummer function should reduce to polynomials for normalisability. From this we get the condition

$$\gamma + \frac{1}{2} + \frac{ab h^2}{\lambda} = -n, \tag{6.63}$$

where $n = 0, 1, 2, \cdots$. To satisfy this condition, either a or b should be negative. In such a case only we get bound states. The expression for the bound state energy is

$$E_n = \pm\left[M^2 + k^2a^2 - \left(abk^2/(n+\tfrac{1}{2}+\gamma)\right)^2\right] . \tag{6.64}$$

There are other types of higher angular momentum states obtainable from (6.60b) and (6.60c). To deduce this we set

$$Y_J \pm Z_J = W_J^{\pm} . \tag{6.65}$$

(6.60c) and (6.60b) now become

$$\frac{d^2W_J^{\pm}}{dr^2} + \frac{2}{r}\frac{dW_J^{\pm}}{dr} + \frac{1-j^2}{r^2}W_J^{\pm} + \left[E^2 + \left(\frac{J(r)}{r}\right)^2 - M^2 - \left(\frac{kH(r)}{r}\right)^2\right]W_J^{\pm}$$

$$= \pm\left[\frac{2E\,J(r) - G\,H(r)}{r}\right]W_J^{\pm} . \tag{6.66}$$

Substituting for $J(r)$ and $H(r)$,

$$\left[\frac{d^2}{dr^2} + \frac{\frac{1}{4}-\gamma^2}{r^2} + \frac{\beta_{\pm}}{r} + \epsilon_{\pm}^2 - m_{\pm}^2\right]\frac{W_J^{\pm}}{r} = 0 , \tag{6.67}$$

where

$$\begin{aligned} \gamma^2 &= j^2 + k^2b^2 - d^2 - \tfrac{3}{4} \\ \beta_{\pm} &= 2cd - 2k^2ab \mp 2Ed \pm Gb \\ \epsilon_{\pm} &= E \mp c \\ m_{\pm} &= M^2 + k^2a^2 \pm Ga . \end{aligned} \tag{6.68}$$

The solutions are given by

$$W_J^{\pm} = \alpha^{\pm} e^{-\rho_{\pm}/2}\rho_{\pm}^{\gamma+3/2}\,{}_1F_1\left(\gamma + \tfrac{1}{2} - \frac{\beta_{\pm}}{2\lambda_{\pm}} , 2\gamma+1 , \rho_{\pm}\right) . \tag{6.69}$$

For normalisability

$$\gamma + \tfrac{1}{2} - \beta_{\pm}/2\lambda_{\pm} = -n \ . \tag{6.70}$$

For a bound state $\beta_{\pm}$ should be positive. This fact will be used to calculate the energy levels which follow from (6.70). The energy corresponding to the upper sign solution in (6.70) is

$$E_n^+ = c + \Big\{ d\Big(\frac{Gb}{2} - k^2 ab\Big) \pm \Big[\Big(-\frac{G^2b^2}{4} - k^4a^2b^2 + Gk^2ab^2\Big)\cdot(4n_\gamma^2)^{-1} + \tfrac{1}{4}(M^2 + k^2a^2 + Ga)\Big(\tfrac{1}{4} + \frac{d^2}{n_\gamma^2}\Big)\Big]^{1/2} \Big\} \Big[\tfrac{1}{4} + \frac{d^2}{n_\gamma^2}\Big]^{-1} \tag{6.71a}$$

and that corresponding to the lower sign solution is

$$E_n^- = -c + \Big\{ d\Big(\frac{Gb}{2} + k^2 ab\Big) \pm \Big[\Big(-\frac{G^2b^2}{4} - k^4a^2b^2 - Gk^2ab^2\Big)\cdot(4n_\gamma^2)^{-1} + \tfrac{1}{4}(M^2 + k^2a^2 - Ga)\Big(\tfrac{1}{4} + \frac{d^2}{n_\gamma^2}\Big)\Big]^{1/2} \Big\} \Big[\tfrac{1}{4} + \frac{d^2}{n_\gamma^2}\Big]^{-1}, \tag{6.71b}$$

where $n_\gamma = 4(n + \frac{1}{2} + \gamma)$. The sign before the square root is chosen in such a way that $\beta_{\pm}$ given in (6.68) is positive. As in the isospinor case, it is not possible to satisfy the two conditions (6.70) simultaneously. So we should set $\alpha^+ \neq 0$, $\alpha^- = 0$ for solutions having energy given in (6.71a) and the solutions corresponding to the energy level (6.71b) have $\alpha^+ = 0$, $\alpha^- \neq 0$.

