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Quantile based reliability aspects of partial moments

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ABSTRACT

Partial moments are extensively used in literature for modeling and analysis of lifetime data. In this paper, we study properties of partial moments using quantile functions. The quantile based measure determines the underlying distribution uniquely. We then characterize certain lifetime quantile function models. The proposed measure provides alternate definitions for ageing criteria. Finally, we explore the utility of the measure to compare the characteristics of two lifetime distributions.

1. Introduction

Let X be a random variable with distribution function $F(x)$ and finite moment of order r. Then the rth upper partial moment about x is defined as

$$\alpha_r(x) = E(X-x)_+^r = \int_x^\infty (t-x)^r dF(t) \tag{1.1}$$

where $(X-x)_+ = \max[(X-x), 0]$. The quantity $(X-x)_+$ is interpreted as the residual age in the context of lifelength studies (Lin, 2003) and the first two moments and variance of $(X-x)_+$ are used in actuarial studies for the analysis of risks (Denuit, 2002). In the assessment of income, x can be taken as the tax exemption level so that $(X-x)_+$ becomes the taxable income.

Gupta and Gupta (1983) have discussed general properties of partial moments. They proved that (1.1) determines the underlying distribution uniquely for any positive real r. Also when r is a positive integer there exists a recurrence relation between two consecutive partial moments. Abraham, Nair, and Sankaran (2007), Chong (1977) and Lin (2003) have characterized the exponential, beta and Pareto II distributions by relationships among various moments. The survival function $\bar{F}(x)$ of X can be written in terms of $\alpha_r(x)$ as (Navarro, Franco, & Ruiz, 1998; Sunoj, 2004)

$$\bar{F}(x) = \frac{(-1)^r}{r!}\frac{d^r\alpha_r(x)}{dx^r}.$$

Gupta (2007) and Sunoj (2004) obtained partial moments and their properties in respect of length biased and equilibrium distributions.

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All these theoretical results and applications thereof use the definition (1.1) based on distribution function. A probability distribution can be specified either in terms of the distribution function or by the quantile function.

$$Q(u) = \inf_{x}\{x : F(x) \geq u\}, \quad 0 \leq u \leq 1$$

and recently there has been substantial interest in modeling statistical data using quantile functions. Many of the quantile functions used in statistical theory and applications like various forms of lambda distributions (Freimer, Mudholkar, Kollia, & Lin, 1998; Gilchrist, 2000; Ramberg & Schmeiser, 1974; van Staden & Loots, 2009), the power-Pareto distribution (Gilchrist, 2000; Hankin & Lee, 2006), Govindarajulu distribution (Nair, Sankaran, & Vineshkumar, 2012) etc. do not have tractable distribution functions.

For example in the case of the Govindarajulu distribution specified by

$$Q(u) = \sigma[(\beta+1)u^{\beta} - \beta u^{\beta+1}], \quad 0 \leq u \leq 1,\ \sigma, \beta > 0$$

cannot be inverted from $Q(u) = x$ to $u = F(x)$ analytically. In practice, for purposes of analysis the procedure is to solve for u from the equation

$$x = \sigma((\beta+1)u^{\beta} - \beta u^{\beta+1})$$

using numerical techniques, corresponding to chosen values of x. Such a collection of values of $F(x)$ is insufficient to determine the characteristics of the distribution exactly. The only alternative to resolve the problem is to find quantile-based equivalents of the definitions of the characteristics and then use them for theoretical analysis. An illustration of this approach that led to new methodology, analysis and models in the context of reliability analysis can be seen in Nair and Sankaran (2009), Nair, Sankaran, and Vineshkumar (2011) and Nair and Vineshkumar (2010). These works also indicate that even when the quantile functions are invertible, the approach provides alternative methodologies that have desirable properties and are easier for applications.

Thus a formulation of the definition and properties of partial moments in terms of quantile functions is essential to study them in the context of the quantile function model. Such a discussion has several advantages. Analytical properties of the partial moments obtained in this approach can be used as an alternative tool in modeling statistical data. Sometimes the quantile based approach is better in terms of tractability. New models and characterizations that are unresolvable in the distribution functions approach can be resolved with the aid of the quantile approach. For example, the sum of two quantile functions is again a quantile function. Hence starting with a partial mean of a quantile function, one can add another partial mean to generate a new partial mean and the corresponding new quantile function. This imparts considerable flexibility in modeling problems. See Sections 2 and 3 for a further elucidation of this and other aspects of the advantages of the quantile approach. In view of these, the objective of the present work is to initiate a discussion of quantile based partial moments in the context of reliability analysis.

The text is organized as follows. In Section 2, we present the definition of partial moments in terms of the quantile function. Various properties of the partial moments and their relationships with other basic reliability concepts are discussed. The proposed definition is used, in Section 3, to characterize certain lifetime quantile function models. The potential of quantile-based definition to characterize various notions of ageing is studied in Section 4. In Section 5 we demonstrate the utility of some of the results by applying them to a real data. Finally, Section 6 provides the utility of the new definition to compare characteristics of two life distributions.

2. Basic results

Let X be a a non-negative random variable with absolutely continuous distribution function $F(x)$ and probability density function $f(x)$. When $F(x)$ is strictly increasing, the quantile function $Q(u)$ is obtained from the solution of $F(x) = u$, as $x = Q(u)$. The mean of the distribution, when $Q(0) = 0$ is given by

$$\mu = \int_0^1 Q(p)dp \tag{2.1}$$

$$= \int_0^1 (1-p)q(p)dp \tag{2.2}$$

where $q(u) = \frac{dQ(u)}{du}$ is the quantile density function.