6.4 Results and discussion

In order to obtain more information about the solutions we gauge-transform them to the string gauge. For the isospinor fermions the transformation is given by [79]

$$\psi' = \psi\, u^{T} \tag{6.72}$$

where

$$u = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2)\, e^{-i\phi} \\ -\sin(\theta/2)\, e^{i\phi} & \cos(\theta/2) \end{pmatrix} . \tag{6.73}$$

Substituting for ψ and u^{T} we find, explicitly,

$$\chi^{\pm} = i \begin{pmatrix} \begin{matrix} \Big[-(P_J^{\pm}+G_J^{\pm})\, e^{-i\phi} \sin(\theta/2) \\ + \frac{(B_J^{\pm}+C_J^{\pm})}{j}\, e^{-i\phi} \Big(\cos(\theta/2) \frac{\partial}{\partial\theta} \\ - \frac{i}{2} \operatorname{cosec}(\theta/2) \frac{\partial}{\partial\phi} \Big) \Big] \end{matrix} & \begin{matrix} \Big[(P_J^{\pm}-G_J^{\pm}) \cos(\theta/2) + \frac{(B_J^{\pm}-C_J^{\pm})}{j} \\ \times \Big(\sin(\theta/2) \frac{\partial}{\partial\theta} \\ - \frac{i}{2} \sec(\theta/2) \frac{\partial}{\partial\phi} \Big) \Big] \end{matrix} \\ \begin{matrix} \Big[(P_J^{\pm}+G_J^{\pm}) \cos(\theta/2) + \frac{(B_J^{\pm}+C_J^{\pm})}{j} \\ \times \Big(-\sin(\theta/2) \frac{\partial}{\partial\theta} - \frac{i}{2} \sec(\theta/2) \frac{\partial}{\partial\phi} \Big) \Big] \end{matrix} & \begin{matrix} \Big[(P_J^{\pm}-G_J^{\pm})\, e^{i\phi} \sin(\theta/2) \\ - \frac{(B_J^{\pm}-C_J^{\pm})}{j}\, e^{i\phi} \Big(\cos(\theta/2) \frac{\partial}{\partial\theta} \\ + \frac{i}{2} \operatorname{cosec}(\theta/2) \frac{\partial}{\partial\phi} \Big) \Big] \end{matrix} \end{pmatrix} Y_J^M(\theta,\phi) . \tag{6.74}$$

As already remarked the wave functions corresponding to the energy levels given by (6.34a) satisfy $P_J^{\pm} = G_J^{\pm}$ and $B_J^{\pm} = C_J^{\pm}$. Hence from the explicit form of the wavefunction given above, it is clear that it describes an isospin-up bound state. Similarly, the energy levels given by (6.34b) are $I_3 = -1/2$ levels. If we consider the scattering behaviour of the solutions the above property will lead to charge-conserved scattering. This is the same as in the case of fermions in the point monopole background having lowest angular momentum [82]. But in a regular dyon field the situation is different. Charge exchange scattering occurs in this case [79].

In the isovector case, the gauge transformation is given by

$$\psi' = \psi u^T , \tag{6.75}$$

where

$$u = \begin{pmatrix} \cos\theta\cos^2\phi + \sin^2\phi & \cos\phi\sin\phi(\cos\theta - 1) & -\sin\theta\cos\phi \\ \cos\phi\sin\phi(\cos\theta - 1) & \cos\theta\sin^2\phi + \cos^2\phi & -\sin\theta\sin\phi \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{pmatrix} \tag{6.76}$$

where the isospin components in ψ are written as a 1×3 row matrix:

$$\chi^{\pm} = \Big((F_{1+}^{\pm}\hat{r}_1 + F_{2+}^{\pm} r\partial_1 + F_{3-}^{\pm}L_1)\mathcal{Y}_{JM} \quad (F_{1+}^{\pm}\hat{r}_2 + F_{2+}^{\pm} r\partial_2 + F_{3-}^{\pm}L_2)\mathcal{Y}_{JM} \quad (F_{1+}^{\pm}\hat{r}_3 + F_{2+}^{\pm} r\partial_3 + F_{3-}^{\pm}L_3)\mathcal{Y}_{JM} \Big)$$

$$+ \begin{pmatrix} \text{similar matrix with } F_i^{\pm} \longleftrightarrow F_{i-}^{\pm} \\ \text{with } \mathcal{Y}_{JM} \to \mathcal{Y}'_{JM} \end{pmatrix} . \tag{6.77}$$