Since $F(x)$ is strictly increasing, $f(x) > 0$ so that the quantile density function $q(u)$ exists by virtue of the relation

$$q(u)f(Q(u)) = 1. \tag{2.3}$$

The quantile version of the partial moment is obtained by setting $F(x) = u$ in (1.1), which gives

$$P_r(u) = \alpha_r(Q(u)) = \int_u^1 (Q(p) - Q(u))^r dp. \tag{2.4}$$

Since the first two partial moments are generally in use, we confine the discussions to the cases $r = 1$ and $r = 2$ in (2.4). When $r = 1$,

$$P_1(u) = \int_u^1 (Q(p) - Q(u))\, dp.$$

Integrating by parts we have the equivalent formula in terms of quantile density function given as

$$P_1(u) = \int_u^1 (1-p)q(p)dp \tag{2.5}$$

and also

$$P_1(u) = \int_u^1 Q(p)\, dp - (1-u)Q(u). \tag{2.6}$$

Likewise, the second partial moment is

$$P_2(u) = \alpha_2(Q(u)) = \int_u^1 (Q(p) - Q(u))^2\, dp$$

and

$$P_2'(u) = \frac{2P_1(u)P_1'(u)}{(1-u)}. \tag{2.7}$$

As mentioned earlier $P_1(u)$ is an essential tool in analyzing certain types of quantile functions rather than a mere alternative to $\alpha_1(x)$. However, the former has some advantages over the latter as revealed in the discussions below.

It may be noticed that both $\alpha_1(x)$ $(\alpha_2(x))$ and $P_1(u)$ $(P_2(u))$ represent the same quantities with different interpretations. In fact $P_1(u)$ is the average of the X values that exceed the $100(1-u)\%$ points of the distribution of X and $P_2(u)$ is interpreted similarly. However, by virtue of the properties of quantile functions, $P_1(u)$ enjoys some special features in comparison with $\alpha_1(x)$. Let $Q_1(u)$ and $Q_2(u)$ be two quantile functions with partial means $R_1(u)$ and $S_1(u)$. Then $Q_1(u)+Q_2(u)$ is also a quantile function with quantile density function $q_1(u) + q_2(u)$. Accordingly Eq. (2.5) gives the partial mean of $Q(u) = Q_1(u) + Q_2(u)$

$$P_1(u) = R_1(u) + S_1(u).$$

Since the sum of two distribution functions need not be a distribution function, a corresponding additive property is not shared by $\alpha_1(x)$. Further if $Y = g(X)$ is a non decreasing function of X then $g(Q(u))$ is the quantile function of Y. Hence the partial mean of Y is obtained as $P_Y(u) = g(P_X(u))$. In particular under the linear transformation $Y = aX + b$, $a > 0$ we have $P_Y(u) = aP_X(u)$.

The variance of $(X - x)_+$ is

$$\begin{aligned} V_+(u) &= \int_u^1 (Q(p) - Q(u))^2\, dp - P_1^2(u) \\ &= \int_u^1 Q^2(p)dp - (Q(u) - P_1(u))^2 + uQ^2(u). \end{aligned} \tag{2.8}$$

From (2.5) and (2.7) the derivatives of $P_1(u)$ and $V_+(u)$ are given by

$$P_1'(u) = -(1-u)q(u) \tag{2.9}$$

and

$$V_+'(u) = \frac{2uP_1(u)P_1'(u)}{1-u}. \tag{2.10}$$

Thus $P_1(u)$ determines $V_+(u)$ as

$$V_+(u) = -\int_u^1 (2pP_1(p)P_1'(p))/(1-p)dp = \frac{u}{1-u}P_1^2(u) + \int_u^1 \frac{P_1^2(p)dp}{(1-p)^2}. \tag{2.11}$$

From (2.10), we have

$$2P_1(u)P_1'(u) = \frac{(1-u)V_+'(u)}{u},$$

which gives on integration,

$$P_1^2(u) = -\int_u^1 \frac{(1-p)V_+'(p)}{p}dp. \tag{2.12}$$

Table 1
Quantile partial means of distributions.

Distribution	Quantile function	$P_1(u)$
Exponential	$-\lambda^{-1}\log(1-u)$	$\lambda^{-1}(1-u)$
Generalized lambda	$\lambda_1+\lambda_2^{-1}(u^{\lambda_3}-(1-u)^{\lambda_4})$	$\frac{1}{\lambda_2}\left[\frac{\lambda_4}{1+\lambda_4}(1-u)^{1+\lambda_4}+\frac{1-u^{\lambda_3+1}}{1+\lambda_3}-(1-u)u^{\lambda_3}\right]$
Generalized Pareto	$\frac{b}{a}\left[(1-u)^{-\frac{a}{a+1}}-1\right]$	$b(1-u)^{\frac{1}{a+1}}$
van Staden-Loots	$\lambda_1+\lambda_2\left[\frac{1-\lambda_3}{\lambda_4}(u^{\lambda_4}-1)-\frac{\lambda_3}{\lambda_4}((1-u)^{\lambda_4}-1)\right]$	$\lambda_2\left[\frac{1-\lambda_3}{\lambda_4}\left(\frac{1-u^{1+\lambda_4}}{1+\lambda_4}-(1-u)u^{\lambda_4}\right)+\frac{\lambda_3}{1+\lambda_4}(1-u)^{1+\lambda_4}\right]$
Govindarajulu	$\sigma[(\beta+1)u^\beta-\beta u^{\beta+1}]$	$\frac{\sigma}{\beta+2}\left[2-(\beta+1)(\beta+2)u^\beta+2\beta(\beta+2)u^{\beta+1}-\beta(\beta+1)u^{\beta+2}\right]$
Half logistic	$\sigma\log\frac{1+u}{1-u}$	$2\sigma\log\frac{2}{1+u}$
Exponential geometric	$\frac{1}{\lambda}\log\left(\frac{1-pu}{1-u}\right)$	$\frac{1-p}{\lambda p}\log\left(\frac{1-pu}{1-p}\right)$
Linear hazard quantile	$(a+b)^{-1}\log\left(\frac{a+bu}{a(1+u)}\right)$	$\frac{1}{b}\log\left(\frac{a+b}{a+bu}\right)$
Power	$\alpha u^{\frac{1}{\beta}}$	$\alpha\left[1-u^{\frac{1}{\beta}}-(\beta+1)^{-1}\left(1-u^{1+\frac{1}{\beta}}\right)\right]$