After the transformation we get

$$\chi^{\pm} = \Big(\Big[(F_{2+}^{\pm}\cos\phi - iF_{3-}^{\pm}\sin\phi)\frac{\partial}{\partial\theta} - (F_{2+}^{\pm}\sin\phi + iF_{3-}^{\pm}\cos\phi)\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\Big]\mathcal{Y}_{JM} \quad \Big[(F_{2+}^{\pm}\sin\phi + iF_{3-}^{\pm}\cos\phi)\frac{\partial}{\partial\theta} + (F_{2+}^{\pm}\cos\phi - iF_{3-}^{\pm}\sin\phi)\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\Big]\mathcal{Y}_{JM} \quad F_{1+}^{\pm}\mathcal{Y}_{JM} \Big)$$

$$+ \begin{pmatrix} \text{similar matrix with } F_i^{\pm} \longleftrightarrow F_{i-}^{\pm} \\ \text{with } \mathcal{Y}_{JM} \to \mathcal{Y}'_{JM} \end{pmatrix} . \tag{6.78}$$

From this it is obvious that $F_{1+}^{\pm}$ and $F_{1-}^{\pm}$ correspond to solution with $I_3 = 0$.

In the isovector boson case we have the row matrix

$$U' = \begin{pmatrix} \frac{1}{J}\left[\left(Y\cos\phi - iZ\sin\phi\right)\frac{\partial}{\partial\theta} - \frac{Y\sin\phi + iZ\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right]Y_J^M & \frac{1}{J}\left[\left(Y\sin\phi + iZ\cos\phi\right)\frac{\partial}{\partial\theta} - \frac{Y\cos\phi - iZ\sin\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right]Y_J^M & X\,Y_J^M \end{pmatrix} \tag{6.79}$$

Here also the solution with $X \neq 0$, $Y = Z = 0$ corresponds to a state with $I_3 = 0$

We shall now compare our solutions with those given by Tang [81,83] and prove the equivalence of the two methods used to achieve the angular separation. For this it is enough to consider the isospinor bosons. Tang [83] studied this problem by applying a singular gauge transformation which creates a singular string in the gauge potential. Then using monopole harmonics the separation of radial angular parts was achieved. Our model reduces to that considered in Ref.83 if we set $\hbar = 0$ and take the parameters $a, b, c,$ and d as in (6.5). So we can compare the eigenvalues and eigenfunctions in these cases. Since the eingenfunctions are gauge-dependent this comparison must be made in the same gauge. For this we gauge-transform our solutions to the gauge used in Ref. 83.

$$U' = u\,U, \tag{6.80}$$

where u is the unitary matrix given in (6.73). This transformation yields,

$$U' = \begin{pmatrix} (F_+ + F_-)\left\{\sqrt{\frac{J+M}{2J}}\cos(\theta/2)\,Y_{J-1/2}^{M-1/2} + \sqrt{\frac{J-M}{2J}}\,e^{-i\phi}\sin(\theta/2)\,Y_{J-1/2}^{M+1/2}\right\} \\ (F_+ - F_-)\left\{\sqrt{\frac{J+M}{2J}}\,e^{i\phi}\sin(\theta/2)\,Y_{J-1/2}^{M-1/2} + \sqrt{\frac{J-M}{2J}}\cos(\theta/2)\,Y_{J-1/2}^{M+1/2}\right\} \end{pmatrix} \tag{6.81}$$

where use has been made of the recurrence relations of associated Legendre polynomials [86]. This can be further shown to be equal to (see Appendix 6.B)

$$U' = \begin{pmatrix} \frac{F_+ + F_-}{\sqrt{2}} \; Y_{-1/2,J,M} \\ \frac{F_+ - F_-}{\sqrt{2}} \; Y_{1/2,J,M} \end{pmatrix}, \tag{6.82}$$

where $Y_{q,J,M}$ are monopole harmonics [84]. From the study of radial equations in Section 6.3 we have seen that either $(F_+ + F_-)$ or $(F_+ - F_-)$ is nonzero, but not both. The radial equations obeyed by $(F_+ + F_-)$ and $(F_+ + F_-)$ are the same as studied in Ref. 83. The angular parts are also the same. So we conclude that both the methods are equivalent. From (6.82) we also see that the states having energy values (6.58a) are isospin-up states and those having energy values (6.58b) are isospin-down states.

Finally we comment on the degeneracy of the system. For massless particles in a pure monopole $(M = c = 0)$ field that the pair of quantisation conditions for both fermions and bosons, (6.27a, b), (6.34a, b), (6.57) and (6.70), can be satisfied simultaneously. The degeneracy of the energy levels in this case for fixed n and J is as given in the following table.