This shows that $V_+(u)$ determines $P_1(u)$ also. Expressions of $P_1(u)$ for various distributions that appear in the sequel are presented in Table 1. Detailed proofs of (2.7), (2.8), (2.10) and (2.11) are given in Appendix. Now we discuss properties of $P_1(u)$ and $P_2(u)$ in the context of reliability analysis. As background material, the quantile based definitions of the hazard rate function $h(x) = \frac{f(x)}{\bar{F}(x)}$, mean residual life function $m(x) = \frac{1}{\bar{F}(x)}\int_x^\infty \bar{F}(t)dt$ and the variance residual life function $\sigma^2(x) = \frac{2}{\bar{F}(x)}\int_x^\infty\int_s^\infty \bar{F}(t)dtds - m^2(x)$ are required. By setting $F(x) = u$, the above functions are respectively equivalent to the hazard quantile function

$$H(u) = h(Q(u)) = \frac{1}{(1-u)q(u)} \tag{2.13}$$

the mean residual quantile function

$$M(u) = m(Q(u)) = \frac{1}{(1-u)}\int_u^1 (1-p)q(p)dp \tag{2.14}$$

and the variance residual quantile function

$$\begin{aligned} V(u) &= \sigma^2(Q(u)) = \frac{1}{1-u}\int_u^1 Q^2(p)dp - (M(u)+Q(u))^2 \\ &= \frac{1}{1-u}\int_u^1 M^2(p)dp. \end{aligned} \tag{2.15}$$

For derivations of the formulas (2.13) to (2.15) and various identities connecting them we refer to Nair and Sankaran (2009). It follows from (2.5), (2.9) and (2.15) that

$$H(u) = -\frac{1}{P_1'(u)}, \tag{2.16}$$

$$M(u) = \frac{P_1(u)}{1-u} \tag{2.17}$$

$$\text{and} \quad V(u) = \frac{1}{1-u}\int_u^1 \frac{P_1^2(p)}{(1-p)^2}dp. \tag{2.18}$$

An immediate consequence of the above identities is the potential of $P_1(u)$ to provide alternative definitions of ageing criteria and stochastic orders. These aspects will be discussed in the subsequent sections.

The partial moments can be related to two other types of moments employed in the analysis of quantile function. These are the L-moments defined by

$$L_r = \int_0^1 \sum_{k=0}^{r-1}(-1)^{r-1-k}\binom{r-1}{k}\binom{r-1+k}{k}u^k Q(u)du, \quad r = 1, 2, 3, \ldots$$

and the probability weighted moments

$$\theta_{r,s,t} = \int_0^1 [Q(u)]^r u^s (1-u)^t du.$$

These two sets of moments are mutually related, the relationship for the first four moments being

$$\theta_{1,0,0} = L_1 = E(X) \tag{2.19}$$

$$\theta_{1,1,0} = \frac{1}{2}(L_2 + L_1) \tag{2.20}$$

$$\theta_{1,2,0} = \frac{1}{6}(L_3 + 3L_2 + 2L_1) \tag{2.21}$$

$$\theta_{1,3,0} = \frac{1}{20}(L_4 + 5L_3 + 9L_2 + 5L_1). \tag{2.22}$$

One can also write the L's in terms of θ's from the above. Now from

$$L_1 = \int_0^1 Q(u)du = \int_0^1 (1-u)q(u)du \quad \text{(by integration by parts)}$$

and since $P_1'(u) = (1-u)q(u)$, we obtain

$$L_1 = -\int_0^1 P_1'(u)du = P_1(0).$$

Further

$$\begin{aligned} L_2 &= \int_0^1 (2u-1)Q(u)du = \int_0^1 u(1-u)q(u)du \\ &= -\int_0^1 uP_1'(u)du \\ &= \int_0^1 P_1(u)du. \end{aligned}$$

Similarly

$$\begin{aligned} L_3 &= \int_0^1 (6u^2-6u+1)Q(u)du \\ &= \int_0^1 (4u-1)P_1(u)du \end{aligned}$$

and

$$\begin{aligned} L_4 &= \int_0^1 (20u^3-30u^2+12u-1)Q(u)du \\ &= \int_0^1 (15u^2-10u+1)P_1(u)du. \end{aligned}$$

That the probability weighted moments can also be expressed in terms of $P_1(u)$ now follows from the relationships in (2.19) through (2.22). For a general discussion of probability weighted moments and L-moments, we refer to Hosking and Wallis (1997).

3. Characterizations

In this section we discuss the application of partial moments in characterizing distributions. In the first place it may be noticed from Eqs. (2.9) that the quantile function can be recovered from the functional form of $P_1(u)$ as

$$Q(u) = -\int_0^u \frac{P_1'(p)dp}{1-p}.$$

Hence all the quantile functions given in Table 1 are characterized by the corresponding forms of $P_1(u)$. Further from Eq. (2.7) it is evident that $P_2(u)$ is uniquely determined from $P_1(u)$.

Families of distributions for which variance is a function of the mean is a problem of interest in distribution theory. For example, see Letac and Mora (1990) and Morris (1982) and their references. A somewhat similar problem is addressed in the context of reliability analysis by Gupta and Kirmani (2000) when they seek the functional forms of the coefficient of variation that characterize residual life distribution. In the same way it is possible to characterize a family of distributions by the quantile partial coefficient of variation $C(u)$ defined by

$$g(u) = C^2(u) = \frac{V_+(u)}{P_1^2(u)}. \tag{3.1}$$

Theorem 3.1. *The quantile density function $q(u)$ of X is represented in terms of $g(u)$ as*

$$q(u) = \frac{\mu}{2} \frac{g'(u)}{(1-u)g(u)-u} \exp\left[\frac{1}{2}\int_0^u \frac{(1-p)g'(p)dp}{p-(1-p)g(p)}\right]. \tag{3.2}$$

In this case

$$P_1(u) = \mu \exp\left[\frac{1}{2}\int_0^u \frac{(1-p)g'(p)dp}{p-(1-p)g(p)}\right]. \tag{3.3}$$

Proof. To prove this, we have

$$V_+(u) = g(u)P_1^2(u)$$

which gives

$$V_+'(u) = P_1^2(u)g'(u) + 2P_1(u)P_1'(u)g(u).$$

Also from (2.10)

$$V_+'(u) = \frac{2u}{1-u}P_1(u)P_1'(u).$$

Hence

$$\frac{P_1'(u)}{P_1(u)} = \frac{(1-u)g'(u)}{2[u-(1-u)g(u)]}.$$

Integrating over $(0, u)$, we have (3.3). The expression for $q(u)$ is obtained directly from (2.9). Notice also that (3.3) says that the partial mean is uniquely determined from the coefficient of variation $C(u)$. □

To illustrate Theorem 3.1, we characterize the family of distributions when $g(u)$ has bilinear form.

Theorem 3.2. *The random variable X follows the generalized Pareto distribution*

$$Q(u) = \frac{b}{a}\left[(1-u)^{-\frac{a}{a+1}} - 1\right], \quad a > -1,\ b > 0 \tag{3.4}$$

if and only if

$$g(u) = \frac{u+c}{1-u} \tag{3.5}$$

for some $c > 0$.

Proof. Assuming (3.5), we have

$$g'(u) = \frac{1+c}{(1-u)^2}$$

and

$$\frac{1}{2}\int_0^u \frac{(1-p)g'(p)}{p-(1-p)g(p)}\,dp = \log(1-u)^{\frac{1+c}{2c}}.$$

Hence

$$P_1(u) = \mu(1-u)^{\frac{1+c}{2c}}$$

$$q(u) = \frac{\mu(1+c)}{2c}(1-u)^{\frac{1+c}{2c}-2}$$

and

$$Q(u) = \int_0^u q(p)dp = \frac{\mu}{u}\left[(1-u)^{-\frac{a}{a+1}} - 1\right]$$

with $a = \frac{1-c}{1+c}$. Conversely for the distribution (3.4),

$$P_1(u) = b(1-u)^{\frac{1}{a+1}} \quad \text{and} \quad V_+(u) = b^2\left(u + \frac{1+a}{1-a}\right)(1-u)^{\frac{2}{a+1}-1}$$

verifying (3.5) with $c = \frac{1+a}{1-a}$, which completes the proof. □

Remark 3.1. The generalized Pareto model (3.4) contains the exponential distribution with mean μ as $c \to 1$ ($a \to 0$). When $-1 < a < 0$, we have the rescaled beta distribution with

$$\bar{F}(x) = \left(1 - \frac{x}{R}\right)^{d}, \quad 0 < x < R,\ d > 0$$

where $d = -\frac{2c}{1+c}$ and when $a > 0$, the Pareto distribution

$$\bar{F}(x) = \left(1 - \frac{x}{\beta}\right)^{-d}, \quad x > 0,\ \beta, d > 0$$

with $d = \frac{2c}{1-c}$ results. The uniform distribution is a special case of the rescaled beta when $d = 1$.

Remark 3.2. The relationship

$$P_1(u) = \mu(1-u)^{\frac{1+c}{2c}}$$

characterizes the generalized Pareto family. In particular the linear function

$$P_1(u) = a + bu$$

holds for all u and $b < 0$ if and only if X is exponential. When $b > 0$, there is no distribution since in that case

$$\int_u^1 (1-p)q(p)dp = a + bu$$

leads to

$$q(u) = -\frac{b}{1-u} < 0.$$

The corresponding $Q(u)$ is decreasing and does not qualify as a quantile function.