TABLE - I

Description of particles	Lowest total angular momentum	Degeneracy of bond states for fixed n and J $(n \neq 0)$.	
		Lowest angular momentum	All higher angular momenta
Isospinor fermions	$0 \; (< I)$	$2(2J+1)$	$4(2J+1)$
Isospinor bosons	$\frac{1}{2} \; (= I)$	$2(2J+1)$	$2(2J+1)$
Isovector fermions	$\frac{1}{2} \; (< I)$	$2(2J+1)$	$\geq 4(2J+1)$
Isovector bosons	$0 \; (< I)$	$(2J+1)$	Two types of bound states 1) $(2J+1)$ $(I_3 = 0)$

The inclusion of either a nonzero mass ($M \neq 0$) or dyon degree freedom ($c \neq 0$) or both, changes the degeneracy of the spectrum. The pair of the above-mentioned conditions are not satisfied simultaneously. The degeneracy is half of that given in the above table except for the $I_3 = 0$ state of the isovector fermions. Unlike the $c = 0$, $M = 0$ case the energy levels are not symmetrically distributed on both sides of zero. We also note that the total number of states for the case of monopoles with $M \neq 0$, agree with the counting as given in Ref. 80. For isovector fermions there are no bound states with $I_3 = 0$ and agreement with Ref. 80 is obtained only if we count the unbound solutions along with the bound states.

6.A Appendix

In this section we solve the first order coupled equations

$$\left(\frac{d}{dr} + \frac{1 \pm j}{r}\right) X^{\pm} = \left(\frac{(B \pm D)}{r} + m \pm \epsilon\right) X^{\mp}, \tag{6.83}$$

which is exactly similar to that of the hydrogen atom problem in Dirac theory [85] if the B/r term is absent. This can be solved by a similar procedure. Dividing (6.83) throughout by $2\lambda = 2\sqrt{m^2 - \epsilon^2}$ we get

$$\begin{aligned} \left(\frac{d}{d\rho} + \frac{1+j}{\rho}\right) X^{+} &= \left(\frac{B+D}{\rho} + \frac{1}{2}\sqrt{\frac{m+\epsilon}{m-\epsilon}}\right) X^{-} \\ \left(\frac{d}{d\rho} + \frac{1-j}{\rho}\right) X^{-} &= \left(\frac{B-D}{\rho} + \frac{1}{2}\sqrt{\frac{m-\epsilon}{m+\epsilon}}\right) X^{+} \end{aligned} \tag{6.84}$$

where $\rho = 2\lambda r$. Using the substitution

$$X^{\pm} = \sqrt{m \pm \epsilon}\; e^{-\rho/2} \rho^{\gamma - 1} \left(Q_2 \pm Q_1\right). \tag{6.85}$$

with

$$\gamma = \sqrt{B^2 + j^2 - D^2},$$

(6.84) can be reduced to

$$\left(\rho \frac{d}{d\rho} + \gamma + j\right)(Q_1 + Q_2) - \rho Q_2 = (B + D)\sqrt{\frac{m-\epsilon}{m+\epsilon}}\,(Q_1 - Q_2)$$

$$\left(\rho \frac{d}{d\rho} + \gamma - j\right)(Q_1 - Q_2) + \rho Q_2 = -(B - D)\sqrt{\frac{m+\epsilon}{m-\epsilon}}\,(Q_1 + Q_2)\,. \tag{6.86}$$

The sum and difference of these gives

$$\left(\rho \frac{d}{d\rho} + \gamma + \frac{Bm - D\epsilon}{\lambda}\right) Q_1 = \left(-j + \frac{Dm - B\epsilon}{\lambda}\right) Q_2$$

$$\left(\rho \frac{d}{d\rho} + \gamma - \frac{Bm - D\epsilon}{\lambda} - \rho\right) Q_2 = \left(-j - \frac{Dm - B\epsilon}{\lambda}\right) Q_1\,. \tag{6.87}$$

Now elimination of either Q_2 or Q_1 gives

$$\rho \frac{d^2 Q_1}{d\rho^2} + (2\gamma + 1 - \rho)\frac{dQ_1}{d\rho} - \left(\gamma + \frac{Bm - D\epsilon}{\lambda}\right) Q_1 = 0 \tag{6.88a}$$

and

$$\rho \frac{d^2 Q_2}{d\rho^2} + (2\gamma + 1 - \rho)\frac{dQ_2}{d\rho} - \left(\gamma + 1 + \frac{Bm - D\epsilon}{\lambda}\right) Q_2 = 0 \tag{6.88b}$$

respectively. In deriving this we have used the relation

$$j^2 - \left(\frac{Dm - B\epsilon}{\lambda}\right)^2 = \gamma^2 - \left(\frac{Bm - D\epsilon}{\lambda}\right)^2 . \tag{6.89}$$