Theorem 3.3. *The distribution of X is exponential if and only if $V_+(u)$ is a function of the form*

$$V_+(u) = a + bu^2, \quad b < 0. \tag{3.6}$$

Proof. The exponential distribution has

$$V_+(u) = \frac{1-u^2}{\lambda^2}$$

so that (3.6) satisfies with $a = \frac{1}{\lambda^2}$ and $b = -\frac{1}{\lambda^2} < 0$. Conversely (3.6) implies that

$$2uP_1(u)P_1'(u) = 2bu(1-u)$$

giving

$$P_1^2(u) = -2b(1-u)^2$$

so that

$$P_1(u) = c(1-u); \qquad c = (-2b)^{1/2} > 0.$$

Thus $q(u) = c(1-u)^{-1}$, the quantile function of the exponential law with mean c. $\square$

Apart from identifying the exact distribution that corresponds to a desired functional form of $P_1(u)$, these characterizations have another important application in modeling. If $R_1(u)$ and $S_1(u)$ are partial means of $Q_1(u)$ and $Q_2(u)$ satisfying

$$P_1(u) = R_1(u) + S_1(u)$$

then the quantile function corresponding to $P_1(u)$ is $Q(u) = Q_1(u) + Q_2(u)$. This is the converse of the result stated in Section 2, and is useful in constructing new distributions. The simplest form of partial mean (no nondegenerate distribution has a constant partial mean) is the linear form

$$R_1(u) = a + bu, \quad b < 0$$

of the exponential distribution. Suppose that Y is random variable with uniform distribution $F(x) = \frac{x}{c}$, $0 \le c \le c$, so that the quantile function is $Q(u) = cu$. Then Y has partial mean

$$S_1(u) = \frac{c(1-u)^2}{2}.$$

Accordingly

$$P_1(u) = \frac{c(1-u)^2}{2} + a + bu$$

has quantile function

$$Q(u) = cu + b\log(1-u), \quad c > 0, b > 0,$$

with support as $(0, \infty)$ and partial mean as a quadratic function. Note that in this case an analytic form for $F(x)$ is not available. This process can be applied to any two quantile functions in Table 1, thus generating partial means with a wide variety of shapes. There is always scope for finding reasonable approximation to a desired functional form of $P_1(u)$ and the corresponding distribution. Such a methodology is not shared by the distribution function approach.

4. Ageing properties

As mentioned earlier, $P_1(u)$ can be employed to provide alternative definitions of various ageing criteria used in reliability theory. One of the advantages is that all the ageing concepts can be expressed in terms of $P_1(u)$ instead of having to use different functions $H(u)$, $M(u)$ and $V(u)$ as in the conventional definitions. The following results are proposed, based on the definitions of ageing concepts in terms of quantile functions discussed in Nair, Sankaran, and Vineshkumar (2008) and Nair and Vineshkumar (2011). We say that X has increasing (decreasing) failure rate, IFR (DFR) if $H(u)$ is increasing (decreasing) in u.

Proposition 4.1. *X is IFR (DFR) if and only if $P_1(u)$ is convex (concave). Further, the hazard quantile function is bathtub (upside down bathtub) shaped with a single change point u_0 if $P_1(u)$ is concave (convex) in $(0, u_0)$ and convex (concave) in $(u_0, 1)$.*

Proof follows from (2.16).

Proposition 4.2. *X is IFRA $\Leftrightarrow P_1'(u) \leq \frac{Q(u)}{\log(1-u)}$.*

Proof. X is IFRA $\Leftrightarrow -\frac{(1-u)}{Q(u)}$ is increasing

$$\begin{aligned}
&\Leftrightarrow Q(u) + \log(1-u)((1-u)q(u)) \geq 0\\
&\Leftrightarrow Q(u) - P_1'(u)\log(1-u) \geq 0\\
&\Leftrightarrow P_1'(u) \leq \frac{Q(u)}{\log(1-u)}.
\end{aligned}$$

The lifetime X has decreasing mean residual life (DMRL) if $M(u)$ is decreasing in u. $\square$

Proposition 4.3. *X is DMRL $\Leftrightarrow \frac{P_1(u)}{1-u}$ is decreasing.*

The proof follows from (2.1).

Proposition 4.4. *X is NBU (new better than used) if and only if*

$$\int_v^{u+v-uv} \frac{P_1'(p)}{1-p}dp \geq \int_0^u \frac{P_1'(p)}{1-p}dp.$$

Proof. X is NBU $\Leftrightarrow Q(u+v-uv) \leq Q(u) + Q(v) 0 \leq u, v < 1$.

$$\begin{aligned}
&\Leftrightarrow \int_v^{u+v-uv} q(p)dp \leq \int_0^u q(p)dp\\
&\Leftrightarrow \int_v^{u+v-uv} \frac{P_1'(p)}{1-p}dp \geq \int_0^u \frac{P_1'(p)}{1-p}dp. \quad \square
\end{aligned}$$

Proposition 4.5. *X is NBUE (new better than used in expectation) $\Leftrightarrow P_1(u) \leq M(1-u)$.*

This follows from the fact that X is NBUE $\Leftrightarrow M(u) \leq M(0)$ and (2.17).

Proposition 4.6. *X is HNBUE (harmonically new better than used in expectation) $\Leftrightarrow S(u) \leq e^{-\frac{Q(u)}{\mu}}$ where $S(u) = \frac{P_1(u)}{\mu}$ is the scaled quantile partial mean.*

Proof.

$$\begin{aligned} X \text{ is HNBUE } &\Leftrightarrow \int_u^1 (1-p)q(p)dp \le \mu \exp\left[-\frac{Q(u)}{\mu}\right] \\ &\Leftrightarrow \mu - \int_0^u (1-p)q(p)dp \le \mu \exp\left[-\frac{Q(u)}{\mu}\right] \\ &\Leftrightarrow \mu - \int_0^u P_1'(p)dp \le \mu \exp\left[-\frac{Q(u)}{\mu}\right] \\ &\Leftrightarrow S(u) \le \exp\left[-\frac{Q(u)}{\mu}\right]. \quad \square \end{aligned}$$

Definition 4.1. We say that X is new better than used in hazard rate (NBUHR) if $h(0) \le h(x)$ or $H(0) \le H(u)$.

Proposition 4.7. *X is NBUHR* $\Leftrightarrow P_1'(0) \le P_1'(u)$.

Definition 4.2. We say that X is new better than used in hazard rate average (NBUHR) if $h(0) \le \frac{1}{\alpha}\int_0^x h(t)dt$.

Proposition 4.8. *X is NBUHRA* $\Leftrightarrow P_1'(0) \le \frac{Q(0)}{\log(1-u)}$.

Definition 4.3. The random variable X is said to have IFRA$*t_0$ if $\bar{F}(t_0) \ge \bar{F}^b(x)$ for all $x \ge t_0 > 0$ and $\frac{t_0}{x} \le b < 1$.

Proposition 4.9. *X is IHRA* $*t_0 \Leftrightarrow$

$$-\int_0^u \frac{P_1'(p)}{1-p}dp \ge \frac{Q(u_0)\log(1-u)}{\log(1-u_0)}; \quad 0 \le u_0 < 1 \textit{ and } t_0 = Q(u_0).$$

Definition 4.4. X belongs to decreasing mean residual harmonic average (DMRLHA) if $\left[\frac{1}{x}\int_0^x \frac{dt}{mt}\right]^{-1}$ is decreasing in x.

Proposition 4.10. *X is DMRLH A* $\Leftrightarrow -\log P_1(u) \le (1-u)Q(u)$.

Proof. X is DMRLHA

$$\begin{aligned} &\Leftrightarrow \frac{1}{Q(u)}\int_0^u \frac{q(p)dp}{M(p)} \text{ is increasing in } u. \\ &\Leftrightarrow \frac{1}{Q(u)}\int_0^u \frac{(1-p)q(p)dp}{\int_p^1 (1-s)q(s)ds} \text{ is increasing} \\ &\Leftrightarrow \frac{1}{Q(u)}\int_0^u -\frac{d}{dp}(\log \int_p^1 (1-s)q(s)ds)dp \text{ is increasing} \\ &\Leftrightarrow \frac{1}{Q(u)}\int_0^u -\frac{d}{dp}\log P_1(p)dp \text{ is increasing} \\ &\Leftrightarrow \frac{1}{Q(u)}P_1(u) \text{ is increasing.} \\ &\Leftrightarrow \frac{d}{du}\frac{\log P_1(u)}{Q(u)} \le 0 \\ &\Leftrightarrow -\log P_1(u) \le (1-u)Q(u). \quad \square \end{aligned}$$

Definition 4.5. X is said to be used better than aged (UBA) if and only if $\bar{F}(x+t) \ge \bar{F}(t)\exp\left[-\frac{x}{m(\infty)}\right]$, $x \ge 0, t \ge 0$ and $m(\infty) < \infty$.

Proposition 4.11.

$$X \textit{ is UBA } \Leftrightarrow \int_v^{v+(1-v)u} \frac{P_1'(p)}{1-p}dp \le -P_1'(1)\log(1-p).$$

Proof. $m(\infty) < \infty$ implies $M(1) < \infty$. Now

$$\begin{aligned} M(1) &= \lim_{u\to 1} M(u) = \lim_{u\to 1} \frac{\int_u^1 (1-p)q(p)dp}{1-u} \\ &= \lim_{u\to 1} -\frac{(1-u)q(u)}{-1} = \lim_{u\to 1} \frac{1}{1+(u)} = \lim_{u\to 1} -P_1'(u) \\ &= -P_1'(1). \end{aligned}$$

Noting that $\frac{\bar{F}(x+t)}{\bar{F}(t)}$ has quantile function $Q(v+u-uv)-Q(v)$ where $u = F(x)$ and $v = F(t)$ and $\exp\left(-\frac{x}{m(\infty)}\right)$ has quantile function $-M(1)\log(1-u)$. Hence,

$$\begin{aligned} X \text{ is UBA } &\Leftrightarrow Q(u+v-uv) - Q(v) \geq -M(1)\log(1-u) \quad \text{for all } 0 \leq u, v < 1. \\ &\Leftrightarrow \int_v^{v+u-uv} q(p)dp \geq P_1'(1)\log(1-u) \\ &\Leftrightarrow \int_v^{v+u-uv} \frac{P_1'(p)dp}{1-p} \leq -P_1'(1)\log(1-u). \quad \square \end{aligned}$$

Definition 4.6. We say that X is used better than aged in expectation (UBAE) if $m(x) \geq m(\infty)$.

Proposition 4.12. *X is UBA $\Leftrightarrow P_1(u) \geq -P_1'(1)(1-u)$.*

The proof follows from the fact that

X is UBAE $\Leftrightarrow M(u) \geq M(1)$.

Definition 4.7. X is said to have decreasing variance residual life (DVRL) if and only if $\sigma^2(x)$ is decreasing in x.