The solutions of (6.88) are

$$Q_1 = \alpha \; {}_1F_1\left(\gamma + \frac{Bm - D\epsilon}{\lambda}, 2\gamma + 1, \rho\right)$$

$$Q_2 = \beta \; {}_1F_1\left(1 + \gamma + \frac{Bm - D\epsilon}{\lambda}, 2\gamma + 1, \rho\right), \tag{6.90}$$

where the constants α and β are related :

$$\beta = \frac{\gamma + (Bm - D\epsilon)/\lambda}{-j + (Dm - B\epsilon)/\lambda}\,\alpha\,, \tag{6.91}$$

which is obtained by substituting (6.90) in either of the equations in (6.87) and putting $\rho = 0$.

6.B. Appendix

In this section we will prove the following results :

$$\sqrt{2}\left[\sqrt{\frac{J+M}{2J}}\cos(\theta/2)\, Y^{M-1/2}_{J-1/2} + \sqrt{\frac{J-M}{2J}}\, e^{-i\phi}\sin(\theta/2)\, Y^{M+1/2}_{J-1/2}\right] = Y_{-\frac{1}{2},J,M} \tag{6.92a}$$

$$\sqrt{2}\left[-\sqrt{\frac{J+M}{2J}}\, e^{i\phi}\sin(\theta/2)\, Y^{M-1/2}_{J-1/2} + \sqrt{\frac{J-M}{2J}}\cos(\theta/2)\, Y^{M+1/2}_{J-1/2}\right] = Y_{\frac{1}{2},J,M}\,. \tag{6.92b}$$

Putting $x = \cos\theta$, $j = J - 1/2$, $m = M - 1/2$ the left hand side of (6.92a) can be written as

$$\sqrt{\frac{j+m+1}{2j+1}}\sqrt{1+x}\; Y^{m}_{j}(x,\phi) + \sqrt{\frac{j-m}{2j+1}}\sqrt{1-x}\; e^{-i\phi}\, Y^{m+1}_{j}(x,\phi)$$

$$= (-1)^m\sqrt{\frac{(j-m)}{4\pi(j+m+1)!}}\left\{(j+m+1)\sqrt{1+x}\; P^{m}_{j}(x) - \sqrt{1-x}\; P^{m+1}_{j}(x)\right\} e^{im\phi}$$

$$= (-1)^m\sqrt{\frac{(j+m+1)!}{4\pi\,(j-m)!}}\;\frac{(1-x^2)^{m/2}\sqrt{1+x}}{2^m\, m!}\left\{ {}_2F_1\left(m-j,\, m+j+1\,;\, m+1\,;\, \frac{1-x}{2}\right)\right.$$

$$\left. + \frac{1-x}{2}\,\frac{m-j}{m+1}\; {}_2F_1\left(m-j+1,\, m+j+2\,;\, m+2\,;\, \frac{1-x}{2}\right)\right\} e^{im\phi}$$

$$= (-1)^m\sqrt{\frac{(j+m+1)!}{4\pi(j-m)!}}\;\frac{(1-x^2)^{m/2}\sqrt{1+x}}{2^m\, m!}\; {}_2F_1\left(m-j\,,m+j+2\,;\, m+1\,;\, \frac{1-x}{2}\right) e^{im\phi}$$

$$= (-1)^m\sqrt{\frac{(j+m+1)!\,(j-m)!}{4\pi\; j!\; j!}}\;\frac{(1-x^2)^{m/2}\sqrt{1+x}}{2^m}\; P^{m,\,m+1}_{j-m}(x)$$

$$= (-1)^{M-1/2}\left[\frac{(2J+1)\,(J-M)!\,(J+M)!}{4\pi(J+\frac{1}{2})!\,(J-\frac{1}{2})!}\right]^{1/2}(1-x)^{(M-1/2)/2}\,(1+x)^{(M+1/2)/2}$$

$$\times\; P^{M-1/2,\,M+1/2}_{J-M}(x)\;\; e^{i(M-1/2)\phi}$$

$$= 2^{M}\left[\frac{(2J+1)(J-M)!(J+M)!}{4\pi(J+\frac{1}{2})!(J-\frac{1}{2})!}\right]^{1/2}(1-x)^{-(M-1/2)/2}(1+x)^{-(M+1/2)/2}$$