Proposition 4.13. *X is DVRL $\Leftrightarrow g(u) \leq \frac{1+u}{1-u}$.*

Proof.

$$\begin{aligned} X \text{ is DVRL } &\Leftrightarrow V(u) = \frac{1}{1-u}\int_u^1 m^d(p)dp \text{ is decreasing} \\ &\Leftrightarrow \frac{1}{1-u}\int_u^1 \frac{P_1^2(p)}{(1-p)^2}dp \text{ is decreasing} \\ &\Leftrightarrow \int_u^1 \frac{P_1^2(p)}{(1-p)^2}dp \leq \frac{P_1^2(u)}{1-u} \\ &\Leftrightarrow V_+(u) - \frac{u}{1-u}P_1^2(u) \leq \frac{P_1^2(u)}{1-u} \\ &\Leftrightarrow g(u) \leq \frac{1+u}{1-u}. \quad \square \end{aligned}$$

Definition 4.8. The random variable X has NBU-t_0 if and only if $\bar{F}(x+t_0) \leq \bar{F}(x)\bar{F}(t_0)$ for all x and some t_0.

Proposition 4.14. *X is*

$$\text{NBU-}t_0 \Leftrightarrow \int_{v_0}^{u+v_0-uv_0} \frac{P_1'(p)dp}{1-p} \geq \int_0^u \frac{P_1'(p)dp}{1-p}$$

for some $0 < v_0 < 1$.

Definition 4.9. X is NBU * t_0 if and only if

$$\bar{F}(x+y) \leq \bar{F}(x)\bar{F}(y)$$

for all $x > 0$ and $y \geq t_0$.

Proposition 4.15. *X is NBU $* t_0 \Leftrightarrow$*

$$\int_v^{u+v-uv} \frac{P_1'(p)dp}{1-p} \geq \int_0^u \frac{P_1'(p)}{1-p}dp$$

for all u and $v \geq v_0$.

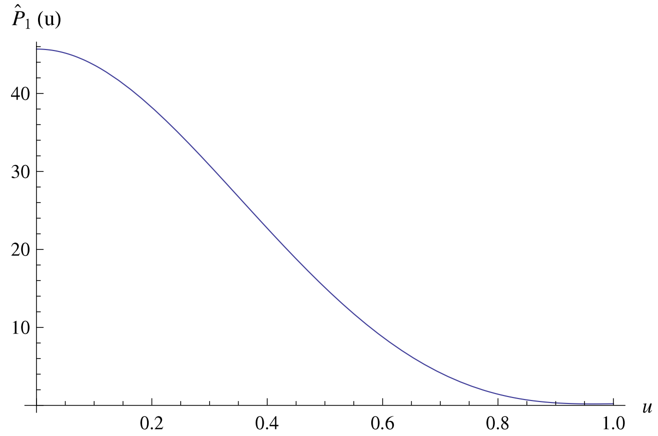

Fig. 1. Plot of $\hat{P}_1(u)$.

5. Application to real data

In this section we demonstrate, with the aid of a real data set, the application of the partial moment in the context of reliability analysis. The Arset data on the failure times of the 50 devices reported in Lai and Xie (2006) is utilized for the purpose; by first examining the Govindarajulu distribution specified by the quantile function in Table 1 as a model. To estimate the parameters of the model we use the method of L-moments by equating the first two L-moments

$$L_1 = \frac{2\sigma}{\beta + 2}$$

and

$$L_2 = \frac{2\sigma\beta}{(\beta + 2)(\beta + 3)}$$

with the sample L-moments

$$\ell_1 = \binom{n}{1}^{-1} \sum_{i=1}^{n} x_{(i)} \tag{5.1}$$

$$\ell_2 = \frac{1}{2}\binom{n}{2}^{-1} \sum_{i=1}^{n} \left\{ \binom{i-1}{1} - \binom{n-i}{1} \right\} x_{(i)} \tag{5.2}$$

where $x_{(i)}$ is the ith order statistic and solve for σ and β from the resulting equations. This gives the estimates

$$\hat{\sigma} = 93.463 \quad \text{and} \quad \hat{\beta} = 2.0915.$$

Dividing the data into 5 groups of 10 observations the goodness of fit test provides a chi-square value of 1.8, which does not reject the model. The estimate of the first partial moment from Table 1 is written as

$$\hat{P}_1(u) = 45.6864 - 288.9407u^{2.0915} + 390.9545u^{3.0915} - 147.4940u^{4.0915} \tag{5.3}$$

A plot of $\hat{P}_1(u)$ can be seen in Fig. 1. A simple differentiation of (5.3) and Proposition 4.1, enables us to find that $P_1(u)$ is concave in (0,0.353) and convex in (0.353,1) so that the hazard quantile function is bathtub-shaped. Similarly, from Proposition 4.3, the mean residual quantile function of the model has a change point if

$$\frac{d}{du}\frac{P_1(u)}{1-u} = \frac{(1-u)P_1'(u) + P_1(u)}{(1-u)^2} = 0. \tag{5.4}$$

Using (5.3), the condition (5.4) simplifies to

$$45.684 - 604.3195u^{1.0915} + 1524.0416u^{2.0915} - 1421.4543u^{3.0915} + 455.9784u^{4.0915} = 0.$$

Solving we find the zero of (5.4) as

$$u_0 = 0.12768$$

so that the mean residual quantile function is initially increasing and then decreasing in u. A fair picture of the ageing properties of the devices are revealed from the analysis in terms of the partial mean $P_1(u)$.

6. Discussion

There are situations in lifetime data analysis to compare the characteristic of two lifetime models. For example, if two manufacturers produce devices that serve the same purpose, the natural interest is to know which is more reliable. Stochastic orders enable us to globally compare two lifetime distributions in terms of their characteristics.