$$\times P_{J+M}^{-(M-1/2),\,-(M+1/2)}(x)\; e^{i(M-1/2)\phi}$$

$$= Y_{-1/2,J,M}(\theta,\phi)\,, \tag{6.93}$$

where ${}_2F_1(\alpha,\beta;\gamma;x)$ are hypergeometric functions and $P_n^{\alpha,\beta}(x)$ are Jacobi polynomials [61, 87]. For deriving (6.93) we have used the relations [61, 87]

$$P_n^m(x) = \frac{(n+m)!}{(n-m)!}\frac{(1-x)^{m/2}}{2^m m!}\,{}_2F_1\left(m-n, m+n+1; m+1; \frac{1-x}{2}\right) \tag{6.94}$$

$$P_n^{\alpha,\beta}(x) = \frac{\Gamma(n+\alpha+1)}{n!\,\Gamma(\alpha+1)}\,{}_2F_1\left(-n, n+\alpha+\beta+1; \gamma+1; \frac{1-x}{2}\right) \tag{6.95}$$

$${}_2F_1(\alpha,\beta;\gamma;z) + \frac{\alpha}{\gamma}z\,{}_2F_1(\alpha+1,\beta+1;\gamma+1;z) = {}_2F_1(\alpha,\beta+1,\gamma;z) \tag{6.96}$$

and [84]

$$P_{n+\alpha+\beta}^{-\alpha,-\beta}(x) = 2^{-\alpha-\beta}(x-1)^{\alpha}(x+1)^{\beta}\,P_n^{\alpha,\beta}(x) \tag{6.97}$$

$$Y_{q,J,M}(x,\phi) = 2^{M}\left[\frac{(2J+1)(J-M)!(J+M)!}{4\pi(J-q)!(J+q)!}\right]^{1/2}(1-x)^{-(q+M)/2}(1+x)^{(q-M)/2}$$
$$\times P_{J+M}^{-(q+M),(q-M)}(x)\,e^{i(M+q)\phi}\,. \tag{6.98}$$

(6.92b) can be proved similarly. In this case instead of (6.96) we should use the relation

$${}_2F_1(\alpha,\beta;\gamma;z) - \frac{\alpha}{\gamma}(1-z)\,{}_2F_1(\alpha+1,\beta+1;\gamma+1;z) = \frac{\gamma-\alpha}{\gamma}\,{}_2F_1(\alpha,\beta+1;\gamma+1;z) \tag{6.99}$$

REFERENCES

1. E. S. Abers and B. W. Lee, Phys.Rep. **9C**(1973) 1
2. P. Langacker, Phys.Rep. **72C**(1981) 185.
3. C. Itzykson and J. B. Zuber, 'Quantum Field Theory', McGraw-Hill, New York, 1980.
4. K. Huang, 'Quarks Leptons & Gauge Fields', World Scientific, Singapore, 1982.
5. T.-P. Cheng and L.-F. Li, 'Gauge Theory of Elementary Particle Physics', Clarendon Press, Oxford, 1984.
6. S. L. Glashow, Nucl.Phys. **22**(1961) 579.
   S. Weinberg, Phys.Rev.Lett. **19**(1967) 1264.
   A. Salam, in 'Elementary Particle Theory' (N. Svartholm Ed.), Almquist and Wicksell, Stockholm, 1968.
7. C. N. Yang and R. L. Mills, Phys.Rev. **96**(1954) 191.
8. R. Utiyama, Phys.Rev. **101**(1956) 1597.
9. J. Goldstone, Nuo.Cim. **19**(1961) 154.
   J. Goldstone, A. Salam and S. Weinberg, Phys.Rev. **127**(1962) 975.
10. F. Englert and R. Brout, Phys.Rev.Lett. **13**(1964) 321.
    G. S. Guralnik, C. R. Hagen and T. W. B. Kibble, Phys.Rev.Lett. **13**(1964) 585.
    P. W. Higgs, Phys.Rev.Lett. **12**(1964) 132; Phys.Rev. **145**(1966) 1156.
11. A. Actor, Rev.Mod.Phys. **51**(1979) 461.
12. J. Arafune, P. G. O. Freund and C. J. Goebel, J.Math.Phys. **16**(1975) 433.
13. A. A. Balavin, A. M. Polyakov, A. S. Schwartz and Yu. S. Tyupkin, Phys.Lett. **59B**(1975) 85
14. P. A. M. Dirac, Proc.R.Soc. **A133**(1931) 60; Phys.Rev. **74**(1948) 817.
15. P. Goddard and D. I. Olive, Rep.Prog.Phys. **41**(1978) 1357.
16. S. Coleman, in 'New Phenomena in Subnuclear Physics' (Proc. 1975 Int. Sch. Subnucl. Phys. Ettore Majorana), (A. Zichichi Ed.), Plenum Press, New York, 1975; in 'The Unity of Fundamental Interactions' (A. Zichichi Ed.), Plenum Press, New York, 1983.
17. N. S. Craigie, P. Goddard and W. Nahm (Eds.), 'Monopoles in Ouantum Field Theory', World Scientific, Singapore, 1982.

- G 3617 -

18. L. O'Raifeartaigh and S. Rouhani, Acta Phys.Austr.Suppl. **XXIII**(1981) 525.

19. P. Rossi, Phys.Rep. **86** (1982) 317.

20. R. A. Carrigan and W. P. Trower (Eds.), 'Magnetic Monopoles', Plenum, New York, 1983.

21. N. S. Craigie (Ed.), 'Theory and Detection of Magnetic Monopoles in Gauge Theories', World Scientific, Singapore, 1986.

22. G. 't Hooft, Nucl.Phys. **B79**(1974) 276.

23. A. M. Polyakov, JETP Lett. **20**(1974) 194.

24. H. Georgi and S. L. Glashow, Phys.Rev.Lett. **28**(1972) 1494; Phys.Rev. **D6**(1972) 2977.

25. L. D. Faddeev, in 'Nonlocal, Nonlinear and Nonrenormalisable Field Theories' (Proc. Int. Symp., Alushta), Dubna: Joint Institute for Nuclear Research, 1976; Lett.Math.Phys. **1**(1976) 289.

26. E. B. Bogomolny, Sov.J.Nucl.Phys. **24**(1976) 449.

27. V. Fateev, A. Svartz and Y. Tyupkin, Teor.Mat.Fiz. **26**(1976) 270.

28. J. Rawnsley, J.Phys. **A10**(1977) L139.

29. M. K. Prasad and C. M. Sommerfield, Phys.Rev.Lett. **35**(1975) 760.

30. B. Julia and A. Zee, Phys.Rev. **D11**(1975) 2227.

31. T. W. Kirkman and C. K. Zachos, Phys.Rev. **D24**(1981) 999.

32. E. B. Bogomolny and M. S. Marinov, Sov.J.Nucl.Phys. **23**(1976) 355.

33. F. A. Bais and J. Primack, Phys.Rev. **D13**(1976) 819.

34. A. P. Protogenov, Phys.Lett. **67B**(1977) 62.

35. I. Ju, Phys.Rev. **D17**(1978) 1637.

36. S. Deser, Phys.Lett. **64B**(1975) 463.
H. Pagels, Phys.Lett. **68B**(1977) 466.

37. T. T. Wu and C. N. Yang, in 'Properties of Matter under Unusual Conditions' (H. Mark and S. Fernbach Eds.), Interscience, New York, 1969.

38. P. H. Frampton, Phys.Rev. **D14**(1976) 528.

39. K. Kermer, Phys.Rev. **D19**(1979) 1243; **D24**(1981) 2336.

40. D. Maison, Nucl.Phys. **B182** (1981) 144.

41. J. Schwinger, Science **165**(1969) 757.

42. E. Tomboulis and G. Woo, Nucl.Phys. **B107**(1976) 221.

43. J. L. Gervais, B. Sakita and S. Wadia, Phys.Lett. **63B**(1976) 55.

44. J. P. Preskill, Phys.Rev.Lett. **43**(1979) 1365.
Ya. B. Zeldovich and M. Y. Khlopov, Phys.Lett. **79B**(1978) 239.

45. G. Giacomelli, in Refs. 17 and 21.

46. T. W. B. Kibble, in Ref. 17.
Q. Shafi in Ref. 21.

47. R. Jackiw and C. Rebbi, Phys.Rev. **D13**(1976) 3398.

48. R. Jackiw and C. Rebbi, Phys.Rev.Lett. **36**(1976) 1116.
P. Hasenfratz and G. 't Hooft, Phys.Rev.Lett. **36**(1976) 1119.
A. S. Goldhaber, Phys.Rev.Lett. **36**(1976) 1122.

49. V. A. Rubakov, JETP Lett. **33**(1981) 644; Nucl.Phys. **B203**(1982) 311.
C. G. Callan Jr, Phys.Rev. **D25**(1982) 2141; **D26**(1982) 2058; Nucl.Phys. **B212**(1983) 391.