Recall that when X and Y are nonnegative random variables with finite expectations. We say that (Shaked & Shanthikumar, 2007)

(i) X is smaller than Y in increasing convex order, $X \leq_{icx} Y$, if and only if,

$$\int_x^\infty \bar{F}_X(t)dt \leq \int_x^1 \bar{F}_Y(t)dt \quad \text{for all } t$$

or equivalently

$$\int_u^1 Q_X(p)dp \leq \int_x^1 Q_Y(p)dp \tag{6.1}$$

and

(ii) X is smaller than Y in excess wealth order $X \leq_{ew} Y$ if and only if,

$$\int_{Q_X(u)}^\infty \bar{F}_X(t)dt \leq \int_{Q_Y(u)}^\infty \bar{F}_Y(t)dt$$

or in terms of quantile functions

$$\begin{aligned} &\int_u^1 [Q_X(p) - Q_X(u)]dp \leq \int_u^1 [Q_Y(p) - Q_Y(u)]dp \\ &\Leftrightarrow \int_u^1 Q_X(p)dp - (1-u)Q_X(u) \leq \int_u^1 Q_Y(p)dp - (1-u)Q_Y(u). \end{aligned} \tag{6.2}$$

It is worthwhile to observe that in the distribution function approach if $\alpha(x)$ and $\beta(x)$ are the partial mean of X and Y, the dominance of Y over X is denoted by $X \leq_P Y$ and defined as $\alpha(x) \leq \beta(x)$ for all x. Hence

$$X \leq_P Y \Leftrightarrow X \leq_{icx} Y.$$

On the other hand in the quantile formulation (6.2) we define a partial order P_1, $X \leq_{P_1} Y$ as

$$\int_u^1 (1-p)q_X(p)dp \leq \int_u^1 (1-p)q_Y(p)dp. \tag{6.3}$$

On integration by parts, (6.3) becomes identical to (6.2). Hence in this case

$$X \leq_{P_1} Y \Leftrightarrow X \leq_{ew} Y.$$

From Corollary 4.A.32 in Shaked and Shanthikumar (2007), the order $\leq_{P_1}$ implies $\leq_P$ if the lower end of the support of X is smaller than that of Y. The same reference also discussed elaborately the properties of $\leq_{icx}$ and $\leq_{ew}$ and its relationships with other orders are useful in reliability.

The quantile functions and characteristics based on them are extensively used in financial mathematics and financial risk management. When X represents the potential loss or risk a firm incurs in a business policy, the maximum possible loss which does not exceed with a high probability α (called the confidence level) is given by the value at risk function given by

$$\text{Va}\, R_\alpha = Q_X(\alpha),$$

the quantile function. On the other hand the expected loss incurred by the firm when the loss exceeds the value at risk is

$$\begin{aligned} C\,\text{Va}\, R_\alpha &= E(X|X > \text{Va}\, R_\alpha) \\ &= Q(\alpha) + \frac{1}{1-\alpha}P_1(\alpha) \end{aligned}$$

which is in terms of the partial mean. In addition, $P_1(u)$ is related to the well known Lorenz curve, Gini index, Bonferroni curve used in economics and to the Leimkuhler curve and the bibliometric analysis. These and other field of applications of partial moments will be reported in a separate work.

Appendix

Proof of (2.7).

$$\begin{aligned} P_2(u) &= \int_u^1 (Q(p)-Q(u))^2 dp \\ &= \int_u^1 Q^2(p)dp - 2Q(u)\int_u^1 Q(p)dp + (1-u)Q^2(u). \end{aligned}$$

Differentiating the above,

$$\begin{aligned} P_2'(u) &= -Q^2(u) - [2Q(u)(-Q(u)) + 2q(u)\int_u^1 Q(p)dp] + 2(1-u)Q(u)q(u) - Q^2(u) \\ &= -2q(u)\int_u^1 Q(p)dp + 2(1-u)Q(u)q(u) \\ &= -2q(u)[P_1(u) + (1-u)Q(u)] + 2(1-u)Q(u)q(u) \\ &= -2q(u)P_1(u) = \frac{2P_1(u)P_1'(u)}{1-u}, \text{ using (2.9).} \quad \square \end{aligned} \tag{A.1}$$

Proof of (2.8).

$$\begin{aligned} V_1(u) &= \int_u^1 [Q(p)-Q(u)]^2 dp - P_1^2(u) \\ &= \int_u^1 Q^2(p)dp - 2Q(u)\int_u^1 Q(p)dp + (1-u)Q^2(u) - P_1^2(u) \\ &= \int_u^1 Q^2(p)dp - 2Q(u)[P_1(u) + (1-u)Q(u)] + (1-u)Q^2(u) - P_1^2(u) \\ &= \int_u^1 Q^2(p)dp + uQ^2(u) - (Q(u)+P_1(u))^2. \quad \square \end{aligned}$$

Proof of (2.10).

$$V_+(u) = P_2(u) - P_1^2(u)$$

$$\begin{aligned} V_+'(u) &= P_2'(u) - 2P_1P_1'(u) \\ &= \frac{2uP_1(u)P_1'(u)}{1-u}, \text{ using (A.1).} \quad \square \end{aligned} \tag{A.2}$$

Proof of (2.11). From (A.2)

$$\begin{aligned} V(u) &= -\int_u^1 \frac{2pP_1(p)P_1'(p)}{1-p} dp \\ &= -\int_u^1 \frac{p}{1-p}\frac{d}{dp}(P_1^2(p))dp \end{aligned}$$

(2.11) is obtained by integrating by parts the right side of the above. $\square$

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