50. R. Jackiw and C. Rebbi, Phys.Rev.Lett. **37**(1976) 172.

51. C. Callan, R. Dashen and J. Gross, Phys.Lett. **63B**(1976) 334.

52. R. Jackiw, Rev.Mod.Phys. **49**(1979) 681.

53. G. 't Hooft, Coral Gables Conference 1977.
R. Jackiw, C. Nohl and C. Rebbi, Phys.Rev. **D15**(1977) 1642.

54. G. 't Hooft, Phys.Rev.Lett. **37**(1976) 8.

55. C. Callan, R. Dashen and J. Gross, Phys. Rev. **D17**(1978) 2717.

56. N. Christ and R. Jackiw, Phys.Lett. **91B**(1979) 228.
R. Jackiw, in 'Field Theory and Strong Interaction' (P. Urban Ed.), Springer-Verlag, Wein, 1980.

57. N. K. Pak and R. Percacci, Nucl.Phys. **B188**(1981) 355.

58. D. J. Gross, R. D. Pisarski and L. G. Yaffe, Rev.Mod.Phys. **53**(1981) 43.

59. R. Hirota, in 'Backlund Transformations' (R.M. Miura Ed.), Springer Verlag, Berlin Heidelberg, 1976; in 'Solitons (R. K. Bullough and P. J. Caudrey Eds.), Springer Verlag, Berlin Heidelberg, 1980.
P. J. Caudrey, in 'Nonlinear Equations in Physics and Mathematics' (A. O. Barut Ed.), R. Reidel Publishing Company, Holland, 1978.

60. R. Hirota and J. Satsuma, Phys.Lett. **85A**(1981) 407.

61. I. S. Gradshteyn and I. M. Ryzhik, 'Table of Integrals, Series and Products', Academic Press, New York, 1965.

62. E. T. Whittaker and G. N. Watson, 'A Course on Modern Analysis', Cambridge, England, 1927.

63. J. L. Richard and A. Rouet, Nucl.Phys. **B183**(1981) 251; **B185**(1981) 47.

64. A. Lapedes and E. Mottola, Nucl.Phys. **B203**(1982) 58.

65. E. Mottola and A. Rouet, Phys.Lett. **119B**(1982) 162.

66. H. Boutaleb-Joutei, A. Chakrabarti and A. Comtet, Phys.Lett. **101B**(1981) 249.

67. W. Mecklenburg and D. P. O'Brein, Phys.Rev. **D18**(1978) 1327.

68. H. Boutaleb-Joutei, A. Chakrabarti and A. Comtet, Phys.Rev. **D20**(1979) 1884.

69. S. Coleman, in 'The Whys of Subnuclear Physics' (A. Zichichi Ed.), Plenum, New York, 1979; Phys.Rev. **D15**(1977) 2929; **D16**(1977) 1248(E).

70. H. Arodz, Phys.Rev. **D27**(1983) 1903.

71. D. Ray, Phys.Rev. **D22**(1980) 2100.

72. E. Witten, Phys.Rev.Lett. **38**(1977) 121.

73. J. P. Hsu and E. Mac, J.Math.Phys. **18**(1977) 100.

74. P. P. Banderet, Helv.Phys.Acta **19**(1946) 503.
Harish Chandra, Phys.Rev. **74**(1948) 883.

75. A. S. Goldhaber, Phys.Rev. **D16** (1977) 1815.

76. C. J. Callias, Phys.Rev. **D16**(1977) 3068.

77. L. Bracci and G. Fiorentini, Phys.Lett. **124B**(1983) 493.

78. T. Dereli, J. H. Swank and L. J. Swank, Phys.Rev. **D11**(1975) 3541.

79. W. J. Marciano and I. J. Muzinich, Phys.Rev.Lett. **50**(1983) 1035; Phys. Rev. **D28**(1983) 973.

80. P. Cox and A. Yildiz, Phys.Rev. **D18**(1978) 1211.

- G 3617 -

81. J.-F. Tang, Phys.Rev. **D26**(1982) 510.

82. A. M. Din and S. M. Roy, Phys.Lett. **129B**(1983) 201.

83. T.-F. Tang, Phys.Lett. **120B**(1983) 364.

84. T. T. Wu and C. N. Yang, Nucl.Phys. **B107** (1976) 365.

85. V. B. Berestetskii, E. M. Lifshitz and L. P. Pitaevskii, 'Relativistic Quantum Theory', Pergamon Press, Oxford, 1971.

86. E. Butkov, 'Mathematical Physics', Addison-Wesley Pub. Co., Reading, Massachusetts, 1968.

87. W. W. Bell, 'Special Functions for Scientists and Engineers', D. Van Nostrand Company Ltd., London, 1965.